Online Appendix to "The Great Housing Boom of China"

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A. Phase Diagram of the Benchmark Economy

The dynamic system can be described by two difference equations in the detrended E-firm capital and housing prices:

$$\widehat{k}^E_{t+1}(1+z)(1+\nu)+\widehat{p}^H_t\overline{H} = \frac{\psi}{1+\beta^{-1}}\left(\widehat{k}^E_t\right)^{\alpha}\left(\chi n^E_t\right)^{1-\alpha} \tag{A1}$$

$$\widehat{p}^H_{t+1}(1+z)(1+\nu) = \rho^E_{t+1}\widehat{p}^H_t \tag{A2}$$

where

$$\rho^E_{t+1}=\begin{cases}\rho^E & \text{if } \widehat{k}^E_t\le\widetilde{k}^E\\ \alpha(1-\psi)\left(\widehat{k}^E_{t+1}/\chi\right)^{\alpha-1} & \text{if } \widehat{k}^E_t>\widetilde{k}^E\end{cases},$$

and $\widetilde{k}^E\equiv\chi/\left\{\left[(1-\psi)\chi\right]^{\frac{1}{\alpha}}(R/\alpha)^{\frac{1}{1-\alpha}}\right\}$ is the minimum level of E-firm capital under which $n^E_t=1$. Accordingly, the constant $\widehat{k}^E$ locus ($\widehat{k}^E_{t+1}=\widehat{k}^E_t$) is characterized by the following step function:

$$\widehat{p}^H_t=\begin{cases}\left[\frac{\rho^E\psi}{(1-\psi)\alpha\left(1+\beta^{-1}\right)}-(1+z)(1+\nu)\right]\widehat{k}^E_t & \text{if } \widehat{k}^E_t<\widetilde{k}^E\\ \frac{\psi}{1+\beta^{-1}}\left(\widehat{k}^E_t\right)^{\alpha}\chi^{1-\alpha}-\widehat{k}^E_t(1+z)(1+\nu) & \text{if } \widehat{k}^E_t\ge\widetilde{k}^E\end{cases}. \tag{A3}$$

Obviously, when $\widehat{k}^E_t<\widetilde{k}^E$, the constant $\widehat{k}^E$ locus is an upward-sloping straight line due to the AK feature of E-firm return to capital during the transition stage. When $\widehat{k}^E_t\ge\widetilde{k}^E$, the constant $\widehat{k}^E$ locus looks like their counterpart in the standard neoclassical economy and is hump-shaped. Moreover, from $(A1)$ and $(A2)$, the constant $\widehat{p}^H$ locus is characterized by

$$\widehat{p}^H_t=\frac{\psi}{1+\beta^{-1}}\left(\widehat{k}^E_t\right)^{\alpha}\chi^{1-\alpha}-(1+z)(1+\nu)\rho^{E-1}\left((1+z)(1+\nu)\right) \tag{A4}$$

where ρ^{E-1} is the inverse function of $\rho^E\left(\widehat{k}^E_t\right)=\alpha(1-\psi)\left(\widehat{k}^E_t/\chi\right)^{\alpha-1}$. Obviously, the constant $\widehat{p}^H$ locus is always upward sloping. Moreover, since $\rho^E>1$, the whole constant $\widehat{p}^H$ locus is on the right side of $\widehat{k}^E_t=\widetilde{k}^E$. Note that the condition for ψ in equation (9) ensures that the constant $\widehat{k}^E$ locus and the constant $\widehat{p}^H_t$ intersect at a point where $\widehat{p}^H_t$ is positive, which is the bubbly steady state.

Figure A-1 plots the phase diagram for $\left\{\widehat{k}_t^E, \widehat{p}_t^H\right\}$. For any initial $\widehat{k}_0^E$, there could be three cases for $\widehat{p}_0^H$. Point A, at which $\widehat{p}_0^H = \widetilde{p}_0^H$ or $h_0^E = \widetilde{h}_0^E$, corresponds to the saddle-path equilibrium. Point B, at which $\widehat{p}_0^H < \widetilde{p}_0^H$ or $h_0^E < \widetilde{h}_0^E$, corresponds to the asymptotic bubbly equilibrium. Point C, at which $h_0^E > \widetilde{h}_0^E$, has an explosive path for a housing bubble, hence is not sustainable.

B. Proof of Propositions and Lemmas

In this section, we prove the various lemmas and propositions.

Proof of Lemma 1. The growth rate of E-firm output is

$$\frac{Y_{t+1}^E}{Y_t^E} = \frac{Y_{t+1}^E}{K_{t+1}^E}\frac{K_{t+1}^E}{K_t^E}\frac{K_t^E}{Y_t^E} = \frac{\rho_{t+1}^E}{(1-\psi)\,\alpha}\frac{K_{t+1}^E}{K_t^E}\frac{(1-\psi)\,\alpha}{\rho_t^E}, \tag{A5}$$

where K_{t+1}^E/K_t^E depends on the entrepreneur's equilibrium portfolio share in physical capital, ϕ_t^E. We now solve for the entrepreneur's equilibrium portfolio share of savings in housing. Using the housing market-clearing condition, $\overline{H} = H_t^E$, we have

$$\left(1-\phi_t^E\right)\frac{1}{1+\beta^{-1}}\psi\left(K_t^E\right)^\alpha\left(A_t\chi n_t^E N_t\right)^{1-\alpha} = P_t^H\overline{H}. \tag{A6}$$

Forwarding $(A6)$ by one period, and with $\left(K_{t+1}^E\right)^\alpha\left(A_{t+1}\chi n_{t+1}^E N_{t+1}\right)^{1-\alpha} = K_{t+1}^E\rho_{t+1}^E/\left[\alpha\,(1-\psi)\right]$, equation $(A6)$ can be rewritten as

$$\left(1-\phi_{t+1}^E\right)\frac{1}{1+\beta^{-1}}\frac{\psi\rho_{t+1}^E K_{t+1}^E}{\alpha\,(1-\psi)} = P_{t+1}^H\overline{H}. \tag{A7}$$

With the law of motion for capital (9), $(A7)$ can be rewritten as

$$\left(1-\phi_{t+1}^E\right)\frac{1}{1+\beta^{-1}}\frac{\psi\rho_{t+1}^E}{\alpha\,(1-\psi)}\frac{\phi_t^E}{1+\beta^{-1}}\psi\left(K_t^E\right)^\alpha\left(A_t\chi n_t^E N_t\right)^{1-\alpha} = P_{t+1}^H\overline{H}. \tag{A8}$$

Dividing $(A8)$ by $(A6)$ for all t, we have

$$\frac{1-\phi_{t+1}^E}{1-\phi_t^E}\frac{\phi_t^E}{1+\beta^{-1}}\frac{\psi\rho_{t+1}^E}{\alpha\,(1-\psi)} = \frac{P_{t+1}^H}{P_t^H} = \rho_{t+1}^E,$$

or simply

$$\frac{1-\phi_{t+1}^E}{1-\phi_t^E}\frac{\phi_t^E}{1+\beta^{-1}}\frac{\psi}{\alpha\,(1-\psi)} = 1. \tag{A9}$$

Equation $(A9)$ is a first-difference equation capturing the dynamics of ϕ_t^E. One solution to equation $(A9)$ is that

$$\phi_t^E = \frac{\alpha\left(1+\beta^{-1}\right)(1-\psi)}{\psi}, \forall t. \tag{A10}$$

To solve for K_{t+1}^E/K_t^E, we substitute $(A10)$ into (9) and obtain

$$K_{t+1}^E = \rho_t^E K_t^E. \tag{A11}$$

Equation $(A11)$ is a variant of the no-arbitrage condition. Comparing (**??**) in the fundamental equilibrium and $(A11)$ in the bubbly equilibrium, we see that in the bubbly equilibrium the optimal portfolio choice by an entrepreneur equalizes the rate of return to capital investment and the rate of return to bubbles by crowding out E-firm capital investment.

Finally, substituting $(A11)$ into $(A5)$, we obtain $Y_{t+1}^E/Y_t^E = \rho_{t+1}^E = \rho_{t+1}^H$.■

Proof of Proposition 1. We first decompose the ratio of housing value to aggregate output as

$$\frac{P_t^H \overline{H}}{Y_t} = \frac{P_t^H \overline{H}}{Y_t^E} \frac{Y_t^E}{Y_t^E + Y_t^F}. \tag{A12}$$

The first argument on the right-hand side of $(A12)$, $P_t^H \overline{H}/Y_t^E$, can be further expressed as

$$\frac{P_t^H \overline{H}}{Y_t^E} = \frac{P_t^H \overline{H}}{K_{t+1}^E} \frac{K_{t+1}^E}{Y_t^E} = \frac{1-\phi_t^E}{\phi_t^E} \frac{K_{t+1}^E}{Y_t^E}. \tag{A13}$$

Equation $(A11)$ implies

$$K_{t+1}^E = (1-\psi)\,\alpha Y_t^E. \tag{A14}$$

With both $(A10)$ and $(A14)$, it is straightforward that $P_t^H \overline{H}/Y_t^E$ is constant. Therefore, by log-differencing $(A12)$, we obtain (25).

Finally, we derive $Y_t^E/\left(Y_t^E + Y_t^F\right)$. Using (6), Y_t^E can be expressed as

$$Y_t^E = \frac{N_t^E}{N_t\,(1-\psi)} \kappa_F^\alpha A_t N_t, \tag{A15}$$

where $\kappa_F \equiv k_{Ft}/\left(n_{Ft} A_t\right) = (\alpha/R)^{\frac{1}{1-\alpha}}$. Similarly, it is easy to show that

$$Y_t^E + Y_t^F = \left(1 + \frac{\psi}{1-\psi} \frac{N_t^E}{N_t}\right) \kappa_F^\alpha A_t N_t. \tag{A16}$$

■

Proof of Proposition 2. To prove this proposition, consider the fundamental equilibrium—that is, $\phi_t^E = 1$ for all t. According to (12), introducing housing reduces the steady-state physical capital. Hence, we only need to show under which condition a marginal reduction in physical capital reduces total entrepreneurial consumption. Aggregating the budget constraints of the young and old entrepreneurs at period t and using the capital market-clearing condition, we obtain

$$\left[N_t c_{1,t}^E + N_{t-1} c_{1,t-1}^E + N_{t+1} k_{t+1}^E\right]/2 = N_t\left[m_t + \rho_t^E k_t^E\right]/2. \tag{A17}$$

With the definition of c_t^E and $m_t + \rho_t^E k_t^E = [\psi + (1-\psi)\,\alpha]\, y_t^E$, a detrended version of $(A17)$ is

$$\widehat{c}_t^E = [\psi + (1-\psi)\,\alpha]\,\widehat{y}_t^E - \widehat{k}_{t+1}^E(1+z)(1+\nu). \tag{A18}$$

Taking the derivative of the right-hand side of $(A18)$ with respect to $\widehat{k}^E$ at the steady state, we can obtain the following sufficient condition for introducing bubbles to reduce aggregate consumption for entrepreneurs:

$$[\psi + (1-\psi)\alpha] MPK^{E*} \mid_{\phi^E=1} > (1+z)(1+\nu). \tag{A19}$$

With (13) and the definition of MPK^E, the inequality $(A19)$ can be rewritten as

$$[\psi + (1-\psi)\alpha]\alpha\left(1+\beta^{-1}\right)(1+z)(1+\nu)/\psi > (1+z)(1+\nu). \tag{A20}$$

Reordering $(A20)$, we obtain (28).

The proof of the welfare implications for entrepreneurs in both the transition and post-transition stages is straightforward. Substituting the detrended version of (9) into $(A18)$, we obtain

$$\widehat{c}_t^E = \left[\psi + (1-\psi)\alpha - \frac{\psi}{\alpha\left(1+\beta^{-1}\right)}\right]\widehat{y}_t^E.$$

Since E-firm capital increases monotonically, we need to prove only

$$\frac{\partial \widehat{c}_t^E}{\partial \widehat{k}_t^E} = \left[\psi + (1-\psi)\alpha - \frac{\psi}{\alpha\left(1+\beta^{-1}\right)}\right] MPK_t^E \mid_{\phi^E=1} > 0. \tag{A21}$$

By assumption (28), $\partial \widehat{c}_t^E / \partial \widehat{k}_t^E > 0$ for all period t. Hence, housing bubbles, by crowding out physical capital, reduce total entrepreneurial consumption.

For entrepreneurs born during the transition, $(A22)$ becomes

$$\begin{aligned} & \log\left(m_t - s_t^E\right) + \beta \log \rho^E s_t^E \\ = \; & (1+\beta)\log k_t^E + \widehat{C}, \end{aligned}$$

where $\widehat{C}$ is a function of parameters. Therefore, it is easy to see that a reduction in capital stock would reduce the welfare of entrepreneurs born during the transition. For the entrepreneur born in the post-transition stage, but before reaching the steady state, the lifetime utility can be expressed as

$$\begin{aligned} & \log\left(m_t - s_t^E\right) + \beta \log \rho_{t+1}^E s_t^E \\ = \; & \log\left(\frac{\psi \rho_t^E k_t^E}{(1+\beta)\alpha(1-\psi)}\right) + \beta \log \frac{\rho_{t+1}^E k_{t+1}^E (1+\nu)}{\phi_t^E}. \end{aligned} \tag{A22}$$

At the steady state, equation $(A22)$, after being detrended, becomes

$$\begin{aligned} & (1+\beta)\log \rho^{E*}\widehat{k}^{E*} - \beta \log \phi^{E*} + C \\ = \; & (1+\beta)\log \alpha \frac{1-\psi}{\psi}\frac{\left(1+\beta^{-1}\right)}{\phi^{E*}}(1+z)(1+\nu)\left[\frac{\psi \phi^{E*}\chi^{1-\alpha}}{\left(1+\beta^{-1}\right)(1+z)(1+\nu)}\right]^{\frac{1}{1-\alpha}} \\ & -\beta \log \phi^{E*} + C \\ = \; & \left[\frac{\alpha(1+\beta)}{1-\alpha} - \beta\right]\log \phi^{E*} + \widetilde{C}, \end{aligned} \tag{A23}$$

where both C and $\widetilde{C}$ are functions of parameters. Hence, introducing housing (i.e., a reduction in ϕ^{E*}) reduces the steady-state welfare if $\alpha\left(1+\beta\right)/\left(1-\alpha\right) > \beta$ or $\alpha\left(1+\beta^{-1}\right) > 1-\alpha$. Note that the joint participation and incentive constraints of young entrepreneurs implies $m = \psi y^{E} > w = \left(1-\alpha\right)\left(1-\psi\right)y^{E}$, which gives the following parameter restriction: $\psi > \left(1-\alpha\right)\left(1-\psi\right)$, or equivalently, $\psi/\left[\psi+\alpha\left(1-\psi\right)\right] > 1-\alpha$. Therefore, with assumption (28), introducing housing reduces the entrepreneurial lifetime utility at the steady state. For entrepreneurs born during the post-transition stage, it is easy to show that equation ($A22$) becomes

$$\begin{aligned} & \log\left(m_t - s_t^{E}\right) + \beta\log\rho_{t+1}^{E}s_t^{E} \\ = \ & \alpha\left(1+\alpha\beta\right)\log k_t^{E} - \left(1-\alpha\right)\beta\log\phi_t^{E} + \overline{C}, \end{aligned} \tag{A24}$$

where $\overline{C}$ is a function of parameters.

We would compare ($A24$) under the fundamental and the bubbly equilibrium as follows. In the bubbly equilibrium, since k_t^{E} is smaller due to the previous cohort's housing investment, a sufficient condition for welfare loss with a reduction in ϕ_t^{E} is

$$\alpha\left(1+\alpha\beta\right) > \left(1-\alpha\right)\beta$$

or $\alpha\left(1+\beta^{-1}\right) > 1-\alpha^2$.■

Proof of Proposition 3: We consider the portfolio choice of an age-j entrepreneur in period t. Suppose that all other entrepreneurs alive in period t hold the same share of savings in housing, $\phi_{k,t}^{E} = \phi_t^{E}$ for $k \neq j$, so that the no-arbitrage condition holds, $P_{t+1}^{H}/P_t^{H} = \rho_{t+1}^{E}$. Accordingly, the age-$j$ entrepreneur is indifferent between housing and physical capital. So $\phi_{j,t}^{E} = \phi_t^{E}$ is an equilibrium solution. The same logic applies for the portfolio choice of other entrepreneurs alive in period t. Hence, there exists a solution that all entrepreneurs hold the same share of housings in their net worth.■

C. Numerical Algorithm

Again, we detrend all per capita variables (except labor inputs and housing) as $\widehat{x}_t = x_t/A_t$. For total labor inputs on both the supply and demand sides, we detrend them by dividing them by the size of the population, N_t. Denote $n_t^{E} \equiv \sum_{j=1}^{J^E-1} n_{j,t}^{E}$ as the total detrended labor demand of E-firms. Since the aggregation holds, the following equation determines total labor allocated to E-firms:

$$n_t^{E} = \left[\left(1-\psi\right)\left(1-\tau_t^{y}\right)\chi\right]^{\frac{1}{\alpha}}\left(\frac{R^{l}-1+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}}\widehat{k}_t^{E}/\chi. \tag{A25}$$

Similarly, denote $n_t^{w} \equiv \sum_{j=1}^{J^R-1} n_{j,t}^{w}$ as the total detrended labor supply of workers. If $n_t^{E} > n_t^{w}$, we have

$$\widehat{w}_t = \left(1-\psi\right)\left(1-\tau_t^{y}\right)\left(1-\alpha\right)\left(\widehat{k}_t^{E}/n_t^{E}\right)^{\alpha}\chi^{1-\alpha}, \tag{A26}$$

$$n_t^{E} = n_t^{w},\ n_t^{F} = \widehat{k}_t^{F} = 0. \tag{A27}$$

Otherwise,

$$\widehat{w}_t = (1-\alpha)\left(\frac{\alpha}{R^l - 1 + \delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{A28}$$

$$n_t^F = n_t^w - n_t^E \tag{A29}$$

$$\widehat{k}_t^F = \left(\alpha / \left(R^l - 1 + \delta\right)\right)^{\frac{1}{1-\alpha}} n_t^F. \tag{A30}$$

Also, we have the following equations for both the transition and post-transition stages:

$$\rho_t^E = \alpha(1-\psi)\left[(1-\alpha)(1-\psi)(1-\tau_t^y)\chi/\widehat{w}_t\right]^{\frac{1-\alpha}{\alpha}} + 1 - \delta, \tag{A31}$$

$$\widehat{m}_t = \psi(1-\tau_t^y)\left(\widehat{k}_t^E\right)^{\alpha}\left(\chi n_t^E\right)^{1-\alpha} / \sum_{j=1}^{J^E-1} n_{j,t}^E, \tag{A32}$$

$$H_t^E = \overline{H}, \tag{A33}$$

$$\widehat{p}_t^H = \widehat{p}_{t+1}^H (1+z)(1+\nu)/\rho_{t+1}^E, \tag{A34}$$

$$\widehat{p}_t^H \overline{H} = \left(1 - \phi_t^E\right) \sum_{j=J^E-1}^{J-1} n_{j,t}^E \widehat{s}_{j,t}^E, \tag{A35}$$

$$\widehat{k}_{t+1}^E = \phi_t^E \sum_{j=J^E-1}^{J-1} n_{j,t}^E \widehat{s}_{j,t}^E. \tag{A36}$$

We assume transition takes T periods. At period T, the economy enters the steady state. The algorithm to solve for the transition takes the following steps:

1. Guess the sequence of $\left\{\phi_t^E, \widehat{k}_{t+1}^E, \widehat{p}_t^H\right\}_{t=1}^{T-1}$.
2. Given $\widehat{k}_1^E$, compute $\left\{n_t^E, \widehat{w}_t, n_t^F, \widehat{k}_{t+1}^F, \rho_t^E, \widehat{m}_t, \widehat{s}_{j,t}^E, \widehat{s}_{j,t}^w, H_t^E\right\}_{t=1}^{T-1}$.
3. Check the following conditions for each period $t = 1, 2, .., T-1$:

$$\phi_t^E = 1 - \frac{\widehat{p}_t^H \overline{H}}{\sum_{j=J^E-1}^{J-1} n_{j,t}^E \widehat{s}_{j,t}^E}, \tag{A37}$$

$$\widehat{p}_t^H = \widehat{p}_{t+1}^H (1+z)(1+\nu)/\rho_{t+1}^E, \tag{A38}$$

$$\widehat{k}_{t+1}^E = \phi_t^E \sum_{j=J^E-1}^{J-1} n_{j,t}^E \widehat{s}_{j,t}^E / \left[(1+z)(1+\nu)\right], \tag{A39}$$

and (since ρ_{T+1}^E is not known)

$$\widehat{k}_{T+1}^E = \widehat{k}^{E*} = \left[\frac{\phi^{E*}\psi(1-\tau^y)\chi^{1-\alpha}}{\left(1+\beta^{-1}\right)(1+z)(1+\nu)}\right]^{\frac{1}{1-\alpha}}. \tag{A40}$$

Figure A-1: Phase Diagram of the Benchmark Economy