Strategic Information Acquisition and Transmission (Online Appendix)

Rossella Argenziano* Sergei Severinov† Francesco Squintani‡

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Abstract

This online appendix accompanies our paper "Strategic Information Acquisition and Transmission" published in the American Economic Journal: Microeconomics.

Proof of convergence of arbitrage condition. We prove that as $n' \to \infty$, for any i, $|p_i| / (n'+1) \to a_i - a_{i-1}$. In fact, condition (7) implies that

$$\frac{4b(n'+2)-2}{n'+1} \leq \frac{|p_{i+1}| - |p_i|}{n'+1} \leq \frac{4b(n+2)+2}{n'+1},$$

and, taking limits for $n' \to \infty$, $4b \leq a_i - a_{i-1} + a_{i+1} - a_i \leq 4b$,which is exactly the arbitrage condition of Crawford and Sobel (1982).

Proposition 1 *For any n' and b, the Pareto-efficient incentive compatible partition is $P^* = \{p_1^*, ..., p_K^*\}$ such that $K = \max\{k \in \mathbb{N} | k + \lceil 4b(n'+2) - 2 \rceil \times \frac{k(k-1)}{2}) \leq n'+1\}$. For all $i = 1, ..., K$, the element p_i^* of the equilibrium partition consists of consecutive types and has cardinality $|p_i^*| = 1 + \lceil 4b(n'+2) - 2 \rceil \times (i-1) + \lfloor \frac{r}{K} \rfloor + \mathbb{I}\left\{ r - \left(\lfloor \frac{r}{K} \rfloor + 1 \right) K + i > 0 \right\}$, where $r \equiv n' + 1 - \left[K + \lceil 4b(n'+2) - 2 \rceil \times \frac{K(K-1)}{2} \right]$, and $\mathbb{I}$ denotes the indicator function.*

*Department of Economics, University of Essex. rargenz@essex.ac.uk
†Department of Economics, University of British Columbia, sseverinov@gmail.com
‡Department of Economics, University of Warwick. f.squintani@warwick.ac.uk

Proof. The equilibrium partition P identified in the Proposition is the one with the largest cardinality K and with the smallest difference in the cardinality of subsequent elements, subject to the incentive compatibility condition (7).

The proof is in three parts. First we show that the negative of the expected residual variance, $E\left[-\left(y_{p_i}^{n'}-\theta\right)^2|P_{n'}\right]$ can be rewritten as $-\frac{1}{3}+E\left[E(\theta|p_i)^2\right]$. Then, we show that among the equilibrium partitions with the largest number of elements, the equilibrium with the smallest difference between the cardinalities of any two subsequent elements minimizes the expected residual variance. Third, we show that, among the equilibrium partitions with the smallest difference in the cardinality of subsequent elements, the one which minimizes the expected residual variance is the one with the largest number of elements.

Part 1: $E\left[-\left(y_{p_i}^{n'}-\theta\right)^2|P_{n'}\right]=-\frac{1}{3}+E\left[E(\theta|p_i)^2\right]$.

By the law of iterated expectations,

$$\begin{aligned}
E\left[-\left(y_{p_i}^{n'}-\theta\right)^2|P_{n'}\right] &= -E_{\theta}\left[\left(E\left[\theta|p_i\right]-\theta\right)^2\right]\\
&= -E_{p_i}\left[E_{\theta}\left[\left(E\left[\theta|p_i\right]-\theta\right)^2|p_i\right]\right]\\
&= -E_{p_i}\left[Var\left[\theta|p_i\right]\right].
\end{aligned}$$

Because $Var\left[\theta\right]=E_{p_i}\left[Var\left[\theta|p_i\right]\right]+Var_{p_i}\left[E(\theta|p_i)\right]$, we thus obtain:

$$\begin{aligned}
E\left[-\left(y_{p_i}^{n'}-\theta\right)^2|P_{n'}\right] &= -Var\left[\theta\right]+Var_{p_i}\left[E(\theta|p_i)\right]\\
&= -Var\left[\theta\right]+E\left[E(\theta|p_i)^2\right]-E\left[E(\theta|p_i)\right]^2\\
&= -Var\left[\theta\right]+E\left[E(\theta|p_i)^2\right]-E\left[\theta\right]^2\\
&= -\frac{1}{12}+E\left[E(\theta|p_i)^2\right]-\left(\frac{1}{2}\right)^2\\
&= -\frac{1}{3}+E\left[E(\theta|p_i)^2\right].
\end{aligned}$$

Part 2: Among the equilibrium partitions with the largest number of elements, the equilibrium with the smallest difference between the cardinalities of any two subsequent elements maxi-

mizes $E\left[-\left(y^{n'}_{p_i}-\theta\right)^2|P_{n'}\right] = -\frac{1}{3}+E\left[E(\theta|p_i)^2\right]$.

Suppose the number of trials is $n+1$ and the number of types is $n'+1$. Consider an equilibrium partition P with I elements $\{k_i,...,k_{i+1}-1\}_{i=1}^{I}$, where $k_{I+1}\equiv n+1$. We obtain:

$$E\left[-\left(y^{n'}_{p_i}-\theta\right)^2|P\right] = -\frac{1}{3}+E\left[E(\theta|p_i)^2\right] = -\frac{1}{3}+\sum_{i=1}^{I}\frac{k_{i+1}-k_i}{n+1}\left(\frac{k_{i+1}+k_i+1}{2\left(n'+2\right)}\right)^2.$$

Next, consider a different equilibrium partition $P'=\left\{k'_i,...,k'_{i+1}-1\right\}_{i=1}^{I}$, such that there is a unique $i\in I$ with $k'_i=k_i+1$, and $k'_j=k_j$ for all $j\neq i$. Denoting the associated expected residual variance by $E\left[-\left(y_p-\theta\right)^2;P\right]$ we obtain:

$$E\left[-\left(y^{n'}_{p'_i}-\theta\right)^2;P'\right]-E\left[-\left(y^{n'}_{p_i}-\theta\right)^2;P\right]$$

$$=\frac{k_{i+1}-(k_i+1)}{n+1}\left(\frac{k_{i+1}+(k_i+1)+1}{2\left(n'+2\right)}\right)^2+\frac{k_i+1-k_{i-1}}{n'+1}\left(\frac{k_i+1+k_{i-1}+1}{2\left(n'+2\right)}\right)^2$$
$$-\frac{k_{i+1}-k_i}{n'+1}\left(\frac{k_{i+1}+k_i+1}{2\left(n'+2\right)}\right)^2-\frac{k_i-k_{i-1}}{n'+1}\left(\frac{k_i+k_{i-1}+1}{2\left(n'+2\right)}\right)^2$$
$$=\frac{\left(k_{i+1}-k_{i-1}\right)\left[\left(k_{i+1}-k_i\right)-\left(k_i+1-k_{i-1}\right)\right]}{4\left(n'+2\right)^2\left(n'+1\right)}>0.$$

where the last inequality holds because P' is an equilibrium partition, hence $k'_{i+1}-k'_i>k'_i-k'_{i-1}$, which implies $k_{i+1}-k_i-1>k_i+1-k_{i-1}$.

Part 3: Among the equilibrium partitions with the smallest difference in the cardinality of subsequent elements, the one which minimizes the expected residual variance is the one with the largest number of elements.

Denoting by $P(m)$ the best equilibrium partition among those with m elements, we prove that $P(j)$ dominates $P(j-1)$. Repeating the argument proves the statement.

To prove that $P(j)$ dominates $P(j-1)$ we describe an algorithm to construct a sequence of partitions with the following features:

(a) the first term of the sequence is $P(j)$

(b) the last term of the sequence is $P(j-1)$

(c) each term of the sequence, except for the last one, is a partition with j elements

(d) each term of the sequence is preferred by both players to the next one (i.e. has a smaller expected residual variance).

The algorithm is the following. Given the n-th term of the sequence (the n-th partition), the $(n+1)$-th is constructed as follows:

(i) If the sub-partition that includes the largest $(j-2)$ elements of n-th partition is identical to the sub-partition that includes the largest $(j-2)$ elements of $P(j-1)$, then let the $n+1$-th partition be $P(j-1)$; i.e., let the first element of the $n+1$-th partition be equal to the union of the first two elements of the n-th partition. This step concludes the algorithm, and satisfies condition (d), because, for any k_1, k_2 with $k_1 > 1$, and $k_2 > k_1 + 1$,

$$\begin{aligned} & \frac{k_2-k_1}{n'+1}\left(\frac{k_2+k_1+1}{2(n'+2)}\right)^2+\frac{k_1-1}{n'+1}\left(\frac{k_1+1+1}{2(n'+2)}\right)^2-\frac{k_2-1}{n'+1}\left(\frac{k_2+1+1}{2(n'+2)}\right)^2 \\ = & \frac{1}{4}\frac{(k_2-k_1)(k_2-1)(k_1-1)}{(n'+2)(n'+1)}>0. \end{aligned}$$

(ii) If the sub-partition that includes the last $(j-2)$ elements of n-th partition is *not* identical to the sub-partition that includes the largest $(j-2)$ elements of $P(j-1)$, then the $(n+1)$-th partition is obtained from the n-th by moving the highest type included in the k-th element p_k^n into the $(k+1)$-th element p_{k+1}^n, where $k < j$ is the highest index that satisfies the following conditions:

(iia) For $l < j-2$, if the sub-partition that includes the last l elements of n-th partition is identical to the sub-partition that includes the last l elements of $P(j-1)$, then $k < j - l$.[1]

(iib) The cardinality of p_{k+1}^n is strictly smaller than the cardinality of the k-th element of $P(j-1)$.

(iic) If the union of p_1^n and p_2^n is equal to the first element of $P(j-1)$, then $k > 2$.

[1]For example, if $j = 10$, if the last three elements of the $n-th$ partition in the sequence are identical to the last three elements of the target partition, then they shouldn't be changed anymore, hence $k < 7$, so that "at most" a type is taken from the 6-th element and moved into the 7-th.

Because the number of types is finite, the algorithm has an end.

The type-(ii) step can be repeated exactly until the condition for the type-(i) step is satisfied because, by construction, the cardinality of the l-th element of $P(j-1)$ is weakly larger than the cardinality of the $(l+1)$-th element of $P(j)$, hence the union of the first two elements of $P(j)$ has cardinality weakly larger than the cardinality of the first element of $P(j-1)$. *Q.E.D.*

Supplementary computations for Example 2: First, let us compute the expert's payoff when he performs $n=2$ trials and fully reveals his information. It is equal to $-E_k(E(\theta - E\theta|k, n=2)^2) - b^2 - 2c$. We may compute:

$$\begin{aligned}&-E_kE((\theta - E\theta)^2|k, n=2) = Prob(k=0|n=2)E((\theta - E\theta)^2|k=0, n=2)\\&+Prob(k=1|n=2)E((\theta - E\theta)^2|k=1, n=2) + Prob(k=2|n=2)E(\theta - E\theta)^2|k=2, n=2)\end{aligned} \quad (1)$$

Note that $Prob(k=0|n=2) = Prob(k=0|n=2) = Prob(k=2|n=2) = \frac{1}{3}$. Also, $E(\theta - E\theta)^2|k=0, n=2) = 2E(\theta - E\theta)^2|k=, n=2)\frac{3}{80}$ and $E(\theta - E\theta)^2|k=1, n=2) = \frac{1}{20}$. Substituting this into (1) we obtain that the expert's total payoff is equal to $-\frac{1}{24} - b^2 - 2c$.

Next, we consider a deviation to $n=1$ trial. Let us show the following: (i) If the trial fails ($k=0$ out of $n=1$), then the expert prefers to induce action 1/4 rather than action 1/2 or action 3/4. It is enough to show that he prefers $\frac{1}{4}$ to $\frac{1}{2}$ (The argument for 3/4 follows by monotonicity), which is so if:

$$-\int_0^1 \left(\frac{1}{4} - \theta - b\right)^2 \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k}d\theta \geq -\int_0^1 \left(\frac{1}{2} - \theta - b\right)^2 \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k}d\theta \quad (2)$$

With $n=1$ and $k=0$, $\frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k} = 2(1-\theta)$, so (2) simplifies to:

$$-2\int_0^1 \left(\frac{1}{4} - \theta - b\right)^2 (1-\theta)d\theta + 2\int_0^1 \left(\frac{1}{2} - \theta - b\right)^2 (1-\theta)d\theta \geq 0 \quad (3)$$

Rearranging terms and integrating, we obtain that (3) is equivalent to $\frac{1}{4}\left(\frac{1}{12} - 2b\right) \geq 0$ which holds because $b \leq \frac{1}{24}$.

(ii) If the trial succeeds ($k=1$ out of $n=1$), then the expert prefers to induce action 3/4 rather than action 1/2 or action 1/4. It is enough to show that he prefers $\frac{3}{4}$ to $\frac{1}{2}$ (the argument for 1/4 follows by monotonicity), which is so if:

$$-\int_0^1 \left(\frac{3}{4}-\theta-b\right)^2 \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k}d\theta \geq -\int_0^1 \left(\frac{1}{2}-\theta-b\right)^2 \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k}d\theta \tag{4}$$

With $n=1$ and $k=1$, $\frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k}=2\theta$, so (4) simplifies to:

$$-2\int_0^1 \left(\frac{3}{4}-\theta-b\right)^2 \theta d\theta + 2\int_0^1 \left(\frac{1}{2}-\theta-b\right)^2 \theta d\theta \geq 0 \tag{5}$$

Rearranging terms and integrating, we obtain that (5) is equivalent to $\frac{1}{8}\left(\frac{1}{12}+2b\right)>0$ which is trivially satisfied.

Extension of Example 1: We show that Example 1 can be extended to show that our overinvestment results hold beyond our parametric statistical model.

Consider an alternative model in which the expert's information acquisition model consists in choosing the fineness of a partition of the state space $[0,1]$, composed of equally sized intervals. I.e., the expert chooses the number n of intervals $[(k-1)/n, k/n]$, $k=1,\ldots,n$, at cost cn, to then observe the interval to which θ belongs. It can be shown that, for $b\leq\frac{7}{60}$ and $c=\frac{1}{35}$, there exists an equilibrium of the covert game such that the decision maker achieves a higher utility than if she acquired information directly.

Consider direct information acquisition first. The decision-maker's payoff for $n=0$ is, again, $-\frac{1}{12}$. If choosing $n=1$, the decision maker pays the cost c, to then take the action 1/4 if $\theta\in[0,1/2]$ and the action 3/4 if $\theta\in(1/2,1]$; thus her expected payoff is $-1/48-c$. Now, suppose $c=1/15$, so that the decision maker chooses $n^*=0$ if acquiring information directly. For $b\leq 7/60$, we now show that there exists an equilibrium in which the expert chooses $n=1$, i.e., "acquires" the partition $\{[0,1/2],(1/2,1]\}$ of the state space, and reveals the interval he observed, inducing action $y=\frac{1}{4}$ if seeing $[0,1/2]$ and $y=\frac{3}{4}$ if seeing $(1/2,1]$. Indeed, if the expert deviates to zero trials, then any message he sends can only induce one of the equilibrium actions, namely $y=\frac{1}{4}$ or $y=\frac{3}{4}$. Because of his upwards bias

$(b > 0)$, he prefers $y = \frac{3}{4}$. The expected utility that the expert obtains by inducing $y = \frac{3}{4}$ is $-b^2 + \frac{1}{2}b - \frac{7}{48}$. For $b \leq \frac{7}{60}$ and $c = \frac{1}{35}$, this is less than $-\frac{1}{48} - b^2 - c$, so this deviation is unprofitable. Again, showing that the expert will not deviate to any $n > 1$ is straightforward and is therefore omitted. Hence, the decision maker achieves a higher utility than if she acquired information directly.