Optimal fiscal and monetary policy with distorting taxes: Appendix

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This appendix provides proofs and modeling details for the paper "Optimal fiscal and monetary policy with distorting taxes".

A. The steady state solution and its uniqueness

We are assuming τ constant, $b > 0$, $C > 0$, $v > 0$. The Lagrange multiplier on the agent's budget constraint is negative if $\tau > 1$, so this is ruled out by the agent's optimization. (There is no reason to work if the after-tax wage is negative.) Negative τ is in principle possible if seigniorage is used to generate revenue that finances a labor subsidy. With $\dot{\tau} = 0$ and $\dot{b} = 0$, the system of five equations, (16), (12), (18), (19) and (20) introduced on page 8 of the main text can be solved recursively to deliver the single quadratic equation in v,

$$\gamma v^2(1 + \tau - 2G) + (\tau - G)v - \beta = 0\,. \tag{A.1}$$

Its roots are

$$v = \frac{G - \tau \pm \sqrt{(G-\tau)^2 + 4\beta\gamma(1 + \tau - 2G)}}{2\gamma \cdot (1 + \tau - 2G)}\,. \tag{A.2}$$

If $\tau > 2G - 1$, the equation has two real roots, one positive and one negative. Since negative v makes no sense in the model, the positive root is the relevant one. If $\tau < 2G - 1$, the roots could be imaginary, in which case they correspond to no equilibrium of the model, or they can be real and of the same sign. For the two roots to be positive, we must have $G < \tau$. But $G < \tau$ and $\tau < 2G - 1$ jointly imply $\tau > 1$, which we have noted is impossible. If $\tau = 2G - 1$, there is only one root, which is negative. So the only case that delivers an equilibrium with positive v is $\tau > 2G - 1$, and in that case the steady state v is unique for each constant value of $\tau > 2G - 1$.

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B. Is the steady state the only equilibrium with constant τ?

The previous section shows there is only one steady state, for any τ for which a steady state exists. But could there be non-steady-state equilibria with τ constant? To check this, we allow for non-zero $\dot{b}$ in the government budget constraint (12). The other equations in the system remain unchanged from the previous section. Our derivation of the previous equation (A.1) can be repeated to result in

$$-\frac{\dot{b}}{b} = \gamma v^2(1 + \tau - 2G) + (\tau - G)v - \beta \,. \tag{B.1}$$

Since

$$b = \frac{C}{v} = \frac{1-\tau}{v \cdot (1 + 2\gamma v)} \,, \tag{B.2}$$

b is monotonically decreasing in v, going to zero as v goes to infinity and to infinity as v goes to zero.

Proposition B.1. *Equilibria with constant τ in which $b \to \infty$ are impossible.*

PROOF:
For an individual agent, taking the path of prices, taxes, and interest rates as given, the only state variable is b, real wealth. In any equilibrium, (9) tells us that $C < 1$ at all times, so discounted utility is bounded above. Suppose there is an equilibrium with $b \to \infty$. It can deliver no greater discounted utility than that provided by the (infeasible) allocation of $C \equiv 1$, $L \equiv 0$, which is finite. If $b \to \infty$ and therefore $v \to 0$, C converges to a positive constant, and the real rate of return on debt, $\beta - \gamma v^2$, converges to β. But if b gets large enough, spending $(\beta - \varepsilon)b$ on consumption plus transactions costs forever (where ε is some small number), while setting $L = 0$, will appear to the competitive private agent to be feasible. Transactions cost $\gamma C^2/b$ increase with increased C, but since $(\beta - \varepsilon)b$ increases linearly with C, velocity $v = b/C$ does not change as we consider higher consumption spending $C(1 + \gamma v) = (\beta - \varepsilon)b$. Thus it will appear to the private agent to be possible, as b grows without bound, to achieve a higher discounted utility than any in allocation that is actually feasible for the whole economy. This shows that an equilbrium with $b \to \infty$ does not exist. □

Proposition B.2. *When there is a constant-tax equilibrium with b constant at $\bar{b}$, there are no constant-tax equilibria with $b > \bar{b}$.*

PROOF:
The right-hand side of (B.1) goes to $-\beta$ as $v \to 0$, which implies that for large enough b, $\dot{b} > 0$. Continuity of that right-hand side, plus our result in appendix A that a constant-tax steady state, when it exists, is unique, implies that when a steady state exists, on any constant-tax equilibrium path with $b > \bar{b}$, $b \to \infty$,

which is impossible, so when a steady state exists, there are no constant-tax equilibria with $b > \bar{b}$. □

Proposition B.3. *When there is no steady state with the constant tax rate τ, there are also no constant-tax equilibria with $\dot{b} > 0$ anywhere along the entire equilibrium time path.*

PROOF:

The fact that $\dot{b}$ is positive for large enough values of b implies, with no steady state, that $\dot{b}$ is positive everywhere and thus (since there is no steady state) that $b \to \infty$, again contradicting the hypothesis that this was an equilibrium. □

Now we consider possible equilibria with constant tax rate and decreasing b. In this model we can rule out such equilibria if we can show that they imply b reaches zero in finite time. Since $b \geq 0$ is our assumption — private agents can't borrow from the government — it may seem obvious that if the equilibrium path implies b reaches zero in finite time and that $\dot{b} < 0$ at that point, the equilibrium is ruled out.

However, as can be seen from (B.2), $b \to 0$ implies $v \to \infty$ and $C \to 0$, while $L \to G + (1-\tau)/2$. In other words, as b hits zero, transactions cost absorb all of spending on consumption. At this point, an individual could stop working, and therefore have no tax obligation (if $\phi = 0$), while still having no consumption. A path converging to this point, while unpleasant, would not seem infeasible to the private agent. Given the time path of prices and interest rates, there would be no incentives to deviate from the path for the private agent.

This paradox can be avoided. For example, if $\phi > 0$, the dynamics of the model are unchanged, except that in (B.1) and equations derived from it G is replaced by $G - \phi/(1-\tau)$. If the values of ϕ and G are thought of by the agent as fixed for all time, even after the agent's b is exhausted, the agent will see these paths as ones on which his tax obligations can't be satisfied. This will increase the agent's initial demand for government debt, reduce the initial price level, and thereby push the equilibrium back to the saddle path. A similar way to avoid the paradox is a trigger policy, where the tax authority promises to introduce a lump-sum tax if the economy starts on an explosive path. This leaves the saddle path with $\phi = 0$ unaffected, while eliminating the explosive paths.

To see how this works out in this model, we again rewrite the $\dot{b}$ equation, multiplying (B.1) through by b and expressing everything on the right as a function of v:

$$-\dot{b} = (1-\tau)\left(\frac{\gamma v(1+\tau-2G)}{1+2\gamma v} + \frac{\tau - G}{1+2\gamma v} - \frac{\beta}{v\cdot(1+2\gamma v)}\right). \tag{B.3}$$

Proposition B.4. *When τ is constant, there are no equilibria with $b \to 0$.*

PROOF:

As v goes to infinity (and b to zero), the right-hand side of (B.3) converges to $(1-\tau)(1+\tau-2G)/2$, a positive number if a steady state exists, implying $\dot{b}$

becomes negative and is bounded away from zero when b becomes small. But this implies that b reaches zero in finite time. Thus a path with b converging to zero cannot be an equilibrium. But we know that constant-tax steady state equilibria, when they exist, are unique, and also that $\dot{b}$ is negative for small enough b. This implies $\dot{b}$ is negative for $b < \bar{b}$ when the constant-tax steady state exists, and thus that there is no equilibrium with constant τ and $b < \bar{b}$.

When $\tau + 1 \leq 2G$, so there is no steady state, it must also be that $\tau < G$, because $G < 1$. In that case the right-hand-side of (B.3) is negative, and therefore $\dot{b} > 0$. Since the right-hand side of (B.3) is continuous, and there is no steady state, this means $\dot{b}$ must be positive for all values of b on any equilibrium path, and therefore that there are no equilibria with constant τ and b decreasing. □

This completes the argument that when τ is constant, the only competitive equilibria that exist are steady states, and that the steady state for a given constant τ is unique.

C. Uniqueness of the initial price level

The nominal government budget constraint is

$$\dot{B} = iB + GP - \tau LP \,. \tag{C.1}$$

If this holds at every moment, including the initial date, it implies that B_0 is fixed and cannot be affected by policy choices. But both our optimal non-stationary and our fixed-τ equilibria, when they exist, imply an initial value for $b = B/P$ that does not depend on B. Thus our analysis implies that at time zero, the price level jumps up or down to match B/P to the equilibrium value of b. Since the equilibrium, when it exists, is unique, the initial price level is uniquely determined.

In our discussion of policy implications in the main text, we considered the possibility of an instantaneous upward jump in B. It does seem plausible that a large upward jump in B could be produced by a brief and very large transfer payment. This would involve mailing checks to the public — which was actually done during the pandemic. Such an action, if followed by a constant τ and G, would affect only the initial price level, not the subsequent real equilibrium path.

The reverse policy action, a discrete downward jump in B_0, seems less plausible. Lump sum transfers are much easier to arrange than large lump-sum taxes or wealth confiscations. In our simple representative agent model, a one-time lump-sum tax, paid for by agents selling nominal bonds, might seem possible. But with heterogeneous holdings of bonds, a uniform lump sum tax might not even be feasible because of the wealth differences, while a uniform lump sum transfer would not face such a problem.

D. Equations defining the non-stationary optimal solution

The utility function $\log C - L$ can be written, using (16) and (18), as

$$U(v,\tau) = \log(1-\tau) - \log(1+2\gamma v) - \frac{(1+\gamma v)(1-\tau)}{1+2\gamma v} - G\,. \tag{D.1}$$

We're maximizing

$$\int_0^\infty e^{-\beta t} U(v_t, \tau_t)\, dt \tag{D.2}$$

by choosing the time path of τ, subject to the constraint (22).

The first order condition with respect to τ produces (25) via straightforward algebra. Because the constraint involves $\dot{v}$, though, the first order condition with respect to v is more complicated. We derive it via a recursive sequence of algebraic expressions.

We already introduced, as R and S, the two factors of the $\dot{v}/v$ expression (22).

(D.3) $$\frac{dS}{dv} = \frac{-2\gamma}{(1+4\gamma v)^2}$$

(D.4) $$\frac{\partial R}{\partial v} = 2\gamma v \cdot (1+\tau-2G) + \tau - G$$

(D.5) $$\frac{\partial R}{\partial \tau} = \gamma v^2 + v$$

(D.6) $$\frac{\partial U}{\partial \tau} = -\omega + \frac{1+\gamma v}{1+2\gamma v}$$

(D.7) $$\omega = \frac{1}{1-\tau} \quad \text{(definition)}$$

(D.8) $$\frac{\partial U}{\partial v} = \frac{-2\gamma}{1+2\gamma v} + \frac{(1-\tau)\gamma}{(1+2\gamma v)^2}$$

(D.9) $$\dot{\eta} = -v\frac{\partial U}{\partial v} + \eta\Big(\beta - v \cdot \big(R\frac{dS}{dv} + S\frac{\partial R}{\partial v}\big)\Big)$$

FOC w.r.t. v in planner's problem. η is multiplier on constraint

(D.10) $$\omega = A + B$$
$$A = \frac{1+\gamma v}{1+2\gamma v}$$
$$B = \frac{(1+\gamma v)(1+2\gamma v)v\eta}{1+4\gamma v}$$

From FOC w.r.t τ. Equivalent to (25) in text.

(D.11) $$\frac{dA}{dv} = \frac{-\gamma}{(1+2\gamma v)^2}$$

(D.12) $$\frac{\partial B}{dv} = B\left(\frac{\gamma}{1+\gamma v} + \frac{1}{v} + \frac{2\gamma}{1+2\gamma v} - \frac{4\gamma}{1+4\gamma v}\right)$$

(D.13) $$\frac{\partial B}{\partial \eta} = v(1+\gamma v)S$$

(D.14) $$\frac{\dot{\omega}}{\omega} = \left(\Big(\frac{\partial A}{\partial v} + \frac{\partial B}{\partial v}\Big)\dot{v} + \frac{\partial B}{\partial \eta}\dot{\eta}\right)(1-\tau)$$

(D.15) $$\dot{\tau} = (1-\tau)\frac{\dot{\omega}}{\omega}$$

This sequence of equations implicitly defines $\dot{\tau}$ as a function of v and τ alone and can be fairly directly translated into computer code to evaluate that function. (In fact, the code came first, and the equations displayed here are translations from the code.) They can be used, though, to obtain some analytical results.

E. Analytically provable properties of the solution path

Proposition E.1. *Properties of the $\dot{v} = 0$ line*
For any $\gamma > 0$, $G \in [0, 1]$ and $\beta > 0$:

i. The $\dot{v} = 0$ locus in v, τ space defined by $R = 0$ defines a continuous, 1-1 function h mapping v to $\tau = h(v)$ and τ to $v = h^{-1}(\tau)$.

ii. $h'(v) < 0$ and $h'' > 0$.

iii. As τ approaches its upper bound $\tau = 1$, the corresponding $v = h(\tau)$ approaches a positive limit, $\bar{v}$, which is the greatest lower bound of v's on the $\dot{v} = 0$ locus.

iv. There is a lower bound $\tau = h(v) > 2G - 1$, and as τ approaches this limit from above $v = h^{-1}(\tau) \to \infty$.

PROOF:
The $R = 0$ equation can be solved to give τ as a function of v:

$$\tau = \frac{\gamma v^2 \cdot (2G - 1) + Gv + \beta}{\gamma v^2 + v} . \tag{E.1}$$

This delivers a value for τ for any $v > 0$ and is continuous and differentiable in v for $v > 0$. Straightforward (though tedious) algebra shows that $h' < 0$ and $h'' > 0$. This proves the first two claims in the proposition. The third follows by observing that (A.2) has a unique positive real solution at $\tau = 1$. The fourth follows from the discussion in Appendix A, where we noted that existence of a real, non-negative solution to $R = 0$ requires $\tau \geq 2G - 1$). □

Proposition E.1 Asserts that the form of the $\dot{v} = 0$ line in Figures 1 and 4 is generic: regardless of the values of $G \in [0, 1)$, $\gamma > 0$, and $\beta > 0$, the slope is negative and the plot is bounded away from the vertical axis. It is also bounded below, but for small G values the lower bound on $h(v)$ may be negative. The upper bound of 1 on G follows from the fact that the $\tau \geq 2G - 1$ condition, which must hold everywhere on the $\tau = h(v)$ locus, implies $\tau \geq 1$ when $G \geq 1$, which is impossible.

Proposition E.2.

1) The $\dot{v} = 0$ locus and the $\dot{\tau} = 0$ locus intersect at $\tau = 1$.

2) Along the $\dot{v} = 0$ locus, as $\tau \to 1$,

a) $\dot{\tau}/(1 - \tau) \to 0$;.

b) ρ, the real return on debt, converges to $\beta - \gamma \bar{v}^2$

PROOF:

The value of v on the $\dot{v}=0$ line at $\tau=1$ can be found by substituting $\tau=1$ into (A.2):

$$\bar{v}=\frac{1}{4\gamma}\left(-1+\sqrt{1+\frac{8\gamma\beta}{1-G}}\,.\right) \tag{E.2}$$

This is always positive, since $\beta>0$, $\gamma>0$, and $G\in[0,1)$. From (D.7) and (D.10), we can see that as $\tau\to 1$ with $v=\bar{v}$, ω, and therefore also η converge to infinity. Using this fact in (D.9), we can conclude that $\dot{\eta}/\eta\to 0$. This follows because the first term on the right of (D.9) goes to zero when divide by η, while the second term, divided by η and evaluated at $v=\bar{v}$, becomes

$$\begin{aligned}\beta-\bar{v}\cdot\frac{1+2\gamma\bar{v}}{1+4\gamma\bar{v}}&(4\gamma v(1-G)+1-G)\\ &=\beta-(2\gamma\bar{v}^2(1-G)+(1-G)v)=\beta-R(\bar{v},1)-\beta=0\,,\end{aligned} \tag{E.3}$$

where $R(\bar{v},1)$ is just our original expression for R in (22), evaluated at $v=\bar{v}$, $\tau=1$ (which of course is by definition zero). This $\dot{\eta}/\eta\to 0$ result implies through (D.10) that $\dot{\tau}/(1-\tau)\to 0$. So, though ω and η are growing without bound as (v,τ) approaches $\bar{v},1$, their growth rates are approaching zero. This proves the first two assertions in the proposition. The third follows from the expression for equilibrium ρ in (10), using the $\dot{\tau}/(1-\tau)\to 0$ result.

Proposition E.3. *On any path in (v,τ) space satisfying the Euler equations and with $v\to 0$,*

1) $\dot{\tau}/(1-\tau)\to\beta$;
2) $\rho\to 0$.

PROOF:

It is easy to see from (D.9) that as $v\to 0$, $\dot{\eta}/\eta\to\beta$. This means from (D.10) that $\dot{\tau}/(1-\tau)\to\beta$ and thus from (10) that $\rho\to 0$.
Note that this result is used in the text as part of the argument that such paths violate the private agent's transversality condition.

Proposition E.4. *Along paths that satisfy the Euler equations, send τ to a finite limit greater than $2G-1$, and make $v\to\infty$, v reaches infinity in finite time.*

PROOF:

Our assumption implies $1+\tau-2G$ is eventually (as v grows), greater than some positive number ε. Then, again for large enough v, from (22),

$$\frac{\dot{v}}{v}>\tfrac{1}{2}\cdot v^2\varepsilon-1-\beta\,.v \tag{E.4}$$

But then again for large enough v the quadratic term in v dominates and

$$\frac{\dot{v}}{v} > \frac{v^2 \varepsilon}{4} \,. \tag{E.5}$$

If we replace the inequality by equality in (E.5), the solution would be

$$-\frac{1}{2v^2} = t + \kappa \,, \tag{E.6}$$

where κ is an arbitary constant. To get a positive solution for v^2, we need $\kappa < 0$, so the solution has the form

$$v = \frac{1}{\sqrt{2 \cdot (-\kappa - t)}} \,, \tag{E.7}$$

which clearly reaches infinity at a finite value of t. Thus, for large enough v, the time path of v satisfying the system dynamics is bounded below by a function that goes to infinity at a finite date and thus itself must go to infinity at a finite date.

F. Conditions for optimality of $\rho = \beta$

Saturating demand for liquidity, as suggested by the Friedman rule, is possible if $f(v)$ is zero over some interval $(0, v^*)$ and then starts increasing. We can see that there are conditions on f that imply that the optimal steady-state equilibrium does not saturate liquidity demand, and also conditions under which saturating liquidity demand is at least a local optimum among steady states. These results assume that ϕ, lump-sum tax revenue, is constrained to be constant and below G, so that positive τ is required in steady state.

Assume that $A = 0$, i.e. all government debt provides liquidity services. Then the government budget constraint (12) implies that in steady state

$$b = \frac{\tau L + \phi - G}{\rho} \,, \tag{F.1}$$

where $\rho = i - \dot{P}/P$ is the real return on government debt. From (10) we have that in steady state

$$\rho = \beta - f' v^2 \,. \tag{F.2}$$

Proposition F.1. *If in a steady state equilibrium with constant $\tau > 0$ and ϕ*

a) $f(v) = 0$ for $0 \le v \le v^*$,

b) $\phi < G$ and constant,

c) $f(v)$ and $f'(v)$ are both positive for $v > v^*$, and

d) $f(v^*) = f'(v^*) = f''(v^*) = 0$,

the demand for liquidity is not saturated (i.e. $v > v^*$) in the optimal steady state.

PROOF:
Equations (9)-(11), (13), (F.1) and the definition $v = C/b$, when they have a solution for C, v, b, L, ρ and L, define a steady-state equilibrium determined by the constant fiscal parameters τ an ϕ. From (9) we can see that, with the assumed conditions on f and its derivatives at v^*,

$$\frac{dC}{d\tau} = \frac{dL}{d\tau} = -1 \quad \text{at} \quad v = v^* \,.$$

The derivative of steady-state utility $\log C - L$ with respect to τ is then, because $\tau > 0$,

$$\frac{-1}{1-\tau} + 1 < 0 \,,$$

which implies that steady state utility can be increased by reducing the labor tax rate.

Reducing taxes raises consumption and labor input, with these effects netting out to increase utility. Reducing taxes also reduces the equilibrium level of real debt b, which reduces $f(v)$. The assumption that $f''(v^*) = 0$ means that this latter effect is negligible for small reductions in taxes.

It is certainly possible to choose a form for f and parameter values so that the optimal steady state equilibrium does set $\beta = \rho$. If the second derivative from the right of f, $f''(v^*)$ is large enough, transactions costs rise so rapidly as v increases that the positive effect on transactions costs of reduced τ dominate, and $v = v^*$ becomes optimal.

Now we consider a version of the model exactly as in the main text, but with the $f(v) = \gamma v$ form for transactions costs changed to $f(v) = \max(0, \gamma(v - v^*))$. This version of f is not differentiable at v^*, suggesting, because of proposition F.1, that optimality of a steady state with $v = v^*$ (or, more precisely, v slightly below v^*, so $f(v) = 0$ and $\rho = \beta$) is possible.

Equations (16)-(20) gather the FOC's from the private sector optimization, the government budget constraint, and the social resource constraint. They define the constraints facing the optimizing tax-rate-setting authority. The change to this new $f(v)$ that allows satiation affects only (16) and (18), which become

$$C \cdot (1 + 2\gamma v - \gamma v^*) \tag{16'}$$

$$C(1 + \gamma(v - v^*)) + G = L \,. \tag{18'}$$

Using these two replacement equations, we can replicate the algebra by which we derived (22) for the original model, which leads to

$$\frac{\dot{v}}{v} = \frac{1 + 2\gamma v - \gamma v^*}{1 + 4\gamma v - \gamma v^*}\left(\gamma v^2(1 + \tau - 2G) + (\tau - G)(1 - \gamma v^*)v - \beta\right). \tag{22'}$$

For small values of v^*, this equation is close to that defining the dynamic constraint in the original model. Furthermore, the condition for the existence of a steady state with fixed τ is the same, independent of v^*, namely $\tau > 2G - 1$. It is then easily checked as in the main model that the $\dot{\tau} = 0$ locus intersects the $\dot{v} = 0$ locus at $\tau = 1$. Thus as before there will be a single path that both satisfies the Euler equations and does not violate feasibility of transversality conditions, and this path will make v approach a positive limit (slightly larger than for the $v^* = 0$ case) and make τ converge to 1.

By setting a constant $v < v^*$, transactions costs disappear from the model, and it might be thought that such a steady state could improve on the interior solution. However, in steady state it must be that $\rho b = \tau L - G$, i.e. that the interest on the debt must be financed by the excess of taxes over spending. With transactions costs gone, we will have $\rho = \beta$ and $C = 1 - \tau$. So what we require is

$$\beta b = \tau(1 - \tau + G) - G = \frac{1 - \tau}{v}, \tag{F.3}$$

which implies $v = \beta/(\tau - G) > \beta/(1 - G)$. Values of v lower than this make constant-tax equilibrium impossible. But this lower bound on v^* is also an upper bound on $\bar{v}$, the limiting value of v along the interior solution path. Thus a steady state with $v < v^*$ is possible only for values of v^* large enough that the interior solution path intersects $v = v^*$. There may be parameter values for which such a steady state is the limit of an optmial path, but it will not be for a "small" value of v^*.

G. This paper's model with Chari-Kehoe initial conditions

The Chari and Kehoe assumption on initial conditions is that the government has neither assets nor liabilities as it enters time 0, but that it can trade liquidity-providing debt B for real bonds issued by the public that do not provide liquidity services. Using A to denote the real bonds (note that earlier in the paper we used A for nominal bonds) that provide no liquidity services, it helps keep notation clear to introduce a net wealth variable

$$W = \frac{B}{P} + A. \tag{G.1}$$

The Chari-Kehoe initial condition is then $W_0 = 0$. This requires, of course, that $A_0 \leq 0$, since we continue to assume $B \geq 0$ is a constraint. The private sector's budget constraint is then

$$\dot{W} = rW + C \cdot (1 + \gamma v) - (1 - \tau)L - (r - \rho)\frac{B}{P}\,, \tag{G.2}$$

where r is the rate of return on A and $\rho = i - \frac{\dot{P}}{P}$ is, as before, the real interest rate on government debt.

The private objective function is (1) as before. The private sector Euler equations then reduce to

$$C = \frac{1-\tau}{1+2\gamma v} \tag{G.3}$$

$$r - \rho = \gamma v^2 \tag{G.4}$$

$$r = \frac{-\dot{\tau}}{1-\tau} + \beta \tag{G.5}$$

These are derived from the Euler equations as in the discussion of the main model in the text.

The government's problem has these private FOC's as constraints, plus a government budget constraint and social resource constraint:

$$GBC: \quad \dot{W} = rW + G - \tau L - (r-\rho)b \tag{G.6}$$

$$SRC: \quad C(1+\gamma v) + G = L\,. \tag{G.7}$$

The constraints allow us to rewrite the whole system in terms of τ, W and v alone:

$$\max_{\tau,v} \int_0^\infty e^{-\beta t}\left(\log(1-\tau) - \log(1+2\gamma v) - \frac{(1-\tau)(1+\gamma v)}{1+2\gamma v} - G\right) dt \tag{G.8}$$

subject to

$$\dot{W} = (\beta - \frac{\dot{\tau}}{1-\tau})W + (1-\tau)\left(G - \frac{\tau(1+\gamma v) - \gamma v}{1+2\gamma v}\right)\,. \tag{G.9}$$

Though the equations look complex, it is easy to verify that the government can, by setting $\dot{\tau} = 0$, $\tau = G$, and $v = 0$, generate a steady-state equilibrium with $L = 1$, $W = 0$, $C = 1 - G$, and $\rho = r$. In other words, a steady state satisfying the Friedman rule ($r = \rho$ and $v = 0$) is possible. With this notation, $v = 0$ no longer appears as an infeasible limit, because we now have A entering only via W and b entering only via v, with both v and W zero in this steady state.

While this stationary equilibrium does not achieve the optimum obtainable with lump-sum taxes and no transactions costs ($\gamma = 0$), which sets $L = 1 + G$, $C = 1$, it does achieve the highest possible level of steady state utility achievable when $\gamma = 0$ and the only tax available is the labor tax τ. In other words, the Friedman rule, in a model with $\gamma > 0$, achieves the same optimal steady state that could be obtained with $\gamma = 0$.

It is possible to check, though we don't go through the details here, that the FOC's for the government's dynamic problem are satisfied at this Friedman-rule steady state. Because the slope of our $f(v) = \gamma v$ function is positive at $v = 0$, the $v \geq 0$ constraint binds and the Euler equation with respect to v doesn't apply, but the W and τ Euler equations are satisfied.

What happens in this equilibrium is that the private sector bonds held by the government provide a steady flow of income rA to the government that is always weakly greater than its interest expenses ρb, allowing $\tau L \leq G$. Raising τ to $\tau = G$ brings the economy to the Friedman rule equilibrium, which is better than any other steady state. Since the initial b can be chosen freely, there is no transition path to the steady state with this policy.

Along this equilibrium path there is no commitment problem, because the optimizing policy maker chooses the same b_t at any start date t. However, Chari and Kehoe motivate the assumption that government net wealth at $t = 0$ is zero by noting that at $t = 0$, if there were any net debt, it would be optimal to repudiate it. If this is possible at dates later than $t = 0$, the government must be committed not to take advantage of this possibility. Welfare can always be improved by an unanticipated, one-time, repudiation of existing b, followed by issuing new b by buying A-type debt, thus increasing W and allowing lower equilibrium τ.