DISCRIMINATION VIA SYMMETRIC AUCTIONS: ONLINE APPENDIX

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1. IMPLEMENTATION USING RANDOMIZATION

In this section, we demonstrate that if the implementation criterion is weakened, the principal can achieve the outcomes corresponding to certain non-hierarchical mechanisms using randomization. The principal can randomize by choosing amongst a set of mechanisms via a lottery. After choosing one such mechanism from the set, the principal can announce it to the buyer. For instance, the principal could toss a coin and choose between a first- and second-price auction. Having chosen, the buyer is informed of the auction format and the game proceeds. Such randomization is appropriate for a principal concerned about expected outcomes (as in **??**).

Randomization can be useful in achieving the outcomes of both unimplementable hierarchical allocation mechanisms and nonhierarchical mechanisms. A simple two-buyer example of a non-hierarchical mechanism is one where irrespective of the values, buyer 1 gets the good 25% of the time and buyer 2 gets it 75%. Clearly, this is not a hierarchical allocation since our definition of the latter requires the equal breaking of ties. Another example is a mechanism in which the seller randomly allocates the good 50% of the time and runs a second price auction the remaining 50%.

A mechanism (a^d, p^d) is defined to be a *randomization* over a set of mechanisms $\mathscr{M}$, if there is a measure ζ defined on $\mathscr{M}$ such that

$$a_i^d(v_i) = \int_{\mathscr{M}} a_i(v_i) d\zeta((a,p)) \qquad \text{and} \qquad p_i^d(v_i) = \int_{\mathscr{M}} p_i(v_i) d\zeta((a,p)).$$

The lemma below shows that all IC and IR direct mechanisms can be obtained as a randomization over hierarchical mechanisms. This lemma follows from results in **?** and **?**.

Lemma 1. *Every IC and IR direct mechanism is a randomization over the set of hierarchical mechanisms.*

Proof. Define the set of non-decreasing interim allocation rules achieved by some index rule as $\mathscr{H}_M$ the set of all feasible, non-decreasing interim allocation rules by $\mathscr{F}_M$ and the set of all feasible interim allocation rules by $\mathscr{F}$. By feasible, we mean that this interim allocation rule can result from some feasible ex-post allocation rule. The proof follows from two observations.

Observation 1. *$\mathscr{F}_M$ is a compact subset of L_2^n in the weak/ weak* topology $\sigma(L_2^n, L_2^n)$.*

Proof. Lemma 8 of **?** shows that the set of feasible interim allocation rules $\mathscr{F}$ is a compact convex subset of L_2^n in this topology.

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By observation, $\mathscr{F}_M$ is convex. We now argue that $\mathscr{F}_M$ is also compact in this topology. By the Eberlein-Smulian theorem (Theorem 6.34, **?**), sequential compactness and compactness coincide in this topology. It is therefore enough to show that if for some sequence $\{a^n\}_{n=1}^{\infty} \subset \mathscr{F}_M$, $a^n \rightharpoonup a$, then $a \in \mathscr{F}_M$. Since each a^n is monotone, it is a function of bounded variation and therefore by Helly's selection theorem, there exists a subsequence which converges pointwise. Therefore a is also non-decreasing, and $a \in \mathscr{F}_M$, concluding our argument. □

Therefore, we have that the closure of the convex hull of $\mathscr{H}_M$ is a subset of $\mathscr{F}_M$ or

$$\overline{\text{conv}(\mathscr{H}_M)} \subseteq \mathscr{F}_M.$$

Observation 2. *For any index function $I : V \to \mathbb{R}^n$, there exists a hierarchical allocation rule $a^h \in \mathscr{H}_M$ which solves*

$$\max_{a \in \mathscr{F}_M} \int_V \left(\sum_j a_j(v) I_j(v) f_j(v) \right) dv. \tag{I-OPT-M}$$

Proof. If I is non-decreasing, i.e. $I_j(v)$ is non-decreasing in v for each $j \in N$, then the solution to (I-OPT-M) is in $\mathscr{H}_M$. This follows easily from the definition of hierarchical allocation rule. Since at every profile of values, the good is allotted to the buyer with the higher index, the rule maximizes the 'index revenue' profile-by-profile. Therefore it solves the maximization problem (I-OPT-M).

So let us consider the solution to (I-OPT-M) for other index functions. We can re-write the problem as

$$\max_{a \in \mathscr{F}} \int_V \left(\sum_j a_j(v) I_j(v) f_j(v) \right) dv,$$
$$a \text{ is non-decreasing.}$$

In this case, we can 'relax' the non-decreasing constraint into the objective function. By the ironing procedure of **?**, there exists an 'ironed' non-decreasing index rule $\hat{I}$ such that the solution to the above problem is the same as

$$\max_{a \in \mathscr{F}} \int_V \left(\sum_j a_j(v) \hat{I}_j(v) f_j(v) \right) dv.$$

Note that the corresponding hierarchical rule for index rule $\hat{I}$ lies in $\mathscr{H}_M$. □

To conclude the proof, suppose by way of contradiction that

$$\overline{\text{conv}(\mathscr{H}_M)} \subsetneq \mathscr{F}_M.$$

Then there exists $a \in \mathscr{F}_M$ such that $a \notin \overline{\text{conv}(\mathscr{H}_M)}$. By Corollary 7.47 of **?** there exists an $I \in L_2^n$ such that

$$\langle a, I \rangle > \max_{a' \in \overline{\text{conv}(\mathscr{H}_M)}} \langle a', I \rangle,$$

where $\langle a, I \rangle$ is the standard inner product $\int_V \left(\sum_j a_j(v) I_j(v) f_j(v) \right) dv$.

By Observation 2, the hierarchical allocation rule corresponding to I solves (I-OPT-M), implying that

$$\langle a, I \rangle > \max_{a' \in \mathscr{F}_M} \langle a', I \rangle.$$

Since $a \in \mathscr{F}_M$, this is a contradiction. It follows that

$$\overline{\text{conv}(\mathscr{H}_M)} = \mathscr{F}_M.$$

The Lemma follows. □

Clearly, the outcome from any mechanism that is a randomization over implementable hierarchical mechanisms can be achieved in such an ex-ante sense. The auctioneer can just randomly choose (using measure ζ) from the symmetric auctions that correspond to the implementable hierarchical mechanisms. Note that, strictly speaking, this is not interim implementation as we defined it. However, for practical applications, it serves the same purpose, as randomization is done before the chosen symmetric auction is announced to the buyers. The next corollary summarizes this discussion, and in it, we use the terminology *outcomes are achievable* to clarify the distinction from interim implementation.

Corollary 1. *The outcomes from an IC and IR direct mechanism are achievable if it is a randomization over implementable hierarchical mechanisms.*

Finally, we discuss the two examples of the unimplementable mechanisms and examine whether their outcomes can be achieved via randomization.

EXAMPLE 2 (Continued): *Recall that, in this example, the seller assigns the good at random (with equal probability), buyer* 1 *is never asked to pay anything, while buyer* 2 *is always asked to pay* 0.25*. The outcome from the mechanism can be achieved by randomizing with equal probability over two implementable hierarchical mechanisms. In the first hierarchical mechanism, buyer 1 is always awarded the good irrespective of value and is not asked to pay anything. In the second hierarchical mechanism, buyer 2 is always given the good irrespective of value and is asked to pay* 0.5*.*

EXAMPLE 3 (Continued): *Recall that, in this example, buyer* 2 *wins the good if and only if her value exceeds that of buyer* 1 *by* 1*. The outcome of this mechanism cannot be achieved using randomization.*

Consider the index function $I_1(v) = I_2(v+1) = v$*. By observation, the allocation rule* a^h *corresponding to these index functions is the unique (almost everywhere) maximizer of*

$$\int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} a_j^d(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v},$$

amongst all IC direct allocations a^d*.*

Therefore, for any hierarchical allocation rule $\tilde{a}^h \neq a^h$ *that differs from* a^h *at a positive measure subset of values, it must be that*

$$\int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} \tilde{a}_j^h(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v} < \int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} a_j^h(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v}.$$

Moreover, any allocation rule that is equal to a^h almost everywhere is not implementable. Therefore, a^h is not a randomization over implementable hierarchical allocations, so its outcome is not achievable.

2. Additional Desiderata

In this appendix, we build on the results in Section IV and show that restrictions in addition to symmetry can be placed on the admissible formats in order to constrain the seller. We separately consider three such desiderata– continuity, monotonicity of the payment rule in the bids and the requirement that the implementation have an ex-post IR equilibrium. Throughout this appendix, we consider the case of two bidders ($n = 2$) primarily for the sake of brevity and tractability.[1] A key takeaway from this section is that the optimal auction is no longer always implementable under these additional requirements.

We provide a short summary of the results before the formal presentation that follows. We show that a hierarchical mechanism has a continuous implementation when, loosely speaking, the interim allocation is continuous. A key consequence of this is that the optimal auction may not have a continuous implementation. This is because, when the distributions are 'irregular' (that is, $\phi_i(\cdot)$ is non-monotone), the optimal allocation rule (after 'ironing') may be discontinuous.

We consider two types of monotonicity of the payment rule separately: the payments are increasing in your opponent's bid (as in a second price auction) and in your own bid (as in a first price auction). In each of these cases, we show that the optimal auction may not implementable with this additional restriction. In fact, this is true even for the simple Example 1 with uniform value distributions.

Finally, for the case of ex-post IR implementation, we begin by observing that if an equilibrium bid is made by two different values of the two bidders, the payment corresponding to this bid can never be higher than the lower of the two values. This provides a simple necessary condition (the complete characterization is far more complex) which can be used to demonstrate that, once again, the optimal auction corresponding to the uniform value distribution case of Example 1 does not have an ex-post IR implementation.

We should note in advance that all the proofs that follow are written discussing the possibility or impossibility of pure strategy implementations satisfying the additional desiderata. One may wonder about mixed strategy implementations. Observation 2 above showed that in any mixed strategy implementation, an at most probability 0 set of values for any buyer can be mixing over bids that achieve different probabilities of winning. It follows that to induce the same interim allocation rule, any mixed strategy implementation must induce the same distribution over bids as some pure strategy index rule implementing the allocation rule. Therefore our results extend to implementation in mixed strategies as well. We omit the details in the interests of brevity.

2.1. *Continuity*

The basic construction we used in the example of Section II consisted of discontinuous payment rules where a buyer i's payment discontinuously changed depending on whether their opponent bid above or below the cutoff bid $\hat{b}$. Of course, conditional on winning, the payments in first- and

[1] Barring Propositions 3 and 4, we can extend the results in this section to more than two bidders.

second-price auctions are continuous in the profile of bids (since losers do not pay, payments in these auctions are not continuous unconditionally). We now examine the effect that the additional requirement of continuity has on the set of implementable mechanisms.

A hierarchical mechanism has a symmetric, *continuous implementation* (a^s, p^s) if $p^s(b_i, b_j)$ is continuous in both b_i, b_j.

We show that the existence of a continuous implementation is equivalent to there being no *non-trivial atoms* in the hierarchical mechanism. A non-trivial atom is one in which there is a positive measure of values of buyer i which have the same index b, and this index also lies in the support of the bid space of buyer j. Formally, a nontrivial atom exists if, for a buyer i, there are two distinct values $v_i, v_i' \in V_i$ such that $I_i(v_i) = I_i(v_i') = b$ and $b \in B_j$. Such atoms can occur in natural applications such as the optimal auction when the buyers' value distributions do not satisfy the monotone hazard rate condition.

The absence of nontrivial atoms is a necessary condition for a continuous implementation. To see this, note that, in any symmetric implementation, at such an atom, it must be the case that $\sigma_i(v_i) = \sigma_i(v_i') = \sigma_j(v_j) = b$ for all $v_j \in I_j^{-1}(b)$. In other words, this says that, in any implementation, it must be that all types at the nontrivial atom make the same bid. However, if the payment p^s is continuous, buyer j has an incentive to bid slightly higher than b. Bidding slightly higher would lead to a continuous increase in payment but a discontinuous increase in the probability of winning, so $\sigma_j(v_j)$ is not a best response for v_j. The result below shows that this is the only additional condition required for the existence of a continuous implementation.

Proposition 1. *Suppose that $n = 2$. An implementable hierarchical mechanism (I, p^h) has a continuous implementation if and only if it has no non-trivial atoms.*

The intuition for this result can be easily seen by revisiting the example of the regular optimal auction of Section II. When the virtual values are increasing, the resulting optimal auction does not have non-trivial atoms. The payment rule we constructed was discontinuous in the opponent's bid b_j but can easily be smoothed around the point of the discontinuity while ensuring that the interim payments remain the same. The simplest way to do this is linearly, which is illustrated below in Figure 1. Additionally, when the hierarchical mechanism has no non-trivial atoms, it is also possible to achieve continuity in b_i. The formal proof is constructive.

Proof of Proposition 2. We argue sufficiency first. Suppose the hierarchical mechanism (I, p^h) has no non-trivial atoms.

To begin, note that there exists a hierarchical mechanism (I', p^h) which implements exactly the same ex-post allocation rule and interim payment rule, such that I_i' is strictly increasing for each $i = 1, 2$. To see this, consider any "atom" of buyer i over bid b. By assumption, $I_j^{-1}(b) = \phi$. Define $I_{i'}'(v'') = I_{i'}'(v'') + \epsilon$ for $i' = 1, 2$ and v'' s.t. $I_{i'}(v'') > b$ and some $\epsilon > 0$. Further, "continuously stretch" the $I_i'(v)$ for $v \in I_i^{-1}(b)$ over $[b, b + \epsilon']$ for some $\epsilon' < \epsilon$. Proceed inductively for each atom in I. Note that by construction the ex-post allocation rule implemented by I' is the same as that by I.

Therefore, we now may now suppose I is strictly increasing wlog. Observe that we do not require the index rules to be continuous in values, and therefore there may be jump discontinuities.

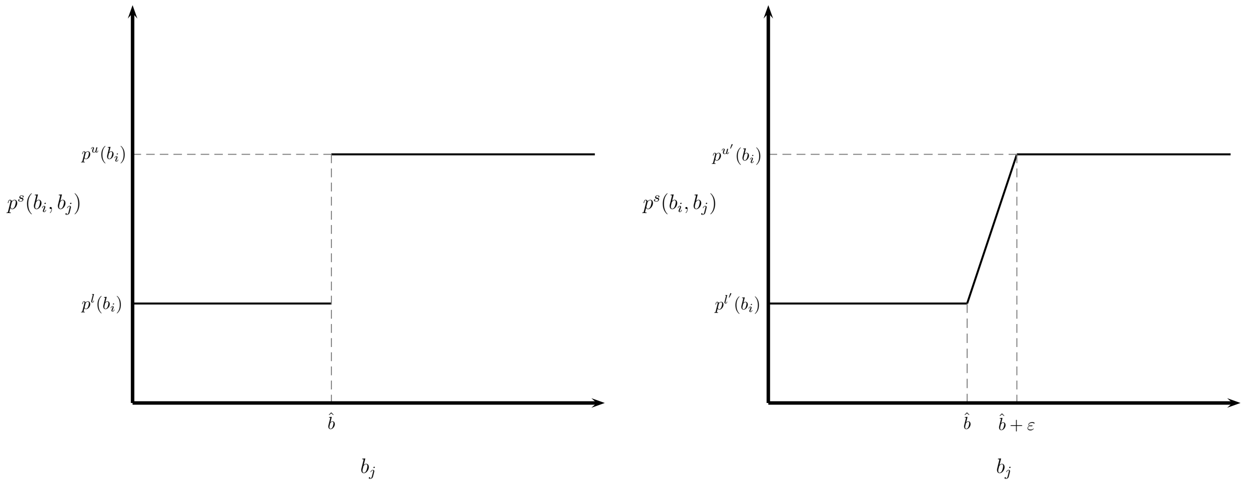

FIGURE 1. Continuous Symmetric Implementation

To ensure continuity in the bid space, we therefore need to be careful about the outcomes at these discontinuities.

Formally, note that since I_1 and I_2 are strictly increasing, G_1 and G_2 do not have any atoms. Further let $\underline{b}_i = I_i(\underline{v}_i)$, $\overline{b}_i = I_i(\overline{v}_i)$. Define $\overline{B}_i = [\underline{b}_i, \overline{b}_i]$, the smallest interval that contains the set of equilibrium bids B_i of buyer i.

Observe that for any $b \in I_i(V_i)$, we require the interim expected payment to be exactly $p_i^h(I_i^{-1}(b))$. We will now extend the desired interim payments for any $b \in \overline{B}_i \backslash B_i$—previously when continuity was not a concern, we set it to be some large payment to deter a deviation. Define it as:

$$p_i^*(b) = \sup_{\beta \in B_i, \beta < b} p_i^h(I_i^{-1}(\beta)) + \overline{v}(b)\left(G_{-i}(b) - \sup_{\beta \in B_i : \beta < b} G_{-i}(b)\right),$$

where $\overline{v}(b) = \inf\{I_i^{-1}(\beta) : \beta \geq b, \beta \in B_i\}$. Note that by construction, $p_i^*(b)$ is continuous in b—the strict increasing-ness of I_1 and I_2 guarantees that G_1 and G_2 are continuous in b. Further note that by construction for $b \in B_i$, $p_i^*(b) = p_i^h(I_i^{-1}(b))$.

We now use this continuous $p_i^*(\cdot)$ to construct an ex-post payment rule that is continuous in both own bid and the opponent's bid. Consider the construction of Theorem 2, with the proviso that the interim expected payment for buyer i bidding $b_i \in \overline{B}_i \backslash B_i$ is $p_i^*(b_i)$.

First consider the case that $b_i \in \overline{B}_i, b_i \notin \overline{B}_j$. In this case the constant payment rule $p^s(b_i, b_j) = p_i^*(b_i)$ for any $b_j \in B_j$ is continuous in both arguments.

Next consider the following payment rule for bids $b_i \in \overline{B}_1 \cap \overline{B}_2$. Let $\hat{b}$ be a bid such that $G_1(\hat{b}) \neq G_2(\hat{b})$. For a given $\varepsilon > 0$, we define $e_i(\varepsilon)$ as

$$e_i(\varepsilon) = \mathbb{E}_i[b_i - \hat{b} \mid \hat{b} \leq b_i \leq \hat{b} + \varepsilon].$$

Consider the payment rule defined as:

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_j \geq \hat{b} + \varepsilon, \\ p^l(b_i) + \frac{b_j - \hat{b}}{\varepsilon}\left(p^u(b_i) - p^l(b_i)\right) & \text{if } \hat{b} \leq b_j < \hat{b} + \varepsilon, \\ p^l(b_i) & \text{if } b_j < \hat{b}. \end{cases}$$

Note that this payment rule is continuous in b_j.

The expected payment for a bid b_i by bidder i is then

$$p^u(b_i)[1 - G_j(\hat{b} + \varepsilon)] + \frac{e_j(\varepsilon)}{\varepsilon}\left(p^u(b_i) - p^l(b_i)\right) + G_j(\hat{b})p^l(b_i),$$

which then yields the following system of equations

$$\begin{bmatrix} 1 - G_2(\hat{b} + \varepsilon) + \frac{e_2(\varepsilon)}{\varepsilon} & G_2(\hat{b}) - \frac{e_2(\varepsilon)}{\varepsilon} \\ 1 - G_1(\hat{b} + \varepsilon) + \frac{e_1(\varepsilon)}{\varepsilon} & G_1(\hat{b}) - \frac{e_1(\varepsilon)}{\varepsilon} \end{bmatrix} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \begin{bmatrix} p_1^*(b_i) \\ p_2^*(b_i) \end{bmatrix}.$$

Note that since $\frac{1-G_2(\hat{b})}{1-G_1(\hat{b})} \neq \frac{G_2(\hat{b})}{G_1(\hat{b})}$ and $e_i(\varepsilon)$ is continuous, there exists a small enough $\varepsilon > 0$ such that

$$\frac{1 - G_2(\hat{b} + \varepsilon) + \frac{e_2(\varepsilon)}{\varepsilon}}{1 - G_1(\hat{b} + \varepsilon) + \frac{e_1(\varepsilon)}{\varepsilon}} \neq \frac{G_2(\hat{b}) - \frac{e_2(\varepsilon)}{\varepsilon}}{G_1(\hat{b}) - \frac{e_1(\varepsilon)}{\varepsilon}}.$$

Finally, note by construction $p_i^*(b_i)$ is continuous in b_i. This in turn implies that p^u and p^l are continuous in b_i which completes the proof of sufficiency.

Next to argue necessity. Consider any hierarchical mechanism (I, p^h) such that I has a non-trival atom. Suppose that buyer 1 has a positive mass on bid b, and some buyer 2 of value $v_2 \in V_2$ also bids b. Bidding $b + \epsilon$ for $\epsilon > 0$, ϵ small will result in a jump in buyer 2's probability of winning the good. However, by continuity of payments requires that buyer 2's payment must increase continuously. This results in a contradiction. □

2.2. *Monotonicity*

Incentive compatibility implies that a buyer's interim payments in any symmetric auction must be nondecreasing in his value. However, the payment rule need not be monotone in an ex-post sense. For instance, in the payment (2) we constructed for the example in Section II, we make neither the restriction that $p^u(b_i) \geq p^l(b_i)$ nor that p^u and p^l are increasing in b_i. In other words, we so far have not restricted ex-post payments from our symmetric implementations to be monotone in either in a buyer's bid or their opponent's bid. Of course, conditional on winning, the payment rules for both first- and second- price auctions are monotone in such an ex-post sense.

We first examine the effect of imposing monotonicity in the opponent's bid, a property of second-price auctions. We say a hierarchical mechanism has a symmetric, *monotone in opponent's bid implementation* (a^s, p^s) if $p^s(b_i, b_j)$ is nondecreasing in b_j. The next result provides necessary and sufficient conditions for such an implementation to exist.

Proposition 2. *Suppose that $n = 2$. An implementable hierarchical allocation mechanism (I, p^h) has a monotone in opponent's bid implementation if and only if whenever G_i first order stochastically dominates G_j, it is the case that $p_i(I_i^{-1}(b)) \leq p_j(I_j^{-1}(b))$ for all $b \in B_1 \cap B_2$.*

Intuition for the sufficiency of the above conditions can be understood by examining payments p^u and p^l in the example of Section II. For this particular construction, monotonicity in the opponent's bid requires that $p^u(b_i) \geq p^l(b_i)$ for all b_i. From (3), this happens if and only if

$$\frac{1}{G_1(\hat{b}) - G_2(\hat{b})}\left(p_1^*\left(I_1^{-1}(b_i)\right) - p_2^*\left(I_2^{-1}(b_i)\right)\right) \geq 0.$$

The sufficiency of our condition is immediate and its necessity is easily established in the proof in the appendix. Observe that if neither distribution first order stochastically dominates the other, there must exist pivot bids $\hat{b}$ and $\hat{b}'$ such that $G_1(\hat{b}) > G_2(\hat{b})$ and $G_1(\hat{b}') < G_2(\hat{b}')$. For bids at which $p_1^*(I_1^{-1}(b_i)) > p_2^*(I_2^{-1}(b_i))$, we can construct the payments p^u and p^l in (3) by pivoting around $\hat{b}$ and similarly we can pivot around $\hat{b}'$ when $p_1^*(I_1^{-1}(b_i)) < p_2^*(I_2^{-1}(b_i))$. This payment rule would satisfy the above inequality and would hence be monotone in the opponent's bid. Note that the condition is not satisfied generically. Intuitively, this is because it is possible to perturb a hierarchical mechanism that violates it to get another mechanism that continues to violate this condition.

Below, we show that Example 1 violates fails this condition. Additionally, we modify Example 3 slightly to show that there are cases where one distribution first order stochastically dominates the other, but the condition is satisfied.

EXAMPLE 1 (Continued): *In this example, $G_1 \sim \mathbb{U}[0,4]$ first order stochastically dominates $G_2 \sim \mathbb{U}[0,2]$. However, for bid $b = 1$ the payments (given by equations 6, 7) are*

$$p_1^*(\phi_1^{-1}(1)) = \frac{9}{8} > \frac{5}{16} = p_2^*(\phi_2^{-1}(1)),$$

violating the condition of the proposition.

EXAMPLE 3 (Continued): *Recall that, in this example, buyer 1's value distribution $F_1 \sim \mathbb{U}[0,1]$ and buyer 2's value distribution $F_2 \sim \mathbb{U}[1,2]$. Now, unlike previously, suppose that the seller subsidizes the bid of buyer 1 by one and a half dollars (instead of one dollar). In this case, $G_1 \sim \mathbb{U}[1.5, 2.5]$ strictly first order stochastically dominates $G_2 \sim \mathbb{U}[1,2]$, so this mechanism is implementable. Additionally, it is easy to show that $p_1(I_1^{-1}(b)) \leq p_2(I_2^{-1}(b))$ for all bids b. Therefore, there is a monotone in opponent's bid implementation for this mechanism.*

Proof of Proposition 3. Consider the payment rule we constructed which pivots around a point $\hat{b}$:

$$\begin{bmatrix} 1 - G_2(\hat{b}) & G_2(\hat{b}) \\ 1 - G_1(\hat{b}) & G_1(\hat{b}) \end{bmatrix} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \begin{bmatrix} p_1^h\left(I_1^{-1}(b_i)\right) \\ p_2^h\left(I_2^{-1}(b_i)\right) \end{bmatrix}.$$

Inverting, we get

$$\begin{aligned} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} &= \frac{1}{G_1(\hat{b}) - G_2(\hat{b})} \begin{bmatrix} G_1(\hat{b}) & -G_2(\hat{b}) \\ -(1 - G_1(\hat{b})) & 1 - G_2(\hat{b}) \end{bmatrix} \begin{bmatrix} p_1^h\left(I_1^{-1}(b_i)\right) \\ p_2^h\left(I_2^{-1}(b_i)\right) \end{bmatrix} \\ &= \frac{1}{G_1(\hat{b}) - G_2(\hat{b})} \begin{bmatrix} G_1(\hat{b}) p_1^h\left(I_1^{-1}(b_i)\right) - G_2(\hat{b}) p_2^h\left(I_2^{-1}(b_i)\right) \\ (1 - G_2(\hat{b})) p_2^h\left(I_2^{-1}(b_i)\right) - (1 - G_1(\hat{b})) p_1^h\left(I_1^{-1}(b_i)\right) \end{bmatrix}. \end{aligned}$$

For monotonicity in the opponents bid, we require that $p^u(b_i) - p^l(b_i) \geq 0$ or, equivalently, that

$$\frac{1}{G_1(\hat{b}) - G_2(\hat{b})}\left(p_1^h\left(I_1^{-1}(b_i)\right) - p_2^h\left(I_2^{-1}(b_i)\right)\right) \geq 0. \tag{1}$$

We can now consider two cases.

Case (1): Neither distribution first order stochastically dominates the other.

This implies that there exist $\hat{b}$ and $\hat{b}'$ such that $G_1(\hat{b}) > G_2(\hat{b})$ and $G_1(\hat{b}') < G_2(\hat{b}')$. This immediately implies that there exists a symmetric implementation.

We can now use these to construct a monotone payment rule. For all b_i where $p_1^h\left(I_1^{-1}(b_i)\right) - p_2^h\left(I_1^{-1}(b_i)\right) > 0$, we pivot the payment around $\hat{b}$. Similarly, for all b_i where $p_1^h\left(I_1^{-1}(b_i)\right) - p_2^h\left(I_1^{-1}(b_i)\right) < 0$, we pivot the payment around $\hat{b}'$. This will ensure that (1) is satisfied and hence that payments are monotone.

Case (2): One of the distributions first order stochastically dominates the other, wlog G_1 first order stochastically dominates G_2.

We first show that in this case there exists a symmetric monotone implementation. If $G_1 = G_2$, then we are done. If not then we can take any $\hat{b}$ such that $G_1(\hat{b}) < G_2(\hat{b})$ and construct the usual payment rule. Clearly, this condition implies that (1) will be satisfied.

We now show the converse. Assume without loss of generality that G_1 strictly first order stochastically dominates G_2. Now suppose, in contradiction, that there is a $b \in B_1 \cap B_2$ such that $p_1^h(I_1^{-1}(b)) > p_2^h(I_2^{-1}(b))$ and that there is a symmetric and monotone implementation p^s. First order stochastic dominance would then imply that

$$p_1^h(I_1^{-1}(b)) = \int_{B_2} p^s(b, b_2)dG_2(b_2) \leq \int_{B_1} p^s(b, b_1)dG_1(b_1) = p_2^h(I_2^{-1}(b)),$$

which isn't possible. This completes the proof. □

We now examine the effect of requiring monotonicity in a buyer's own bid. We say a hierarchical mechanism has a symmetric, *monotone in own bid implementation* (a^s, p^s) if $p^s(b_i, b_j)$ is nondecreasing in b_i. This requirement restricts the relative rates at which the interim payments of both buyers can increase in their bids for any implementable mechanism. Put differently, if one buyer's payment increases very rapidly, then this monotonicity requirement will place a lower bound on the rate at which the other buyer's payment must increase.

For simplicity, the characterization restricts attention to hierarchical mechanisms with strictly increasing and differentiable index functions—this ensures that the implied distribution over bids for any buyer has a density. Moreover, the characterization involves slightly different necessary and sufficient conditions, as we have been unable to derive a single characterizing condition. The sufficient condition involves the slopes $\frac{dp_i^h(I_i^{-1}(b_i))}{db_i}$ of the payments on the common part of the supports of the bid spaces $B_1 \cap B_2$. Since $B_1 \cap B_2$ is a closed interval, these derivatives refer to the left (right) derivative at the upper (lower) bound of the support.

Proposition 3. *Suppose that $n = 2$. Consider an implementable hierarchical mechanism (I, p^h) with differentiable and strictly increasing index functions. (I, p^h) has a monotone in own bid implementation if,* *for all distinct $i, j \in \{1,2\}$ and for all $b \in B_1 \cap B_2$ for which $\frac{dp_i^h(I_i^{-1}(b))}{db} > 0$, we have*

$$\begin{aligned} &\inf\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\} \\ \leq\ & \frac{\frac{dp_j^h(I_j^{-1}(b))}{db}}{\frac{dp_i^h(I_i^{-1}(b))}{db}} \\ \leq\ & \sup\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\}, \end{aligned} \tag{2}$$

with a strict lower (upper) inequality unless the corresponding infimum (supremum) is reached on a set of bids with positive G_j (G_i) mass.

Conversely, (I, p^h) has a monotone in own bid implementation only if, for all distinct $b, b' \in B_1 \cap B_2$, we have

$$\begin{aligned} &\inf\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\} \\ \leq\ & \frac{p_j^h(I_j^{-1}(b')) - p_j^h(I_j^{-1}(b))}{p_i^h(I_i^{-1}(b')) - p_i^h(I_i^{-1}(b))} \\ \leq\ & \sup\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\}, \end{aligned} \tag{3}$$

with a strict lower (upper) inequality unless the corresponding infimum (supremum) is reached on a set of bids with positive G_j (G_i) mass.

The astute reader might observe that, on the surface, it seems that the necessary condition is stronger than the sufficient condition in the above proposition (divide the numerator and denominator of the central term of (3) by $b' - b$ and take the limit $b' \to b$ to get the same central term of (2)). However, consider a case where neither the infinimum nor supremum is achieved on a set of positive mass. Further, suppose both inequalities in (3) are satisfied strictly for every pair b', b. It may still be the case that for some b, one of the inequalities in (2) may be satisfied only as an equality. In this case, the sufficient condition will be violated, while the necessary condition is satisfied. This discussion also demonstrates that the gap between these conditions is (loosely speaking) quite small.

Note that the above conditions are always satisfied whenever neither bid space B_1 or B_2 is a subset of the other. In that case, for distinct $i, j \in \{1,2\}$, we get

$$\inf\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\} = 0 \text{ and } \sup\left\{\frac{g_i(\tilde{b})}{g_j(\tilde{b})} : \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_i(\tilde{b}) + g_j(\tilde{b}) > 0\end{array}\right\} = \infty,$$

because there are bids $b_i \in B_i$, $b_j \in B_j$ that do not lie in the intersection $b_i, b_j \notin B_i \cap B_j$. Note that, once again, the necessary condition is not generically satisfied (the intuition is identical to the monotone in opponent's bid case) and below, we show that it is, in particular, not satisfied by Example 1.

EXAMPLE 1 (Continued): *Since* $G_1 \sim \mathbb{U}[0,4]$ *and* $G_2 \sim \mathbb{U}[0,2]$, *we have*

$$\inf\left\{\frac{g_2(\tilde{b})}{g_1(\tilde{b})} \;:\; \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_1(\tilde{b}) + g_2(\tilde{b}) > 0\end{array}\right\} = 0 \text{ and } \sup\left\{\frac{g_2(\tilde{b})}{g_1(\tilde{b})} \;:\; \begin{array}{l}\tilde{b} \in B_1 \cup B_2, \\ g_1(\tilde{b}) + g_2(\tilde{b}) > 0\end{array}\right\} = 2.$$

For bids $b' = 1$ *and* $b = 0$, *the necessary condition is violated because*

$$\frac{p_1^*(\phi_1^{-1}(1)) - p_1^*(\phi_1^{-1}(0))}{p_2^*(\phi_2^{-1}(1)) - p_2^*(\phi_2^{-1}(0))} = \frac{9/8}{5/16} > 2.$$

Proof of Proposition 4. First note that:

$$\begin{aligned}\frac{\mathrm{d}p_i^h(I_i^{-1}(b))}{\mathrm{d}b} &= \frac{\mathrm{d}p_i^h(v_i)}{\mathrm{d}v_i}\bigg|_{v_i=I_i^{-1}(b)} \frac{\mathrm{d}I_i^{-1}(b)}{\mathrm{d}b} \\ &= v_i\frac{\mathrm{d}a_i(v_i)}{\mathrm{d}v_i}\frac{1}{I_i'(v_i)}\Big|_{v_i=I_i^{-1}(b)} \\ &= v_i\frac{\mathrm{d}G_j(I_i(v_i))}{\mathrm{d}v_i}\frac{1}{I_i'(v_i)}\Big|_{v_i=I_i^{-1}(b)} \\ &= v_i g_j(I_i(v_i))\Big|_{v_i=I_i^{-1}(b)} \\ &= I_i^{-1}(b)g_j(b).\end{aligned}$$

Since I_j is strictly increasing and differentiable by assumption, $g_j(b)$ exists and therefore, so does $\frac{\mathrm{d}p_i^h(I_i^{-1}(b))}{\mathrm{d}b}$.

Sufficiency. To see that the conditions are sufficient, recall that any symmetric implementation $p^s(b,b')$ must be such that for any $b \in B_1 \cap B_2$

$$\begin{aligned}\int_{B_2} p^s(b,b')g_2(b')db' &= p_1^h(I_1^{-1}(b)), \\ \int_{B_1} p^s(b,b')g_1(b')db' &= p_2^h(I_2^{-1}(b)).\end{aligned}$$

Note that $p_i^h(I_i^{-1}(b))$ is non-decreasing in b by assumption for $i = 1,2$. Therefore for an implementation that is monotone in own bid, it is sufficient to find p^s such that:

$$\int_{B_2} \frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b}g_2(b')db' = \frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}, \tag{4a}$$

$$\int_{B_1} \frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b}g_1(b')db' = \frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}, \tag{4b}$$

where $\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} \geq 0$.

If both $\frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}$ and $\frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}$ equal 0, setting $\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} = 0$ solves (4). So suppose not. Without loss suppose $\frac{\mathrm{d}p_2(I_2^{-1}(b)}{\mathrm{d}b} \neq 0$, the other case follows symmetrically.

Pick a set $\underline{B}$ with $G_1(\underline{B}) > 0$ such that $\frac{g_2(b')}{g_1(b')} \leq \frac{\frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}}$ for all $b' \in \underline{B}$. Similarly pick a set $\overline{B}$ with $G_2(\overline{B}) > 0$ such that $\frac{\frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}} \geq \frac{g_2(b')}{g_1(b')}$ for all $b' \in \underline{B}$.

Finally consider a p^s s.t.:

$$\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} = \begin{cases} \bar{x} & b' \in \overline{B} \\ \underline{x} & b' \in \underline{B} \\ 0 & \text{otherwise} \end{cases}$$

Substituting into (4):

$$\underline{x}G_2(\underline{B}) + \overline{x}G_2(\overline{B}) = \frac{\mathrm{d}p_1^h(I_1^{-1}(b))}{\mathrm{d}b},$$
$$\underline{x}G_1(\underline{B}) + \overline{x}G_1(\overline{B}) = \frac{\mathrm{d}p_2^h(I_2^{-1}(b))}{\mathrm{d}b}.$$

By construction, therefore $\underline{x}$ and $\overline{x}$ must be positive. Formally, notice that if this system does not have a non-negative solution, then the Farkas alternative of:

$$y_1\frac{G_2(\underline{B})}{G_1(\underline{B})} + y_2 \geq 0,$$
$$y_1\frac{G_2(\overline{B})}{G_1(\overline{B})} + y_2 \geq 0,$$
$$y_1\frac{\frac{\mathrm{d}p_1(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2(I_2^{-1}(b))}{\mathrm{d}b}} + y_2 < 0,$$

must have a solution. Clearly this is impossible by assumption since $\frac{\frac{\mathrm{d}p_1(I_1^{-1}(b))}{\mathrm{d}b}}{\frac{\mathrm{d}p_2(I_2^{-1}(b))}{\mathrm{d}b}} \in \left[\frac{G_2(\underline{B})}{G_1(\underline{B})}, \frac{G_2(\overline{B})}{G_1(\overline{B})}\right]$.

Therefore, by construction, we have shown that $\frac{\mathrm{d}p^s(b,b')}{\mathrm{d}b} \geq 0$ for all b'.

Necessity. Once again, recall that any symmetric implementation $p^s(b',b)$ must be such that for any $b' \in B_1 \cap B_2$

$$\int_{B_2} p^s(b',b)g_2(b)db = p_1^h(I_1^{-1}(b')),$$
$$\int_{B_1} p^s(b',b)g_1(b)db = p_2^h(I_2^{-1}(b')).$$

Suppose the condition (3) is violated for some $b'' > b'$. Any symmetric implementation must satisfy:

$$\int_{B_2} (p^s(b'',b) - p^s(b',b))g_2(b)db = p_1^h(I_1^{-1}(b'') - p_1^h(I_1^{-1}(b')), \tag{5a}$$
$$\int_{B_1} (p^s(b'',b) - p^s(b',b))g_1(b)db = p_2^h(I_2^{-1}(b'') - p_2^h(I_2^{-1}(b')). \tag{5b}$$

Analogous to the Farkas Lemma argument above, when (3) is violated, there cannot exist a solution to (5) such that $p^s(b'',b) - p^s(b',b) \geq 0$, for all $b \in B_1 \cup B_2$. □

2.3. *Ex-Post IR*

While the symmetric implementations we construct for Theorem 2 are by definition IR, they are IR in an interim sense. As we have argued above, the equilibrium however need not be IR in an ex-post sense: certain bid profiles may result in losing bidders having to make payments or winners having to pay more than their valuation. This is unappealing and may result in certain bidders choosing not to participate. Perhaps more importantly, this may result in non-payment by budget-constrained bidders. This is because a bidder's valuation may reflect her ability to pay for the good. Additionally, certain bidders who plan to pay by obtaining a loan may be unable to obtain credit upon losing the auction.[2]

Formally, we say that a hierarchical mechanism (I, p^h) has a symmetric, *ex-post IR implementation* (a^s, p^s) with associated equilibrium strategies σ, if for all $\mathbf{v} \in \mathbf{V}$ and $i \in N$, we have $p^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right) \leq v_i a^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right).$

This states that at any bid profile that occurs in equilibrium, winning buyers are never charged more than their value and losers do not have to make payments, although they may receive subsidies (which implies that losers may not be inactive). Notice that, when there are ties, the above inequality implies that buyers only have to pay in the event that they win.

The ex-post IR requirement places a bound on the payments that the symmetric auction can require buyers to make at both winning and losing bids. As in the case with inactive losers implementation, the optimal auction may not have an ex-post IR implementation. In fact, this can be demonstrated by once again revisiting Example 1.

EXAMPLE 1 (Continued): *Recall that buyer* 1 *with value* $v_1 = 3$ *has a virtual value of* $\phi_1(3) = 2$*, always wins the good, and pays* $p_1^*(3) = \frac{5}{2}$*. For there to be a symmetric ex-post IR implementation, there must exist a symmetric payment* p^s *such that*

$$\int_0^2 p^s(2, b_2) dG_2(b_2) = \frac{5}{2}$$

which in turn implies that there must exist at least one $b \in [0, 2]$ *such that*

$$p^s(2, b) \geq \frac{5}{2}.$$

However, note that a buyer 2 *with value* $v_2 = 2$ *also has the virtual value* $\phi_2(2) = 2$*. Since there is a* $b \in [0, 2]$ *such that* $p^s(2, b) \geq \frac{5}{2}$*, there will be a bid profile in the support of the equilibrium bids at which buyer 2 is paying more than her value. This violates the ex-post IR requirement.*

We derive necessary and sufficient conditions for a hierarchical mechanism to admit a symmetric ex-post IR implementation. Due to the complexity of the characterization, we need to make two additional assumptions. As in Proposition 3, we first restrict attention to hierarchical mechanisms (I, p^h) in which the index functions I are differentiable and strictly increasing. Second, we further restrict attention to the case where the lower bounds of the supports of the bid space do not coincide, or $I_1(\underline{v}_1) \neq I_2(\underline{v}_2)$. The characterization for this case is easier to state. In the appendix, we present the characterization for allocation rules in which $I_1(\underline{v}_1) = I_2(\underline{v}_2)$.

[2] If we were to take the procurement interpretation of our model, the ex-post IR requirement would ensure that firms can cover their costs and complete the project.

Without loss of generality, we assume that bidder 1's bid space has the lower support, or

$$I_1(\underline{v}_1) = \underline{b}_1 < \underline{b}_2 = I_2(\underline{v}_2).$$

Additionally, we define $I_i(\overline{v}_i) = \overline{b}_i$ for $i \in \{1, 2\}$ and

$$v(b) \equiv \min\{I_1^{-1}(b), I_2^{-1}(b)\} \qquad \text{for } b \in B_1 \cap B_2,$$

as the lower of the values of the two buyers corresponding to a bid b that lies in both bid spaces. Recall that, since we have restricted attention to strictly increasing index functions, this inverse is well defined.

We can now state a simple first necessary condition that a hierarchical mechanism (I, p^h) must satisfy to have an ex-post IR implementation.

Condition C1: The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2 : \quad v(b)G_2(b) \geq p_1^h\left(I_1^{-1}(b)\right). \tag{C1}$$

This is an intuitive necessary condition. $v(b)$ is the maximum amount that can be charged to a winning buyer who bids $b \in B_1 \cap B_2$ and whose opponent bids $b' \in B_1 \cap B_2$, $b' \leq b$. Since the auction is symmetric, such a profile of bids will not reveal the identity of the winning bidder, so the ex-post IR requirement restricts the payment to be lower than both possible values of the winning bidder. Hence, bidder 1's interim payment $p_1^h\left(I_1^{-1}(b)\right)$ cannot be higher than $v(b)G_2(b)$ for any bid $b \in B_1 \cap B_2$. Note that the necessity of this condition does not hinge on the lower bounds of the supports of the bid spaces being different, and C1 will continue to remain necessary when $\underline{b}_1 = \underline{b}_2$. We revisit Example 1 yet again and show that it violates this condition.

EXAMPLE 1 (Continued): *Once again, consider buyer* 1 *with value* $v_1 = 3$*, at which the interim payment is* $p_1^*(3) = \frac{5}{2}$*. At the bid* $\phi_1(3) = 2$*, Condition C1 is violated because*

$$v(2) = \min\{\phi_1^{-1}(2), \phi_2^{-1}(2)\} = \min\{3, 2\} = 2,$$

and hence,

$$v(2)G_2(2) = 2 < p_1^*\left(\phi_1^{-1}(2)\right) = \frac{5}{2}.$$

It remains to derive a similar condition for the interim payment of buyer 2, which accounts for the fact that the lower bounds of the supports of the bid distributions differ ($\underline{b}_1 < \underline{b}_2$). Suppose that one buyer bids $b \in B_1 \cap B_2$ while the other bid is in $[\underline{b}_1, \underline{b}_2)$. Then, it is clear that the buyer bidding b is buyer 2, so payments in this range of bids can be chosen to be up to her value $I_2^{-1}(b)$ which may be higher than $v(b)$. By contrast, when buyer 1 bids b, she can never be charged more than $v(b)$ even if her value $I_1^{-1}(b)$ is strictly greater. This argument yields an analogous necessary condition for buyer 2.

Condition C1': The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2 : \quad v(b)\left(G_1(b) - G_1(\underline{b}_2)\right) + I_2^{-1}(b)G_1(\underline{b}_2) \geq p_2^h\left(I_2^{-1}(b)\right). \tag{C1'}$$

However, conditions C1 and C1′ together need not be sufficient. This is because ensuring the appropriate interim payment for buyer 1 places a bound on the amount that can be extracted from buyer 2 from bids that lie in the common support $B_1 \cap B_2$. Suppose that, at a bid b, the interim payment $p_1^h\left(I_1^{-1}(b)\right)$ of buyer 1 is substantially lower than that of buyer 2, which is $p_2^h\left(I_2^{-1}(b)\right)$. This may prevent the seller from extracting the entire expected payment $v(b)[G_1(b) - G_1(\underline{b}_2)]$ from buyer 2 when buyer 1's bids lie in the range $[\underline{b}_2, b]$.

Hence, we need to derive the maximum payment $\eta(b) \leq v(b)[G_1(b) - G_1(\underline{b}_2)]$ that can be extracted symmetrically from buyer 2 when (i) she bids $b \in B_1 \cap B_2$, (ii) positive payments are only taken when b is the winning bid (i.e., the other buyer's bids are in the range $[\underline{b}_2, b]$) and (iii) buyer 1's expected payment from bid b is $p_1^h\left(I_1^{-1}(b)\right)$.

In words, we need to define payments for bids $b \in B_1 \cap B_2$ in a way that maximizes the amount extracted from buyer 2 while ensuring that buyer 1's expected payment remains $p_1^h\left(I_1^{-1}(b)\right)$. If this amount extracted is greater than the required payment $p_2^h\left(I_2^{-1}(b)\right)$ for buyer 2, subsidies can always be provided when buyer 1's bids lie in the range $[\underline{b}_1, \underline{b}_2)$ because, in equilibrium, such bids can only come from buyer 1.

We now need some additional notation. First, we define the following function for $b \in B_2$, which depends on the ratios of the densities:

$$L(b) = \begin{cases} \infty & \text{if } g_1(b) = g_2(b) = 0, \\ \frac{g_1(b)}{g_2(b)} & \text{otherwise.} \end{cases}$$

That is, $L(\cdot)$ is the likelihood ratio of a buyer bidding b being buyer 1 versus buyer 2. Further, we define

$$\underline{\ell} \equiv \min_{b \in B_2} \{ L(b) \}.$$

This is the lowest value of the likelihood ratio for bids in B_2. Since index functions are assumed to be differentiable and strictly increasing, densities g_1 and g_2 are well defined and continuous on B_1 and B_2 respectively. As a result, $\underline{\ell}$ is well defined and positive when $\overline{b}_2 \leq \overline{b}_1$ and 0 when $\overline{b}_2 > \overline{b}_1$.

Additionally, we define the sets

$$\gamma(\ell) \equiv \left\{ b \in B_2 \,\middle|\, L(b) \leq \ell \right\}$$

as the set of bids less than b where the likelihood ratio is at most ℓ, and

$$\bar{\bar{\gamma}}(\ell) \equiv \left\{ b \in B_2 \,\middle|\, L(b) = \ell \right\}$$

similarly as the set of bids less than b where the likelihood ratio is exactly ℓ. These sets will be useful to describe payment rules that derive $\eta(b)$. To obtain $\eta(b)$, we concentrate the maximum payment $v(b)$ on bids that are more likely to lie in the bid space of buyer 1 relative to that of buyer 2, and buyer 1's interim payment is then guaranteed by providing a subsidy at bids that are least likely.

When Condition C1 holds, that is, when $v(b)G_2(b) \geq p_1^h\left(I_1^{-1}(b)\right)$, the following two cases are mutually exclusive and exhaustive for any $b \in B_1 \cap B_2$.[3]

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) > 0 \qquad \text{OR} \qquad v(b)G_2(b) = p_1^h\left(I_1^{-1}(b)\right). \tag{Case 1}$$

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \qquad \text{AND} \qquad v(b)G_2(b) > p_1^h\left(I_1^{-1}(b)\right). \tag{Case 2}$$

$\eta(b)$ needs to be derived separately for each of these two cases, and hence, we analyze them separately below.

Case 1. Let $\hat{B}$ be a subset of $\bar{\bar{\gamma}}(\underline{\ell})$ such that

$$v(b)G_2\left([\underline{b}_2, b]\backslash\hat{B}\right) \geq p_1^h\left(I_1^{-1}(b)\right).$$

If $v(b)G_2(b) = p_1^h\left(I_1^{-1}(b)\right)$, then $\hat{B}$ must be a G_2-null set else consider any set $\hat{B}$ that satisfies the above inequality and has a strictly positive measure.

We now define a payment rule,

$$\hat{p}(b,b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b]\backslash\hat{B}, \\ s & \text{for } b' \in \hat{B}, \\ 0 & \text{for } b' \in B_2 \text{ and } b' \notin \left([\underline{b}_2, b] \cup \hat{B}\right). \end{cases} \tag{C2,P1}$$

where s is chosen to solve

$$v(b)G_2\left([\underline{b}_2, b]\backslash\hat{B}\right) + sG_2\left(\hat{B}\right) = p_1^h\left(I_1^{-1}(b)\right).$$

Note that s here is a subsidy. We set

$$\eta(b) = \int_{\underline{b}_2}^{\bar{b}_2} \hat{p}(b,b')dG_1(b'). \tag{6}$$

Observe that $\eta(b)$ does not depend on the choice of $\hat{B}$. In addition, observe that, when $\bar{b}_2 > \bar{b}_1$, then $\hat{B} \subset (\bar{b}_1, \bar{b}_2]$ and $\eta(b) = v(b)$.

Case 2. Since $G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0$, it must be that $\bar{b}_2 \leq \bar{b}_1$. Here, we define the payment rule $\hat{p}_\ell$ for $\ell > \underline{\ell}$ as follows:

$$\hat{p}_\ell(b,b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b]\backslash\gamma(\ell), \\ s & \text{for } b' \in \gamma(\ell), \\ 0 & \text{for } b' \in B_2 \text{ and } b' \notin \left([\underline{b}_2, b] \cup \hat{B}\right). \end{cases} \tag{C2,P2}$$

where s is chosen to solve

$$v(b)G_2\left([\underline{b}_2, b]\backslash\gamma(\ell)\right) + sG_2(\gamma(\ell)) = p_1^h\left(I_1^{-1}(b)\right).$$

Note that, for ℓ close to $\underline{\ell}$, s is negative, so the payment rule $\hat{p}_\ell$ is ex-post IR. Define:

$$\eta_\ell(b) = \int_{\underline{b}_2}^{\bar{b}_2} \hat{p}_\ell(b,b')dG_1(b'), \tag{7}$$

[3] Recall that we use G_i to represent both a measure and a CDF.

and let

$$\eta(b) = \lim_{\ell \downarrow \underline{\ell}} \left[\eta_\ell(b)\right].$$

We can now define the second condition.

Definition 2.1 (Condition C2)**.** The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2 : \quad \eta(b) + I_2^{-1}(b) G_1(\underline{b}_2) \geq p_2^h\left(I_2^{-1}(b)\right), \tag{C2}$$

with the inequality holding strictly for any b such that

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \text{ and } v(b) G_2(b) > p_1^h\left(I_1^{-1}(b)\right).$$

The following proposition states that the two conditions C1 and C2 are necessary and sufficient for a symmetric ex-post IR implementation.

Proposition 4. *Suppose that $n = 2$. Consider an implementable hierarchical allocation mechanism (I, p^h) with differentiable and strictly increasing index functions such that the lower bounds of the supports of the bid distributions differ; that is, $\underline{b}_1 < \underline{b}_2$. Then, Conditions C1 and C2 are necessary and sufficient for there to exist a symmetric, ex-post IR implementation of (I, p^h).*

We end this section by observing that Proposition 4 can be adapted to accommodate entry fees. In many practical situations, auctions are often conducted in two steps: buyers first pay to participate, following which the auction is conducted. Such entry fees can relax ex-post IR constraints of the auction itself, as buyers are making a part of the payment before participating. In particular, if the seller could charge a high enough entry fee, he would not need the buyers to make payments in the auction and could offer rebates instead. Having sunk the entry cost, ex-post IR would then be obtained automatically. Conditions C1 and C2 can be appropriately weakened to accommodate a given entry fee; the construction in this section can simply be altered so that winning bidder never pays more than her value plus the fee and the loser never has to pay more than the fee.

Proof of Proposition 5. We will first prove the theorem as stated for differentiable and strictly increasing index rules (that is, no atoms in G_i). Later, we will extend to the more general case.

Proof. We first demonstrate sufficiency, and then argue necessity.

Sufficiency. For simplicity, we will only define payments for equilibrium bids; off-equilibrium bids can be discouraged in the same way as in the proof of Theorem 2. We consider the two cases of Condition C2 separately. For (Case 1) we consider the following payment rule:

$$p^s(b, b') = \begin{cases} \frac{1}{G_1(\underline{b}_2)}\left[p_2^h(I_2^{-1}(b)) - \eta(b)\right] & \text{for } b' \in [\underline{b}_1, \underline{b}_2) \\ \hat{p}(b, b') & \text{for } b' \notin [\underline{b}_1, \underline{b}_2) \end{cases}$$

where $\hat{p}$ is given by (C2,P1), and $\eta(b)$ is given by (6).

Similarly, (Case 2) we consider the following payment rule:

$$p^s(b,b') = \begin{cases} \frac{1}{G_1(\underline{b}_2)}\left[p_2^h(I_2^{-1}(b)) - \eta_l(b)\right] & \text{for } b' \in [\underline{b}_1, \underline{b}_2) \\ \hat{p}_l(b,b') & \text{for } b' \notin [\underline{b}_1, \underline{b}_2) \end{cases}$$

where in this case $\hat{p}_\ell$ is given by (C2,P2) for a given $\ell > \underline{\ell}$, and $\eta_l(b)$ is as defined in (7). From Condition C2 and continuity there is a ℓ close enough to $\underline{\ell}$ for which $p^s(b,b') < v(b)$ for $b' \in [\underline{b}_1, \underline{b}_2)$.

By construction, for each buyer, his expected payment will equal his interim payment in the hierarchical mechanism. Condition (C2) guarantees that in the range $b' \in [\underline{b}_1, \underline{b}_2)$, the implementation still satisfies ex-post IR, $p^s(b,b') \leq I_2^{-1}(b)$. Indeed it might be the case that $p^s(b,b') < 0$.

Necessity. We first verify these conditions are necessary for implementation in a symmetric auction where bidding the actual index rule is each buyer's equilibrium strategy. We then show that the same conditions also rule out other implementations as well.

Let us verify that C1 is necessary. So suppose not, i.e. suppose: $v(b)G_2(b) < p_1^h(I_1^{-1}(b))$ for some $b \in B_1 \cap B_2$. Note that if a buyer bids b, and the other bidder bids $b' \in B_1 \cap B_2$, the maximum she can be asked to pay without violating ex-post IR is $v(b)$. But now, for bidder 1, it follows that the maximum expected payment that she can be asked to make is $v(b)G_2(b)$. If her required payment, $p_1^h(I_1^{-1}(b))$, exceeds this, then there cannot be a symmetric, ex-post IR implementation.

With buyer 2, there is a little more 'wriggle room.' When buyer 2 bids $b \in B_1 \cap B_2$, she could be a winner in some 'asymmetric' profiles; i.e. when buyer 1 bids in the range $[\underline{b}_1, \underline{b}_2)$. At these bid profiles, a potentially higher payment (up to $I_2^{-1}(b)$) can be extracted from buyer 2. Condition (C2) then guarantees that the required interim payment, $p_2^h(I_2^{-1}(b))$ can be extracted.

Note that in the construction of either $\hat{p}$ or $\hat{p}_\ell$, either the maximum permissible amount $v(b)$ is being paid by the *winning* buyer, or a rebate of s is being returned to the buyer who bids b. The rebates are being paid when the other buyer's bid b' has the lowest possible value of $L(b')$. This means that the rebates are worth the lowest possible in expectation to a winning buyer 2, because they occur where $L(\cdot)$ is minimized.

We begin by considering the following maximization problem for $b \in B_1 \cap B_2$ and any given $s \leq 0$:

$$m_s(b) = \max_{\varrho(\cdot)} \int_{\underline{b}_2}^{\bar{b}_1} \varrho(b') dG_1(b'), \tag{Max-P}$$

$$\text{s.t. } \int_{\underline{b}_2}^{\bar{b}_2} \varrho(b') dG_2(b') = p_1^h(I_1^{-1}(b)), \tag{λ}$$

$$s \leq \varrho(b') \leq v(b), \qquad \forall b' \in [\underline{b}_2, b], \tag{$\delta(b'), \kappa(b')$}$$

$$s \leq \varrho(b') \leq 0, \qquad \forall b' \in (b, \max\{\bar{b}_1, \bar{b}_2\}]. \tag{$\delta(b'), \kappa(b')$}$$

To understand this optimization program in words, fix a bid b. Think of $\varrho(\cdot)$ as the payment made by the buyer in this case as a function of the other buyer's bid. The program asks what the maximum expected payment that can be extracted from buyer 2 is subject to constraints we describe next. The first constraint requires that the expected payment of buyer 1 under $\varrho(\cdot)$ is his correct interim payment. The latter two constraints require that $\varrho(\cdot)$ is pointwise bounded below by s and bounded above by the maximum possible ex-post IR payment $v(b)$ when winning and 0 when losing. The terms in the parentheses to the right of the constraints denote the corresponding dual (co-state) variables.

We claim that $\lim_{s\downarrow-\infty} m_s(b) = \eta(b)$. When $v(b)G_2(b) = p_1^h(I_1^{-1}(b))$, then $\varrho(b') = v(b)$ for all $b' \in [\underline{b}_2, b]$ is the only feasible function, so this case is trivial. Hence, we focus on the case $v(b)G_2(b) > p_1^h(I_1^{-1}(b))$.

The Hamiltonian in this case is:

$$\begin{aligned} & g_1(b') - \lambda g_2(b') + \delta(b') - \kappa(b') = 0, \\ \Longrightarrow\ & \frac{g_1(b')}{g_2(b')} - \lambda + \frac{1}{g_2(b')}(\delta(b') - \kappa(b')) = 0 \end{aligned}$$

with complementary slackness conditions:

$$\begin{aligned} & \delta(b')(s - \varrho(b')) = 0, \\ \text{for } b' \in [\underline{b}_2, b], \quad & \kappa(b')(v(b) - \varrho(b')) = 0, \\ \text{for } b' \in (b, \max\{\overline{b}_1, \overline{b}_2\}], \quad & \kappa(b')\varrho(b') = 0, \\ \text{and} \quad & \delta(b'), \kappa(b') \geq 0. \end{aligned}$$

By observation, the solution to this for any s is 'bang bang', i.e.

$$\varrho(b') = \begin{cases} s & \text{if } L(b') \leq \lambda^\star, \\ v(b) & \text{if } L(b') > \lambda^\star, b' \in [\underline{b}_2, b], \\ 0 & \text{otherwise.} \end{cases}$$

with $\lambda^\star$ selected such that the corresponding primal equation binds for $\varrho(\cdot)$ selected thus. The corner case that needs care is when $G_2(\bar{\bar{\gamma}}(\underline{\ell})) > 0$. In this case, there is a positive measure of $b' \in [\underline{b}_2, \overline{b}_2]$ such that $L(b') = \underline{\ell}$. Here, the solution is bang bang, but possibly (depending on s), there is $\hat{B} \subseteq \bar{\bar{\gamma}}(\underline{\ell})$ such that

$$\varrho(b') = \begin{cases} s & \text{if } b' \in \hat{B} \subseteq \bar{\bar{\gamma}}(\underline{\ell}), \\ v(b) & \text{if } b' \in [\underline{b}_2, b] \backslash \hat{B}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows by construction, therefore, that $\lim_{s\downarrow-\infty} m_s(b) = \eta(b)$. Therefore, subject to the payment rule extracting the appropriate interim payment $p_1^h(I_1^{-1}(b))$ when buyer 1 bids b, $\eta(b)$ is the maximum expected payment that can be extracted from buyer 2 when she bids b and buyer 1 makes a bid higher than $\underline{b}_2$. It follows therefore that if inequality (C2) is violated, there cannot be an implementation satisfying both symmetry and ex-post individual rationality.

Next, consider any other mechanism (I', p^h) with a differentiable index rule, that implements the same mechanism. Then, it must be that

$$I_i'(v_i) = \Gamma(I_i(v_i)).$$

for some differentiable and strictly increasing function $\Gamma : \mathbb{R} \to \mathbb{R}$. Note that the resulting distribution on bids, which we shall denote by G_i', is

$$G_i'(\Gamma(b)) = G_i(b).$$

Note that this implies that

$$g_i'(\Gamma(b))\Gamma'(b) = g_i(b).$$

Our previous arguments already imply that Conditions C1 and C2, written in terms of G_i''s are necessary for an implementation. By the equations above, we see that for $b \in B_1 \cap B_2$

$$v(b)G_2'(\Gamma(b)) \geq p_1^h(I_1^{-1}(b)) \implies v(b)G_2(b) \geq p_1^h(I_1^{-1}(b)).$$

Also, for any $b \in B_2$,

$$L(b) = \frac{g_1(b)}{g_2(b)} = \frac{g_1'(\Gamma(b))}{g_2'(\Gamma(b))}.$$

Therefore our conditions in terms of the original G_i's are necessary for any pure strategy implementation. □

Weakly increasing index rules. So far we have only considered strictly increasing index rules. If the index rules are not strictly increasing, the corresponding bid distributions will have atoms. Denote by $\mathcal{B}_i$ the atoms in G_i. For $b_i \in \mathcal{B}_i$, the size of the atom is $G_i(\{b_i\})$—recall that this is a measure and not a density. Further, $I_i^{-1}(\cdot)$ may be correspondence— $v(\cdot)$ may not be well defined. Redefine $v(b)$ as

$$v(b) = \inf\{v \in I_1^{-1}(b) \cup I_2^{-1}(b)\}.$$

Note that when $I_1^{-1}(b)$ and $I_2^{-1}(b)$ are singletons, this is the same as the old definition of $v(b)$. Now Condition C1 will be as before with this extended definition of $v(\cdot)$.

Next, note that Condition C2 depends on g_1/g_2, which again may not be well defined. We redefine $L(\cdot)$ as follows

$$L(b) = \begin{cases} \frac{g_1(b)}{g_2(b)} & b \in B_2 \text{ and } b \notin \mathcal{B}_1 \cup \mathcal{B}_2, \\ \frac{G_1(\{b\})}{G_2(\{b\})} & b \in \mathcal{B}_1 \cap \mathcal{B}_2, \\ 0 & b \in \mathcal{B}_2 \backslash \mathcal{B}_1. \end{cases}$$

We can now redefine $\eta(b)$ with this definition $L(b)$. It should be clear that Conditions C1 and C2 thus extended are necessary and sufficient. □

2.4. *Symmetric Ex-Post IR Implementation with Common Lower Bound of Bid Space Support*

We now use the previous intuition to derive axioms for the case where $\underline{b}_1 = \underline{b}_2$. This adds a little more complexity to our analysis. To see why, recall that our previous implementation 'heavily' used the fact that $\underline{b}_1 < \underline{b}_2$. In particular, profiles of the sort (b, b') where $b \in B_1 \cap B_2$ and $b' < \underline{b}_2$ were used as a sort of residual claimant. The payment of the winning buyer in profiles could be set as high v_2 to make up for any 'shortfall' in buyer 2's expected payment vis-a-vis her interim payment. Conversely, she can be given a rebate to make up for any surplus.

Since $G_1(\underline{b}_2) = 0$, Condition C2 rewritten in this case reflects the fact that there is no such region to make up for any shortfall:

Definition 2.2 (Condition C2′)**.** Condition C2′ requires that for all b in $B_1 \cap B_2$, with $\underline{b}_1 = \underline{b}_2$

$$\eta(b) \geq p_2^h(I_2^{-1}(b)) \tag{8}$$

with the inequality holding strictly for any b such that:

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \text{ and } v(b)G_2(b) > p_1^h(I_1^{-1}(b)).$$

Intuitively, Condition $C2'$ requires that the maximum expected payment $\eta(b)$ that can be extracted from buyer 2 when she bids b, among all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation, is more than $p_2^h(I_2^{-1}(b))$. In the previous section this was enough, because any excess $\eta(b) - p_2^h(I_2^{-1}(b))$ can be rebated to buyer 2 when the other buyer bids in the range $[\underline{b}_1, \underline{b}_2)$. Now, this is no longer enough.

We need an additional condition to account for the fact that there is no lower region to 'rebate' any surplus to. We now write down the exact analog condition, i.e. that the minimum expected payment $\zeta(b)$ that can be extracted from buyer 2 when she bids b, among all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation, is at most $p_2^h(I_2^{-1}(b))$.

If both conditions hold, there clearly exists a payment rule which will achieve the required implementation, since the set of all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation is convex.

We consider two cases depending on the ordering of the upper bound of the possible bids, $\bar{b}_1$ and $\bar{b}_2$.

If $\bar{b}_1 > \bar{b}_2$, we can rebate money to buyer 2 similarly as before—in this case when the other bidder bids in the range $(\bar{b}_2, \bar{b}_1]$. In this case define $\zeta(b) = 0$ for all $b \in B_1 \cap B_2$.

Now let us consider the other case, i.e. that $\bar{b}_1 \leq \bar{b}_2$—in this case $B_1 \subseteq B_2$. We need some additional notation. First, we define

$$\bar{\ell} = \max_{b' \in ([\underline{b}_2, \bar{b}_2] \setminus \{b : g_1(b) = g_2(b) = 0\})} L(b').$$

As before $\bar{\ell}$ is well defined. As before, there are two sub-cases. The first sub-case is when

$$G_2(\bar{\bar{\gamma}}(\bar{\ell})) > 0.$$

Let $\hat{B} \subset \bar{\bar{\gamma}}(\bar{\ell}(b)) > 0$ be a (potentially empty) subset such that:

$$v(b)G_2\left([\underline{b}, b] \setminus \hat{B}\right) \geq p_1^h(I_1^{-1}(b)).$$

We now define a payment rule

$$\hat{p}'(b, b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b] \setminus \hat{B}, \\ s & \text{for } b' \in \hat{B}, \\ 0 & \text{o.w.} \end{cases} \tag{C3,P1}$$

where s is chosen to solve

$$v(b)(G_2([\underline{b},b]\backslash\hat{B}))+sG_2(\hat{B})=p_1^h(I_1^{-1}(b)).$$

Notice that s here is a subsidy. We set:

$$\zeta(b)=\int_{\underline{b}_2}^{\bar{b}_2}\hat{p}'(b,b')dG_1(b').$$

The second sub-case is when

$$G_2(\bar{\bar{\gamma}}(\bar{\ell}))=0,$$

we define the payment rule for $\ell<\bar{\ell}$

$$\hat{p}'_\ell(b,b')=\begin{cases} v(b) & \text{for } b'\in[\underline{b},b],\ L(b')\leq\ell,\\ s & \text{for } L(b')>\ell,\\ 0 & \text{otherwise.}\end{cases} \tag{C3,P2}$$

where s is chosen to solve

$$v(b)G_2(\{b':b'\in[\underline{b},b],L(b')\leq\ell\})+sG_2(\{b':L(b')>\ell\})=p_1^h(I_1^{-1}(b)).$$

Here we set

$$\zeta(b)=\lim_{\ell\uparrow\bar{\ell}}\left[\int_{\underline{b}_2}^{b}\hat{p}'_\ell(b,b')dG_1(b')\right].$$

Definition 2.3 (Condition C3)**.** Condition C3 requires that

$$\zeta(b)\leq p_2^h(I_2^{-1}(b))$$

with the inequality holding strictly when:

$$G_2(\bar{\bar{\gamma}}(\bar{\ell}))=0 \text{ and } v(b)G_2(b)>p_1^h(I_1^{-1}(b)).$$

We can now state the proposition

Proposition 5. *Suppose there are 2 buyers. Consider a hierarchical allocation mechanism (I,p^h) with differentiable and strictly increasing index functions such that the lower bounds of the supports of the bid distributions are the same, that is, $\underline{b}_1=\underline{b}_2$. Then Conditions C1, C2′ and C3 are necessary and sufficient for there to exist a symmetric, ex-post IR implementation of (I,p^h).*

The proof follows from also considering the analogous minimization problem to (Max-P) and is omitted.