Bank Leverage and Social Welfare, Online Technical Appendix[*]

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Contents

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1. Introduction

The first two sections below describe the solution to the model analyzed in our papers and proceedings paper. In section 2, we apply the Lagrangian approach which in effect assumes an interior solution. In section 3 we develop an approach that works even if the solution is on the boundary. This is important in an environment like this, because there can be discontinuities on the boundary with the possible consequence that a point which satisfies sufficient conditions for local optimality in fact are not globally optimal. The approach we develop converts the banker problem into a maximization problem in one variable alone, after substituting out the other variables using the constraints. The calculations reported in Figure 1 are based on this latter approach. Section 4 discusses the computation of the equilibrium with leverage. The MATLAB software to reproduce Figure 1 is called run_scaled.m, and appears in code_P&P.zip.

Section 5 considers the version of the model without scaling effort cost by $N + d$. We show the Francesco Ferrante and Andrea Prestipino (private communication) results that the equilibrium in that case is not interesting. Depending on parameter values, either there is no deposit contract equilibrium (i.e., the only equilibrium has $d = 0$) or there is one, but it involves $R = R^g$ and $p(e) = 1$. This analysis uses our piecewise linear specification of $p(e)$ and quadratic form for the cost of e. In section 6 we consider a general class of such functions and conclude that these do not offer a way out.

In section 7 we consider an example in which the banker experiences diminishing returns to its investments. In this case, we identify model parameterizations in which we identify a candidate equilibrium in which e and d satisfy sufficient conditions to be a global maximum for the banker's problem. However, the cases we examined were not deposit contract equilibria because the banker's outside option dominated the value of the deposit contract. We searched without success for parameterizations with the property that taking a bank deposit dominates the banker's outside option. The code for this exercise is diminish.m, also contained in code_P&P.zip.

2. Solving the Model Using Lagrangian Methods

Here, we solve the model by representing the banker problem in Lagrangian form. By representing all the equilibrium conditions (including the incentive constraint) as equality constraints, the approach in effect focuses on interior equilibria.

Because effort is not observable, whatever terms the banker receives he will ex post always exert effort to maximize his criterion, subject to the given values of R_d^g, R_d^b and d. That is,

ex post, the effort level set by the banker that takes deposits is:

$$e\left(N+d\right)=\beta p'\left(e\right)\left[\left(R^g-R^b\right)\left(N+d\right)-\left(R^g_d-R^b_d\right)d\right].$$

Because the mutual fund understands that this is how banker effort will be set, the above equation is included among the conditions that characterize the loan contract. The equation is referred to as the incentive constraint. The surplus received by the mutual fund, S^{mf}, is its profits:

$$S^{mf}=p\left(e\right)R^g_d+\left(1-p\left(e\right)\right)R^b_d-R.$$

We assume that the banker makes a take-it-or-leave-it offer to the mutual fund, which drives the mutual fund $S^{mf}=0$. This is the mutual fund's zero profit condition. It is convenient to write this constraint in the following form:

$$R^g_d-R^b_d=\frac{R-R^b_d}{p\left(e\right)}.$$

We assume that the cash constraint in the bad state is binding. The Lagrangian representation of the problem is:

$$\max_{e,d,R^b_d}\beta\left\{p\left(e\right)R^g\left(N+d\right)+\left(1-p\left(e\right)\right)R^b\left(N+d\right)-Rd\right\}-\frac{1}{2}e^2\left(N+d\right)$$

$$+\eta\left\{e\left(N+d\right)^\sigma-\beta p'\left(e\right)\left[\left(R^g-R^b\right)\left(N+d\right)-\frac{R-R^b_d}{p\left(e\right)}d\right]\right\}+v\left[R^b\left(N+d\right)-R^b_d d\right]$$

Here, we have substituted out the mutual fund's zero profit condition in the objective and in the incentive constraint. The first order conditions are:

$$e:\beta p'\left(e\right)\left(R^g-R^b\right)\left(N+d\right)-e\left(N+d\right)+\eta\left\{\left(N+d\right)-\beta\left(\frac{p'\left(e\right)}{p\left(e\right)}\right)^2\left(R-R^b_d\right)d\right\}=0$$

$$d:\beta\left\{p\left(e\right)R^g+\left(1-p\left(e\right)\right)R^b-R\right\}-\frac{1}{2}e^2$$

$$+\eta\left\{e-\beta p'\left(e\right)\left[\left(R^g-R^b\right)-\frac{R-R^b_d}{p\left(e\right)}\right]\right\}+v\left(R^b-R^b_d\right)\quad=\quad0$$

$$R^b_d:-\eta\beta\frac{p'\left(e\right)}{p\left(e\right)}d-vd=0\ \left(d>0\right)$$

$$\eta:e\left(N+d\right)=\beta p'\left(e\right)\left[\left(R^g-R^b\right)\left(N+d\right)-\frac{R-R^b_d}{p\left(e\right)}d\right]$$

$$v:R^b\left(N+d\right)-R^b_d d$$

Substituting the R^b_d condition into the d condition:

$$0\quad=\quad\beta\left\{p\left(e\right)R^g+\left(1-p\left(e\right)\right)R^b-R\right\}-\frac{1}{2}e^2$$

$$+\eta\left\{e-\beta p'\left(e\right)\left[\left(R^g-R^b\right)-\frac{R-R^b_d}{p\left(e\right)}\right]\right\}-\eta\beta\frac{p'\left(e\right)}{p\left(e\right)}\left(R^b-R^b_d\right)$$

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or, after rearranging,

$$d : \beta \left(1 - \eta \frac{p'(e)}{p(e)} \right) \left[p(e) R^g + (1 - p(e)) R^b - R \right] = e \left(\frac{1}{2} e - \eta \right).$$

Then, the 5 equations in the following 5 unknowns, e, d, R_d^b, η, v, are

$$e \quad : \quad \beta p'(e) \left(R^g - R^b \right) (N + d) - e(N + d) + \eta \left\{ (N + d) - \beta \left(\frac{p'(e)}{p(e)} \right)^2 \left(R - R_d^b \right) d \right\} = 0$$

$$d \quad : \quad \beta \left(1 - \eta \frac{p'(e)}{p(e)} \right) \left[p(e) R^g + (1 - p(e)) R^b - R \right] = e \left(\frac{1}{2} e - \eta \right)$$

$$R_d^b \quad : \quad v = -\eta \beta \frac{p'(e)}{p(e)}$$

$$\eta \quad : \quad e(N + d) = \beta p'(e) \left[\left(R^g - R^b \right) (N + d) - \frac{R - R_d^b}{p(e)} d \right]$$

$$v \quad : \quad R^b (N + d) - R_d^b d$$

The households also provide an equilibrium condition in the form of savings supply:

$$R = \frac{u'(Y - d)}{\beta} = \frac{1}{\beta(Y - d)}.$$

Here, Y is the first period endowment of the household.

To compute the equilibrium for the model, consider first the possibility that the two constraints are slack, so that $v = \eta = 0$ and the problem is (taking into account the zero profit condition of the mutual funds):

$$\max_{e, d, R_d^b} \beta \left\{ p(e) R^g (N + d) + (1 - p(e)) R^b (N + d) - Rd \right\} - \frac{1}{2} e^2 (N + d).$$

Optimality for e and d, respectively, implies:

$$(a) \quad : \quad e = \beta \bar{b} \left(R^g - R^b \right)$$

$$(b) \quad : \quad \beta \left\{ p(e) R^g + (1 - p(e)) R^b - R \right\} = \frac{1}{2} e^2$$

Then, d is obtained by solving (b) with the equilibrium condition on R:

$$d = Y - \frac{1}{\beta \left(p(e) R^g + (1 - p(e)) R^b \right) - \frac{1}{2} e^2}.$$

If the bad-state cash constraint is violated, then solve the constrained version of the model with $v > 0$ (and therefore also $\eta < 0$, by equation R_d^b). To solve the constrained version of the model, set $p \in [\bar{a}, 1]$ and compute $e = (p - \bar{a}) / \bar{b}$. The η equation can be solved for d:

$$e(N + d) = \beta \bar{b} \left[\left(R^g - R^b \right) (N + d) - \frac{\frac{d}{\beta(Y - d)} - (N + d) R^b}{p} \right], \tag{2.1}$$

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where use has been made of the equilibrium condition, $R = u'(Y - d)/\beta$ and the assumption that the cash constraint in the bad state is binding. Then, solve the v equation for R_d^b:

$$R_d^b = \frac{N + d}{d} R^b.$$

Use the e equation to solve for η

$$\eta = \frac{e(N + d) - \beta p'(e)(R^g - R^b)(N + d)}{(N + d) - \beta \left(\frac{p'(e)}{p(e)}\right)^2 (R - R_d^b) d}. \tag{2.2}$$

All the terms in the d equation can be evaluated. Adjust p until that equation is satisfied.

If the model has been solved in which there is a loan contract with $d > 0$ then it is necessary to verify that this contract dominates the bank's outside option, in which it sets $d = 0$. In this case,

$$e^* = \beta \bar{b}(R^g - R^b) \tag{2.3}$$

$$p^* = \bar{a} + \bar{b}e^* \tag{2.4}$$

$$u^* = \beta(p^* R^g + (1 - p^*) R^b) N - \frac{1}{2}(e^*)^2 N. \tag{2.5}$$

It is necessary to verify that

$$u \geq u^*,$$

where

$$\begin{aligned} u &= \beta \{p(e) R^g(N + d) + (1 - p(e)) R^b(N + d) - Rd\} - \frac{1}{2}e^2(N + d) \\ &= \beta[p(e) R^g + (1 - p(e)) R^b] N \\ &\quad + \beta[p(e) R^g + (1 - p(e)) R^b - R] d - \frac{1}{2}e^2(N + d) \end{aligned}$$

Finally, we need to verify the second order conditions for the banker's problem. This involves the determinant of a particular bordered Hessian. To define the bordered Hessian, let

$$\begin{aligned} V(e, d, R_d^b) &= \beta\left(p(e)R^g(N + d) + (1 - p(e)) R^b(N + d) - Rd\right) - \frac{1}{2}e^2(N + d) \\ &\quad + \eta\left(e(N + d)^\sigma - \beta\bar{b}\left[(R^g - R^b)(N + d) - \frac{R - R_d^b}{p(e)}d\right]\right) \\ &\quad + v\left(R^b(N + d) - R_d^b d\right). \end{aligned}$$

and

$$\begin{aligned} g(e, d, R_d^b) &= e(N + d) - \beta\bar{b}\left[(R^g - R^b)(N + d) - \frac{R - R_d^b}{p(e)}d\right] \\ h(e, d, R_d^b) &= R^b(N + d) - R_d^b d. \end{aligned}$$

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The bordered Hessian is:

$$
\begin{bmatrix}
V_{ee} & V_{ed} & V_{eR_d^b} & g_e & h_e \\
V_{ed} & V_{dd} & V_{dR_d^b} & g_d & h_d \\
V_{eR_d^b} & V_{dR_d^b} & V_{ee} & g_{R_d^b} & h_{R_d^b} \\
g_e & g_d & g_{R_d^b} & 0 & 0 \\
h_e & h_d & h_{R_d^b} & 0 & 0
\end{bmatrix}.
$$

The solution to the first order conditions is a local maximum if the determinant of the bordered Hessian matrix is negative (see the appendix in Henderson and Quandt (1971) or Mas Colell, Whinston and Green (1995)). In the numerical example featured in the paper, this determinant is negative.

3. Solving the Banker Problem by Substituting Out the Constraints

In the approach here, we allow for the possibility of a corner solution where the banker's incentive constraint is not represented as an equality. The approach we take is to substitute out the constraints in the problem, including the incentive constraint. The computation of the equilibrium mainly focuses on the banker problem, since the other equilibrium condition is just the household intertemporal condition associated with the deposit decision.

Bankers have three options: (a) they may take a deposit contract from a mutual fund and invest what they have in assets; (b) they could forego a deposit contract and simply invest their net worth in assets (the 'outside option'); or (c) they could take their net worth and deposit it at the interest rate, R, in a mutual fund. Although we allow all three choices, we focus on equilibria in which bankers choose (a). We first describe the banker's outside option, (b). We then consider (a). The value of option (c) is simply, βNR.

3.1. The Outside Option

A banker that takes no deposits and sets e to e^*, where

$$
e^* = \beta \bar{b} \left(R^g - R^b \right).
$$

Then, the value of the outside option is:

$$
V^* = \left(\beta \left[p^* R^g + (1 - p^*) R^b \right] - \frac{1}{2} (e^*)^2 \right) N.
$$

An individual banker in any equilibrium in which bankers take deposit contracts must prefer that to the outside option.

3.2. The Banker's Deposit Contract Problem

We now investigate the actions of a banker conditional on taking a deposit contract from a mutual fund. To select a loan contract, a banker solves:

$$V = \max_{e,d,R_d^b,R_d^g} \beta \left\{ p\left(e\right) \left[R^g\left(N+d\right) - R_d^g d\right] + \left(1 - p\left(e\right)\right) \left[R^b\left(N+d\right) - R_d^b d\right] \right\} - \frac{1}{2}e^2\left(N+d\right),$$
(3.1)

subject to R given, $d \geq 0$ and

'incentive': $e(N+d) - \beta\bar{b}\left[\left(R^g - R^b\right)\left(N+d\right) - \left(R_d^g - R_d^b\right)d\right] \begin{cases} = 0 & 0 < e < \left(1 - \bar{a}\right)/\bar{b} \\ \leq 0 & e = \left(1 - \bar{a}\right)/\bar{b} \\ \geq 0 & e = 0 \end{cases}$

'zero profit': $R = p\left(e\right)R_d^g + \left(1 - p\left(e\right)\right)R_d^b$

'good-state cash constraint': $R^g\left(N+d\right) \geq R_d^g d,$

'bad-state cash constraint': $R^b\left(N+d\right) \geq R_d^b d.$

3.2.1. Non binding Cash and Incentive Constraints

It is instructive to begin with the assumption that the incentive and cash constraints are non-binding, while the zero-profit condition is binding. The zero-profit is obviously always binding, for otherwise the solution is $R_d^b = R_d^g = -\infty$. Substituting the zero profit condition into (3.1), we obtain

$$\max_{e,d} \beta \left\{ p\left(e\right)R^g\left(N+d\right) + \left(1 - p\left(e\right)\right)R^b\left(N+d\right) - Rd \right\} - \frac{1}{2}e^2\left(N+d\right).$$

Interior optimality of e and d imply, respectively,

$$e = \beta p'\left(e\right)\left(R^g - R^b\right)$$
$$0 = \beta\left\{p\left(e\right)R^g + \left(1 - p\left(e\right)\right)R^b - R\right\} - \frac{1}{2}e^2.$$

The first expression states that the marginal cost of effort must equal the benefit of shifting probability in the direction of the good outcome. The second expression says that the marginal benefit of extra deposits, in terms of extra profits, must equal the marginal cost. Substituting this second expression into banker's objective, we see that it equals βRN, which is what a banker can obtain by depositing his net worth into another bank, making no effort and earning the return on deposits.

We compute the equilibrium by solving for e using the first of the two optimality conditions. We can then solve for R from the second optimality condition. In effect, the demand for loans is infinitely elastic at the interest rate,

$$R = p\left(e\right)R^g + \left(1 - p\left(e\right)\right)R^b + \frac{1}{2\beta}e^2.$$

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For a higher R, demand for d is zero. For a lower R, demand is infinite. So, the equilibrium quantity of deposits must be where this horizontal demand intersects with the upward-sloped supply:

$$d = Y - \frac{1}{\beta R}.$$

Next, we verify whether it was correct to ignore the cash and incentive constraints. Because we ignored the incentive constraint in solving the problem, the only way that the e decision can be reconciled with 'incentive' is for deposit returns to be state-independent:

$$R_d^g = R_d^b = R.$$

Here, the last equality reflects the zero profit condition. This ways of the setting the deposit returns across states is consistent with the cash constraints if

$$(N + d) R^g \geq dR, \quad (N + d) R^b \geq dR.$$

If this is the case, then the banker problem is solved.

3.2.2. Binding Bad-state Cash and Incentive Constraints

3.2.2.1.Overview If N is sufficiently low, then the solution in the previous subsection can lead to a violation of the cash constraint in the bad state. In this case, solving the banker problem requires setting $R_d^g > R_d^b$. But, this in turn has an impact on e via 'incentive' and so it is necessary to incorporate 'incentive' explicitly into the solution of the banker problem.[1] In this way, we are led to incorporate three constrains: (i) the zero profit condition of mutual funds ('zero profit'), (ii) 'incentive' and (iii) the assumption that the bad-state cash constraint binds:

$$R^b (N + d) = dR_d^b.$$

We use our three constraints, (i), (ii) and (iii), to substitute out three of the four decision variables in the banker problem, (3.1). As a consequence, we reduce the original four dimensional maximization problem in e, d, R_d^g, R_d^b to a one dimensional maximization problem in e alone. Substituting out the incentive constraint is somewhat tricky because it is characterized by inequalities.

To proceed, we find it convenient to express the banker's objective, (3.1), in terms of leverage, $L = (N + d)/N$, instead of d. Making use of the zero profit condition, the objective is:

$$V(e, L; R) = \left\{ \beta \left[p(e) R^g + (1 - p(e)) R^b - R \right] - \frac{1}{2}e^2 \right\} NL + \beta RN. \qquad (3.2)$$

---

[1] This is the intuition underlying the result in section 2, showing that the bad-state cash constraint binds if, and only if, the incentive constraint binds.

Define

$$\tilde{V}(e; R) = \max_{L} V(e, L; R), \tag{3.3}$$

where the maximization is restricted by our three constraints, (i)-(iii). The level of effort that solves the banker's deposit contract problem is the value of e that solves:

$$\max_{0 \le e \le e_{\max}} \tilde{V}(e; R), \tag{3.4}$$

where

$$e_{\max} \equiv \frac{1 - \bar{a}}{\bar{b}}.$$

To ensure that (5.2) is well defined for each e, we make several assumptions. Not surprisingly, if R is too low, the demand for deposits by banks is infinite. To ensure finite demand, R must satisfy

$$R > \frac{1}{4\beta} \left(\beta \bar{b} \left[R^g - R^b \right] - \frac{\bar{a}}{\bar{b}} \right)^2 + \left[\bar{a} R^g + (1 - \bar{a}) R^b \right]. \tag{3.5}$$

Values of R which violate (3.5) imply that $0 < e < e_{\max}$ can be found, for which the solution to (5.2) implies $\tilde{V}(e; R) = +\infty$ and $L = +\infty$. Such a value of R cannot be an equilibrium because it is inconsistent with clearing in the market for deposits, where supply is bounded above by Y. In addition, the following restriction is helpful to guarantee $d \ge 0$:

$$\beta R^g \ge \frac{1 - \bar{a}}{\bar{b}^2}. \tag{3.6}$$

Finally, we assume

$$\frac{1 - \bar{a}}{\bar{b}} > \frac{1}{2} \left(\beta \bar{b} \left(R^g - R^b \right) - \frac{\bar{a}}{\bar{b}} \right) > 0. \tag{3.7}$$

This is a necessary condition for the existence of an interior solution to (5.2).

Suppose that R also satisfies the upper bound constraint,

$$R^g - \frac{1}{2\beta} \left(\frac{1 - \bar{a}}{\bar{b}} \right)^2 \ge R. \tag{3.8}$$

We show that,

$$\tilde{V}(e; R) = V(e; R), \tag{3.9}$$

for all $0 \le e \le e_{\max}$. Here, $V(e; R)$ a ratio of two second order polynomials in e:

$$V(e; R) \equiv \left(\frac{p(e)}{p'(e)} e - \frac{1}{2} e^2 \right) \frac{\beta R}{l(e; R)} N, \tag{3.10}$$

$$l(e; R) \equiv \frac{p(e)}{p'(e)} e - \beta \left[p(e) R^g + (1 - p(e)) R^b - R \right]. \tag{3.11}$$

Under our assumptions, $V(e; R)$ is nicely behaved. For example, we show that $l(e; R)$ strictly positive for all e. So, if (3.8) is satisfied, then the solution to the contract problem, (5.2),

is straightforward. For example, the maximum is guaranteed to exist because $V(e; R)$ is a continuous function and e is restricted to lie in a compact set.

Suppose (3.8) is not satisfied. We show that in this case, (3.9) holds for all e except $e = e_{\max}$. At $e = e_{\max}$, $\tilde{V}(e; R)$ takes a discontinuous jump up to $\infty > \tilde{V}(e_{\max}; R) > V(e_{\max}; R)$. Let e^* denote value of e that solves (5.2) when $\tilde{V}(e; R)$ is replaced by $V(e; R)$. If $\tilde{V}(e_{\max}; R) \leq V(e^*; R)$, then e^* solves (5.2). If $\tilde{V}(e_{\max}; R) > V(e^*; R)$, then the solution to (5.2) is $e_{\max}$. Interestingly, when (3.8) is violated, then the banker's outside option dominates taking a loan contract. It follows that (3.8) will be satisfied in any deposit contract equilibrium.

All the results just described are derived below. We state them in the form of propositions.

3.2.2.2. Formal Results   We begin by studying the properties of the polynomial, l, in (3.11). Substituting out the functional form for $p(e)$,

$$l(e; R) = \frac{\bar{a}}{\bar{b}} e + e^2 - \beta \left[\left(\bar{a} + \bar{b}e \right) R^g + \left(1 - \left(\bar{a} + \bar{b}e \right) \right) R^b - R \right].$$

Collecting terms:

$$l(e; R) = \overbrace{-\beta \left[\bar{a} R^g + (1 - \bar{a}) R^b - R \right]}^{\gamma} - \left(\overbrace{\beta \bar{b} \left[R^g - R^b \right]}^{\xi} - \frac{\bar{a}}{\bar{b}} \right) e + e^2$$

$$= \gamma - \xi e + e^2.$$

Evidently, $l(e; R)$ is a 'U' shape function of e. This function achieves its unique minimum at

$$e = \frac{1}{2} \xi > 0.$$

This value of e is strictly interior to the interval, $[0, e_{\max}]$ by (3.7). The value of l at the minimum is:

$$l\left(\frac{1}{2} \xi; R \right) = \gamma - \frac{1}{4} \xi^2.$$

If R satisfies (3.5), then $\gamma > \xi^2/4$, so

$$l(e; R) > 0, \text{ for all } e.$$

We now develop an upper bound on l. Note from the definition of γ, that

$$\gamma + \beta \left[\bar{a} R^g + (1 - \bar{a}) R^b \right] = \beta R, \tag{3.12}$$

so

$$\frac{\beta R}{l(e; R)} = \frac{\gamma + \beta \left[\bar{a} R^g + (1 - \bar{a}) R^b \right]}{\gamma - \xi e + e^2}.$$

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From the latter expression and the fact, $\gamma > 0$ (see (3.5)), we conclude that $l(0; R) < \beta R$. In addition, because the derivative of l with respect to e is negative at $e = 0$, $l(e; R)$ is decreasing at $e = 0$ until it reaches its minimum at $e = \xi/2 > 0$. For $e > \xi/2$, $l(e; R)$ increases monotonically in e. So, $l(e; R)$ is guaranteed to satisfy $l(e; R) \leq \beta R$ for all $0 \leq e \leq e_{\max}$ if $l(e_{\max}; R) \leq \beta R$. It can be verified that $l(e_{\max}; R) \leq \beta R$ is implied by (3.6).[2]

We summarize the argument as follows. The function, $l(e; R)$ achieves a minimum in the interior of its domain, $0 \leq e \leq e_{\max}$ by (3.7). That l is strictly positive at its minimum is guaranteed by the lower bound, (3.5), on R. Finally, the result that $l(e; R) \leq \beta R$ for all $0 \leq e \leq e_{\max}$ is implied by (3.6). We summarize these results in the form of a proposition:

Proposition 3.1. *Suppose R and the model parameters satisfy (3.5), (3.6) and (3.7). Then, $\beta R \geq l(e; R) > 0$ for all $0 \leq e \leq e_{\max}$.*

For each $0 \leq e \leq e_{\max}$, we now identify the set of values of banker leverage, $L \geq 1$, that is consistent with the three constraints, (i)-(iii). Using the binding bad-state cash constraint and the implication of the zero profit condition, $R - R_d^b = p(e)\left(R_d^g - R_d^b\right)$, the incentive constraint, 'incentive', reduces to:

$$\left\{ e - \beta \bar{b} \left[\left(R^g - R^b\right) - \frac{R\left(\frac{L-1}{L}\right) - R^b}{p(e)} \right] \right\} \begin{cases} = 0, & 0 < e < e_{\max} \\ \leq 0, & e = e_{\max} \\ \geq 0, & e = 0. \end{cases} \qquad (3.13)$$

Here, we have made use of the definition of L and rearranged terms. Multiplying the object in braces in (3.13) by $p(e)/\bar{b}$ and rearranging, we obtain

$$\frac{p(e)}{\bar{b}} \left\{ e - \beta \bar{b} \left[\left(R^g - R^b\right) - \frac{R\left(\frac{L-1}{L}\right) - R^b}{p(e)} \right] \right\}$$
$$= l(e; R) - \frac{\beta R}{L},$$

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[2]To see this, consider

$$\beta R^g \geq \frac{1 - \bar{a}}{\bar{b}^2}.$$

add $-\beta\left[R^g - R^b\right] + \beta\left[R^g - R^b\right]\bar{a}$ to both sides of the previous expression, to obtain:

$$\beta R^b + \beta\left[R^g - R^b\right]\bar{a} \geq \frac{1 - \bar{a}}{\bar{b}^2} - \beta\left[R^g - R^b\right] + \beta\left[R^g - R^b\right]\bar{a},$$

or,

$$\beta\left[\bar{a}\left(R^g - R^b\right) + R^b\right] \geq -\beta\left[R^g - R^b\right] + \beta\left[R^g - R^b\right]\bar{a} + \frac{1}{\bar{b}}\frac{1 - \bar{a}}{\bar{b}}.$$

This in turn is equivalent to

$$\beta\left[\bar{a}R^g + (1 - \bar{a})R^b\right] \geq -\left(\beta\bar{b}\left[R^g - R^b\right] - \frac{\bar{a}}{\bar{b}}\right)\frac{1 - \bar{a}}{\bar{b}} + \left(\frac{1 - \bar{a}}{\bar{b}}\right)^2.$$

The result follows by adding γ to both sides and taking into account (3.12) and the definition of ξ.

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where $l(e;R)$ is defined in (3.11). Then, using the Proposition 3.1 result, $l(e;R) > 0$, we express (3.13) as follows:

$$L = \frac{\beta R}{l(e;R)}, \text{ for } 0 < e < e_{\max} \tag{3.14}$$

$$L \leq \frac{\beta R}{l(e_{\max};R)}, \text{ for } e = e_{\max}$$

$$L \geq \frac{\beta R}{l(0;R)}, \text{ for } e = 0. \tag{3.15}$$

Proposition 3.1 and (3.14), (3.15) imply $L \geq 1$ for $0 \leq e < e_{\max}$. In the condition that pertains to $e = e_{\max}$, we combine the restriction, $L \geq 1$, with the incentive constraint to obtain the following restriction:

$$1 \leq L \leq \frac{\beta R}{l(e_{\max};R)}. \tag{3.16}$$

Proposition 3.1 guarantees that the interval in (3.16) is non-empty. We state our findings in the form of a proposition:

Proposition 3.2. *Suppose that the conditions of Proposition 3.1 hold. Suppose we have an e satisfying $0 \leq e \leq e_{\max}$. The set of values of banker leverage, L, consistent with the zero profit condition, the bad-state cash constraint, the incentive constraint and $L \geq 1$ is provided by (3.14), (3.15), (3.16).*

We now investigate the relationship between $\tilde{V}(e;R)$ in (3.3) and $V(e;R)$ in (3.10). Consider an interior value of e, $0 < e < e_{\max}$. For each such value of e, (3.14) indicates that there is only one value of L that is consistent with our three constraints, (i)-(iii). So, the solution to (3.3) is

$$\begin{aligned} \tilde{V}(e;R) &= \left(\beta \left[p(e)R^g + (1-p(e))R^b - R\right] - \frac{1}{2}e^2\right) N\frac{\beta R}{l(e;R)} + \beta RN \\ &= \frac{\left(\beta \left[p(e)R^g + (1-p(e))R^b - R\right] - \frac{1}{2}e^2 + l(e;R)\right)}{l(e;R)} N\beta R \\ &= \left(\frac{p(e)}{p'(e)}e - \frac{1}{2}e^2\right) \frac{\beta R}{l(e;R)} N, \end{aligned}$$

after rearranging. We conclude,

$$\tilde{V}(e;R) = V(e;R), \text{ for } 0 < e < e_{\max}.$$

Next, consider $e = e_{\max}$. Suppose the object in braces in (3.2) is weakly positive, so that R satisfies (3.8). Then, $V(e_{\max}, L; R)$ is weakly increasing in L. In this case, optimality in

(3.3) is consistent with setting L to its upper bound of $\beta R/l\left(e_{\max}; R\right)$. We conclude that if (3.8) holds, then

$$\tilde{V}\left(e_{\max}; R\right) = V\left(e_{\max}; R\right).$$

Now suppose the object in braces in (3.2) is strictly negative, so that R violates (3.8). In this case, $\tilde{V}\left(e_{\max}, L; R\right)$ is strictly decreasing in L and so the solution to (3.3) puts L at its lower bound of unity. Thus,

$$\tilde{V}\left(e_{\max}; R\right) = \left(\beta R^g - \frac{1}{2}\left(\frac{1-\bar{a}}{\bar{b}}\right)^2\right) N > V\left(e_{\max}; R\right). \tag{3.17}$$

It is obvious that when the object in braces in (3.2) is strictly negative, then $V\left(e_{\max}; R\right) < \beta RN$. We see that

$$\lim_{e \to e_{\max}} \tilde{V}\left(e; R\right) \to V\left(e_{\max}; R\right),$$

while $\tilde{V}\left(e; R\right)$ takes on a value higher than $V\left(e_{\max}; R\right)$ at $e = e_{\max}$. We conclude that when R violates (3.8), $\tilde{V}\left(e; R\right)$ jumps discontinuously to $\tilde{V}\left(e_{\max}; R\right) < +\infty$ at the upper bound for e.

Now consider the lower bound on e, $e = 0$. Substituting into (3.2), we obtain,

$$V\left(0, L; R\right) = \beta\left[\bar{a}R^g + (1-\bar{a})R^b - R\right]NL + \beta RN,$$

where the term in square brackets is negative, by (3.5). In this case, $V\left(0, L; R\right)$ is decreasing in L. According to (3.15), the solution to (3.3) implies $L = \beta R/l\left(0; R\right)$, so that

$$\begin{aligned}
\tilde{V}\left(0; R\right) &= \beta\left[\bar{a}R^g + (1-\bar{a})R^b - R\right]N\frac{\beta R}{l\left(0; R\right)} + \beta RN \\
&= V\left(0; R\right).
\end{aligned}$$

We summarize our results as follows:

Proposition 3.3. *Suppose that the conditions of Proposition 3.1 hold. Suppose R satisfies (3.8). Then, $\tilde{V}\left(e; R\right) = V\left(e; R\right)$ for all $0 \le e \le e_{\max}$. Suppose R does not satisfy (3.8). Then: (a) $\tilde{V}\left(e; R\right) = V\left(e; R\right)$ for $0 \le e < e_{\max}$; (b) $\tilde{V}\left(e; R\right)$ jumps up discontinuously to the finite value, (3.17), at $e = e_{\max}$ and (c) $\tilde{V}\left(e_{\max}; R\right) < \beta RN$.*

This proposition justifies the strategy for solving the banker's deposit contract problem, (5.2), described in the previous subsection. Begin by finding e^*, the value of e, $0 \le e \le e_{\max}$, that maximizes $V\left(e; R\right)$. If R satisfies (3.8), then e^* is the solution to the contract problem. Suppose R does not satisfy (3.8). If $\tilde{V}\left(e_{\max}; R\right) \le V\left(e^*; R\right)$, then e^* solves the contract problem after all. If $\tilde{V}\left(e_{\max}; R\right) > V\left(e^*; R\right)$, then $e_{\max}$ is the solution to the contract

problem. Notice that, regardless of whether (3.8) holds, optimizing $V(e; R)$ is central to the strategy for solving the contract problem.

The practical problem of finding the maximum of $V(e; R)$ is simple. For example, one can begin by constructing an extremely fine grid of values of e covering the closed interval, $[0, e_{\max}]$. One then graphs $V(e; R)$ over the grid points and identifies the point on the grid, e^*, where the function is maximized. With this good guess for the maximum in hand, one can proceed by driving V_e, the derivative of $V(e; R)$ with respect to e, to zero. This derivative is:

$$V_e(e; R) = \frac{p(e)}{p'(e)} L(e; R) N + \left[\frac{p(e)}{p'(e)} e - \frac{1}{2} e^2 \right] L_e(e; R) N = 0.$$

From this expression we see that if there is an interior equilibrium then it must be that $L_e < 0$. This is because all the other terms are positive for $e > 0$. For example, although the expression in square brackets is zero for $e = 0$, its derivative is strictly positive for $e \geq 0$. Differentiating (3.11),

$$L_e = \frac{\beta R \left(\beta p'(e) \left(R^g - R^b \right) - e - \frac{p(e)}{p'(e)} \right)}{\left(\frac{p(e)}{p'(e)} e - \beta \left[p(e) R^g + (1 - p(e)) R^b - R \right] \right)^2} = \frac{2L^2}{\beta R} \left(\frac{1}{2} \left[\beta \bar{b} \left(R^g - R^b \right) - \frac{\bar{a}}{\bar{b}} \right] - e \right).$$

Note that (3.7) guarantees the existence of $e \in [0, e_{\max}]$ such that $L_e < 0$. Using the previous expression to substitute out for L_e in V_e, we obtain the following very simple expression:

$$\frac{V_e(e; R)}{NL} = \frac{p(e)}{p'(e)} + \left(\frac{p(e)}{p'(e)} e - \frac{1}{2} e^2 \right) \frac{2L}{\beta R} \left(\frac{1}{2} \left[\beta \bar{b} \left(R^g - R^b \right) - \frac{\bar{a}}{\bar{b}} \right] - e \right) = 0.$$

An interior maximum requires $V_e = 0$. Evidently, for $e = 0$, $V_e > 0$. Under (3.7) it is possible to have $V_e < 0$ for e high enough, in which case there is an interior solution. In several numerical examples, we found that the solution is interior.

It is particularly simple to find an e that sets V_e to zero numerically. After multiplying the previous expression by $l(e; R)$ (recall that $l(e; R) > 0$) we find that setting $V_e = 0$ requires a value of e that solves:

$$\left(\left[\frac{\bar{a}}{\bar{b}} + e \right] e - \beta \left[p(e) R^g + (1 - p(e)) R^b - R \right] \right) \left(\frac{\bar{a}}{\bar{b}} + e \right) \tag{3.18}$$

$$+ 2e \left(\frac{\bar{a}}{\bar{b}} + \frac{1}{2} e \right) \left(\frac{1}{2} \left[\beta \bar{b} \left(R^g - R^b \right) - \frac{\bar{a}}{\bar{b}} \right] - e \right) = 0.$$

This quadratic equation can be solved for two values of e by the well-known quadratic formula. The desired value of e is the one that is closest to e^*. As long as the initial grid for e was fine enough, this strategy is guaranteed to identify the global maximum of $V(e; R)$.

3.2.2.3.Coherence of the Analysis We have described two ways to solve the banker contract problem. One uses Lagrange multipliers and appears in section 2. The other appears in the previous subsection and substitutes out the constraints in the banker's problem. Coherence of the analysis requires that both approaches provide the same solution, when the solution is interior. Here, we show that the solutions are indeed the same.

Section 2 displays five first order conditions for the Lagrangian problem, including the constraints. Substituting out the two multipliers, deposits and R_d^b, that system can be expressed as follows:

$$e = \beta\bar{b}(R^g - R^b) - \frac{\left\{\beta\left[p\left(e\right)R^g + (1-p\left(e\right))R^b - R\right] - \frac{1}{2}e^2\right\}\left[\frac{p(e)+e\bar{b}}{\beta\bar{b}} - \bar{b}(R^g - R^b)\right]}{\frac{p(e)e}{\beta\bar{b}} - [p(e)R^g + (1-p(e))R^b - R]}$$

We now show that the preceding equation coincides with (3.18). Rearranging, we obtain,

$$e = \beta\bar{b}(R^g - R^b) - \frac{\left\{\beta\left[p\left(e\right)R^g + (1-p\left(e\right))R^b - R\right] - \frac{1}{2}e^2\right\}\left[\frac{p(e)}{\bar{b}} + e - \beta\bar{b}(R^g - R^b)\right]}{\frac{p(e)e}{\bar{b}} - \beta\left[p(e)R^g + (1-p(e))R^b - R\right]}.$$

Define $X \equiv \beta\left[p\left(e\right)R^g + (1-p\left(e\right))R^b - R\right]$. Then, the previous condition can be written as follows:

$$e = \beta\bar{b}(R^g - R^b) - \frac{\left(X - \frac{1}{2}e^2\right)\left[\frac{p(e)}{\bar{b}} + e - \beta\bar{b}(R^g - R^b)\right]}{\frac{p(e)e}{\bar{b}} - X}.$$

Rearranging, we obtain:

$$\left[e - \beta\bar{b}(R^g - R^b)\right]\left[\frac{p(e)e}{\bar{b}} - X\right] = -\left(X - \frac{1}{2}e^2\right)\left[\frac{p(e)}{\bar{b}} + e - \beta\bar{b}(R^g - R^b)\right],$$

$$\left[e - \beta\bar{b}(R^g - R^b)\right]\frac{p(e)e}{\bar{b}} + \frac{p(e)}{\bar{b}}X = \frac{1}{2}e^2\left[\frac{p(e)}{\bar{b}} + e - \beta\bar{b}(R^g - R^b)\right],$$

$$\left[e - \beta\bar{b}(R^g - R^b)\right]\left[\frac{p(e)e}{\bar{b}} - \frac{1}{2}e^2\right] + \frac{p(e)}{\bar{b}}X = \frac{1}{2}e^2\left(\frac{p(e)}{\bar{b}}\right). \tag{3.19}$$

Rewriting (3.18), we obtain,

$$0 = \left(\frac{p(e)e}{\bar{b}} - X\right)\frac{p(e)}{\bar{b}} + 2e\left(\frac{p(e)}{\bar{b}} - \frac{1}{2}e\right)\left[\frac{1}{2}\beta\bar{b}(R^g - R^b) - \left(e + \frac{1}{2}\frac{p(e) - \bar{b}e}{\bar{b}}\right)\right],$$

$$0 = \left(\frac{p(e)e}{\bar{b}} - X\right)\frac{p(e)}{\bar{b}} + \left(\frac{p(e)e}{\bar{b}} - \frac{1}{2}e^2\right)\left[\beta\bar{b}(R^g - R^b) - \left(2e + \frac{p(e) - \bar{b}e}{\bar{b}}\right)\right],$$

$$0 = \left(\frac{p(e)e}{\bar{b}} - X\right)\frac{p(e)}{\bar{b}} + \left(\frac{p(e)e}{\bar{b}} - \frac{1}{2}e^2\right)\left[\beta\bar{b}(R^g - R^b) - e - \frac{p(e)}{\bar{b}}\right],$$

$$0 = -\left[e - \beta\bar{b}(R^g - R^b)\right]\left(\frac{p(e)e}{\bar{b}} - \frac{1}{2}e^2\right) - \frac{p(e)}{\bar{b}}X + \frac{1}{2}e^2\frac{p(e)}{\bar{b}},$$

which is identical (3.19).

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4. Socially Optimal Leverage

In this section, we explore the implications of the model for leverage restrictions. In the first section we describe a strategy for finding the best leverage restriction, which allows for the possibility that the solution is on a corner. In the example we work with, this strategy leads us to the conclusion that the solution is, in fact, interior. We then turn to a Lagrangian approach to the problem, which implicitly assumes an interior solution. The Lagrangian approach has the advantage that it reveals the economic reason that social welfare may be improved by imposing a leverage restriction.

4.1. Socially Optimal Leverage: An Approach Based on Substituting out the Constraints

Consider a regulation that restricts leverage to be $f < 1$ times what it is in an unregulated equilibrium. We assume that the leverage restriction is strictly binding for the banker. That is, the banker is required to select a lower value of d than is privately optimal in the unregulated economy. Suppose that the leverage restriction is $\bar{L} = fL$, where L denotes the level of leverage in the unregulated equilibrium. The equilibrium rate of interest, $\bar{R}$, is determined by the household Euler equation for deposits:

$$\bar{R} = \frac{1}{\beta \left(Y - \left(\bar{L} - 1 \right) N \right)}.$$

One 'parameter' associated with the banker's choice of deposit contract has been taken off the table, namely, d. The banker may still choose the other parameters, e, R_d^g, R_d^b. But, these must be consistent with the incentive constraint, the zero profit condition and the cash constraints (i.e., constraints (i)-(iii) in section 3.2.2.1). Suppose, to begin, that the cash and incentive constraints are non-binding. In this case, the banker selects the following level of effort:

$$e = \beta \bar{b} \left(R^g - R^b \right).$$

For this level to be consistent with the incentive constraint, it must be that

$$R_d^g = R_d^b = \bar{R}.$$

The requirement that the cash constraints be satisfied is:

$$R^g \geq \frac{\bar{L} - 1}{\bar{L}} \bar{R}, \ R^b \geq \frac{\bar{L} - 1}{\bar{L}} \bar{R}.$$

If the bad-state cash constraint is violated, then we proceed under the assumption that it is binding, along with the incentive constraint. The banker's contract problem is now to simply choose a level of effort for the given $\bar{L}$:

$$\max_{e \in E} V \left(e, \bar{L}; \bar{R} \right), \tag{4.1}$$

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where $V(e, L; R)$ is defined in (3.2). The set E represents the values of e that are consistent with $\bar{L}$ and the three constraints, (i)-(iii), in section section 3.2.2.1. That is, the elements of E are the values of e that solve (3.14), (3.15) and (3.16) for the given $\bar{L}$. Consider the values of e that solve (3.14):

$$l\left(e; \bar{R}\right) = \frac{\beta \bar{R}}{\bar{L}}. \tag{4.2}$$

Substituting out for l from (3.11) and using the functional form for $p(e)$, (4.2) reduces to:

$$\left[\frac{\bar{a}}{\bar{b}} + e\right] e - \beta \left[\left(\bar{a} + \bar{b}e\right) R^g + \left(1 - \left(\bar{a} + \bar{b}e\right)\right) R^b - \bar{R}\right] = \frac{\beta \bar{R}}{\bar{L}}.$$

Collecting terms,

$$\overbrace{-\beta\left[\bar{a}R^g + (1-\bar{a})R^b - \bar{R}\right]}^{\gamma} - \overbrace{\left(\beta\bar{b}\left(R^g - R^b\right) - \frac{\bar{a}}{\bar{b}}\right)}^{\xi} e + e^2 = \frac{\beta \bar{R}}{\bar{L}}$$

so that (4.2) reduces to:

$$\gamma - \xi e + e^2 = \frac{\beta \bar{R}}{\bar{L}}.$$

The values of e that solve this equation are:

$$e = \frac{\xi \pm \sqrt{\xi^2 - 4\left(\gamma - \frac{\beta \bar{R}}{\bar{L}}\right)}}{2}.$$

The intersection of the above two values of e with the interval, $[0, e_{\max}]$, is included in the set, E.

Condition (3.15) implies that $e_{\max}$ also belongs to E if

$$\bar{L} \leq \frac{\beta \bar{R}}{l\left(e_{\max}; \bar{R}\right)}.$$

Condition (3.16) implies that $0 \in E$ if

$$\bar{L} \geq \frac{\beta \bar{R}}{l\left(0; \bar{R}\right)}.$$

We conclude that E contains at most 4 elements.

Once the set, E, is available, the maximization problem, (4.2), is straightforward to solve. In the examples that we studied, we found that $e = e_{\max}$ and $e = 0$ are not elements of E. That is, the solution to the banker problem is interior.

We can compute the optimal f by placing a fine grid on $[f_l, 1]$, where f_l is a number, $0 < f_l < 1$. We then evaluate (4.1) for each f on the grid and identifying the maximum. We make f_l small enough until we identify an interior maximum for f. In the example that we developed, $f_l = 0.9$ works.

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4.2. Socially Optimal Leverage: A Lagrangian Approach

To explore the economic role of leverage restrictions, it is convenient to express the banker problem in Lagrangian form. The first subsection states the banker problem with a leverage constraint in this form. We assume that the banker incentive constraint and bad-state cash constraints are binding in the unregulated economy, for otherwise the leverage constraint would be counterproductive. In the second subsection we consider the decision of a benevolent regulator who has the power to restrict the amount of banker leverage. Again, for gaining economic intuition, it is useful to express the regulator problem in Lagrangian form. We then show that when the banker incentive constraint is binding, then a leverage restriction is desirable because it forces the banker to internalize a pecuniary externality.

4.2.1. Private Sector

We modify the banker problem to include the requirement,

$$L \leq \bar{L}.$$

The banker's objective is (3.2). We substitute out for R_d^g, R_d^b by imposing the binding cash constraint in the bad state and the zero profit condition. The banker's problem in Lagrangian form is:

$$
\begin{aligned}
V \;=\; & \max_{e,L} \left\{ \beta \left[p\left(e\right) R^g + \left(1 - p\left(e\right)\right) R^b - R \right] - \frac{1}{2} e^2 \right\} LN + \beta RN \\
& + \eta \left(\beta \bar{b} \left[\left(R^g - R^b\right) - \frac{R \frac{L-1}{L} - R^b}{p\left(e\right)} \right] - e \right) \\
& + \xi \left(\bar{L} - L \right).
\end{aligned}
$$

Here, $\xi \geq 0$ represents the Lagrange multiplier on the leverage constraint and $\eta \geq 0$ represents the multiplier on the incentive constraint. Note that the above problem corresponds to the same problem we studied in the previous section, with one exception. The exception is that by representing the incentive constraint as an equality constraint we implicitly assume an interior solution. In the numerical example that we studied, we verified the interiority assumption by also taking the approach in the previous section, which does not assume interiority of the solution.

The first order condition for L is:

$$
[p(e)R^g + (1 - p(e))R^b] - \frac{1}{2\beta} e^2 = R + \eta \left(\frac{R}{N} \frac{p'(e)}{p(e)} \frac{1}{L^2} \right) + \frac{\xi}{\beta N}. \tag{4.3}
$$

According to this expression, the banker equates the net marginal benefit of deposits (left side of equality) to the privately experienced cost of deposits (right side). The benefit is

the expected earnings on assets financed by deposits, net of the marginal cost of the effort expended to manage the assets. The cost of extra funds has three components: (i) the expected marginal cost of the deposits, R; plus (ii) the cost, measured by η, of tightening the incentive constraint; plus (iii) the cost, measured by ξ, of tightening the leverage constraint.

The first order condition for e is:

$$0 = \left[\beta p'(e)(R^g - R^b) - e\right] LN + \eta \left[\beta \left(\frac{p'(e)}{p(e)}\right)^2 \left(R\frac{L-1}{L} - R^b\right) - 1\right]. \qquad (4.4)$$

4.2.2. Benevolent Regulator

We assume there is a regulator that has the power to compel the banker to issue fewer deposits. In effect, the regulator takes over the choice of deposits from the banker, leaving the banker only with a decision over the other parameters of the loan contract, e, R_d^g, R_d^b. The regulator is assumed to be benevolent in the sense of wishing to maximize social welfare. Of course, the regulator must respect all the private sector equilibrium conditions. There are four such conditions. The equilibrium conditions are the one associated with the household deposit decision:

$$\beta R = u'\left(Y - \left(\bar{L} - 1\right)N\right), \qquad (4.5)$$

where we have replaced d with $(L-1)N$; the resource constraint in each period:

$$
\begin{aligned}
c &= Y - \left(\bar{L} - 1\right)N \\
C &= \left[p(e)R^g + (1 - p(e))R^b\right]\bar{L}N,
\end{aligned}
$$

and the incentive constraint of the bankers:

$$e - \beta\bar{b}\left[R^g - R^b - \frac{R\frac{L-1}{L} - R^b}{p(e)}\right] = 0.$$

At the same time, we have five unknowns:

$$R, \bar{L}, c, C, e.$$

In choosing a value for $\bar{L}$, the regulator in effect chooses values for all five economic variables.[3] So, we can think of the regulator's problem as simply choosing all the variables subject to the four private sector equilibrium conditions, with the objective of maximizing social welfare.

---

[3]Actually, there are two other unknowns, R_d^g and R_d^b. We have substituted these out of the problem by apply the binding bad-state cash constraint and the zero profit condition on mutual funds.

After substituting out for c, C and R, the regulator's problem is:

$$\max_{\{\bar{L},e\}} \beta u(Y - (\bar{L}-1)N) + \left\{ \beta \left[p(e)R^g + (1-p(e))R^b \right] - \frac{1}{2}e^2 \right\} \bar{L}N$$

$$+ \eta \left[\beta p'(e) \left(R^g - R^b - \frac{\frac{u'(Y-(\bar{L}-1)N)}{\beta}\frac{\bar{L}-1}{\bar{L}} - R^b}{p(e)} \right) - e \right].$$

The first order condition with respect to $\bar{L}$ is, after rearranging,

$$\left[p(e)R^g + (1-p(e))R^b \right] - \frac{1}{2}e^2 \tag{4.6}$$

$$= R + \eta \left(\frac{R}{N}\frac{p'(e)}{p(e)}\frac{1}{\bar{L}^2} \right) + \eta \frac{p'(e)}{p(e)} \frac{\left[-u''(Y-(\bar{L}-1)N) \right]}{\beta} \frac{\bar{L}-1}{\bar{L}},$$

where $u'' \left(Y - (\bar{L}-1)N \right)/\beta < 0$ denotes the first derivative of R with respect to $\bar{L}$, taking into account (4.5). On the left side of (4.6) we have the marginal benefit of an increase in leverage, which coincides with the assessment of that benefit by the banker (see (4.3)). On the right the regulator includes three costs: (i) R, which measures the marginal cost of extra leverage to the household in the form of decreased consumption; (ii) the direct tightening in the banker incentive constraint due to increased leverage; and (iii) the indirect tightening in the banker incentive constraint that occurs as extra leverage raises R (see the u'' term in (4.6)). By comparing (4.6) and (4.3), we see that the banker correctly internalizes costs (i) and (ii). However, the banker does not internalize the general equilibrium effect on the market interest rate, R, of issuing deposits. This effect constitutes an externality, a *pecuniary externality* because it operates through the price system. Obviously the banker can be made to correctly internalize the pecuniary externality if the banker's multiplier is given by:

$$\xi = \eta \frac{p'(e)}{p(e)} \frac{\left[-u''(Y-(\bar{L}-1)N) \right]}{\beta} \frac{\bar{L}-1}{\bar{L}} \beta N. \tag{4.7}$$

The first order condition associated with the effort decision is:

$$\left[\beta p'(e) \left(R^g - R^b \right) - e \right] \bar{L}N + \eta \left[\beta \left(\frac{\bar{b}}{p(e)} \right)^2 \left(\frac{u'(Y-(\bar{L}-1)N)}{\beta} \frac{\bar{L}-1}{\bar{L}} - R^b \right) - 1 \right] = 0. \tag{4.8}$$

Note that this first order condition is identical to the first order condition of the banker (see (4.4)).

It is easy to verify that if the regulator's choice of $\bar{L}$ is imposed as a restriction on the banker, then the equilibrium values of $R, \bar{L}, c, C, e$ are the ones that solve the regulator's problem. Moreover, the banker's multipliers, ξ and η, are the ones that satisfy (4.7) and (4.8), respectively. It is also easy to verify using this type of argument, that the regulator does not want to restrict leverage in case the incentive and bad-state cash constraints are not binding.

5. Detailed Analysis of a Version of the Model With No Scaling of Effort Cost

Here, we study the properties of a simple version of the model in which the banker's cost of effort is replaced with

$$\frac{1}{2}e^2.$$

This is the version of the model studied by Christiano and Ikeda (2013,2014). To keep the analysis simple, we focus on the special case, $R^b = 0$. We describe the observations of Francesco Ferrante and Andrea Prestipino, who brought to our attention an error in our earlier analysis of this model. In our analysis, we characterized the solution to the banker's deposit contract problem by the solution to the first order conditions of the Lagrangian representation of the problem. Ferrante and Prestipino show that the correct characterization leads to very different model properties that are not interesting. They recommend replacing the cost of effort with (??). This is a recommendation whose wisdom is verified in the preceding sections. Interestingly, the properties reported for the model in Christiano and Ikeda (2013,2014) are in fact the properties of the model with the effort cost specification recommended by Ferrante and Prestipino.

In this version of the model, the mutual fund zero profit condition is:

$$p\left(e\right)R_d^g = R.$$

Conditional on R_d^g and d the banker effort choice, after receiving a deposit contract, solves:

$$\max_e V\left(e, d, R_d^g\right) = \beta p\left(e\right)\left[R^g\left(N + d\right) - R_d^g d\right] - \frac{1}{2}e^2. \tag{5.1}$$

We adopt the following piecewise-linear specification of $p\left(e\right)$:

$$p\left(e\right) = \min\left\{\bar{a} + \bar{b}e, 1\right\}, \ e \geq 0. \tag{5.2}$$

To visualize the problem in (5.1), construct a graph with e on the horizontal axis and the return $\beta\left[\bar{a} + \bar{b}e\right]\left[R^g\left(N + d\right) - R_d^g d\right]$ and the utility cost, $e^2/2$ on the vertical axis. The problem is to choose e to maximize the vertical distance between the return and the utility cost. These functions are defined over $e \geq 0$. The cost function is at its minimum at zero, has zero slope there and is increasing and convex for $e > 0$. The return function is piecewise linear, has value $\beta\bar{a}\left[R^g\left(N + d\right) - R_d^g d\right]$ at the intercept and slope $\beta\bar{b}\left[R^g\left(N + d\right) - R_d^g d\right]$ for $0 \leq e < \left(1 - \bar{a}\right)/\bar{b}$. Thereafter, it has slope zero. The value of e that solves the problem is:

$$e = \min\left[\tilde{e}, \frac{1 - \bar{a}}{\bar{b}}\right], \tag{5.3}$$

where
$$\tilde{e} = \beta \bar{b} \left[R^g \left(N + d \right) - R_d^g d \right],$$

The banker's deposit contract problem is to choose a contract, e, d, R_d^g, to maximize $V\left(e, d, R_d^g\right)$ subject to the zero profit condition and the incentive constraint, (5.3). After solving the deposit contract problem, the banker decides whether to take the contact, or go with its outside option. Here, we specify that the banker's outside option is to take no bank contract and simply invest his net worth, N, choosing e to solve (5.1) with $d = 0$.

5.1. Lagrangian Approach to the Problem

The approach proceeds under the (alas, possibly false) assumption that the solution to the problem puts e in the following set:

$$0 < e < \frac{1 - \bar{a}}{\bar{b}},$$

with the incentive constraint,

$$e = \beta \bar{b} \left[R^g \left(N + d \right) - R_d^g d \right].$$

The Lagrangian representation of the problem is:

$$\max_{e,d} \beta \left[p\left(e\right) R^g \left(N + d \right) - Rd \right] - \frac{1}{2} e^2 \qquad (5.4)$$
$$+ \eta \left(e - \beta \bar{b} \left[R^g \left(N + d \right) - \frac{R}{p\left(e\right)} d \right] \right),$$

where η denotes the multiplier on the incentive constraint. Also, the zero profit condition of the mutual fund has been imposed by replacing R_d^g with $R/p\left(e\right)$.

The first order condition with respect to d:

$$d \quad : \quad \beta \left[p\left(e\right) R^g - R \right] - \frac{\eta \beta \bar{b}}{p\left(e\right)} \left[p\left(e\right) R^g - R \right] = 0$$

$$d \quad : \quad \beta \left(1 - \frac{\eta \bar{b}}{p\left(e\right)} \right) \left[p\left(e\right) R^g - R \right] = 0,$$

and with respect to the incentive constraint:

$$\eta \quad : \quad e - \beta \bar{b} \left[R^g \left(N + d \right) - \frac{R}{p\left(e\right)} d \right] = 0$$

$$\eta \quad : \quad e - \beta \bar{b} R^g N - \frac{\beta \bar{b}}{p\left(e\right)} \left(p\left(e\right) R^g - R \right) d = 0$$

All together:

$$e \quad : \quad \beta \bar{b} R^g (N + d) - e + \eta \left(1 - \beta \left(\frac{\bar{b}}{p(e)} \right)^2 d \right) = 0$$

$$d \quad : \quad \beta \left(1 - \frac{\eta \bar{b}}{p(e)} \right) [p(e) R^g - R] = 0$$

$$\eta \quad : \quad e - \beta \bar{b} R^g N - \frac{\beta \bar{b}}{p(e)} (p(e) R^g - R) d = 0.$$

This represents three equations in e, d, η. We can identify at least two solutions. First, suppose that the object in parentheses in d is non-negative, so that $p(e) R^g = R$. Then, d and η become two seemingly incompatible equations:

$$d \quad : \quad p(e) R^g = R$$

$$\eta \quad : \quad e = \beta \bar{b} R^g N.$$

These equations cannot be solved for arbitrary R. They only have a solution when R takes on the following value:

$$R = \overbrace{\left[\bar{a} + \bar{b}^2 \beta R^g N \right]}^{\equiv \tilde{p}} \times R^g.$$

We assume that model parameters have been chosen such that

$$0 < \bar{a} + \bar{b}^2 \beta R^g N < 1.$$

Given the value of R described above, then we can back out equilibrium d using the saving supply curve:

$$R = \frac{1}{\beta (Y - d)}.$$

Finally, η can be backed out to solve e :

$$\eta = \frac{e - \beta \bar{b} R^g (N + d)}{\left(1 - \beta \left(\frac{\bar{b}}{\tilde{p}} \right)^2 d \right)} = \frac{-\beta \bar{b} R^g d}{\left(1 - \beta \left(\frac{\bar{b}}{\tilde{p}} \right)^2 d \right)}.$$

There appears to be a second solution to the bank deposit equations. In particular, set $e \in \left[0, (1 - \bar{a}) / \bar{b} \right]$. Compute η that solves:

$$\eta = \frac{p(e)}{\bar{b}}.$$

Solve the e equation for d.

$$d = \frac{e - \eta - \beta \bar{b} R^g N}{\beta \bar{b} R^g - \beta \eta \left(\frac{\bar{b}}{p(e)} \right)^2} = \frac{e - \frac{p(e)}{\bar{b}} - \beta \bar{b} R^g N}{\beta \bar{b} R^g - \beta \frac{p(e)}{\bar{b}} \left(\frac{\bar{b}}{p(e)} \right)^2}$$

Then adjust e until the η equation is satisfied.

We considered the following parameterization:

$$\bar{a} = 0.7, \ \bar{b} = 0.4, \ \beta = 0.99, \ R^g = 1.20, \ N = 1, \ Y = 2.$$

Then, we found the following two solutions to the Lagrangian equations:

solution #1: $e = 0.4752, \ p = 0.89, \ \eta = -0.635, \ d = 1.054$

solution #2: $p = 0.89, \ \eta = 2.2252, \ d = -73.45, \ e = 0.4752$

Evidently, the second solution is not economically interesting because $d < 0$.

We now consider the second order conditions associated with the Lagrangian problem. Let

$$V(e,d) \ = \ \beta\left[p(e) R^g (N+d) - Rd\right] - \frac{1}{2}e^2 + \eta\left(e - \beta\bar{b}\left[R^g(N+d) - \frac{R}{p(e)}d\right]\right)$$

$$g(e,d) \ = \ e - \beta\bar{b}\left[R^g(N+d) - \frac{R}{p(e)}d\right].$$

Consider the following bordered Hessian:

$$\begin{bmatrix} V_{ee} & V_{ed} & g_e \\ V_{ed} & V_{dd} & g_d \\ g_e & g_d & 0 \end{bmatrix}.$$

The solution to the first order conditions of the Lagrangian problem represent a local maximum (minimum) if the above matrix has a positive (negative) determinant. It is easy to verify that $V_{dd} = 0$ in each solution. Then, the determinant is

$$(2V_{ed}g_e - V_{ee}g_d)\, g_d.$$

But, it is also easy to verify that $g_d = 0$ in the case of both solutions to the first order conditions. The Lagrangian analysis provides no basis for thinking that the solution to the first order conditions is either a local minimum or a local maximum of the Lagrangian problem.

5.2. Substituting Out the Constraint

We now proceed with an analysis that does not rule out corner solutions. Let $e_{\max}$ denote the level of effort that makes success certain, i.e., $p(e_{\max}) = 1$:

$$e_{\max} = \frac{1 - \bar{a}}{\bar{b}}.$$

What levels of bank debt have the property that $e_{\max}$ is incentive compatible? Suppose there exists a value of d such that

$$e_{\max} \leq \beta \bar{b} R^g N + \beta \bar{b} \left[R^g - R \right] d. \tag{5.5}$$

With such a d the banker would not have an incentive to deviate to a higher value of e. This deviation results in a higher cost of effort and produces no benefit because $p(e)$ remains unchanged at unity. The banker also has no incentive to deviate to a lower value of e. The marginal benefit of such a deviation in terms of reduced utility cost of effort is $e_{\max}$. But, the cost in terms of reduced revenues is greater (the slope of $p(e)$ is $\bar{b}$ for reductions in e). We summarize this result in the form of a proposition:

Proposition 5.1. *The effort level, $e_{\max}$, is incentive compatible for any d that satisfies (5.5).*

The banker's objective is:

$$\max_{e,d} \beta p(e) R^g N + \beta p(e) \left[R^g - \frac{R}{p(e)} \right] d - \frac{1}{2} e^2,$$

subject to e being incentive compatible, ex post, with d. We now consider three cases: $R^g > R$, $R^g = R$ and $R^g < R$.

Consider $R^g > R$, so the set of d's that satisfies (5.5) is unbounded above. Then, by setting $e = e_{\max}$ the banker's return,

$$\beta R^g N + \beta \left[R^g - R \right] d - \frac{1}{2} e_{\max}^2,$$

can be made arbitrarily large by choosing an arbitrarily large value for d. Thus, the banker's demand for d is infinite for $R^g > R$.

Suppose that $R = R^g$. In this case, the return from taking a bank deposit is:

$$\max_{e,d} \beta p(e) R^g N + \beta p(e) \left[1 - \frac{1}{p(e)} \right] R d - \frac{1}{2} e^2.$$

When contemplating $e < e_{\max}$ the banker would always prefer the outside option, $d = 0$, since $d > 0$ generates a negative return to deposits. So, we only need to consider what the banker wants to do conditional on $e = e_{\max}$. The Proposition indicates that any $d \geq 0$ lies in the banker's choice set because they all satisfy incentive-compatibility. For each $d \geq 0$, the banker's return is:

$$V = \beta R^g N - \frac{1}{2} e_{\max}^2.$$

Under the outside option, the banker sets

$$e^* = \min \left\{ \beta \bar{b} R^g N, e_{\max} \right\},$$

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and receives the return

$$V^* = p^* \beta R^g N - \frac{1}{2} (e^*)^2,$$
$$p^* = p(e^*) = \bar{a} + \bar{b} e^*.$$

Then,

$$V - V^* = \beta R^g N \bar{b} (e_{\max} - e^*) + \frac{1}{2} \left[(e^*)^2 - e_{\max}^2 \right]$$

Since,

$$(e^*)^2 - e_{\max}^2 = (e^* - e_{\max})(e^* + e_{\max}),$$

we have,

$$V - V^* = \left[\beta R^g N \bar{b} + \frac{1}{2} (e^* + e_{\max}) \right] (e^* - e_{\max}).$$

Since $e^* \le e_{\max}$, it follows that $V \le V^*$ and

$$V - V^* = \begin{cases} 0 & \beta \bar{b} R^g N \ge e_{\max} \\ < 0 & \beta \bar{b} R^g N < e_{\max} \end{cases}.$$

If the banker's effort under the outside option is less than $e_{\max}$, then the outside option is strictly preferred. If the banker exerted the same effort in a bank contract and issued deposits, he would lose money on the deposits. He would have to raise effort to $e_{\max}$ simply to get to the point of earning zero on deposits. But, at that point the banker's situation is equivalent to taking the outside option and choosing a suboptimal level of effort. If the banker's effort under the outside option is equal to $e_{\max}$, then the banker is indifferent between taking a deposit contract and the outside option. We assume that under indifference, the banker takes a deposit contract. We conclude that when $R = R^g$, then

$$d \in [0, +\infty), \text{ for } \beta \bar{b} R^g N \ge e_{\max}$$
$$d = 0, \text{ for } \beta \bar{b} R^g N < e_{\max}.$$

Finally, we consider the case, $R^g < R$. In this case, it is trivial that the outside option is preferred to the deposit contract. Whatever effort the banker exerts (whether incentive compatible or not), the banker loses money on deposits. We summarize the previous results in the following proposition:

Proposition 5.2. *Consider four cases:*

(i) If $R^g > R$, then the banker chooses a deposit contract with $d = +\infty$.

(ii) If $R^g = R$ and $\beta \bar{b} R^g N \ge \frac{1-\bar{a}}{\bar{b}}$, then the banker is indifferent over any deposit contract with $d \ge 0$.

(iii) If $R^g = R$ and $\beta \bar{b} R^g N < \frac{1-\bar{a}}{\bar{b}}$ then the banker strictly prefers the outside option, so that $d = 0$.

(iv) If $R^g < R$, then the banker strictly prefers the outside option, so that $d = 0$.

The previous proposition displays what is in effect the demand curve from the financial system, for deposits, as a function of R. Suppose

$$\beta \bar{b} R^g N \geq \frac{1 - \bar{a}}{\bar{b}}.$$

In this case, the demand for d is represented in a graph with R on the vertical axis and d on the horizontal, as a horizontal line at $R = R^g$. The upward-sloped supply curve is provided by household optimization:

$$R = \frac{1}{\beta (Y - d)}.$$

At $d = 0$, we have $R = 1/(\beta Y)$. So, if

$$\frac{1}{\beta Y} < R^g,$$

then there is a unique intersection of demand and supply. At this point, the equilibrium level of demand is:

$$d = Y - \frac{1}{\beta R^g}.$$

We summarize the preceding results in the form of two propositions, which distinguish between two cases. In the first, the return on the banker's project is high and the in the second it is relatively low.

Proposition 5.3. *Suppose $\beta Y R^g > 1$ and $\beta \bar{b} R^g N \geq (1 - \bar{a})/\bar{b}$. Then, there exists a unique deposit equilibrium with*
$$R = R^g, \, d = Y - (\beta R^g)^{-1}, \, e = (1 - \bar{a})/\bar{b}, \, p(e) = 1.$$

The proposition describes a situation in which the banker's return is high enough, that he exerts sufficient effort to guarantee the good outcome. The interest rate on deposits is the same as the return on the banker's project, so that he makes no profits in this equilibrium. The banker makes the same level of effort by taking a deposit contract as he would under his outside option, so he is indifferent between the bank contract and the outside option.

Proposition 5.4. *Suppose $\beta \bar{b} R^g N < (1 - \bar{a})/\bar{b}$. Then there exists a unique equilibrium with zero deposits.*

In this case, the demand for deposits is $+\infty$ for $R > R^g$ and then drops discontinuously to $d = 0$ for $R \leq R^g$. As a result, there is no equilibrium with positive deposits.

6. Generalizing The Probability and Cost Functions, No Scaling

We now keep to the case in which effort is not scaled and $R^b = 0$. But, we consider fairly general cost and probability functions. We would like it to be the case that with $R^g > R$ there are versions of the model in which the banker finds it optimal to choose a deposit contract in which the probability of success is less than unity. Here, we expand the set of cost and probability functions to a broader class. We show that, within this class, it is not possible to find functions that have the desired property.

The probability function satisfies:

$$
\begin{aligned}
0 &\leq\ p\left(e\right) \leq 1, \\
p'\left(e\right) &>\ 0 \text{ for all } e_{\max} \geq e \geq 0,
\end{aligned}
$$

where $[0, e_{\max}]$ denotes the domain of p. In addition, the cost of effort function is $f\left(e\right)$. Here, we assume $f\left(e\right)$ is increasing and convex. We assume both p and f are twice differentiable. The ex post problem of the banker (i.e., after R_d^g and d have been set) is

$$
\max_{0 \leq e \leq e_{\max}} \beta p\left(e\right)\left[R^g\left(N + d\right) - R_d^g d\right] - f\left(e\right).
$$

The solution is:

$$
f'\left(e\right) = \beta p'\left(e\right)\left[R^g\left(N + d\right) - R_d^g d\right], \tag{6.1}
$$

for $e < e_{\max}$ and

$$
f'\left(e_{\max}\right) \leq \beta p'\left(e_{\max}\right)\left[R^g\left(N + d\right) - R_d^g d\right].
$$

At $e_{\max}$ a further increase in e is not possible. So, if the left side of the previous equation is less than the right, the banker has no incentive to adjust e. The banker certainly would not want to reduce e below $e_{\max}$ because doing so would have a smaller marginal benefit, $f'\left(e_{\max}\right)$, than the marginal cost, $\beta p'\left(e_{\max}\right)\left[R^g\left(N + d\right) - R_d^g d\right]$ (at $e_{\max}$ the derivatives of f and g are interpreted as left derivatives). We seek restrictions on f and g which guarantee that there is a unique value of e that solves (6.1). Define

$$
\begin{aligned}
\frac{d}{de}\left(\frac{f'\left(e\right)}{p'\left(e\right)}\right) &= \frac{f''\left(e\right)}{p'\left(e\right)} - \frac{f'\left(e\right)}{\left[p'\left(e\right)\right]^2}p''\left(e\right) \\
&= \frac{f'\left(e\right)}{ep'\left(e\right)}\left[\frac{ef''\left(e\right)}{f'\left(e\right)} - \frac{ep''\left(e\right)}{p'\left(e\right)}\right] \\
&= \frac{f'\left(e\right)}{ep'\left(e\right)}\left[\epsilon_f\left(e\right) - \epsilon_p\left(e\right)\right],
\end{aligned}
$$

where

$$
\epsilon_f\left(e\right) = \frac{ef''\left(e\right)}{f'\left(e\right)}, \quad \epsilon_p\left(e\right) = \frac{ep''\left(e\right)}{p'\left(e\right)}.
$$

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Following is a set of sufficient conditions for there to be a unique solution to (6.1):

$$\epsilon_f(e) > \epsilon_p(e), \quad 0 < e < e_{\max} \tag{6.2}$$

$$\frac{f'(0)}{p'(0)} = 0. \tag{6.3}$$

According to (6.2) there is more curvature in f than there is in p. The simple example in which $f(e) = e^2/2$ and $p(e)$ is linear obviously satisfies (6.2) and (6.3).

The ex ante problem of the banker is

$$\max_{e,d.R_d^g} \beta p(e)\left[R^g(N+d) - R_d^g d\right] - f(e),$$

subject to the incentive constraint and the zero profit condition for mutual funds. Let $d(e)$ denote the value(s) of d for which e is incentive compatible. That is, $d(e)$ solves (6.1) or,

$$\beta\left[R^g - \frac{R}{p(e)}\right]d(e) = \frac{f'(e)}{p'(e)} - \beta R^g N.$$

Let e_d be defined by

$$p(e_d)R^g - R.$$

Note that $e_d < e_{\max}$ because of the assumption, $R^g > R$. For $e > e_d$, $d(e)$ is single-valued. For $e = e_d$, $d(e)$ is composed of every $d \geq 0$. Replacing d in the banker's objective with $d(e)$:

$$\begin{aligned}
V(e) &= \beta p(e)R^g N + p(e)\beta\left[R^g - \frac{R}{p(e)}\right]d - f(e) \\
&= \beta p(e)R^g N + p(e)\frac{f'(e) - \beta p'(e)R^g N}{p'(e)} - f(e) \\
&= \frac{p(e)}{p'(e)}f'(e) - f(e).
\end{aligned}$$

The first order condition is:

$$\begin{aligned}
V'(e) &= f'(e) - f'(e) + p(e)\frac{d}{de}\left(\frac{f'(e)}{p'(e)}\right) \\
&= p(e)\frac{f'(e)}{ep'(e)}\left[\epsilon_f(e) - \epsilon_p(e)\right].
\end{aligned}$$

To have a unique interior solution for e, we require that $V'(e)$ switch sign exactly one time on the interior of $0 \leq e \leq e_{\max}$. This is inconsistent with (6.2). If we satisfy (6.2), then the optimal value of e is $e = e_{\max}$.

We conclude that we cannot find a p function that has the property that we have an interior solution for e.

7. Model with Diminishing Returns on Bank Balance Sheet

Here, we explore a version of the model in which the banker has diminishing returns to investment. This could happen for two reasons: diminishing returns in the technology, or market power. The example emphasized here assumes diminishing returns in technology.

7.1. Case with $R^b = 0$

Here, we consider the special case, $R^b = 0$. We begin with an interpretation of the diminishing returns according to which it comes from market power. We then consider the case where it is due to the investment technology itself. We then proceed to do the calculation for that case.

7.1.1. Market Power

We bring in market power in the usual Dixit-Stiglitz way. The mutual funds are 'retail banks'. They collect the deposits and lend them to the investment banks. The investment banks issue securities to specialized investment projects, which have diminishing returns. The ith investment bank earns r_i on its assets and receives the following profits:

$$\beta p(e)\left[r_i l_i - R_d^g d_i\right] - \frac{1}{2}e^2, \ l_i = n + d_i$$

A fraction, $1 - p$, of l_i turns into zero. This is an unsuccessful project. The l_i's that don't turn into zero are loaned to an aggregator:

$$l = \left[\int_0^p l_i^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} di,$$

where a fraction, $p\left(e\right)$, of $l_i = 0$. The aggregator takes all these funds and lends them for a profit,

$$R^l l - \int_0^p r_i l_i di$$

The aggregator is competitive and earns R^l on l. The first order conditions are:

$$R^l \left(\frac{l}{l_i}\right)^{\frac{1}{\varepsilon}} = r_i, \ i \in [0, p].$$

or,

$$l_i = l \left(\frac{R^l}{r_i}\right)^{\varepsilon}.$$

Then,

$$l = \left[\int_0^p \left(l \left(\frac{R^l}{r_i} \right)^\varepsilon \right)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} di$$

$$= l \left(R^l \right)^\varepsilon \left[\int_0^p \left(\frac{1}{r_i} \right)^{\varepsilon-1} di \right]^{\frac{\varepsilon}{\varepsilon-1}} di,$$

or,

$$\left(R^l \right)^{-\varepsilon} = \left[\int_0^p (r_i)^{1-\varepsilon} di \right]^{\frac{\varepsilon}{\varepsilon-1}} di$$

$$R^l = \left[\int_0^p (r_i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}} di.$$

The banker's ex ante objective is now

$$\beta p(e) \left[r_i (n + d_i) - R_d^b d_i \right] - \frac{1}{2} e^2$$

or,

$$\max_{e, R_d^b} \beta p(e) \left[R^l l^{1/\varepsilon} (n + d_i)^{\frac{\varepsilon-1}{\varepsilon}} - R_d^b d_i \right] - \frac{1}{2} e^2$$

The banker goes to get a loan contract, e, R_d^b, d, just like before. Ex post, the banker chooses e given R_d^b, d to solve

$$\max_e \beta p(e) \left[R^l l^{1/\varepsilon} (n + d_i)^{\frac{\varepsilon-1}{\varepsilon}} - R_d^b d_i \right] - \frac{1}{2} e^2,$$

which implies a first order condition:

$$e = \beta \bar{b} \left[R^l l^{1/\varepsilon} (n + d_i)^{\frac{\varepsilon-1}{\varepsilon}} - R_d^b d_i \right].$$

It is understood that the banker will do this, and so ex ante the banker's problem must take the previous equation as given.

7.1.2. Diminishing Return on Investment

7.1.2.1.Lagrangian Approach   The ex ante problem of the banker is

$$\max_{e, R_d^b} \beta \left(p(e) A (n + d)^\theta - Rd \right) - \frac{1}{2} e^2$$

$$+ \eta \left(\beta \bar{b} \left[A (n + d)^\theta - \frac{R}{p(e)} d \right] - e \right),$$

in Lagrangian form. In terms of the market power interpretation of the previous subsection,

$$A \equiv R^l l^{1/\varepsilon}, \ \theta \equiv \frac{\varepsilon - 1}{\varepsilon}.$$

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The first order conditions are:

$$e \quad : \quad \beta \bar{b} A (n + d)^\theta - e + \eta \left(\beta \left(\frac{\bar{b}}{p(e)} \right)^2 Rd - 1 \right) = 0$$

$$d \quad : \quad \beta \left[1 + \eta \frac{\bar{b}}{p(e)} \right] \left[p(e) \theta A (n + d)^{\theta - 1} - R \right] = 0$$

$$\eta \quad : \quad \beta \bar{b} \left[A (n + d)^\theta - \frac{R}{p(e)} d \right] - e = 0$$

These equations at first appear to have two solutions, each corresponding to setting one of the two objects in square brackets in the d equation to zero. As it turns out, the first object cannot be zero, for to suppose so entails a contradiction. In particular, suppose it is zero. Substitute out for e in the e equation from the η equation:

$$\beta \bar{b} \frac{R}{p(e)} d = -\eta \left[\beta \left(\frac{\bar{b}}{p(e)} \right)^2 Rd - 1 \right]$$

According to the first solution, $\eta = -p(e) / \bar{b}$, so multiplying the above by $p(e) / \bar{b}$:

$$\beta Rd = \left(\frac{p(e)}{\bar{b}} \right)^2 \left[\beta \left(\frac{\bar{b}}{p(e)} \right)^2 Rd - 1 \right] = \beta Rd - \left(\frac{p(e)}{\bar{b}} \right)^2,$$

or,

$$0 = \left(\frac{p(e)}{\bar{b}} \right)^2.$$

But, $p(e) \geq \bar{a} > 0$, contradiction. So, we cannot solve the first order conditions of the Lagrangian problem by setting the first object in brackets in the d equation to zero. Now consider setting the second object in d to zero:

$$p(e) \theta A (n + d)^{\theta - 1} = R. \tag{7.1}$$

We can solve this equation for e :

$$e = \frac{R - \bar{a} \theta A (n + d)^{\theta - 1}}{\bar{b} \theta A (n + d)^{\theta - 1}}.$$

Using this to substitute out for e in the η equation, that equation reduces to a nonlinear equation in d and R. We want a general equilibrium, so R is replaced by $R = 1 / (\beta (Y - d))$ so that the η equation is still an equation in d alone. With d and e in hand, η can then be solved from the e equation:

$$\eta = \frac{\beta \bar{b} A (n + d)^\theta - e}{1 - \beta \left(\frac{\bar{b}}{p(e)} \right)^2 Rd} = \frac{\beta \frac{\bar{b}}{p(e)} Rd}{1 - \beta \left(\frac{\bar{b}}{p(e)} \right)^2 Rd}.$$

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We now check whether the solution to e and d we have found satisfy the second order conditions for e and d to be a (local) maximum for the banker's deposit problem. The banker problem holds R fixed at its equilibrium value. Define

$$
\begin{aligned}
V(e,d;R) &= \beta\left(p(e)A(n+d)^\theta - Rd\right) - \frac{1}{2}e^2 + \eta\left(\beta\bar{b}\left[A(n+d)^\theta - \frac{R}{p(e)}d\right] - e\right) \\
g(e,d;R) &= \beta\bar{b}\left[A(n+d)^\theta - \frac{R}{p(e)}d\right] - e.
\end{aligned}
$$

We construct the bordered Hessian in the usual way:

$$
\begin{bmatrix}
V_{ee} & V_{ed} & g_e \\
V_{ed} & V_{dd} & g_d \\
g_e & g_d & 0
\end{bmatrix}.
$$

Here, $V_{i,j}$ denotes the derivative of V with respect to i, j, where $i = e, d$ and $j = e, d$. Similarly for g_i. Condition (7.1) implies that $g_d = 0$. It then follows that the determinant is:

$$
-g_e^2 V_{dd},
$$

where

$$
\begin{aligned}
V_{dd} &= \beta\left(1 + \frac{\eta\bar{b}}{p(e)}\right)\theta(\theta-1)p(e)A(n+d)^{\theta-2} \\
g_e &= \beta\left(\frac{\bar{b}}{p(e)}\right)^2 Rd - 1.
\end{aligned}
$$

Note that, using the solution for η provided above,

$$
g_e = -\frac{\beta\frac{\bar{b}}{p(e)}Rd}{\eta}.
$$

Intuitively, we expect $\eta > 0$ for two reasons. First, the incentive constraint is assumed to be binding. Second, suppose η is zero, then the value of the constraint multiplying η in the banker's Lagrangian is negative. So, under a penalty function interpretation of the multiplier, we need $\eta > 0$ to push the Lagrangian down. Still, we have not found a direct proof that $\eta > 0$.

We can see that if we want the second order sufficient conditions for a local maximum to be satisfied, then we need to have

$$
1 + \frac{\eta\bar{b}}{p(e)} > 0.
$$

This does not rule out a negative value of η.

We considered the following model parameters:

$$
A = 1,\ \bar{a} = 0.1,\ \bar{b} = 1,\ \beta = 0.99,\ \theta = 0.80,\ Y = 1.396,\ N = 0.766,
$$

implying the following solution to the first order conditions:

$$d = 0.050, \ R = 0.750, \ e = 0.80, \ p = 0.90, \ \eta = 0.043.$$

The bordered Hessian is:

$$\begin{bmatrix} -1.0 & 0.86 & 0.95 \\ 0.86 & -0.19 & 0.0 \\ 0.95 & 0.0 & 0 \end{bmatrix}.$$

The determinant of this matrix is

$$0.17,$$

which is positive. So, the solution to the first order conditions is a local maximum for the banker's deposit choice problem.

We have found that the above scenario is not a deposit contract equilibrium. This is because the return on deposits, 0.40, is dominated by the return on depositing net worth in the bank, whose an option whose period 1 value is $\beta R N = 0.569$. After a not-very-systematic search, we were not able to find model parameters with the property that the solution to the banker problem satisfies the determinant condition and dominates the banker's outside option.

7.1.2.2. Substituting Out the Constraints We now investigate whether we have a global maximum. We pursue this by substituting out the incentive constraint. The incentive constraint is:

$$f(d; e, R) \equiv \frac{\beta \bar{b}}{p(e)} \left[p(e) A (n + d)^\theta - Rd \right] - e \begin{cases} = 0 & 0 < e < (1 - \bar{a})/\bar{b} \\ \geq 0 & e = (1 - \bar{a})/\bar{b} \\ \leq 0 & e = 0 \end{cases}.$$

We seek $d(e; R)$, the set of d such that the incentive constraint is satisfied for given e and R. The set, $d(e; R)$, could be empty. Then, define

$$\tilde{V}(e; R) = \max_{d \in d(e)} \left[\beta \left(p(e) A (n + d)^\theta - Rd \right) - \frac{1}{2} e^2 \right].$$

The problem we seek to solve is

$$\max_{0 \leq e \leq e_{\max}, \ d(e) \text{ non-empty}} \tilde{V}(e; R).$$

We can write the objective in terms of f :

$$\frac{p(e)}{\bar{b}} [f + e] = \beta \left[p(e) A (n + d)^\theta - Rd \right]$$

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so that

$$\beta \left(p(e)A (n + d)^\theta - Rd \right) - \frac{1}{2}e^2$$
$$= \frac{p(e)}{\bar{b}} [f (d; e, R) + e] - \frac{1}{2}e^2.$$

If there is a d that solves the IC constraint for given e, then this reduces to:

$$\tilde{V} (e; R) = \frac{p(e)}{\bar{b}}e - \frac{1}{2}e^2 = \frac{\bar{a}}{\bar{b}}e + \frac{1}{2}e^2.$$

Superficially, this suggests that whatever e we have, a higher value of e is superior. Again, superficially, this suggests that at the solution identified above, where $0 < e < (1 - \bar{a})/\bar{b}$, a higher value dominates. This is a mistake, because $\tilde{V}$ can only be evaluated for an e that has the property that $d(e)$ is non-empty. The fact that the incentive constraint is satisfied implies:

$$f (d, e) = \beta\bar{b} \left[A (n + d)^\theta - \frac{R}{p(e)}d \right] - e = 0.$$

At the same time, (7.1) indicates that d is set to the value that makes f as big as possible, given e. If $f = 0$ and (7.1) is satisfied, then if e is increased to $e + \varepsilon$, where $\varepsilon > 0$, then there is no d such that $f (d, e + \varepsilon) = 0$. That is, $d(e)$ is empty, no matter how small is ε. Thus, when we substitute out the banker's constraint, the solution is on a corner. It is not a corner where $e = (1 - \bar{a})/\bar{b}$ or $d = Y$. It is not a corner from the point of view of the Lagrangian problem.

7.1.3. Properties of Equilibrium

$$\beta \left(p(e)A (n + d)^\theta - R (N + d) \right) - \frac{1}{2}e^2 + \beta RN$$
$$= \beta \left[p(e)A (n + d)^{\theta-1} - R \right] (N + d) - \frac{1}{2}e^2 + \beta RN$$
$$= \beta p(e) (1 - \theta) A (n + d)^\theta - \frac{1}{2}e^2 + \beta RN,$$

using (7.1). Suppose the bank take on no deposits. Then, its effort is:

$$e^* = \beta\bar{b}AN^\theta,$$

and its objective takes on the value,

$$\beta p^* An^\theta - \frac{1}{2}e^2$$

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7.2. Case with $R^b > 0$

Here, we set up the problem with diminishing returns when $R^b > 0$. We did not investigate numerical examples of this case. First, we evaluate the outside option of the banker, which is to not take a deposit contract at all. In this case the banker solves

$$\max_e \beta \left[p(e) R^g N^\theta + (1 - p(e)) R^b N^\theta \right] - \frac{1}{2} e^2,$$

and optimality implies (assuming an interior solution):

$$e^* = \beta p'(e) \left(R^g - R^b \right) N^\theta.$$

The value of the objective is:

$$V^* = \beta \left[p^* R^g N^\theta + (1 - p^*) R^b N^\theta \right] - \frac{1}{2} (e^*) e^2.$$

Next, we consider a banker who takes a deposit contract. At the time that the banker takes out a loan with parameters, e, d, R_d^g, R_d^b, the objective is:

$$\beta \left[p(e) \left(R^g (N + d)^\theta - R_d^g d \right) + (1 - p(e)) \left(R^b (N + d)^\theta - R_d^b d \right) \right] - \frac{1}{2} e^2.$$

Ex post, since effort is not observed the banker selects the best possible e subject to the given R_d^g, R_d^b, d :

$$e = \beta p'(e) \left[\left(R^g - R^b \right) (N + d)^\theta - \left(R_d^g - R_d^b \right) d \right].$$

The mutual funds make zero profits:

$$p(e) R_d^g + (1 - p(e)) R_d^b = R.$$

The cash constraints are:

$$R^g (N + d)^\theta \geq R_d^g d, \quad R^b (N + d)^\theta \geq R_d^b d.$$

To solve this model we first assume that the cash and incentive constraints are not binding, while only the zero profit condition is binding.

Substituting out the zero profit condition from the objective, we conclude that the banker's problem is:

$$\max_{d,e} \beta \left[p(e) R^g (N + d)^\theta + (1 - p(e)) R^b (N + d)^\theta - Rd \right] - \frac{1}{2} e^2.$$

Optimality implies:

$$
\begin{aligned}
e &= \beta p'(e) \left(R^g - R^b \right) (N + d)^\theta \\
R &= \theta \left[p(e) R^g + (1 - p(e)) R^b \right] (N + d)^{\theta - 1}.
\end{aligned}
$$

The equilibrium can be found by directly replacing R with $1/\left(\beta\left(Y-d\right)\right)$. This gives two equations in the two unknowns, e, d. Now, we need to construct values of R_d^g, R_d^b such that the incentive and cash constraints are satisfied. The incentive constraint requires $R_d^g = R_d^b$. The zero profit condition then implies

$$R_d^g = R_d^b = R.$$

We verify that cash constraints are satisfied. For interesting parameter values, they are not satisfied. In particular, we can expect that the cash constraint in the bad state is not satisfied. That is, the bank does not have enough net worth to insulate its creditor from a bad outcome on its balance sheet.

In this case, we move to the case where the cash constraint in the good state is ignored and the cash constraint in the bad state is binding. This is a constraint on the loan contract agreed on by the mutual fund and the bank. It is easy to substitute out of the problem, using

$$R - R_d^b = p\left(e\right)\left(R_d^g - R_d^b\right).$$

At the time that the contract, e, d, R_d^g, R_d^b, is agreed on the zero profit condition, the effort equation and the cash constraint are restrictions that the contract must satisfy. We use the zero profit condition to solve out for R_d^g and R_d^b in the incentive constraint:

$$e = \beta p'\left(e\right)\left[\left(R^g - R^b\right)\left(N + d\right)^\theta - \frac{R - R_d^b}{p\left(e\right)}d\right].$$

Using the cash constraint we simplify this further:

$$e = \frac{\beta p'\left(e\right)}{p\left(e\right)}\left[p\left(e\right)\left(R^g - R^b\right)\left(N + d\right)^\theta - \left(Rd - \left(N + d\right)R^b\right)\right].$$

Let $d\left(e\right)$ denote the value of $d \in [0, Y]$ for the incentive constraint is satisfied given a particular value of e. A value of e for which $d\left(e\right)$ is empty is not admissible. If there is more than one $d \in [0, Y]$, then $d\left(e\right)$ is the d that maximizes

$$\beta\left[p(e)R^g\left(N + d\right)^\theta + \left(1 - p\left(e\right)\right)R^b\left(N + d\right)^\theta - Rd\right] - \frac{1}{2}e^2.$$

Then, the problem is:

$$\max_{0 \le e \le (1-\bar{a})/\bar{b},\, d(e)\text{ not empty}} \beta\left[p(e)R^g\left(N + d\left(e\right)\right)^\theta + \left(1 - p\left(e\right)\right)R^b\left(N + d\left(e\right)\right)^\theta - Rd\left(e\right)\right] - \frac{1}{2}e^2.$$

This objective can be graphed over all e satisfying the indicated constraint.

The solution to the banker problem provides a mapping from R to e and $d\left(e\right)$. The household fonc provides a mapping from d to R. The intersection represents an equilibrium. It is possible to investigate whether a deposit restriction is desirable. Once the equilibrium is solved, then reduce d by 5% and compute the new R from the household fonc. Feed that R to the banker who then (taking d as given) solves for e.