An Online Appendix to
"Quadratic Voting:
How Mechanism Design Can Radicalize Democracy"

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Abstract

This appendix proves the main theorem of "Quadratic Voting: How Mechanism Design Can Radicalize Democracy"' in the 2017 *American Economic AssociationPapers and Proceedings* and available at `http://ssrn.com/abstract=2790624`.

First consider the "if" direction. The general quadratic rule is $c(x) = kx^2$ for some $k > 0$. By price-taking, voters maximize $2pu_iv_i - kv_i^2$. A necessary condition for maximization is that $2pu_i = 2kv_i$ or

$$v_i^\star = \frac{pu_i}{k}.$$

Thus

$$\text{sign}\left(\sum_i v_i^\star\right) = \text{sign}\left(\sum_i \frac{pu_i}{k}\right) = \text{sign}\left(\sum_i u_i\right)$$

as $k, p > 0$.

For the "only if" direction, consider any cost c. Then by strict convexity and differentiability, voters will chose the unique $v_i^\star$ solving

$$2pu_i = c'(v_i) \iff v_i^\star = \gamma(2pu_i),$$

where γ is the inverse function of c', which is well-defined by strict convexity. Consider the special case of the robust optimality requirement in which $p = 1/2$; for the "only if" direction this is without loss of generality. In this case we have $v_i^\star = \gamma(u_i)$. The only homogeneous of degree one functions of a single variable are linear, so either γ is linear or it is not homogeneous of degree one. In the first case, inversion and integration yields that c takes the form claimed. In the second case, there must exist some $u' > 0, \kappa > 1$ such that $\gamma(\kappa u') \neq \kappa\gamma(u')$. Let $\Delta \equiv \frac{\gamma(\kappa u')}{\kappa\gamma(u')} - 1$.

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Again we can break this into two cases: $\Delta > 0$ and $\Delta < 0$. In the first case, let $N^\star$ be the least integer strictly greater than $\frac{2\kappa(1+\Delta)}{\Delta}$ and let $N^{\star\star}$ be the greatest integer strictly less than $\frac{N^\star}{\kappa}$.

Consider a collective decision problem where $N^{\star\star}$ voters have value $-\kappa u'$, $N^\star$ voters have value u' and there are no other voters. Then

$$\sum_i u_i = N^\star u' - N^{\star\star}\kappa u' > N^\star u' - \frac{N^\star}{\kappa}\kappa u' = 0.$$

However, by the oddness of γ derived from the evenness of c,

$$\sum_i v_i^\star = N^\star \gamma(u') - N^{\star\star}\gamma(\kappa u') = \gamma(u')\left[N^\star - N^{\star\star}\kappa(1+\Delta)\right] \leq$$

$$\gamma(u')\left[N^\star - (N^\star - \kappa)(1+\Delta)\right] = \kappa\gamma(u')\left[1 + \Delta - \Delta\frac{N^\star}{\kappa}\right] < \kappa\gamma(u')\left[1 + \Delta - 2(1+\Delta)\right] < 0.$$

Here we used the fact that $\kappa, \gamma > 0$ for all non-zero arguments of γ by the strict monotonicity of γ. Thus $c$cannot in this case be robustly optimal.

Now consider the case when $\Delta < 0$. Let $\hat{N}$ be the greatest integer strictly less than $-\frac{2\kappa(1+\Delta)}{\Delta}$ and let $\tilde{N}$ be the least integer strictly greater than $\frac{\hat{N}}{\kappa}$. Consider a collective decision problem where $\tilde{N}$ voters have value $-\kappa u'$, $\hat{N}$ voters have value u' and there are no other voters. Then

$$\sum_i u_i = \hat{N}u' - \tilde{N}\kappa u' < \hat{N}u' - \frac{\hat{N}}{\kappa}\kappa u' = 0.$$

However, by the oddness of γ derived from the evenness of c,

$$\sum_i v_i^\star = \hat{N}\gamma(u') - \tilde{N}\gamma(\kappa u') = \gamma(u')\left[\hat{N} - \tilde{N}\kappa(1+\Delta)\right] \geq$$

$$\gamma(u')\left[\hat{N} - \left(\hat{N} + \kappa\right)(1+\Delta)\right] = -\kappa\gamma(u')\left[1 + \Delta + \Delta\frac{\hat{N}}{\kappa}\right] > \kappa\gamma(u')\left[2(1+\Delta) - 1 - \Delta\right] > 0.$$

Here we used the fact that $k, \gamma > 0$ for all non-zero arguments of γ by the strict monotonicity of γ and thus that $\Delta > -1$. Thus c cannot in this case be robustly optimal or thus in any case when γ is not homogeneous of degree one, completing the proof.