Unifying Identity-Specific and Financial Externalities in Auction Design*

Jingfeng Lu
National University of Singapore
December 2007

Abstract Coexistence of identity-specific and financial externalities among bidders is a salient feature of auctions with buyers who are cross shareholders or competing firms in an oligopoly. This paper unifies these two types of externalities in revenue-maximizing auction design. Our main findings are the following. **First**, these two types of externalities can be unified through the framework of Myerson (1981). Both affect the winning probabilities through their impact on players' externality-augmented virtual values, while their impact on buyers' payments is reflected by an externality-correcting component for each type of externalities, which equals the respective externalities. These components eliminate strategic bidding that would arise from the existence of externalities. **Second**, the two types of externalities interact fundamentally through shaping players' virtual values. At the optimum, the player with the highest externality-augmented virtual value wins given that it is positive, otherwise seller physically destroys the item. Financial externalities amplify the impact of the identity-specific externalities on winning probabilities. **Third**, our approach is applicable to revenue-maximizing auction design with cross shareholding. **Fourth**, our finding renders an approach for revenue-maximizing auction design with asymmetric financial externalities. Particularly, when seller does not value the object, a revenue-maximizing auction can be obtained from any revenue-maximizing auction for a regular setting without externalities by solely transforming the payments.

Keywords: Auction design; Financial Externalities; Identity-Specific Externalities.
JEL classifications: D44, D82.

* I thank Murali Agastya, Olivier Bos, Subir Bose, Isabelle Brocas, Parkash Chander, Olivier Compte, Jeff Ely, Claude d'Aspremont, George Deltas, Atsushi Kajii, Emiel Massland, Preston McAfee, Benny Moldovanu, Tymofiy Mylovanov, Sander Onderstal, Thomas Palfrey, Ariel Rubinstein, Aggey Semenov, Sergei Severinov, Yeneng Sun, Asher Wolinsky, Licun Xue, Nicholas Yannelis and Zaifu Yang for helpful comments and suggestions. All errors are mine. Earlier versions have been presented at the 2007 Econometrics Society Summer Meeting, the 2007 SAET Conference, the 17th International Game Theory Conference. Early versions have been circulated under the title "When and How to Dismantle Nuclear Weapons: Auction Design with Externalities". Financial support from National University of Singapore (R-122-000-088-112) is gratefully acknowledged.
Address: Department of Economics, National University of Singapore, Singapore 117570. Tel : (65) 6516-6026. Fax : (65) 6775-2646. Email: ecsljf@nus.edu.sg.

1 Introduction

Many auctions are featured by the prevalence of allocative externalities among players (including seller and bidders). An allocation outcome in an auction setting can be described by the winner of the object and the payments of buyers (Myerson (1981)). Thus, allocative externalities on any player can depend on both the identity of the winner and the buyers' payments to allow for the most generality. The externalities are identity-specific if they depend on only the identity of winner, while they are called financial externalities if they are contingent only on buyers' payments. While licensing an innovation among competitors and selling nuclear weapons are employed to exemplify identity-specific externalities (e.g.,Jehiel, Moldovanu and Stacchetti (1996)), charity auctions (e.g., Goeree, Maasland, Onderstal and Turner (2005), Maasland and Onderstal (2007) and Engers and McManus (2007)) and bidding rings (e.g., McAfee and McMillan (1992) and Deltas (2002)) are more concerned with financial externalities . While these two types of externalities do not always go together, auction situations are not rare where they do. An salient example is an auction where buyers are cross shareholders who compete for a scarce resource. Since each buyer holds a share of other's profits, each buyer suffers a negative financial externality that equals a share of other buyer's payments. On top of this, each buyer enjoys an identity-specific externality that equals a share of the winner's added value from winning the auction (Dasgupta and Tsui (2004)). Though licensing an innovation among competitors and selling nuclear weapons are mainly used to exemplify identity-specific externalities, these examples actually also involve financial externalities. For example, in the North Korea's nuclear weapon case, while the seller (North Korea) puts great identity-specific externalities on the buyers (China, Japan, Russia, South Korea, US) if it keeps its nuclear arsenal, some buyers may also experience financial externalities due to the payments of buyers. Some parties (e.g., South Korea) may prefer that North Korea raises more money to support itself in order to alleviate the burden on them to support North Korea, while others (e.g., US) may not like North Korea to consolidate its power by raising a lot of money. In the technical innovation licensing example, while the winner imposes negative identity-specific externalities on losers' profits, the buyers could also gain utility from other competitors' payments, since the higher payments of other buyers certainly decrease their resources available for other dimensions of competition, such as advertising expenses.

Revenue-maximizing auctions focusing on a single type of externalities have been studied by a number of papers.[1] Jehiel, Moldovanu and Stacchetti (1996, 1999), Varma (2002), and Brocas (2003, 2005) among others consider identity-specific externalities imposed on losers by the winning buyer, while Goeree, Maasland, Onderstal and Turner (2005), Maasland and Onderstal (2007) and Engers and McManus (2007) study the cases where financial externalities among buyers depend on payments of themselves. The approaches adopted for auction design with different types of externalities show little similarities and connections, so do the derived revenue-maximizing auctions. Clearly, identity-specific and financial externalities differ completely in nature. It remains in question whether auction design with these two types of externalities can be unified in an integrated framework. Furthermore, the existing insights say little on how to design revenue-maximizing auction when both types of externalities prevail, since nothing has been revealed in the literature so far on how they might interact. In this paper, we adopt a setting where both identity-specific and financial externalities exist and study the revenue-maximizing selling mechanism. This study illustrates that both types of externalities indeed affect the revenue-maximizing allocation rule (the winning probabilities and payments) through the same channel. In this sense, we provide a unified approach for obtaining the revenue-maximizing mechanisms with pure identity-specific and/or financial externalities. More importantly, our study further identifies the interaction between the two types of externalities in shaping the revenue-maximizing mechanism. This leads to a complete characterization of the revenue-maximizing auction for the case where both types of externalities matter.

Following Goeree, Maasland, Onderstal and Turner (2005), we begin with a setting where financial externalities among buyers are proportional to the total payments of all buyers. Diverging from their setting, we allow these proportions of externalities to differ across buyers to accommodate more generality.[2] While the literature largely focuses on identity-specific externalities among buyers, in this paper we allow identity-specific externalities among **all** players, including the seller and buyers. Situations abound where the existence of identity-specific externalities between the seller and buyers is the major

[1] Dasgupta and Tsui (2004) is the only paper that derives the equilibrium strategy for standard auctions with cross-shareholding. However, they have not touched the revenue-maximizing auction in this setting.

[2] A most general setup of financial externalities of linear form is addressed when auction with cross shareholders is also studied in Section 5.

concern. One recent example is the North Korea's nuclear weapon case, where the seller (North Korea) puts great externalities on the buyers (China, Japan, Russia, South Korea, US) if it keeps its nuclear arsenal. Other examples where externalities exist between the seller and buyers include selling retaliation in the WTO by a member country or selling a soccer player by a club. We allow the bidders to be heterogeneous in both their value distributions and identity-specific externalities that they enjoy/suffer. In particular, our analysis does not require the identity-specific externalities to be uniformly positive or negative.

One feature of our analysis lies in the option for the seller to physically destroy the item. One should note that "destroying the item" differs from "not-selling". In the context of selling nuclear weapons, "dismantling" means "destroying" in this paper. In previous auction design literature, destroying the auctioned item has not been formulated as a possible outcome or as a nonparticipation threat. The significance of introducing this option is the following. First, we can explicitly address under what conditions should the seller destroy the object and what actions should be taken to maximize his revenue if he destroys the object. Second, allowing this new option enlarges the freedom of auction design with externalities. Destroying the item can be an optimal allocation outcome for the seller or be used by the seller as an optimal nonparticipation threat, since it eliminates the identity-specific externalities imposed on buyers. Specifically, eliminating these identity-specific externalities has two effects. First, seller's threat to a buyer who refuses to participate can be made more severe. This happens when a buyer enjoys positive identity-specific externalities whoever else keeps the object. In this case, the most severe nonparticipation threat is to destroy the object. Second, the seller may extract higher rent when he destroys the object if it is unsold. This can occur when the buyers suffer highly negative identity-specific externalities when the seller keeps the item. In this situation, the seller can be better off by committing to destroy the object and collecting a payment from each buyer.

The revenue-maximizing selling mechanism is fully characterized in terms of the non-participation threats, the winning probabilities, the probability of destroying the item, and the payments of buyers. The two types of externalities interact fundamentally through shaping players' externality-augmented virtual values, which are obtained from the regular virtual values by adjusting for the externalities and seller's destroying cost.[3] If only buyer i does not participate, the item is then assigned to the one (including the seller) generating buyer i the smallest identity-specific externality provided that this externality is nonpositive. Otherwise the seller destroys the object. These nonparticipation threats induce full participation of bidders. The winning probabilities are determined by the players' augmented virtual values. Both types of externalities affect the winning probabilities solely through the same channel of modifying players' virtual values. The player with the highest augmented virtual value wins given that it is positive. If no buyer wins, the seller may keep the item or destroy it. The seller destroys the item if and only if his augmented value is negative. A unique feature of the payment schedule is that every buyer's payment includes two externality-correcting components that equal the allocative identity-specific and financial externalities, respectively.

The general findings are applied to various settings with pure identity-specific and/or financial externalities. For symmetric settings in particular, we establish that modified second-price and/or first-price auctions with appropriately set entry fee and reserve price are revenue maximizing. Each buyer's payments include an externality-correcting component that equals the allocative externalities to him. Our study also leads to further results on auction design for settings with pure financial externalities. Useful linkages between revenue-maximizing auctions for settings with and without externalities are discovered. In particular, for the case where the seller does not value the item,[4] we establish one-to-one correspondences between revenue-maximizing auctions with and without externalities. We find that the revenue-maximizing auctions for a regular setting without externalities need only be modified in the payments in order to be revenue-maximizing in settings with financial externalities.[5]

The methodology developed is further applied to revenue-maximizing auction design

[3]Please refer to Lemma 2 for detailed definitions of players' externality-augmented virtual values.

[4]This assumption is commonly adopted in the literature of auction design with financial externalities, such as Maasland and Onderstal (2002, 2007) and Goeree, Maasland, Onderstal and Turner (2005) and Engers and McManus (2007).

[5]Goeree, Maasland, Onderstal and Turner (2005) show that a lowest-price all-pay auction is revenue-maximizing in symmetric settings of financial externalities. Applying our findings to their setting leads to alternative revenue-maximizing first-price or second-price auctions. One advantage of the alternative second price auction lies in that the maximal expected revenue is implemented through weakly dominant strategy.

with cross shareholding buyers who compete for a scarce resource. As each buyer holds a share of other's profits, each buyer suffers a negative financial externality that equals a share of other buyer's payments while enjoying an identity-specific externality that equals a share of the winner's added value from winning the auction (Dasgupta and Tsui (2004)). Thus, the identity-specific externalities are perfectly correlated to the private information of the winner. Furthermore, financial externalities in this setting takes the most general form in the linear class. The general applicability of the methodology developed is further evidenced through this application.

The remainder of the paper is organized as follows. Section 2 introduces the basic model with both identity-specific and financial externalities. Section 3 derives the revenue-maximizing mechanism for this setting. Section 4 applies the general findings to further study revenue-maximizing auctions in settings with pure identity-specific and/or financial externalities, respectively. Section 5 derives revenue-maximizing auction with cross shareholding buyers. Section 6 concludes.

2 The Model

2.1 Motivating Examples

Jehiel, Moldovanu and Stacchetti (1996) present an example with N firms competing in an oligopoly. A technical innovation will be licensed to only one firm. Whoever wins the license will increase his market share. Thus the adoption of the innovation by the winner will increase his profit, however, it will also decrease the profit of the losers. In this sense, the winner imposes negative identity-specific externalities on losers. If the seller is also a competitor, his adoption of the innovation also imposes identity-specific externalities on the buyers. We can further imagine that the buyers could gain utility from other buyers' payments as they are competitors, since higher payments of the competitors in the auction certainly decrease their resources available for other dimensions of competition, such as advertising expense. The utility component from other buyers' payments can thus be modeled as financial externalities. These financial externalities effect could be very significant especially when buyers are financially constrained as the auctioned object could be very expensive such as those spectrum auctions.

The North Korea nuclear weapon case also mimics the above situation. China, Japan, Russia, South Korea, US and others (such as Iran) could be considered as potential buyers, while North Korea can be considered as the seller. Clearly, there are identity-specific externalities among players as some may feel threatened if a particular country holds the nuclear weapon. Moreover, one can see the existence of financial externalities for the following arguments. Some parties (e.g., South Korea) may benefit from the money that North Korea could raise to support itself since it can help to alleviate the burden on them to support North Korea, while others (e.g., US) may not like to see that North Korea consolidates its power by raising a lot of money.

A third example where both identity-specific and financial externalities prevail emerges simply due to cross shareholding among buyers. Consider a situation where buyers are cross shareholders who compete for a scarce resource. Obtaining the resource increases the value of the winner but does not affect the value of the losers. Since each buyer holds a share of other's profits, each buyer suffers a negative financial externality that equals a share of other buyer's payments. On top of this, each buyer enjoys an identity-specific externality that equals a share of the winner's added value due to winning the auction (Dasgupta and Tsui (2004)). A salient feature of this example lies in that the identity-specific externalities are rather determined by the winner's added value that is his private information. In this example, the identity-specific externalities are perfectly correlated to the private information of the winner.

2.2 The Basic Setting

In this section, we first focus on a basic setting that accommodate the first two examples. We will further study the setting of the third example with cross shareholding in Section 5. There is a seller who wants to sell one indivisible object to N potential buyers through an auction. We use $\mathcal{N} = \{1, 2, \cdots, N\}$ to denote the set of all potential buyers, where $\mathcal{N}$ is public information. The seller's value for the object is $v_0(\geq 0)$, which is public information. Hereafter, we represent the seller as player 0 and bidder i as player i. The ith buyer's private value of the object is v_i, which is his private information. These values $v_i,\ i \in \mathcal{N}$ are independently distributed on intervals $[\underline{v}_i, \overline{v}_i]$ respectively following cumulative distribution function $F_i(\cdot)$ with density function $f_i(\cdot)(> 0)$. We assume the regularity condition that the virtual value functions $J_i(v) = v - (1 - F_i(v))/f_i(v)$ increase on intervals $[\underline{v}_i, \overline{v}_i]$. The density $f_i(\cdot)$ is assumed to be public information. The seller and the buyers are assumed to be risk neutral.

Player i enjoys/suffers an externality e_{ij} when player j obtains the item, $i, j = 0, 1, \cdots, N$. By definition, $e_{ii} = 0$, $i = 0, 1, \cdots, N$. These externalities are public information.[6] The auctioned item can be destroyed by the seller at a cost of $c_0 \geq 0$. If the item is destroyed, no player enjoys/suffers any identity-specific externality. In addition, there exist financial externalities among the bidders. Specifically, bidder i enjoys/suffers an externality $\alpha_i \sum_{j \in \mathcal{N}} x_j$ from every bidder' payments x_j, $j \in \mathcal{N}$.[7] We assume that $\alpha_i \in [0, 1)$, $\forall i \in \mathcal{N}$ and $\sum_{i \in \mathcal{N}} \alpha_i < 1$.[8]

As a result, buyer i's payoff is $v_i - x_i + \alpha_i \sum_{j \in \mathcal{N}} x_j$ if he wins and pays x_i; his payoff is $e_{ij} + \alpha_i \sum_{j \in \mathcal{N}} x_j - x_i$ if he pays x_i while another player j (seller or buyer) wins; and his payoff is $\alpha_i \sum_{j \in \mathcal{N}} x_j - x_i$ if he pays x_i while the item is destroyed. The seller's payoff is $v_0 + \sum_{j \in \mathcal{N}} x_j$ if he keeps the item; his payoff is $e_{0i} + \sum_{j \in \mathcal{N}} x_j$ if bidder i wins; his payoff is $\sum_{j \in \mathcal{N}} x_j - c_0$ if the item is destroyed by him.

The game extends as follows. At **time 0**, the proportions α_i, the seller's value v_0, the destroying cost c_0, the identity-specific externalities e_{ij} and the distributions of $v_i, i \in \mathcal{N}$ are revealed by Nature as public information. Every buyer i, $i \in \mathcal{N}$ observes his private value v_i. At **time 1**, the seller announces the rule of the selling mechanism. The possibility of destroying the item by the seller is allowed. We assume that the seller has the power of committing to the proposed rule. At **time 2**, the buyers simultaneously and confidentially make their participation decisions and announce their bids if they decide to participate. At **time 3**, the payoffs of the seller and buyers are determined according to the announced rule at time 1.

Externalities lead to an auction design problem in which the buyers have mechanism-dependent reservation utilities. Jehiel and Moldovanu (1996) have pointed out that for

[6] Jehiel, Moldovanu and Stacchetti (1996) allow e_{ij} to be private information of player j. They however found that this additional complication does not deliver additional insight. In Section 5, we consider a case with cross shareholding where identity-specific externalities are private information of the party who imposes the externalities.

[7] A most general setting where bidder i enjoys $\alpha_{ij} x_j$ externalities from bidder j's payments x_j, $j \in \mathcal{N}$ will be considered in Section 5 when revenue maximization with cross shareholders is addressed.

[8] Similar restrictions on the magnitude of financial externalities have been adopted by Goeree, Maasland, Onderstal and Turner (2005), Maasland and Onderstal (2007) and Engers and McManus (2007). This restriction guarantees the existence of revenue-maximizing mechanism. This will be clearer when Lemma 2 is shown.

a first price auction the best strategy of some bidders is simply not to participate to the auction, although doing so does not avoid the negative externalities they may experience. For this reason, we explicitly deal with the revenue-maximizing endogenous participation and derive the revenue-maximizing mechanism while allowing endogenous entry.

Based on the "semirevelation" principle established by Stegeman (1996) that allows no participation, we only need to consider truthful direct semirevelation mechanisms, which require buyers to submit signals if and only if they participate, and reveal truthfully their types if they participate. Following Stegeman (1996), we introduce a null message $\emptyset$ to denote the signal of a nonparticipant.[9] Let $\mathbf{m} = (m_1, m_2, \cdots, m_N)$, where m_i is the signal of buyer i and it takes values in $\mathcal{M}_i = [\underline{v}_i, \overline{v}_i] \cup \{\emptyset\},\ \forall i \in \mathcal{N}$. Define $\mathcal{M} = \prod_{i=1}^{N} \mathcal{M}_i$. The seller determines how to allocate the object and how much each buyer pays, using a set of outcome functions that accommodates all participation possibilities. These outcome functions announced by the seller consist of the probability $p_0(\mathbf{m})$ for the seller to keep the item, the winning probability functions $p_i(\mathbf{m})$ and payment functions $x_i(\mathbf{m})$ of buyer $i, \forall i \in \mathcal{N}$. Note that $1 - \sum_{i=0}^{N} p_i(\mathbf{m})$ is the probability of destroying the item by the seller. This set of allocation functions is denoted by $(\mathbf{p}, \mathbf{x})$. Following Jehiel, Moldovanu and Stacchetti (1996), we assume that the buyers who do not participate have no chance to win the object and their payments to the seller are zero, i.e., $p_i(\mathbf{m}) = 0$ and $x_i(\mathbf{m}) = 0$ if $m_i = \emptyset,\ \forall i \in \mathcal{N},\ \forall \mathbf{m} \in \mathcal{M}$.[10] In addition, clearly the feasibility of mechanism $(\mathbf{p}, \mathbf{x})$ requires $\sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}$.[11]

Denote by $\mathcal{E}$ a pure-strategy entry pattern, in which $\mathcal{T}_i \subset [\underline{v}_i, \overline{v}_i]$ is the participating type of bidder i. We say $(\mathbf{p}, \mathbf{x})$ is a truthful direct semirevelation mechanism implementing entry $\mathcal{E}$ if and only if the following conditions hold:

(a) Buyer i with private values belonging to $\mathcal{T}_i$ participates and reveals truthfully his value, i.e., if he participates, he gets expected utility which is equal to or higher than his expected utility from nonparticipation.

(b) Buyer i with private values belonging to $[\underline{v}_i, \overline{v}_i] \setminus \mathcal{T}_i$ does not participates, i.e., if he participates, he gets expected utility which is equal to or lower than his expected utility

[9] Unlike the revelation principle whose applicability requires full participation of buyers, the "semirevelation" principle accommodates all entry patterns including the full participation.

[10] This assumption is consistent with the **no passive reassignment** (NPR) assumption adopted by Stegeman (1996).

[11] It deserves to be pointed out that the mechanism $(\mathbf{p}, \mathbf{x})$ includes in itself all nonparticipation threats.

from nonparticipation.

(c) $p_i(\mathbf{m}) \geq 0,\ \forall 0 \leq i \leq N$, with $\sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}$.

(d) $p_i(\mathbf{m}) = 0$ and $x_i(\mathbf{m}) = 0$ if $m_i = \emptyset,\ \forall i \in \mathcal{N},\ \forall \mathbf{m} \in \mathcal{M}$.

When $\mathcal{T}_i = [\underline{v}_i, \overline{v}_i], \forall i \in \mathcal{N}$, we have the case of full participation. Any allocation outcome (the winning probabilities and payments) implemented by a truthful direct semirevelation mechanism inducing any entry $\mathcal{E}$ is replicable through a **full-participation** truthful direct semirevelation mechanism that treats signals in $[\underline{v}_i, \overline{v}_i] \setminus \mathcal{T}_i$ as null signal $\emptyset$. We can always modify the original truthful direct revelation mechanism that implements entry $\mathcal{E}$ by treating bidder i's signal $m_i \in [\underline{v}_i, \overline{v}_i] \setminus \mathcal{T}_i$ as $\emptyset$. The modified mechanism is thus a truthful direct revelation mechanism that induces full participation.[12] Therefore, there is no loss of generality to consider only the truthful direct semirevelation mechanisms that induce full participation for the revenue-maximizing mechanism.

It deserves to be pointed out that in our setting with allocative externalities, the mechanisms should accommodate the null signal $\emptyset$ even though full participation should be induced at the optimum. In particular, introducing the null signal $\emptyset$ is necessary for describing the nonparticipation threats.[13]

3 The Revenue-Maximizing Mechanism

In this section we derive the revenue-maximizing mechanism when both identity-specific and financial externalities exist. For any truthful direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ implementing full participation, the seller's expected revenue is given by:

$$\begin{aligned} R(\mathbf{p}, \mathbf{x}) &= E_{\mathbf{v}}\Big\{(v_0 + e_{0,0})p_0(\mathbf{v}) + \sum_{i=1}^{N} e_{0i}\ p_i(\mathbf{v}) - c_0\Big(1 - \sum_{i=0}^{N} p_i(\mathbf{v})\Big) + \sum_{i=1}^{N} x_i(\mathbf{v})\Big\} \\ &= E_{\mathbf{v}}\Big\{(v_0 + c_0 + e_{0,0})p_0(\mathbf{v}) + \sum_{i=1}^{N}(e_{0i} + c_0)\ p_i(\mathbf{v}) + \sum_{i=1}^{N} x_i(\mathbf{v})\Big\} - c_0, \qquad (1) \end{aligned}$$

where $\mathbf{v} = (v_1, v_2, \cdots, v_N)$. The support of $\mathbf{v}$ is $\mathcal{V} = \prod_{i=1}^{N} [\underline{v}_i, \overline{v}_i]$.

For buyer i with private value v_i, if he submits signal $m_i \in \mathcal{M}_i$, his interim expected payoff is given by:

$U_i(v_i, m_i; \mathbf{p}, \mathbf{x})$

[12] Detailed proof is available from the author upon request.

[13] Condition (10) in Lemma 1 will further illustrate this point.

$$= E_{\mathbf{v}_{-i}}\left(v_i \; p_i(m_i, \mathbf{v}_{-i}) + \sum_{j \geq 0} e_{ij} \; p_j(m_i, \mathbf{v}_{-i}) - x_i(m_i, \mathbf{v}_{-i}) + \alpha_i \sum_{j \in \mathcal{N}} x_j(m_i, \mathbf{v}_{-i})\right), \quad (2)$$

where $\mathbf{v}_{-i} = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_N)$. The support of $\mathbf{v}_{-i}$ is $\mathcal{V}_{-i} = \prod_{j=1, j \neq i}^{N} [\underline{v}_j, \overline{v}_j]$.

The seller's optimization problem is to find the revenue-maximizing truthful direct semirevelation mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ that implements full participation, i.e.,

$$\max_{(\mathbf{p}, \mathbf{x})} \; R(\mathbf{p}, \mathbf{x}) \quad (3)$$

Subject to:

$$(i) \; U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) \geq U_i(v_i, \emptyset; \mathbf{p}, \mathbf{x}); \; \forall v_i \in [\underline{v}_i, \overline{v}_i], \; \forall i \in \mathcal{N}, \quad (4)$$

$$(ii) \; U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) \geq U_i(v_i, v_i'; \mathbf{p}, \mathbf{x}); \; \forall v_i \in [\underline{v}_i, \; \overline{v}_i], \; v_i' \in [\underline{v}_i, \; \overline{v}_i], \; \forall i \in \mathcal{N}, \quad (5)$$

$$(iii) \; p_i(\mathbf{m}) = x_i(\mathbf{m}) = 0 \; \textit{if} \; m_i = \emptyset, \; p_i(\mathbf{m}) \geq 0, \; \forall i \in \mathcal{N}, \; \sum_{i=0}^{N} p_i(\mathbf{m}) \leq 1, \; \forall \mathbf{m} \in \mathcal{M}. \quad (6)$$

Restrictions (4)-(6) come from conditions (a)-(d) of Section 2.2.

For any direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$, we define

$$Q_i(v_i; \mathbf{p}) = E_{\mathbf{v}_{-i}} p_i(\mathbf{v}). \quad (7)$$

If $(\mathbf{p}, \mathbf{x})$ is a truthful direct semirevelation mechanism implementing full participation, then $Q_i(v_i; \mathbf{p})$ is the conditional expected probability that buyer i wins the object if his private value is v_i.

Following similar procedure of Myerson (1981), we can show the following necessary and sufficient conditions for a direct semirevelation mechanism to be a **truthful** one that implements full participation.[14]

Lemma 1: *Direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *is a* **truthful** *direct semirevelation mechanism that implements full participation, if and only if* $\forall i \in \mathcal{N}$ *the following conditions and (6) hold:*

$$Q_i(s_i; \mathbf{p}) \leq Q_i(v_i; \mathbf{p}), \; \forall \underline{v}_i \leq s_i \leq v_i \leq \overline{v}_i, \quad (8)$$

$$U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) = U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + \int_{\underline{v}_i}^{v_i} Q_i(s_i; \mathbf{p}) ds_i, \; \forall v_i \in [\underline{v}_i, \overline{v}_i], \quad (9)$$

$$U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) \geq U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}). \quad (10)$$

[14] The proof is omitted as it follows Myerson (1981) closely.

Note that (10) differs from its counterpart in Lemma 2 of Myerson (1981). In Myerson (1981), the utility level $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ that buyer i obtains if he does not participate is not mechanism dependent. In particular, $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ is exogenous and fixed at zero in Myerson (1981). However, in our setting with allocative externalities, $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ must be determined by the mechanism adopted and thus can differ from zero.

Define $\gamma_{ij} = -\alpha_i$ if $i \neq j$ and $\gamma_{ii} = 1 - \alpha_i$. Define $\Gamma_{N\times N} = (\gamma_{ij})_{i,j\geq 1}$ the payment coefficient matrix. When $\sum_{i\in\mathcal{N}} \alpha_i < 1$, Γ is nonsingular as $|\Gamma| = 1 - \sum_{i\in\mathcal{N}} \alpha_i > 0$. Define $\mathbf{b} = (b_i)_{N\times 1} = (\Gamma')^{-1}\mathbb{1}_{N\times 1}$ where all elements in $\mathbb{1}_{N\times 1}$ are 1. We thus have $b_i = \frac{1}{1-\sum_{j\in\mathcal{N}} \alpha_j} > 0,\ \forall i \in \mathcal{N}$.[15]

Before we proceed to derive the expression for the seller's expected revenue from a truthful direct semirevelation mechanism, we further introduce the following definitions of generalized virtual values of players.

Definition 0: (Externality-Augmented Virtual Values) $\tilde{J}_i(v_i) = b_i J_i(v_i) + \sum_{j\in\mathcal{N}} b_j e_{ji} + e_{0i} + c_0,\ i \in \mathcal{N}$ *are defined as the buyers' externality-augmented virtual value functions; and* $\tilde{J}_0(v_0) = v_0 + \sum_{j\in\mathcal{N}} b_j e_{j0} + c_0$ *is defined as the sellers' externality-augmented virtual value.*

Note that the externality-augmented virtual values cover the virtual values coined by Myerson (1981) as a special case, while accommodating the flexibilities of identity-specific and financial externalities as well as costs of destroying the object. When $e_{ij} = 0,\ c_0 = 0,\ \alpha_{ij} = 0, \forall i, j$, the externality-augmented virtual values degenerate to the standard case. Clearly, since $J_i(\cdot)$ is an increasing function, $\tilde{J}_i(\cdot)$ must be an increasing function as $b_i \geq 0, \forall i \in \mathcal{N}$.

Based on Lemma 1, we can replace (4) and (5) by (8), (9) and (10) in the seller's optimization problem. As a result, the expected revenue of the seller from a mechanism $(\mathbf{p}, \mathbf{x})$ satisfying conditions (4)-(6) is given in the following Lemma.

Lemma 2: *For a truthful direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *that implements full*

[15] For a setting of financial externalities where buyer $i,\ i \in \mathcal{N}$ enjoys an externality that equals a proportion (denoted by $\varphi_i \in [0, 1)$) of the total payments of the **other** buyers. We assume $\sum_{i\in\mathcal{N}} \frac{\varphi_i}{1+\varphi_i} < 1$. Then we have $b_i = \frac{1}{(1+\varphi_i)(1-\sum_{j\in\mathcal{N}} \frac{\varphi_j}{1+\varphi_j})} > 0$. I thank sander Onderstal who pointed out that this setting of financial externalities is isomorphic to that of Section 2.2.

participation, the seller's expected revenue can be written as

$$R(\mathbf{p},\mathbf{x}) = E_{\mathbf{v}}\Big\{\sum_{i=0}^{N} p_i(\mathbf{v})\tilde{J}_i(v_i)\Big\} - \sum_{i=1}^{N} b_i U_i(\underline{v}_i,\underline{v}_i;\mathbf{p},\mathbf{x}) - c_0. \tag{11}$$

Proof: See Appendix.

Lemma 2 differs from its counterpart in Myerson (1981) in the following aspects. First, a generalized version of virtual values appears in (11); Second, due to the financial externalities, constants $b_j,\ j \in \mathcal{N}$ appear both in $\tilde{J}_i(\cdot)$ and before U_i; Third, there is no term c_0 in myerson (1981). From Lemma 2, we immediately have the following revenue equivalence theorem.

Proposition 1: *The seller and bidders' expected payoffs from a mechanism that implements full participation are completely determined by the expected payoffs of the lowest types of* $\underline{v}_i,\quad i \in \mathcal{N}$ *and the players' winning probabilities for all* $\mathbf{v} \in \mathcal{V}$.

Proof: See Appendix.

We are now ready to characterize the optimal mechanism. Before we present the revenue-maximizing selling mechanism, we first introduce the following definitions.

Definition 1: (Nonparticipation Threats) *If only buyer* $i,\ i \in \mathcal{N}$ *does not participate, the item is assigned to the one (including the seller) who brings him the smallest identity-specific externality provided that it is nonpositive, otherwise the seller destroys the item. All buyers pay zero.*

The nonparticipation threats of Definition 1 share the same spirit with those of Jehiel, Moldovanu and Stacchetti (1996). As we do not employ transfers between seller and participating buyers to further push down the nonparticipants' payoffs, the threats of Definition 1 might not be the strongest possible. Clearly, variety of threats are feasible that differ in requirement for seller's commitment power and the degree of penalty. Nevertheless, no matter what threats are adopted, there is no loss of generality to focus on full-participation mechanism for revenue-maximizing auctions, though the optimal revenue depends on the strongness of the threats.

The nonparticipation threats of Definition 1 can be written equivalently as follows. $\forall i \in \mathcal{N}$, let $j_0 = argmin_{j \geq 0, j \neq i} e_{ij}$. If $e_{i,j_0} \leq 0$, then set $p^*_{j_0}(m_i, \mathbf{m}_{-i}) = 1$ where $m_i = \emptyset$ and $\mathbf{m}_{-i} \in \mathcal{V}_{-i}$. If $e_{i,j_0} > 0$, then set $p^*_j(m_i, \mathbf{m}_{-i}) = 0,\ \forall j \geq 0$, where $m_i = \emptyset$ and $\mathbf{m}_{-i} \in \mathcal{V}_{-i}$. In addition, $\forall j \in \mathcal{N}$, $x^*_j(m_i, \mathbf{m}_{-i}) = 0$ where $m_i = \emptyset$ and $\mathbf{m}_{-i} \in \mathcal{V}_{-i}$.

Definition 2: (Full Participation Winning Probabilities I) *If all buyers participate and buyer i, $\forall i \in \mathcal{N}$ submits signal $m_i \in [\underline{v}_i, \overline{v}_i]$, the object is assigned to the player (including the seller) whose signal renders the highest "augmented virtual value", provided this value is nonnegative.*[16] *Ties are broken randomly. If this value is negative, the object is destroyed by the seller.*

The full participation winning probabilities of Definition 2 follows closely the insight of Myerson (1981). There are two major differences. First, the virtual values adopted in Definition 2 have been much generalized to reflect the impact of identity-specific and financial externalities as well as costs of destroying the object on optimal auction design. Second, the possibility of destroying the object is modeled in our analysis for the first time in the auction design literature to our best knowledge.

The full participation winning probabilities of Definition 2 can be written equivalently as follows. $\forall \mathbf{m} \in \mathcal{V},\ \forall i \in \{0, 1, \cdots, N\}$,

$$p_i^*(\mathbf{m}) = \begin{cases} 1, & \text{if } \tilde{J}_i(m_i) > \max_{j=0, j\neq i}^{N} \tilde{J}_j(m_j) \text{ and } \tilde{J}_i(m_i) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Definition 3: (Full Participation Payments I) *Every bidder i, $i \in \mathcal{N}$ pays an entry fee $E_i = -\min_{j\geq 0} e_{ij}$. In addition, the winning buyer i pays $\tilde{J}_i^{-1}(\max\{0, \max_{j\neq i} \tilde{J}_j(m_j)\})$;*[17] *each losing buyer pays an externality-correcting payment (positive or negative) that equals the allocative identity-specific externality to him.*

Let $\mathbf{x}^\dagger(\mathbf{m}) = (x_i^\dagger(\mathbf{m}))$. We thus have $\mathbf{x}^\dagger(\mathbf{m})$ is the payments functions defined following Myerson (1981) that is incentive compatible with the winning probabilities (12) while ignoring the financial externalities. The full participation payments of Definition 3 can be written equivalently as follows. $\forall \mathbf{m} \in \mathcal{V},\ \forall i \in \mathcal{N}$,

$$x_i^\dagger(\mathbf{m}) = \begin{cases} \tilde{J}_i^{-1}(\max\{0, \max_{j=0, j\neq i}^{N} \tilde{J}_j(m_j)\}) + E_i, & \text{if } i \text{ wins}, \\ e_{ij} + E_i, & \text{if } j (\geq 0) \text{ wins, where } j \neq i, \\ E_i, & \text{if the object is destroyed.} \end{cases} \tag{13}$$

Payments $\mathbf{x}^\dagger(\mathbf{m})$ have accommodated the impact of identity-specific externalities on revenue-maximizing auction. To incorporate the impact of financial externalities, we have to modify $\mathbf{x}^\dagger(\mathbf{m})$ and define the following set of payment functions $\mathbf{x}^*(\mathbf{m}) = (x_i^*(\mathbf{m}))$.

[16] We treat the seller's signal as v_0.

[17] We use $\tilde{J}_i^{-1}(\cdot)$ to denote the inverse function of $\tilde{J}_i(\cdot),\ i \in \mathcal{N}$. If $x < \tilde{J}_i(\underline{v}_i)$, $\tilde{J}_i^{-1}(x)$ is defined as $\underline{v}_i$; if $x > \tilde{J}_i(\overline{v}_i)$, $\tilde{J}_i^{-1}(x)$ is defined as $\overline{v}_i$.

Definition 4: (Full Participation Payments II) $\mathbf{x}^*(\mathbf{m}) = \Gamma^{-1} \cdot \mathbf{x}^\dagger(\mathbf{m})$, *i.e.*, $\forall \mathbf{m} \in \mathcal{V}$, $\forall i \in \mathcal{N}$,

$$x_i^*(\mathbf{m}) = x_i^\dagger(\mathbf{m}) + \frac{\alpha_i}{1 - \sum_{j=1}^N \alpha_j} \sum_{j=1}^N x_j^\dagger(\mathbf{m}). \tag{14}$$

As $\mathbf{x}^*(\mathbf{m}) = \Gamma^{-1} \cdot \mathbf{x}^\dagger(\mathbf{m})$, we have $\mathbf{x}^\dagger(\mathbf{m}) = \Gamma \cdot \mathbf{x}^*(\mathbf{m})$. Based on Lemma 2 and the above definitions, we are then able to present the revenue-maximizing mechanism as in the following proposition.

Proposition 2: *The nonparticipation threats of Definition 1, the full participation winning probabilities and payments of Definitions 2 and 4 constitute a revenue-maximizing truthful direct semirevelation mechanism. The mechanism implements full participation of bidders. The expected payoff of bidder i of type $\underline{v}_i$ is $\min_{j \geq 0} e_{ij}$, which is nonpositive.*

Proof: See Appendix.

According to Proposition 2, the two types of externalities interact fundamentally through shaping players' augmented virtual values, which completely determine the winning probabilities. Both types of externalities affect the winning probabilities solely through this channel of modifying players' virtual values. From the definitions of the augmented virtual values in Lemma 2, we see that the existence of the financial externality amplifies the impact of the players' values and the identity-specific externalities on the winning probabilities.

From Definition 4, we see a unique feature of the payment schedule. Every buyer's payments include externality-correcting components that equal the allocative identity-specific externalities (e_{ij} if player j wins, zero otherwise) and financial externalities ($\alpha_i \sum_{j=1}^N x_j^*(\mathbf{m})$), respectively. This is clear as $x_i^*(\mathbf{m}) = x_i^\dagger(\mathbf{m}) + \alpha_i \sum_{j=1}^N x_j^*(\mathbf{m})$.

Proposition 2 answers the questions of when the object is destroyed by the seller and how the seller should proceed to maximize his expected revenue if the item is to be destroyed. From Proposition 2, we have the following results regarding the probability of destroying the object.

Corollary 1: *If $\tilde{J}_0(v_0) \geq 0$, the object is never destroyed by the seller. If instead $\tilde{J}_0(v_0) < 0$, the object is destroyed by the seller with probability $\prod_{i=1}^N F_i(J_i^{-1}(-\sum_{j \in \mathcal{N}} e_{ji} - (1 - \sum_{j \in \mathcal{N}} \alpha_j)(e_{0i} + c_0)))$.*[18]

[18] We use $J_i^{-1}(\cdot)$ to denote the inverse function of $J_i(\cdot)$. If $x < \underline{v}_i - \frac{1}{f_i(\underline{v}_i)}$, $J_i^{-1}(x)$ is defined as $\underline{v}_i$; if

From Corollary 1, we see a necessary condition for the seller to destroy the item is that $\sum_{j\geq 0} e_{j0} < 0$, i.e., the total identity-specific externalities on the bidders if the seller keeps the item is negative. Another observation is that the existence of financial externalities can be a force that contributes to destroying the item by the seller. This happens when $v_0 + \sum_{j\geq 0} e_{j0} + c_0 \geq 0$, but $v_0 + \frac{\sum_{j\geq 0} e_{j0}}{1-\sum_{j\in\mathcal{N}} \alpha_j} + c_0 < 0$.

We use $(\mathbf{p^0}, \mathbf{x^0})$ to denote the revenue-maximizing mechanism when the identity-specific externalities are $e^0_{ij},\ i, j \in \{0, 1, \cdots, N\}$. When $e^0_{i0}, \forall i \in \mathcal{N}$ are negative enough, we have $\min_{j\geq 0} e^0_{ij} = e^0_{i0}, \forall i \in \mathcal{N}$. Thus $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p^0}, \mathbf{x^0}) = e^0_{i0}, \forall i \in \mathcal{N}$ from Proposition 2. From (11), the optimal expected revenue is

$$R(\mathbf{p^0}, \mathbf{x^0}) = -c_0 - \frac{1}{1-\sum_{j\in\mathcal{N}} \alpha_j} \sum_{j\geq 0} e^0_{j0} + \int_{\mathcal{V}} \Big\{ p^0_0(\mathbf{v})(v_0 + c_0 + \frac{\sum_{j\geq 0} e^0_{j0}}{1-\sum_{j\in\mathcal{N}} \alpha_j}) \\ + \sum_{i=1}^{N} p^0_i(\mathbf{v})(\frac{J_i(v_i) + \sum_{j\in\mathcal{N}} e^0_{ji}}{1-\sum_{j\in\mathcal{N}} \alpha_j} + e^0_{0i} + c_0) \Big\} \mathbf{f}(\mathbf{v}) d\mathbf{v}.$$

Let $R'(\mathbf{p^0}, \mathbf{x^0})$ denote the value of the right-hand-side of $R(\mathbf{p^0}, \mathbf{x^0})$ when e^0_{i0} decreases to $e'_{i0},\ i \in \mathcal{N}$. Clearly $R'(\mathbf{p^0}, \mathbf{x^0}) \geq R(\mathbf{p^0}, \mathbf{x^0})$ as $p^0_0(\mathbf{v}) \in [0, 1]$. Suppose when e^0_{i0} decreases to $e'_{i0},\ i \in \mathcal{N}$, the corresponding revenue-maximizing auction rule changes to $(\mathbf{p'}, \mathbf{x'})$. Denote the optimal expected revenue by $R'(\mathbf{p'}, \mathbf{x'})$ when externalities are $e'_{i0},\ i \in \mathcal{N}$. We must have $R'(\mathbf{p'}, \mathbf{x'}) \geq R'(\mathbf{p^0}, \mathbf{x^0})$. Therefore, $R'(\mathbf{p'}, \mathbf{x'}) \geq R(\mathbf{p^0}, \mathbf{x^0})$, i.e., the seller's optimal expected revenue increase as externalities $e^0_{i0},\ i \in \mathcal{N}$ decrease. This helps to explain why North Korea tries to convince the relevant countries that it owns very powerful nuclear weapons. This result holds whether or not financial externalities exist.

So far, we have focused on financial externalities where all bidders' payments carry symmetric weights. Maasland and Onderstal (2002, 2007) and Engers and McManus (2007) rather allow other bidders' payment have different impact than that of own payments. These cases can be similarly analyzed. We will consider the most general form of linear financial externalities in Section 5.

$x > \overline{v}_i$, $J_i^{-1}(x)$ is defined as $\overline{v}_i$.

4 Applications to One Type of Externalities

Clearly, Proposition 2 applies to setting with one type of externalities. Revenue-maximizing auctions with a single type of externalities have been well studied. In the case of pure identity-specific externalities, Jehiel, Moldovanu and Stacchetti (1996) has fully derived the winning probabilities, though the payments schedule has not been fully characterized. In the case of pure financial externalities, the literature (Maasland and Onderstal (2002, 2007) and Goeree, Maasland, Onderstal and Turner (2005) and Engers and McManus (2007)) so far has rather focused on **symmetric** bidders. In this section, we apply Proposition 2 and further present some more complete characterizations on revenue-maximizing auctions for settings with one type of externalities.

4.1 The Case of Pure Identity-Specific Externalities

4.1.1 The General Setting

We first consider a general setting with pure identity-specific externalities, which allows asymmetry across bidders. In other words, we consider the case where $\alpha_i = 0,\ \forall i \in \mathcal{N}$ in the setting of Section 2.2. In this case, we have $\tilde{J}_i(v_i) = J_i(v_i) + \sum_{j\geq 0} e_{ji} + c_0,\ i \in \mathcal{N}$ and $\tilde{J}_0(v_0) = v_0 + \sum_{j\geq 0} e_{j0} + c_0$. Applying Proposition 2 leads to the following result.

Corollary 2: *The nonparticipation threats of Definition 1, the full participation winning probabilities and payments of Definitions 2 and 3, where* $\tilde{J}_i(v_i) = J_i(v_i) + \sum_{j\geq 0} e_{ji} + c_0,\ i \in \mathcal{N}$ *and* $\tilde{J}_0(v_0) = v_0 + \sum_{j\geq 0} e_{j0} + c_0$*, constitute a revenue-maximizing truthful direct semirevelation mechanism.*

4.1.2 Symmetric Setting

We now show that the revenue-maximizing mechanism of Corollary 2 reduces to a modified second price auction in a symmetric setting. In this symmetric setting, $F_i(\cdot) = F(\cdot),\ f_i(\cdot) = f(\cdot)$ on support $[\underline{v}, \overline{v}],\ \forall i \in \mathcal{N}$. In addition, $e_{i0} = e_{10},\ e_{0i} = e_{01},\ e_{ij} = e, \forall i, j \in \mathcal{N}$ and $i \neq j$. As usual, we assume the regularity condition that $J(v) = v - \frac{1-F(v)}{f(v)}$ is an increasing function. The augmented virtual value function of a representative buyer is defined as $\tilde{J}(\cdot) = J(\cdot) + c_0 + \sum_{j\geq 0} e_{j1}$. The inverse function of $\tilde{J}(\cdot)$ is denoted by $\tilde{J}^{-1}(\cdot)$. The seller's augmented value is $\tilde{J}_0(v_0) = v_0 + \sum_{j\geq 0} e_{j0} + c_0$.

Based on (12) and (13), $\forall \mathbf{m} \in \mathcal{V}$, the full participation winning probability of buyer i, $\forall i \in \mathcal{N}$ is written as

$$p_i^{s*}(\mathbf{m}) = \begin{cases} 1 & \text{if} \quad m_i \geq z_i(\mathbf{m}_{-i}), \\ 0 & \text{if} \quad m_i < z_i(\mathbf{m}_{-i}), \end{cases} \tag{15}$$

and his full participation payment is written as

$$x_i^{s\dagger}(\mathbf{m}) = \begin{cases} z_i(\mathbf{m}_{-i}) + E, & \textit{if } \ i \ \textit{wins}, \\ e + E, & \textit{if } \ j(\geq 0) \ \textit{wins},, \ \textit{where } j \neq i, \\ E, & \textit{if the object is destroyed}, \end{cases} \tag{16}$$

where $z_i(\mathbf{m}_{-i}) = \max\{\max_{j \neq i, j \in \mathcal{N}} m_j, \tilde{J}^{-1}(\max\{0, \tilde{J}_0(v_0)\})\}$ and $E = -\min_{j \geq 0} e_{1j}$. In addition, the seller keeps the object with probability of

$$p_0^{s*}(\mathbf{m}) = \begin{cases} 1, & \textit{if } \tilde{J}_0(v_0) > \tilde{J}(\max_{j=1}^{N} m_j) \textit{ and } \tilde{J}_0(v_0) \geq 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{17}$$

(16) means that every buyer pays an entry fee of $E = -\min_{j \geq 0} e_{1j}$. Moreover, if buyer i wins, he pays an additional $z_i(\mathbf{m}_{-i})$. If he loses, he pays an externality-correcting payment that equals the identity-specific externality he enjoys or suffers. From (17), it is optimal for the seller to destroy the unsold object if and only if $\tilde{J}_0(v_0) < 0$, i.e., the sum of the seller's value, the destroying cost of the seller and the total externalities to the buyers is negative when the seller keeps the item. When $\tilde{J}_0(v_0) < 0$, the seller is better off by committing to destroy the object (eliminating the externalities) and collecting a payment from each buyer that equals the externality to him.

Note that the nonparticipation threats (Definition 1) and (15)-(17) constitute a truthful direct semirevelation mechanism that induces full participation. Clearly, this mechanism is equivalent to the following modified second price auction.

Definition 5: (Auction $\mathcal{A}_0$) *Every bidder pays an entry fee $E = -\min_{j \geq 0} e_{1j}$. The highest buyer wins if his bid is higher than the reserve price $\tilde{J}^{-1}(\max\{0, \tilde{J}_0(v_0)\})$. If no buyer bids higher than the reserve price, then the seller destroys the item if and only if $\tilde{J}_0(v_0) < 0$. The winning bidder pays the second highest bid or the reserve price, whichever is higher. Every losing buyer pays an externality-correcting payment that equals the allocative identity-specific externality to him.*

Based on the above results, we have the following proposition that describes the revenue-maximizing auction.

Proposition 3: *In a symmetric setting with pure identity-specific externalities, the modified second-price sealed-bid auction $\mathcal{A}_0$ of Definition 5 together with the nonparticipation threats of Definition 1 is revenue-maximizing.*

Each buyer's payment is adjusted by the amount of allocative externality to him, while he suffers or enjoys this externality at the same time. This creates a situation where buyers bid as if there is no externality on them. Based on similar arguments for the standard second-price auction, bidding his true value is a weakly dominant strategy for every buyer in the second price auction $\mathcal{A}_0$. This is why a modified second-price auction with the externality-correcting payments is revenue-maximizing, provided that the reserve price and entry fee are properly set. In auction $\mathcal{A}_0$, the entry fee is set at the highest possible level which can be supported by the nonparticipation threats, and the optimal reserve price is determined by the seller's augmented value $\tilde{J}_0(v_0)$.

Define $\mathcal{B}_0$ as a modified first-price auction, which differs from $\mathcal{A}_0$ only in terms of the payments of the winning bidder, i.e., the winning bidder pays his own bid or the reserve price $\tilde{J}^{-1}(\max\{0, \tilde{J}_0(v_0)\})$, whichever is higher. Based on Proposition 3 and the revenue equivalence theorem of Proposition 1, we have that auction $\mathcal{B}_0$ with the threats of Definition 1 is also revenue-maximizing.

4.2 The Case of Pure Financial Externalities

In this section, we conduct further studies on revenue-maximizing auctions for settings with pure financial externalities. Useful linkages between revenue-maximizing auctions for settings with and without externalities are established based on the insights of Section 3. Specifically, when seller does not value the item, we will establish one-to-one correspondences between revenue-maximizing auctions with and without financial externalities. Therefore, revenue-maximizing auctions for various settings of financial externalities can be obtained through transforming the revenue-maximizing second-price and/or first-price auctions for regular settings without externalities. In this sense, our findings provide a general approach for deriving the revenue-maximizing auctions in a variety of settings with financial externalities.

4.2.1 General Settings

So far the literature on auction with financial externalities has focused on symmetric settings. Proposition 2 rather applies to general settings allowing asymmetry among bidders. We first apply Proposition 2 and present the revenue-maximizing mechanism for a general setting. We consider the case where $e_{ij} = 0,\ \forall 0 \leq i, j \leq N$ in the settings of Section 2.2. The augmented virtual values of the players can be simplified as $\overline{J}_i(v_i) = \frac{J_i(v_i)}{d} + c_0,\ i \in \mathcal{N}$ and $\overline{J}_0(v_0) = v_0 + c_0$, where $d = 1 - \sum_{j \in \mathcal{N}} \alpha_j$. Note that $d = 1$ corresponds to the case of no externalities. The inverse function of $\overline{J}_i(\cdot)$ is denoted by $\overline{J}_i^{-1}(\cdot)$. In this case, the nonparticipation threats of Definition 1 take the following form in this setting.

Definition 1′: (Nonparticipation Threats) *If at least one bidder does not participate, the seller keeps the item by himself, all participating bidders pay zero.*

Following Definitions 2-4, we introduce the following full participation winning probability and payment functions.

$\forall \mathbf{m} \in \mathcal{V},\ \forall i \in \{0, 1, \cdots, N\}$,

$$\overline{p}_i^*(\mathbf{m}) = \begin{cases} 1, & \textit{if } \overline{J}_i(m_i) > \max_{j=0, j \neq i}^N \overline{J}_j(m_j) \textit{ and } \overline{J}_i(m_i) \geq 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{18}$$

and $\forall i \in \mathcal{N}$,

$$\overline{x}_i^\dagger(\mathbf{m}) = \begin{cases} \overline{J}_i^{-1}(\max_{j=0, j \neq i}^N \overline{J}_j(m_j)), & \textit{if } i \textit{ wins}, \\ 0, & \textit{Otherwise}. \end{cases} \tag{19}$$

Let $\overline{\mathbf{x}}^\dagger(\mathbf{m})_{N \times 1} = (\overline{x}_i^\dagger(\mathbf{m}))$. We next define another set of full participation payments functions $\overline{\mathbf{x}}^*(\mathbf{m})_{N \times 1} = (\overline{x}_i^*(\mathbf{m}))$:

$$\overline{\mathbf{x}}^*(\mathbf{m}) = \Gamma^{-1} \cdot \overline{\mathbf{x}}^\dagger(\mathbf{m}). \tag{20}$$

According to Propositions 2, we have the following result.

Corollary 3: *The nonparticipation threats of Definition 1′, full participation winning probabilities (18) and payments (20) constitute a revenue-maximizing truthful direct semi-revelation mechanism.*

According to (18), if $v_0 > 0$, then higher externalities (lower $d \in (0, 1]$) lead to higher winning probabilities for all buyers and lower probability for the seller to keep the item.

Note that the item is never destroyed as $\overline{J}_0(v_0) \geq 0$ in this case. Clearly, when $d = 1$, (18) and (19) give a revenue-maximizing truthful direct semirevelation mechanism in a setting without externalities.

4.2.2 When Seller Does Not Value the Object

The assumption of $v_0 = 0$ has been commonly adopted in the literature of auction design with financial externalities, such as Maasland and Onderstal (2002, 2007) and Goeree, Maasland, Onderstal and Turner (2005) and Engers and McManus (2007).

When $v_0 = 0$, neither (18) or (19) depends on $d \in [0, 1)$. Therefore, we can replace the $\overline{J}_i(\cdot)$ in (18) and (19) by $J_i(\cdot)$ and Corollary 3 still holds. In addition, according to Myerson (1981), (18) and (19) with $\overline{J}_i(\cdot)$ replaced by $J_i(\cdot)$ constitute a revenue-maximizing mechanism without financial externalities.[19] Based on these observations, the relation between the seller's optimal expected revenue with and without financial externalities is discovered as in the next proposition.

Proposition 4: *Suppose $v_0 = 0$. The seller's optimal expected revenue with financial externalities equals $\frac{1}{d}$ times of that without externalities. Thus, seller's optimal expected revenue increases with the financial externalities among the buyers.*

Proof: See Appendix.

Based on the above discussions, the revenue-maximizing mechanisms for the cases with and without externalities can be obtained from each other solely through transforming the payment functions. In the next proposition, we show that this statement holds in a context of general mechanisms.

Proposition 5: *There exists a one-to-one correspondence between revenue-maximizing mechanisms with and without financial externalities. Every mechanism can be obtained from its counterpart solely through transforming the payment functions using matrix Γ. Specifically, $\tilde{\mathbf{x}} = \Gamma^{-1}\mathbf{x}$ where $\tilde{\mathbf{x}}$ is the payment functions for the setting with externalities and $\mathbf{x}$ is the payments for the setting without externalities.*

Proof: See Appendix.

Thus, a useful connection between the revenue-maximizing mechanisms with and without financial externalities is disclosed. This result thus provides a general approach for revenue-maximizing auction design with financial externalities when $v_0 = 0$. Provided

[19] This result is also implied by Corollary 3 with $d = 1$.

that we know a revenue-maximizing mechanism without externalities and the specific form of financial externalities (i.e., we know Γ), then a revenue-maximizing mechanism with financial externalities can simply be obtained through transforming the payment functions appropriately by adding a term that adjusts for the financial externalities.

Next, we present some further results on auction design with financial externalities. To begin, we have the following result from Proposition 5.

Corollary 4: *Seller's optimal expected revenue does not depend on the distribution of the externalities across the buyers. In other words, only the sum of all α_i counts.*

In addition, when $v_0 = 0$, the Corollary 3 revenue-maximizing mechanism can be implemented through the following auction.

(a.1) There is no entry fee, the reserve price for buyer i, $\forall i \in \mathcal{N}$ is $\hat{v}_i$ ($\geq \underline{v}_i$), which is the unique solution of $J_i(\hat{v}_i) = 0$;

(a.2) If at least one buyer does not participate, the seller keeps the item, no bidder pays;

(a.3) If all participate, we denote buyer i's bid by b_i, $\forall i \in \mathcal{N}$. Buyer i wins if $J_i(b_i)$ is the highest among all $J_j(b_j)$, $\forall j \in \mathcal{N}$ and $b_i \geq \hat{v}_i$. Ties are broken randomly. Suppose buyer i, $\forall i \in \mathcal{N}$ is the winner. The payments are the following. First, buyer i pays z_1, which is $J_i^{-1}(\max_{j=1,j\neq i}^{N} J_j(b_j))$ or the reserve price $\hat{v}_i$ ($\geq \underline{v}_i$), whichever is higher. Second, every buyer $j \in \mathcal{N}$ pays $z_2 = \frac{\alpha_j z_1}{1-\sum_{i=1}^{N}\alpha_i}$. If no bidder wins, the seller keeps the item, and no one pays.

This result can also be derived from Proposition 5. From Myerson (1981), we have that in the setting without externalities, the auction defined in (a.1) to (a.3) with $z_2 = 0$ is revenue-maximizing. It follows from Proposition 5 that the mechanism defined in (a.1) to (a.3) is revenue-maximizing with financial externalities.

Goeree, Maasland, Onderstal and Turner (2005) study a **symmetric** independent private value setting where buyers' values follow cumulative distribution function $F(\cdot)$ on $[\underline{v}, \overline{v}]$ and the seller's value is zero. They assume that each buyer enjoys a positive externality which equals a **common** proportion (denoted by $\alpha < \frac{1}{N}$ where N is the number of buyers) of the total payments of all buyers. They show that a two-stage lowest-price all-pay auction with proper entry fee E_0 and reserve price R is revenue-maximizing. In the first stage, buyers make the decision whether or not to pay the entry fee and participate. All types of buyers participate, however, there exists a bidding threshold $\hat{v}$ ($\geq \underline{v}$) which is also the threshold of winning type. The bidding threshold $\hat{v}$ is the unique solution of

$J(\hat{v}) = 0$, the reserve price R equals $\frac{\hat{v}F(\hat{v})^{N-1}}{1-\alpha}$ and the entry fee E_0 equals $\frac{\alpha R(N-1)(1-F(\hat{v}))}{1-N\alpha}$.[20]

For this symmetric setting, the revenue-maximizing auction defined by $(a.1)$ to $(a.3)$ can be described as follows.

(b.1) There is no entry fee, the reserve price is $\hat{v}$ $(\geq \underline{v})$;

(b.2) Same as (a.2);

(b.3) If all participate, the highest buyer wins if his bid is no less than the reserve price $\hat{v}$, and his payment consists of two components. First, he pays z_1, which is the second highest bid or the reserve price $\hat{v}$ $(> \underline{v})$, whichever is higher. Second, every buyer pays z_2, where $z_2 = \frac{\alpha z_1}{1-\alpha N} > 0$. If the highest bid is less than $\hat{v}$, the seller keeps the item, and no one pays.

Since the auction defined by $(b.1)$, $(b.2)$ and $(b.3)$ is also revenue-maximizing, we thus have the following result.

Corollary 5: *The modified second price auction defined by* $(b.1) - (b.3)$ *is revenue equivalent to the revenue-maximizing two-stage lowest-price all-pay auction established by Goeree, Maasland, Onderstal and Turner (2005).*

Note that in a symmetric setting without externalities, a first-price auction with reserve price $\hat{v}$ is also revenue-maximizing. Thus a revenue equivalent modified first price auction can also be constructed following Proposition 5.

Interestingly, in the Section 4.2.2 auctions, the optimal reserve prices are set in the same way as in Myerson (1981) where no externalities is involved. This result holds because $v_0 = 0$. In this case, the augmented virtual values of the players (including seller) are ranked in the same order as the regular virtual values. Thus, the optimal reserve price should be set in the same way as in Myerson (1981).

5 Revenue-Maximization with Cross Shareholders

We now turn to the setting of the third example in Section 2.1 where the externalities are rather due to cross shareholding among buyers. While we set $c_0 = 0, e_{ij} = 0$ and $\alpha_i = 0$, $\forall i, j$ in the setting of Section 2.2 to eliminate externalities denoted by these parameters, we allow cross shareholding among bidders who compete for a scarce resource. In this section, v_i denotes the added value of buyer i if he is the winner. Following

[20] Please refer to Proposition 5 in Goeree, Maasland, Onderstal and Turner (2005) for details.

Dasgupta and Tsui (2004),[21] we assume that player $i(\geq 0)$ owes a fraction of $s_{ij} \in [0,1]$ of the player $j(\geq 0)$ where $\sum_{j\geq 0} s_{ij} \leq 1$. We normalize $s_{00} = 1,\ s_{i0} = s_{0i} = 0, \forall i \in \mathcal{N}$, i.e., there is no cross shareholding between buyers and seller.

Bidder i thus enjoys an identity-specific externalities $s_{ij}v_j$ if bidder $j(\neq i)$ wins due to the shareholding.[22] This component $s_{ij}v_j$ is rather the private information of the winner. Due to cross shareholding, bidder i suffers a negative externality $-\sum_{j\in\mathcal{N}, j\neq i} s_{ij}x_j$ from other bidder's payments $x_j,\ j \in \mathcal{N}$. When $s_{ii} = 1, \forall i \in \mathcal{N}$, we have the standard setting without cross shareholding.

The players' payoffs are as follows. Buyer i's payoff is $s_{ij}v_j - \sum_{k\in\mathcal{N}} s_{ik}x_k$ if player j wins and payments of bidders are $x_k, \forall k \in \mathcal{N}$. The seller's payoff is $v_0 + \sum_{j\in\mathcal{N}} x_j$ if he keeps the item; his payoff is $\sum_{j\in\mathcal{N}} x_j$ if bidder i wins. The game extends as in Section 2.2. Shares s_{ij} are public information, which is revealed at time 0.

For any truthful direct semirevelation mechanism $(\mathbf{p}, \mathbf{x})$ implementing full participation, the seller's expected revenue is given by:

$$\tilde{R}(\mathbf{p}, \mathbf{x}) \quad = \quad E_{\mathbf{v}}\Big\{v_0 p_0(\mathbf{v}) + \sum_{i=1}^{N} x_i(\mathbf{v})\Big\}. \tag{21}$$

For buyer i with private value v_i, if he submits signal $m_i \in \mathcal{M}_i$, his interim expected payoff is given by:

$$\tilde{U}_i(v_i, m_i; \mathbf{p}, \mathbf{x}) = E_{\mathbf{v}_{-i}}\Big\{\sum_{j\geq 0} s_{ij}v_j\ p_j(m_i, \mathbf{v}_{-i}) - \sum_{j\in\mathcal{N}} s_{ij}x_j(m_i, \mathbf{v}_{-i})\Big\}. \tag{22}$$

The seller's optimization problem is to find the revenue-maximizing truthful direct semirevelation mechanism $(\tilde{\mathbf{p}}^*, \tilde{\mathbf{x}}^*)$ that implements full participation. In other words, the seller maximizes $\tilde{R}(\mathbf{p}, \mathbf{x})$ subject to constrains (4)-(6) where $U_i(\cdot, \cdot; \cdot, \cdot)$ functions are replaced by $\tilde{U}_i(\cdot, \cdot; \cdot, \cdot)$. This optimization program can be solved following the same method as in Section 3. A counterpart of Lemma 1 is the following.[23]

Lemma 3: *Direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *is a* **truthful** *direct semirevelation mechanism that implements full participation, if and only if* $\forall i \in \mathcal{N}$ *the following condi-*

[21] They studied standard first price and second price auctions.

[22] We can allow the identity-specific externalities take a form of $h_i(v_j)$, which can alternatively be interpreted as informational externalities.

[23] The proof is omitted as it follows Myerson (1981) closely.

tions and (6) hold:

$$Q_i(v_i'; \mathbf{p}) \leq Q_i(v_i; \mathbf{p}),\ \forall \underline{v}_i \leq v_i' \leq v_i \leq \overline{v}_i, \tag{23}$$

$$\tilde{U}_i(v_i, v_i; \mathbf{p}, \mathbf{x}) = \tilde{U}_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + s_{ii} \int_{\underline{v}_i}^{v_i} Q_i(s_i; \mathbf{p}) ds_i,\ \forall v_i \in [\underline{v}_i, \overline{v}_i], \tag{24}$$

$$\tilde{U}_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) \geq \tilde{U}_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x}). \tag{25}$$

Note that only (24) differs from its counterpart in Lemma 1. In Lemma 1, $s_{ii} = 1, \forall i$. Define payment coefficient matrix $S_{N\times N} = (s_{ij})_{i,j\geq 1}$. We assume S is nonsingular. When $s_{ij} = \frac{1}{N}$ (symmetric cross shareholding), S actually is singular. We will discuss this case at a later stage.

Define $\tilde{\mathbf{b}} = (\tilde{b}_i)_{N\times 1} = (S')^{-1}\mathbb{1}_{N\times 1}$ where all elements in $\mathbb{1}_{N\times 1}$ are 1. In this paper, we focus on the case where $b_i \geq 0,\ \forall i \in \mathcal{N}$. When there is a b_i which is negative, the optimal mechanism does not exist. This will be clear after we present Lemma 4.

Before we proceed to derive the expression for the seller's expected revenue from a truthful direct semirevelation mechanism, we further introduce the following definitions of generalized virtual values of players.

Definition $0'$: (Generalized Virtual Values) $\ddot{J}_i(v_i) = v_i \sum_{j\in\mathcal{N}} \tilde{b}_j s_{ji} - \tilde{b}_i s_{ii} \frac{1-F(v_i)}{f(v_i)},\ i \in \mathcal{N}$ *are defined as the buyers' generalized virtual value functions; and* $\ddot{J}_0(v_0) = v_0$ *is defined as the sellers' generalized virtual value.*

Note that the generalized virtual values cover the virtual values coined by Myerson (1981) as a special case. When $s_{ii} = 1,\ s_{ij} = 0,\ \forall i, j$, we have $\tilde{b}_i = 1,\ \forall i$. Thus the generalized virtual values degenerate to the standard case. Clearly, $\ddot{J}_i(\cdot)$ is increasing when $J_i(\cdot)$ is increasing and $b_i \geq 0,\ \forall i \in \mathcal{N}$. Similar to Lemma 2, we have the following Lemma.[24]

Lemma 4: *For a truthful direct semirevelation mechanism* $(\mathbf{p}, \mathbf{x})$ *that implements full participation, the seller's expected revenue can be written as*

$$\tilde{R}(\mathbf{p}, \mathbf{x}) = E_{\mathbf{v}}\Big(\sum_{i\geq 0} p_i(\mathbf{v}) \ddot{J}_i(v_i)\Big) - \sum_{i=1}^{N} \tilde{b}_i \tilde{U}_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}). \tag{26}$$

[24]The proof is similar to that of Lemma 2. Note that $\tilde{\mathbf{b}}' S\mathbf{x} = \mathbb{1}'\mathbf{x} = \sum_{i\in\mathcal{N}} x_i$, where $\mathbf{x}_{1\times N} = (x_i)$. This fact and (24) leads to Lemma 4. The proof is available from the author upon request.

Lemma 4 differs from its counterpart in Myerson (1981) in the following aspects. First, a generalized version of virtual values appears in (26); Second, due to the cross shareholding, constants $\tilde{b}_j,\ j \in \mathcal{N}$ appear both in $\ddot{J}_i(\cdot)$ and before $\tilde{U}_i$. From Lemma 4, it is clear that when there is a $\tilde{b}_i < 0$, then a higher $\tilde{U}_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$ always benefits the seller. Thus, an optimal mechanism must not exist. In addition, Lemma 4 immediately leads to that a version of revenue equivalence theorem like Proposition 1 holds in a setting with cross shareholding among buyers.

We are now ready to characterize the optimal mechanism. Before we present the revenue-maximizing selling mechanism, we first introduce the following definitions.

Definition $2'$: (Full Participation Winning Probabilities I) *If all buyers participate and buyer i, $\forall i \in \mathcal{N}$ submits signal $m_i \in [\underline{v}_i, \overline{v}_i]$, the object is assigned to the player (including the seller) whose signal renders the highest "generalized virtual value".[25] Ties are broken randomly.*

The winning probabilities of Definition $2'$ can be written equivalently as follows. $\forall \mathbf{m} \in \mathcal{V},\ \forall i \in \{0, 1, \cdots, N\}$,

$$\tilde{p}_i^*(\mathbf{m}) = \begin{cases} 1, & \textit{if } \ddot{J}_i(m_i) > \max_{j=0, j\neq i}^{N} \ddot{J}_j(m_j) \textit{ and } \ddot{J}_i(m_i) \geq 0, \\ 0, & \textit{otherwise.} \end{cases} \tag{27}$$

Definition $3'$: (Full Participation Payments I) *The winning buyer i pays $s_{ii}\ddot{J}_i^{-1}(\max\{0, \max_{j\neq i} \ddot{J}_j(m_j)\})$;[26] each losing buyer pays an externality-correcting payment (positive or negative) that equals the allocative identity-specific externality ($s_{ij}m_j$) to him.*

The full participation payments of Definition 3' can be written equivalently as follows. $\forall \mathbf{m} \in \mathcal{V},\ \forall i \in \mathcal{N}$,

$$\tilde{x}_i^\dagger(\mathbf{m}) = \begin{cases} s_{ii}\ddot{J}_i^{-1}(\max_{j=0, j\neq i}^{N} \ddot{J}_j(m_j)), & \textit{if } i \textit{ wins}, \\ s_{ij}m_j, & \textit{if } j (\geq 0) \textit{ wins, where } j \neq i. \end{cases} \tag{28}$$

Let $\tilde{\mathbf{x}}^\dagger(\mathbf{m}) = (\tilde{x}_i^\dagger(\mathbf{m}))$. We next define another set of full participation payment functions $\tilde{\mathbf{x}}^*(\mathbf{m}) = (\tilde{x}_i^*(\mathbf{m}))$.

Definition $4'$: (Full Participation Payments II) $\tilde{\mathbf{x}}^*(\mathbf{m}) = S^{-1} \cdot \tilde{\mathbf{x}}^\dagger(\mathbf{m})$.

[25] We treat the seller's signal as v_0.

[26] We use $\ddot{J}_i^{-1}(\cdot)$ to denote the inverse function of $\ddot{J}_i(\cdot),\ i \in \mathcal{N}$. If $x < \ddot{J}_i(\underline{v}_i)$, $\ddot{J}_i^{-1}(x)$ is defined as $\underline{v}_i$; if $x > \ddot{J}_i(\overline{v}_i)$, $\ddot{J}_i^{-1}(x)$ is defined as $\overline{v}_i$.

Based on Lemma 4 and the above definitions, we are then able to present the revenue-maximizing mechanism as in the following proposition.[27]

Proposition 6: *The nonparticipation threats of Definition* $1'$*, the full participation winning probabilities and payments of Definitions* $2'$ *and* $4'$ *constitute a revenue-maximizing truthful direct semirevelation mechanism. The mechanism implements full participation of bidders.*

Proposition 6 reveals that cross shareholding fundamentally affect the revenue-maximizing auction through shaping players' generalized virtual values, which completely determine the winning probabilities. From Definition $4'$, we see a unique feature of the payment schedule. A proportion (s_{ii}) of every buyer's payments ($s_{ii}\tilde{x}_i^*(\mathbf{m})$) include externality-correcting components that equal the allocative identity-specific externalities ($s_{ij}m_j$ if player j wins, zero otherwise) and financial externalities ($-\sum_{j\neq i}^N s_{ij}\tilde{x}_j^*(\mathbf{m})$), respectively. This is clear as $s_{ii}\tilde{x}_i^*(\mathbf{m}) = \tilde{x}_i^\dagger(\mathbf{m}) - \sum_{j\neq i}^N s_{ij}\tilde{x}_j^*(\mathbf{m})$.

Proposition 6 means that the methodology of Section 3 is applicable to the case with cross shareholding among buyers, where the identity-specific externalities are private information of the winner, and the financial externalities take the most general linear form.

We now turn to the case with symmetric cross shareholding among buyers. In this case, $s_{ij} = \frac{1}{N}$ in matrix S, thus S is singular and Proposition 6 does not apply. Though we can set $\tilde{b}_i = 1$ such that a result of (26) can still be obtained, the payments schedule $\tilde{\mathbf{x}}^*(\mathbf{m})$ that is incentive compatible with the optimal winning probabilities $\tilde{\mathbf{p}}^*(\mathbf{m})$ does not exist. This means that the optimal winning probabilities can not be defined as $\tilde{\mathbf{p}}^*(\mathbf{m})$ of Definition $2'$. These results are rather surprising as one may expect symmetric cross shareholding to be the simplest case. Further work remains to be done in this direction.

6 Conclusion

This paper studies auction design with identity-specific and financial externalities. We find that these two types of externalities interact fundamentally through shaping players' externality-augmented virtual values, which are obtained from the regular virtual values by adjusting for the externalities and seller's destroying cost. At the optimum, the winning probabilities of the players are determined by their augmented virtual values. The

[27]The proof is similar to that of Proposition 2. It is available from the author upon request.

player with the highest externality-augmented virtual value wins given that it is positive. Both types of externalities affect the winning probabilities completely through the same channel of modifying players' virtual values. A unique feature of the revenue-maximizing mechanism is that buyers' payments consist of externality-correcting components that equal the two types of allocative externalities to them. These components eliminate the impact of the externalities on the strategic bidding behavior. Our study provides a unified treatment for revenue-maximizing mechanism design with pure identity-specific or financial externalities. In symmetric settings, modified second-price and first-price auctions with externality-correcting payments are established to be revenue-maximizing.

We find that introducing the possibility for the seller to destroy the item enlarges the freedom of revenue-maximizing mechanism design, when there exist identity-specific externalities between seller and buyers. At the optimum, the seller destroys the unsold item if and only if his augmented value is negative. Jehiel, Moldovanu and Stacchetti (1996) point out that the seller is better off by not selling at all if the total identity-specific externalities generated by a sale is larger than total values. Our analysis reveals that the seller can be further better off by physically destroying the item while extracting payments from all buyers, if his augmented value is negative. This reveals that the crucial force driving the dismantling result is the identity-specific externalities on the buyers imposed by the seller instead of those among the buyers.

When buyers suffer highly negative identity-specific externalities if the seller holds the item, the seller's optimal expected revenue increases as these externalities become more negative. This provides an alternative explanation to why North Korea tries to convince relevant countries that its nuclear weapons are powerful.

Financial externalities amplify the impact of the identity-specific externalities. Particularly, when the total identity-specific externalities on the buyers is negative when the seller keeps the item, the existence of financial externalities among buyers further decreases seller's augmented value. Thus, financial externalities may lead the seller to destroy the item, which may not be destroyed otherwise.

Our study leads to interesting findings on auction design for settings with pure financial externalities. Especially for the case where the seller does not value the item, we establish one-to-one correspondence between revenue-maximizing auctions with and without externalities. As a result, the revenue-maximizing auctions for a regular setting without externalities need only be properly modified in the payments to be revenue-maximizing in settings with financial externalities.

The methodology and insights developed apply to the case with cross shareholding among buyers, where the identity-specific externalities are private information of the winner, and the financial externalities take the most general linear form. However, when financial externalities take nonlinear forms, generally our methodology no longer applies. More works need to be done in this direction.

Appendix

Proof of Lemma 2: From (2),

$$
\begin{aligned}
\int_{\underline{v}_i}^{\overline{v}_i} & U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) f_i(v_i) dv_i \\
= & \int_{\underline{v}_i}^{\overline{v}_i} \left(\int_{\mathcal{V}_{-i}} \Big(v_i \ p_i(\mathbf{v}) + \sum_{j \geq 0} e_{i,j} \ p_j(\mathbf{v}) - x_i(\mathbf{v}) + \alpha_i \sum_{j \in \mathcal{N}} x_j(\mathbf{v}) \Big) \mathbf{f}_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \right) f_i(v_i) dv_i \\
= & \int_{\mathcal{V}} \Big(v_i \ p_i(\mathbf{v}) + \sum_{j \geq 0} e_{i,j} \ p_j(\mathbf{v}) - x_i(\mathbf{v}) + \alpha_i \sum_{j \in \mathcal{N}} x_j(\mathbf{v}) \Big) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \qquad \text{(A.1)}
\end{aligned}
$$

where $\mathbf{f}_{-i}(\mathbf{v}_{-i}) = \prod_{j=1, j \neq i}^{N} f_j(v_j)$ is the density of $\mathbf{v}_{-i}$, and $\mathbf{f}(\mathbf{v}) = \prod_{i=1}^{N} f_i(v_i)$ is the density of $\mathbf{v}$. From (A.1), we have

$$
\begin{aligned}
& \frac{1}{1 - \sum_{j \in \mathcal{N}} \alpha_j} \sum_{i=1}^{N} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) f_i(v_i) dv_i \\
= & \int_{\mathcal{V}} \Big(p_0(\mathbf{v}) \sum_{i \in \mathcal{N}} \frac{e_{i,0}}{1 - \sum_{j \in \mathcal{N}} \alpha_j} + \sum_{i=1}^{N} \frac{v_i + \sum_{j \in \mathcal{N}} e_{j,i}}{1 - \sum_{j \in \mathcal{N}} \alpha_j} \ p_i(\mathbf{v}) - \sum_{i=1}^{N} x_i(\mathbf{v}) \Big) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \qquad \text{(A.2)}
\end{aligned}
$$

Note that $e_{i,i} = 0, \ \forall i \geq 0$. From (1) and (A.2),

$$
\begin{aligned}
R(\mathbf{p}, \mathbf{x}) = & -c_0 - \frac{1}{1 - \sum_{j \in \mathcal{N}} \alpha_j} \sum_{i=1}^{N} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) f_i(v_i) dv_i \\
& + \int_{\mathcal{V}} \Big(p_0(\mathbf{v})(v_0 + c_0 + \sum_{j \geq 0} \frac{e_{j,0}}{1 - \sum_{j \in \mathcal{N}} \alpha_j}) + \sum_{i=1}^{N} p_i(\mathbf{v}) (\frac{v_i + \sum_{j \in \mathcal{N}} e_{j,i}}{1 - \sum_{j \in \mathcal{N}} \alpha_j} + e_{0,i} + c_0) \Big) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \text{(A.3)}
\end{aligned}
$$

From (9), we have

$$
\begin{aligned}
& \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) f_i(v_i) dv_i = \int_{\underline{v}_i}^{\overline{v}_i} [U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + \int_{\underline{v}_i}^{v_i} Q_i(s_i; \mathbf{p}) ds_i] f_i(v_i) dv_i \\
& = U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + \int_{\underline{v}_i}^{\overline{v}_i} [\int_{\underline{v}_i}^{v_i} Q_i(s_i; \mathbf{p}) ds_i] f_i(v_i) dv_i \\
& = U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + \int_{\underline{v}_i}^{\overline{v}_i} [\int_{s_i}^{\overline{v}_i} f_i(v_i) dv_i] Q_i(s_i; \mathbf{p}) ds_i \\
& = U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + \int_{\underline{v}_i}^{\overline{v}_i} [1 - F_i(s_i)] Q_i(s_i; \mathbf{p}) ds_i. \qquad \text{(A.4)}
\end{aligned}
$$

From (7), we have

$$
\begin{aligned}
&\int_{\underline{v}_i}^{\overline{v}_i} [1 - F_i(s_i)] Q_i(s_i; \mathbf{p}) ds_i \\
&= \int_{\underline{v}_i}^{\overline{v}_i} [1 - F_i(s_i)] \left\{ \int_{\mathcal{V}_{-i}} p_i(s_i, \mathbf{v}_{-i}) \mathbf{f}_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} \right\} ds_i \\
&= \int_{\mathcal{V}} p_i(\mathbf{v}) \frac{1 - F_i(v_i)}{f_i(v_i)} \mathbf{f}(\mathbf{v}) d\mathbf{v}. \qquad \text{(A.5)}
\end{aligned}
$$

From (A.4) and (A.5), we have

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i, v_i; \mathbf{p}, \mathbf{x}) f_i(v_i) dv_i \\
&= \sum_{i=1}^{N} U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}) + \int_{\mathcal{V}} \Big(\sum_{i=1}^{N} p_i(\mathbf{v}) \frac{1 - F_i(v_i)}{f_i(v_i)} \Big) \mathbf{f}(\mathbf{v}) d\mathbf{v}. \qquad \text{(A.6)}
\end{aligned}
$$

From (A.3) and (A.6), we have (11). □

Proof of Proposition 1: According to the semirevelation principle, for any mechanism that implements full participation, there must exist an equivalent truthful direct semirevelation mechanism that delivers the same participation and allocation for any $\mathbf{v} \in \mathcal{V}$, including the winning probability for every player and payment for every bidder. The result in this proposition immediately comes from applying Lemmas 1 and 2 to this equivalent mechanism. □

Proof of Proposition 2: From (11), a truthful direct semirevelation mechanism that induces full participation must be revenue-maximizing if it satisfies the following two conditions. First, it minimizes $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$, $\forall i \in \mathcal{N}$. Second, it also maximizes $\sum_{i=0}^{N} p_i(\mathbf{v}) \tilde{J}_i(v_i)$, $\forall \mathbf{v} \in \mathcal{V}$. We show that the direct semirevelation mechanism $(\mathbf{p}^*, \mathbf{x}^*)$ of Proposition 2 satisfies the above criteria and thus maximizes the seller's expected revenue. We then verify that $(\mathbf{p}^*, \mathbf{x}^*)$ is truthful.

First, the nonparticipation threats of Definition 1 push $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ to take the lowest possible value $\min_{j \geq 0} e_{i,j}$. Note that we assume that the seller is cashless. Thus he cannot create negative financial externalities to the nonparticipant by manipulating the payments of the participants.

Second, the set of full-participation winning probability functions of Definition 2 clearly maximizes $\sum_{i=0}^{N} p_i(\mathbf{v}) \tilde{J}_i(v_i), \forall \mathbf{v} \in \mathcal{V}$.

Third, the set of full-participation payment functions of Definition 4 drive $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$ to exactly equal $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$, which in turn equals $\min_{j\geq 0} e_{i,j}$. Note that $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x})$ cannot be lower than $U_i(\underline{v}_i, \emptyset; \mathbf{p}, \mathbf{x})$ from (10).

From Definitions 2, 3 and 4, we can verify that

$$\begin{aligned} &x_i^*(\mathbf{v}) - \alpha_i \sum_{j\in\mathcal{N}} x_j^*(\mathbf{v}) = x_i^\dagger(\mathbf{v}) \\ &= v_i\ p_i^*(\mathbf{v}) + \sum_{j\geq 0} e_{i,j}\ p_j^*(\mathbf{v}) - \min_{j\geq 0} e_{i,j} - \int_{\underline{v}_i}^{v_i} p_i^*(s_i, \mathbf{v}_{-i}) ds_i,\ \forall i \in \mathcal{N}. \end{aligned} \tag{A.7}$$

Therefore, for $p_i^*(\cdot),\ 0 \leq i \leq N$ and $x_i^*(\cdot),\ i \in \mathcal{N}$, (2) leads to

$$\begin{aligned} &U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}^*, \mathbf{x}^*) \\ &= E_{\mathbf{v}_{-i}}\left(\underline{v}_i\ p_i^*(\underline{v}_i, \mathbf{v}_{-i}) + \sum_{j\geq 0} e_{i,j}\ p_j^*(\underline{v}_i, \mathbf{v}_{-i}) - x_i^*(\underline{v}_i, \mathbf{v}_{-i}) + \alpha_i \sum_{j\in\mathcal{N}} x_j^*(\underline{v}_i, \mathbf{v}_{-i})\right) \\ &= \min_{j\geq 0} e_{i,j}. \end{aligned} \tag{A.8}$$

Thus, we have shown that the Proposition 2 mechanism minimizes $U_i(\underline{v}_i, \underline{v}_i; \mathbf{p}, \mathbf{x}),\ \forall i \in \mathcal{N}$, and it also maximizes $\sum_{i=0}^N p_i(\mathbf{v}) \tilde{J}_i(v_i)$, $\forall \mathbf{v} \in \mathcal{V}$.

The full-participation winning probabilities and payments $\mathbf{p}^*(\cdot)$ and $\mathbf{x}^*(\cdot)$ together with the nonparticipation threats of Definition 1 lead to a Nash equilibrium in which every type of buyers participates and reveals truthfully their types, because the conditions in lemma 1 are satisfied. We thus have that the full-participation winning probabilities and payments $\mathbf{p}^*(\cdot)$ and $\mathbf{x}^*(\cdot)$ together with the nonparticipation threats constitute a **truthful** direct semirevelation mechanism that maximizes the seller's expected revenue. In the same spirit of Jehiel, Moldovanu and Stacchetti (1996), there is no need to consider the joint deviation from the Nash equilibrium.[28] Thus all the other winning probabilities and payments functions which are not relevant to the equilibrium path can be specified in any way. □

Proof of Proposition 4: Since the winning probabilities (18) of all players do not depend on $d \in (0, 1]$ if $v_0 = 0$, the result follows from (11) and (15) immediately. Note that the lowest types $\underline{v}_i,\ i \in \mathcal{N}$ always get zero payoff at the optimum in this setting. □

Proof of Proposition 5: Suppose $(\mathbf{p}, \mathbf{x})$ is a revenue-maximizing general mechanism for the case of no externalities, where $\mathbf{p} = (p_i(\cdot))$ and $\mathbf{x} = (x_i(\cdot))$. Define $\tilde{\mathbf{x}} = \Gamma^{-1}\mathbf{x}$. Thus if $(\mathbf{p}, \tilde{\mathbf{x}})$ is

[28] Footnote 11 in Jehiel, Moldovanu and Stacchetti (1996) points out that joint deviations of buyers are irrelevant since full-participation Nash equilibrium is studied.

adopted for the case with financial externalities, then every bidder i has to pay the externalities he enjoys/suffers on top of $x_i(\cdot)$ by the construction of $\tilde{x}_i(\cdot)$. Assume $s_i(v_i)$ is the equilibrium strategy of bidder i when $(\mathbf{p}, \mathbf{x})$ is adopted in the setting without externalities, then clearly $s_i(v_i)$ is also the equilibrium strategy of bidder i when $(\mathbf{p}, \tilde{\mathbf{x}})$ is adopted in the setting with financial externalities.

Let $\mathbf{s}(\mathbf{v}) = (s_1(v_1), \cdots, s_N(v_N))$. Note that $E_{\mathbf{v}}(\sum_{j=1}^N x_j(\mathbf{s}(\mathbf{v})))$ is the seller's expected revenue from $(\mathbf{p}, \mathbf{x})$ in the setting without externalities, and $E_{\mathbf{v}}(\sum_{j=1}^N \tilde{x}_j(\mathbf{s}(\mathbf{v})))$ is the seller's expected revenue from $(\mathbf{p}, \tilde{\mathbf{x}})$ in the setting with financial externalities.[29] Since $\sum_{j=1}^N x_j(\mathbf{s}(\mathbf{v})) = d\sum_{j=1}^N \tilde{x}_j(\mathbf{s}(\mathbf{v}))$, we have $E_{\mathbf{v}}(\sum_{j=1}^N x_j(\mathbf{s}(\mathbf{v}))) = dE_{\mathbf{v}}(\sum_{j=1}^N \tilde{x}_j(\mathbf{s}(\mathbf{v})))$. As a result, Proposition 5 implies that $(\mathbf{p}, \tilde{\mathbf{x}})$ must be revenue-maximizing in the setting with financial externalities.

If $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ is a revenue-maximizing general mechanism for the case with externalities, where $\tilde{\mathbf{p}} = (\tilde{p}_i(\cdot))$ and $\tilde{\mathbf{x}} = (\tilde{x}_i(\cdot))$. Define $\mathbf{x} = \Gamma\tilde{\mathbf{x}}$. Similarly, we can show that $(\tilde{\mathbf{p}}, \mathbf{x})$ is a revenue-maximizing general mechanism for the case without externalities. □

[29] Note that destroying the item is never desired in the case of financial externalities.

References

[1] I. Brocas, Endogenous entry in auctions with negative externalities, Theory and Decision **54**, 125-149 (2003)

[2] I. Brocas, Auctions with type-dependent and negative externalities: the optimal mechanism, Working paper, 2005

[3] S. Dasgupta and K. Tsui, Auctions with cross-shareholdings, Economic Theory 24, 163-194, 2004.

[4] George Deltas, Determining damages from the operation of bidding rings: An analysis of the post-auction "knockout" sale, Economic Theory 19, 243-269 (2002)

[5]M. Engers and B. McManus, Charity Auctions, International Economic Review, 48 (Aug. 2007).

[6] Jacob K. Goeree, Emiel Maasland, Sander Onderstal, and John L. Turner, How (Not) to raise money, Journal of Political Economy, **113(4)**, 897-926 (2005)

[7] P. Jehiel, B. Moldovanu, Strategic nonparticipation, Rand Journal of Economics, 27 (1), 84-98, 1996.

[7] P. Jehiel, B. Moldovanu and E. Stacchetti, How (not) to sell nuclear weapons, The American Economic Review **86**, 814-929 (1996)

[8] P. Jehiel, B. Moldovanu and E. Stacchetti, Multidimensional mechanism design for auctions with externalities, Journal of Economic Theory **85**, 258-293 (1999)

[9] Emiel Maasland, Sander Onderstal, Optimal auctions with financial externalities, Working paper, 2002.

[10] Emiel Maasland, Sander Onderstal, Auctions with financial externalities, Economic Theory, 2007, 32(3), 551-574.

[11] P. McAfee and J. McMillan, "Bidding Rings," American Economic Review, 82(3), 579-99, 1992.

[12] R. B. Myerson, Optimal auction design, Mathematics of Operation Research, **6**, 58-73 (1981)

[13] M. Stegeman, Participation costs and efficient auctions, Journal of Economic Theory, **71**, 228-259 (1996).

[14] G. D. Varma, Standard auctions with identity-dependent externalities, Rand Journal of Economics **33(4)**, 689-708 (2002)