Bid Caps in Noisy Contests

ONLINE APPENDIX

(Not Intended for Publication)

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In this online appendix, we collect the analyses and discussions omitted from the main text.[1] Online Appendix A provides sufficient condition under which a flexible cap or no cap can be optimal in a two-player Tullock contest setting. Online Appendix B characterizes the optimal cap schemes in a multi-player contest with two player types. Online Appendix C collects the proofs of propositions.

A Optimal Cap Schemes in Two-Player Contests

Proposition A1 ***(Flexible Cap vs. No Cap in Two-player Tullock Contests)*** *Suppose that $n = 2$, $\lambda \in [0, 1]$, and $r \in (0, 1]$. The following statements hold.*

(i) If

$$\frac{r(1-v^r)}{1+v^r} + \frac{(1-v)\lambda - 1}{1+v} > 0, \tag{A1}$$

then the optimal contest imposes a flexible cap.

(ii) If

$$v\left[(2+r)v^r - r\right] > \lambda(1-v^r)(r-v), \tag{A2}$$

then the optimal contest imposes no cap.

Remark 1 follows immediately from Proposition A1.

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[1]This note is not self-contained; it is the online appendix of the paper "Bid Caps in Noisy Contests."

B Optimal Cap Schemes in Multi-player Contests with Two Player Types

The two-player example in Section 3.4 and Figure 1 provide an intuitive account of the fundamental trade-off between the cost and competition effects in asymmetric contests, as well as how the optimum depends on players' type differential and the noisiness of the winner-selection mechanism. However, a multi-player contest differs substantially from its bilateral counterpart. In a two-player contest, player heterogeneity can be captured by a single parameter, $v \equiv v_2/v_1$. In contrast, heterogeneity is inherently multidimensional with three or more players, which cannot readily be defined or measured without imposing a specific structure on the profile of prize valuations $(v_1, \ldots, v_n)$. This nuance prevents handy comparative statics.

We consider a simple Tullock contest setting with a two-type distribution—i.e., stronger and weaker—to demonstrate the complications. There are $n_s \geq 1$ stronger players and $n_w \geq 1$ weaker players, with $n_s + n_w = n \geq 3$. The former type values the prize at v_s, while the latter values it at v_w, with $v_s \geq v_w > 0$. Despite the vast simplification, it is difficult to provide a simple account of the heterogeneity between players, as in the previous section: This depends on prize valuations across types—i.e., the ratio between v_s and v_w—and also the composition of types within the pool, i.e., (n_s, n_w). We analyze two simple cases, which demonstrate that a variation in either dimension may change the optimum fundamentally.

Case I: $n_s = 1$. We first assume one stronger player vs. $n-1$ weaker opponents. The following result can be obtained.

Proposition A2 ***(Optimal Contest with One Strong Player)*** *Suppose that $n_s = 1$, $n_w \geq 2$, and $\lambda + r > 1$. There exist two cutoffs $\hat{v}_h(\lambda, r) \in (0, 1)$ and $\hat{v}_l(\lambda, r) \in (0, 1)$ such that a flexible cap is optimal if $v_w/v_s < \hat{v}_l(\lambda, r)$ and no cap is optimal if $v_w/v_s > \hat{v}_h(\lambda, r)$.*

The prediction is largely in line with that of Proposition A1 in a two-player setting. When v_w/v_s is sufficiently small, a flexible cap plays a more significant equalizing role. Conversely, the optimum requires no cap when v_w/v_s is sufficiently large: The direct discount on bidding incentives outweighs the limited equalizing role of a bid cap; as a result, the contest needs no intervention.

Case II: $n_s \geq 2$. The prediction drastically differs in the case of two or more stronger players, and the optimum with respect to the ratio v_w/v_s can be nonmonotone.

Proposition A3 ***(Optimal Contest with Two or More Strong Players)*** *Suppose that $n_s \geq 2$ and $n_w \geq 1$. Fixing $\lambda < 1$ and $r < 1$, there exists a lower threshold $\underline{v}(\lambda, r) \in (0, 1)$*

and an upper threshold $\bar{v}(\lambda, r) \in (0,1)$, with $\bar{v}(\lambda, r) \geq \underline{v}(\lambda, r)$, such that no cap is optimal if $v_w/v_s < \underline{v}(\lambda, r)$ or $v_w/v_s > \bar{v}(\lambda, r)$.

Although a sufficiently large ratio of v_w/v_s—i.e., $v_w/v_s > \bar{v}(\lambda, r)$—implies no policy intervention, as in Propositions A1 and A2, no cap also emerges as the optimum when v_w/v_s is sufficiently small, i.e., $v_w/v_s < \underline{v}(\lambda, r)$, which overturns the predictions of Propositions A1 and A2. Proposition A3 suggests that a flexible cap can be optimal only if v_w/v_s is in an intermediate range. This result reveals the complexity involved in a multi-player setting.

The competition effect loses its appeal when multiple stronger players are present. Suppose that $(n_s, n_w) = (2, 1)$. In this case, a stronger player has to outperform his equally competent peer to secure the prize, which may help discipline him from shirking regardless of the prevailing cap scheme. Meanwhile, a cap that handicaps the stronger may not effectively revive the weaker's momentum, as a win is difficult regardless when outnumbered by more competent opponents. A smaller v_w/v_s turns out to elevate the cost of a flexible cap: To level the playing field and incentivize the single underdog, a sufficiently high marginal tax rate is required to offset the initial asymmetry, which may cause excessive incentive loss from the two stronger players. In this scenario, contest design involves a hidden selection problem: The designer may simply "abandon" the weaker, while sustaining the competition between the stronger. This effect would not come into play in a bilateral contest.

C Proofs

Proof of Proposition A1

Proof. Clearly, with $n = 2$, both players are active in equilibrium and the set $\mathcal{P}$ defined in (23) can be simplified as

$$\mathcal{P} = \left\{ (p_1^*, p_2^*) : p_1^* + p_2^* = 1, \frac{1}{2} \leq p_1^* \leq \frac{1}{1+v^r} \right\}.$$

For notational convenience, define $p_1^\dagger := 1/(1+v^r)$. Substituting $p_2^* = 1 - p_1^*$ into the contest objective (22), the maximization problem degenerates to a single-variable optimization problem as follows:

$$\max_{p_1^* \in [1/2, p_1^\dagger]} \mathcal{F}(p_1^*),$$

where

$$\mathcal{F}(p_1^*) = r\Bigg\{ (1-\lambda) v p_1^* (1-p_1^*)^{1-\frac{1}{r}} \left[(p_1^*)^{\frac{1}{r}} + (1-p_1^*)^{\frac{1}{r}} \right]$$

$$+\lambda\left[2vp_1^*(1-p_1^*)+(1-p_1^*)\left[p_1^*-(p_1^*)^{1-\frac{1}{r}}(1-p_1^*)^{\frac{1}{r}}\right]\right]\Bigg\}.$$

Carrying out the algebra, we can obtain that

$$\mathcal{F}'(p_1^*)=(1-p_1^*)\mathcal{G}(\eta),$$

where $\eta := p_1^*/(1-p_1^*)\in[1,v^{-r}]$ and

$$\mathcal{G}(\eta):=r\Bigg\{(1-\lambda)v\left[\left(1+\frac{1}{r}\right)\eta^{\frac{1}{r}}+\left(\frac{1}{r}-1\right)\eta^{1+\frac{1}{r}}+1-\eta\right]$$
$$+\lambda\left[2v\left(1-\eta\right)+\left(1-\eta+\left(\frac{1}{r}-1\right)\left(\frac{1}{\eta}\right)^{\frac{1}{r}}+\left(\frac{1}{r}+1\right)\left(\frac{1}{\eta}\right)^{\frac{1}{r}-1}\right)\right]\Bigg\}.$$

It can be verified that $p_1^*=p_1^\dagger=1/(1+v^r)$, or equivalently, $\eta=v^{-r}$, in a two-player contest without a cap. Therefore, a sufficient condition for a flexible cap to be optimal is $\mathcal{F}'(p_1^\dagger)<0$, or equivalently, $\mathcal{G}(v^{-r})<0$. Carrying out the algebra, we can obtain that

$$\begin{aligned}\mathcal{G}(v^{-r})=&v^{-r}\times\Big\{(1-\lambda)\left[(r+1)v^r+1-r+rv^{r+1}-rv\right]\\&+\lambda\times\left[(r+1)v^{r+1}+rv^r+(1-r)v-r\right]\Big\}\\=&v^{-r}\times\left[\lambda(v^r+1)(v-1)+r(v+1)(v^r-1)+(v^r+1)\right]\\=&-(1+v^{-r})(v+1)\times\left[\frac{r(1-v^r)}{1+v^r}+\frac{(1-v)\lambda-1}{1+v}\right].\end{aligned}$$

It is evident that $\mathcal{G}(v^{-r})<0$ if

$$\frac{r(1-v^r)}{1+v^r}+\frac{(1-v)\lambda-1}{1+v}>0,$$

which corresponds to (A1) in Proposition A1(i).

Next, note that $\mathcal{G}(\eta)$ can be bounded from below by

$$\mathcal{G}(\eta)=(1-\lambda)v\left[\left(1+\frac{1}{r}\right)\eta^{\frac{1}{r}}+\left(\frac{1}{r}-1\right)\eta^{\frac{1}{r}+1}+1-\eta\right]$$

$$
\begin{aligned}
&+\lambda\left[2v(1-\eta)+1-\eta+\left(\frac{1}{r}-1\right)\eta^{-\frac{1}{r}}+\left(\frac{1}{r}+1\right)\eta^{1-\frac{1}{r}}\right]\\
\geq&(1-\lambda)v\left[\left(1+\frac{1}{r}\right)+\left(\frac{1}{r}-1\right)+1-v^{-r}\right]\\
&+\lambda\left[2v(1-v^{-r})+1-v^{-r}+\left(\frac{1}{r}-1\right)v+\left(\frac{1}{r}+1\right)v^{1-r}\right]\\
=&\frac{v^{-r}}{r}\left\{v\left[(2+r)v^{r}-r\right]+\lambda(v^{r}-1)(r-v)\right\},
\end{aligned}
$$

where the inequality follows from $\eta\in[1,v^{-r}]$. Clearly, $\mathcal{G}(\eta)>0$ for all $\eta\in[1,v^{-r}]$, or equivalently, $\mathcal{F}'(p_1^*)>0$ for all $p_1^*\in[\frac{1}{2},p_1^\dagger]$, if

$$
v\left[(2+r)v^{r}-r\right]>\lambda(1-v^{r})(r-v),
$$

which implies that $\mathcal{F}(p_1^*)$ is uniquely maximized at $p_1^*=p_1^\dagger$ on $[\frac{1}{2},p_1^\dagger]$ and it is optimal to have no cap. Note that the above inequality corresponds to (A2) in Proposition A1(ii). This completes the proof. ■

Proof of Proposition A2

Proof. Note that players of the same type must win with equal probabilities in equilibrium. Therefore, the winning probability distribution $\boldsymbol{p}^*\equiv(p_1^*,\ldots,p_n^*)$ is fully characterized by (p_s^*,p_w^*), where p_s^* and p_w^* respectively represent the stronger players' and the weaker players' equilibrium winning probabilities. With slight abuse of notation, the set $\mathcal{P}$ defined in (23) can then be simplified as

$$
\mathcal{P}=\left\{(p_s^*,p_w^*):n_sp_s^*+n_wp_w^*=1,\ 1/n\geq p_w^*\geq p_w^\dagger\right\},
$$

where $p_w^\dagger$ is the equilibrium winning probability of each weaker player under no cap. Normalizing v_s to 1 without loss of generality and substituting $p_s^*=(1-n_wp_w^*)/n_s$ into the contest objective (22), the designer's optimization problem boils down to

$$
\max_{p_w^*\in[p_w^\dagger,1/n]}\mathcal{F}(p_w^*),
$$

where $\mathcal{F}(\cdot)$ is given by

$$
\mathcal{F}(p_w^*):=(1-\lambda)v_w(p_w^*)^{1-\frac{1}{r}}(1-p_w^*)\left[n_s\left(\frac{1-n_wp_w^*}{n_s}\right)^{\frac{1}{r}}+n_w(p_w^*)^{\frac{1}{r}}\right]
$$

$$+\lambda\left\{n_s\left(\frac{1-n_wp_w^*}{n_s}\right)^{1-\frac{1}{r}}\left[1-\left(\frac{1-n_wp_w^*}{n_s}\right)\right]\left[\left(\frac{1-n_wp_w^*}{n_s}\right)^{\frac{1}{r}}-(p_w^*)^{\frac{1}{r}}\right]+nv_wp_w^*(1-p_w^*)\right\}. \tag{A3}$$

Carrying out the algebra, we can obtain that

$$\begin{aligned}\mathcal{F}'(p_w^*)=&(1-\lambda)v_w\times\left\{\left(1-\frac{1}{r}\right)(p_w^*)^{-\frac{1}{r}}(1-p_w^*)\left[n_s(p_s^*)^{\frac{1}{r}}+n_w(p_w^*)^{\frac{1}{r}}\right]\right.\\&\left.-(p_w^*)^{1-\frac{1}{r}}\left[n_s(p_s^*)^{\frac{1}{r}}+n_w(p_w^*)^{\frac{1}{r}}\right]+(p_w^*)^{1-\frac{1}{r}}(1-p_w^*)n_w\frac{1}{r}\left[-(p_s^*)^{\frac{1}{r}-1}+(p_w^*)^{\frac{1}{r}-1}\right]\right\}\\&+\lambda\times\left\{\left(\frac{1}{r}-1\right)n_w(p_s^*)^{-\frac{1}{r}}(1-p_s^*)\left[(p_s^*)^{\frac{1}{r}}-(p_w^*)^{\frac{1}{r}}\right]+n_w(p_s^*)^{1-\frac{1}{r}}\left[(p_s^*)^{\frac{1}{r}}-(p_w^*)^{\frac{1}{r}}\right]\right.\\&\left.-n_s(p_s^*)^{1-\frac{1}{r}}(1-p_s^*)\frac{1}{r}\left[\frac{n_w}{n_s}(p_s^*)^{\frac{1}{r}-1}+(p_w^*)^{\frac{1}{r}-1}\right]+nv_w(1-2p_w^*)\right\}.\end{aligned} \tag{A4}$$

Recall that $p_w^\dagger$ is the equilibrium winning probability of each weaker player under no cap. Therefore, for a flexible cap to be optimal, it suffices to show that $\mathcal{F}'(p_w^\dagger)>0$ when v_w is sufficiently small.

Denote the equilibrium winning probability of each strong player by $p_s^\dagger$. We first take a closer look at the equilibrium winning probability $(p_s^\dagger,p_w^\dagger)$ under no cap. From the first-order conditions for each type of players, we have that

$$(p_s^\dagger)^{1-\frac{1}{r}}(1-p_s^\dagger)=v_w(p_w^\dagger)^{1-\frac{1}{r}}(1-p_w^\dagger). \tag{A5}$$

Note that $n_s=1$ by assumption. Therefore, we have that $p_s^\dagger=1-n_wp_w^\dagger$. Substituting the expression of p_s into the above condition, for a sufficiently small v_w, we can obtain that

$$p_w^\dagger=\left(\frac{v_w}{n_w}\right)^r\left[1+o(1)\right].$$

Carrying out the algebra, for a sufficiently small v_w, we have that

$$\begin{aligned}\mathcal{F}'(p_w^\dagger)=&(1-\lambda)\times\left\{v_w\left(1-\frac{1}{r}\right)\left(\frac{v_w}{n_w}\right)^{-1}\left[1+o(1)\right]+o(1)\right\}\\&+\lambda\times\left\{n_w\left[1+o(1)\right]+o(1)\right\}\\=&\frac{n_w}{r}(\lambda+r-1)+o(1)>0,\end{aligned}$$

where the strict inequality follows from the condition $\lambda + r > 1$ assumed in Proposition A2. In other words, there exists a threshold $\hat{v}_l(\lambda, r) > 0$ such that imposing a flexible cap is optimal to the designer for all $v_w/v_s < \hat{v}_l(\lambda, r)$.

Next, we show that having no cap is optimal if v_w is sufficiently large. It is evident that $p_s^\dagger = 1/n + o(1)$ and $p_w^\dagger = 1/n + o(1)$ in this case. Therefore, $\mathcal{F}'(p_w^*)$ in (A4) can be bounded from above by

$$\begin{aligned}\mathcal{F}'(p_w^*) =&(1-\lambda) \times n \times \left[\left(1-\frac{1}{r}\right)\left(1-\frac{1}{n}\right) - n \times \frac{1}{n} + o(1)\right] \\ &+ \lambda \times \left[-n \times \frac{1}{r}\left(1-\frac{1}{n}\right) + n \times \left(1-\frac{2}{n}\right) + o(1)\right] < 0, \text{ for all } p_w^* \in [p_w^\dagger, 1/n].\end{aligned}$$

Therefore, there exists a threshold $\hat{v}_h(\lambda, r) > 0$ such that having no cap is optimal for all $v_w/v_s > \hat{v}_h(\lambda, r)$. This concludes the proof. ■

Proof of Proposition A3

Proof. Similar to the proof of Proposition A2, we normalize v_s to 1 without loss of generality.

We first consider the case in which v_w is sufficiently small. It is evident that $p_w^\dagger = o(1)$ and $p_s^\dagger = 1/n_s + o(1)$. It follows from the first-order conditions (A5) that

$$p_w^\dagger = \frac{1}{n_s}\left(\frac{v_w n_s}{n_s - 1}\right)^{\frac{r}{1-r}}\left[1 + o(1)\right].$$

By the above equation and (A3), when v_w is sufficiently small, we can obtain that

$$\begin{aligned}\mathcal{F}(p_w^\dagger) =&(1-\lambda)v_w \left\{\frac{1}{n_s}\left(\frac{v_w n_s}{n_s-1}\right)^{\frac{r}{1-r}}\left[1+o(1)\right]\right\}^{1-\frac{1}{r}} n_s^{1-\frac{1}{r}}\left[1+o(1)\right] \\ &+ \lambda \times \left\{n_s (p_s^\dagger)^{-1}(1-p_s^\dagger)\left[1+o(1)\right] + o(1)\right\} \\ =&(1-\lambda)\left(1-\frac{1}{n_s}\right) + \lambda\left(1-\frac{1}{n_s}\right) + o(1) = 1 - \frac{1}{n_s} + o(1).\end{aligned}$$

For $p_w^* > v_w^{\frac{2r}{2-r}}$, we have that

$$\begin{aligned}\mathcal{F}(p_w^*) =&(1-\lambda)v_w (p_w^*)^{1-\frac{1}{r}}(1-p_w^*)\left[n_s (p_s^*)^{\frac{1}{r}} + n_w (p_w^*)^{\frac{1}{r}}\right] \\ &+ \lambda\left\{n_s (p_s^*)^{1-\frac{1}{r}}(1-p_s^*)\left[(p_s^*)^{\frac{1}{r}} - (p_w^*)^{\frac{1}{r}}\right] + n v_w p_w^*(1-p_w^*)\right\} \\ \leq&(1-\lambda)v_w (p_w^*)^{1-\frac{1}{r}}(n_s p_s^* + n_w p_w^*) + \lambda\left[n_s (p_s^*)^{1-\frac{1}{r}}(p_s^*)^{\frac{1}{r}}(1-p_s^*) + n v_w p_w^*\right]\end{aligned}$$

$$
\begin{aligned}
&=(1-\lambda)v_w(p_w^*)^{1-\frac{1}{r}} + \lambda\left[n_s p_s^*(1-p_s^*) + n v_w p_w^*\right] \\
&\leq (1-\lambda)v_w^{\frac{1}{2-r}} + \lambda\left(1-\frac{1}{n_s} + n v_w^{\frac{2+r}{2-r}}\right) \\
&= \lambda\left(1-\frac{1}{n_s}\right) + o(1) < \mathcal{F}(p_w^\dagger),
\end{aligned}
$$

where the last inequality follows from $\lambda < 1$.

For $p_w^* \leq v_w^{\frac{2r}{2-r}}$, it follows from (A4) that

$$
\begin{aligned}
\mathcal{F}'(p_w^*) &= (1-\lambda) \times \left\{\left(1-\frac{1}{r}\right) v_w (p_w^*)^{-\frac{1}{r}} n_s^{1-\frac{1}{r}}\left[1+o(1)\right]\right\} + \lambda \times O(1) \\
&\leq (1-\lambda)\left(1-\frac{1}{r}\right) n_s^{1-\frac{1}{r}} v_w^{-\frac{r}{2-r}}\left[1+o(1)\right] < 0.
\end{aligned}
$$

To summarize, $\mathcal{F}(p_w^*)$ is strictly decreasing in p_w^* for $p_w^* \in [p_w^\dagger, v_w^{\frac{2r}{2-r}}]$ and $\mathcal{F}(p_w^*) < \mathcal{F}(p_w^\dagger)$ for all $p_w^* \in (v_w^{\frac{2r}{2-r}}, 1/n]$ if v_w is sufficiently small, which in turn implies that there exists a threshold $\underline{v}(\lambda, r) > 0$ such that having no cap is optimal for all $v_w/v_s < \underline{v}(\lambda, r)$.

Next, we consider the case where v_w is sufficiently large. In this case, we have that $p_w^\dagger = 1/n + o(1)$ and $p_s^\dagger = 1/n + o(1)$. Therefore, for all $p_w^* \in [p_w^\dagger, 1/n]$, we have that

$$
\begin{aligned}
\mathcal{F}'(p_w^*) =& (1-\lambda) \times n \times \left[\left(1-\frac{1}{r}\right)\left(1-\frac{1}{n}\right) - n \times \frac{1}{n} + o(1)\right] \\
&+ \lambda \times \left[-n \times \frac{1}{r}\left(1-\frac{1}{n}\right) + n \times \left(1-\frac{2}{n}\right) + o(1)\right] < 0,
\end{aligned}
$$

and thus $\mathcal{F}(p_w^*)$ is strictly decreasing in p_w^*, which implies the optimality of imposing no cap on the contest. Therefore, there exists $\bar{v}(\lambda, r)$ such that having no cap is optimal for all $v_w/v_s > \bar{v}(\lambda, r)$. This concludes the proof. ■