Online Appendix of
Structural Change with Long-Run Income and Price Effects

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A Equilibrium Dynamics and Asymptotics

This section characterizes the dynamic and asymptotic properties of a general version of the economy introduced in Section 2. Our main result, stated in Proposition 2, shows that the economy converges to a path of constant real consumption growth under milder conditions than the ones analyzed in Section 2. It also provides a sharp characterization of the constant growth path as a function of sectoral income elasticities, sectoral capital shares and productivity growth of different sectors. Moreover, we show that these results hold for a wider class of models. In particular, we introduce a generalization of the baseline nonhomthetic CES aggregator that allows us to establish a precise connection to empirical measures of the growth rate of real aggregate consumption as defined in the Penn World Table 8.0 (Feenstra et al., 2013).

Notation Henceforth, we denote by η_f^x the elasticity of a (potentially multivariate) function f with respect to its argument x, i.e., $\eta_f^x \equiv \partial \log f / \partial \log x$. When function f is defined over a single variable x, we simply refer to the elasticity function as η_f. Moreover, we use bold face notation to indicate a collection of sectoral variables. For instance, $\boldsymbol{p}(t)$ stands for the set of sectoral prices at time t, that is, $\{p_i\,(t)\}_{i=1}^I$.

A.1 Household Preferences and Demand

Consider a unit mass of households with identical preferences over a stream of real consumption per capita $[c\,(t)]_{t=0}^\infty$, defined as

$$U(0) \equiv \int_0^\infty e^{-(\rho-n)t} u\,(c\,(t))\,dt, \tag{A.1}$$

where $u(\cdot)$ is the instantaneous utility function, $\rho > 0$ is the discount rate, and $n \geq 0$ denotes population growth. We make the standard assumption that $n < \rho$, and that the instantaneous utility function u is asymptotically isoelastic, that is, $\lim_{c\to\infty} \eta_u \equiv cu'/u = 1-\theta$ with $\theta > 0$. Households inelastically supply labor, $L\,(t) \equiv L\,(0)\,e^{nt}$.

Per capita real consumption $c\,(t)$ aggregates cosumption of a bundle $\boldsymbol{c}(t) = \{c_i\,(t)\}_{i=1}^I$ of goods according to generalized nonhomothetic CES preferences defined implicitly through

$$\sum_{i=1}^I \left[g\,(c\,(t))^{-\varepsilon_i}\,c_i\,(t)\right]^{\frac{\sigma-1}{\sigma}} = 1, \tag{A.2}$$

where each consumption good is characterized by an income elasticity parameter $\varepsilon_i > 0$ and $\sigma \in (0,1)$ denotes the elasticity of substitution. Function $g(\cdot)$ is a positive-valued and monotonic function and will be characterized in what follows.[1]

When consumers face prices $\boldsymbol{p} \equiv \{p_i\}_{i=1}^{I}$, the expenditure function corresponding to the preferences above is be given by (see derivations in the main Appendix A)

$$e(c; \boldsymbol{p}) \equiv \left(\sum_{i=1}^{I} \left[g(c)^{\varepsilon_i} p_i \right]^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \tag{A.3}$$

Correspondingly, the consumption expenditure share of good i is given by

$$\omega_i(c; \boldsymbol{p}) \equiv \frac{p_i c_i}{e(c; \boldsymbol{p})} = \left(\frac{g(c)^{\varepsilon_i} p_i}{e(c; \boldsymbol{p})} \right)^{1-\sigma}. \tag{A.4}$$

For any variable that varies across sectors, e.g., income elasticity parameters ε_i, we can define expenditure-weighted averages. For instance, we define the economy-wide average income elasticity paramter as

$$\overline{\varepsilon(c; \boldsymbol{p})} \equiv \sum_{i=1}^{I} \omega_i(c; \boldsymbol{p}) \varepsilon_i. \tag{A.5}$$

In the remainder of the paper, we will drop the dependence of expenditure-weighted average functions on the prices whenever the corresponding prices are clear from the context.

The real consumption aggregator defined by equation (A.2) characterizes the consumer's instantaneous utility function up to a monotonic transformation implied by $g(\cdot)$. In order for our concept of real aggregate consumption $c(t)$ to correspond to empirical measures of real income, the choice of $g(\cdot)$ should express consumer utility in terms of equivalent expenditure in constant prices.[2] Accordingly, we choose function $g(\cdot)$ in such a way that aggregate consumption per capita c is expressed in terms of constant prices $\boldsymbol{q}$, i.e., in real terms. We define $g(\cdot)$ implicitly through:

$$c \equiv \left(\sum_{i=1}^{I} \left[g(c)^{\varepsilon_i} q_i \right]^{1-\sigma} \right)^{\frac{1}{1-\sigma}}, \tag{A.6}$$

that is, we let $g \equiv e^{-1}(\cdot; \boldsymbol{q})$. With this definition, Equation (A.2) defines $c(t)$ as an aggregator of a bundle $\{c_i(t)\}_{i=1}^{I}$ expressed in terms of the cost of an optimal bundle when consumers face given constant prices $\boldsymbol{q}$.

[1]To find the basic isoelastic nonhomothetic CES aggregator presented in Section 2, one can simply replace $g(c) \to c$ and $\varepsilon_i \to \frac{\varepsilon_i - \sigma}{1-\sigma}$. We focus our attention to the empirically relevant case (at least, for three sectors) where the elasticity of substitution is not greater than unity to avoid a taxonomical analysis, but these preferences are well defined for $\sigma > 1$ (see main Appendix A) and, as it will become clear in the analysis that follows, the theoretical results extend to $\sigma > 1$.

[2]This point becomes critical only when we aim to characterize the asymptotic behavior of our economy. Measures of real consumption constructed based on chained Fisher indices provide a local approximation of the utility for any smooth utility function. Therefore, in so far as the local approximation holds, the choice of $g(\cdot)$ does not bear on our empirical results in Section 3. We refer the readers to the online Appendix for more details on this point.

The next proposition, characterizes the solution to the household problem.

Proposition 1. *Consider the household's problem of maximizing (A.1) where the aggregator is defined by Equations (A.2) and (A.6), subject to the flow budget constraint*

$$\dot{a}(t) = w(t) + [r(t) - n]\, a(t) - e(c(t); \boldsymbol{p}(t)), \tag{A.7}$$

and the No-Ponzi condition

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t \left(r(t') - n\right) dt'\right) \geq 0, \tag{A.8}$$

for some path of wage $w(\cdot)$, interest rate $r(\cdot)$, and sectoral prices $\boldsymbol{p}(\cdot)$. Then, the Euler Equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \overline{\frac{\dot{p}(t)}{p(t)}} - (1-\sigma) Cov\left(\frac{\varepsilon_i}{\overline{\varepsilon(c(t))}}, \frac{\dot{p}_i(t)}{p_i(t)}\right)}{\eta^c_{e_c}(c(t); t) - \eta_{u'}(c(t))}, \tag{A.9}$$

along with the transversality condition

$$\lim_{t\to\infty} a(t)\, e^{-(\rho-n)t} \frac{c(t)}{e(c(t); \boldsymbol{p}(t))} \frac{\overline{\varepsilon(c; \boldsymbol{q})}}{\overline{\varepsilon(c; \boldsymbol{p}(t))}} = 0, \tag{A.10}$$

characterize necessary conditions for any paths of consumption and assets to be the solution to the household problem. In the Euler Equation above $\overline{\dot{p}/p}$ and $\bar{\varepsilon}$ are expenditure-weighted sectoral averages at price $\boldsymbol{p}(t)$, and the elasticity of marginal expenditure is given by

$$\begin{aligned} \eta^c_{e_c}(c(t); t) &= \left(\frac{\overline{\varepsilon(c; \boldsymbol{p}(t))}}{\overline{\varepsilon(c; \boldsymbol{q})}} - 1\right)\left[1 + (1-\sigma)\left(\frac{Var(\varepsilon; c, \boldsymbol{p}(t))}{\overline{\varepsilon(c; \boldsymbol{p}(t))}^2}\right)\right], \\ &\qquad + (1-\sigma)\left[\frac{Var(\varepsilon; c, \boldsymbol{p}(t))}{\overline{\varepsilon(c; \boldsymbol{p}(t))}^2} - \frac{Var(\varepsilon; c, \boldsymbol{q})}{\overline{\varepsilon(c; \boldsymbol{q})}^2}\right]. \end{aligned} \tag{A.11}$$

Furthermore, let $\Delta \equiv \varepsilon_{max}/\varepsilon_{min}$ be the ratio of the largest to the smallest sectoral elasticity parameters correpsonding to the household preference's aggregator in Equation (A.2), and suppose the elasticity of intertemporal substition is bounded above by $1/\underline{\theta}$, that is, $\eta_{u'}(c) < -\underline{\theta}$ for all c. If the following inequality is satisfied

$$\underline{\theta} > (\Delta - 1)\left[\frac{1}{\Delta} + \frac{1-\sigma}{4}(\Delta - 1)\right], \tag{A.12}$$

then the household problem has a unique solution, fully characterized by the Euler equation and the transversality conditions above.

The proposition above establishes that under mild conditions on the concavity of instantaneous utility function $u(\cdot)$, the household problem has a unique optimum that can be found by solving an Euler equation. First, note that if all income elasticity parameters are the same, then the real consumption elasticity of the marginal expenditure in Equation (A.11) becomes zero and the Euler equation reduces to its familiar form of standard homothetic preferences with heterogenous technological growth growth rates across consumption sectors (see, e.g., Ngai and Pissarides, 2007).

When income elasticity parameters are heterogenous across sectors, we have two distinct modifications to the standard Euler equation. To unpack these two different modifications, note that $\left(\eta^c_{e_c} - \eta_{u'}\right)\frac{\dot{c}}{c}$ is the rate of decline in the marginal utility of consumer spending (expenditure) at constant prices, which in turn depends on the concavity of the instantaneous utility function and the expenditure function. Accordingly, the numerator of equation (A.9) describes the growth rate of the marginal utility of consumer spending, while the denominator expresses the ratio between growth rates of consumer spending and real aggreate consumption in terms of base prices $\boldsymbol{q}$.

First, due to nonhomotheticity, the expenditure function is a nonlinear function of real consumption. Therefore, the denominator in the right hand side of the Euler equation (A.9) includes an adjustment term that reflects the convexity of the expenditure function. The adjustment effectively increases the concavity of the instantaneous utility function by the degree of the convexity of the expenditure function. Equation (A.11) shows that this nonhomotheticity adjustment depends on the mean and the variance of the income elasticity parameters with distributions implied by expenditure shares, under current and base prices. The larger the mean and the variance of the income elasticity parameters under current prices relative to base prices, the larger is this adjustment. Intuitively, the components of consumer's expenditure corresponding to goods with higher income elasticity have higher convexity in real aggregate consumption. When the income of a typical consumer grows, she spends a larger share of her income on more income elastic goods. As a result, the expenditure function as a whole becomes a more convex function of real aggregate consumption.

The last term on the numerator of Euler equation (A.9) accounts for the interaction of income elasticity and growth rates of sectoral prices. If the rates of growth in sectoral prices are positively correlated with income elasticity parameters, when (real) income grows consumers have to shift a larger share of their expenditure toward *more expensive* goods. This effectively reduces the growth rate of their real consumption.

We emphasize that equations (A.9), (A.10), (A.11), and (A.12) are all invariant to common scaling of sectoral elasticity parameters ε_i's. Therefore, our choice of cardinality for function $g(\cdot)$ in equation (A.6) highlights (and pins down) the one degree of freedom that we face in our choice of sectoral elasticity parameters.

A.2 Production

The production side of our model includes sectoral heterogeneity in rates of technological progress, analyzed first by Ngai and Pissarides (2007), as well as sectoral heterogeneity in factor intensities, studied first in a two-sector setting by Acemoglu and Guerrieri (2008). We show that our growth model remains fully tractable when we incoporate both these supply side channels.

Capital is accumulated using investment goods produced by sector $i = 0$,

$$\dot{K}(t) = Y_0(t) - \delta K(t). \tag{A.13}$$

Labor and capital are combined by producers of consumption good sectors $i \in \{1, \dots, I\}$ to produce output using a Cobb-Douglas technology

$$Y_i(t) = A_i(t) L_i(t)^{1-\alpha_i} K_i(t)^{\alpha_i}, \qquad \text{for } i \in \{0, \dots, I\}, \tag{A.14}$$

where the production function in sector i is characterized with sector-specific capital intensity $\alpha_i \in (0, 1)$. The Hicks-neutral technological progress $A_i(t)$ grows exogenously with the (potentially time varying) rate $\gamma_i(t)$ that asymptotically becomes constant, i.e., rate $\lim_{t\to\infty} \dot{A}_i(t) / A_i(t) = \gamma_i > 0$ for all sectors i.

Since we assume competitive labor and capital markets, the marginal revenue product of labor and capital have to equate their respective prices, that is,

$$w(t) = (1 - \alpha_i) \frac{p_i(t) Y_i(t)}{L_i(t)}, \tag{A.15}$$

$$R(t) = \alpha_i \frac{p_i(t) Y_i(t)}{K_i(t)}. \tag{A.16}$$

We define sectoral capital-labor ratios as

$$\kappa_i(t) \equiv \frac{K_i(t)}{L_i(t)} = \frac{\alpha_i}{1 - \alpha_i} \frac{w(t)}{R(t)}, \qquad \text{for } i \in \{0, \dots, I\}. \tag{A.17}$$

Equation (A.17) shows that sectoral capital-labor ratios are proportional to each other, and to the relative price of labor to capital.

As before, we normalize the price of the investment sector in each period to unity, $p_0(t) \equiv 1$. We find the prices of sectoral consumption goods by equalizing marginal revenue products of capital from equation (A.16)

$$\begin{aligned} p_i(t) &= \frac{\alpha_0^{\alpha_0}(1-\alpha_0)^{1-\alpha_0}}{\alpha_i^{\alpha_i}(1-\alpha_i)^{1-\alpha_i}} \cdot \frac{A_0(t)}{A_i(t)} \cdot \left(\frac{w(t)}{R(t)}\right)^{\alpha_0-\alpha_i}, \\ &= \left(\frac{\alpha_0}{\alpha_i}\right)^{\alpha_i}\left(\frac{1-\alpha_0}{1-\alpha_i}\right)^{1-\alpha_i} \frac{A_0(t)}{A_i(t)} \kappa_0(t)^{\alpha_0-\alpha_i}, \end{aligned} \tag{A.18}$$

where in the second equality we have substituted for relative price of inputs from equation (A.17), and κ_0 denotes the capital-labor ratio in the investment sector. Equation (A.18) shows that consumption good prices depend only on sectoral TFPs and the capital-labor ratio in the investment sector. As expected, goods produced by sectors with lower TFPs are more expensive. The dependence of prices on capital-labor ratio in the investment sector in equation (A.18) is a proxy for their dependence on the rental price of capital. Goods produced by sectors with higher capital intensities become more expensive as capital-labor ratios rise and the rental price of capital falls.

Equation (A.18) illustrates how both supply-side forces driving structural change appear through sectoral prices. A higher rate of technological progress in sector i (relative to the investment sector) is a force lowering the price in this sector, the mechanism featured in the model of Ngai and Pissarides (2007). As the economy accumulates capital, the capital-labor ratio grows proportionally in all sectors. A higher capital intensity in sector i (relative to the investment sector) is an alternative force lowering the price in this sector, one formalized in the model by Acemoglu and Guerrieri (2008).

A.3 Competitive Equilibrium and Equilibrium Dynamics

Market clearing implies that for all $t \geq 0$,

$$L(t) = \sum_{i=0}^{I} L_i(t) \tag{A.19}$$

$$K(t) = \sum_{i=0}^{I} K_i(t) = a(t)L(t), \tag{A.20}$$

$$p_i(t)Y_i(t) = \omega_i(t)\, e(t)\, L(t), \tag{A.21}$$

where $\omega_i(t) = \omega_i(c(t); \boldsymbol{p}(t))$ and $e(t) = e(c(t); \boldsymbol{p}(t))$ are the expenditure share and expenditure functions as defined in equations (A.3) and (A.4). Equation (A.21) connects the production allocations to the nonhomothetic CES demand system. In particular, this equation characterizes the total sectoral outputs that, together with prices of labor and capital, pin down equilibrium sectoral allocations of labor and capital from equations (A.15) and (A.16). Moreover, the total value of output at time t is given by

$$\begin{aligned} Y(t) &\equiv \sum_{i=0}^{I} p_i(t) Y_i(t), \\ &= Y_0(t) + L(t) e(t). \end{aligned}$$

An equilibrium path in our economy consists of the path $[c(\cdot), a(\cdot), \boldsymbol{\omega}(\cdot), \boldsymbol{Y}(\cdot), \boldsymbol{K}(\cdot), \boldsymbol{L}(\cdot)]_{t=0}^{\infty}$ of per capita real consumption and assets, sectoral shares in consumption expenditure, and sectoral outputs and allocations of capital and employment, along with the path $[r(\cdot), R(\cdot), w(\cdot), \boldsymbol{p}(\cdot)]_{t=0}^{\infty}$ of real interest rate, rental price of capital, wage, and sectoral output prices satisfying equations (A.4), (A.9), (A.10), (A.13), (A.14), (A.19), (A.20), (A.21) and $r(t) = R(t) - \delta$ for all $t \geq 0$.

The next proposition characterizes the asymptotic properties of the competitve equilibrium of our economy.

Proposition 2. *Let constant γ^* be defined as*

$$\gamma^* \equiv \min_{i \in \mathcal{I}/\{0\}} \frac{(1-\alpha_0)\gamma_i + \alpha_i \gamma_0}{(1-\alpha_0)\varepsilon_i/\varepsilon_{max}}, \tag{A.22}$$

where ε_{max} is the maximum among all income elasticity parameters. Assume that condition (A.12) is satisfied (which ensures that the instantaneous utility function defined in equations (A.1), (A.6), and (A.2) is concave in real consumption c), and that

$$\rho > n + (1-\theta)\gamma^*. \tag{A.23}$$

Then, there exists a unique competitive equilibrium path for our economy, along which per capita real consumption asymptotically grows at a constant rate

$$\lim_{t\to\infty} \frac{\dot{c}(t)}{c(t)} = \gamma^*. \tag{A.24}$$

Let $\mathcal{I}^*$ be the set of consumption sectors achieving the minimum in equation (A.22). Asymptotically, the share of sectors belonging to this set in employment and consumption expenditure converges to a time-constant distribution. The employment and consumption expenditure shares of any sector i outside the set $\mathcal{I}^*$ vanishes at a rate

$$\lim_{t\to\infty} \frac{\dot{l}_i(t)}{l_i(t)} = \lim_{t\to\infty} \frac{\dot{\omega}_i(t)}{\omega_i(t)} = -(1-\sigma)\left(\gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0 - \frac{\varepsilon_i}{\varepsilon_{max}}\gamma^*\right). \tag{A.25}$$

Proposition 2 states the key asymptotic properties of our economy. First, as with standard growth models, the equilibrium is unique and the rate of growth of real consumption and real interest rate coverge to constant values. More notably, the economy asymptotically features realloction of labor across consumption sectors: while some sectors converge to comprising constant shares, others shrink at a constant rate. Crucially, the rate of growth of real consumption and the set of sectors that do not vanish asymptotically are determined through a combination two forces: 1) the supply-side subsitition forces, as captured by the sectoral rates of technical growth and capital intensities in the numerator of equation (A.22), and 2) the demand-side income forces, as captured by the preference elasticity parameter in the denominator of the same equation. This relation generalizes and encompasses the results of both Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008) and unifies them with our account of long-run demand nonhomotheticity.

Here, we offer some basic intuition for the results and leave a detailed proof of the proposition to Appendix B. Define the productivity-adjusted capital-labor ratio in the investment sector as

$$\tilde{\kappa}_0(t) \equiv \frac{K_0(t)}{A_0(t)^{\frac{1}{1-\alpha_0}} L_0(t)}. \tag{A.26}$$

This variable has a one-to-one relationship with real interest rate through $r(t) = \alpha_0 \tilde{\kappa}_0(\cdot) t^{\alpha_0 - 1} - \delta$ and, as we will explain below, constitutes the main state variable of the economy. As we expect from a path of asymptotically constant growth, this variable has to converge to a constant. Now, substituting the normalized capital-labor ratio of the investment sector in equation (A.18), we observe that the growth rates of consumption good prices take the form

$$\frac{p_i(t)}{p_i(0)} = \left(\frac{\tilde{\kappa}_0(t)}{\tilde{\kappa}_0(0)}\right)^{\alpha_0 - \alpha_i} e^{-\left(\gamma_i - \frac{1-\alpha_i}{1-\alpha_0}\gamma_0\right)t}. \tag{A.27}$$

Therefore, if the rental price of capital remains constant, price of consumption good i falls at the rate $(\gamma_i - \gamma_0) + \frac{\alpha_i - \alpha_0}{1-\alpha_0}\gamma_0$, where the first terms in the parantheses captures technical growth in sector i and the second term captures the extent to which technical growth in the investment sector differentially impacts growth in sector i through differences in capital intensity.

Since households invest optimally, both investement and household expenditure comprise non-negligible values of the total value as the economy grows. Therefore, total consumption expenditure of households asymptotically grows at the same rate as the rate of growth of the investment sector. Combining these insights and equations (A.27) and (A.4) yields

$$\lim_{t\to\infty} \frac{\dot{\omega}_i(t)}{\omega_i(t)} = (1-\sigma)\left[\eta_g \varepsilon_i \gamma^* - \left(\gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0\right)\right]. \tag{A.28}$$

Since all shares are bounded above by one, the rate of growth of sectoral shares cannot remain asymptotically positive. Therefore, for the equilibrium to be well defined we need to ensure that these rates of growth do not asymptotically exceed 0. This simple rule in fact pins down the asymptotic rate of growth of real consumption per capita γ^* in equation (A.22), which in general will be different from the rate of growth of per capita consumption expenditure $\frac{\gamma_0}{1-\alpha_0}$.

While Proposition 2 provides a simple account of the asymptotic properties of our model, the dynamics of the equilibrium path in this economy may generally be more complex. However, the model still remains fully tractable and the dynamic equations can be written in terms of a state variable $\tilde{\kappa}_0$, investment-sector capital-labor ratio, and a control variable c, per capita real consumption. Appendix C provides the full derivation of dynamic equations characterizing the competitive equilibrium everywhere along an path with an initial condition $(c(0), \tilde{\kappa}_0(0))$. A system of two linear equations in $\frac{\dot{c}(t)}{c(t)}$ and $\frac{\dot{\tilde{\kappa}}_0(t)}{\tilde{\kappa}_0(t)}$ determines rates of growth of the two variables for $(c(t), \tilde{\kappa}_0(t))$ at time t. Our unified model includes heterogeneity across sectors along three different dimensions, i.e., income elasticity of demand for sectoral outputs, capital intensitiy, and rate of technical growth, the interactions between all these sources of heterogeneity appear in the dynamic equations.

Here, for the sake of brevity, we present the dynamic equations for the special case where $\alpha_i \equiv \alpha$ for all sectors i. This case parallels the workhorse model analyzed by Buera and Kaboski (2009) and Herrendorf et al. (2013) including two competing forces: income nonhomotheticity and heterogeneous rates of technological growth. When capital intensities are identical, capital-labor ratios equalize across all sectors in equilibrium and $\tilde{\kappa}_0$ equals the economy-wide capital-labor ratio. Dropping the subscript 0 to reflect this fact, the dynamics of equilibrium paths take the following form:

$$
\begin{aligned}
\frac{\dot{c}}{c} &= \frac{\alpha\tilde{\kappa}^{\alpha-1} - (\delta+\rho) + \bar{\gamma}\left(1 + (1-\sigma)\,\rho_{\varepsilon,\gamma}\right) - \gamma_0}{-\eta_{u'} - 1 + \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}'} - 1\right)\left(1 + (1-\sigma)\,Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\right) + (1-\sigma)\left(Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right) - Var'\left(\frac{\varepsilon}{\bar{\varepsilon}'}\right)\right)}, \\
\dot{\tilde{\kappa}} &= \tilde{\kappa}^{\alpha} - \frac{\tilde{e}}{\tilde{\kappa}} - \left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right)\tilde{\kappa},
\end{aligned}
$$

where $\tilde{\kappa}$ denotes the normalized capital-labor ratio (which is the same across all sectors), $\bar{\cdot}$, $Var(\cdot)$, and $\rho_{\cdot,\cdot}$ indicate mean, variance, and correlation coefficient of sectoral variables with distribution implied by expenditure shares under current prices, while $\bar{\cdot}'$ and $Var'(\cdot)$ denote mean and variance of sectoral variables with distribution implied by expenditure shares under base prices.

B Proofs of Section A

Proof of Proposition 1.
Using an argument similar to the one used for Lemma 1, we can decompose the problem into two intra-temporal and intertemporal problems. To avoid repetition, we focus on the latter, using the definition of the expenditure function in Equation (A.3) as the cost of real consumption $c(t)$ for the representative consumer in terms of the price of investment good at time t.

For a given path of wages $[w(t)]_{t=0}^{\infty}$, rental prices of capital $[r(t)]_{t=0}^{\infty}$, and sectoral good prices $[\boldsymbol{p}(t)]_{t=0}^{\infty}$, the current-value Hamiltonian for the consumer problem (A.1) may be written as:

$$\hat{\mathcal{H}}(t, c(t), a(t), \lambda(t)) \equiv u(c(t)) + \lambda(t)(w(t) + [r(t) - n]a(t) - e(c(t); \boldsymbol{p}(t))).$$

Let us start with the necessary conditions. The FOCs for the Hamiltonian are as follows:

$$\frac{\partial \hat{\mathcal{H}}}{\partial c} = 0 \quad \Rightarrow u'(c) - \lambda \frac{\partial e}{\partial c} = 0, \tag{B.1}$$

$$\frac{\partial \hat{\mathcal{H}}}{\partial a} = (\rho - n)\lambda - \dot{\lambda} \quad \Rightarrow \frac{\dot{\lambda}}{\lambda} = -(r - \rho). \tag{B.2}$$

In addition, we impose that the solution satisfy the transversality condition:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda(t)a(t) = 0. \tag{B.3}$$

Equations (B.1) and (B.2) together with the law of evolution of assets (A.10) and the transversality equation (B.3) characterize paths of per capita real aggregate consumption and asset holdings $[c(\cdot), a(\cdot)]$, and costate $\lambda(\cdot)$ that satisfy necessary conditions for optimality.

Next, we show conditions that ensure the solution above indeed corresponds to the unique solution to the household utility maximization problem. A standard argument (using (B.2) and the No-Ponzi constraint) shows that *for all feasible pairs* $[c(\cdot), a(\cdot)]$, we have that $\lim_{t\to\infty} \exp(-(\rho - n)t)\lambda(t)a(t) \geq 0$. Therefore, we can establish that the pair characterized by Equations (B.1), (B.2), and (B.3) indeed correspond to the optimum if the Hamiltonian is concave in c. Furthermore, since the Hamiltonian is separable in (c, a) and linear a, strict concavity in c implies the uniqueness of the optimum for the household problem. We will come back to the characterization of conditions ensuring the concavity of e at the end of the proof.

From Equation (B.1), we find that

$$\lambda(t) = \frac{u'(c(t))}{\frac{\partial e(c(t);\boldsymbol{p}(t))}{\partial c}} = \frac{u'(c(t))}{\frac{e}{c}\eta_e^c}, \tag{B.4}$$

where $\frac{\partial e}{\partial c}$ is the marginal (dollar) cost of consumption.We can compute the consumption elasticity of expenditure:

$$\begin{aligned}
\eta_e^c = \frac{\partial \log e}{\partial \log c} &= \frac{c}{1-\sigma}\frac{\partial}{\partial c}\log\sum_{i=1}^{I}\left(g^{\varepsilon_i}p_i\right)^{1-\sigma}, \\
&= \eta_g \cdot \sum_{i=1}^{I}\varepsilon_i\omega_i,
\end{aligned}$$

where the last equation can be explicitly expressed as $\eta_g\left(c\right)\cdot\overline{\varepsilon\left(c;\boldsymbol{p}\right)}$ as a function of real aggregate consumption c and current prices $\boldsymbol{p}$.

To translate the equations above into the Euler format, we need to compute $\dot{\lambda}\left(t\right)/\lambda\left(t\right)$, which corresponds to the growth in value of income at time t. Using Equation (B.4), we can write the growth in utility value of income as the sum of the contribution of growth of real consumption, decline in the price index, and the decline in the average income elasticity parameter, that is,

$$\frac{\dot{\lambda}}{\lambda} = \eta_{u'}^c\frac{\dot{c}}{c} - \frac{\dot{\overline{(e/c)}}}{e/c} - \frac{\dot{\eta_e^c}}{\eta_e^c}.$$

First, we simplify the growth of consumption expenditure:

$$\begin{aligned}
\frac{\dot{e}}{e} &= \eta_e^c\frac{\dot{c}}{c} + \sum_i\frac{\dot{p}_i}{p_i}\omega_i, \\
&= \eta_g\overline{\varepsilon}\cdot\frac{\dot{c}}{c} + \overline{\frac{\dot{p}_i}{p_i}},
\end{aligned}$$

where we have used the fact that $\eta_e^{p_i} = \omega_i$ from Lemma 1 (see below) and have defined $\overline{\frac{\dot{p}_i}{p_i}}$ to be the average rate of growth of prices across sectors, as weighted by their corresponding consumption expenditures. Next, we compute the growth of the real consumption elasticity of expenditure:

$$\begin{aligned}
\frac{\dot{\eta_e^c}}{\eta_e^c} &= \frac{\dot{\eta_g}}{\eta_g} + \frac{\dot{\overline{\varepsilon}}}{\overline{\varepsilon}}, \\
&= \eta_{\eta_g}\cdot\frac{\dot{c}}{c} + \eta_{\overline{\varepsilon}}^c\cdot\frac{\dot{c}}{c} + \sum_i\eta_{\overline{\varepsilon}}^{p_i}\left(\frac{\dot{p}_i}{p_i}\right), \\
&= \eta_{\eta_g}\cdot\frac{\dot{c}}{c} + \left(1-\sigma\right)\eta_g\overline{\varepsilon}\cdot Var\left(\frac{\varepsilon}{\overline{\varepsilon}}\right) + \left(1-\sigma\right)\sum_i\left(\frac{\epsilon_i}{\overline{\epsilon}}-1\right)\left(\frac{\dot{p}_i}{p_i}\right)\omega_i, \\
&= \eta_{\eta_g}\cdot\frac{\dot{c}}{c} + \left(1-\sigma\right)\eta_g\overline{\varepsilon}\cdot Var\left(\frac{\varepsilon}{\overline{\varepsilon}}\right)\cdot\frac{\dot{c}}{c} + \left(1-\sigma\right)Cov\left(\frac{\varepsilon}{\overline{\varepsilon}},\frac{\dot{p}}{p}\right),
\end{aligned}$$

where in the third equality, we have used the results of Lemma 1 substituting for the elasticities of average income elasticity parameter $\bar{\varepsilon}$.

Combining all of the above, we find the Euler equation to be:

$$\frac{\dot{c}}{c} = \frac{r - \rho - \overline{\frac{\dot{p}_i}{p_i}} - (1-\sigma)\, Cov\left(\frac{\varepsilon_i}{\bar{\varepsilon}}, \frac{\dot{p}_i}{p_i}\right)}{-\eta_{u'} - 1 + \eta_{\eta_g} + \eta_g \bar{\varepsilon}\left(1 + (1-\sigma)\, Var\left(\frac{\varepsilon_i}{\bar{\varepsilon}}\right)\right)}. \tag{B.5}$$

The Euler Equation (B.5) is expressed for a general function $g(\cdot)$. Specializing this result to the specific function defined in Equation (A.6), we will now derive the Euler Equation for the real consumption stated in terms of constant prices $\boldsymbol{q}$. Since g is the inverse of the expenditure function at prices $\boldsymbol{q}$, from Equation (B.5), we have:

$$1 = \eta_g(c) \cdot \overline{\varepsilon(c; \boldsymbol{q})},$$

suggesting $\eta_g = \overline{\varepsilon(c; \boldsymbol{q})}^{-1}$. It follows that:

$$\eta_e^c = \frac{\overline{\varepsilon\,(c; \boldsymbol{p})}}{\overline{\varepsilon\,(c; \boldsymbol{q})}},$$

which is positive if we have $\varepsilon_i > 0$ for all sectors. This ensures that the function e is monotonically increasing and one-to-one. This elasticity, which may in general be different from zero, characterizes the way income effects shifts expenditure across sectors with different prices. In particular, whenever the average elasticity parameter $\bar{\varepsilon}$ is higher than the one at base prices, marginal cost of increasing real consumption exceeds the current aggregate price index.

Similarly, substituting for $\boldsymbol{q}$ in Equation (B.10) from Lemma 1 below, we find:

$$\eta_{\eta_g} = -\frac{\partial \log \overline{\varepsilon(c; \boldsymbol{q})}}{\partial \log c} = -(1-\sigma)\left(\frac{\overline{\varepsilon^2(c; \boldsymbol{q})}}{\left(\overline{\varepsilon(c; \boldsymbol{q})}\right)^2} - 1\right),$$

To ensure the sufficiency of the FOCs and the uniqueness of the solution, we need to find conditions under which the Hamiltonian in strictly concave. The second order condition for c is

$$u'' - \lambda \frac{\partial^2 e}{\partial c^2} = \frac{u}{c^2}\eta_u\left(\eta_{u'} - \eta_{\partial e/\partial c}\right),$$

where we have substituted for λ from Equation (B.4). This expression has to always be negative. We need to only focus on the term within the parentheses. The first term on the right hand side, by assumption, is always greater than $\underline{\theta}$. To compute the second term, note that

$$
\begin{aligned}
\eta^c_{e_c} \equiv \eta^c_{\partial e/\partial c} &= \frac{c}{e \cdot \eta^c_e / c} \frac{\partial^2 e}{\partial c^2}, \\
&= \eta^c_e + \eta^c_{\eta^c_e} - 1, \qquad \text{(B.6)}
\end{aligned}
$$

and $\eta^c_{\eta^c_e}$ is given by Equation (A.14) from Section A.

Finally, we can find η_{η_g} in Equation (A.14) by using the definition of function $g(\cdot)$ in Equation (A.6). Since $e(c;\boldsymbol{q}) = c$, we have that $\eta^c_{\eta^c_e} = 0$ at prices $\boldsymbol{q}$ and therefore:

$$
0 = \eta_g \overline{\varepsilon_i} \left[\eta_{\eta_g} + (1-\sigma)\, Var\left(\frac{\varepsilon_i}{\overline{\overline{\varepsilon_i(c;\boldsymbol{q})}}} \right) \right], \qquad \text{(B.7)}
$$

which implies $\eta_{\eta_g} = -(1-\sigma)\, Var\left(\frac{\varepsilon_i}{\overline{\overline{\varepsilon_i(c;\boldsymbol{q})}}}\right)$.

Combining all the results above, we find the second order condition to require:

$$
\theta + \frac{\overline{\varepsilon\,(c;\boldsymbol{p})}}{\overline{\overline{\varepsilon\,(c;\boldsymbol{q})}}} - 1 + (1-\sigma) \left[\frac{\overline{\varepsilon\,(c;\boldsymbol{p})}}{\overline{\overline{\varepsilon\,(c;\boldsymbol{q})}}} \left(\frac{Var\,(\varepsilon; c, \boldsymbol{p})}{\overline{\varepsilon\,(c;\boldsymbol{p})}^2} \right) - \frac{Var\,(\varepsilon; c, \boldsymbol{q})}{\overline{\varepsilon\,(c;\boldsymbol{q})}^2} \right] > 0,
$$

for all c. Remembering that the variance of any distribution on $\{\varepsilon_i\}_{i=1}^I$ is bounded above by $\frac{1}{4}\left(\varepsilon_{max} - \varepsilon_{min}\right)^2$, it immediately follows that condition (A.12) is a sufficient condition for the FOC to always characterize an optimal solution.

∎

Lemma 1. *The real consumption and sectoral price elasticities of the expenditure function are given by:*

$$
\begin{aligned}
\eta^c_e &= \eta_g \overline{\varepsilon}, \qquad \text{(B.8)} \\
\eta^{p_i}_e &= \omega_i. \qquad \text{(B.9)}
\end{aligned}
$$

Furthermore, define the average $\overline{x}$ of some sectoral parameters x_i across the consumption sector weighted by expenditure shares:

$$
\bar{x} \equiv \sum_{i=1}^{I} x_i \omega_i.
$$

The elasticities of this function are given by:

$$
\begin{aligned}
\eta^c_{\overline{x}} &= \frac{1-\sigma}{\bar{x}} \eta_g Cov\,(\varepsilon, x)\,, \qquad \text{(B.10)} \\
\eta^{p_i}_{\overline{x}} &= (1-\sigma) \left(\frac{x_i}{\overline{x}} - 1 \right) \omega_i. \qquad \text{(B.11)}
\end{aligned}
$$

Proof. We can compute the consumption elasticity of expenditure:

$$\begin{aligned}
\eta_e^c = \frac{\partial \log e}{\partial \log c} &= \frac{c}{1-\sigma}\frac{\partial}{\partial c}\log \sum_{i=1}^{I}\left(g^{\varepsilon_i}p_i\right)^{1-\sigma}, \\
&= \eta_g \cdot \sum_{i=1}^{I}\varepsilon_i\omega_i, \\
&= \eta_g\overline{\varepsilon}.
\end{aligned}$$

where the last equation can be explicitly expressed as $\eta_g(c)\cdot\overline{\epsilon(c;\boldsymbol{p})}$ as a function of real aggregate consumption c. Similarly, we can compute the price elasticity of the expenditure function:

$$\begin{aligned}
\eta_e^{p_i} = \frac{\partial \log e}{\partial \log p_i} &= \frac{p_i}{1-\sigma}\frac{\partial}{\partial p_i}\log \sum_{j=1}^{I}\left(g^{\varepsilon_j}p_j\right)^{1-\sigma}, \\
&= \left(\frac{g^{\varepsilon_i}p_i}{e}\right)^{1-\sigma}, \\
&= \omega_i.
\end{aligned}$$

Next, we use the expressions above to compute the elasticities of the sectoral shares in consumption expenditure. From Equation (A.4) we have:

$$\begin{aligned}
\eta_{\omega_i}^c &= (1-\sigma)\left(\eta_g\varepsilon_i - \eta_e^c\right) = (1-\sigma)\,\eta_g\left(\varepsilon_i - \overline{\varepsilon}\right), \\
\eta_{\omega_i}^{p_j} &= (1-\sigma)\left(\delta_{ij} - \eta_e^{p_j}\right) = (1-\sigma)\left(\delta_{ij} - \omega_j\right),
\end{aligned}$$

where δ_{ij} stands for the kronecker delta function.

Finally, we use the elasticities of the expenditures shares to compute elasticities of a general function $\overline{x}$. We find:

$$\begin{aligned}
\eta_{\overline{x}}^c &= \frac{1}{\overline{x}}\sum_j x_j\omega_j\eta_{\omega_j}^c, \\
&= \frac{1-\sigma}{\overline{x}}\eta_g\sum_j x_j\omega_j\left(\varepsilon_j - \overline{\varepsilon}\right), \\
&= \frac{1-\sigma}{\overline{x}}\eta_g Cov\left(\varepsilon_i, x_i\right) \\
\eta_{\overline{x}}^{p_i} &= \frac{1}{\overline{x}}\sum_j x_j\omega_j\eta_{\omega_j}^{p_i}, \\
&= \frac{1-\sigma}{\overline{x}}\sum_j x_j\omega_j\left(\delta_{ij} - \omega_i\right), \\
&= (1-\sigma)\left(\frac{x_i}{\overline{x}} - 1\right)\omega_i.
\end{aligned}$$

For instance, for $x_i \equiv \varepsilon_i$, we find that:

$$\eta^c_{\bar{\varepsilon}} = (1-\sigma)\,\eta_g Cov\left(\frac{\varepsilon_i}{\bar{\varepsilon}}, \varepsilon_i\right) = (1-\sigma)\,\eta_g \bar{\varepsilon} Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right). \tag{B.12}$$

∎

Proof of Proposition 2

Our strategy for the proof of this proposition is as follows. To establish the existence and uniqueness of the competitive equilibrium, we invoke the second Welfare Theorem. We formulate the social planner's problem, whose potential solutions have to correspond to different competitive equilibria in our economy. We solve the social planner's problem and show that it has a unique solution, and further establish a direct correspondence between this solution and the competitive equilibrium, which thus has to also be unique.

Let $\hat{u}(c_1, \cdots, c_I) \equiv u(c)$, where c is defined through Equation (A.2), denote the instantaneous utility of the representative household over a bundle of $\boldsymbol{c} = (c_i)_{i=1}^I$ per capita consumption of I different goods. The social planner's problem can be stated as the following maximization problem:

$$\max_{\{k_i(\cdot), l_i(\cdot)\}_{i=0}^I} \int_0^\infty e^{-(\rho-n)t} \hat{u}(c_1(t), \cdots, c_I(t)),$$

where

$$c_i = A_i k_i^{\alpha_i} l_i^{1-\alpha_i}, \qquad 1 \le i \le I, \tag{B.13}$$

$$\dot{k} = A_0 k_0^{\alpha_0} l_0^{1-\alpha_0} - (\delta + n)\,k, \tag{B.14}$$

subject to constraints:

$$\sum_{i=0}^I l_i = 1, \tag{B.15}$$

$$\sum_{i=0}^I k_i = k, \tag{B.16}$$

and $k(0) = K(0)/L(0)$ and $k(t) \ge 0$ for all t.

The corresponding present value Hamiltonian is given by:

$$\mathcal{H} = \hat{u} + \mu\left(A_0 L k_0^{\alpha_0} l_0^{1-\alpha_0} - (\delta+n)\,k\right),$$

where we substitute for per capita consumption, i.e., Equations (B.13) and (B.16), in the expression for $\hat{u}$ making the latter a function of vectors of sectoral per capita stocks of capital and employment shares $(\boldsymbol{k},\boldsymbol{l})$.[3]

We can show that the function $\mathcal{M}(k)=\max_{(\boldsymbol{k},\boldsymbol{l})}\hat{\mathcal{H}}$ under the constraints suggested by Equations (B.15) and (B.16) is a strictly concave function of k, if $\hat{u}$ is a strictly concave function of its arguments. To see why, let us define functions $\mathcal{F}(\boldsymbol{k},\boldsymbol{l})\equiv\hat{u}(\boldsymbol{k},\boldsymbol{l})+\mu A_0 L k_0^{\alpha_0} l_0^{1-\alpha_0}$ and $\hat{\mathcal{F}}(\boldsymbol{k})=\max_{\boldsymbol{l}}\mathcal{F}(\boldsymbol{k},\boldsymbol{l})$. First, it is straightforward to see that $\mathcal{F}$ is jointly strictly concave in $(\boldsymbol{k},\boldsymbol{l})$ for $\mu\geq 0$, which implies that $\hat{\mathcal{F}}$ is a strictly concave function of vector $\boldsymbol{k}$. The strict concavity of the latter then implies that $\mathcal{M}(k)=\max_{\sum_i k_i=k}\hat{\mathcal{F}}(\boldsymbol{k})$ is a strictly concave function of k.

Now let us find a candidate solution for the social planner's problem that satisfies the following conditions:

$$-\frac{\partial\mathcal{H}}{\partial k} = (\delta+n)\mu-\varsigma=\dot{\mu}-(\rho-n)\mu, \tag{B.17}$$

$$\frac{\partial\mathcal{H}}{\partial k_i} = \alpha_i\frac{\partial\hat{u}}{\partial c_i}A_i\left(\frac{k_i}{l_i}\right)^{\alpha_i-1}=\varsigma, \qquad 1\leq i\leq I, \tag{B.18}$$

$$\frac{\partial\mathcal{H}}{\partial l_i} = (1-\alpha_i)\frac{\partial\hat{u}}{\partial c_i}A_i\left(\frac{k_i}{l_i}\right)^{\alpha_i}=\xi, \qquad 1\leq i\leq I, \tag{B.19}$$

$$\frac{\partial\mathcal{H}}{\partial k_0} = \alpha_0\mu A_0\left(\frac{k_0}{l_0}\right)^{\alpha_0-1}=\varsigma, \tag{B.20}$$

$$\frac{\partial\mathcal{H}}{\partial l_0} = (1-\alpha_0)\mu A_0\left(\frac{k_0}{l_0}\right)^{\alpha_0}=\xi, \tag{B.21}$$

and the transversality condition

$$\lim_{t\to\infty}e^{-(\rho-n)t}\mu(t)\,k(t)=0, \tag{B.22}$$

where ξ and ς are Lagrange multipliers corresponding to the two constraints Equations (B.13) and (B.16), respectively.

From Equations (B.17) and (B.20) we find:

$$\mu(t)=\mu(0)\exp\left(-\int_0^t\left(\alpha_0\tilde{\kappa}_0(t')^{\alpha_0-1}-(\rho+\delta)\right)dt'\right), \tag{B.23}$$

where we have used the productivity adjusted definition of capital-labor ratio:

$$\tilde{\kappa}_0(t)\equiv A_0(t)^{-1/(1-\alpha_0)}\frac{k_0(t)}{l_0(t)}. \tag{B.24}$$

Similarly, we define the economy-wide aggregate capital-labor ratio $\tilde{k}(t)\equiv A_0(t)^{-1/(1-\alpha_0)}k(t)/l(t)$

[3] Let us remember that according to our vector notation $\boldsymbol{k}\equiv(k_0,\cdots,k_I)$ denotes the vector of sectoral per capita stocks of capital, which is distinct from k, the economy-wide total per capita stock of capital.

and rewrite the transversality condition as:

$$\lim_{t\to\infty} \tilde{k}\left(t\right)\mu\left(0\right)\exp\left(-\int_0^t\left(\alpha_0\tilde{\kappa}_0\left(t'\right)^{\alpha_0-1}-\left(\delta+n+\frac{\gamma_0}{1-\alpha_0}\right)\right)dt'\right)=0, \tag{B.25}$$

where we have used the fact, which we will show later, that asymptotically $k\left(t\right)\to\tilde{k}^* e^{\frac{\gamma_0}{1-\alpha_0}t}$. Note that with this transformation Equation (B.14) can be written as:

$$\dot{\tilde{k}}\left(t\right)=l_0\tilde{\kappa}_0\left(t\right)^{\alpha_0}-\left(n+\delta+\frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}\left(t\right). \tag{B.26}$$

Henceforth, we use the notation that $\tilde{x}\left(t\right)$ denotes variable $x\left(t\right)$ adjusted by productivity in the investment sector, i.e., $\tilde{x}\left(t\right)\equiv A_0\left(t\right)^{-1/(1-\alpha_0)}x\left(t\right)$. Dividing (B.19) by (B.18), we find:

$$\frac{1-\alpha_i}{\alpha_i}\tilde{\kappa}_i\left(t\right)=\frac{\tilde{\xi}\left(t\right)}{\varsigma\left(t\right)}, \tag{B.27}$$

suggesting that capital-labor ratios in all sectors are proportional to each other. This relation echoes Equation (A.17), suggesting, as we show below, that ξ and ς correspond to the wage and rental price of capital in a competitive equilibrium.

Starting from any initial value $\mu(0)$, Equation (B.23) along with conditions (B.18) to (B.21) define a unique path for the allocations. The argument follows from the strict concavity of function $\mathcal{F}$ defined earlier, and the intuition is as follows. Equation (B.23) determines $\mu(t)$ and therefore function $\mathcal{F}$ at time t. The optimal allocation at this point it time is simply given by maximizing function $\mathcal{F}$ under constraints (B.15) and (B.15), which has to be unique.

Note that any candidate path that satisfies the conditions above has to further satisfy the following two asymptotic conditions:

$$\lim_{t\to\infty}\tilde{\kappa}_0\left(t\right) = \tilde{\kappa}_0^*>0, \tag{B.28}$$

$$\lim_{t\to\infty} l_0\left(t\right) = l_0^*, \qquad 0<l_0^*<1, \tag{B.29}$$

implying that the asymptotic capital-labor ratio and employment in the investment sector are constant and interior. If, on the contrary, we asymptotically have $\tilde{\kappa}_0\to 0$, we can show that $\xi\to 0$ and Equation (B.19) has to be violated.[4] If $\tilde{\kappa}_0\to\infty$, Equation (B.26) implies that the tranversality condition B.25 has to be violated. Therefore, condition (B.28), has to hold. Now from Equation (B.27) we learn that all sectoral capital-labor ratios asymptote to nonzero constants and therefore $\lim_{t\to\infty}\tilde{k}\left(t\right)=\tilde{k}^*=\sum_{i=0}^I l_i^*\tilde{\kappa}_i^*>0$. Hence, from Equation (B.26)

[4] Assume $\tilde{\kappa}_0$ converges to zero exponentially at a constant rate. From Equation (B.23), μ has to converge to zero at the rate of $-\infty$, which implies the same has to be the case for ξ. From Equation (B.19), we need to have that $\partial\hat{u}/\partial c_i$ converges to zero at the rate of $-\infty$, which can hold only if c_i's all grow at an ever increasing rate. This contradicts the initial assumption that capital-labor ratios converge to zero.

we know that $l_0^* > 0$, since otherwise $\tilde{k} \to 0$. Finally, assuming $l_0^* = 1$ would suggest that $\tilde{k}^* = \tilde{\kappa}_0^*$ and $(\tilde{\kappa}_0^*)^{\alpha_0 - 1} = n + \delta + \frac{\gamma_0}{1-\alpha_0}$. Substituting this in Equation (B.25) would violate the transversality condition. Therefore, any candidate path satisfying the conditions above will asymptotically be an interior candidate solution, in the sense that labor-capital ratios grow at the same rate as the rate of technological progress in the investment sector and there is an interior split of employment between the investment and the consumption sector. This implies, as we will see shortly, that the per capita consumption expenditure in the corresponding competitive equilibrium also grows at the same rate as the rate of growth of technology in the investment sector.

Next, we show that the growth of real consumption per capita along any candidate path satisfying the conditions above has to be asymptotically constant. To see this, note that combining Equations (B.13) and (B.27), we find that $c_i = A_0^{\frac{\alpha_i}{1-\alpha_0}} A_i l_i \tilde{\kappa}_0^{\alpha_i}$ for $i \in \{1, \cdots, I\}$. Equation (B.28) then implies:

$$\lim_{t \to \infty} \frac{\dot{c}_i(t)}{c_i(t)} = \gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0 + \hat{\gamma}_i^C, \tag{B.30}$$

where we defined the asymptotic rate of growth of the share of employment in sector i as:[5]

$$\hat{\gamma}_i^C \equiv \lim_{t \to \infty} \frac{\dot{l}_i(t)}{l_i(t)} \leq 0. \tag{B.31}$$

Now, from Equation (A.2), we have that for all $t \geq 0$:

$$\sum_{i=1}^{I} \nu_i \left(\gamma_i + \frac{\alpha_i}{1-\alpha_0}\gamma_0 + g_i - \eta_g \varepsilon_i \frac{\dot{c}}{c} \right) = 0,$$

where we have defined the effective share of sector i in consumption as $\nu_i \equiv \left(g(c)^{-\varepsilon_i} c_i\right)^{(\sigma-1)/\sigma}$.[6] The rate of growth of per capita real consumption is then given by

$$\frac{\dot{c}}{c} = \frac{1}{\eta_g \overline{\varepsilon}} \left(\overline{\gamma} + \frac{\overline{\alpha}}{1-\alpha_0}\gamma_0 + \overline{\hat{\gamma}^C} \right), \tag{B.32}$$

where averages are taken with respect to the distribution implied by $\{\nu_i\}_{i=1}^I$. Equation (B.19) suggests that, up to a constant, consumption share ν_i's are the same as employment shares

[5] Note that Equation (B.29) establishes the total employment share in the consumption sector asymptotically converges to a constant $l_C \to 1 - l_0^*$. However, within the aggregate consumption sector, some sectors could continue to shrink asymptotically which can result in Equation (B.31) taking nonzero values.

[6] As a reminder, share ν_i equals the share of sector i in consumption expenditure $\omega_i \equiv (g(c)^{\varepsilon_i} p_i/e)^{1-\sigma}$ for the corresponding prices in the competitive equilibrium. We introduce a different variable here solely to respect the conceptual distinction between the formulation of the social planner's problem and the competitive equilibrium where the prices are implicit in the former (and the current formulation).

l_i's. To see this, rewrite Equation (B.19) as:

$$\begin{aligned}\xi &= (1-\alpha_i)\, u'(c)\, \frac{\partial c}{\partial c_i} \frac{c_i}{l_i}, \\ &= (1-\alpha_i)\, u'(c)\, \frac{c}{\eta_g \overline{\varepsilon}} \frac{\nu_i}{l_i},\end{aligned}$$

where we have used $\eta_c^{c_i} = \nu_i / \eta_g \overline{\varepsilon}$ from Equation (A.8).

Define set $\mathcal{I}^* \subset \{1, \cdots, I\}$ as the set of consumption sectors with nonzero asymptotic employment shares, i.e., $\mathcal{I}^* = \left\{ i | \hat{\gamma}_i^C = 0 \right\}$. Consider some sector $i \notin \mathcal{I}^*$, for which $\hat{\gamma}_i^C < 0$ implying that the asymptotic employment share of this sector is zero, i.e., $\lim_{t\to\infty} l_i = 0$. Since consumption shares ν_i's have to grow proportionally to employment shares as we showed above, for any such sector i we have $\lim_{t\to\infty} \nu_i = 0$. It then follows that $\lim_{t\to\infty} \overline{\hat{\gamma}^C} = \sum_{i\in\mathcal{I}^*} \hat{\gamma}_i^C = 0$, and taking the limit of expression (B.32), we find the asymptotic rate of growth of per capital consumption as:

$$\gamma^* \equiv \lim_{t\to\infty} \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta_g \overline{\varepsilon}^*} \left(\overline{\gamma}^* + \frac{\overline{\alpha}^*}{1-\alpha_0} \gamma_0 \right),$$

where averages are taken with respect to the distribution implied by $\{\nu_i^*\}_{i=1}^I$ the limit of distribution $\{\nu_i\}_{i=1}^I$, with support $\mathcal{I}^*$.

In order to characterize the set $\mathcal{I}^*$, we need to compute the asymptotic rate of growth of ν_i for each sector i. Substituting Equation (B.27) in Equation (B.19), we find:

$$u'(c)\, \frac{\partial c}{\partial c_i} = \frac{1}{A_i} \left(\frac{\varsigma}{\alpha_i} \right)^{\alpha_i} \left(\frac{\xi}{1-\alpha_i} \right)^{1-\alpha_i}, \tag{B.33}$$

where the right hand side captures the social cost of producing good i in marginal utility terms at time t. From Equations (B.20) and (B.21), we have:

$$\varsigma = \mu \alpha_0 \tilde{\kappa}_0^{\alpha_0 - 1}, \tag{B.34}$$

$$\xi = \mu A_0^{\frac{1}{1-\alpha_0}} (1-\alpha_0)\, \tilde{\kappa}_0^{\alpha_0}. \tag{B.35}$$

Substituting in the expression above we find:

$$\frac{u'(c)}{\mu} \frac{c}{c_i} \frac{\nu_i}{\eta_g \overline{\varepsilon}} = \left(\frac{\alpha_0}{\alpha_i} \right)^{\alpha_i} \left(\frac{1-\alpha_0}{1-\alpha_i} \right)^{1-\alpha_i} \frac{A_0^{\frac{1-\alpha_i}{1-\alpha_0}}}{A_i} \tilde{\kappa}_0^{\alpha_0 - \alpha_i}, \tag{B.36}$$

where we have again used $\eta_c^{c_i} = \nu_i / \eta_g \overline{\varepsilon}$.

We proceed by establishing that the asymptotic rate of growth of the term $\frac{u'(c)}{\mu} \frac{c}{\eta_g \overline{\varepsilon}}$ is $\frac{\gamma_0}{1-\alpha_0}$.

To see this, let us rewrite Equation (B.19) as:

$$\frac{cu'(c)}{\eta_g \overline{\varepsilon}} \nu_i = \frac{l_i}{1-\alpha_i} \xi,$$

which we can then sum over $i \in \{1, \cdots, I\}$ to find:

$$\frac{cu'(c)}{\eta_g \overline{\varepsilon}} = \xi \sum_{i=1}^{I} \frac{l_i}{1-\alpha_i}.$$

Since $\sum_{i=1}^{I} l_i$ converges to a nonzero value $1 - l_0^*$, the rate of growth of the expression on the left hand side is the same as ξ, which we know, from Equation (B.35), grows at the same rate as $\mu A_0^{1/(1-\alpha_0)}$. We use this fact and the fact that $\frac{\nu_i}{c_i} = v_i^{1/(1-\sigma)} g(c)^{\varepsilon_i}$ to conclude from Equation (B.36) that:

$$\lim_{t \to \infty} \frac{\dot{v}_i}{v_i} = (1-\sigma) \left(\eta_g \varepsilon_i \gamma^* - \gamma_i - \frac{\alpha_i}{1-\alpha_0} \gamma_0 \right). \tag{B.37}$$

Crucially, Equation (B.37) implies that for any sector $i^* \in \mathcal{I}^*$, whose employment share does not asymptotically vanish, we should have:

$$\gamma^* = \frac{1}{\eta_g \varepsilon_{i^*}} \left(\gamma_{i^*} + \frac{\alpha_{i^*}}{1-\alpha_0} \gamma_0 \right).$$

Now, consider the set $\hat{\mathcal{I}}$, defined as follows:

$$\hat{\mathcal{I}} \equiv \operatorname{argmin}_{i \in \mathcal{I}/\{0\}} \frac{(1-\alpha_0)\gamma_i + \alpha_i \gamma_0}{(1-\alpha_0)\varepsilon_i}.$$

Consider the rate of growth of employment share in a consumption sector $\hat{i} \in \hat{\mathcal{I}}$:

$$\begin{aligned} \lim_{t \to \infty} \frac{\dot{v}_{\hat{i}}}{v_{\hat{i}}} &= (1-\sigma) \left(\eta_g \varepsilon_{\hat{i}} \gamma^* - \gamma_{\hat{i}} - \frac{\alpha_{\hat{i}}}{1-\alpha_0} \gamma_0 \right), \\ &= (1-\sigma) \varepsilon_{\hat{i}} \left[\frac{1}{\varepsilon_{i^*}} \left(\gamma_{i^*} + \frac{\alpha_{i^*}}{1-\alpha_0} \gamma_0 \right) - \frac{1}{\varepsilon_{\hat{i}}} \left(\gamma_{\hat{i}} + \frac{\alpha_{\hat{i}}}{1-\alpha_0} \gamma_0 \right) \right]. \end{aligned}$$

The expression shows that $\hat{\mathcal{I}} = \mathcal{I}^*$. If $\hat{i} \notin \mathcal{I}^*$, this expression has to be strictly negative, violating the assumption that $\hat{i} \in \hat{\mathcal{I}}$. If $i^* \notin \hat{\mathcal{I}}$, will have to be negative violating the assumption that $i^* \in \mathcal{I}^*$. It then follows that:

$$\gamma^* = \min_{i \in \mathcal{I}/\{0\}} \frac{(1-\alpha_0)\gamma_i + \alpha_i \gamma_0}{\eta_g (1-\alpha_0) \varepsilon_i}.$$

Finally, we need to check that the transversality condition is indeed satisfied. Since we know that the asymptotic rate of growth of $\frac{cu'(c)}{\mu\eta_g\overline{\varepsilon}}$ is $\frac{\gamma_0}{1-\alpha_0}$, we find that:

$$\lim_{t\to\infty}\frac{\dot{\mu}(t)}{\mu(t)}=(1-\theta)\gamma^*-\frac{\gamma_0}{1-\alpha_0},$$

where we have used the fact that $\lim_{t\to\infty}\overline{\varepsilon}(t)=\overline{\varepsilon}^*$ and $\lim_{t\to\infty}\eta_g(c(t))=\eta_g$ are both constants. Therefore, in order to satisfy condition B.25, we need to ensure that

$$\rho>n+(1-\theta)\gamma^*.$$

From Equations (B.17) and (B.20), $\mu(t)\geq 0$ for all t. Therefore, from strict concavity of $\mathcal{F}$ it follows that these equations together give a unique path of $[\boldsymbol{k}(\cdot),\boldsymbol{l}(\cdot),k(\cdot),\mu(\cdot)]_{t=0}^{\infty}$. Due to the strict concavity of $\mathcal{M}$, we conclude that the resulting path corresponds to the unique solution to the social planner's problem (see Theorem 7.14 in Acemoglu, 2008). This completes the proof of the proposition.

For completeness, we state the correspondence between the variables above and the variables characterizing the competitive equilibrium. Comparing expressions (B.34) and (B.35) with expressions (A.15) and (A.16) in the derivations of the competitive equilibrium, we find that $\varsigma\equiv\mu R$ and $\xi\equiv\mu w$. Substituting in the expression (B.33) yields:

$$\frac{u'(c)}{\mu}\frac{\partial c}{\partial c_i}=\frac{1}{A_i}\left(\frac{R}{\alpha_i}\right)^{\alpha_i}\left(\frac{w}{1-\alpha_i}\right)^{1-\alpha_i},$$

where the right hand side corresponds to the expression for unit price of goods in sector i in Equation (A.18). We use $\eta_c^{c_i}=\nu_i/\eta_g\overline{\varepsilon}$ to rewrite this expression as

$$\frac{cu'(c)}{\mu}\frac{\nu_i}{\eta_g\overline{\varepsilon}}=p_ic_i,$$

which, once we sum over $i\in\{1,\cdots,I\}$, implies

$$\frac{cu'(c)}{\mu\eta_g\overline{\varepsilon}}\equiv e.$$

In light of this connection, the key step of the proof above establishing the rate of growth of the expression on the left hand side has a stright forward interpretation: per capita consumption expenditure grows at the same rate as the output of the investment sector.

∎

C Equilibrium Dynamic Equations

In this section, we characterize the dynamics of this economy along the equilibrium path. Define investment sector productivity-adjusted aggregate and investment variables:

$$\tilde{k}\left(t\right)\equiv\frac{K\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)L\left(t\right)},\qquad\tilde{y}_{0}\left(t\right)\equiv\frac{Y_{0}\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)L\left(t\right)}.\qquad\tilde{e}\left(t\right)\equiv\frac{e\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)},\qquad\tilde{\kappa}_{0}\left(t\right)\equiv\frac{\kappa_{0}\left(t\right)}{A_{0}^{\frac{1}{1-\alpha_{0}}}\left(t\right)},\tag{C.1}$$

and denote the share of labor employed in the investment sector be $l_0(t) \equiv L_0(t)/L$. With this normalization, the rental price of capitaly and the capital-labor ratio in the investment sector have the following one-to-one relationship $R\left(t\right) = \alpha_0 \tilde{\kappa}_0(t)^{\alpha_0 - 1}$ and we can use them interchangably to characterize the path of the aggregate economy.

Per capita consumption and productivity-adjusted capital-labor ratio in the investment sector $(c(t), \tilde{\kappa}_0(t))$ fully characterize the state of the economy at time t. First, note that productivity-adjusted per capita consumption expenditure $\tilde{e}(t)$ is a function of these two variables and time:

$$\tilde{e}\left(t\right)=\tilde{e}\left(c\left(t\right),\tilde{\kappa}_{0}\left(t\right),t\right)=\left(\sum_{i=1}^{I}\left[\varphi_{i}(t)g\left(c\left(t\right)\right)^{\varepsilon_{i}}\tilde{\kappa}_{0}\left(t\right)^{\alpha_{0}-\alpha_{i}}\right]^{1-\sigma}\right)^{\frac{1}{1-\sigma}},\tag{C.2}$$

where we have substituted from Equations (A.18) and (C.1) in the definition of Equation (A.3) and have defined a (exogenously given) time-dependent function

$$\varphi_{i}(t)\equiv\left(\frac{\alpha_{0}}{\alpha_{i}}\right)^{\alpha_{i}}\left(\frac{1-\alpha_{0}}{1-\alpha_{i}}\right)^{1-\alpha_{i}}A_{0}\left(t\right)^{-\frac{\alpha_{i}}{1-\alpha_{0}}}A_{i}\left(t\right)^{-1}.$$

The direct dependence on time is due to the impact of time-varying sectoral technologies on prices. Similarly, we can write the share of sector i in consumption expenditure as a function of $(c(t), \tilde{\kappa}_0(t))$ and time as $\omega_i\left(t\right) = \omega_i\left(c\left(t\right), \tilde{\kappa}_0\left(t\right), t\right)$. Averages of the income elasticity parameter and capital share in the consumption sector of the economy $\bar{\varepsilon}$ and $\bar{\alpha}$ then also become functions of the two state variables (and time). We emphasize in the special case where capital intensities are identical across all consumption sectors, the expenditure function in Equation (A.3) becomes independent of capital-labor ratios and solely depends on real consumption per capita c and time.

In addition to the expenditure and sectoral shares, the duplet $(c(t), \tilde{\kappa}_0(t))$ also pins down total investment and total per-capita stock of capital at time t along any equilibrium path. To see that, we first compute the employment share of the investment sector, dropping the dependence on time to simplify notation:

$$
\begin{aligned}
l_0 \equiv \frac{L_0}{L} &= \frac{1}{1+\frac{L_C}{L_0}}, \\
&= \frac{(1-\alpha_0)\,\tilde{y}_0}{(1-\alpha_0)\,\tilde{y}_0+(1-\bar{\alpha})\,\tilde{e}},
\end{aligned} \tag{C.3}
$$

where in the second line we have used the equality $\frac{L_C(t)}{L_0(t)} = \frac{1-\bar{\alpha}}{1-\alpha_0}\frac{e}{\tilde{y}_0}$. Combining Equation (C.3) with $\tilde{y}_0 = l_0 \tilde{\kappa}_0^{\alpha_0}$, we can write both the normalized output and the employment share of the investment sector as:

$$
\tilde{y}_0 = \tilde{\kappa}_0^{\alpha_0} - \frac{1-\bar{\alpha}}{1-\alpha_0}\tilde{e}, \tag{C.4}
$$

$$
l_0 = 1 - \frac{1-\bar{\alpha}}{1-\alpha_0}\frac{\tilde{e}}{\tilde{\kappa}_0^{\alpha_0}}. \tag{C.5}
$$

Therefore, since average capital share $\bar{\alpha}$ and normalized expenditure $\tilde{e}$ are both functions of $(c(t), \tilde{\kappa}_0(t))$ and time, so are the employment share and normalized output of the investment sector. Finally, the following lemma establishes that the total per-capita stock of capital is also a function of the same pair of variables.

Lemma 2. *Along any equilibrium path and for all times t, the aggregate productivity-adjusted per-capita stock of capital $\tilde{k}$ and the productivity-adjusted capital-labor ratio in the investment sector $\tilde{\kappa}_0$ satisfy the following equation:*

$$
\tilde{k} = \tilde{\kappa}_0 \left[1 + \frac{\tilde{e}}{\tilde{\kappa}_0^{\alpha_0}}\left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\right)\right], \tag{C.6}
$$

where $\tilde{e}$ and $\bar{\alpha}$ are functions of $(c, \tilde{\kappa}_0)$ and time as defined by Equation (C.2). Moreover, for any level of per capita real consumption $c > 0$ at time t, Equation (C.6) defines a monotonically increasing and one-to-one mapping between $\tilde{k}$ and $\tilde{\kappa}_0$.

Proof. Along any equilibrium path, the output of all consumption goods are strictly positive and therefore $\kappa_i > 0$ for all $i \geq 1$. From Equation (A.16), we know that $R \geq \alpha_0 A_0 \kappa_0^{\alpha_0 - 1}$ and therefore along any equilibrium path $\kappa_0 > 0$. Hence, the allocations of labor and capital to all sectors are always interior.

Aggregate capital to labor ratio in the economy may be written as:

$$
\begin{aligned}
k = \frac{K}{L} &= \frac{L_0}{L}\kappa_0 + \frac{L_C}{L}\kappa_C, \\
&= l_0\kappa_0 + (1-l_0)\,\kappa_0\frac{\kappa_C}{\kappa_0}, \\
&= \kappa_0\left[l_0 + (1-l_0)\,\frac{\bar{\alpha}/\,(1-\bar{\alpha})}{\alpha_0/\,(1-\alpha_0)}\right], \\
&= \kappa_0\left[1 + \frac{\tilde{e}}{\tilde{\kappa}_0^{\alpha_0}}\left(\frac{\bar{\alpha}}{\alpha_0} - \frac{1-\bar{\alpha}}{1-\alpha_0}\right)\right],
\end{aligned} \tag{C.7}
$$

where in the second equality, we have defined L_C and κ_C as the total employment and captial-labor ratio in the consumption sector, and in the third equality, we have used the expressions for capital-to-labor ratios in Equations (A.17) as well as

$$
\kappa_C = \frac{\overline{\alpha}}{1-\overline{\alpha}}\frac{w}{R}.
$$

In the last equality, we have used the expression for the share of employment in the investment sector from Equation (C.5). Adjusting both sides of Equation (C.7) with respect to the productivity in the investment sector yields the desired result.

We will now show that the function defined by Equation (C.7) is one-to-one and monotonically increasing, mapping values of $\tilde{k}$ to $\tilde{\kappa}_0$ everywhere along any equilibrium path. To show this, it is sufficient to establish that the derivative of this function with respect to $\tilde{\kappa}_0$ is everywhere strictly positive.

$$
\begin{aligned}
\frac{\partial \tilde{k}}{\partial \tilde{\kappa}_0} &= 1 + \left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\right)\frac{\partial}{\partial\tilde{\kappa}_0}\left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right) + \left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right)\frac{\partial}{\partial\tilde{\kappa}_0}\left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\right), \\
&= 1 + \left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0\,(1-\alpha_0)}\right)\tilde{\kappa}_0^{-\alpha_0}\tilde{e}\frac{\partial\log\left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right)}{\partial\log\tilde{\kappa}_0} + \left(\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\right)\frac{\bar{\alpha}}{\alpha_0\,(1-\alpha_0)\,\tilde{\kappa}_0}\frac{\partial\log\bar{\alpha}}{\partial\log\tilde{\kappa_0}}, \\
&= 1 + \left(\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}}{\alpha_0\,(1-\alpha_0)}\right)\left[(\bar{\alpha}-\alpha_0)\left(1-\alpha_0-\sum_i \eta_{\tilde{e}}^{p_i}\eta_{p_i}^{\tilde{\kappa}_0}\right) + \bar{\alpha}\sum_i \eta_{\bar{\alpha}}^{p_i}\eta_{p_i}^{\tilde{\kappa}_0}\right], \\
&= 1 + \left(\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}}{\alpha_0\,(1-\alpha_0)}\right)\left[(\bar{\alpha}-\alpha_0)\,(1-\alpha_0-(\alpha_0-\overline{\alpha})) + \overline{\alpha}\,(1-\sigma)\sum_i \omega_i\left(\frac{\alpha_i}{\overline{\alpha}}-1\right)(\alpha_0-\alpha_i)\right], \\
&= 1 + \left(\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}}{\alpha_0\,(1-\alpha_0)}\right)\left[(\bar{\alpha}-\alpha_0)\,(1-\overline{\alpha}) - (1-\sigma)\,Var\,(\alpha)\right],
\end{aligned}
$$

where in the third equality we have invoked the results of Lemma 1. Recalling the expression for the employment share of the investment sector l_0 from Equation C.5, we can now rewrite this as follows:

$$\begin{aligned}\frac{\partial \tilde{k}}{\partial \tilde{\kappa}_0} &= 1+(1-l_0)\left(\frac{\bar{\alpha}-\alpha_0}{\alpha_0}-\frac{(1-\sigma)\,Var\,(\alpha)}{\alpha_0\,(1-\bar{\alpha})}\right), \\ &= l_0+\frac{1-l_0}{\alpha_0\,(1-\bar{\alpha})}\left[\bar{\alpha}\,(1-\bar{\alpha})-(1-\sigma)\,Var\,(\alpha)\right].\end{aligned}$$

Finally, note that the expression within the square bracket on the right hand side is always positive. This is because:

$$\begin{aligned}\bar{\alpha}\,(1-\bar{\alpha})-(1-\sigma)\,Var\,(\alpha) &\geq \bar{\alpha}\,(1-\bar{\alpha})-Var\,(\alpha)\,, \\ &= \bar{\alpha}-\overline{\alpha^2}>0.\end{aligned}$$

where the inequality in the second line follows from the fact that for all sectors i, we have $0<\alpha_i<1$. This completes the proof that the mapping of $\tilde{k}$ to $\tilde{\kappa}_0$ is monotonic and one-to-one. ∎

Equation (C.6) shows that whenever average capital share of the consumption sector exceeds that of the investment sector, the economy-wide capital-labor ratio k is greater than the capital-labor ratio in the investment sector κ_0. Furthermore, the lemma ensures us that the relationship that the two ratios is one-to-one. This point is critical since it allows us to use the investment sector capital-labor ratio $\tilde{\kappa}_0$ as the state variable fully characterizing the path of capital accumulation in the economy.

The next proposition characterizes the dynamics of competitive equilibria in our economy in terms of the two state variables $(c,\tilde{\kappa_0})$. Before introducing the dynamic equations, let us introduce a function a function χ of the state variables:

$$\chi\,(c,\tilde{\kappa}_0;t)\equiv\frac{\tilde{\kappa}_0^{-\alpha_0}\tilde{e}\,(c,\tilde{\kappa}_0;t)}{\alpha_0\,(1-\alpha_0)}. \tag{C.8}$$

This function has the property that $\chi=\frac{l_C}{\alpha_0(1-\bar{\alpha})}=\frac{\tilde{k}/\tilde{\kappa}_0-1}{\bar{\alpha}-\alpha_0}$ and will greatly simplify the exposition of the forthcoming lemma.

Proposition 3. *The following system of two equations characterizes the dynamics of state* *variables* $(c, \tilde{\kappa}_0)$ *in any competitive equilibria of our economy:*

$$\begin{aligned}
&\left[-\eta_{u'} - 1 + \eta_{\eta_g} + \eta_g \overline{\varepsilon} \left(1 + (1-\sigma) Var\left(\frac{\varepsilon}{\overline{\varepsilon}}\right)\right)\right] \frac{\dot{c}}{c} \\
&\qquad + \left[\alpha_0 - \overline{\alpha}\left(1 + (1-\sigma)\rho_{\varepsilon,\alpha}\right)\right] \frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} = \alpha_0 \tilde{\kappa}_0^{\alpha_0 - 1} - (\delta + \rho) + \bar{\gamma}\left(1 + (1-\sigma)\rho_{\varepsilon,\gamma}\right) \\
&\qquad\qquad - \frac{1 - \overline{\alpha}\left(1 + (1-\sigma)\rho_{\varepsilon,\alpha}\right)}{1-\alpha_0}\gamma_0,
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
&\eta_g \overline{\varepsilon}\left[\overline{\alpha}\left(1 + (1-\sigma)\rho_{\varepsilon,\alpha}\right) - \alpha_0\right]\chi \frac{\dot{c}}{c} \\
&+ \left[1 + (1-\overline{\alpha})(\overline{\alpha} - \alpha_0)\chi - (1-\sigma) Var(\alpha)\chi\right] \frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} = \left(1 - \alpha_0 (1-\overline{\alpha})\chi\right) \tilde{\kappa}_0^{\alpha_0 - 1} \\
&\qquad - \left(1 + (\overline{\alpha} - \alpha_0)\chi\right)\left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right) \\
&\qquad + \left[\overline{\alpha}(\overline{\alpha} - \alpha_0) + (1-\sigma) Var(\alpha)\right]\chi\gamma_0 \\
&\qquad + \left[\overline{\alpha}\left(1 + (1-\sigma)\rho_{\alpha,\gamma}\right) - \alpha_0\right]\chi\overline{\gamma},
\end{aligned} \tag{C.10}$$

where $\overline{x}$ *and* $Var(x)$ *denote the average and variance of a sector-specific set of parameters* $\boldsymbol{x}$ *and* $\rho_{x,x'}$ *denotes the correlation coeffcient between this parameter and another set of parameters* $\boldsymbol{x}'$*, all according to the distribution implied by sectoral expenditures shares at time* t*, and* χ *is defined by Equation* (C.8)*.*

If the condition (A.12) *is satisfied (the instantaneous utility function defined in Equations* (A.1)*,* (A.6)*, and* (A.2) *is concave in real consumption* c*), then the system above uniquely determines* $(\dot{c}/c, \dot{\tilde{\kappa}}_0/\tilde{\kappa}_0)$ *at time* t*.*

Proof. First, let us express the Euler Equation (B.5) in terms of the variables $(c, \tilde{\kappa}_0, t)$ by substituting for the growth of sectoral prices based on the production side of our economy. From Equation (A.18) we can write sectoral prices as

$$p_i = \tilde{\varphi}_i A_i^{-1} A_0^{\frac{1-\alpha_i}{1-\alpha_0}} \tilde{\kappa}_0^{\alpha_0 - \alpha_i},$$

where φ_i is a constant sector-specific parameter. This implies that the rate of growth of sectoral prices is given by:

$$\frac{\dot{p}_i}{p_i} = \frac{1-\alpha_i}{1-\alpha_0}\gamma_0 - \gamma_i + \frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}(\alpha_0 - \alpha_i).$$

This allows us to compute the average of growth rates of sectoral prices and their covariance with income elasticity parameters under the distribution implied by expenditure shares:

$$\begin{aligned}
\overline{\left(\frac{\dot{p}_i}{p_i}\right)} &= \frac{1-\overline{\alpha}}{1-\alpha_0}\gamma_0 - \overline{\gamma} + (\alpha_0 - \overline{\alpha})\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}, \\
Cov\left(\varepsilon_i, \frac{\dot{p}_i}{p_i}\right) &= -Cov(\varepsilon_i, \gamma_i) - \left(\frac{\gamma_0}{1-\alpha_0} + \frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right) Cov(\varepsilon_i, \alpha_i).
\end{aligned}$$

Substituting this relation and the fact that $r = R - \delta = \alpha_0 \tilde{\kappa}_0^{\alpha_0 - 1} - \delta$ in the Euler equation, yields:

$$\begin{aligned}
&\left[-\eta^c_{u'} - 1 + \eta_{\eta_g} + \eta_g \bar{\varepsilon}\left(1 + (1-\sigma) Var\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\right)\right]\frac{\dot{c}}{c} \\
&\qquad + \left[\alpha_0 - \overline{\alpha}\left(1 + (1-\sigma)\rho_{\varepsilon,\alpha}\right)\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} = \alpha_0 \tilde{\kappa}_0^{\alpha_0 - 1} - (\delta + \rho) + \bar{\gamma}\left(1 + (1-\sigma)\rho_{\varepsilon,\gamma}\right) \\
&\qquad\qquad - \frac{1 - \overline{\alpha}\left(1 + (1-\sigma)\rho_{\varepsilon,\alpha}\right)}{1-\alpha_0}\gamma_0,
\end{aligned}$$

where $\rho_{\varepsilon,\gamma}$ and $\rho_{\varepsilon,\alpha}$ denote correlation coefficients between the income elasticity parameters and the technological rates of growth and capital shares at the sectoral levels, both under the distributions implied by expenditure shares.

The equation governing the evolution of aggregate capital stock can be written as follows

$$\begin{aligned}
\dot{\tilde{k}} &= \left(\frac{\dot{K}}{K} - n - \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \\
&= \frac{Y_0 \tilde{k}}{K} - \left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \\
&= \tilde{y}_0 - \left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \\
&= \tilde{\kappa}_0^{\alpha_0}\left(1 - \alpha_0 (1 - \overline{\alpha})\chi\right) - \left(n + \delta + \frac{\gamma_0}{1-\alpha_0}\right)\tilde{k}, \qquad \text{(C.11)}
\end{aligned}$$

where we have used $\dot{K} = Y_0 - \delta K$ in the second equality and Equations (C.4) and (C.8) in the fourth equality. Next, we need to transform this equation into one described in terms of the per-capita stock of capital in the investment sector $\tilde{\kappa}_0$. This will complete the characterization of the dynamics of the pair $(c, \tilde{\kappa}_0)$.

Lemma 2 established that along any equilibrium path a one-to-one mapping exists that relates a level of (productivity-adjusted) per capita stock of capital $\tilde{k}$ to a corresponding level of (productivity-adjusted) capital-per-worker $\tilde{\kappa}_0$ in the investment sector. Taking the derivative of this function (from Equation (C.6)) yields:

$$\begin{aligned}\dot{\tilde{k}} &= \dot{\tilde{\kappa}}_0 + (1-\alpha_0)\,\tilde{\kappa}_0^{-\alpha_0}\tilde{e}\frac{\overline{\alpha}-\alpha_0}{\alpha_0(1-\alpha_0)}\dot{\tilde{\kappa}}_0 + \tilde{\kappa}_0^{1-\alpha_0}\frac{\overline{\alpha}-\alpha_0}{\alpha_0(1-\alpha_0)}\dot{\tilde{e}} + \tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\frac{1}{\alpha_0(1-\alpha_0)}\dot{\overline{\alpha}}, \\ &= \left[\tilde{\kappa}_0 + (1-\alpha_0)\,\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}\frac{\overline{\alpha}-\alpha_0}{\alpha_0(1-\alpha_0)}\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} + \frac{\tilde{\kappa}_0^{1-\alpha_0}\tilde{e}}{\alpha_0(1-\alpha_0)}\left[(\overline{\alpha}-\alpha_0)\frac{\dot{\tilde{e}}}{\tilde{e}} + \overline{\alpha}\frac{\dot{\overline{\alpha}}}{\overline{\alpha}}\right]. \qquad \text{(C.12)}\end{aligned}$$

Therefore, we need to compute the growth rates of expenditure $\dot{\tilde{e}}/\tilde{e}$ and average (consumption-sector) capital intensities $\dot{\overline{\alpha}}/\overline{\alpha}$ in terms of the growth rates of real consumption $\dot{c}/c$ and investment-sector capital-to-labor ratio $\dot{\tilde{\kappa}}_0/\tilde{\kappa}_0$.

Now, we can use the expressions for the elasticity of function $\overline{\alpha}$ with respect to real consumption and price, from Lemma 1, to find:

$$\begin{aligned}\frac{\dot{\overline{\alpha}}}{\overline{\alpha}} &= \eta^c_{\overline{\alpha}}\cdot\frac{\dot{c}}{c} + \sum_i \eta^{p_i}_{\overline{\alpha}}\frac{\dot{p}_i}{p_i}, \\ &= \frac{1-\sigma}{\overline{\alpha}}\eta_g Cov\,(\varepsilon,\alpha)\cdot\frac{\dot{c}}{c} + (1-\sigma)\sum_i\left(\frac{\alpha_i}{\overline{\alpha}}-1\right)\omega_i\left[\frac{1-\alpha_i}{1-\alpha_0}\frac{\dot{A}_0}{A_0} - \frac{\dot{A}_i}{A_i} + (\alpha_0-\alpha_i)\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right], \\ &= (1-\sigma)\,\eta_g\overline{\varepsilon}\rho_{\varepsilon,\alpha}\cdot\frac{\dot{c}}{c} - \frac{1-\sigma}{\overline{\alpha}}\left[Cov\,(\alpha,\gamma) + \left(\frac{\gamma_0}{1-\alpha_0} + \frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right)Var\,(\alpha)\right].\end{aligned}$$

Similarly, we can write the growth rate of consumption expenditure as:

$$\begin{aligned}\frac{\dot{\tilde{e}}}{\tilde{e}} &= \eta^c_e\frac{\dot{c}}{c} + \sum_i \eta^{p_i}_e\frac{\dot{p}_i}{p_i} - \frac{1}{1-\alpha_0}\frac{\dot{A}_0}{A_0}, \\ &= \eta_g\overline{\varepsilon}\frac{\dot{c}}{c} + \sum_i\omega_i\left[\frac{1-\alpha_i}{1-\alpha_0}\frac{\dot{A}_0}{A_0} - \frac{\dot{A}_i}{A_i} + (\alpha_0-\alpha_i)\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0}\right] - \frac{1}{1-\alpha_0}\frac{\dot{A}_0}{A_0}, \\ &= \eta_g\overline{\varepsilon}\frac{\dot{c}}{c} - \overline{\gamma} + (\alpha_0-\overline{\alpha})\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} - \frac{\overline{\alpha}\gamma_0}{1-\alpha_0}.\end{aligned}$$

We can write Equation (C.12) as:

$$\begin{aligned}\frac{\dot{\tilde{k}}}{\tilde{\kappa}_0} &= [1 + (1-\alpha_0)(\overline{\alpha}-\alpha_0)\chi]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} + \chi\left[(\overline{\alpha}-\alpha_0)\frac{\dot{\tilde{e}}}{\tilde{e}} + \overline{\alpha}\frac{\dot{\overline{\alpha}}}{\overline{\alpha}}\right], \\ &= [1 + (1-\overline{\alpha})(\overline{\alpha}-\alpha_0)\chi - (1-\sigma)\,Var\,(\alpha)]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} + \eta_g\overline{\varepsilon}\,[\overline{\alpha}\,(1+(1-\sigma)\,\rho_{\varepsilon,\alpha}) - \alpha_0]\,\chi\frac{\dot{c}}{c} \\ &\qquad - [\overline{\alpha}\,(1+(1-\sigma)\,\rho_{\alpha,\gamma}) - \alpha_0]\,\chi\overline{\gamma} - [\overline{\alpha}\,(\overline{\alpha}-\alpha_0) + (1-\sigma)\,Var\,(\alpha)]\,\chi\gamma_0.\end{aligned}$$

Finally, substituting this expression into Equation (C.11) give us:

$$\begin{aligned}
\left[1+(1-\overline{\alpha})(\overline{\alpha}-\alpha_0)\chi-(1-\sigma)Var(\alpha)\right]\frac{\dot{\tilde{\kappa}}_0}{\tilde{\kappa}_0} & \\
+\eta_g\overline{\varepsilon}\left[\overline{\alpha}\left(1+(1-\sigma)\rho_{\varepsilon,\alpha}\right)-\alpha_0\right]\chi\frac{\dot{c}}{c} &= \left(1-\alpha_0(1-\overline{\alpha})\chi\right)\tilde{\kappa}_0^{\alpha_0-1} \\
& \quad -\left(1+(\overline{\alpha}-\alpha_0)\chi\right)\left(\rho+\delta+\frac{\gamma_0}{1-\alpha_0}\right) \\
& \quad +\left[\overline{\alpha}(\overline{\alpha}-\alpha_0)+(1-\sigma)Var(\alpha)\right]\chi\gamma_0 \\
& \quad +\left[\overline{\alpha}\left(1+(1-\sigma)\rho_{\alpha,\gamma}\right)-\alpha_0\right]\chi\overline{\gamma}.
\end{aligned}$$

where we have used the fact that $l_0 = 1 - l_C = 1 - \alpha_0(1-\overline{\alpha})\chi$ and $\tilde{k}/\tilde{\kappa}_0 = 1 + (\overline{\alpha}-\alpha_0)\chi$.

To ensure that the system of Equations above indeed has a solution, we need to establish that the determinant of the following matrix is nonzero:

$$M = \begin{bmatrix} -\eta^c_{u'} - 1 + \eta_{\eta_g} + \eta_g\overline{\varepsilon}\left(1+(1-\sigma)Var\left(\frac{\varepsilon}{\overline{\varepsilon}}\right)\right) & \alpha_0 - \overline{\alpha}\left(1+(1-\sigma)\rho_{\varepsilon,\alpha}\right) \\ \eta_g\overline{\varepsilon}\left[\overline{\alpha}\left(1+(1-\sigma)\rho_{\varepsilon,\alpha}\right)-\alpha_0\right]\chi & 1+(1-\overline{\alpha})(\overline{\alpha}-\alpha_0)\chi-(1-\sigma)Var(\alpha)\chi \end{bmatrix}.$$

A necessary condition for the Euler equation to have a unique solution is that the expression $M_{1,1}$ is strictly positive. Since $M_{1,2}\cdot M_{2,1}\le 0$ and $M_{2,2} = \frac{\partial k}{\partial \kappa_0} > 0$ (from Lemma 2), this is sufficient to ensure that the determinant of matrix M is indeed positive.

∎

Starting from initial level of per capita stock of capital $k(0)$, Equation (C.6) and a choice of $c(0)$, give the corresponding allocation of capital to the investment sector κ_0. For all $t \ge 0$, Equations (C.9) and (C.10) describe the dynamics of the economy in terms of $(c, \tilde{\kappa}_0)$.

The two state variables $(c, \tilde{\kappa}_0)$ at time t are sufficient to fully specify the economy. As we discussed, both $\tilde{e}$ and $\bar{\alpha}$ are functions of c and $\tilde{\kappa}_0$, with the dependence on the latter going through the dependence of functions (A.4) on prices, as specified by Equation (A.18). Knowing the two state variables at any given time t, capital and labor employed in each consumption good sector i may be written as:

$$\begin{aligned}
K_i &= \left(\frac{\alpha_i}{\alpha_0}\omega_i(c,\tilde{\kappa}_0)\cdot\frac{\tilde{e}(c,\tilde{\kappa}_0)}{\tilde{\kappa}_0^{\alpha_0}}\right)\cdot LA_0^{\frac{1}{1-\alpha_0}}, \\
L_i &= \left(\frac{1-\alpha_i}{1-\alpha_0}\omega_i(c,\tilde{\kappa}_0)\cdot\frac{\tilde{e}(c,\tilde{\kappa}_0)}{\tilde{\kappa}_0^{\alpha_0}}\right)\cdot L,
\end{aligned}$$

which completes the characterization of the economy.

D Stone-Geary Estimation

We follow the estimation procedure described in Herrendorf et al. (2013). The identification of the parameters in the utility function is based on the estimation of the first order conditions of the intra-period problem to estimate the parameters of the aggregator (28),

$$\frac{L^c_{it}}{L^c_t} = \frac{\Omega^c_i p^{c\,1-\sigma}_{it}}{\sum_{i\in\{a,m,s\}}\Omega^c_i p^{c\,1-\sigma}_{it}}\left(1+\frac{\sum_{i\in\{a,s\}} p^c_{it}\bar{c}_i}{\sum_{i\in\{a,m,s\}} p^c_{it}C^c_{it}}\right) - \frac{p^c_{it}\bar{c}_i}{\sum_{i\in\{a,m,s\}} p^c_{it}C^c_{it}} \tag{D.1}$$

with the understanding that $\bar{c}_m = 0$. We perform the estimation with and without sectoral trade controls for net exports. As with nonhomothetic CES preferences, we estimate three parameters that are common across countries $\{\sigma, \bar{c}_a, \bar{c}_s\}$ that govern the price and income elasticities, and $\{\Omega^c_i\}_{i\in\mathcal{I},c\in C}$ which are country specific parameters.

Table G.6 (also in the online appendix) reports the estimates. As expected, we find that $\bar{c}_a < 0$ and $\bar{c}_s > 0$. However, we cannot reject that the point estimates are statistically different from zero at conventional levels when clustering the standard errors at the country level.[7]

To have a better grasp of the magnitude of the income effects, we compute the values of $-\frac{p^c_{a,t}\bar{c}_a}{\sum_{i\in\{a,m,s\}} p^c_{it}C^c_{it}}$ and $\frac{p^c_{s,t}\bar{c}_s}{\sum_{i\in\{a,m,s\}} p^c_{it}C^c_{it}}$. For the U.S., they are never higher (in absolute terms) than .05%, which suggests that non-homtheticities are insignificant when companred to aggregate consumption. The highest values of the non-homotheticities are found in Venezuela where they reach 11% for agriculture and 22% for services.

E Stata Code for Monte-Carlo Simulations and Estimation

This Stata code generates time series of prices, sectoral expenditure for one country as chosen by a representative agent with nonhomothetic CES.[8] The code prints on screen the estimated elasticities using the true price index, a chained Fisher price index and a simple CPI constructed using weighted prices by expenditure shares. The bottom line is that all of them provide very similar estimates.

Note that we have shocks to growth rates of relative prices and aggregate consumption to avoid colinearity among regressors and allow for identification. All shocks can be dispensed with if two (or more) countries are generated, as this provides additional variation to identify the elasticities.

```
** Create a fake dataset using non-homothetic CES and estimate it
clear all
```

[7] We control for net sectoral exports in agriculture and manufacturing in each regression. We also report the log-likelihood of the overall fit and the Akaike information criterion in Table G.6.

[8] We thank Tomasz Swiecki for a fruitful discussion on this topic.

```
set more off

******* Basic Parameters *********
set obs 100 /*number of observations*/
global gc .01 /*growth of real c*/
global sigma .75 /* Price Elasticity -- Change here to obtain different resilts*/
global ga .022
global gm .017
global gs .016
global ea 0.5 /*Income Elasticities */
global em 1.1
global es 1.4


******* Generate Variables ********
gen t = _n
gen rc = rnormal(0,.01)
gen C = 100*(1+$gc+rc)^t

gen ra = rnormal(0,.01)
gen rm = rnormal(0,.01)
gen rs = rnormal(0,.01)

gen pa= 100*(1+$ga+ra)^(-t)
gen pm= 100*(1+$gm+rm)^(-t)
gen ps= 100*(1+$gs+rs)^(-t)

gen P = (C^($ea-1)*pa^(1-$sigma)+C^($em-1)*pm^(1-$sigma) ///
+C^($es-1)*ps^(1-$sigma))^(1/(1-$sigma))
label var P "Ideal Price Index"

gen expsh_a = C^($ea-1)*(pa/P)^(1-$sigma)
gen expsh_m = C^($em-1)*(pm/P)^(1-$sigma)
gen expsh_s = C^($es-1)*(ps/P)^(1-$sigma)

gen cpi =expsh_a*pa+ expsh_m*pm + expsh_s*ps
gen fisher = sqrt(expsh_a*pa[t-1]+ expsh_m*pm[t-1] + expsh_s*ps[t-1]) ///
             *sqrt(expsh_a[t-1]*pa+ expsh_m[t-1]*pm + expsh_s[t-1]*ps)

gen expenditure = P*C
gen log_cpi_inv = log(expenditure/cpi)
gen log_fisher_inv=log(expenditure/fisher)
```

```
****** Generate Logs of Variables ********
gen log_expsh_a = log(expsh_a)
gen log_expsh_m = log(expsh_m)
gen log_expsh_s = log(expsh_s)

gen log_sh_am   = log(expsh_a)-log(expsh_m)
gen log_sh_sm   = log(expsh_s)-log(expsh_m)
gen log_pam     = log(pa/pm)
gen log_psm     = log(ps/pm)
gen log_c       = log(C)

****** Regression Analysis *********
* True Regression
reg log_sh_am log_pam log_c
reg log_sh_sm log_psm log_c

* Regression with Regular CPI and Fisher Indices
reg log_sh_am log_pam log_cpi_inv
reg log_sh_sm log_psm log_cpi_inv

* Regressions run in the paper (SUR)
local eq1 log_sh_am log_pam log_cpi_inv,
local eq2 log_sh_sm log_psm log_cpi_inv,

constraint 1 [log_sh_sm]log_psm=[log_sh_am]log_pam
sureg (`eq1') (`eq2') ,  const(1)

local eq1 log_sh_am log_pam log_fisher_inv,
local eq2 log_sh_sm log_psm log_fisher_inv,

constraint 1 [log_sh_sm]log_psm=[log_sh_am]log_pam
sureg (`eq1') (`eq2') ,  const(1)

********* Display Results **********
quietly {

noisily: di "*******************************"_n"SIGMA"  _n ///
 "True: $sigma" _n "Estimated: "(1-_b[log_sh_sm:log_psm])
noisily: di _n "RELATIVE INCOME ELASTICITIES"
noisily: di "True: " (($ea-$em)/($es-$em))
noisily: di "Estimated: "(_b[log_sh_am:log_fisher_inv]/_b[log_sh_sm:log_fisher_inv]) ///
 _n  "*******************************"
}
```

F Further Discussion on the Definition and Use of Real Consumption

In this paragraph, we outline the justification for the choice of the chained-Fisher price index as a deflator to construct our empirical measures of real consumption. A more detailed explanation is given in subsection F.1. The justification comes from the following two facts. First, the exact ideal price index for any non-homothetic continuously differentiable utility function is the Törnqvist-Theil Index up to a second order approximation. Second, the Fisher index and the Törnqvist-Theil Index approximate each other up to second order. Thus, the using a Fisher index provides a second order approximation to our ideal price index. Diewert (1976, 1978, 2002) provides a formal proof of these statements and a quantitative analysis. To assess the accuracy of this second order estimation we have also run Monte-Carlo simulations using the true price index and the Fisher price index and the estimation results are almost identical. For example, when generating data consistent with the U.S., the estimates were identical up to the fifth decimal. Also, using a simple non-chained deflator did not make a big quantitative difference. This is indeed a mere reproduction of the results reported in Diewert (1976, 1978) and Hill (2006). Online Appendix E contains a template of the code used.

F.1 On the Use of Price Deflators to Construct Real Consumption Measures

In this section we discuss the results from Diewert (1976, 1978) and cover in more detail the discussion on the Use of Price Deflators to Construct Real Consumption Measures discussed in the first paragraph paragraph. Let the distance function be defined as $d(\mathbf{c}, u) = \max_d\{\delta : U(\mathbf{c}/\delta) \geq u\}$, where bold fonts denote vectors and $U(\mathbf{c})$ is consumer's direct utility function. This function is the dual of the cost function implied by U, $\kappa(\mathbf{p}, u)$ (see Deaton, 1979). Note that by definition $U(\mathbf{c}/d(\mathbf{c}, u)) = u$. Thus, d is simply a quantity index defined relative to scalar value u.

To see more clearly the quantity index nature of the distance function, studying the dual of the distance function is useful. Let $\psi(\mathbf{p}, E) = \psi(1, \mathbf{p}/E) \equiv \psi(\mathbf{p}/E)$ denote the indirect utility associated with U, where we have used homogeneity of degree zero of the indirect utility function. Suppose that at given prices $\mathbf{p}$ and expenditure E, we have that $\psi(\mathbf{p}, E) = u$. Define the dual distance $d^*(\mathbf{p}, u)$. Then,

$$\psi\left(\frac{\mathbf{p}}{d^*(\mathbf{p}, u)}\right) = u. \tag{F.1}$$

Thus, $d^*(\mathbf{p}, u) = E$. In other words,

$$d(\mathbf{q}, u)\kappa(\mathbf{p}, u) = E. \tag{F.2}$$

It is in this sense that $d \equiv C$ is interpreted as a quantity index (real consumption in our case) and $\kappa \equiv P$ as a price index. The previous expression can be written in terms of log-changes as

$$\Delta \ln d + \Delta \ln \kappa = \Delta \ln E, \tag{F.3}$$

where Δ denotes a change between, say, period 0 and 1. The Malmquist Quantity Index is defined the difference in distances holding the reference utility constant, $\ln Q_{01}^M(\mathbf{q}_0, \mathbf{q}_1, u) = \ln d(\mathbf{q}_1, u) - \ln d(\mathbf{q}_0, u)$. Intuitively, it compares two different consumption baskets measuring the change in the distance to an indifference curve that generates utility u. Theorems 2.16 and 2.17 in Diewert (1976) state the following:

Theorem Let the distance function generated by U be a general translog function of the form

$$\ln d(\mathbf{c}, U) = a_0 + \mathbf{a}' \ln \mathbf{c} + \frac{1}{2} \ln \mathbf{c}' \mathcal{A} \ln \mathbf{c} + b_0 \ln U + \ln U \mathbf{b}' \ln \mathbf{c} + \frac{1}{2} c_0 (\ln U)^2 \tag{F.4}$$

where bold fonts denote vectors and calligraphic, matrices.[9] Suppose that the quantity vector $\mathbf{c}_i$ is a solution to the maximization problem $\max_{\mathbf{c}}\{U(\mathbf{c_i}) : \mathbf{p_i c} = \mathbf{p_i c_i}\}$ then

$$Q_{01}(\mathbf{c_0}, \mathbf{c_1}, \mathbf{p}_0, \mathbf{p}_1) = \prod_{i=1}^{N} \left(\frac{c_{1i}}{c_{0i}}\right)^{\frac{\omega_{0i}+\omega_{1i}}{2}} = \frac{d(\mathbf{c}_1, u^*)}{d(\mathbf{c}_0, u^*)} = Q_{01}^M(\mathbf{c}_0, \mathbf{c}_1, u^*) \tag{F.5}$$

where c_{ji} and ω_{ji} denote consumption and expenditure shares of good i at time t and $u^* = \sqrt{U(\mathbf{q_o})U(\mathbf{q_1})}$. Let the cost function generated by U, $\kappa(\mathbf{p}, u)$, be a translog function (analogous to F.4). Then,

$$P_{01}(\mathbf{c_0}, \mathbf{c_1}, \mathbf{p}_0, \mathbf{p}_1) = \prod_{i=1}^{N} \left(\frac{p_{1i}}{p_{0i}}\right)^{\frac{\omega_{0i}+\omega_{1i}}{2}} = \frac{\kappa(\mathbf{p}_1, u^*)}{\kappa(\mathbf{p}_0, u^*)} \tag{F.6}$$

The left hand side of (F.5) and (F.6) correspond to the Törnqvist indeces (which can be computed empirically), while the right hand side of (F.5) is the Malquist Index and the Konüs index or "True Cost of Living" (Diewert, 1976), for (F.6). Diewert (1976) shows that there exists functional specifications of (F.4) other than the translog for which this result also holds exactly. In other words, this is not an if and only if result. The appeal of the translog function is that it approximates to a second order any continuously differentiable function. In particular, for the cost function implied by (A.3) is continuously differentiable, so this theory applies to the preferences in our paper.

Diewert (1978) shows that the Törnqvist index and the Fisher price index coincide with each other up to the second order terms of a Taylor expansion around any arbitrary pair of prices and quantities $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{c}_0, \mathbf{c}_1)$. In other words, Törnqvist and Fisher price indeces have exactly the same first and second order derivatives evaluated at $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{c}_0, \mathbf{c}_1)$. Thus, they approximate each other up to a second order. Quantitatively, the difference between these indices is very small. For example, Hill (2006) studies a time series data set covering 64 components of United States gross domestic product from 1977 to 1994. He finds that making all possible bilateral comparisons between any two years, the Fisher and Törnqvist price indices differed by only 0.1 per cent on average. This close correspondence is consistent with the results of other empirical studies using annual time series data (Diewert, 1976, 1978). Thus, the use of both price indices seems to make very little difference in practice.

Diewert (1978) argues that chaining the indices overtime is preferable to hold a constant basket as long as quantities change less between two adjacent periods than in two distant-in-time periods.

[9]There exists some regularity restrictions for the distance function in order to be a well defined object from the point of view of consumer theory. See Diewert (1971, 1976).

Indeed, Alterman et al. (1999) (page 61, Proposition 1) showed that if the logarithmic price ratios $\ln(p_{it}/p_{it-1})$ trend linearly with time t and the expenditure shares also trend linearly with time, then the Törnqvist chained-index is exact in the sense that it does not introduce any intertemporal bias. That is, chaining by year or defining a price index between two arbitrary years is equivalent. They also show that quantitatively using Fisher or Törnqvist makes little difference (the theoretical reason being again that one approximates the other very well, as we have already discussed).

We conclude from this analysis that using the chained Fisher price indexes to deflate nominal expenditure constitutes a very good approximation of real consumption.

F.2 Income Elasticities and Utility Cardinalization

Finally, concerning the identification of the income elasticities, as noted by Hanoch (1975), there is one degree of freedom that is not pinned down by the nonhomothetic CES, (2). This is why only the relative slopes of the Engel curves, $\varepsilon_i - \varepsilon_m$ can be identified. However, this normalization does not affect the real allocations, just the level of utility. To see that, consider the monotonic transformation our real consumption measure as $\tilde{C}_t = C_t^{\zeta}$ in the definition of the utility function, (2). This would not change the real allocations in our economy nor the expenditure elasticities as defined in (14) (see Appendix A). However, the implied real consumption elasticities would change from ε_i to $\zeta\varepsilon_i$. Thus, the level of the estimates we obtain in our estimation $\varepsilon_i - \varepsilon_j$ depend on the choice of the definition of real consumption. As we have discussed, when we deflate using the chained-Fisher price index, our aggregate consumption is defined relative to a baseline utility level (that is not known).[10] This is the reason why in the theory section A we choose a different normalization of utility that can be interpreted in meaningful monetary terms. We follow the definition of Feenstra et al. (2013) for real aggregate variables. We select a cardinalization of the utility such that it informs us of the monetary cost of obtaining a given level of utility for a given set of reference prices. In practice, this definition introduces a normalization to income elasticities (dividing all of them by the elasticity with highest value, $\varepsilon_{\max}$). Thus, this pins down the degree of freedom in defining income elasticities.

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[10] An alternative possibility to circumvent this under-identification problem, as suggested by Hanoch (1975), is to pin down the elasticity level for one of the sectors and express the rest relative to them. As, we do not have ex-ante information on what one of this values may be we prefer not to pursue this approach.

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G Additional Figures and Tables

Table G.1: Residual Relative Log-Expenditure Shares on Residual Aggregate Consumption

Dep. Var.:	$\log\left(\frac{\text{Agriculture}}{\text{Manufact.}}\right)$		$\log\left(\frac{\text{Services}}{\text{Manufact.}}\right)$	
Residual Log-Expenditure	(1)	(2)	(3)	(4)
Residual Log Aggregate Income	-0.34	-0.42	0.27	0.23
	(0.11)	(0.12)	(0.08)	(0.07)
Residual Log Agg. Income below Median Income		0.19		0.09
		(0.12)		(0.13)
R^2	0.27	0.29	0.20	0.21
Observations	513	513	513	513

Note: Standard errors clustered by country. Residual Aggregate Income is constructed by taking the residuals of the following OLS regresssion: $\log Y_t^c = \alpha \log p_{at}^c + \beta \log p_{mt}^c + \gamma \log p_{st}^c + \xi^c + \nu_t^c$ where superscript c denotes country, and subscript t, time. p_{at}^c denotes price of agriculture in country c at time t. Likewise p_{mt}^c and p_{st}^c denote the prices of manufacturing and services, respectively. ξ^c denotes a country fixed effect and ν_t^c the error term. Residual log-expenditures are constructed in an analogous manner.

Table G.2: Contribution of Relative Prices and Consumption

Specification	Log-Likelihood	LR Test χ^2	LR Test p-value	AIC	BIC
FE Only	-324.28	–	–	754.56	1010.02
FE + Prices	-270.30	107.96	0.00	648.61	908.89
FE + Consumption	363.34	1375.25	0.00	-616.68	-351.58
Full Specification	412.08	1472.71 97.47	0.00 0.00	-712.15	-442.23

Note: AIC refers to the Akaike Information Criterion, BIC refers to the Bayesian Information Criterion. The first two Likelihood Ratio Tests are done against the model that has only country-(relative)sector fixed effects. The last Likelihood Ratio Test compares the full model against one with fixed effects and consumption.

H Plots of Fit with Regional Estimates for All Countries

Figures H.4, H.5 and H.6 starting on page 48 report the fit for all countries in our sample using the region estimates for OECD, Asian and Latin American countries separately $\{\sigma_r, \varepsilon_{s,r} - \varepsilon_{m,r}, \varepsilon_{a,r} - \varepsilon_{m,r}\}$ where r denotes the regional estimation .

Table G.3: Partial Correlations

Regression Equation	Consumption		Relative Prices	
	Partial Corr.	Partial Corr.^{2}	Partial Corr.	Partial Corr.^{2}
L_a/L_m	-0.85	0.72	0.11	0.01
L_s/L_m	0.46	0.21	0.17	0.03

Note: Country*(relative)sector fixed effects included. Suppose that y is determined by $x_1, x_2, \ldots, x_k$. The partial correlation between y and x_1 is estimates the correlation that would be observed between y and x_1 if the other x's did not vary. The squared correlations estimate the proportion of the variance of y that is explained by each.

Table G.4: Heterogeneous Price Elasticity of Substitution

Dep. Var: Emp. Shares	Agri.-Manu. (1)	Serv.-Manu (2)
σ_{am}	0.67	
	(0.12)	
σ_{sm}		0.78
		(0.18)
$\varepsilon_a - \varepsilon_m$	-1.01	
	(0.13)	
$\varepsilon_s - \varepsilon_m$		0.33
		(0.13)
$c \cdot sm$ Fixed Effects	Y	Y
Trade Controls	Y	Y

Note: Standard errors clustered by country.

Table G.5: Value Added Regressions

Dep. Var: Value Added	(1)	(2)
σ	0.91	0.51
	(0.25)	(0.14)
$\varepsilon_a - \varepsilon_m$	-1.00	-1.17
	(0.21)	(0.15)
$\varepsilon_s - \varepsilon_m$	0.22	0.10
	(0.08)	(0.15)
Observations	1043	1043
$c \cdot sm$ Fixed Effects	N	Y

Note: Standard errors clustered by country.

Table G.6: Stone-Geary Estimation Results

Dep. Var.: Emp. Shares	(1)	(2)
σ	0.21	0.21
	(0.05)	(0.05)
$\bar{c}_a$	-208	-201
	(205)	(194)
$\bar{c}_s$	415	395
	(475)	(452)
Log-Likelihood	173677	173645
Akaike IC	-347348	-347274
Bayesian IC	-347333	-347236
Sectoral Trade Control	N	Y
Measures of Non-homotheticity		
Entire Sample		
$\max \frac{p_s \bar{c}_s}{\sum_{i \in \{a,m,s\}} p_i C_i}$	-0.00	-0.00
$\min \frac{p_s \bar{c}_s}{\sum_{i \in \{a,m,s\}} p_i C_i}$	-0.11	-0.11
$\max \frac{p_a \bar{c}_a}{\sum_{i \in \{a,m,s\}} p_i C_i}$	-0.00	-0.00
$\min \frac{p_a \bar{c}_a}{\sum_{i \in \{a,m,s\}} p_i C_i}$	-0.11	-0.11
United States		
$\max \frac{p_s^{U.S.} \bar{c}_s}{\sum_{i \in \{a,m,s\}} p_i^{U.S.} C_i^{U.S.}}$	4.9e-04	4.9e-04
$\min \frac{p_a^{U.S.} \bar{c}_a}{\sum_{i \in \{a,m,s\}} p_i^{U.S.} C_i^{U.S.}}$	-5.4e-04	-5.4e-04

Note: Standard errors clustered by country.

Table G.7: Contribution of Prices and Consumption for OECD Countries

Specification	Log-Likelihood	LR Test χ^2	LR Test p-value	AIC	BIC
FE Only	43.84	–	–	-45.68	33.91
FE + Prices	119.21	150.74	0.00	-194.41	-111.03
FE + Consumption	428.92	770.16	0.00	-811.84	-724.67
Full Specification	441.01	794.35 24.18	0.00 0.00	-834.03	-743.07

Note: AIC refers to the Akaike Information Criterion, BIC refers to the Bayesian Information Criterion. The first two Likelihood Ratio Tests are done against the model that has only country-(relative)sector fixed effects. The last Likelihood Ratio Test compares the full model against one with fixed effects and consumption.

Regression Equation	Consumption Partial Corr.	Consumption Partial Corr.^{2}	Relative Prices Partial Corr.	Relative Prices Partial Corr.^{2}
L_a/L_m	-0.52	0.27	-0.24	0.06
L_s/L_m	0.88	0.77	0.02	0.00

Note: Country*(relative)sector fixed effects included. Suppose that y is determined by $x_1, x_2, \ldots, x_k$. The partial correlation between y and x_1 is estimates the correlation that would be observed between y and x_1 if the other x's did not vary. The squared correlations estimate the proportion of the variance of y that is explained by each.

Figure G.1: Predicted Evolution of Expenditure Shares for the U.S.

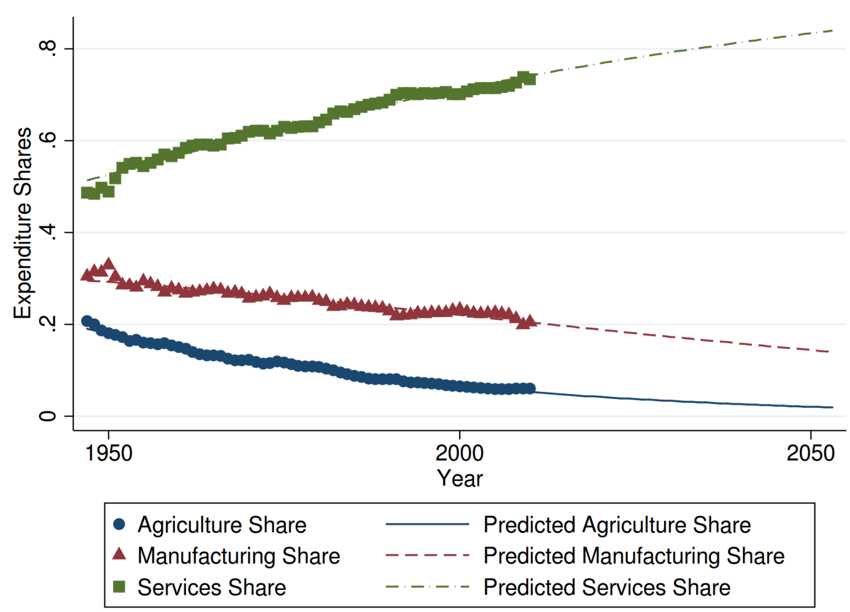

Table G.8: Contribution of Prices and Consumption for Non-OECD Countries

Specification	Log-Likelihood	LR Test χ^2	LR Test p-value	AIC	BIC
FE Only	-186.07	–	–	442.13	595.38
FE + Prices	-182.83	6.47	0.01	437.66	595.28
FE + Consumption	261.90	895.94	0.00	-449.81	-287.80
Full Specification	280.09	932.32	0.00	-484.19	-317.81
		36.38	0.00		

Note: AIC refers to the Akaike Information Criterion, BIC refers to the Bayesian Information Criterion. The first two Likelihood Ratio Tests are done against the model that has only country-(relative)sector fixed effects. The last Likelihood Ratio Test compares the full model against one with fixed effects and consumption.

Regression Equation	Consumption Partial Corr.	Consumption Partial Corr.^{2}	Relative Prices Partial Corr.	Relative Prices Partial Corr.^{2}
L_a/L_m	-0.88	0.78	0.17	0.03
L_s/L_m	0.22	0.05	0.00	0.00

Note: Country*(relative)sector fixed effects included. Suppose that y is determined by $x_1, x_2, \ldots, x_k$. The partial correlation between y and x_1 is estimates the correlation that would be observed between y and x_1 if the other x's did not vary. The squared correlations estimate the proportion of the variance of y that is explained by each.

Table G.9: Growth Rates of Relative Prices in the Country Panel

$$\log\left(\frac{p^c_{i,t}}{p^c_{m,t}}\right) = \alpha^c_{im} + \beta_i \cdot \text{Year} + \varepsilon^c_{im,t}, \qquad i = \{s, a\}$$

	$\log\left(\frac{p^c_a}{p^c_m}\right)$	$\log\left(\frac{p^c_s}{p^c_m}\right)$
Year	-0.59	0.13
	(0.05)	(0.04)
Country-Sector FE	Yes	Yes
R^2	0.49	0.41
Observations	1680	1680

Note: Year has been re-scaled to Year/100.

Figure G.2: Country Fit Using Stone-Geary Preferences

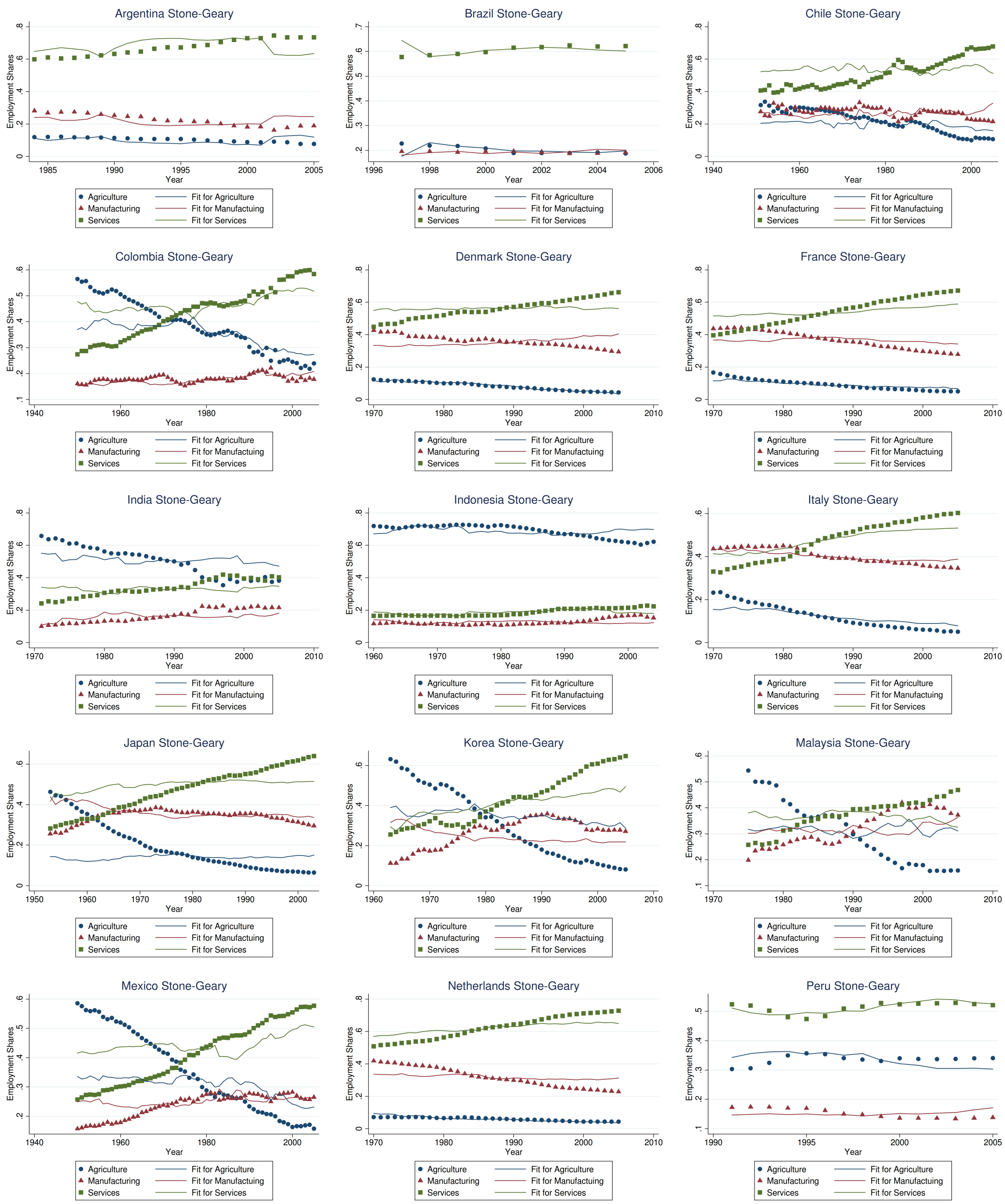

Figure G.3: Country Fit Using Stone-Geary Preferences

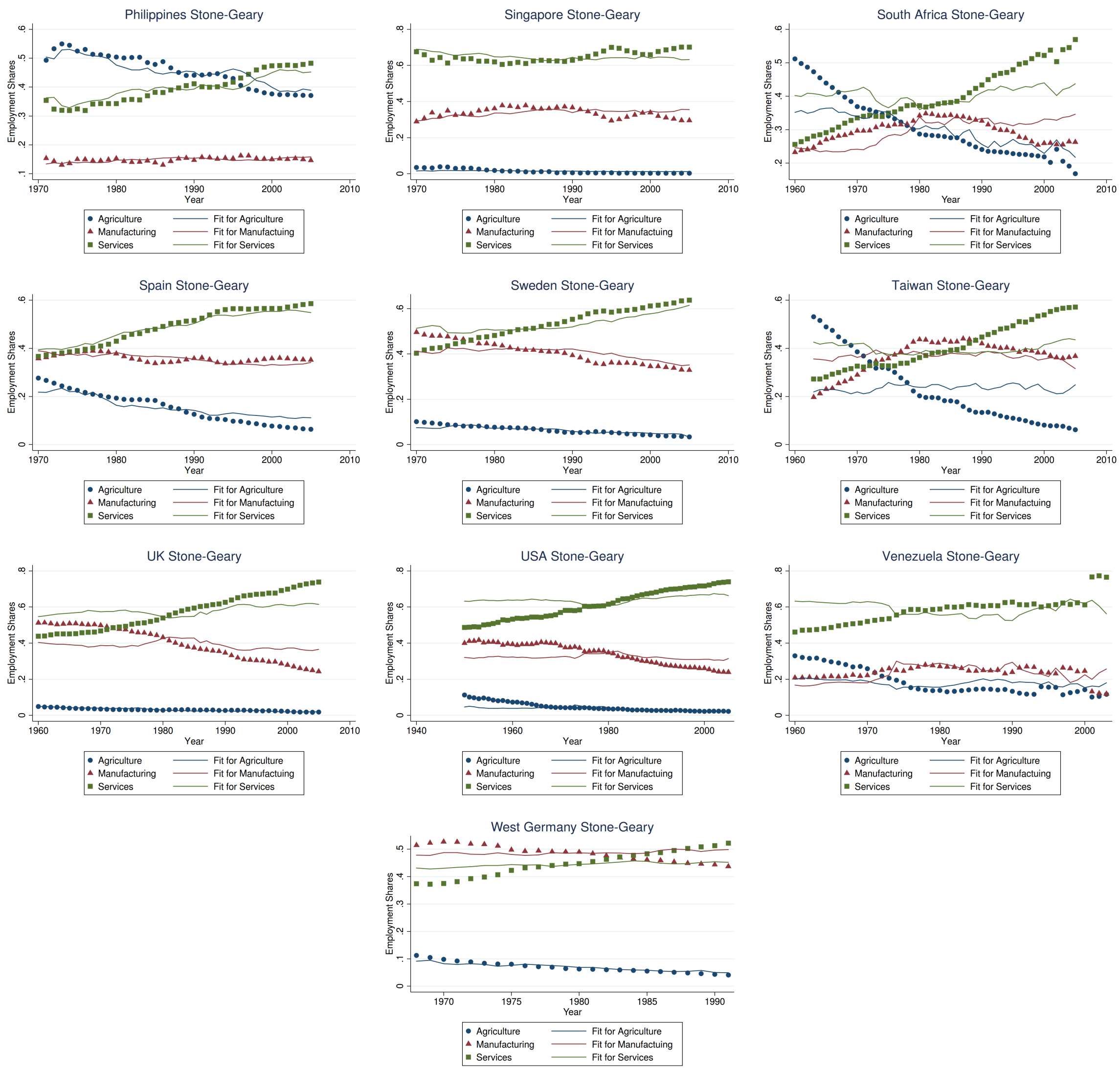

Figure H.1: Fit of Baseline Model Parameters for OECD countries

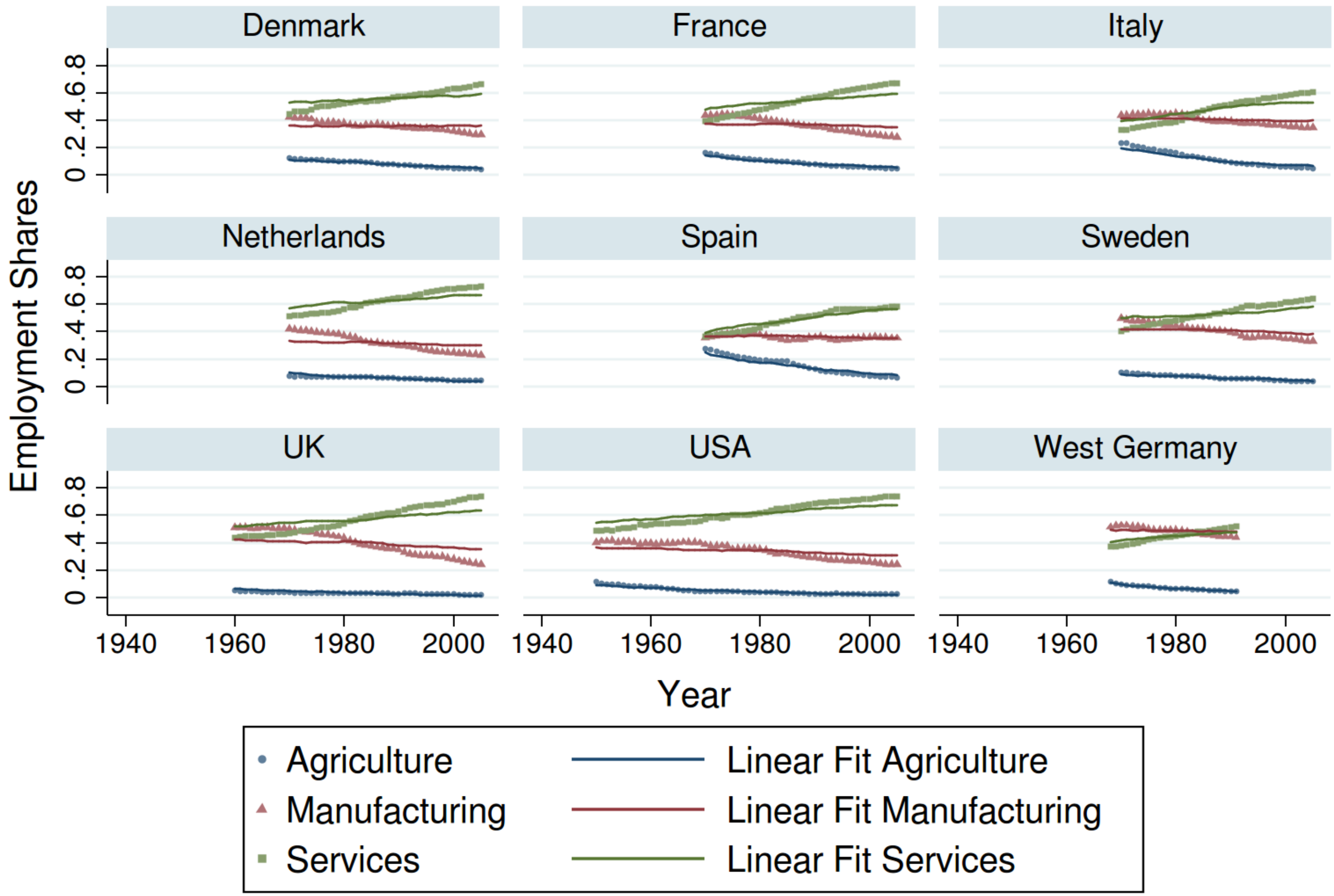

Figure H.2: Fit of Baseline Model Parameters for Asian countries

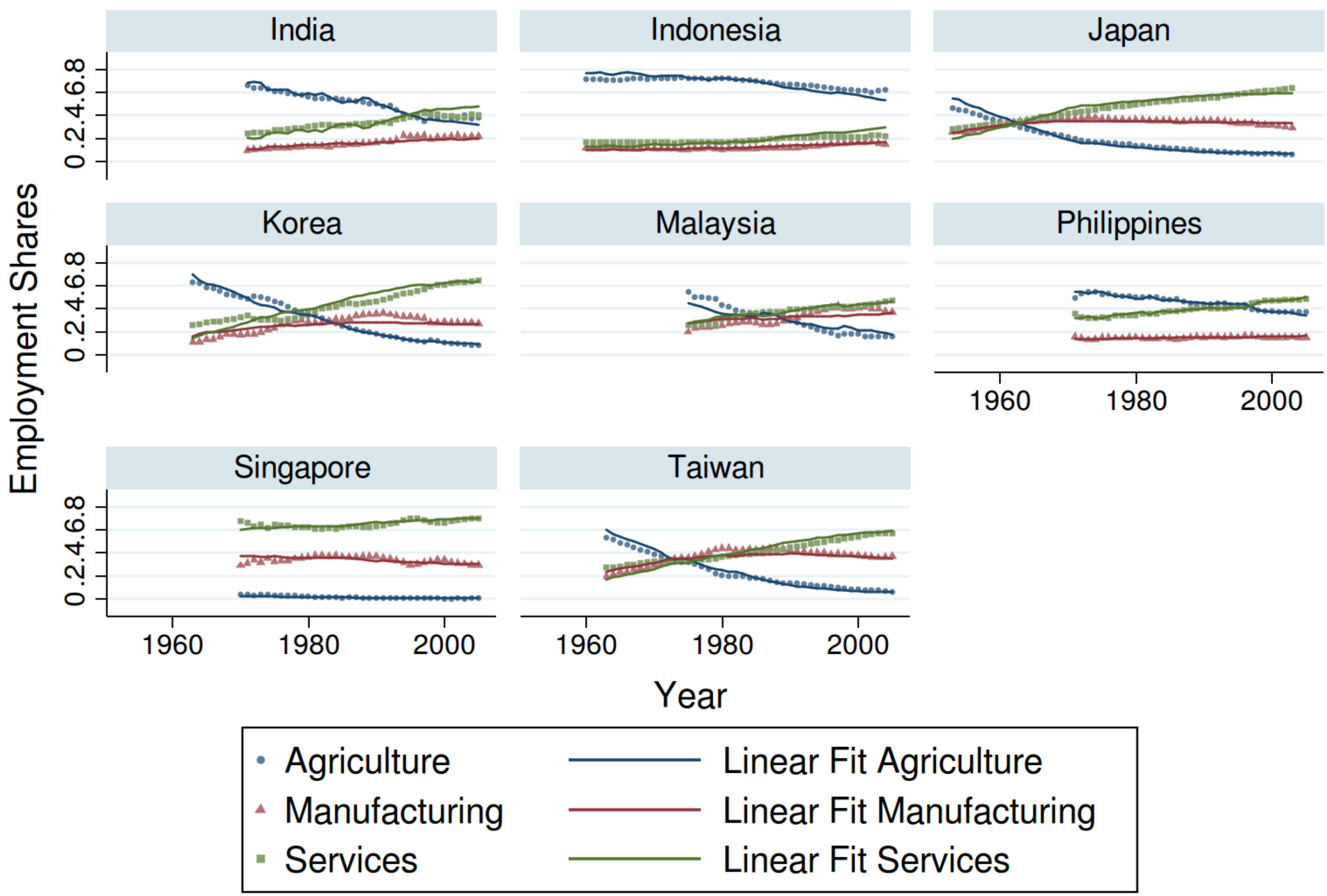

Figure H.3: Fit of Baseline Model Parameters for Latin American countries and South Africa

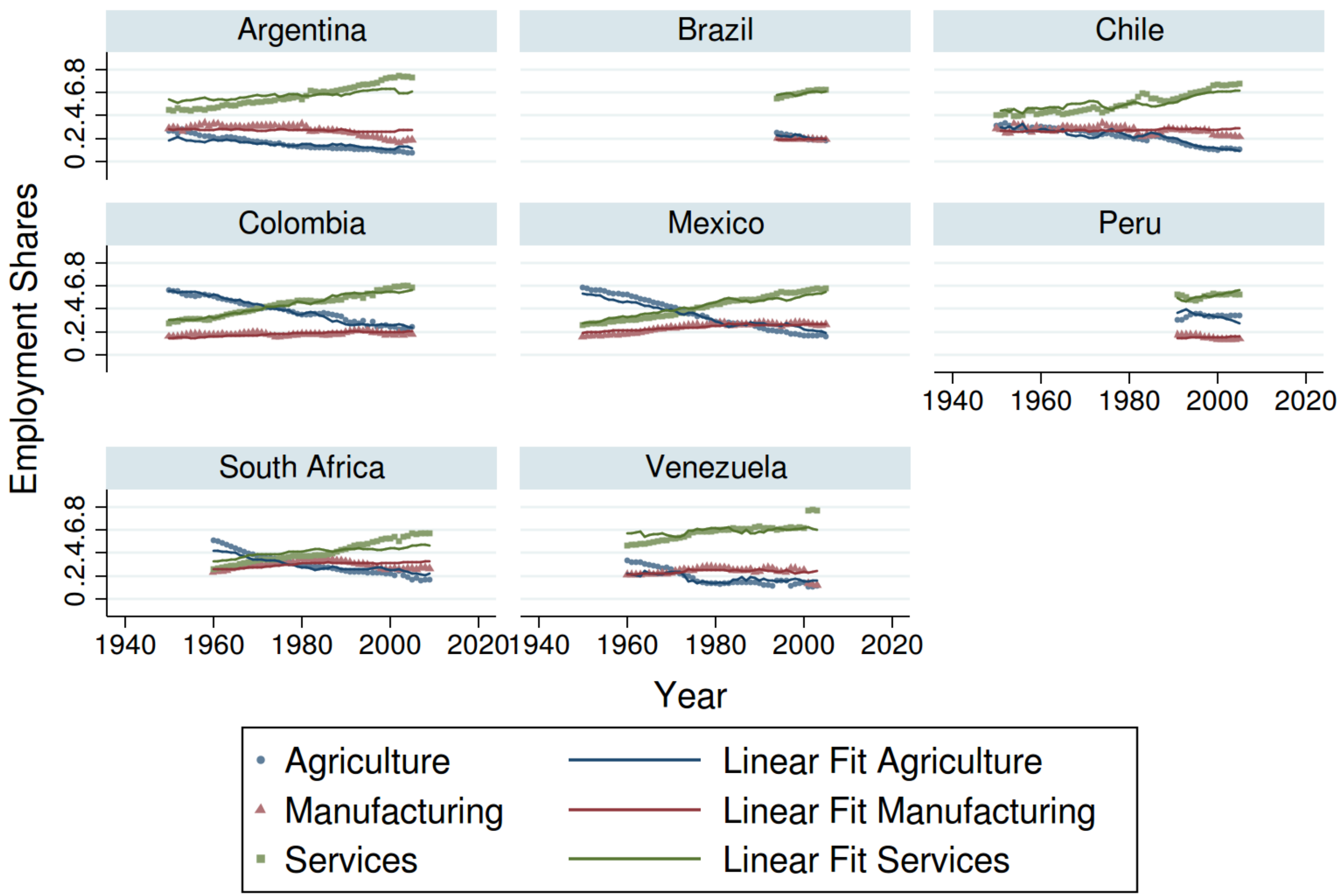

Figure H.4: Fit Using Regional Estimates for OECD countries

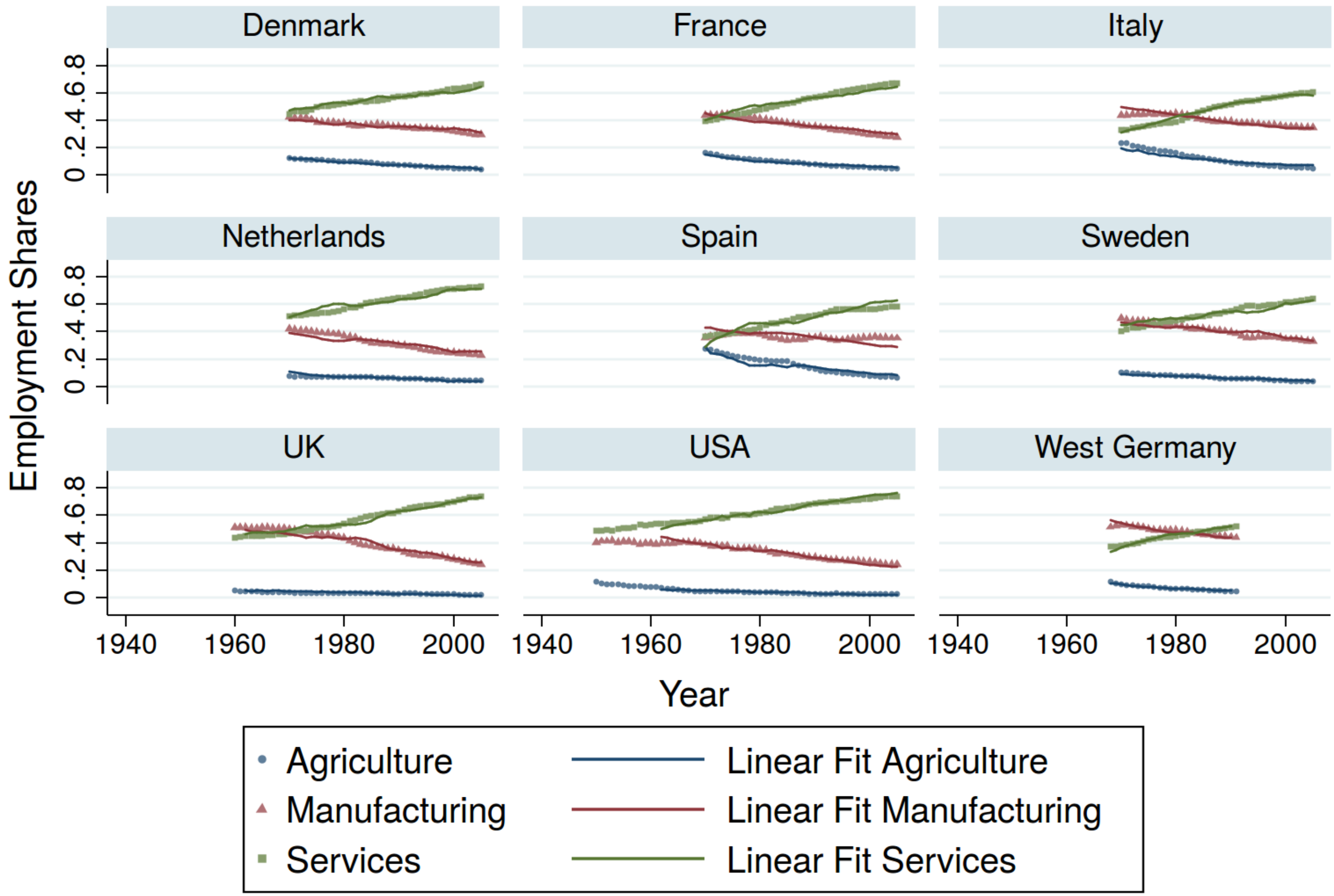

Figure H.5: Fit Using Regional Estimates for Asian countries

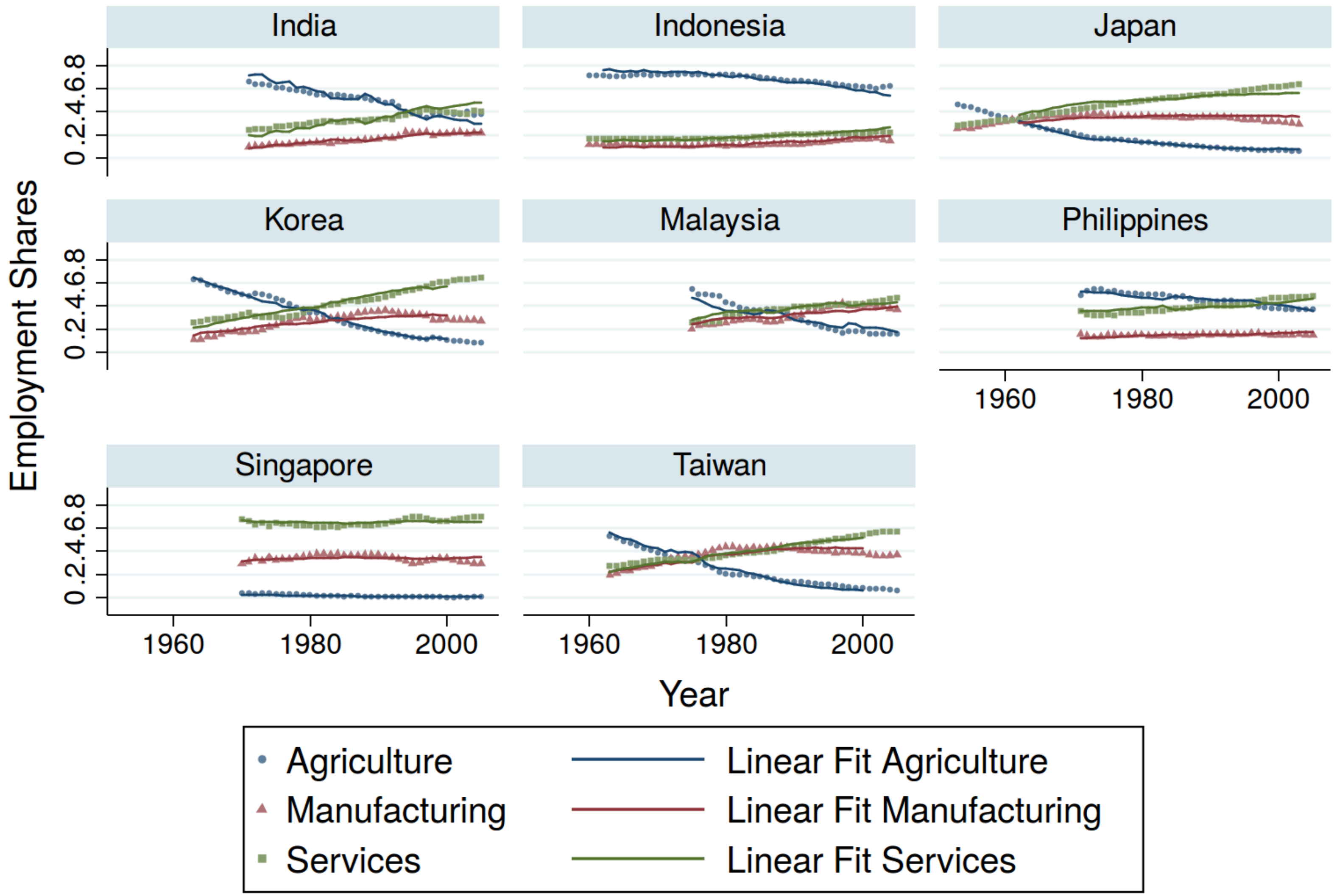

Figure H.6: Fit Using Regional Estimates for Latin American countries and South Africa