Online Appendix: Monopolization with Must-Haves (Not for Publication)

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1 More Extensions

1.1 Preliminaries

In discussing some of the extensions that follow, it is convenient to scale k's demand. Formally, suppose that k's demand can be written as $Q_k(p_k) = \lambda_k \tilde{Q}_k(p_k)$, where $\lambda_k \in [0, \infty)$ capture its scale.

With this in mind, two observations are worth highlighting for what follows. First, we converge to the case where only B is present (Propositions 1 and 5) by taking $\lambda_A \to 0$ (with $\lambda_B > 0$). Second, in Section 2, we introduced the concept of *vertical differentiation disadvantage*, defined as the loss in relative quality experienced by a distributor that does not carry A against her rival that does. This was epitomized by the expression $v_A(p_A^{lf}) + v_B(p_B^{lf}) - v_B(c_B)$. As we also showed in that section, this concept is important since it determines the extent of monopolization of B.

What is the connection between the scale of products' demands and the disadvantage experienced by a distributor that does not carry A? Notice that $v_k(p) = \lambda_k \tilde{v}_k(p)$, where $\tilde{v}_k(p) = \int_p^\infty \tilde{Q}_k(x)dx$. Hence:

(i) The higher λ_A (keeping everything else constant), the higher the disadvantage from losing A (holding prices fixed), since $v_A(p_A^{lf})$ is uniformly higher for a given p_A^{lf}. This makes it easier to monopolize B.

(ii) The higher λ_B (keeping everything else constant), the lower the disadvantage (holding prices fixed), since $v_B(c_B) - v_B(p_B^{lf})$ is uniformly higher for any given $p_B^{lf} > c_B$. This makes it harder to monopolize B.

Intuitively, the higher λ_A is, the more important is product A for consumers. Hence, a distributor that does not carry A is in a weaker competitive position against her rivals that do. Conversely, the higher λ_B is, the more important is for consumers to have a good deal for B. As a result, the loss in relative quality from losing A is less important if the fringe provides B at a lower cost.

1.2 More Products

Adding more products does not qualitatively affect our main results. Suppose, for instance, that there is another product Z that is also perfectly monopolized by M. If the latter can tie the sales of A and Z, this is equivalent to increasing $v_A(p_A^{lf})$ in the laissez-faire scenario, so M will be able to achieve higher wholesale and retail prices for B. In other words, when M can tie his entire portfolio of products, what matters for monopolization is the degree of vertical differentiation disadvantage induced by losing M's overall portfolio, not the one induced by losing each of M's products individually.[1]

Alternatively, suppose that Z is produced by M and an equally efficient fringe of perfectly competitive producers. M will then attempt to monopolize both B and Z. This is equivalent to increasing the scale of B's demand in our baseline setting, which makes monopolization more difficult. Finally, suppose that a fringe supplies Z but not by M. This case is qualitatively similar to the previous case, except that M will not be able to soften downstream competition for Z. The extent of monopolization will also be lower since distributors can now lower the prices of B and Z to overcome the loss of A, decreasing the must-have nature of this last product.

1.3 More Efficient Fringe and Different Varieties of B

A strictly more efficient fringe does not affect the paper's main qualitative results. The only minor difference is that when the fringe is equally efficient, M can start monopolizing B in the laissez-faire as soon as $v_A(p_A^{lf}) > 0$. In contrast, when the fringe is strictly more efficient, M will be able to monopolize B only when $v_A(p_A^{lf})$ is significant enough to offset the fringe's efficiency advantage.

Another way of seeing it is as follows. By Propositions 1 and 5, we know that if $\lambda_A = 0$, then there is no monopolization of B. When the fringe is equally efficient than M, the latter can start monopolizing B as soon as $\lambda_A > 0$ (though the extent of monopolization will be small when λ_A is small). In contrast, when the fringe is strictly more efficient than M in producing B, M will have to continue offering B at cost for $\lambda_A \in [0, \bar{\lambda}_A]$ with $\bar{\lambda}_A > 0$.

Something similar occurs when the varieties of B produced by M and the fringe are imperfect substitutes (so far, we have assumed that the two varieties are perfect substitutes). Again, M will be able to soften competition for B only if the surplus in A can overcome the fact that there is some intrinsic value (from an industry perspective) in carrying the fringe's variety.

1.4 Weak Complements/Substitutes

So far, we have assumed that A and B are neither complements nor substitutes in consumption. However, using continuity arguments, it is clear that our results extend to the case in which A and B are weak complements or weak substitutes. However, the "weak" qualifier is important. For instance, if A and B are very close substitutes, then distributors could use B to substitute for the loss of A, substantially decreasing the must-have nature of A. On the other hand, if A and B are perfect complements, then M will be able to monopolize both markets even when tying provisions,

[1] Note that the possibility of tying A and Z is important. The reason is that if M cannot tie these two products (but can tie, say, A with B or Z with B), then a distributor can partially use Z (or A) to mitigate the vertical differentiation of losing A (or Z).

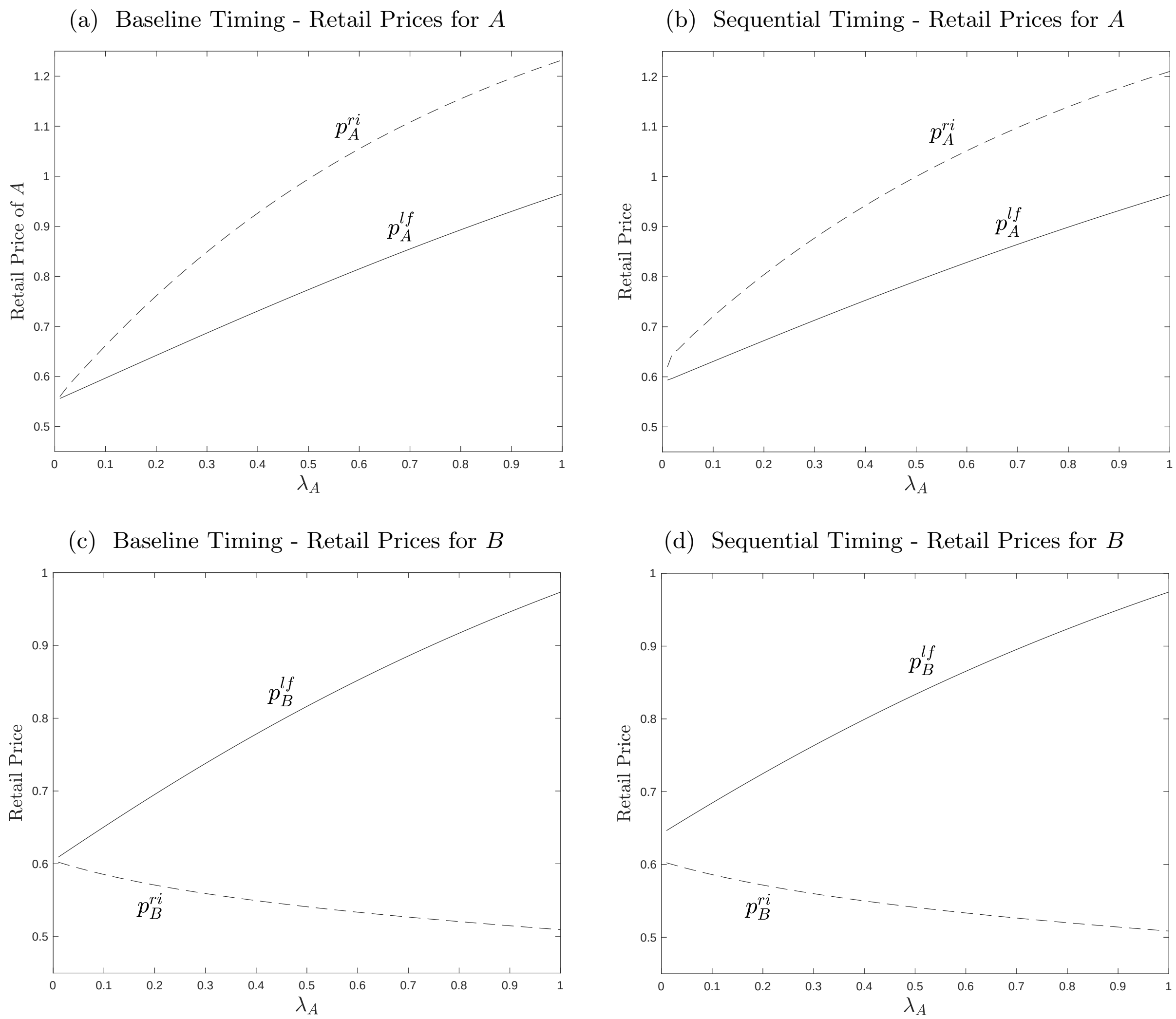
exclusivity clauses, distributor discrimination, and refusal to deal are all banned.

1.5 Alternative Timing

In the main text, we assumed that distributors simultaneously accept contracts and set retail prices. We make this assumption mainly for tractability since it allows us to avoid recomputing equilibrium retail prices after deviation by one of the distributors. An alternative ("sequential") timing would be to assume that distributors first make observable contract acceptance decisions and then set retail prices downstream.

As shown in Figure 1.1, the results are qualitatively similar under both timings. The figure

Figure 1.1: Different Timings - The Effect of λ_A



Notes. Demands: $Q_k(p) = \lambda_k(1-(1-\rho_k)p)^{\frac{\rho_k}{1-\rho_k}}$. Distribution: $G(x) = e^{-e^{-x}}$. Population size: 100.000. Parameters: $n = 3$, $\gamma = 5$, $c_A = 1/4$, $c_B = 1/4$, $\rho_A = 3/2$, $\lambda_B = 1$, $\rho_B = 1$.

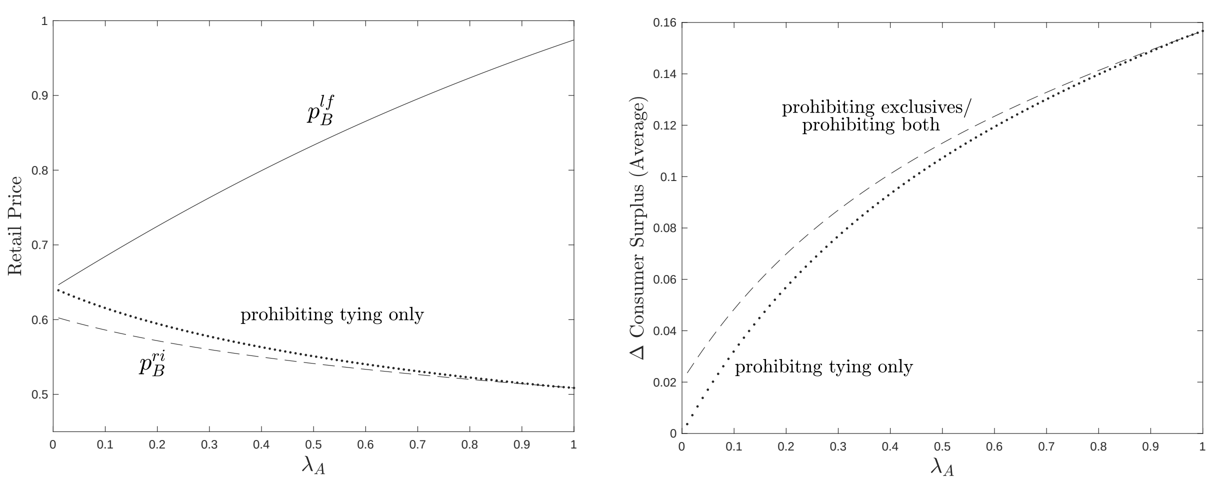
plots the equilibrium retail prices as a function of λ_A for the laissez-faire and robust-intervention outcomes according to the model with horizontally differentiated distributors.[2] The only significant difference that arises is that unlike in Propositions 1 and 5 of the main text, under this alternative timing, M can now monopolize market B even if A is not present (i.e., $p_B^{lf} > p_B^{ri}$ even when $\lambda_A = 0$).

Intuitively, under this alternative sequential timing, M's exclusive distributors adjust their retail pricing downward after observing deviation by a rival. As a result, distributors' reservation payoffs are lower than in the baseline model of the main text and less sensitive to increases in the wholesale price of rival distributors. The latter allows M to increase wholesale and retail prices of B with exclusives even when $\lambda_A = 0$.

The fact that under the sequential timing, there is some softening of competition in market B—despite A's absence—also implies that banning exclusives is no longer equivalent to prohibiting tying when M is forced to make non-discriminatory offers and must deal with all distributors. This is shown in Figure 1.2, which builds upon Figure 1.1 but also plots the effects of banning tying, distributor discrimination, and refusal to deal while allowing exclusivity provisions. As the figure shows, banning exclusives makes banning tying irrelevant, although the converse is not true. This implies that banning exclusives leads to larger drops in the retail price of B and larger gains in consumer surplus than only forbidding tying.[3]

Figure 1.2: Sequential Timing - Banning Exclusives vs. Forbidding Tying

(a) Retail Prices of B (b) Change in $u = v_A(p_A) + v_B(p_B)$



Notes. Demands: $Q_k(p) = \lambda_k(1-(1-\rho_k)p)^{\frac{\rho_k}{1-\rho_k}}$ - Distribution: $G(x) = e^{-e^{-x}}$ - Population size: 100.000
Remaining parameters: $n = 3$, $\gamma = 5$, $c_A = 1/4$, $c_B = 1/4$, $\rho_A = 3/2$, $\lambda_B = 1$, $\rho_B = 1$.

[2] Check the Supplementary Material for the `MATLAB` codes used to create the different plots found throughout the paper and the online Appendix.

[3] Intuitively, simply forbidding tying eliminates the multiproduct anticompetitive strategy but still allows M to pursue a single-product anticompetitive strategy. In contrast, banning exclusives eliminates both the multiproduct and single-product anticompetitive strategies; M can no longer soften downstream competition, given that he cannot make his offer mutually exclusive to the fringe.

It is important to highlight, however, that the difference between banning exclusives and prohibiting tying becomes negligible as λ_A increases (see Figure 1.2). Accepting the fringe's low-cost offer for B becomes increasingly attractive as the complementarity between the two products increases with λ_A. As a result, M's single-product anticompetitive strategy becomes increasingly more difficult the larger the scale of A's demand is.

1.6 Simpler Contracts: Upstream Contractual Frictions with Public Offers

In the main text, we showed that our must-have mechanism continues to operate with upstream contractual frictions in the case of private offers. What happens, however, if offers are public (as in Sections 2–5 of the main text)?

To explore this possibility, let us follow Calzolari, Denicolò and Zanchettin (2020) and capture contractual frictions in a reduced-form way. More precisely, suppose that extracting rents by means of fixed fees creates deadweight losses: With a lump-sum payment of T_i, M earns T_i, but the distributor loses $(1+\kappa)T_i$, with $\kappa > 0$.[4] We then converge to the case in which M is forced to make linear-price contracts by taking $\kappa \to \infty$.

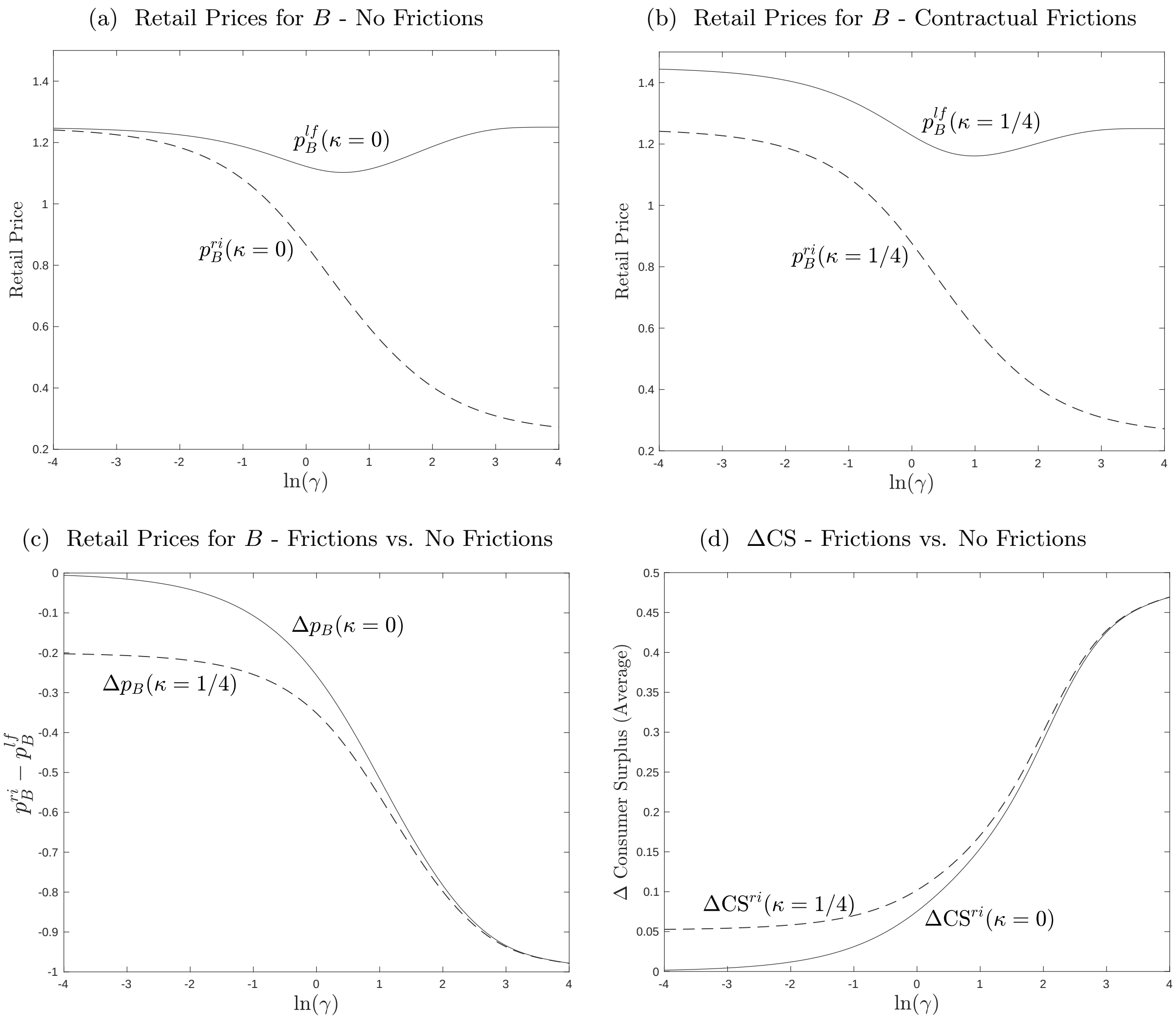
Figure 1.3 shows the effects of introducing contractual frictions in the model with public offers and horizontally differentiated distributors. Panels (a) and (b) plot the equilibrium retail prices for B in the laissez-faire scenario (lf) and under the robust-intervention benchmark (ri). Building on these prices, Panel (c) plots the change in the retail price of B due to the robust intervention, while Panel (d) plots the consumer welfare gains following this intervention.

Two results stand out in these figures. The first, captured in Figure 1.3(b), is that in contrast to our baseline model (see Proposition 6), the robust intervention now has an effect even when distributors are local monopolies. This effect is completely unrelated to our theory of must-haves. Indeed, as explained by Greenlee, Reitman and Sibley (2008) and Calzolari, Denicolò and Zanchettin (2020) in the context of a single monopoly distributor, contractual frictions force M to sell A and B above their marginal costs of production. As a result, M does not completely internalize downstream profits, creating contractual externalities even in the absence of scale economies, downstream competition, and one-stop shopping. Ultimately, this makes it profitable for M to use tying provisions to monopolize market B, which explains why the intervention now also produces gains when $\gamma = 0$.

The second result, illustrated in Figures 1.3(c) and 1.3(d), is that this monopolization based on contractual frictions weakens with the introduction of downstream competition and is eventually fully replaced by the monopolization based on must-haves that we have presented in this paper. The reason was already discussed in the Related Literature: contractual frictions matter less with more downstream competition because there is less profit to extract from distributors and hence,

[4] It is assumed that this friction appears only when $T_i > 0$. This guarantees that lump-sum subsidies do not entail any special benefit.

Figure 1.3: The Effects of Contractual Frictions



Notes. Demands: $Q_k(p) = \lambda_k(1-(1-\rho_k)p)^{\frac{\rho_k}{1-\rho_k}}$. Distribution: $G(x) = e^{-e^{-x}}$. Population size: 100.000. Parameters: $n = 3$, $c_A = 1/4$, $c_B = 1/4$, $\lambda_A = 1$, $\lambda_B = 1$, $\rho_A = 3/2$, $\rho_B = 1$.

less need for fixed fees (or, more generally, within-product nonlinearities in the contracts).

1.7 More Complex Contracts

In the main text, we showed that M is unable to profitably monopolize B if A is not present. This is because when M tries to induce distributors to accept contracts with wholesale prices for B that are higher than the fringe's cost, the sum of distributors' reservation payoffs increases more rapidly than the corresponding increase in industry profits.

This result, however, raises the question: What types of contracts are necessary and sufficient for M to maximize industry profits (i.e., to fully monopolize B) in the single-product case? In this online Appendix, we use the simple model with two Bertrand competitors of Section 2 of the main text to show that multilateral contracts are necessary and sufficient. This implies that if M has these types of contracts at his disposal, then the presence of an additional product does not affect the extent of monopolization of market B (as this market is already fully monopolized), and must-have products play no role in the market outcome.

1.7.1 Multilateral Contracts Are Sufficient For Single-Product Full Monopolization

Suppose that M offers the following multilateral contracts to $i = 1, 2$ (where $\epsilon > 0$):

- $(w_{Bi} = p_B^m, T_{Bi} = -\epsilon/2, e_i = 1)$ if $j \neq i$ also accepts M's offers.
- $(w_{Bi} = c_B, T_{Bi} = -\epsilon/2, e_i = 1)$ if $j \neq i$ deviates to the fringe's contract.

Because a distributor that rejects M's exclusive offer is always weakly better off accepting the fringe's contract (rather than not accepting a contract at all), the continuation game accepts the following equilibrium candidates:

1. Both distributors accept M's offer and reject the fringe's contract. If so, both distributors charge p_B^m on-path and obtain profits of $\epsilon/2$. This candidate is an equilibrium for if distributor i deviates to the fringe's contract, she knows that j will have a wholesale price for B equal to c_B, so she expects zero profits due to Bertrand competition downstream.

2. One distributor, say D_1, accepts M's offer, while the other distributor accepts the fringe's offer. This cannot be an equilibrium. By taking the fringe's offer, D_2 obtains zero profits since D_1 will have a wholesale price for B equal to c_B. As a result, D_2 deviates and takes M's contract to obtain at least $\epsilon/2$.

3. Both distributors reject M's offer and take the fringe's offer for B. This cannot be an equilibrium either. Both distributors are making zero profits on-path. Consequently, a distributor has incentives to deviate to M's offer to secure at least $\epsilon/2$.

Thus following the aforementioned offers, the unique equilibrium of the continuation game is for both distributors to accept M's offers. As a result, equilibrium retail prices will be $p_{Bi}^* = p_{Bj}^* = p_B^m$ and M's profits $\Pi_M^{lf} = \pi_B(p_B^m; c_B) - \epsilon \to \pi_B(p_B^m; c_B)$ as $\epsilon \to 0$. Hence, multilateral contracts are sufficient for single-product full monopolization. □

1.7.2 Contingent Contracts Are Necessary For Single-Product Full Monopolization

We will prove the contrapositive statement: if M can only offer the most general bilateral contracts (even if they are in the most general form $\{T_i(q_{Bi}), e_i\}$), there cannot be single-product full monopolization (in fact, we will show that there will be no monopolization at all).

So suppose M offers contracts of the form $\{T_i(q_{Bi}), e_i\}$ to $i = 1, 2$. As in the proof of Proposition 1, in equilibrium is without loss of generality to (i) focus on the case where M offers all distributors contracts with exclusivity provisions, $\{T_i(q_{Bi}), e_i = 1\}$, and (ii) assume that every distributor accepts M's offer. Denote then by p^*_{Bi} and p^*_{Bj} distributors on-path retail prices following M's offers,[5] and without loss of generality assume that $p^*_{B1} \leq p^*_{B2}$. Assume, furthermore, that (i) the tie-breaking rule is such that all consumers buy from D_1 if retail prices are the same (the proof can be easily generalized to an arbitrary tie-breaking rule), and (ii) that $p^*_{B2} \leq p^m_B$ (the proof for when $p^*_{B2} > p^m_B$ follows the exact same logic).

It is then an equilibrium for both distributors to accept M's offers if and only if the following participation constraints are satisfied:

$$\begin{aligned} p^*_{B1}Q_B(p^*_{B1}) - T_1(Q_B(p^*_{B1})) &\geq \max\{0, (p^*_{B2} - c_B)Q_B(p^*_{B2})\} \\ -T_2(0) &\geq \max\{0, (p^*_{B1} - c_B)Q_B(p^*_{B1})\} \end{aligned}$$

But since M's equilibrium profits are $\Pi^*_M = T_1\big(Q_B(p^*_{B1})\big) + T_2(0) - c_BQ_B(p^*_{B1})$, distributors' participation constraints imply that:

$$\Pi^*_M \leq (p^*_{B1} - c_B)Q_B(p^*_{B1}) - \max\{0, (p^*_{B2} - c_B)Q_B(p^*_{B2})\} - \max\{0, (p^*_{B1} - c_B)Q_B(p^*_{B1})\}$$

This expression is strictly negative unless $p^*_{B1} \geq c_B$ and $p^*_{B2} \leq c_B$. Thus, and given our presumption that $p^*_{B1} \leq p^*_{B2}$, with bilateral contracts M will always induce distributors to charge $p^*_{B1} = p^*_{B2} = c_B$ (for example, offering $(w_{Bi} = c_B, T_{Bi} = 0, e_i = 1)$) and obtain zero profits.

Hence, it is clear that the best M can do with bilateral contracts is to let distributors buy from the fringe, i.e., there is no monopolization in B whatsoever. □

1.8 Single-Product Strategic Rival

Suppose that M's rival in the provision of B is not a fringe of competitive suppliers, as assumed thus far, but a single-product strategic rival, say S, with the same per-unit cost c_B. To see why our results remain qualitatively (and sometimes quantitatively) unchanged, consider two cases regarding S's space of available contracts.

Start with the case in which S can only offer two-part tariffs (exclusives are not available for S, but they are for M). Since S cannot prevent distributors from accepting his offer while buying everything from M, S will have no choice but to discard any negative transfers in his offers.[6] It is easy to see then that, in equilibrium, S will offer product B at cost and set a fixed fee of zero, just

[5] We assume that $T_1(\cdot)$ and $T_2(\cdot)$ are such that an equilibrium in pure strategies exists at the pricing stage.

[6] The only reason to offer a negative fee is to entice a distributor to accept a higher wholesale price than that being offered by the rival upstream supplier. However, since S cannot include exclusivity provisions in his contracts, a distributor does not need to decide between his offer and that of M; she can accept both to obtain the negative fee and obtain B at the lowest per-unit cost. Anticipating the latter, S will never offer negative fees.

as the fringe does. Consequently, substituting the fringe for a strategic rival, in this case, has no effect on the outcome of the game.

Consider now the alternative case in which S is also allowed to offer exclusive contracts for B. When the scale of A's demand is small (i.e., $\lambda_A \approx 0$), S still operates very similarly to the fringe in that he makes no profit in equilibrium. The only difference with the laissez-faire equilibrium (see Propositions 2 and 5) is that S makes offers with wholesale prices above cost that also include negative fixed fees. However, as λ_A increases, it is possible to prove that wholesale prices and fixed fees in S's offers converge to c_B and zero from above and below, respectively, to a point where λ_A is sufficiently large, where S stops offering exclusives altogether and sells B at cost. In this case—again—substituting the fringe for a strategic rival has no effect whatsoever on the outcome of the game.

1.9 Nash-in-Nash Bargaining

So far, we have assumed that M makes take-it-or-leave-it offers to the distributors. In this section, we show that if upstream contractual frictions are present (as in the case of private offers), our results extend to the case where M and the distributors bilaterally negotiate according to a Nash-in-Nash Bargaining Protocol.[7]

Consider the model with horizontally differentiated distributors of Sections 4 and 5 of the main text, except that now M and the distributors negotiate according to a Nash-in-Nash Bargaining protocol (the fringe continues to offer distributors product B at a constant per-unit wholesale price equal to c_B). That is, M engages with each distributor in a bilateral Nash bargain, anticipating that M and the other distributors will also reach an agreement and taking the terms of those agreements as given. The bargaining weight of M relative to D_i is assumed to be equal to $\beta_i \in (0, 1)$.

We will first show that if fixed fees are available, the same type of supplier opportunism that arises in the take-it-or-leave-it protocol with private offers emerges here. Thus, M is unable to raise products' wholesale prices above their marginal cost of production, so there is no monopolization in B.

Indeed, suppose that M offers tying-exclusive contracts $(w_{Ai}, w_{Bi}, T_i, e_i = 1)$. Given the bargaining protocol, the offers of other distributors are given, so the retail prices charged by D_i only depend on his offer (as in the case of private offers):

$$\begin{aligned} \boldsymbol{p}_i^*(\boldsymbol{w}_i) &= \underset{(p_{Ai}, p_{Bi})}{\arg\max}\, \Pi_D(\boldsymbol{p}_i, \boldsymbol{p}_{-i}^e; w_{Ai}, w_{Bi}) \\ &= \underset{(p_{Ai}, p_{Bi})}{\arg\max} \left\{ (\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi})) s(\boldsymbol{p}_i, \boldsymbol{p}_{-i}^e) \right\} \end{aligned} \tag{1}$$

[7] The Nash-in-Nash bargaining protocol was first proposed by Horn and Wolinsky (1988). Collard-Wexler, Gowrisankaran and Lee (2019) later provided a noncooperative foundation for this bargaining solution based on a model of alternating offer bargaining. This protocol has been playing an ever more prominent role in the empirical literature and the antitrust practice analyzing the effect of upstream negotiations on market outcomes (see, e.g., Farrell et al., 2011; Crawford and Yurukoglu, 2012).

where $\boldsymbol{p}^e_{-i}$ is D_i's conjecture about the retail prices charged by her rivals. With some abuse of notation, define then:[8]

$$\begin{aligned}\hat{\Pi}_M(\mathcal{N},\boldsymbol{w}_i,\boldsymbol{w}_{-i}) \equiv &\big((w_{Ai}-c_A)Q_A(p^*_{Ai}(\boldsymbol{w}_i))+(w_{Bi}-c_B)Q_B(p^*_{Bi}(\boldsymbol{w}_i))\big)s(\boldsymbol{p}^*_i(\boldsymbol{w}_i),\boldsymbol{p}^e_{-i})\\ &+\textstyle\sum_{j\neq i}\left[\big((w_{Aj}-c_A)Q_A(p^e_{Aj})+(w_{Bj}-c_B)Q_B(p^e_{Bj})\big)s(\boldsymbol{p}^e_j,(\boldsymbol{p}^*_i(\boldsymbol{w}_i),\boldsymbol{p}^e_{-ij}))\right]\end{aligned}$$

$$\begin{aligned}&\hat{\Pi}_M(\mathcal{N}\setminus\{i\},\boldsymbol{w}_i=\boldsymbol{\infty},\boldsymbol{w}_{-i})\\ &\qquad\equiv\textstyle\sum_{j\neq i}\left[\big((w_{Aj}-c_A)Q_A(p^e_{Aj})+(w_{Bj}-c_B)Q_B(p^e_{Bj})\big)s(\boldsymbol{p}^e_j,(\boldsymbol{p}_i=\boldsymbol{\infty},\boldsymbol{p}^e_{-ij}))\right]\end{aligned}$$

where $\mathcal{N}=\{1,...,n\}$ is set of distributors operating in the downstream market. That is, $\hat{\Pi}_M(\mathcal{N},\boldsymbol{w}_i,\boldsymbol{w}_{-i})$ are M's profits before fixed fees if he reaches an agreement with D_i, while $\hat{\Pi}_M(\mathcal{N}\setminus\{i\},\boldsymbol{w}_i=\boldsymbol{\infty},\boldsymbol{w}_{-i})$ are his profits if he does not (taking all other agreements as given).

The Nash protocol then implies that (w_{Ai},w_{Bi},T_i) must solve:

$$\begin{aligned}\max_{(w_{Ai},w_{Bi},T_i)}&\left[\hat{\Pi}_M(\mathcal{N},\boldsymbol{w}_i,\boldsymbol{w}_{-i})-\hat{\Pi}_M(\mathcal{N}\setminus\{i\},\boldsymbol{w}_i=\boldsymbol{\infty},\boldsymbol{w}_{-i})+T_i\right]^{\beta_i}\\ &\times\left[\Pi_D(\boldsymbol{p}^*_i(\boldsymbol{w}_i),\boldsymbol{p}^e_{-i};w_{Ai},w_{Bi})-\max_{\tilde{p}_{Ai}=\infty,\tilde{p}_{Bi}}\Pi_D(\tilde{\boldsymbol{p}}_i,\boldsymbol{p}^e_{-i};\infty,c_B)-T_i\right]^{1-\beta_i}\end{aligned}$$

subject to:

$$\begin{aligned}&\hat{\Pi}_M(\mathcal{N},\boldsymbol{w}_i,\boldsymbol{w}_{-i})-\hat{\Pi}_M(\mathcal{N}\setminus\{i\},\boldsymbol{w}_i=\boldsymbol{\infty},\boldsymbol{w}_{-i})+T_i\geq 0\\ &\Pi_D(\boldsymbol{p}^*_i(\boldsymbol{w}_i),\boldsymbol{p}^e_{-i};w_{Ai},w_{Bi})-\max_{\tilde{p}_{Ai}=\infty,\tilde{p}_{Bi}}\Pi_D(\tilde{\boldsymbol{p}}_i,\boldsymbol{p}^e_{-i};\infty,c_B)-T_i\geq 0\end{aligned}$$

The first-order conditions of this problem are then:

$$\begin{aligned}\frac{\partial}{\partial T_i}=&\frac{\beta_i}{\hat{\Pi}_M(\mathcal{N},\boldsymbol{w}_i,\boldsymbol{w}_{-i})-\hat{\Pi}_M(\mathcal{N}\setminus\{i\},\boldsymbol{w}_i=\boldsymbol{\infty},\boldsymbol{w}_{-i})+T_i}\\ &-\frac{1-\beta_i}{\Pi_D(\boldsymbol{p}^*_i(\boldsymbol{w}_i),\boldsymbol{p}^e_{-i};w_{Ai},w_{Bi})-\max_{\tilde{p}_{Ai}=\infty,\tilde{p}_{Bi}}\Pi_D(\tilde{\boldsymbol{p}}_i,\boldsymbol{p}^e_{-i};\infty,c_B)-T_i}=0\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial w_{ki}}=&\frac{\beta_i\frac{\partial}{\partial w_{ki}}\hat{\Pi}_M(\mathcal{N},\boldsymbol{w}_i,\boldsymbol{w}_{-i})}{\hat{\Pi}_M(\mathcal{N},\boldsymbol{w}_i,\boldsymbol{w}_{-i})-\hat{\Pi}_M(\mathcal{N}\setminus\{i\},\boldsymbol{w}_i=\boldsymbol{\infty},\boldsymbol{w}_{-i})+T_i}\\ &-\frac{(1-\beta_i)\frac{\partial}{\partial w_{ki}}\Pi_D(\boldsymbol{p}^*_i(\boldsymbol{w}_i),\boldsymbol{p}^e_{-i};w_{Ai},w_{Bi})}{\Pi_D(\boldsymbol{p}^*_i(\boldsymbol{w}_i),\boldsymbol{p}^e_{-i};w_{Ai},w_{Bi})-\max_{\tilde{p}_{Ai}=\infty,\tilde{p}_{Bi}}\Pi_D(\tilde{\boldsymbol{p}}_i,\boldsymbol{p}^e_{-i};\infty,c_B)-T_i}=0\end{aligned}$$

[8] Note the difference with expression (**??**) of the main text (M's profits in the take-it-or-leave-it protocol with private offers). Here, M and D_i are equally informed about the terms that parties may sign in other bilateral relationships. This explains why M's profits depend only on $\boldsymbol{p}^e_j$ (rather than $\boldsymbol{p}^e_j$ and $\boldsymbol{p}^*_j(\boldsymbol{w}_j)$).

Combining these first-order conditions, we obtain that (w_{Ai}^*, w_{Bi}^*) must be such that:

$$\frac{\partial}{\partial w_{ki}}\hat{\Pi}_M(\mathcal{N}, \boldsymbol{w}_i, \boldsymbol{w}_{-i}) + \frac{\partial}{\partial w_{ki}}\Pi_D(\boldsymbol{p}_i^*(\boldsymbol{w}_i), \boldsymbol{p}_{-i}^e; w_{Ai}, w_{Bi}) = 0 \text{ for } k = A, B$$

That is,

$$(w_{Ai}^*, w_{Bi}^*) = \underset{(w_{Ai}, w_{Bi})}{\arg\max} \left\{\hat{\Pi}_M(\mathcal{N}, \boldsymbol{w}_i, \boldsymbol{w}_{-i}) + \Pi_D(\boldsymbol{p}_i^*(\boldsymbol{w}_i), \boldsymbol{p}_{-i}^e; w_{Ai}, w_{Bi})\right\}$$

Hence, the equilibrium wholesale prices under the Nash-in-Nash protocol are also equal to the products' marginal costs, i.e., $\boldsymbol{w}_i^{\text{NiN}} = (c_A, c_B)$ for all $i = 1, ..., n$ (where NiN stands for "Nash-in-Nash").

This result should not be surprising: we know from Collard-Wexler, Gowrisankaran and Lee (2019) that the Nash-in-Nash Protocol is a type of contract equilibrium. Hence, when fixed fees are available, the same type of supplier opportunism that arises under take-it-or-leave-it private offers also emerges here, precluding M from raising products' wholesale prices above their marginal costs of production. This implies that the only effect that the Nash-in-Nash protocol has compared to the take-it-or-leave-it private offers is that the negotiated fixed fees between M and the distributors are different; T_i now depend on the relative bargaining weight of M vis-a-vis D_i.

Similarly to the case of private offers, we can restore the must-have monopolization mechanism by introducing contractual frictions. Suppose, for instance, that M is forced to offer linear contracts to the distributors so that M offers tying-exclusives contracts of the form $(w_{Ai}, w_{Bi}, e_i = 1)$ to all the distributors. Then, the negotiated terms between M and D_i must now solve:

$$\max_{(w_{Ai}, w_{Bi})} \left[\hat{\Pi}_M(\mathcal{N}, \boldsymbol{w}_i, \boldsymbol{w}_{-i}) - \hat{\Pi}_M(\mathcal{N} \setminus \{i\}, \boldsymbol{w}_i = \boldsymbol{\infty}, \boldsymbol{w}_{-i})\right]^{\beta_i} \times \left[\Pi_D(\boldsymbol{p}_i^*(\boldsymbol{w}_i), \boldsymbol{p}_{-i}^e; w_{Ai}, w_{Bi}) - \max_{\tilde{p}_{Ai}=\infty, \tilde{p}_{Bi}} \Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}^e; \infty, c_B)\right]^{1-\beta_i}$$

subject to:

$$\begin{aligned}
&\hat{\Pi}_M(\mathcal{N}, \boldsymbol{w}_i, \boldsymbol{w}_{-i}) - \hat{\Pi}_M(\mathcal{N} \setminus \{i\}, \boldsymbol{w}_i = \boldsymbol{\infty}, \boldsymbol{w}_{-i}) \geq 0 \\
&\Pi_D(\boldsymbol{p}_i^*(\boldsymbol{w}_i), \boldsymbol{p}_{-i}^e; w_{Ai}, w_{Bi}) - \max_{\tilde{p}_{Ai}=\infty, \tilde{p}_{Bi}} \Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}^e; \infty, c_B) \geq 0
\end{aligned}$$

From here it is clear that $w_{ki}^{\text{NiN}} > c_k$ for $k = A, B$. Moreover, just as in the case of take-it-or-leave-it private offers, if A is not present, there will be no monopolization in B since $\Pi_D(\boldsymbol{p}_i^*(\boldsymbol{w}_i), \boldsymbol{p}_{-i}^e; w_{Ai}, w_{Bi}) \geq \max_{\tilde{p}_{Ai}=\infty, \tilde{p}_{Bi}} \Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}^e; \infty, c_B)$ is violated for all $w_{Bi} > c_B$. A similar argument can also be used to show that if tying provisions are forbidden, there cannot be monopolization in B either.

In sum, the Nash-in-Nash case is almost identical to the take-it-or-leave-it private offers case: when fixed fees are available, there is no monopolization due to supplier opportunism. However, upstream contractual frictions ameliorate M's opportunism, restoring the must-have monopolization mechanism.

1.10 Nonlinear Pricing Downstream

The main text assumes that distributors compete in linear prices, whereas pay-TV subscriptions rely instead on fixed fees. In this online appendix, we consider the consequences of allowing two-part tariffs (or, more generally, nonlinear contracts) in the downstream market using the baseline model of Section 2.

We will show two results. First, with the model as currently constructed, there is no monopolization in B if distributors can offer nonlinear contracts/two-part tariffs downstream. This is because two-part tariffs allow M to fully extract the surplus in product A, so the outcome is the same "as if" there is a unit-inelastic demand for product A. That said, we show, secondly, that A's must-have nature is restored (and, therefore, so is the monopolization of market B) once we introduce heterogeneity in consumers' valuations for product A. This is because such heterogeneity now precludes M from extracting all the surplus in A.

1.10.1 Two-Part Tariffs with Homogenous Consumers

Consider the baseline model of Section 2, except that distributors can now offer a two-part tariffs to consumers. That is, D_i announces a schedule (p_{Ai}, p_{Bi}, f_i) where p_{ki} is the marginal price that D_i sets for product $k = A, B$, and f_i is a fixed/"entry" fee.[9] We will characterize M's preferred SPNE and show that it involves no monopolization in market B.[10]

As usual, when characterizing M's preferred SPNE it is without loss of generality to (i) focus on the case where M offers all distributors tying-exclusive contracts of the form $(w_{Ai}, w_{Bi}, T_i, e_i = 1)$ and (ii) assume that every distributor accepts M's offer. Suppose then that M offers contracts with a vector of wholesale prices $(w_{A1}, w_{B1}, w_{A2}, w_{B2})$, and without loss of generality assume that $v_A(w_{A1}) + v_B(w_{B1}) \geq v_A(w_{A2}) + v_B(w_{B2})$. To further simplify the proof, assume—as in the proof of Proposition 2—that if both distributors offer the same surplus to consumers, the tie-breaking rule is such that all consumers buy from D_1 (the proof can easily be generalized to an arbitrary tie-breaking rule).

Given M's offers, Bertrand competition with two-part tariffs implies that the equilibrium downstream is given by $p^*_{Ai} = w_{Ai}$, $p_{Bi} = w^*_{Bi}$ for $i = 1, 2$ and:

$$f^*_1 = v_A(w_{A1}) + v_B(w_{B1}) - v_A(w_{A2}) - v_B(w_{B2}) \qquad \text{and} \qquad f^*_2 = 0$$

This implies that on-path only D_1 will be visited by consumers and selling strictly positive units.

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[9]Note that because consumers are one-stop shoppers, it is without loss to focus on a single contract (p_{Ai}, p_{Bi}, f_i) (rather than in two individual two-part tariff contracts (p_{Ai}, f_{Ai}) and a (p_{Bi}, f_{Bi})).

[10]Using similar arguments as in Proposition 2 of the main text, it is possible to prove that all equilibria are outcome equivalent to M's preferred SPNE.

Both distributors accepting M's offers is then a Nash equilibrium if and only if the following participation constraints are satisfied:

$$f_1^* - T_1 \geq \max\{0, v_B(c_B) - v_A(w_{A2}) - v_B(w_{B2})\}$$
$$-T_2 \geq \max\{0, v_B(c_B) - v_A(w_{A2}) - v_B(w_{B2})\}$$

That is, if D_i's on-path payoff is greater than her reservation payoff. Note that both distributors' reservation payoffs are the same and equal to $\max\{0, v_B(c_B) - v_A(w_{A2}) - v_B(w_{B2})\}$ in this case.[11]

In M's preferred SPNE, both distributors' participation constraints must be binding. Obtaining the equilibrium transfers from the binding constraints, substituting them into M's payoff function, and using the fact that D_1 is the only distributor selling on-path implies that M's payoff in his preferred SPNE is the solution to the following problem:

$$\begin{aligned}
\hat{\Pi}_M^{\text{pSPNE}} &= \max_{\boldsymbol{w}} \Big\{ \pi_A(w_{A1}; c_A) + \pi_B(w_{B1}; c_B) + T_1^* + T_2^* \Big\} \\
&= \max_{\boldsymbol{w}} \Big\{ \big[\pi_A(w_{A1}; c_A) + v_A(w_{A1})\big] + \big[\pi_B(w_{B1}; c_B) + v_B(w_{B1})\big] \\
&\qquad - v_A(w_{A2}) - v_B(w_{B2}) - 2\max\{0, v_B(c_B) - v_A(w_{A2}) - v_B(w_{B2})\} \Big\}
\end{aligned}$$

subject to our premise that $v_A(w_{A1}) + v_B(w_{B1}) \geq v_A(w_{A2}) + v_B(w_{B2})$. It is then straightforward to prove that the solution to this problem involves $w_{A1}^{lf} = c_A$, $w_{B1}^{lf} = c_B$ and any pair $(w_{A2}^{lf}, w_{B2}^{lf})$ such that:

$$v_A(w_{A2}^{lf}) + v_B(w_{B2}^{lf}) = v_B(c_B)$$

Thus, M's profits are $\hat{\Pi}_M^{\text{pSPNE}} = v_A(c_A)$, both distributors obtain profits equal to zero, and consumer surplus is $\text{CS}^{lf} = v_B(c_B)$. That is, the solution involves no monopolization: (i) M's profits are equal to A's monopoly profit (which is equal to $v_A(c_A)$ under nonlinear pricing), and (ii) consumer surplus is given by the surplus of purchasing B at a marginal price equal to the fringe's marginal cost.

1.10.2 Two-Part Tariffs with Heterogeneous Consumers

As it is evident from the previous example, the problem with allowing nonlinear pricing downstream in the presence of homogeneous consumers is that M can fully extract all the surplus that A creates. This issue, however, has an easy fix: in reality, consumers are likely to differ in how much they value a product. When this is the case, distributors will be unable to extract the surplus from

[11] To see why, suppose first that D_1 deviates to the fringe's contract. Since she expects D_2 to charge $(p_{A2}^*, p_{B2}^*, f_2^*) = (w_{A2}, w_{B2}, 0)$, D_1's optimal deviation is $w_{B1}' = c_B$ and $v_B(c_B) - f_1' = v_A(w_{A2}) + v_B(w_{B2})$ (as long as f_1' is non-negative). On the other hand, if D_2 deviates, and given that she expects D_1 to charge $(p_{A1}^*, p_{B1}^*, f_1^*) = (w_{A1}, w_{B1}, f_1^*)$, her optimal deviation is $w_{B2}' = c_B$ and $v_B(c_B) - f_2' = v_A(w_{A1}) + v_B(w_{B1}) - f_1^*$ (as long as f_2' is non-negative). Using that $f_1^* = v_A(w_{A1}) + v_B(w_{B1}) - v_A(w_{A2}) - v_B(w_{B2})$, this last condition can be rewritten as $v_B(c_B) - f_2' = v_A(w_{A2}) + v_B(w_{B2})$. Hence, for both distributors, the optimal deviation is the same, and so are reservation payoffs.

high-valuation consumers, restoring, once again, A's "musthavedness" and M's monopolization mechanism.

As a simple example, consider the same setting as in the previous subsection, but suppose consumers are heterogeneous in their valuation for A: a fraction μ of consumers have a demand $\theta_L Q_A(p_A)$ for this product, while a fraction $1-\mu$, a demand $\theta_H Q_A(p_A)$, with $0<\theta_L<\theta_H$. As in the previous section, distributors announce a single two-part tariff for all consumers.[12] Due to space constraints, we will not characterize the equilibrium in detail; we will simply show that for a certain range of parameters, M can secure strictly more than A's monopoly profits in his preferred SPNE.

A's monopoly profit.— We begin by characterizing A's monopoly profit. To do this, suppose that A is the only product in the market and that M deals directly with final consumers using a two-part tariff of the form (p_A, f_A). To obtain the optimal two-part tariff scheme, we have two cases to consider: (i) M optimally decides to serve both types of consumers, or (ii) M only serves the high-value consumers.

If M decides to sell to both types, then he sets $f_A=\theta_L v_A(p_A)$ so type-H consumers end up with strictly positive surplus. M profits, in this case, are $[\mu\theta_L+(1-\mu)\theta_H]\pi_A(p_A;c_A)+\theta_L v_A(p_A)$, so the optimal price, which we denote by p_A^{lh}, is strictly greater than c_A and satisfies:

$$\left[\mu\theta_L+(1-\mu)\theta_H\right]\left[Q_A(p_A^{lh})+(p_A^{lh}-c_A)Q_A'(p_A^{lh})\right]-\theta_L Q_A(p_A^{lh})=0 \tag{2}$$

An alternative strategy is to sell only to the high types. In that case, it is straightforward to prove that M charges $(p_A,f_A)=(c_A,\theta_H v_A(c_A))$ for a total profit of $(1-\mu)\theta_H v_A(c_A)$.

The first strategy is obviously better than the second one when μ is not too low, i.e., when $\mu>\mu^*$, where μ^* equalizes the profits of the two different cases. Hence, A's "monopoly profit" in this case is:

$$\pi_A^m=\begin{cases}(1-\mu)\theta_H v_A(c_A) & \text{if } \mu<\mu^* \\ [\mu\theta_L+(1-\mu)\theta_H]\pi_A(p_A^{lh};c_A)+\theta_L v_A(p_A^{lh}) & \text{if } \mu\geq\mu^*\end{cases}$$

From here on, we will assume that $\mu\geq\mu^*$, hence $\pi_A^m=[\mu\theta_L+(1-\mu)\theta_H]\pi_A(p_A^{lh};c_A)+\theta_L v_A(p_A^{lh})$.

For certain parameter values, M can secure more than π_A^m in his preferred SPNE.— Consider now the true game. We will show that for a certain range of parameters, M can secure strictly more than A's monopoly profit in his preferred SPNE.

To do this, suppose that M offers $\big(w_{A1}=c_A, w_{B1}=c_B, T_1, e_1=1\big)$ and $\big(w_{A2}, w_{B2}, T_2, e_2=1\big)$, with $w_{A2}\geq c_A$ and $\theta_L v_A(w_{A2})+v_B(w_{B2})=v_B(c_B)-\epsilon$, for $\epsilon>0$. Suppose then that it is an equilibrium of the continuation game for both distributors to accept M's offer. It is possible to prove

[12]This is not without loss in this setting, but it serves to explain the gist of the argument in the simplest possible way. As it will become evident below, the key ingredient is not the specific type of nonlinear pricing scheme considered but rather the existence of nonextractable surplus in product A.

that the equilibrium downstream is then given by $p^*_{A1} = p^{lh}_A$, $p^*_{B1} = c_B$, $p^*_{A2} = w_{A2}$, $p^*_{B2} = w_{B2}$, $f^*_2 = 0$, and:

$$f^*_1 = \theta_L v_A(p^{lh}_A) + v_B(c_B) - \theta_L v_A(w_{A2}) - v_B(w_{B2}) \Longrightarrow f^*_1 = \theta_L v_A(p^{lh}_A) + \epsilon$$

Hence, in the downstream market only D_1 is serving consumers on-path, leading to profits equal to $\pi^m_A + \epsilon - T_1$ after fixed fees, while D_2's on-path profits after fixed fees are equal to $-T_2$.

Now let us find the maximum upstream fees T_1 and T_2 so that both distributors accepting the offers is indeed an equilibrium. Starting from a situation where both distributors are expected to accept M's offer, consider first a deviation by D_2 to the fringe's contract. It is easy to see that following this deviation, D_2 will set B's marginal price at the fringe's cost. However, to determine the optimal deviation fee, f'_2, the distributor must decide whether to sell to all types or just to the low-types.[13] The low-types will switch to D_2 whenever:

$$\theta_L v_A(p^*_{A1}) + v_B(p^*_{B1}) - f^*_1 \leq v_B(c_B) - f'_2 \Longleftrightarrow f'_2 \leq \epsilon$$

The high-types, in turn, will switch whenever $\theta_H v_A(p^*_{A1}) + v_B(p^*_{B1}) - f^*_1 \leq v_B(c_B) - f'_2$, or equivalently, whenever:

$$f'_2 \leq \epsilon - (\theta_H - \theta_L) v_A(p^*_{A1}) = \epsilon - (\theta_H - \theta_L) v_A(p^{lh}_A)$$

Note, crucially, that inducing high-valuation consumers to switch is more costly for D_2 than inducing the low-valuation consumers to do the same. Moreover, if $\epsilon < (\theta_H - \theta_L) v_A(p^{lh}_A)$, D_2 cannot profitably induce type-H consumers to switch, as this would entail offering a negative entry fee, i.e., $f'_2 < 0$. When that is the case, D_2 will necessarily set $f'_2 = \epsilon$, her reservation payoff will then be equal to $\mu\epsilon$, and the maximum upstream fee that M can charge while still inducing D_2 to accept the contract is $T^*_2 = -\mu\epsilon$.

We can repeat a similar procedure for the case where D_1 deviates. We then find that D_1 will be unable to profitably induce type-H consumers to switch when $\epsilon < (\theta_H - \theta_L) v_A(w_{A2})$. When this parametric condition holds, D_1's reservation payoffs are also $\mu\epsilon$, and the maximum upstream fee that M can charge while still inducing her to accept the contract is:

$$\pi^m_A + \epsilon - T^*_1 = \mu\epsilon \Longrightarrow T^*_1 = \pi^m_A + \epsilon(1 - \mu)$$

In consequence, if (i) M makes the offers described above, (ii) M can coordinate distributors' acceptance decision into his most favorable outcome (as in his preferred SPNE), and (iii) $\epsilon <$

[13] Given that D_2 lost access to A, then if he entices the high-type values to go to her store, she automatically entices the low-type values to follow as well.

$(\theta_H - \theta_L)\min\{v_A(p_A^{lh}), v_A(w_{A2})\}$, then M's payoffs are equal to:

$$\Pi_M = T_1^* + T_2^* = \pi_A^m + (1-2\mu)\epsilon$$

This is strictly greater than π_A^m so long as $\mu \in [\mu^*, 1/2)$.[14]

Finally, since here always exists a triple $(\epsilon, w_{A2}, w_{B2})$ such that (i) $w_{A2} \geq c_A$, (ii) $\theta_L v_A(w_{A2}) + v_B(w_{B2}) = v_B(c_B) - \epsilon$, and (iii) $0 < \epsilon < (\theta_H - \theta_L)\min\{v_A(p_A^{lh}), v_A(w_{A2})\}$, the previous result implies that if $\mu \in [\mu^*, 1/2)$, then M must be obtaining strictly more than A's monopoly profit in this preferred SPNE. This is because M can always make the offers described above, coordinate distributors, and obtain strictly more than π_A^m as we showed above. $\square$

2 Proofs Omitted from Section 4

2.1 Characterization and Properties of Equilibrium Retail Prices

In this appendix, we show the existence of an equilibrium in the retail pricing subgame given some vector of on-path equilibrium wholesale prices $\boldsymbol{w} = (\boldsymbol{w}_1, ..., \boldsymbol{w}_n)$. Consider D_i's pricing problem:

$$\begin{aligned}&\max_{\boldsymbol{p}_i} \big(\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi})\big)s\big(\boldsymbol{p}_i, \boldsymbol{p}_{-i}\big) \\ &\quad = \max_{\boldsymbol{p}_i} \big(\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi})\big)\tilde{s}(u(\boldsymbol{p}_i), \boldsymbol{u}_{-i}(\boldsymbol{p}_{-i}))\end{aligned} \tag{3}$$

where $u(\boldsymbol{p}_i) = v_A(p_{Ai}) + v_B(p_{Bi})$, $\boldsymbol{u}_{-i}(\boldsymbol{p}_{-i}) = \big(u(\boldsymbol{p}_j)\big)_{j\neq i}$, and $\tilde{s}(u_i, \boldsymbol{u}_{-i}) = \mathbb{P}(u_i + \xi_i^\ell/\gamma \geq \max_{j\neq i}\{u_j + \xi_j^\ell/\gamma\})$. We begin with the following claim:

Claim 2.1.1. *Let $\boldsymbol{p}_i^{BR}$ be a solution of (3); then $w_{ki} < p_{ki}^{BR} < p_k^m(w_{ki})$, where $p_k^m(w_{ki}) \equiv \arg\max_{p_k} \pi_k(p_k; w_{ki})$.*

Proof. The first-order conditions of (3) can be written as:

$$\frac{\pi_k'(p_{ki}; w_{ki})}{Q_A(p_{ki})} - \big(\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi})\big)\left.\frac{\partial \ln \tilde{s}}{\partial u_i}\right|_{(u(\boldsymbol{p}_i), \boldsymbol{u}_{-i}(\boldsymbol{p}_{-i}))} = 0, \text{ for } k = A, B \tag{4}$$

First, we will show that $p_{ki}^{BR} \geq w_{ki}$. Combine both first-order conditions to obtain $(p_{Ai} - w_{Ai})\varepsilon_A(p_{Ai})/p_{Ai} = (p_{Bi} - w_{Bi})\varepsilon_B(p_{Bi})/p_{Bi}$, for which we use the fact that $\pi_k'(p; w)/Q_k(p) = 1 + \varepsilon_k(p)(p - w)/p$. Since $\text{sign}(\varepsilon_A(p_{Ai})) = \text{sign}(\varepsilon_B(p_{Bi}))$, then $\text{sign}(p_{Ai} - w_{Ai}) = \text{sign}(p_{Bi} - w_{Bi})$ necessarily. But if so, then $p_{Ai} < w_{Ai}$ if and only if $p_{Bi} < w_{Bi}$, so the distributor that sets $p_{ki} < w_{ki}$ would be earning strictly negative profits. This, however, cannot be optimal since the distributor can always deviate and set arbitrarily high retail prices to earn zero profits. Thus, $p_{ki}^{BR} \geq w_{ki}$.

[14] This, of course, requires that $\mu^* < 1/2$. However, it is easy to build parametric examples in which this condition is fulfilled.

Now we will show that $p_{ki}^{BR} > w_{ki}$. Note that the condition $(p_{Ai} - w_{Ai})\varepsilon_A(p_{Ai})/p_{Ai} = (p_{Bi} - w_{Bi})\varepsilon_B(p_{Bi})/p_{Bi}$ also implies that $p_{Ai} = w_{Ai}$ if and only if $p_{Bi} = w_{Bi}$. However, that requires $\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi}) = 0$, so the first-order conditions of the distributor's problem imply that $p_{ki}^{BR} = p_k^m(w_{ki})$ for $k = A, B$. But if so, then $\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi}) > 0$, contradicting that $\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi}) = 0$.

Finally, we show that $p_{ki}^{BR} < p_k^m(w_{ki})$. Given that $p_{ki}^{BR} > w_{ki}$, in the optimum $\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi}) > 0$. Since $\partial \ln \tilde{s}/\partial u_i > 0$, the fact that $\pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi}) > 0$ implies that the left-hand side of (4) is strictly negative for all $p_{ki} \geq p_k^m(w_{ki})$. Hence, $p_{ki}^{BR} < p_k^m(w_{ki})$. $\square$

The claim states that irrespective of what other distributors do, the retail price a distributor charges for product $k = A, B$ is always between its wholesale price and its monopoly price (corresponding to that wholesale price). Therefore, on-path equilibrium prices $\boldsymbol{p}^*(\boldsymbol{w})$ will be such that $w_{ki} < p_{ki}^*(\boldsymbol{w}) < p_k^m(w_{ki})$.

Now, notice that in problem (3), the prices charged by distributor $j \neq i$ affects D_i's profit function only through the consumer traffic function $\tilde{s}(u_i(\boldsymbol{p}_i), \boldsymbol{u}_{-i}(\boldsymbol{p}_{-i}))$. This function, furthermore, depends on the vector of overall surplus each distributor leaves in the hands of final consumers, not on the individual prices charged.

This fact implies that the original retail pricing subgame is strategically equivalent to one where distributors solve a two-step problem: first, they select their optimal mix of prices subject to leaving visitors at least a surplus of u_i, and then they compete for consumer traffic by deciding how much surplus u_i to actually leave their visitors. Consequently, studying the equilibrium set of the original pricing subgame is equivalent to studying the equilibrium set of an auxiliary surplus subgame where distributors solve $\max_{u_i \in [\underline{u}_i, \bar{u}_i]} Y(u_i; w_{Ai}, w_{Bi})\tilde{s}(u_i, \boldsymbol{u}_{-i})$, where $\underline{u}_i = v_A(p_A^m(w_{Ai})) + v_B(p_B^m(w_{Bi}))$, $\bar{u}_i = v_A(w_{Ai}) + v_B(w_{Bi})$, and:

$$Y(u_i; w_{Ai}, w_{Bi}) = \left\{ \max_{p_{Ai}, p_{Bi}} \pi_A(p_{Ai}; w_{Ai}) + \pi_B(p_{Bi}; w_{Bi}) \text{ s.t. } v_A(p_{Ai}) + v_B(p_{Bi}) = u_i \right\} \tag{5}$$

With this in mind, we then have following result (due to Quint, 2014):

Claim 2.1.2. *$\tilde{s}(u_i, \boldsymbol{u}_{-i})$ is log-concave in u_i, and $\ln \tilde{s}$ satisfies strictly increasing differences in u_i and u_j for $i \neq j$*

Proof. This result follows from the log-concavity of the distribution of consumer-specific shocks $G(x)$. For the formal proof see Quint (2014), Theorem 1. $\square$

Claim 2.1.2 implies that $\ln Y(u_i; w_{Ai}, w_{Bi}) + \ln \tilde{s}(u_i, \boldsymbol{u}_{-i})$ is supermodular in $\boldsymbol{u} = (u_i, \boldsymbol{u}_{-i})$. Hence the downstream surplus game is a smooth and strictly log-supermodular game. Consequently, we have the following result:

Lemma 2.1.1. *Suppose that on-path wholesale prices are given by* $\boldsymbol{w}$*. Then there exists a vector of equilibrium retail prices* $\boldsymbol{p}^*(\boldsymbol{w})$*. Moreover, if distributors have the same wholesale prices for each product, then* $p^*_{ki} = p^*_{kj}$ *for all* $i, j = 1, ..., n$ *and* $k = A, B$*.*

Proof. Existence of $\boldsymbol{p}^*(\boldsymbol{w})$ follows because the equivalent surplus game is log-supermodular (e.g., Vives, 1999). The result that equilibrium retail prices are symmetric if distributors have the same wholesale prices follows because symmetric supermodular games, where strategy spaces are single-dimensional, have only symmetric equilibria (Vives, 1999, 2005). This implies that if distributors have the same wholesale prices, then all distributors necessarily offer the same equilibrium surplus. The symmetry of prices then follows because the solution to (5) is unique. □

Now suppose that distributors have the same on-path wholesale prices w_A and w_B for A and B, respectively. By the previous lemma, we know that on-path, all distributors charge the same retail prices for each product, i.e., $p^*_{ki} = p^*_k$ for $i = 1, ..., n$ and $k = A, B$. This immediately implies that equilibrium retail prices must satisfy:

$$(p^*_A, p^*_B) \in \underset{p_{Ai}, p_{Bi}}{\arg\max} \; \big(\pi_A(p_{Ai}; w_A) + \pi_B(p_B; w_B)\big) H\big(\gamma(v_A(p_{Ai}) + v_B(p_{Bi}) - v_A(p^*_A) - v_B(p^*_B))\big) \quad (6)$$

where $H(x)$ is the cumulative distribution function of $\max_{j \neq i}\{\xi^\ell_j\} - \xi^\ell_i$. The following lemma states some of the properties these equilibrium retail prices satisfy (we will use these properties in the proof of Lemma 2):

Lemma 2.1.2. *If* $(w_{Ai}, w_{Bi}) = (w_A, w_B)$ *for* $i = 1, ..., n$*, then equilibrium prices* p^*_A *and* p^*_B *are unique and characterized by the following first-order conditions:*

$$\frac{\pi'_A(p^*_A; w_A)}{Q_A(p^*_A)} = \frac{\pi'_B(p^*_B; w_B)}{Q_B(p^*_B)} \quad (7)$$

$$\frac{\pi'_B(p^*_B; w_B)}{Q_B(p^*_B)} = \gamma\big[\pi_A(p^*_A; w_A) + \pi_B(p^*_B; w_B)\big]\left(\frac{H'(0)}{H(0)}\right) \quad (8)$$

Proof. Obtaining the first-order conditions of (6) and imposing symmetry yields (7) and (8). To show uniqueness, recall that $\pi'_k(p; w)/Q_k(p) = 1 + \varepsilon_k(p)(p - w)/p$ is strictly decreasing in p for $k = A, B$. Hence, condition (7) determines a unique p_A as a function of p_B, $p_A = \psi(p_B)$, which, furthermore, is strictly increasing in p_B. Evaluating this on (8) we get $\pi'_B(p^*_B; w_B)/Q_B(p^*_B) - z(\gamma, n)\big[\pi_A(\psi(p^*_B); w_A) + \pi_B(p^*_B; w_B)\big] = 0$, where $z(\gamma, n) \equiv \gamma H'(0)/H(0)$.

Define then the function $\Gamma(p_B) \equiv \pi'_B(p_B; w_B)/Q_B(p_B) - z(\gamma, n)\big[\pi_A(\psi(p_B); w_A) + \pi_B(p_B; w_B)\big]$. Because the pair (p^*_A, p^*_B) exists, we know there is at least one p^*_B such that $\Gamma(p^*_B) = 0$. Now, note that:

$$\Gamma'(p_B) = \frac{\partial}{\partial p_B}\left(\frac{\pi'_B(p_B; w_B)}{Q_B(p_B)}\right) - z(\gamma, n)\left[\pi'_A(\psi(p_B); w_A)\psi'(p_B) + \pi'_B(p_B; w_B)\right]$$

which is strictly less than zero in the relevant range of prices (i.e., those prices such that $w_k < p^*_k < p^m_k(w_k)$ given Claim 2.1.1). The latter follows since: (i) $\pi'_k(p; w)/Q_k(p) = 1 + \varepsilon_k(p)(p - w)/p$ is strictly decreasing in p, (ii) $\pi_k'(p_k; w_k) > 0$ in the relevant range (since $\pi_k(p_k; w_k)$ is strictly concave, and, therefore, strictly quasiconcave), and (iii) $\psi'(p_B) > 0$. Consequently, there can be at most one p_B^* (in the relevant range of prices) such that $\Gamma(p_B^*) = 0$, which implies that p_B^* is unique, and so is $p_A^* = \psi(p_B^*)$. □

2.2 Proof of Lemma 2

Let $\boldsymbol{w}^{lf}$ be the solution to M's laissez-faire problem:

$$\max_{\boldsymbol{w}} \hat{\Pi}_M^{lf} = \sum_i \big(\pi_A(p_{Ai}^*(\boldsymbol{w}); c_A) + \pi_B(p_{Bi}^*(\boldsymbol{w}); c_B)\big) s(\boldsymbol{p}_i^*(\boldsymbol{w}), \boldsymbol{p}_{-i}^*(\boldsymbol{w})) - \sum_i \left(\max_{\tilde{p}_{Ai}=\infty, \tilde{p}_{Bi}} \Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}^*(\boldsymbol{w}); \infty, c_B) \right) \quad (9)$$

In this Appendix, we show that $p_{ki}^*(\boldsymbol{w}^{lf}) = p_k^{lf}$ for $k = A, B$, where:

$$(p_A^{lf}, p_B^{lf}) \equiv \arg\max_{p_A, p_B} \{\pi_A(p_A; c_A) + \pi_B(p_B; c_B) - nR(p_A, p_B)\} \quad (10)$$

and:

$$R(p_A, p_B) \equiv \max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B) H\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A) - \gamma v_B(p_B)\big)$$

Moreover, we will also show that M's profit in the laissez-faire equilibrium are equal to $\hat{\Pi}^{lf} = \pi_A(p_A^{lf}; c_A) + \pi_B(p_B^{lf}; c_B) - nR(p_A^{lf}, p_B^{lf})$.

The idea behind the proof.— The proof is somewhat involved, which is why we begin with an outline of it. Note from (9) that wholesale prices $\boldsymbol{w}$ only enter into the objective function indirectly, through the equilibrium retail prices $\boldsymbol{p}^*(\boldsymbol{w}) \equiv (\boldsymbol{p}_i^*(\boldsymbol{w}), \boldsymbol{p}_{-i}^*(\boldsymbol{w}))$. Consider, therefore, the auxiliary problem where M maximizes over retail prices directly:

$$\max_{\boldsymbol{p}} \hat{\Pi}_M^{lf\star} = \sum_i \big(\pi_A(p_{Ai}; c_A) + \pi_B(p_{Bi}; c_B)\big) s(\boldsymbol{p}_i, \boldsymbol{p}_{-i}) - \sum_i \left(\max_{\tilde{p}_{Ai}=\infty, \tilde{p}_{Bi}} \Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}; \infty, c_B) \right) \quad (11)$$

Given that M has full control over prices in (11), while it controls them only imperfectly in (9), $\hat{\Pi}_M^{lf\star} \geq \hat{\Pi}_M^{lf}$. We will then show that problem (11) has a unique global maximum involving symmetric prices $p_{ki}^{lf} = p_k^{lf}$ for $k = A, B$ and $i, j = 1, .., n$ and that those prices are the same as the ones given by (10). This also implies that:

$$\hat{\Pi}_M^{lf\star} = \pi_A(p_A^{lf}; c_A) + \pi_B(p_B^{lf}; c_B) - nR(p_A^{lf}, p_B^{lf})$$

We then return to our original problem (9) and show that these symmetric prices can be implemented with non-discriminatory tying offers with exclusivity provisions $(w_A^{lf}, w_B^{lf}, T^{lf}, e^{lf} = 1)$.

Because $\hat{\Pi}_M^{lf\star}$ is an upper bound of M's "true" profits, $\hat{\Pi}_M^{lf}$, this immediately implies that these nondiscriminatory offers are optimal for M in the laissez faire. Hence, the solution to problem (9) is such that $p_{ki}^*(\boldsymbol{w}^{lf}) = p_k^{lf}$ for $k = A, B$, where the latter are given by (10), and M's equilibrium profits are equal to $\hat{\Pi}_M^{lf} = \hat{\Pi}_M^{lf\star} = \pi_A(p_A^{lf}; c_A) + \pi_B(p_B^{lf}; c_B) - nR(p_A^{lf}, p_B^{lf})$.

2.2.1 The Proof

Step 1: Transforming Problem (11).— Note first that:

$$\Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}; \infty, c_B) = \max_{\tilde{p}_{Bi}} \pi_B(\tilde{p}_{Bi}; c_B)\mathbb{P}\big(v_B(\tilde{p}_{Bi}) + \xi_i^\ell/\gamma \geq \max_{j \neq i}\{v_A(p_{Aj}) + v_B(p_{Bj}) + \xi_j^\ell/\gamma\}\big)$$

Thus, $\boldsymbol{p}_{-i}$ only enters indirectly through $\boldsymbol{u}_{-i}(\boldsymbol{p}_{-i}) \equiv \big(v_A(p_{Aj}) + v_B(p_{Bj})\big)_{j \neq i}$. In consequence, $\Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}; \infty, c_B)$ can be written as $\Pi_D(\tilde{\boldsymbol{p}}_i, \boldsymbol{p}_{-i}; \infty, c_B) = \mathcal{R}(\boldsymbol{u}_{-i}(\boldsymbol{p}_{-i}))$, where:

$$\mathcal{R}(\boldsymbol{u}_{-i}) = \max_{\tilde{p}_{Bi}} \pi_B(\tilde{p}_{Bi}; c_B)\mathbb{P}\big(v_B(\tilde{p}_{Bi}) + \xi_i^\ell/\gamma \geq \max_{j \neq i}\{u_j + \xi_j^\ell/\gamma\}\big)$$

We can thus rewrite problem (11) as $\max_{\boldsymbol{u}} \hat{\Pi}_M^{lf\star} = \mathcal{V}^{lf}(\boldsymbol{u}) - \sum_i \mathcal{R}(\boldsymbol{u}_{-i})$, where:

$$\mathcal{V}^{lf}(\boldsymbol{u}) = \max_{\boldsymbol{p}} \left\{ \sum_i \big(\pi_A(p_{Ai}; c_A) + \pi_B(p_{Bi}; c_B)\big) s_i(\boldsymbol{p}_i, \boldsymbol{p}_{-i}) \quad \text{s.t. } u_i = v_A(p_{Ai}) + v_B(p_{Bi}) \text{ for } i = 1, ..., n \right\}$$

Using the fact that $\pi_k(p_{ki}; c_k)$ is strictly concave in p_{ki}, and given that $\sum_{i=1}^n s_i(u_i, \boldsymbol{u}_{-i}) = 1$ and $0 \leq s_i(u_i, \boldsymbol{u}_{-i}) \leq 1$ (so $s_1(u_1, \boldsymbol{u}_{-1})$, $s_2(u_2, \boldsymbol{u}_{-2})$, ..., $s_n(u_n, \boldsymbol{u}_{-n})$ can be thought as the weights of a convex combination), it is then straightforward to prove that $\mathcal{V}^{lf}(\boldsymbol{u})$ is strictly concave in $\boldsymbol{u}$.

Step 2: Symmetric Surpluses are a Local Maximum.— Note that the function $\hat{\Pi}_M^{lf\star} = \mathcal{V}^{lf}(\boldsymbol{u}) - \sum_i \mathcal{R}(\boldsymbol{u}_{-i})$, is exchangeable upon permutation of distributors. Hence by Waterhouse (1983) the problem has a symmetric critical/stationary point $u_i = u^{lf}$ for all $i = 1, ..., n$ (i.e., $u_i = u^{lf}$ for all i satisfies the first-order conditions of the problem).

We now demonstrate that this symmetric critical point is a local maximum by showing that $\mathcal{V}^{lf}(\boldsymbol{u}) - \sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ is locally concave around that point. Since $\mathcal{V}^{lf}(\boldsymbol{u})$ is strictly concave in $\boldsymbol{u}$, a sufficient condition for this is that $\mathcal{R}(\boldsymbol{u}_{-i})$ is locally convex in $\boldsymbol{u}$. This is exactly the case, as the following claim states.

Claim 2.2.1. *Let $B_\epsilon(\boldsymbol{x})$ denote an ϵ-ball around $\boldsymbol{x}$. If $\boldsymbol{u}_{-i} \in B_\epsilon(u^{lf}, ..., u^{lf})$, then $\mathcal{R}(\boldsymbol{u}_{-i})$ is strictly convex in $\boldsymbol{u}_{-i}$.*

Proof. Let $\tilde{R}(u) \equiv \mathcal{R}(u, ..., u) = \pi_B(\tilde{p}_B^*(u); c_B)H(\gamma v_B(\tilde{p}_B^*(u)) - \gamma u)$ and $V^{lf}(u) \equiv \mathcal{V}(u, ..., u) =$

$\pi_A(p^*_A(u); c_A) + \pi_B(p^*_B(u); c_B)$, where:

$$(p^*_A(u), p^*_B(u)) = \arg\max_{p_A, p_B} \left\{\pi_A(p_A; c_A) + \pi_B(p_B; c_B) \text{ s.t. } v_A(p_A) + v_B(p_B) = u\right\}$$
$$\tilde{p}^*_B(u) = \arg\max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B) H(\gamma v_B(\tilde{p}_B) - \gamma u)$$

and $H(x)$ is the cumulative distribution function of $\max_{j\neq i}\{\xi^\ell_j\} - \xi^\ell_i$. To simplify notation, it is convenient to work with the CDF of $(\max_{j\neq i}\{\xi^\ell_j\} - \xi^\ell_i)/\gamma$, which we denote by $\tilde{H}(x)$, instead of the CDF of $\max_{j\neq i}\{\xi^\ell_j\} - \xi^\ell_i$. This allows us to omit γ from our notation. Under this alternative notation, we have that $\tilde{p}^*_B(u) = \arg\max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B)\tilde{H}(v_B(\tilde{p}_B) - u)$.

Consider then the following problem: $\max_u V^{lf}(u) - n\tilde{R}(u)$. Denoting by u^{lf} the solution to this problem, we have that u^{lf} must satisfy the following first-order condition:

$$\frac{\pi'_B(p^*_B(u^{lf}); c_B)}{Q_B(p^*_B(u^{lf}))} = \frac{\pi'_B(\tilde{p}^*_B(u^{lf}); c_B)}{Q_B(\tilde{p}^*_B(u^{lf}))} n\tilde{H}\big(v_B(\tilde{p}^*_B(u^{lf})) - u^{lf}\big) \tag{12}$$

where we are using the fact that:

$$-\frac{dV^{lf}}{du} = \frac{\pi'_A(p^*_A(u); c_A)}{Q_A(p^*_A(u))} = \frac{\pi'_B(p^*_B(u); c_B)}{Q_B(p^*_B(u))} \quad \text{and} \quad -\tilde{R}'(u) = \frac{\pi'_B(\tilde{p}^*_B(u); c_B)}{Q_B(\tilde{p}^*_B(u))}\tilde{H}\big(v_B(\tilde{p}^*_B(u)) - u\big)$$

Using the first-order condition (12), it is possible to prove that $v_B(\tilde{p}^{lf}_B) < u^{lf}$. With this inequality in mind, consider $\tilde{R}(u) = \max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B)\tilde{H}(v_B(\tilde{p}_B) - u)$. Note that when $u \in B_\epsilon(u^{lf})$, then $v_B(\tilde{p}^*_B(u)) < u$ (since $v_B(\tilde{p}^{lf}_B) < u^{lf}$), so when solving this problem we can restrict attention to the set of $\tilde{p}_B$ such that $v_B(\tilde{p}_B) < u$. In this region, moreover, the function $f(u) \equiv \pi_B(\tilde{p}_B; c_B)\tilde{H}(v_B(\tilde{p}_B) - u)$ is strictly convex in u:

$$f''(u) = \pi_B(\tilde{p}_B; c_B)\tilde{H}''(v_B(\tilde{p}_B) - u) > 0$$

where the inequality follows because $\tilde{H}''(v_B(\tilde{p}_B) - u) > 0$ given that (i) $v_B(\tilde{p}_B) - u < 0$ and (ii) $\tilde{H}(x)$ is a log-concave cumulative distribution function with mode greater than or equal to zero (i.e., it is the CDF of $(\max_{j\neq i}\{\xi^\ell_j\} - \xi^\ell_i)/\gamma$). Thus, when $u \in B_\epsilon(u^{lf})$, the problem that determines $\tilde{R}(u)$ satisfies the Convex Maximum Theorem conditions (see, e.g., Carter, 2001, p. 342), implying that $\tilde{R}(u)$ is strictly convex in u.

Consequently, the function $\mathcal{R}(\boldsymbol{u}_{-i})$ is strictly convex in $\boldsymbol{u}_{-i}$ when moving in the direction $u_i = u_j$ for all $j \neq i$ (i.e., when imposing symmetry ex-ante) in a neighborhood of $(u^{lf}, ..., u^{lf})$. Thus, by continuity, $\mathcal{R}(\boldsymbol{u}_{-i})$ is strictly convex in $\boldsymbol{u}_{-i}$ in a neighborhood of $(u^{lf}, ..., u^{lf})$ when moving in any direction. That is, $\mathcal{R}(\boldsymbol{u}_{-i})$ is strictly convex in $\boldsymbol{u}_{-i}$ in a neighborhood of $(u^{lf}, ..., u^{lf})$. □

Step 3: Symmetric Surpluses are the Unique Global Maximum.— Even though $\mathcal{V}^{lf}(\boldsymbol{u})$ is strictly concave in $\boldsymbol{u}$, the objective function $\mathcal{V}^{lf}(\boldsymbol{u}) - \sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ is not strictly concave in the entire domain because the term $\sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ is not necessarily convex in $\boldsymbol{u}_{-i}$. Hence, we cannot show that the symmetric point is a global optimum using global strict concavity. Instead, we follow an alternative approach that relies on convexifying the term $\sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ and using the fact that $\sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ is convex in $\boldsymbol{u}_{-i}$ around the symmetric critical point $\boldsymbol{u}^{lf} = (u^{lf}, ..., u^{lf})$.

Here is the formal proof. First, we know that $\mathcal{V}^{lf}(\boldsymbol{u}) - \sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ has a symmetric critical point $\boldsymbol{u}^{lf} = (u^{lf}, ..., u^{lf})$, and that at such point $\mathcal{R}(\boldsymbol{u}_{-i})$ is strictly convex in $\boldsymbol{u}_{-i}$ (see Claim 2.2.1), so $\sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ is convex in $\boldsymbol{u}_{-i}$ also. Define $f(\boldsymbol{u}) \equiv \sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ and let $\mathrm{cv}\, f$ be the convex envelope of f in $\boldsymbol{u}$, i.e., $\mathrm{cv}\, f = \inf\{t \,|\, (t, \boldsymbol{u}) \in \textbf{conv epi}\, f\}$. That is, $\mathrm{cv}\, f$ is the greatest convex function in $\boldsymbol{u}$ which is less than f. By construction, we have that $\mathcal{V}^{lf}(\boldsymbol{u}) - \mathrm{cv}\, f(\boldsymbol{u}) \geq \mathcal{V}^{lf}(\boldsymbol{u}) - f(\boldsymbol{u})$. Furthermore, since $\mathcal{V}^{lf}(\boldsymbol{u})$ is strictly concave in $\boldsymbol{u}$ and $\mathrm{cv}\, f(\boldsymbol{u})$ is convex $\boldsymbol{u}$, then $\mathcal{V}^{lf}(\boldsymbol{u}) - \mathrm{cv}\, f(\boldsymbol{u})$ is strictly concave in $\boldsymbol{u}$. This implies that the latter has a unique interior global optimum $\boldsymbol{u}^*$:

$$\mathcal{V}^{lf}(\boldsymbol{u}^*) - \mathrm{cv}\, f(\boldsymbol{u}^*) > \mathcal{V}^{lf}(\boldsymbol{u}) - \mathrm{cv}\, f(\boldsymbol{u}) \;\; \forall\; \boldsymbol{u} \neq \boldsymbol{u}^*$$

which is a critical point of $\mathcal{V}^{lf}(\boldsymbol{u}) - \mathrm{cv}\, f(\boldsymbol{u})$.

Note then that since $\mathcal{R}(\boldsymbol{u}_{-i})$ is convex in $\boldsymbol{u}_{-i}$ around $\boldsymbol{u}^{lf}$, then $f(\boldsymbol{u})$ is convex in $\boldsymbol{u}$ around $\boldsymbol{u}^{lf}$ also. This implies that at $\boldsymbol{u}^{lf}$, $\mathrm{cv}\, f$ and f coincide, so $\mathcal{V}^{lf}(\boldsymbol{u}^{lf}) - \mathrm{cv}\, f(\boldsymbol{u}^{lf}) = \mathcal{V}^{lf}(\boldsymbol{u}^{lf}) - f(\boldsymbol{u}^{lf})$. But since $\boldsymbol{u}^{lf}$ is a critical point of $\mathcal{V}^{lf}(\boldsymbol{u}) - f(\boldsymbol{u})$, this implies that it is also a critical point of $\mathcal{V}^{lf}(\boldsymbol{u}) - \mathrm{cv}\, f(\boldsymbol{u})$. Furthermore, given that $\mathcal{V}^{lf}(\boldsymbol{u}) - \mathrm{cv}\, f(\boldsymbol{u})$ has a unique critical point, then it must be that $\boldsymbol{u}^{lf} = \boldsymbol{u}^*$. However, if so, then:

$$\mathcal{V}^{lf}(\boldsymbol{u}^{lf}) - f(\boldsymbol{u}^{lf}) = V(\boldsymbol{u}^{lf}) - \mathrm{cv}\, f(\boldsymbol{u}^{lf}) > \mathcal{V}^{lf}(\boldsymbol{u}) - \mathrm{cv}\, f(\boldsymbol{u}) \geq \mathcal{V}^{lf}(\boldsymbol{u}) - f(\boldsymbol{u}) \;\; \forall\; \boldsymbol{u} \neq \boldsymbol{u}^{lf}$$

Hence,

$$\mathcal{V}^{lf}(\boldsymbol{u}^{lf}) - f(\boldsymbol{u}^{lf}) > \mathcal{V}^{lf}(\boldsymbol{u}) - f(\boldsymbol{u}) \;\; \forall\; \boldsymbol{u} \neq \boldsymbol{u}^{lf}$$

that is, $\boldsymbol{u}^{lf}$ is the unique global optimum of $\mathcal{V}^{lf}(\boldsymbol{u}) - f(\boldsymbol{u})$ □

Step 4: The vector of surpluses $\boldsymbol{u}^{lf} = (u^{lf}, ..., u^{lf})$ is a solution of M's "true" problem.— We know that the solution to $\max_{\boldsymbol{u}} \mathcal{V}^{lf}(\boldsymbol{u}) - \sum_i \mathcal{R}(\boldsymbol{u}_{-i})$ is unique and involves symmetric surpluses, i.e., $\boldsymbol{u}^{lf} = (u^{lf}, ..., u^{lf})$. Moreover, from step 2, we also know that u^{lf} satisfies condition (12). Because the vector $(p_A^*(u), p_B^*(u))$ is unique for any given u, we can then recover the unique set of on-path prices consistent with u^{lf}, i.e., $p_k^{lf} \equiv p_k^*(u^{lf})$. Note that by construction, these prices satisfy the following first-order conditions:

$$\frac{\pi_k'(p_k^{lf}; c_k)}{Q_k(p_k^{lf})} = \frac{\pi_B'(\tilde{p}_B^{lf}; c_B)}{Q_B(\tilde{p}_B^{lf})} nH\big(\gamma v_B(\tilde{p}_B^{lf}) - \gamma v_A(p_A^{lf}) - \gamma v_B(p_B^{lf})\big), \text{ for } k = A, B$$

where $\tilde{p}_B^{lf}$ denotes the solution to $\max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B) H\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A^{lf}) - \gamma v_B(p_B^{lf})\big)$. That is, $p_k^{lf} \equiv p_k^*(u^{lf})$ are given by the solution of problem (10).

From here, it is relatively easy to see that M can implement (p_A^{lf}, p_B^{lf}) by suitably choosing the wholesale prices of his offer, implying that M's true problem (9) has the same solution as M's auxiliary problem (11). Indeed, by Lemma 2.1.1 of the online Appendix 2.1, we know that if distributors have the same wholesale prices, they will charge the same prices for each product. Furthermore, by Lemma 2.1.2 of the same appendix, we also know that the induced equilibrium prices $p_A^*(w_A, w_B)$ and $p_B^*(w_A, w_B)$ for a given set of wholesale prices are unique and characterized by the following conditions:

$$\frac{\pi_A'(p_A^*; w_A)}{Q_A(p_A^*)} = \frac{\pi_B'(p_B^*; w_B)}{Q_B(p_B^*)} \quad \text{and} \quad \frac{\pi_B'(p_B^*; w_B)}{Q_B(p_B^*)} = z(\gamma, n)\big[\pi_A(p_A^*; w_A) + \pi_B(p_B^*; w_B)\big]$$

Using these two conditions plus the first-order conditions that determine (p_A^{lf}, p_B^{lf}), it is straightforward to prove that there exists a $w_A^{lf} > c_A$ and a $w_B^{lf} > c_B$ that implements (p_A^{lf}, p_B^{lf}). Hence, the solution to M's true problem

$$\max_{\boldsymbol{w}} \hat{\Pi}_M^{lf} = \sum_i \big(\pi_A(p_{Ai}^*(\boldsymbol{w}); c_A) + \pi_B(p_{Bi}^*(\boldsymbol{w}); c_B)\big) s(\boldsymbol{p}_i^*(\boldsymbol{w}), \boldsymbol{p}_{-i}^*(\boldsymbol{w})) - \sum_i R(\boldsymbol{p}_{-i}^*(\boldsymbol{w}))$$

induces the same retail prices as the solution to problem (10), i.e., $p_{ki}^*(\boldsymbol{w}^{lf}) = p_k^{lf}$ for $k = A, B$, also implying that M's equilibrium profits are equal to $\hat{\Pi}_M^{lf} = \hat{\Pi}_M^{lf\star} = \pi_A(p_A^{lf}; c_A) + \pi_B(p_B^{lf}; c_B) - nR(p_A^{lf}, p_B^{lf})$. □

2.3 Proof of Proposition 5

Recall that by Lemma 2, the laissez-faire retail prices can be characterized by solving the following problem $\max_{(p_A, p_B)} \{\pi_A(p_A; c_A) + \pi_B(p_B; c_B) - nR(p_A, p_B)\}$ where:

$$R(p_A, p_B) \equiv \max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B) H\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A) - \gamma v_B(p_B)\big)$$

Moreover, M's equilibrium profits are equal to $\hat{\Pi}_M^{lf} = \pi_A(p_A^{lf}; c_A) + \pi_B(p_B^{lf}; c_B) - nR(p_A^{lf}, p_B^{lf})$.

2.3.1 When A Does Not Exist

When A does not exists, M's problem becomes:[15]

$$\max_{p_B} \hat{\Pi}_M(p_B) = \pi_B(p_B; c_B) - n \max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B) H\big(\gamma v_B(\tilde{p}_B) - \gamma v_B(p_B)\big) \tag{13}$$

[15] Mathematically speaking, this case is obtained by multiplying A's demand by a constant z, $zQ_A(p_A)$, and setting $z = 0$.

Let p_B^{of} be the retail price of B that would ensue if all distributors in equilibrium buy B from the fringe, i.e.,

$$p_B^{of} = \arg\max_{p_B} \pi_B(p_B; c_B)H\big(\gamma v_B(p_B) - \gamma v_B(p_B^{of})\big)$$

and denote by $\tilde{p}_B^*(p_B)$ the solution to $\max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B)H\big(\gamma v_B(\tilde{p}_B) - \gamma v_B(p_B)\big)$. Note that by construction $\tilde{p}_B^*(p_B^{of}) = p_B^{of}$. This implies that M can always guarantee himself a payoff of zero by setting $p_B^{lf}(\text{without } A) = p_B^{of}$, as $\pi_B(p_B^{of}; c_B) = n\pi_B(p_B^{of}; c_B)H(0)$, given that $H(0) = 1/n$.

Now suppose by contradiction that $p_B^{lf}(\text{without } A) \neq p_B^{of}$. Notice that a distributor that rejects M's offer always has the option to obtain B from the fringe and charge $p_B^{lf}(\text{without } A)$. Consequently,

$$\max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B)H\big(\gamma v_B(\tilde{p}_B) - \gamma v_B(p_B^{lf}(\text{without } A))\big) \geq \pi_B(p_B^{lf}(\text{without } A); c_B)/n$$

with strictly inequality unless $p_B^{lf}(\text{without } A) = p_B^{of}$. Hence, if $p_B^{lf}(\text{without } A) \neq p_B^{of}$, then $\hat{\Pi}_M(p_B^{lf}(\text{without } A)) < 0$, which contradicts the premise that $p_B^{lf}(\text{without } A)$ was optimal in the first place. Consequently, when A does not exist, M induces distributors to charge the same retail prices for B as if they all source the product from the fringe, i.e., $p_B^{lf}(\text{without } A) = p_B^{of}$. □

2.3.2 When A is Present

We prove this part of the proposition in two steps. First, we show that $\hat{\Pi}_M^{lf} > \pi_A(p_A^m; c_A)$. Then, we show that $p_B^{lf}(\text{without } A) < p_B^{lf}(\text{with } A) < p_B^m$.

Proof that $\hat{\Pi}_M^{lf} > \pi_A(p_A^m; c_A)$.—Suppose that M induces distributors to charge $(p_A^{lf}, p_B^{lf}) = (p_A^m, c_B + \epsilon)$ where $\epsilon > 0$ but small enough. Note then that $R(p_A^{lf}, p_B^{lf}) = R(p_A^m, c_B + \epsilon) = 0$ since $v_B(\tilde{p}_B) < v_A(p_A^m) + v_B(c_B + \epsilon)$ for all $\tilde{p}_B \geq c_B$, as $v_A(p_A^m) > 0$ given that $Q_A(p)$ is strictly downward sloping. Hence, M's profits, in this case, are equal to:

$$\begin{aligned}\Pi_M^{lf} &= \pi_A(p_A^m; c_A) + \pi_B(c_B + \epsilon; c_B) - nR(p_A^m, c_B + \epsilon) \\ &= \pi_A(p_A^m; c_A) + \pi_B(c_B + \epsilon; c_B) > \pi_A(p_A^m; c_A)\end{aligned}$$

This implies that M can always guarantee himself profits that are strictly greater than A's single-monopoly profits. □.

Proof that $p_B^{lf}(\text{without } A) < p_B^{lf}(\text{with } A) < p_B^m$.—Because the laissez-faire retail prices are given by $\max_{(p_A, p_B)} \{\pi_A(p_A; c_A) + \pi_B(p_B; c_B) - nR(p_A, p_B)\}$, then (p_A^{lf}, p_B^{lf}) satisfy the following first-order conditions:

$$\frac{\pi_k'(p_k^{lf}; c_k)}{Q_k(p_k^{lf})} - \frac{\pi_B'(\tilde{p}_B^{lf}; c_B)}{Q_B(\tilde{p}_B^{lf})} nH\big(\gamma v_B(\tilde{p}_B^{lf}) - \gamma v_A(p_A^{lf}) - \gamma v_B(p_B^{lf})\big) = 0, \text{ for } k = A, B$$

where $\tilde{p}_B^{lf} = \arg\max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B) H\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A^{lf}) - \gamma v_B(p_B^{lf})\big)$. This implies that we can find the equilibrium retail prices p_A^{lf}(with A) and p_B^{lf}(with A) by solving for $(p_A, p_B, \tilde{p}_B)$ from the following system of nonlinear equations:

$$\frac{\pi_A'(p_A; c_A)}{Q_A(p_A)} - \frac{\pi_B'(p_B; c_B)}{Q_B(p_B)} = 0 \tag{14}$$

$$\frac{\pi_B'(p_B; c_B)}{Q_B(p_B)} - \frac{\pi_B'(\tilde{p}_B; c_B)}{Q_B(\tilde{p}_B)} nH\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A) - \gamma v_B(p_B)\big) = 0 \tag{15}$$

$$\frac{\pi_B'(\tilde{p}_B; c_B)}{Q_B(\tilde{p}_B)} - \pi_B(\tilde{p}_B; c_B)\left(\frac{\gamma H'\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A) - \gamma v_B(p_B)\big)}{H\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A) - \gamma v_B(p_B)\big)}\right) = 0 \tag{16}$$

Note then that (16) implies that $\pi_B'(\tilde{p}_B; c_B) > 0$. Hence, we immediately have that p_B^{lf}(with A) $< p_B^m$; otherwise, $\pi_B'(p_B^{lf}(\text{with } A); c_B) \leq 0$ and (15) would be violated:

$$\frac{\pi_B'(p_B^{lf}(\text{with } A); c_B)}{Q_B(p_B^{lf}(\text{with } A))} - \frac{\pi_B'(\tilde{p}_B; c_B)}{Q_B(\tilde{p}_B)} nH\big(\gamma v_B(\tilde{p}_B) - \gamma v_A(p_A^{lf}(\text{with } A)) - \gamma v_B(p_B^{lf}(\text{with } A))\big) < 0$$

On the other hand, proving that p_B^{lf}(without A) $< p_B^{lf}$(with A) is a bit more involved. To simplify notation in what follows, let $p_A^{lf} \equiv p_A^{lf}$(with A), $p_B^{lf} \equiv p_B^{lf}$(with A), and $\tilde{p}_B^{lf} \equiv \tilde{p}_B^{lf}$(with A), while use p_B^{of} to denote p_B^{lf}(without A), i.e., $p_B^{of} \equiv p_B^{lf}$(without A). We start with the following two results:

Claim 2.3.1. $\delta^{lf} \equiv v_A(p_A^{lf}) + v_B(p_B^{lf}) - v_B(\tilde{p}_B^{lf}) > 0$

Proof. Suppose then by contradiction that $\delta^{lf} \leq 0$; then, $H(-\gamma\delta^{lf}) \geq 1/n$ given that $H(0) = 1/n$. Because p_A^{lf}, p_B^{lf} and $\tilde{p}_B^{lf}$ satisfy (14)–(16), that fact that $H(-\gamma\delta^{lf}) \geq 1/n$ implies that $\pi_B'(p_B^{lf}; c_B)/Q_B(p_B^{lf}) \geq \pi_B'(\tilde{p}_B^{lf}; c_B)/Q_B(\tilde{p}_B^{lf})$ Consequently, since $\pi_k'(p; w)/Q_k(p)$ is strictly decreasing in p, we obtain that $p_B^{lf} \leq \tilde{p}_B^{lf}$. But if so, then $v_A(p_A^{lf}) + v_B(p_B^{lf}) > v_B(\tilde{p}_B^{lf})$, that is, $\delta > 0$; a contradiction. □

Claim 2.3.2. $p_B^{of} > \tilde{p}_B^{lf}$.

Proof. We know that p_B^{of} is given by the following condition $p_B^{of} = \arg\max_{\tilde{p}_B} \pi_B(\tilde{p}_B; c_B) H\big(\gamma v_B(\tilde{p}_B) - \gamma v_B(p_B^{of})\big)$. Taking the first-order condition of this problem, the condition can then be rewritten as:

$$\frac{\pi_B'(p_B^{of}; c_B)}{Q_B(p_B^{of})} = \pi_B(p_B^{of}; c_B)\left(\frac{\gamma H'(0)}{H(0)}\right) \tag{17}$$

On the other hand, $\tilde{p}_B^{lf}$ must satisfy (16), that is:

$$\frac{\pi_B'(\tilde{p}_B^{lf}; c_B)}{Q_B(\tilde{p}_B^{lf})} = \pi_B(\tilde{p}_B^{lf}; c_B)\left(\frac{\gamma H'\big(-\gamma\delta^{lf}\big)}{H\big(-\gamma\delta^{lf}\big)}\right) \tag{18}$$

Since $g(x)$ was strictly log-concave, so is $G(x)$ and $1-G(x)$, and, therefore, so is $H(x)$. The latter implies, given that $\delta^{lf}>0$ (due to Claim 2.3.1), that $H'(0)/H(0) < H'(-\gamma\delta^{lf})/H(-\gamma^{lf})$. Consequently, since $\pi'_B(p;w)/Q_B(p)$ is strictly decreasing in p and $\pi_B(p;c_B)$ is strictly increasing in p,[16] conditions (17) and (18) imply that $p_B^{of} > \tilde{p}_B^{lf}$. $\square$

We are now ready to prove that $p_B^{lf}(\text{without } A) = p_B^{of} < p_B^{lf}(\text{with } A) = p_B^{lf}$. Suppose by contradiction that $p_B^{of} \geq p_B^{lf}$. Since $\pi'_B(p;w)/Q_B(p)$ is strictly decreasing in p, the premise implies that:

$$\frac{\pi'_B(p_B^{lf};c_B)}{Q_B(p_B^{lf})} \geq \frac{\pi'_B(p_B^{of};c_B)}{Q_B(p_B^{of})} = \pi_B(p_B^{of};c_B)\left(\frac{\gamma H'(0)}{H(0)}\right)$$

However, p_B^{lf} must satisfy (15). Hence, the above condition can be written as:

$$\frac{\pi'_B(\tilde{p}_B^{lf};c_B)}{Q_B(\tilde{p}_B^{lf})}H(-\gamma\delta^{lf}) \geq \pi_B(p_B^{of};c_B)\gamma H'(0) \tag{19}$$

where I am using the fact that $H(0)=1/n$ to simplify terms. However, $\tilde{p}_B^{lf}$ must also satisfy (16). Using this last condition on (19) and simplifying terms, we have that:

$$\pi_B(\tilde{p}_B^{lf};c_B)H'(-\gamma\delta^{lf}) \geq \pi_B(p_B^{of};c_B)H'(0) \tag{20}$$

Note than that because $H(x)$ is a strictly log-concave CDF with mode greater than or equal to zero (i.e., it is the CDF of $\max_{j\neq i}\{\xi_j^\ell\} - \xi_i^\ell$), then $H(x)$ is strictly convex when $x \leq 0$. Thus, $H'(0) > H'(-\gamma\delta^{lf})$ since $\delta^{lf}>0$. However, if so, a necessary condition to satisfy (20) is that $\pi_B(\tilde{p}_B^{lf};c_B) > \pi_B(p_B^{of};c_B)$, that is, $\tilde{p}_B^{lf} > p_B^{of}$ (since $\pi_B(p;c)$ is strictly concave in p and both p_B^{of} and $\tilde{p}_B^{lf}$ are strictly less than p_B^m). This contradicts our premise that $p_B^{of} \geq p_B^{lf}$. $\square$

2.4 Proof of Proposition 6

Recall that by Lemma 2, the laissez-faire retail prices can be characterized by solving the following problem $\max_{(p_A,p_B)}\{\pi_A(p_A;c_A)+\pi_B(p_B;c_B)-nR(p_A,p_B)\}$ where:

$$R(p_A,p_B) \equiv \max_{\tilde{p}_B} \pi_B(\tilde{p}_B;c_B)H\big(\gamma v_B(\tilde{p}_B)-\gamma v_A(p_A)-\gamma v_B(p_B)\big)$$

This implies that p_A^{lf}, p_B^{lf}, and $\tilde{p}_B^{lf} \equiv \tilde{p}_B^*(p_A^{lf},p_B^{lf})$ must satisfy:

$$\frac{\pi'_B(\tilde{p}_B^{lf};c_B)}{Q_B(\tilde{p}_B^{lf})} = \gamma\pi_B(\tilde{p}_B^{lf};c_B)\left(\frac{H'(-\gamma\delta^{lf})}{H(-\gamma\delta^{lf})}\right) \tag{21}$$

$$\frac{\pi'_k(p_k^{lf};c_k)}{Q_k(p_k^{lf})} = \frac{\pi'_B(\tilde{p}_B^{lf};c_B)}{Q_B(\tilde{p}_B^{lf})}nH(-\gamma\delta^{lf}),\text{ for } k=A,B \tag{22}$$

[16]This follows because $\pi_B(p;c_B)$ is strictly concave in p and both p_B^{of} and $\tilde{p}_B^{lf}$ are strictly less than p_B^m.

where $\delta^{lf} \equiv v_A(p_A^{lf}) + v_B(p_B^{lf}) - v_B(\tilde{p}_B^{lf}) > 0$ (see Claim 2.3.1 in Section 2.3 of this online Appendix). From these conditions it is straightforward to prove that $p_k^{lf} \in [c_k, p_k^m]$ and $\tilde{p}_B^{lf} \in [c_B, p_B^m]$ necessarily (which implies that $\pi_k'(p_k^{lf}; c_k)/Q_k(p_k^{lf}) \in [0,1]$ and $\pi_B'(\tilde{p}_B^{lf}; c_B)/Q_B(\tilde{p}_B^{lf}) \in [0,1]$). Moreover, Lemma 2 also implies that M's equilibrium profits are equal to $\hat{\Pi}_M^{lf} = \pi_A(p_A^{lf}; c_A) + \pi_B(p_B^{lf}; c_B) - nR(p_A^{lf}, p_B^{lf})$.

2.4.1 Equilibrium as $\gamma \to 0$

We want to prove the following result:

Proposition 6 (Part 1). *If $\gamma \to 0$, $(p_A^{lf}, p_B^{lf}) \to (p_A^m, p_B^m)$ and $\hat{\Pi}_M^{lf} \to \pi_A(p_A^m; c_A)$.*

Proof.—Taking $\gamma \to 0$ in (21) and (22) and using the fact that $H(0) = 1/n$ immediately leads to $p_A^{lf} = p_A^m$ and $p_B^{lf} = \tilde{p}_B^{lf} = p_B^m$. This implies that M's equilibrium profits are then:

$$\begin{aligned}\hat{\Pi}_M^{lf} &= \pi_A(p_A^m; c_A) + \pi_B(p_B^m; c_B) - n\pi_B(p_B^m; c_B) \lim_{\gamma \to 0} H\big(\gamma v_B(p_B^m) - \gamma v_A(p_A^m) - \gamma v_B(p_B^m)\big) \\ &= \pi_A(p_A^m; c_A) + \pi_B(p_B^m; c_B) - \pi_B(p_B^m; c_B) = \pi_A(p_A^m; c_A)\end{aligned}$$

2.4.2 Equilibrium as $\gamma \to +\infty$

We want to prove the following result:

Proposition 6 (Part 2). *If $\gamma \to \infty$, then:*

$$\begin{aligned}&(p_A^{lf}, p_B^{lf}) \to \underset{(p_A, p_B)}{\arg\max} \Big\{\pi_A(p_A; c_A) + \pi_B(p_B; c_B) \text{ s.t. } v_A(p_A) + v_B(p_B) \geq v_B(c_B)\Big\} \\ &\hat{\Pi}_M^{lf} \to \pi_A\big(\lim_{\gamma \to \infty} p_A^{lf}; c_A\big) + \pi_B\big(\lim_{\gamma \to \infty} p_B^{lf}; c_B\big)\end{aligned}$$

As in the main text, let $\Delta \equiv v_A(p_A^m) + v_B(p_B^m) - v_B(c_B)$. Note that the limit prices in the above proposition can be equivalently written as:

- If $\Delta \geq 0$, then $(p_A^{lf}, p_B^{lf}) \to (p_A^m, p_B^m)$ as $\gamma \to \infty$.
- If $\Delta < 0$, then $(p_A^{lf}, p_B^{lf}) \to (\bar{p}_A, \bar{p}_B)$ as $\gamma \to \infty$, where $(\bar{p}_A, \bar{p}_B)$ is given by:

$$v_A(\bar{p}_A) + v_B(\bar{p}_B) = v_B(c_B) \qquad \text{and} \qquad \frac{\pi_A'(\bar{p}_A; c_A)}{Q_A(\bar{p}_A)} = \frac{\pi_B'(\bar{p}_B; c_B)}{Q_B(\bar{p}_B)} \tag{23}$$

Note that in the $\Delta < 0$, we are using the fact that $(p - c)Q_k'(p)/Q_k(p)$ is strictly decreasing in p for $k = A, B$, to ensure that the solution to the problem is unique and interior (this is where the last set of conditions come from: they are the result of the first-order conditions of the problem).

Proof.—Because $\delta^{lf} > 0$ and $H(x)$ is log-concave, we have that $\lim_{\gamma \to \infty} H'(-\gamma\delta^{lf})/H(-\gamma\delta^{lf}) > 0$. This implies that $\gamma H'(-\gamma\delta^{lf})/H(-\gamma\delta^{lf}) \to \infty$ as $\gamma \to \infty$. However, if so, then $\lim_{\gamma \to \infty} \pi_B(\tilde{p}_B^{lf}; c_B) = 0$, or equivalently, $\tilde{p}_B^{lf} \to c_B$ as $\gamma \to \infty$, since otherwise the right-hand side of (21) would diverge violating the fact that $\pi'_B(\tilde{p}_B^{lf}; c_B)/Q_B(\tilde{p}_B^{lf}) \in [0,1]$ for all γ. Thus, $\lim_{\gamma\to\infty} \tilde{p}_B^{lf} = c_B$. This immediately implies that $\lim_{\gamma\to\infty} R(p_A^{lf}, p_B^{lf}) = 0$, so:

$$\hat{\Pi}_M^{lf} \to \pi_A\big(\lim_{\gamma\to\infty} p_A^{lf}; c_A\big) + \pi_B\big(\lim_{\gamma\to\infty} p_B^{lf}; c_B\big)$$

Now, the fact that $\lim_{\gamma\to\infty} \tilde{p}_B^{lf} = c_B$ implies that $\pi'_B(\tilde{p}_B^{lf}; c_B)/Q_B(\tilde{p}_B^{lf}) \to 1$ as $\gamma \to \infty$. Hence, applying the limit to (22), we get $\lim_{\gamma\to\infty} \pi'_k(p_k^{lf}; c_k)/Q_k(p_k^{lf}) = \lim_{\gamma\to\infty} nH(-\gamma\delta^{lf})$. Since $\delta^{lf} > 0$ and $H(0) = 1/n$, we have that $\lim_{\gamma\to\infty} nH(-\gamma\delta^{lf}) \equiv K \in [0,1]$. Thus, we have two cases to consider: $K = 0$ and $K \in (0,1]$.

Suppose first that $K = 0$. Then (22) implies that $p_k^{lf} \to p_k^m$ for $k = A, B$. Since $\tilde{p}_B^{lf}$ is also converging to c_B, $\lim_{\gamma\to\infty} \delta^{lf} = v_A(p_A^m) + v_B(p_B^m) - v_B(c_B) = \Delta$. However, for $K = 0$ we also require that $\gamma\delta^{lf} \to \infty$ (otherwise $nH(-\gamma\delta^{lf})$ would converge to something strictly positive). Thus $K = 0$ (i.e., $p_k^{lf} \to p_k^m$) can be the limit equilibria only if $\Delta > 0$ or if $\Delta = 0$ and δ^{lf} converges to zero more slowly than $\gamma \to \infty$ does.

Consider next $K \in (0,1]$. This implies that $p_k^{lf} \to \bar{p}_k$ for $k = A, B$, where $\pi'_k(\bar{p}_k; c_k)/Q(\bar{p}_k) = K$. However, $K \in (0,1]$ also requires $\gamma\delta^{lf}$ to be converging to a constant, so:

$$\lim_{\gamma\to\infty} \delta^{lf} = v_A(\bar{p}_A) + v_B(\bar{p}_B) - v_B(c_B) = 0 \tag{24}$$

and at the same rate as $\gamma \to \infty$. Hence, the values of $(K, \bar{p}_A, \bar{p}_B)$ are determined by solving the system of equations given by (24) and $\pi'_k(\bar{p}_k; c_k)/Q(\bar{p}_k) = K$ for $k = A, B$. These are exactly the conditions stated in (23). □

2.5 Proof of Proposition 7

The strategy of proof is the following. First, we show that the equilibrium retail outcome following the highly discriminatory offers stated in the proposition converges to an equivalent outcome (in terms of prices paid by final consumers) to that of the laissez-faire's when $\gamma \to \infty$. We then use this convergence result to show that the offers of the proposition are indeed ϵ-optimal for M when $\gamma \to \infty$.

Lemma 2.5.1. *As in the main text, let $\Delta \equiv v_A(p_A^m) + v_B(p_B^m) - v_B(c_B)$, and suppose M makes the discriminatory offers of Proposition 7. Then, the $n-1$ distributors that carry only B charge the same retail price for that product. Let $(p_{Ai}^\epsilon, p_{Bi}^\epsilon)$ and p_B^ϵ be the equilibrium retail prices charged by distributor i and her $n-1$ rivals, respectively. As $\gamma \to \infty$, $p_B^\epsilon \to c_B$ and only D_i makes positive sales. Moreover,*

- *If $\Delta \geq 0$, then $(p_{Ai}^\epsilon, p_{Bi}^\epsilon) \to (p_A^m, p_B^m)$ as $\gamma \to \infty$.*
- *If $\Delta < 0$, then $(p_{Ai}^\epsilon, p_{Bi}^\epsilon) \to (\bar{p}_A, \bar{p}_B)$ as $\gamma \to \infty$, where $(\bar{p}_A, \bar{p}_B)$ is given by (23).*

Proof. The first part of the proof (that the $n-1$ distributors that carry only B charge the same retail price for that product) follows from the same arguments of supermodular games used to show that if all distributors have the same wholesale prices, equilibrium retail prices must be symmetric (see Section 2.1 of this online Appendix).

For the second part of the proof, we first characterize the retail equilibrium for an arbitrary $\gamma \in (0,\infty)$ when $(w_{Ai}, w_{Bi}) = (c_A, c_B)$ and $(w_{Aj}, w_{Bj}) = (\infty, c_B)$ for all other $j \neq i$, and then take $\gamma \to \infty$. To start, notice that if $(p^{\epsilon}_{Aj} = \infty, p^{\epsilon}_{Bj} = p^{\epsilon}_{B})$ for all $j \neq i$, then D_i's problem is:

$$\max_{p_{Ai},p_{Bi}} \big(\pi_A(p_{Ai};c_A) + \pi_B(p_{Bi};c_B)\big) \int G\big(\zeta + \gamma(v_A(p_{Ai}) + v_B(p_{Bi}) - v_B(p^{\epsilon}_B))\big)^{n-1} g(\zeta)d\zeta$$

The first-order conditions of the this problem yield:

$$\frac{\pi'_k(p^{\epsilon}_{ki};c_k)}{Q_k(p^{\epsilon}_{ki})} = \gamma\big(\pi_A(p^{\epsilon}_{Ai};c_A) + \pi_B(p^{\epsilon}_{Bi};c_B)\big)J(\gamma\delta^{\epsilon}) \quad \text{for } k = A, B \tag{25}$$

where $\delta^{\epsilon} \equiv v_A(p^{\epsilon}_{Ai}) + v_B(p^{\epsilon}_{Bi}) - v_B(p^{\epsilon}_B)$ and:

$$J(x) \equiv \frac{(n-1)\int G(\zeta+x)^{n-2} g(\zeta+x) g(\zeta) d\zeta}{\int G(\zeta+x)^{n-1} g(\zeta)d\zeta}$$

On the other hand, D_j's problem when D_i charges $(p^{\epsilon}_{Ai}, p^{\epsilon}_{Bi})$ and all other distributors charge $(p^{\epsilon}_{Am} = \infty, p^{\epsilon}_{Bm} = p^{\epsilon}_B)$ for $m \neq j$ and $m \neq i$ is:

$$\max_{p_{Bj}} \pi_B(p_{Bi};c_B) \int G\big(\zeta + \gamma(v_B(p_{Bj}) - v_B(p^{\epsilon}_B))\big)^{n-2} G\big(\zeta + \gamma(v_B(p_{Bj}) - v_A(p^{\epsilon}_{Ai}) - v_B(p^{\epsilon}_{Bi}))\big) g(\zeta)d\zeta$$

Differentiating with respect to p_{Bj} and imposing symmetry (i.e., $p^{\epsilon}_{Bj} = p^{\epsilon}_B$) yields:

$$\frac{\pi'_B(p^{\epsilon}_B;c_B)}{Q_B(p^{\epsilon}_B)} = \gamma\pi_B(p^{\epsilon}_B;c_B) M(-\gamma\delta^{\epsilon}) \tag{26}$$

where:

$$M(x) \equiv \frac{\int \Big[(n-2)\frac{g(\zeta)}{G(\zeta)} + \frac{g(\zeta+x)}{G(\zeta+x)}\Big] G(\zeta)^{n-2} G(\zeta+x) g(\zeta)d\zeta}{\int G(\zeta+x)G(\zeta)^{n-2}g(\zeta)d\zeta}$$

Hence, for any given $\gamma \in (0,+\infty)$, $(p^{\epsilon}_{Ai}, p^{\epsilon}_{Bi})$ and p^{ϵ}_B are given the system of equations comprised by (25) and (26).

We now move on to the second part: equilibrium convergence. Note that $J(x)$ and $M(x)$ are the reversed hazard rate function (i.e., the ratio of the density to the distribution function) of two log-concave distributions with support over $\mathbb{R}$. As a result, $J(x) > 0$, $M'(x) > 0$, $J'(x) < 0$, $M'(x) < 0$, and $xJ(x) \to 0$ and $xM(x) \to 0$ as $x \to \infty$ (this last part follows because log-concave distributions have sub-exponential tails). Moreover, given that $(w_{Ai} = c_A, w_{Bi} = c_B)$ and $(w_{Aj}, w_{Bj}) = (\infty, c_B)$ for all other $j \neq i$, then in any retail equilibrium $p^*_{ki} \in [c_k, p^m_k]$ and $p^*_B \in [c_B, p^m_B]$ (see Claim

2.1.1 in Section 2.1 of this online Appendix). This implies that $\pi'_k(p_k; c_k)/Q_k(p_k) \in [0,1]$ and $\pi'_B(p_B; c_B)/Q_B(p_B) \in [0,1]$.

We first deal with the convergence of p^ϵ_B; in particular, we show that $p^\epsilon_B \to c_B$ as $\gamma \to \infty$. Suppose not, i.e., $p^\epsilon_B \to p \in (c_B, p^m_B]$. Then by (26) it must be that $\lim_{\gamma\to\infty} \gamma\pi_B(p^\epsilon_B; c_B)M(-\gamma\delta^\epsilon) = C \in [0,1)$. Since $\pi_B(p; c_B) > 0$, this implies that $M(-\gamma\delta^\epsilon) \to 0$ as $\gamma \to \infty$, which requires $\gamma\delta^\epsilon \to -\infty$. The latter then implies that $\gamma J(\gamma\delta^\epsilon) \to \infty$, so $p^\epsilon_{Ai} \to c_A$ and $p^\epsilon_{Bi} \to c_B$ (otherwise, the right-hand side of (25) would diverge). However, if so, then $\delta^\epsilon \to v_A(c_A)+v_B(c_B)-v_B(p) > 0$ which implies that $\gamma\delta^\epsilon \to \infty$ as $\gamma \to \infty$. This contradicts that $\gamma\delta^\epsilon \to -\infty$. Thus, $p^\epsilon_B \to c_B$ necessarily.

We now deal with the convergence of $(p^\epsilon_{Ai}, p^\epsilon_{Bi})$. To do this, consider $\lim_{\gamma\to\infty} \gamma J(\gamma\delta^\epsilon)$. There are three possibilities: (i) $\gamma J(\gamma\delta^\epsilon) \to \infty$, (ii) $\gamma J(\gamma\delta^\epsilon) \to K \in (0,\infty)$, and (iii) $\gamma J(\gamma\delta^\epsilon) \to 0$ as $\gamma \to \infty$.

Note first that (i) leads to a contradiction. Indeed, if that is the case, then $p^\epsilon_{Ai} \to c_A$ and $p^\epsilon_{Bi} \to c_B$, which would imply that $\delta^\epsilon \to v_A(c_A)$ as $\gamma \to \infty$. The latter implies that $\gamma\delta^\epsilon \to \infty$ faster than a logarithmic rate of convergence. Consequently, $\gamma J(\gamma\delta^\epsilon) \to 0$ as $\gamma \to \infty$ given the sub-exponential tails of the log-concave distribution. This contradicts our original premise that $\gamma J(\gamma\delta^\epsilon) \to \infty$.

Consider next possibility (ii). Then, $(p^\epsilon_{Ai}, p^\epsilon_{Bi}) \to (\bar{p}_A, \bar{p}_B)$ where the latter are given by:

$$\frac{\pi'_A(\bar{p}_A; c_A)}{Q_A(\bar{p}_A)} = \frac{\pi'_B(\bar{p}_B; c_B)}{Q_B(\bar{p}_B)} \quad \text{and} \quad \frac{\pi'_B(\bar{p}_B; c_B)}{Q_B(\bar{p}_B)} = K\big(\pi_A(\bar{p}_A; c_A) + \pi_B(\bar{p}_B; c_B)\big)$$

On the other hand, for $\gamma J(\gamma\delta^\epsilon) \to K \in (0,\infty)$, it must be that $\gamma\delta^\epsilon \to \infty$ at the same rate as $-\ln(K/\gamma)$. Consequently, it must be that $\lim_{\gamma\to\infty} \delta^\epsilon = v_A(\bar{p}_A) + v_B(\bar{p}_B) - v_B(c_B) = 0$ and at a rate equal to $-\ln(K/\gamma)/P(\gamma)$, where $P(\gamma)$ is a polynomial with finite degree. Thus, possibility (ii) is feasible when $(\bar{p}_A, \bar{p}_B, K)$ satisfy:

$$\frac{\pi'_A(\bar{p}_A; c_A)}{Q_A(\bar{p}_A)} = \frac{\pi'_B(\bar{p}_B; c_B)}{Q_B(\bar{p}_B)} \qquad \frac{\pi'_B(\bar{p}_B; c_B)}{Q_B(\bar{p}_B)} = K\big(\pi_A(\bar{p}_A; c_A) + \pi_B(\bar{p}_B; c_B)\big) \tag{27}$$

plus $v_A(\bar{p}_A) + v_B(\bar{p}_B) = v_B(c_B)$ and $\delta^\epsilon \to 0$ at a rate $-\ln(K/\gamma)/P(\gamma)$. It is then easy to prove that the system (27) has a solution if and only if $\Delta < 0$. When that is the case, furthermore, the solution is unique.

Finally, consider possibility (iii). Then, $p^\epsilon_{ki} \to p^m_k$. If so, $\lim_{\gamma\to\infty} \delta^\epsilon = v_A(p^m_A) + v_B(p^m_B) - v_B(c_B) = \Delta$. Hence, for $\gamma J(\gamma\delta^\epsilon) \to 0$ as $\gamma \to \infty$, it must be that $\Delta > 0$ or that $\Delta = 0$ and $\delta^\epsilon \to 0$ more slowly than the rate of convergence of a logarithmic function.

Consequently, there is a unique outcome consistent with the equilibrium conditions for the different values of Δ. When $\Delta > 0$, $p^\epsilon_B \to c_B$ and $p^\epsilon_{ki} \to p^m_k$ as $\gamma \to \infty$. When, on the other hand, $\Delta = 0$, $p^\epsilon_B \to c_B$, $p^\epsilon_{ki} \to p^m_k$, and $\delta^\epsilon \to 0$ at a slower rate than a logarithmic rate of convergence. Finally, when $\Delta < 0$, $p^\epsilon_B \to c_B$, $p^\epsilon_{ki} \to \bar{p}_k$ (where $(\bar{p}_A, \bar{p}_B)$ are given by solving (27)), and $\delta^\epsilon \to 0$ at a rate $-\ln(K/\gamma)/P(\gamma)$. Since we know that the equilibrium exists for all $\gamma \in (0,\infty)$, the equilibrium must be converging to this unique outcome as $\gamma \to \infty$. □

Lemma 2.5.2. *The discriminatory offers in Proposition 7 are ϵ-optimal for M when $\gamma \to \infty$.*

Proof. Since $p_B^\epsilon \to c_B$ as $\gamma \to 0$, D_i's reservation payoff when dealing with M is approximately zero for γ sufficiently high (the payoff of rejecting M's offer and just procuring B from the fringe is zero in the limit). Consequently, when $\gamma \to \infty$, M can extract all of D_i's profits, and therefore obtain the entire industry surplus using the fixed fee on A's contract, T_{Ai}. This is an upper bound for M's profit (it is the payoff he receives in the laissez-faire outcome when $\gamma \to \infty$), implying that making these highly discriminatory offers is (approximately) optimal for M when γ is arbitrarily large. □

2.6 Proof of Proposition 8

2.6.1 Preliminaries

Let $\boldsymbol{w}^{lf}$ be the vector of wholesale prices offered by M in the laissez-faire equilibrium. By Lemma 2, we know that $p_{ki}^*(\boldsymbol{w}^{lf}) = p_k^{lf}$ for $k = A, B$, where:

$$(p_A^{lf}, p_B^{lf}) \equiv \arg\max_{p_A, p_B} \{\pi_A(p_A; c_A) + \pi_B(p_B; c_B) - nR(p_A, p_B)\}$$

Thus, M is inducing distributors to charge the same prices for each product, so he must be making nondiscriminatory offers in the laissez-faire equilibrium. This implies that M's problem in the laissez-faire, can be equivalently written as:

$$\max_{w_A, w_B} \hat{\Pi}_M^{lf} = \pi_A(p_A^*(w_A, w_B); c_A) + \pi_B(p_B^*(w_A, w_B); c_B) - nR(p_A^*(w_A, w_B), p_B^*(w_A, w_B))$$

where $p_A^*(w_A, w_B)$ and $p_B^*(w_A, w_B)$ are given by Lemma 2.1.2.

Now suppose that M is forced to make non-discriminatory offers and prohibited from engaging in refusal to deal. In an earlier version of this paper, we showed that in this case banning exclusivity provisions produces the same effect as barring tying clauses. No matter the type of prohibition, the robust-intervention ("ri") game has a unique SPNE in which M solves the following problem:

$$\max_{w_A, w_B = c_B} \hat{\Pi}_M^{ri} = \pi_A(p_A^*(w_A, w_B); c_A) + \pi_B(p_B^*(w_A, w_B); c_B) - nR(p_A^*(w_A, w_B), p_B^*(w_A, w_B)) \quad (28)$$

where, again, $p_A^*(w_A, w_B)$ and $p_B^*(w_A, w_B)$ are given by Lemma 2.1.2. Note that this is the same problem as in the laissez-faire but with the additional restriction that $w_B = c_B$. Denoting by w_A^{ri} and $w_B^{ri} = c_B$ the solution to this problem, the retail prices that consumers pay on-path following this intervention are $p_k^{ri} = p_k^*(w_A^{ri}, w_B^{ri})$ for $k = A, B$.

The idea of the proof is the following. Note that both in the laissez-faire and in the robust-intervention scenario, M deals with all distributors in a nondiscriminatory way. This implies that in both scenarios, consumers visit their preferred distributor. Hence, to show that $\Delta \mathrm{CS}^{ri} \equiv \mathrm{CS}^{ri} - \mathrm{CS}^{lf} > 0$, it suffices to show that $u^{ri} = v_A(p_A^{ri}) + v_B(p_B^{ri}) > v_A(p_A^{lf}) + v_B(p_B^{lf}) = u^{lf}$.

To do this, we first show that M's laissez-faire problem can be rewritten as M directly choosing u^{lf} (instead of choosing (w_A^{lf}, w_B^{lf}) or (p_A^{lf}, p_B^{lf})), and that the same can be said about M's problem in the robust-intervention, i.e., we can work as if M is deciding u^{ri} directly. We then encompass both scenarios into a single maximization in which M chooses u and there is parameter $b \in \{0, 1\}$ indexes whether the robust intervention is in place or not (so $u^{lf} = u^*(b = 0)$ and $u^{re} = u^*(b = 1)$). Finally, we use monotone comparative statics to establish that $u^*(b)$ is strictly increasing in b.

2.6.2 The Proof

Step 1: Rewriting M's problems.— Recall that M's laissez-faire problem is:

$$\max_{w_A, w_B} \hat{\Pi}_M^{lf} = \pi_A(p_A^*(w_A, w_B); c_A) + \pi_B(p_B^*(w_A, w_B); c_B) - nR(p_A^*(w_A, w_B), p_B^*(w_A, w_B))$$

where $R(p_A, p_B)$ is given by $R(p_A, p_B) = \max_{p_{Bi}} \pi_B(p_{Bi}; c_B) H\big(v_B(p_{Bi}) - v_A(p_A) - v_B(p_B)\big)$. This is also M's problem in the robust intervention but with the additional restriction that $w_B = c_B$.

Note next that $R(p_A, p_B)$ does not depend on p_A and p_B individually, but rather on $u(p_A, p_B)$. Hence, distributors' outside options can be written as $R(p_A, p_B) = \tilde{R}(u(p_A, p_B))$, where $\tilde{R}(u) = \max_x \pi_B(x; c_B) H(v_B(x) - u)$ is strictly decreasing in u. Using this fact, we can write M's problem in both scenarios as $\max_u V^{lf}(u) - n\tilde{R}(u)$ and $\max_u V^{ri}(u) - n\tilde{R}(u)$, respectively, where:

$$\begin{aligned} V^{lf}(u) = \max_{w_A, w_B} \ & \pi_A(p_A^*(w_A, w_B); c_A) + \pi_B(p_B^*(w_A, w_B); c_B) \\ & \text{subject to } v_A(p_A^*(w_A, w_B)) + v_B(p_B^*(w_A, w_B)) = u \end{aligned} \tag{29}$$

and $V^{ri}(u)$ is defined by the same maximization with the additional constraint that $w_B = c_B$.

Intuitively, since distributors' reservation payoffs in either scenario depend only on the surplus consumers are obtaining from purchasing both products, u, we can decompose M's problem into two steps. First, he chooses the optimal mix of prices that maximize industry profits subject to leaving consumers with some given surplus u. This is akin to a Ramsey-pricing problem. Second, M decides how much u to leave consumers, determining his profits (which are equal to the total industry profit minus distributors' reservation payoffs).

Rewriting both problems this way, we can encompass both scenarios into a single maximization:

$$\max_u \ (1 - b)V^{lf}(u) + bV^{ri}(u) - n\tilde{R}(u) \tag{30}$$

where parameter $b \in \{0, 1\}$ indexes whether the authority is implementing the robust intervention. Letting $u^*(b)$ be the solution to (30), the intervention necessarily leads to a gain in consumer welfare if $u^*(b)$ is strictly increasing in b, which by standard monotone comparative statics is the case whenever $dV^{lf}/du < dV^{ri}/du$.

Step 2: Proof that $\Delta\text{CS}^{ri} > 0$.—To prove that $dV^{lf}/du < dV^{ri}/du$, we will make use of the following two claims whose proofs can be found in the next subsection:

Claim 2.6.1. *$V^{lf}(u)$ and $V^{ri}(u)$ exhibit the following properties:*

- *They are bounded, twice continuously differentiable, strictly concave functions.*
- *There exists a $\bar{u} \in (0, +\infty)$ such that:*

$$V^{lf}(\bar{u}) = V^{ri}(\bar{u}) \quad \textit{and} \quad \left.\frac{dV^{lf}}{du}\right|_{u=\bar{u}} = \left.\frac{dV^{ri}}{du}\right|_{u=\bar{u}}$$

- *$V^{lf}(u) \geq V^{ri}(u)$ for all $u \in [0, \bar{u}]$. Moreover, $\arg\max_u V^{lf}(u) < \bar{u}$, $\arg\max_u V^{ri}(u) < \bar{u}$, and $\arg\max_u V^{lf}(u) \neq \arg\max_u V^{ri}(u)$.*

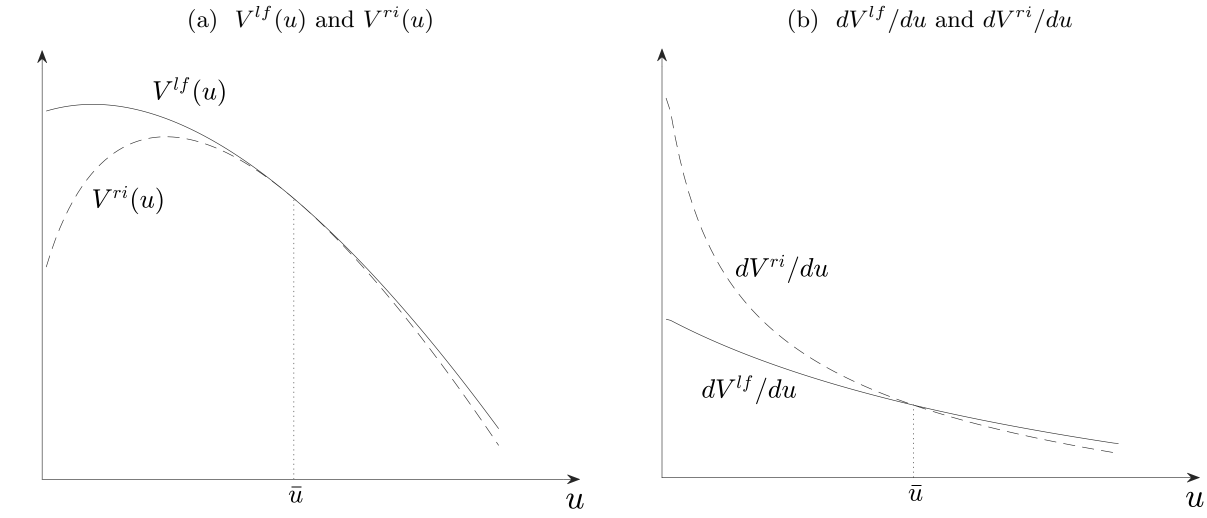
Claim 2.6.2. *If $V^{lf}(u)$ and $V^{ri}(u)$ satisfy the conditions of Claim 2.6.1, then $dV^{lf}/du < dV^{ri}/du$ for all $u \in [0, \bar{u})$.*

With these results in hand, we can prove that $u^*(b)$ is strictly increasing in $b \in \{0, 1\}$ (implying that $\Delta\text{CS} > 0$). By Claim 2.6.1, we know there exists a $\overline{u} \in (0, +\infty)$ such that $V^{lf}(\overline{u}) = V^{ri}(\overline{u})$ and $(dV^{lf}/du)|_{u=\bar{u}} = (dV^{ri}/du)|_{u=\bar{u}}$. The latter implies that at $u = \bar{u}$, the solutions to $V^{lf}(\bar{u})$ and $V^{ri}(\bar{u})$ coincide and are equal to $w_A = c_A$ and $w_B = c_B$.

Now, we know from Proposition 5 in the main text that $w_A^{lf} > c_A$ and $w_B^{lf} > c_B$. Furthermore, it is not difficult to prove that $w_A^{ri} > c_A$ also (while $w_B^{ri} = c_B$ as we know). Since equilibrium retail prices are strictly increasing in wholesale prices, it must be that u^{lf} and u^{ri} are strictly less than $\bar{u}$. Hence, when solving $\max_u\ (1-b)V^{lf}(u) + bV^{ri}(u) - n\tilde{R}(u)$ it is without loss of generality to restrict the domain to $u \in [0, \bar{u})$. But by Claim 2.6.2 we know that $dV^{lf}/du < dV^{ri}/du$; hence, $u^*(b)$ is strictly increasing in $b \in \{0, 1\}$ □

Before moving on, we provide a graphical explanation for why $dV^{lf}/du < dV^{ri}/du$ when $u \in [0, \bar{u})$. Claim 2.6.2 is the key result, as Figure 2.1 attempts to convey. The proof of the claim uses the properties stated in Claim 2.6.1 to show—using the contraction mapping theorem—that $f(u) \equiv dV^{lf}/du - dV^{ri}/du$ vanishes only at $\bar{u}$ over $[0, \bar{u}]$. This last result, plus the fact that $V^{lf}(u) \geq V^{ri}(u)$, $V^{lf}(\bar{u}) = V^{ri}(\bar{u})$, and $(dV^{lf}/du)|_{u=\bar{u}} = (dV^{ri}/du)|_{u=\bar{u}}$ immediately imply that $dV^{lf}/du < dV^{ri}/du$ for all $u \in [0, \bar{u})$.

Figure 2.1: Graphical Illustration of Claim 2.6.2



2.7 Proof of Claim 2.6.1

For ease of exposition, we have divided the proof of this claim into several shorter claims.

Claim 2.7.1. *$V^{lf}(u)$ and $V^{ri}(u)$ are bounded, twice continuously differentiable, strictly concave functions.*

Proof. Given that demands are thrice continuously differentiable, so are the objective functions $V^{lf}(u)$ and $V^{ri}(u)$. Hence, by the smooth maximum theorem (see, e.g., Carter, 2001, p. 603), $V^{lf}(u)$ and $V^{ri}(u)$ are bounded, twice continuously differentiable functions.

Regarding strict concavity, we already argued that $V^{lf}(u)$ is strictly concave in u when we showed that non-discriminatory contracts were optimal in the laissez faire. Thus, we focus on proving that $V^{ri}(u)$ is strictly concave in u too. To do this, recall that

$$V^{ri}(u) = \max_{w_A} \pi_A(p_A^*(w_A, c_B); c_A) + \pi_B(p_B^*(w_A, c_B); c_B) \quad \text{subject to } \; v_A(p_A^*(w_A, c_B)) + v_B(p_B^*(w_A, c_B)) = u \tag{31}$$

where $p_A(w_A, c_B)$ and $p_B(w_A, c_B)$ are strictly increasing in w_A and are such that $p_A(w_A, c_B) < p_A^m(w_A)$ and $p_B(w_A, c_B) < p_B^m(c_B)$, where $p_k^m(w) \equiv \arg\max_p (p-w)Q_k(p)$. Since the dependence of prices on c_B will not play any role in what follows, to avoid cluttering notation, we will omit it altogether, i.e., $p_k(w_A, c_B) \equiv p_k(w_A)$.

Now, let $w_A^*(u)$ be the solution to (31). Consequently, $w_A^*(u)$ must satisfy $v_A(p_A(w_A^*(u))) + v_B(p_B(w_A^*(u))) = u$. By the implicit function theorem we have that:

$$\frac{\partial w_A^*}{\partial u} = -\left(Q_A(p_A(w_A))\frac{\partial p_A}{\partial w_A} + Q_B(p_B)\frac{\partial p_B}{\partial w_A} \right)^{-1} \Bigg|_{w_A = w_A^*(u)} < 0$$

With this in mind, consider the Lagrangian of (31):

$$\max_{w_A,\lambda} \mathcal{L}(w_A, \lambda; u) = \pi_A(p_A(w_A); c_A) + \pi_B(p_B(w_A); c_B) + \lambda\big[v_A(p_A(w_A)) + v_B(p_B(w_A)) - u\big]$$

and let $(w_A^*(u), \lambda^*(u))$ be the solution to this problem. By the envelope theorem $dV^{ri}/du = -\lambda^*(u)$, so proving that $V^{ri}(u)$ is strictly concave in u is equivalent as showing that $\lambda^*(u)$ is strictly increasing in u. The first-order condition of the Lagrangian with respect to w_A tells us that $\lambda^*(u) = \Psi(p_A(w_A^*(u)), p_B(w_A^*(u)))$, where:

$$\Psi(p_A(w_A), p_B(w_A)) = \frac{(\partial p_A/\partial w_A)\pi_A'(p_A(w_A); c_A) + (\partial p_B/\partial w_A)\pi_B'(p_B(w_A); c_B)}{(\partial p_A/\partial w_A)Q_A(p_A(w_A)) + (\partial p_B/\partial w_A)Q_B(p_B(w_A))}$$

Consequently,

$$\frac{\partial \lambda^*}{\partial u} = \frac{\partial w_A^*}{\partial u}\left(\frac{\partial \Psi}{\partial p_A}\frac{\partial p_A}{\partial w_A} + \frac{\partial \Psi}{\partial p_B}\frac{\partial p_B}{\partial w_A}\right)\bigg|_{w_A = w_A^*(u)}$$

Since $\partial w_A^*/\partial u < 0$, to demonstrate that $\partial \lambda^*/\partial u > 0$ it suffices to show that:

$$\left(\frac{\partial \Psi}{\partial p_A}\frac{\partial p_A}{\partial w_A} + \frac{\partial \Psi}{\partial p_B}\frac{\partial p_B}{\partial w_A}\right)\bigg|_{w_A = w_A^*(u)} < 0$$

In doing so, note that $\max_{w_A,\lambda} \mathcal{L}(w_A, \lambda, u) = \max_\lambda \mathcal{L}(\tilde{w}_A(\lambda, u), \lambda, u)$, where $\tilde{w}_A(\lambda, u)$ is defined as $\tilde{w}_A(\lambda, u) \equiv \arg\max_{w_A} \mathcal{L}(w_A, \lambda, u)$. By construction, we have that $w_A^*(u) = \tilde{w}_A(\lambda^*(u), u)$. Note, further, that the first-order condition that determines $\tilde{w}_A(\lambda, u)$ is independent of u, which implies that $\tilde{w}_A(\lambda, u) = \tilde{w}_A(\lambda)$:

$$\tilde{w}_A(\lambda, u):\ \lambda = \Psi(p_A(\tilde{w}_A(\lambda, u)), p_B(\tilde{w}_A(\lambda, u))) \Longrightarrow \lambda = \Psi(p_A(\tilde{w}_A(\lambda)), p_B(\tilde{w}_A(\lambda)))$$

Consequently, we have that $w_A^*(u) = \tilde{w}_A(\lambda^*(u))$. Now, notice that:

$$\frac{\partial^2}{\partial \lambda \partial w_A}\mathcal{L}(w_A, \lambda, u) = -Q_A(p_A(w_A))\frac{\partial p_A}{\partial w_A} - Q_B(p_B(w_A))\frac{\partial p_B}{\partial w_A} < 0$$

Hence, $\mathcal{L}(w_A, \lambda; u)$ has strictly decreasing differences in (w_A, λ). This implies that $\tilde{w}_A(\lambda)$ is strictly decreasing in λ. However, we know that $\tilde{w}_A(\lambda)$ satisfies $\lambda = \Psi(p_A(\tilde{w}_A(\lambda)), p_B(\tilde{w}_A(\lambda)))$. Consequently, by the implicit function theorem we have that:

$$\frac{\partial \tilde{w}_A}{\partial \lambda} = \left(\frac{\partial \Psi}{\partial p_A}\frac{\partial p_A}{\partial w_A} + \frac{\partial \Psi}{\partial p_B}\frac{\partial p_B}{\partial w_A}\right)^{-1}\bigg|_{w_A = \tilde{w}_A(\lambda)} < 0 \tag{32}$$

given that we know that $\tilde{w}_A(\lambda)$ is strictly decreasing in λ. Since the latter holds, in particular for $w_A = \tilde{w}_A(\lambda^*(u)) = w_A^*(u)$, then $\partial \lambda^*/\partial u > 0$, or, equivalently, $V^{ri}(u)$ is strictly concave in u. □

Claim 2.7.2. *$\exists\ \bar{u} \in (0, +\infty)$ such that $V^{lf}(\bar{u}) = V^{ri}(\bar{u})$ and $(dV^{lf}/du - dV^{ri}/du)|_{u=\bar{u}} = 0$.*

Proof. Consider first $V^{lf}(u)$, i.e., $V^{lf}(u) = \max_{w_A,w_B} \pi_A(p_A^*(w_A,w_B);c_A) + \pi_B(p_B^*(w_A,w_B);c_B)$ subject to $v_A(p_A^*(w_A,w_B)) + v_B(p_B^*(w_A,w_B)) = u$. Denoting by (w_A^*,w_B^*) the solution to this problem, the corresponding first-order conditions are:

$$\frac{\pi_A'(p_A^*(w_A^*,w_B^*);c_A)}{Q_A(p_A^*(w_A^*,w_B^*))} = \frac{\pi_B'(p_B^*(w_A^*,w_B^*);c_B)}{Q_A(p_B^*(w_A^*,w_B^*))} \tag{33}$$

$$v_A(p_A^*(w_A^*,w_B^*)) + v_B(p_B^*(w_A^*,w_B^*)) = u \tag{34}$$

These two first-order conditions plus the equilibrium conditions leading to on-path prices

$$\frac{\pi_A'(p_A^*;w_A)}{Q_A(p_A^*)} = \frac{\pi_B'(p_B^*;w_B)}{Q_B(p_B^*)} \tag{35}$$

$$\frac{\pi_B'(p_B^*;w_B)}{Q_B(p_B^*)} = \left[\pi_A(p_A^*;w_A) + \pi_B(p_B^*;w_B)\right]\gamma n(n-1)\int_{-\infty}^{+\infty} g(\zeta)^2 G(\zeta)^{n-2} d\zeta \tag{36}$$

determine (w_A^*,w_B^*) and $p_k^*(w_A^*,w_B^*)$ for $k = A,B$, for a given level of u. Moreover, using the envelope theorem, we have that:

$$-\frac{dV^{lf}}{du} = \frac{\pi_A'(p_A^*(w_A^*,w_B^*);c_A)}{Q_A(p_A^*(w_A^*,w_B^*))} = \frac{\pi_B'(p_B^*(w_A^*,w_B^*);c_B)}{Q_B(p_B^*(w_A^*,w_B^*))}$$

Now, because $p_k^*(w_A,w_B)$ is strictly increasing in w_A and w_B, by the implicit function theorem w_k^* is strictly decreasing in u. Hence, there exists $\bar{u} \in (0,\infty)$ such that $w_A^*(\bar{u}) = c_A$. But, if so, then conditions (33) and (35) imply that $w_B^*(\bar{u}) = c_B$. Consequently, $\bar{u} = v_A(p_A^*(c_A,c_B)) + v_B(p_B^*(c_A,c_B))$, $V^{lf}(\bar{u}) = \pi_A(p_A^*(c_A,c_B);c_A) + \pi_B(p_B^*(c_A,c_B);c_B)$, and

$$-\frac{dV^{lf}}{du}\bigg|_{u=\bar{u}} = \frac{\pi_A'(p_A^*(c_A,c_B);c_A)}{Q_A(p_A^*(c_A,c_B))} = \frac{\pi_B'(p_B^*(c_A,c_B);c_B)}{Q_B(p_A^*(c_A,c_B))}$$

Now consider $V^{ri}(u)$, i.e., $V^{ri}(u) = \max_{w_A} \pi_A(p_A^*(w_A,c_B);c_A) + \pi_B(p_B^*(w_A,c_B);c_B)$ subject to $v_A(p_A^*(w_A,c_B)) + v_B(p_B^*(w_A,c_B)) = u$, where, $p_k^*(w_A,c_B)$ for $k = A,B$ must satisfy conditions (35) and (36) evaluated at $w_B = c_B$. The solution to this problem, which we denote by w_A^{**}, is given by the implicit equation $v_A(p_A^*(w_A^{**},c_B)) + v_B(p_B^*(w_A^{**},c_B)) = u$.

Note that since $p_A^*(w_A,c_B)$ is strictly increasing in w_A and $\bar{u} = v_A(p_A^*(c_A,c_B)) + v_B(p_B^*(c_A,c_B))$, then it must hold that $w_A^{**}(\bar{u}) = c_A$. Hence, $V^{ri}(\bar{u}) = V^{lf}(\bar{u}) = \pi_A(p_A^*(c_A,c_B);c_A) + \pi_B(p_B^*(c_A,c_B);c_B)$. Moreover, in the proof of Claim 2.7.1, we already showed that $-dV^{ri}/du = \Psi(p_A^*(w_A^{**},c_B), p_B^*(w_A^{**},c_B))$, where:

$$\Psi(p_A^*(w_A,c_B), p_B^*(w_A,c_B)) = \frac{(\partial p_A/\partial w_A)\pi_A'(p_A^*(w_A,c_B);c_A) + (\partial p_B/\partial w_A)\pi_B'(p_B^*(w_A,c_B);c_B)}{(\partial p_A/\partial w_A)Q_A(p_A^*(w_A,c_B)) + (\partial p_B/\partial w_A)Q_B(p_B^*(w_A,c_B))}$$

But when $u = \bar{u}$ and, therefore, $w_A^{**} = c_A$ $(w_B = c_B)$, the retail equilibrium pricing condition (35) implies that $\pi_A'(p_A^*(c_A,c_B);c_A)/Q_A(p_A^*(c_A,c_B)) = \pi_B'(p_B^*(c_A,c_B);c_B)/Q_B(p_B^*(p_B^*(c_A,c_B))$, which, in turn, implies that:

$$\Psi(p_A^*(c_A,c_B),p_B^*(c_A,c_B)) = \frac{\pi_A'(p_A^*(c_A,c_B);c_A)}{Q_A(p_A^*(c_A,c_B))} = \frac{\pi_B'(p_B^*(c_A,c_B);c_B)}{Q_B(p_B^*(c_A,c_B))}$$

Thus,

$$-\frac{dV^{lf}}{du}\bigg|_{u=\bar{u}} = -\frac{dV^{ri}}{du}\bigg|_{u=\bar{u}} = \frac{\pi_A'(p_A^*(c_A,c_B);c_A)}{Q_A(p_A^*(c_A,c_B))} = \frac{\pi_B'(p_B^*(c_A,c_B);c_B)}{Q_B(p_B^*(c_A,c_B))}$$

□

Claim 2.7.3. *$V^{lf}(u) \geq V^{ri}(u)$ for all $u \in [0,\bar{u}]$. Moreover, $\arg\max_u V^{lf}(u) < \bar{u}$, $\arg\max_u V^{ri}(u) < \bar{u}$, and $\arg\max_u V^{lf}(u) \neq \arg\max_u V^{ri}(u)$.*

Proof. That $V^{lf}(u) \geq V^{ri}(u)$ follows because the maximand of both problems is the same but $V^{ri}(u)$ involves an additional constraint. That $\arg\max_u V^{lf}(u) < \bar{u}$ follows because $\arg\max_u V^{lf}(u) = v_A(p_A^m(c_A)) + v_B(p_B^m(c_B))$, which also implies that $\arg\max_u V^{lf}(u) \neq \arg\max_u V^{ri}(u)$ since implementing $p_k^m(c_k)$ for $k = A, B$ requires $w_k > c_k$ for $k = A, B$ and the solution to $V^{ri}(u)$ imposes that $w_B = c_B$. Finally, that $\arg\max_u V^{ri}(u) < \bar{u}$ follows because:

$$\frac{dV^{ri}}{du}\bigg|_{u=\bar{u}} = -\frac{\pi_A'(p_A^*(c_A,c_B);c_A)}{Q_A(p_A^*(c_A,c_B))} = -\frac{\pi_B'(p_B^*(c_A,c_B);c_B)}{Q_B(p_B^*(c_A,c_B))} < 0$$

and $V^{ri}(u)$ is strictly concave in u. □

2.8 Proof of Claim 2.6.2

This claim is a direct application of the following theorem.

Theorem 1. *Let φ_1 and φ_2 be bounded class $C^{(2)}$ strictly concave functions over $[0,+\infty)$ that satisfy the following conditions: (i) there exists a $\bar{u} \in (0,+\infty)$ such that $\varphi_1(\bar{u}) = \varphi_2(\bar{u})$ and $\varphi_1'(\bar{u}) = \varphi_2'(\bar{u})$; (ii) $\varphi_1(u) \geq \varphi_2(u)$ for all $u \in [0,\bar{u}]$, and (iii) $u_1 < \bar{u}$, $u_2 < \bar{u}$, and $u_1 \neq u_2$, where $u_i \equiv \arg\max_u \varphi_i(u)$. Then $\varphi_1'(u) < \varphi_2'(u) \;\; \forall u \in [0,\bar{u})$.*

Preliminaries.— To prove the theorem, we will make use of the following preliminary results:

Theorem (Bennett and Fisher, 1974). *If h is a continuous function from a compact set $K \subset \mathbb{R}$ into itself satisfying the condition:*

$$|h(x) - h(y)| < \frac{1}{2}\,||x - h(y)| + |y - h(x)|| \;\; \textit{for all } x, y \in K \textit{ with } x \neq y$$

Then h has a unique fixed point.

Lemma 2.8.1. *Let $g : [0,\bar{u}] \to [0,\bar{u}+\varepsilon]$ be a contraction and $\bar{u}$ its unique fixed point. Then there is no other $u^* \in [0,\bar{u}]$ such that $g(u^*) = \bar{u}$.*

Proof. Define $h : [0, \bar{u}] \to [0, \bar{u}]$ as $h(x) := x - g(x) + g(\bar{u})$. Note that $h(\bar{u}) = \bar{u}$, so $\bar{u}$ is a fixed point of h. Moreover, given that:

$$\begin{aligned}\frac{1}{2}\left||x-h(y)|+|y-h(x)|\right| = \frac{1}{2}\left||h(y)-x|+|h(x)-y|\right| \geq \frac{1}{2}\left|h(x)-y+h(y)-x\right| \\ \geq \frac{1}{2}\left|g(x)+g(y)+2\bar{u}\right| > \frac{1}{2}\left|g(x)+g(y)+2(x-y)\right| > \frac{1}{2}\left|2(g(y)-g(x))+2(x-y)\right| \\ > \left|(x-g(x)+(y-g(y))\right| > \left|h(x)-h(y)\right|\end{aligned}$$

then by Bennett and Fisher (1974), $\bar{u}$ is the unique fixed point of h. Suppose then by contradiction that there exists a $u^* \neq \bar{u}$ in the interval $[0, \bar{u}]$ such that $g(u^*) = \bar{u}$. Then $h(u^*) = u^* - g(u^*) - g(\bar{u}) = u^*$, implying that $u^* \neq \bar{u}$ is also a fixed point of h. This contradicts the fact that $\bar{u}$ is the unique fixed point of h. □

Proof of the theorem.— It is a direct implication of the following two propositions:

Proposition 2.8.2. *Define $f(u) \equiv \varphi_1'(u) - \varphi_2'(u)$, then $f(u)$ vanishes only at $\bar{u}$ over $[0, \bar{u}]$.*

Proof. Since φ_1 and φ_2 are bounded and class $C^{(2)}$, then φ_1'' and φ_2'' are continuous and bounded. Hence, for a constant $R > 2$, $K \equiv R \cdot \max\{\max|\varphi_1''|, \max|\varphi_2''|\}$ is well defined. For $\varepsilon > 0$, define $\mathcal{D}_\varepsilon \equiv [0, \bar{u} + \varepsilon]$. Because $\mathbb{R}$ is complete and $\mathcal{D}_\varepsilon$ is closed, $\mathcal{D}_\varepsilon$ is complete. Define, moreover, $g(u) \equiv \frac{1}{K} f(u) + \bar{u}$ and let $D \equiv \max_{u \in \mathcal{D}_\varepsilon} f(u)$ and $F := \min_{u \in \mathcal{D}_\varepsilon} f(u)$.

Note that for R large enough, we have that $D/K < \varepsilon$ and $F/K + \bar{u} \geq 0$. Thus, $g(\mathcal{D}_\varepsilon) \subseteq \mathcal{D}_\varepsilon$. Moreover:

$$\left|g'(u)\right| = \frac{1}{K}\left|f'(u)\right| = \frac{1}{K}\left|\varphi_1''(u) - \varphi_2''(u)\right| \leq \frac{1}{K}\left|\varphi_1''(u)\right| + \frac{1}{K}\left|\varphi_2''(u)\right| < \frac{1}{2} + \frac{1}{2} < 1$$

Thus, $g(u)$ is a contraction. Define $\mathcal{J} := \{u \in [0, \bar{u}] : g(u) = \bar{u}\}$. Clearly, $g(\bar{u}) = \frac{1}{K} f(\bar{u}) + \bar{u} = \frac{1}{K} \cdot 0 + \bar{u} = \bar{u}$, so $\bar{u} \in \mathcal{J}$. Moreover, by Lemma 2.8.1, it must be that $\bar{u}$ is the only element in $\mathcal{J}$.

Define $\mathcal{I} \equiv \{u \in [0, \bar{u}] : f(u) = 0\}$, i.e., $\mathcal{I}$ is the set of all the points over $[0, \bar{u}]$ such that $f(u)$ vanishes. To prove that $\mathcal{I} = \mathcal{J}$ we proceed by double contention. Take $u \in \mathcal{I}$, then $g(u) = \frac{1}{K} f(u) + \bar{u} = \frac{1}{K} \cdot 0 + \bar{u} = \bar{u}$. Hence $u \in \mathcal{J}$. On the other hand, take $u \in \mathcal{J}$, then:

$$g(u) = \bar{u} \implies \frac{1}{K} f(u) + \bar{u} = \bar{u} \implies \frac{1}{K} f(u) = 0 \implies f(u) = 0$$

Therefore $u \in \mathcal{I}$. Consequently, since $\mathcal{I} = \mathcal{J}$ and $\mathcal{J}$ is a singleton, then $\mathcal{I}$ is also a singleton. Thus, over $[0, \bar{u}]$, $f(u)$ vanishes only at $\bar{u}$. □

Proposition 2.8.3. *Define $f(u) \equiv \varphi_1'(u) - \varphi_2'(u)$. If over $[0, \bar{u}]$ the function $f(u)$ vanishes only at $\bar{u}$, then $\varphi_1'(u) < \varphi_2'(u) \quad \forall u \in [0, \bar{u})$.*

Proof. We will first show that statement necessarily implies that $u_1 < u_2$. We will then use this fact to show that $\varphi_1'(u) < \varphi_2'(u)$ for all $u \in [0, \bar{u})$.

Suppose by contradiction that $u_1 > u_2$. Given that $\varphi_1(u)$ is strictly concave in u, this implies that $f(u_2) = \varphi_1'(u_2) > 0$. Because $f(u)$ cannot change of sign over $[0, \bar{u})$ ($f(u)$ vanishes only at $\bar{u}$ over $[0, \bar{u}]$), we then conclude that $f(u) > 0$ for all $u \in [0, \bar{u})$, i.e., $\varphi_1'(u) > \varphi_2'(u)$ for all $u \in [0, \bar{u})$. However, if so, then for $u \in (\bar{u} - \epsilon, \bar{u})$ we have that $\varphi_1(u) < \varphi_2(u)$ given that $\varphi_1(\bar{u}) = \varphi_2(\bar{u})$. Contradiction.

Thus, $u_1 \leq u_2$. Moreover, since $u_1 \neq u_2$ by assumption, then $u_1 < u_2$. However, this implies that $f(u_1) = -\varphi_2'(u_1) < 0$ given that $\varphi_2(u)$ is strictly concave in u. Because $f(u)$ does not change sign over $[0, \bar{u})$, then $f(u) < 0$ for all $u \in [0, \bar{u})$. That is, $\varphi_1'(u) < \varphi_2'(u)$ for all $u \in [0, \bar{u})$. □

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