Online Appendix – Competing to Commit: Markets with Rational Inattention

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Proof of Lemma 1

Proof. ("Only if" direction.) As a first step, we show that any RVP best response of the consumer must display an adjusted multinomial logit formulation *everywhere*.

Claim 1. *Suppose β is a RVP best response to μ. Then, for every $w \in V$ and $y_1, y_2 \geq 0$, it holds that*

$$\beta_i(w, y_1, y_2) = \frac{\pi_i \cdot e^{\frac{w-y_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{w-y_j}{k}} + 1 - \pi_1 - \pi_2}, \ (i \in I) \tag{7}$$

where (π_1, π_2) is a solution to problem

$$\max_{\pi_1', \pi_2' \geq 0} \mathbb{E}_\mu \left[\log \left(\pi_1' \cdot e^{\frac{v-x_1}{k}} + \pi_2' \cdot e^{\frac{v-x_2}{k}} + (1 - \pi_1' - \pi_2') \right) \right] \ \textit{s.t.}\ \pi_1' + \pi_2' \leq 1. \tag{8}$$

In particular, for every $i \in I$, it holds that

$$\pi_i = \mathbb{E}_\mu[\beta_i]. \tag{9}$$

Proof of Claim 1. Let μ be consistent with a strategy profile σ of the sellers. Suppose that β is a best response to μ and that it is RVP. Since β is a best response to μ, known results show that[30] (7) holds μ-almost surely, where (π_1, π_2) is a solution to problem (8). Now, fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. Since β is RVP, there exists a sequence $(\mu^n, \tilde{\sigma}^n)$ with the desired properties such that β is a best response to μ^n for every $n \in \mathbb{N}$. Once again, we know that β must take the following logit functional form in (7) μ^n-a.s. for every $n \in \mathbb{N}$, where each (π_1, π_2) is replaced by (π_1^n, π_2^n), which is a solution to

$$\max_{\pi_1', \pi_2' \geq 0} \mathbb{E}_{\mu^n} \left[\log \left(\pi_1' \cdot e^{\frac{v-x_1}{k}} + \pi_2' \cdot e^{\frac{v-x_2}{k}} + (1 - \pi_1' - \pi_2') \right) \right] \ \text{subject to}\ \pi_1' + \pi_2' \leq 1.$$

[30] See Matějka and McKay (2015), and Denti, Marinacci, and Montrucchio (2020).

Since β is a best reply to all μ^n, and $\mu^n(v, x_1, x_2) > 0$ for all n by assumption, it must be that $(\pi_1^n, \pi_2^n) = (\bar{\pi}_1, \bar{\pi}_2)$ for some $\bar{\pi}_1, \bar{\pi}_2 \in [0, 1]$.

Now, let (v', x_1', x_2') be a generic element in the support of μ. Since $\tilde{\sigma}^n \to \sigma$ implies $\mu^n \to \mu$ in the topology of strong convergence, we have that $\mu^n(\text{Supp}(\mu)) > 0$ for large n. Because β has to be a best response at all n, we know that

$$\beta_i(v', x_1', x_2') = \frac{\bar{\pi}_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \bar{\pi}_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \bar{\pi}_1 - \bar{\pi}_2} = \frac{\pi_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \pi_1 - \pi_2},$$

for all $i \in I$. Therefore, $\bar{\pi}_i = \pi_i$ for all $i \in I$. Lastly, that equation (9) holds follows from standard results. See, e.g., Matějka and McKay (2015), Corollary 2. □

Note that Claim 1 did not use symmetry. We now impose symmetry to prove points $(ii), ..., (v)$ of Lemma 1, thus concluding the "only if" direction.[31] Since β is symmetric by assumption, this implies that $\pi_1 = \pi_2 =: \pi$, i.e., condition (ii) is satisfied. Points (iii), (iv) and (v) now follow from a standard analysis of problem (8), once the constraint $\pi_1' = \pi_2'$ is imposed. We omit the details.

("If" direction.) Let $\beta = (\beta_1, \beta_2)$ be given by (2) and satisfy points $(i), ..., (v)$. Clearly, β is symmetric. Furthermore, given the symmetry of μ, we know that β is *a* best response to μ.[32] Fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. To prove that β is indeed RVP, we distinguish three cases. For each case, we define a symmetric perturbation $\tilde{\sigma}'(\cdot|\cdot) \in \Delta(\mathbb{R}_+^2)^V$ such that $\tilde{\sigma}'(x_1, x_2|v) > 0$. Then, for each $n \in \mathbb{N}$, we let $\tilde{\sigma}^n = \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly and $\tilde{\sigma}^n(x_1, x_2|v) > 0$ for every $n \in \mathbb{N}$. Let μ^n be consistent with $\tilde{\sigma}^n$. It remains to define $\tilde{\sigma}'(\cdot|\cdot)$ and to show that β is indeed a best response to μ^n for each $n \in \mathbb{N}$ in each case.

Case 1. Suppose $\pi = \mathbb{E}_\mu[\beta_i] \in (0, 1/2)$. At the end of this proof, Lemma 6 shows that the other conditions displayed in point (v) of Lemma 1 are redundant. Let $A := \frac{1}{2}\beta_1(v, x_1, x_2) + \frac{1}{2}\beta_1(v, x_2, x_1) > 0$. Note that $A = \frac{1}{2}\beta_2(v, x_1, x_2) + \frac{1}{2}\beta_2(v, x_2, x_1)$ due to symmetry. Also, $\beta_i(v, x, x)$ is strictly decreasing in $x \geq 0$,

[31] That equation (2) and point (i) of Lemma 1 hold follows already from Claim 1.

[32] We say that μ is *symmetric* if $\mu(A) = \mu(A^{\text{sym}})$ for every measurable $A \subseteq V \times \mathbb{R}_+^2$, where $A^{\text{sym}} := \bigcup\{(v, x_1, x_2) : (v, x_2, x_1) \in A\}$ is the symmetric conjugate of A. If μ is part of a symmetric assessment, then it is symmetric.

and that $\beta_i(v,v,v)=\pi$. There are two possibilities:

1) Suppose $A<\pi$. Then, there exists $\alpha\in(0,1)$ and $\varepsilon>0$ such that (i) $v-\varepsilon\geq 0$ and (ii) $\alpha A+(1-\alpha)\beta_i(v,v-\varepsilon,v-\varepsilon)=\pi$ for each $i\in I$. Let $\tilde{\sigma}'(\cdot|\cdot)\in\Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v')=\delta_{(v',v')}$ if $v'\neq v$, and $\tilde{\sigma}'(\cdot|v')=\alpha\left(\frac{1}{2}\delta_{(x_1,x_2)}+\frac{1}{2}\delta_{(x_2,x_1)}\right)+(1-\alpha)\delta_{(v-\varepsilon,v-\varepsilon)}$ when $v'=v$. Since $\mathbb{E}_{\mu^n}[\beta_i]=\pi$ for each $i\in I$, β is a best response to μ^n for each $n\in\mathbb{N}$ by Lemma 6 below and Corollary 2 of Matějka and McKay (2015).

2) Suppose $A\geq\pi$. Then, there exists $\alpha\in(0,1)$ and $\varepsilon\geq 0$ such that $\alpha A+(1-\alpha)\beta_i(v,v+\varepsilon,v+\varepsilon)=\pi$ for each $i\in I$. Let $\tilde{\sigma}'(\cdot|\cdot)\in\Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v')=\delta_{(v',v')}$ if $v'\neq v$, and $\tilde{\sigma}'(\cdot|v')=\alpha\left(\frac{1}{2}\delta_{(x_1,x_2)}+\frac{1}{2}\delta_{(x_2,x_1)}\right)+(1-\alpha)\delta_{(v+\varepsilon,v+\varepsilon)}$ when $v'=v$. Like before, $\mathbb{E}_{\mu^n}[\beta_i]=\pi$ for each $i\in I$ implies that β is a best response to μ^n for each $n\in\mathbb{N}$ as required.

Case 2. Suppose $\pi=0$, so that $\beta_1=\beta_2=0$. Let $A:=\frac{1}{2}e^{\frac{v-x_1}{k}}+\frac{1}{2}e^{\frac{v-x_2}{k}}>0$. There are two possibilities:

1) If $A\leq 1$, let $\tilde{\sigma}'(\cdot|\cdot)\in\Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v')=\delta_{(v',v')}$ if $v'\neq v$, and $\tilde{\sigma}'(\cdot|v')=\frac{1}{2}\delta_{(x_1,x_2)}+\frac{1}{2}\delta_{(x_2,x_1)}$ when $v'=v$. By construction, for every $n\in\mathbb{N}$, we have $\mathbb{E}_{\mu^n}\left[e^{\frac{v-x_i}{k}}\right]\leq 1$ for each $i\in I$. This implies that β is a best reply to μ^n for all $n\in\mathbb{N}$ as required.

2) If $A>1$, there exists $\alpha\in(0,1)$ and $\varepsilon>0$ such that $\alpha A+(1-\alpha)e^{-\varepsilon/k}\leq 1$. Let $\tilde{\sigma}'(\cdot|\cdot)\in\Delta(\mathbb{R}^2_+)^V$ be such that $\tilde{\sigma}'(\cdot|v')=\delta_{(v',v')}$ if $v'\neq v$, and $\tilde{\sigma}'(\cdot|v')=\alpha\left(\frac{1}{2}\delta_{(x_1,x_2)}+\frac{1}{2}\delta_{(x_2,x_1)}\right)+(1-\alpha)\delta_{(v+\varepsilon,v+\varepsilon)}$ when $v'=v$. For all $n\in\mathbb{N}$, we have $\mathbb{E}_{\mu^n}\left[e^{\frac{v-x_i}{k}}\right]\leq 1$ for all $i\in I$. This implies that β is a best reply to μ^n for all $n\in\mathbb{N}$ as required.

Case 3. The proof for the case $\pi=1/2$ is similar to the one for $\pi=0$ (see Case 2). We omit the details. □

Lemma 6. *Let μ be symmetric, and β be given by (2). If $\pi=\mathbb{E}_\mu[\beta_i]\in(0,1/2)$ for all $i\in I$, then $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right]\geq 1$ for all $i\in I$, and $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}}\right)^{-1}\right]\geq 1/2$.*

Proof. For every $y>0$ and $\gamma\in(0,1/2)$, let

$$g(y,\gamma)=\frac{1-2\gamma}{\gamma y+(1-2\gamma)}\quad\text{and}\quad h(y,\gamma)=\frac{\gamma}{\gamma+(1-2\gamma)y}.$$

Note that g and h are strictly decreasing and convex in $y > 0$ for every $\gamma \in (0, 1/2)$. Moreover, $g(y, \gamma) = 1 - 2\gamma$ iff $y = 2$, and $h(y, \gamma) = 2\gamma$ iff $y = 1/2$.

From Jensen's inequality, we have

$$2\pi = \mathbb{E}_\mu[\beta_1+\beta_2] = \mathbb{E}_\mu \left[h \left(\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}} \right)^{-1}, \pi \right) \right] \geq h \left(\mathbb{E}_\mu \left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}} \right)^{-1} \right], \pi \right),$$

which implies that $\mathbb{E}_\mu \left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}} \right)^{-1} \right] \geq 1/2$. Similarly,

$$1-2\pi = 1-\mathbb{E}_\mu[\beta_1+\beta_2] = \mathbb{E}_\mu \left[g \left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}, \pi \right) \right] \geq g \left(\mathbb{E}_\mu \left[e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}} \right], \pi \right),$$

which implies that $\mathbb{E}_\mu \left[e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}} \right] \geq 2$. Since μ is symmetric, it holds that $\mathbb{E}_\mu \left[e^{\frac{v-x_1}{k}} \right] = \mathbb{E}_\mu \left[e^{\frac{v-x_2}{k}} \right]$. Therefore, $\mathbb{E}_\mu \left[e^{\frac{v-x_i}{k}} \right] \geq 1$ for every $i \in I$. □

Proof of Proposition 1

Proof. We show that colluding firms offer the same price as a monopolist facing the aggregate trade engagement level. Since the functional form of the monopolist's best response is enough to characterize the unique trading equilibrium in Ravid (2020), this proves that monopoly and collusion are equilibrium outcome-equivalent. Given this fact, the proposition follows from Theorem 1 and Corollary 1 in Ravid (2020).

Fix $v \in V$. Suppose firms face individual demands given by

$$Q^i(x_1, x_2) := \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2} \quad (i \in I).$$

Colluding firms solve the problem (P): $\max_{x_1,x_2 \geq 0} \Pi^M(x_1, x_2)$, where $\Pi^M(x_1, x_2) := \sum_{i=1,2} Q^i(x_1, x_2) \cdot x_i$. If $0 = \pi_i < \pi_j$ for some $i \in I$, the problem (P) is identical to the problem solved by the monopolist in Ravid (2020). The equilibrium outcome equivalence between monopoly and collusion is, therefore, immediate. Thus, suppose $\pi_i > 0$ for all $i \in I$, and let $D := \sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2 > 0$. The FOCs associated to problem (P):

$$D = \frac{\pi_j e^{\frac{v-x_j}{k}}}{k} \cdot (x_i - x_j) + \frac{x_i}{k}(1 - \pi_i - \pi_j), \quad \forall i \in I. \tag{10}$$

An interior solution exists,[33] and is characterized by the FOCs. Combing equations (10) across $i \in I$ yields $\frac{x_1 - x_2}{k} \cdot D = 0$, which is true iff $x_1 = x_2$. Let $x_1 = x_2 = x$. Equation (10) becomes $D = \frac{x}{k}(1 - \pi_1 - \pi_2)$, and admits unique solution x^* given by

$$x^* = k\left(1 + W\left(\frac{\pi_1 + \pi_2}{1 - \pi_1 - \pi_2} e^{\frac{v-k}{k}}\right)\right), \tag{11}$$

where $y \mapsto W(y)$ is the Lambert function.[34] Equation (11) is identical to equation (6) in Ravid (2020), characterizing the monopolist optimal equilibrium pricing when the consumer's overall trade engagement level is $\pi_1 + \pi_2 = \pi^M$. Thus, monopoly and collusion models are equilibrium outcome-equivalent, as required. $\square$

Proof of Lemma 2

Proof. A profile (μ, β, σ) is a competitive trading equilibrium if and only if it is symmetric trading equilibrium. That is, (a) μ is consistent with σ, (b) β is given by (2) with $\pi > 0$, and (c) σ is symmetric and σ_i is a best response to σ_{-i} given β.

We focus on the equilibrium behavior of the firms. Fix $v \in V$ arbitrarily. From Milgrom and Roberts (1990), for fixed symmetric logit demand β of the consumer, the unique NE of the pricing game played by the firms is pure and symmetric. To characterize it, suppose (b) holds and let $\sigma_i(\cdot|v) = \delta_{x_i(v)}$ for all $i \in I$. Then, taking $\pi \in (0, 1/2]$ and $-i$'s offer $x_{-i}(v) = x_{-i}$ as given, firm i solves:

$$\max_{x_i \geq 0} \frac{x_i \cdot \pi e^{\frac{v-x_i}{k}}}{\pi \cdot \left(e^{\frac{v-x_i}{k}} + e^{\frac{v-x_{-i}}{k}}\right) + 1 - 2\pi}$$

[33]For all $x_j \geq 0$, if $x_i = 0$, the LHS of (10) is strictly greater than the RHS. This implies that any solution to (P) (if it exists) must be interior. Conversely, there exists a $\bar{x}_i > 0$ such that, for all $x_i \geq \bar{x}_i$, the RHS of (10) is strictly greater than the LHS for all $x_j \geq 0$. This means that $\Pi^M(x_i, x_j)$ is eventually decreasing in x_i for all $x_j \geq 0$, implying that a bounded solution exists.

[34]The Lambert function is defined as the inverse of $z \mapsto ze^z$.

The first-order condition can be written as

$$\frac{d\Pi_i^C(v,x_1,x_2)}{dx_i} = \beta_i(v,x_1,x_2)\left(1 - \frac{x_i}{k}\cdot[1-\beta_i(v,x_1,x_2)]\right) = 0. \qquad (12)$$

Note that $\beta_i > 0$ for all $x_i \geq 0$ and $\lim_{x_i\to\infty}\beta_i = 0$. Therefore, $x_i \mapsto \frac{d\Pi_i^C(v,x_1,x_2)}{dx_i}$ crosses zero exactly once from above. It follows that $x_i \mapsto \Pi_i^C(v,x_i,x_{-i})$ admits a unique (interior) global maximum characterized by the FOC. We rearrange (12) and use symmetry to see that, in equilibrium, $x_i(v) = x_{-i}(v) =: x^C(v)$ satisfies:

$$x^C(v;\pi) = k\cdot\left[1 + \frac{\pi e^{\frac{v-x^C(v;\pi)}{k}}}{\pi e^{\frac{v-x^C(v;\pi)}{k}} + 1 - 2\pi}\right].$$

Define $\phi(v;\pi) := \pi e^{\frac{v-x^C(v;\pi)}{k}} / \left(\pi e^{\frac{v-x^C(v;\pi)}{k}} + 1 - 2\pi\right)$. Then, the equilibrium firm behavior is given by $x^C(v;\pi) = k\cdot[1+\phi(v;\pi)]$, where optimality requires that

$$\left(1 + e^{\phi(v;\pi)}\frac{1-2\pi}{\pi e^{\frac{v-k}{k}}}\right)\phi(v;\pi) = 1.$$

The above equation uniquely pins down $\phi(v;\pi) > 0$ for fixed $\pi \in (0,1/2]$: The LHS is continuously increasing in ϕ, it goes to 0 as $\phi \downarrow 0$ and goes to ∞ as $\phi \uparrow \infty$. This concludes the proof of the lemma. □

Proof of Theorem 1

Proof. We start with the proof of the if and only if statement. The necessity proof is as follows. As we argued in Lemma 2, in any competitive trading equilibrium, the sellers charge a price $x_i(v) = x^C(v)$ strictly above k for each $v \in V$. Now, suppose by way of contradiction that a trading equilibrium exists but $k \geq k^t$, i.e., $\mathbb{E}_\lambda\left[e^{v/k-1}\right] \leq 1$. In equilibrium, we would have $\mathbb{E}_\mu\left[e^{\frac{v-x(v)}{k}}\right] < \mathbb{E}_\lambda\left[e^{v/k-1}\right] \leq 1$. This is in contradiction with our hypothesis of on-path equilibrium trade.[35] Thus, $k < k^t$ is necessary for the existence of a competitive trading equilibrium.

We now turn to the sufficiency direction. We split the proof in two parts. First,

[35] From Lemma 1, the consumer's trade engagement level with each firm would equal zero.

we restrict attention to values of k for which trade occurs with probability 1. We then consider the remaining parameter values. To this end, we need to introduce some further notation. Let k^e be the unique solution to $\mathbb{E}_\lambda\left[e^{2-v/k^e}\right] = 1$. Notice that

$$\mathbb{E}_\lambda\left[e^{2-v/k^t}\right] > \mathbb{E}_\lambda\left[e^{1-v/k^t}\right] = \mathbb{E}_\lambda\left[\frac{1}{e^{v/k^t-1}}\right] \geq \frac{1}{\mathbb{E}_\lambda\left[e^{v/k^t-1}\right]} = 1.$$

Thus, $0 < k^e < k^t$.

Suppose first that $k \leq k^e$. Take $x^C(v) = 2k$ for all $v \in V$. Observe that this configuration of prices is an equilibrium of the pricing game played by the firms when they face a symmetric logit demand with $\pi = 1/2$. At the same time, from Lemma 1, a symmetric trade engagement level $\pi = 1/2$ is consistent with this configuration of prices if and only if $\mathbb{E}_\lambda\left[e^{2-v/k}\right] \leq 1$, or equivalently, $k \leq k^e$. Therefore, a symmetric efficient equilibrium exists. Since a symmetric efficient equilibrium is indeed a competitive trading equilibrium, we are done.

Now, consider the case where $k \in (k^e, k^t)$, or equivalently, $\mathbb{E}_\lambda\left[e^{2-v/k}\right] > 1$ and $\mathbb{E}_\lambda\left[e^{v/k-1}\right] > 1$. Define the functions $\phi = \phi(v; p)$ and $x = x^C(v; p)$ as in Lemma 2 with π replaced by p. Let $F = F(p)$ be defined as

$$F(p) := \mathbb{E}_\lambda\left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p)}\right]. \tag{13}$$

Since the function $F(\cdot)$ satisfies $F(p) = \frac{1}{p}\mathbb{E}[\beta_i(v, x^C(v;p), x^C(v;p))]$ for every $i \in I$, a symmetric RVP trading equilibrium where trade occurs with probability strictly between 0 and 1 exists if $F(p^*) = 1$ for some $p^* \in (0, 1/2)$.[36] We prove this by relying on the Intermediate Value Theorem, hence exploiting the continuity of $F(\cdot)$ in $p \in (0, 1/2]$. In particular, we show that there exists $0 < p_0 < p_1 < 1/2$ such that for all $p \in (0, p_0)$, we have $F(p) > 1$, and for all $p \in (p_1, 1/2)$, we have $F(p) < 1$.

Existence of $0 < p_1 < 1/2$*:* We exploit the fact that $F(\cdot)$ is continuously differentiable. This follows from the Implicit Function Theorem that guarantees that

[36] See Lemma 6 above.

$\phi(v;p)$ is continuously differentiable in $p \in (0, 1/2]$ for all $v \in V$.[37] Given that V is finite and $\phi(v;p) \uparrow 1$ as $p \uparrow 1/2$, for every $\varepsilon > 0$ there exists a $\bar{p}_1 \in (0, 1/2)$ such that $\phi(v;p) > 1 - \varepsilon$ for all $v \in V$ and $p \in (\bar{p}_1, 1/2)$. Fix $\varepsilon > 0$ and $\delta > 0$ small enough so that $\mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k}\right] - \delta > 1$, and let $\bar{p}_1$ be the p-threshold corresponding to ε.[38] For every $v \in V$, define

$$A(v) := \max_{p \in [\bar{p}_1, 1/2]} e^{1-v/k+\phi(v;p)} \cdot \frac{\partial}{\partial p}\phi(v;p) \cdot \frac{D_{\max}(p)}{D_{\min}(p)}$$

where

$$D_{\max}(p) := \max_{v \in V} \left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k}\right)^2 > 0$$

$$D_{\min}(p) := \min_{v \in V} \left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k}\right)^2 > 0.$$

We make two observations.

Obs. 1: Each $A(v)$ is a well-defined real number since it is the maximum value of a continuous function on a compact support. Again by the finiteness of V, there exists $\bar{p}_2 \in (0, 1/2)$ such that $(1-2p) \cdot A(v) \leq \delta$ for all $v \in V$ and $p \in (\bar{p}_2, 1/2)$.

Obs. 2: Since $D_{\max}(p), D_{\min}(p) \to 1$ as $p \uparrow 1/2$, we have that $D_{\max}(p)/D_{min}(p) \to 1$ as $p \uparrow 1/2$. Therefore, there exists $\bar{p}_3 \in (0, 1/2)$ such that $D_{\max}(p)/D_{min}(p) \leq 1 + \delta/2$ for all $p \in (\bar{p}_3, 1/2)$.

Now, let $\bar{p} = \max\{\bar{p}_1, \bar{p}_2, \bar{p}_3\} < 1/2$. For all $p \in (\bar{p}, 1/2)$, we have:

$$\begin{aligned} F'(p) &= \mathbb{E}_\lambda \left[\frac{2 \cdot \left(e^{1+\phi(v;p)-v/k} - 1\right) - (1-2p) \cdot e^{\phi(v;p)+1-v/k} \cdot \frac{\partial}{\partial p}\phi(v;p)}{\left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k}\right)^2}\right] \\ &\geq \mathbb{E}_\lambda \left[\frac{2}{D_{\max}(p)} \cdot e^{1+\phi(v;p)-v/k} - \frac{2}{D_{\min}(p)} - (1-2p) \cdot \frac{e^{\phi(v;p)+1-v/k}}{D_{\min}(p)} \cdot \frac{\partial}{\partial p}\phi(v;p)\right] \end{aligned}$$

[37] More formally, for $\phi \in (0, \infty)$, $v \in V$, and $p \in (0, 1/2 + \tau)$, let

$$G(p, \phi, v) := \phi \cdot \left(1 + e^{\phi} \cdot (1-2p) / \left[p \cdot e^{\frac{v-k}{k}}\right]\right) - 1.$$

For $\tau > 0$ small enough, the assumptions of the Implicit Function Theorem are satisfied by G. Thus, there exists continuously differentiable $\bar{\phi}(v;p)$ on $(0, 1/2+\tau) \times V$ such that $G(p, \bar{\phi}(v;p), v) = 0$ for all $v \in V$ and $p \in (0, 1/2 + \tau)$. Let $\bar{\phi}(v;p) = \phi(v;p)$ on $V \times (0, 1/2]$.

[38] Such $\varepsilon, \delta > 0$ exist because $\mathbb{E}_\lambda \left[e^{2-v/k}\right] > 1$ by assumption.

$$
\begin{aligned}
&= \frac{1}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[2 \cdot e^{1+\phi(v;p)-v/k} - \frac{D_{\max}(p)}{D_{\min}(p)} \left(2 + (1-2p) \cdot e^{\phi(v;p)+1-v/k} \cdot \frac{\partial}{\partial p}\phi(v;p) \right) \right] \\
&\geq \frac{1}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[2 \cdot e^{1+\phi(v;p)-v/k} - 2 \cdot \frac{D_{\max}(p)}{D_{\min}(p)} - (1-2p) \cdot A(v) \right] \\
&\geq \frac{2}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[e^{1+\phi(v;p)-v/k} - 1 - \delta \right] \geq \frac{2}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} - 1 - \delta \right] > 0,
\end{aligned}
$$

where the first inequality comes from the fact that $\frac{\partial}{\partial p}\phi(v;p) \geq 0$.[39] Since $F(1/2) = 1$ and $F'(p) > 0$ for all $p \in (0, 1/2)$ close to 1/2, the existence of p_1 follows.

Existence of $0 < p_0 < p_1 < 1/2$*:* Given that V is finite, $\phi(v;p) \downarrow 0$ and $2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p) \to 1$ as $p \downarrow 0$, for every $\varepsilon > 0$ there exists a $\underline{p} \in (0, 1/2)$ such that $\phi(v;p) < \varepsilon$ and $2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p) < 1 + \varepsilon$ for all $v \in V$ and $p \in (0, \underline{p})$. Let $\varepsilon > 0$ be small enough so that $\mathbb{E}_\lambda \left[e^{v/k-1-\varepsilon} \right] / (1+\varepsilon) > 1$. Such an $\varepsilon > 0$ exists because $\mathbb{E}_\lambda \left[e^{v/k-1} \right] > 1$. For all $p \in (0, \underline{p})$, we have:

$$
F(p) = \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p)} \right] \geq \frac{\mathbb{E}_\lambda \left[e^{\frac{v-k}{k}} \cdot e^{-\phi} \right]}{1+\varepsilon} \geq \frac{\mathbb{E}_\lambda \left[e^{v/k-1-\varepsilon} \right]}{1+\varepsilon} > 1.
$$

Thus, a $p_0 \in (0, p_1)$ with the desired properties exists. This concludes the proof of existence of a competitive equilibrium.

Uniqueness: Once again, we distinguish between two cases. First, suppose $k \leq k^e$, or equivalently, $\mathbb{E}_\lambda \left[e^{2-v/k} \right] \leq 1$. From the proof of existence, we know that a competitive efficient equilibrium exists. We want to show that no other symmetric trading equilibrium can exist. For each $p \in (0, 1/2]$ and $v \in V$, let $\phi = \phi(v;p)$, $x = x^C(v;p)$, and $F = F(p)$ be defined as above. Note that $F(1/2) = 1$. To prove that no other symmetric trading equilibrium exists, it is sufficient to show that $F(p) \neq 1$ for all $p \in (0, 1/2)$. With this goal in mind, first note that $\phi(v;p)$ is strictly increasing in $p \in (0, 1/2]$ for every $v \in V$, and that $\phi(v; 1/2) = 1$. Thus, given that V is finite, when p is strictly below 1/2, there exists $\varepsilon > 0$ small enough such that $\phi(v;p) < 1 - \varepsilon$ for all $v \in V$. Second, observe that $\mathbb{E}_\lambda \left[e^{2-c-v/k} \right] < 1$ for

[39] See the proof of Lemma 8.

any constant $c > 0$. Now, fix $p \in (0, 1/2)$ and its corresponding $\varepsilon > 0$. We have

$$F(p) = \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p)} \right] = \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{\phi + 1 - v/k}} \right]$$
$$> \mathbb{E}_\lambda \left[\frac{1}{2p + (1-2p) \cdot e^{2-\varepsilon - v/k}} \right] \geq \frac{1}{2p + (1-2p) \cdot \mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} \right]} > 1.$$

where the first strict inequality comes from $\phi = \phi(v; p) < 1 - \varepsilon$ for all $v \in V$, the weak inequality is an application of Jensen's inequality, and the last inequality is implied by $\mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} \right] < 1$. Hence, $F(p) \neq 1$ for all $p < 1/2$ as required.

Now, consider the case where $k \in (k^e, k^t)$. Suppose towards a contradiction that there exist $0 < p^* < p^{**} < 1/2$ such that $F(p^*) = F(p^{**}) = 1$. Define $\gamma \in (0, 1)$ implicitly by $p^{**} = \gamma p^* + (1-\gamma)1/2$. We have

$$F(p^{**}) = \mathbb{E}_\lambda \left[\frac{1}{2p^{**} + (1-2p^{**}) \cdot e^{\phi(v;p^{**}) + 1 - v/k}} \right]$$
$$= \mathbb{E}_\lambda \left[\frac{1}{2\gamma p^* + 1 - \gamma + \gamma(1-2p^*) \cdot e^{\phi(v;p^{**}) + 1 - v/k}} \right]$$
$$< \mathbb{E}_\lambda \left[\frac{1}{1 - \gamma + \gamma \left(2p^* + (1-2p^*) \cdot e^{\phi(v;p^*) + 1 - v/k} \right)} \right] \leq 1.$$

The first inequality follows from the fact that $p^{**} > p^*$ and that $\phi(v; p)$ is strictly increasing in $p \in (0, 1/2)$ for all $v \in V$. In order to prove the second inequality, we define $g(\gamma) := \mathbb{E}_\lambda \left[\frac{1}{1-\gamma+\gamma\left(2p^*+(1-2p^*)\cdot e^{\phi(v;p^*)+1-v/k}\right)} \right]$. Note that $g(0) = 1$ and $g(1) = F(p^*) = 1$. It remains to show that $g(\gamma)$ is convex for all $\gamma \in [0, 1]$. Taking the second derivative, we get

$$g''(\gamma) = \mathbb{E}_\lambda \left[\frac{2 \left(2p^* + (1-2p^*) \cdot e^{\phi(v;p^*) + 1 - v/k} - 1 \right)^2}{\left(1 - \gamma + \gamma \left(2p^* + (1-2p^*) \cdot e^{\phi(v;p^*) + 1 - v/k} \right) \right)^3} \right] \geq 0.$$

Thus, we reached the contradiction that $F(p^{**}) < 1$. We conclude that there is at most one $p \in (0, 1/2)$ such that $F(p) = 1$, and the proof of uniqueness. □

Proof of Lemma 3

Proof. From Lemma 2, we know that the firms price according to $x^C(v;\pi^C) = k \cdot (1+\phi(v;\pi^C))$. Under collusion,[40] for every $v \in V$, each active firm plays a strategy $\sigma^M(\cdot|v) = \delta_{x^M(v)}$ such that $x^M(v;\pi^M) = k \cdot \left(1 + W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)\right)$, and $W(\cdot)$ is the Lambert function. Compared to the equilibrium price formula of the competition model, we note that the only difference is that $W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)$ is replaced by $\phi(v;\pi^C)$. Fix $p \in (0,1/2)$ arbitrarily. According to Lemma 2, $\phi(v;p)$ is the unique solution to equation (5), where π is replaced by p. Note that (5) is equivalent to $\frac{p}{1-2p}e^{v/k-1} = \phi \cdot \frac{p}{1-2p}e^{v/k-1} + \phi e^{\phi}$. Therefore

$$\frac{2p}{1-2p}e^{v/k-1} > \frac{p}{1-2p}e^{v/k-1} = \phi(v;p)\left(\frac{p}{1-2p}e^{v/k-1} + e^{\phi(v;p)}\right) > \phi(v;p)e^{\phi(v;p)}.$$

Applying the Lambert function on both sides yields $\phi(v;p) < W\left(\frac{2p}{1-2p}e^{v/k-1}\right)$, which implies the result. □

Proof of Proposition 2

Proof. Fix $k \in (0,k^t)$, and let $(\mu^M, \sigma^M, \beta^M)$ and $(\mu^C, \beta^C, \sigma^C)$ be the unique symmetric equilibrium under collusion and competition respectively associated with the cost parameter k. Set $\pi^M = \mathbb{E}_{\mu^M}[\beta_1^M + \beta_2^M]$ and $\pi^C = \mathbb{E}_{\mu^C}[\beta_i^C]$ for each $i \in I$.

If $k \leq k^e$, the result follows from Proposition 3 and Corollary 1 of Ravid (2020). In words, while an efficient equilibrium cannot exist under collusion, it is the only competitive trading equilibrium outcome. Hence, $0 < \pi^M < 1 = 2\pi^C$, as required.

Now assume that $k \in (k^e, k^t)$. Let $W(v;2p) := W\left(\frac{2p}{1-2p}e^{v/k-1}\right)$ for all $v \in V$ and $p \in (0,1/2)$. From Lemma 3, we know $\phi(v;p) < W(v;2p)$. Following the proof of Theorem 1 in Ravid (2020), the overall equilibrium engagement level in the collusion benchmark is given by $\pi^M = 2p^M$, where p^M the unique solution in $(0,1/2)$ to the equation:

$$G(2p) := \mathbb{E}_\lambda\left[\frac{e^{\frac{v-k}{k}} \cdot e^{-W(v;2p)}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-W(v;2p)} + (1-2p)}\right] = 1. \tag{14}$$

[40] See the proof of Proposition 1, and Proposition 2 in Ravid (2020).

Let F be defined as in the proof of Theorem 1. We have

$$1 = G(2p^M) = \mathbb{E}_\lambda\left[\frac{1}{2p^M + (1-2p^M)\cdot e^{W(v;2p^M)+1-v/k}}\right]$$
$$< \mathbb{E}_\lambda\left[\frac{1}{2p^M + (1-2p^M)\cdot e^{\phi(v;p^M)+1-v/k}}\right] = F(p^M),$$

where the strict inequality follows from Lemma 3. From the proof of Theorem 1, we conclude that $p^M < \pi^C$. This is equivalent to $\pi^M < 2\pi^C$. □

Proof of Proposition 3

Proof. Follows directly from the proof of Theorem 1. □

Proof of Corollary 1

Proof. Follows directly from the proof of Lemma 2. □

Proof of Theorem 2

Preliminary analysis for the collusion benchmark. For each $k \in (0, k^t]$, let $F_k^M : [0,1) \to \mathbb{R}_+$ be defined as $F_k^M(p) := \mathbb{E}_\lambda\left[\frac{1}{p+(1-p)\cdot e^{W(p,v,k)+1-v/k}}\right]$. Again, we abuse notation and write $W(p,v,k)$ for $W\left(\frac{p}{1-p}e^{v/k-1}\right)$, where $W(\cdot)$ is the Lambert function. We are interested in the solution $p^M(k)$ to $F_k^M(p) = 1$. By the Implicit Function Theorem, we know that whenever this solution exists, it is continuously differentiable. In his Theorem 1, Ravid (2020) shows that $p^M(k)$ exists uniquely in $(0,1)$ whenever $k \in (0, k^t)$. The following Lemma characterizes additional properties that $p^M(k)$ satisfies as k ranges in $(0, k^t)$.

Lemma 7. *We have:*

(i) $\lim_{k\uparrow k^t} p^M(k) = 0$.

(ii) $\lim_{k\uparrow k^t} \frac{\partial}{\partial k}p^M(k) = -\mathbb{E}_\lambda\left[\frac{v}{(k^t)^2}\cdot e^{v/k^t-1}\right] / \mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^t}}{e^{2\cdot(1-v/k^t)}}\right]$.

Proof. (i): Recall from Ravid (2020) that $F_k^M(\cdot)$ crosses the line $y=1$ only once from above.[41] Therefore, it is sufficient to show that (#): for every $p \in (0,1)$, there

[41] This is shown by Ravid (2020) in the proof of Theorem 1.

exists $k_p \in (0, k^t)$ such that for all k strictly between k_p and k^t, $F_k^M(p) < 1$.

Since the Lambert function $W(\cdot)$ is strictly increasing, $W(p,v,k)$ is strictly decreasing in k for every $p \in (0,1)$ and $v \in V$. It further satisfies $W(p,v,k) > 0$ for all $p \in (0,1)$, $v \in V$ and $k > 0$. Fix $p \in (0,1)$ arbitrarily. Given the finiteness of V, there exists $c_p > 0$ such that $W(p,v,k) > c_p$ for all $v \in V$ and $k \in (0, k^t)$. Since $\mathbb{E}_\lambda\left[e^{v/k^t-1}\right] = 1$, we have $\mathbb{E}_\lambda\left[e^{v/k^t-1-c_p}\right] < 1$. Therefore, continuity implies that there exists k_p strictly between 0 and k^t so that $\mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1$ for all $k \in (k_p, k^t)$. Fix any such k. We have:

$$\begin{aligned} F_k^M(p) &= \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{W(p,v,k)+1-v/k}}\right] \leq \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{c_p+1-v/k}}\right] \\ &\leq 2p + (1-2p)\cdot \mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1. \end{aligned}$$

Thus, (#) holds.

(ii): For each $k \in (0, k^t)$, we totally differentiate the equation $F_k^M(p^M(k)) = 1$ to obtain:[42]

$$\frac{\partial}{\partial k} p^M(k) = -\frac{A_M}{B_M} \tag{15}$$

where

$$A_M = \mathbb{E}_\lambda\left[\frac{v \cdot (1-p^M(k)) \cdot e^{W(p^M(k),v,k)+1-v/k}}{k^2 \cdot D_M^2 \cdot (1 + W(p^M(k),v,k))}\right],$$

$$B_M = \mathbb{E}_\lambda\left[\frac{1}{D_M^2}\cdot\left(1 - e^{W(p^M(k),v,k)+1-v/k} + (1-p^M(k))\cdot\frac{e^{W(p^M(k),v,k)}\cdot W'\left(\frac{p^M(k)}{1-p^M(k)}e^{v/k-1}\right)}{(1-p^M(k))^2}\right)\right],$$

and

$$D_M = p^M(k) + (1-p^M(k))\cdot e^{W(p^M(k),v,k)+1-v/k}.$$

As $k \uparrow k^t$, we know from (i) that $p^M(k) \to 0$. Therefore, $A_M \to \mathbb{E}_\lambda\left[\frac{v}{(k^t)^2}\cdot e^{v/k^t-1}\right]$ and $B_M \to \mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^t}}{e^{2\cdot(1-v/k^t)}}\right]$.[43] This concludes the proof of Lemma 7. □

Preliminary analysis for the competition model. For each $k \in (k^e, k^t]$, we define $F_k^C : [0, 1/2) \to \mathbb{R}_+$ as $F_k^C(p) := \mathbb{E}_\lambda\left[\frac{1}{2p+(1-2p)\cdot e^{\phi(v;p,k)+1-v/k}}\right]$, where for

[42] To derive equation (18), we used the fact that $W'(x) = \frac{W(x)}{x\cdot(1+W(x))}$ for all $x > 0$.

[43] Here, we used the fact that $W'(x) = 1$ as $x \downarrow 0$.

$p > 0$, we let $\phi(v; p, k)$ be defined as the unique solution to equation (5), and we set $\phi(v; 0, k) := 0$ for all $v \in V$ and $k \in (k^e, k^t]$. Let $p^C(k)$ be a solution to $F_k^C(p) = 1$. From Theorem 1, we know that $p^C(k)$ exists and is unique for all $k \in (k^e, k^t)$. Again, by the Implicit Function theorem we know that $p^C(k)$ is continuously differentiable on (k^e, k^t). The next Lemma provides additional properties that $p^C(k)$ satisfies.

Lemma 8. *We have:*

(i) $\lim_{k \uparrow k^t} p^C(k) = 0$.

(ii) $\lim_{k \uparrow k^t} \frac{\partial}{\partial k} p^C(k) = -\mathbb{E}_\lambda \left[\frac{v}{(k^t)^2} \cdot e^{v/k^t - 1} \right] / \mathbb{E}_\lambda \left[\frac{2(1 - e^{1 - v/k^t}) + 1}{e^{2 \cdot (1 - v/k^t)}} \right]$.

Proof. (i): We show that (#): for every $p \in (0, 1/2)$, there exists $k_p \in (k^e, k^t)$ such that for all k strictly between k_p and k^t, $F_k^C(p) < 1$. Given our proof of Theorem 1, (#) implies that for all k sufficiently close to k^t, $p^C(k) < p$, proving the statement.

From equation (5), $\phi(v; p, k)$ is strictly decreasing in k for every $p \in (0, 1/2)$ and $v \in V$, and satisfies $\phi(v; p, k) > 0$ for all $p \in (0, 1/2)$, $v \in V$ and $k > 0$. Fix $p \in (0, 1/2)$ arbitrarily. Given the finiteness of V, there exists $c_p > 0$ such that $\phi(v; p, k) > c_p$ for all $v \in V$ and $k \in (k^e, k^t]$. Since $\mathbb{E}_\lambda \left[e^{v/k^t - 1} \right] = 1$, we have $\mathbb{E}_\lambda \left[e^{v/k^t - 1 - c_p} \right] < 1$. Therefore, continuity implies that there exists k_p strictly between k^e and k^t so that $\mathbb{E}_\lambda \left[e^{v/k - 1 - c_p} \right] < 1$ for all $k \in (k_p, k^t)$. Fix any such k. We have:

$$
\begin{aligned}
F_k^C(p) &= \mathbb{E}_\lambda \left[\frac{1}{2p + (1 - 2p) \cdot e^{\phi(v; p, k) + 1 - v/k}} \right] \le \mathbb{E}_\lambda \left[\frac{1}{2p + (1 - 2p) \cdot e^{c_p + 1 - v/k}} \right] \\
&\le 2p + (1 - 2p) \cdot \mathbb{E}_\lambda \left[e^{v/k - 1 - c_p} \right] < 1.
\end{aligned}
$$

Thus, (#) holds.

(ii): We first totally differentiate equation (5) to find the partial derivatives of ϕ with respect to p and k. That is, $\phi_p(v; p, k) := \frac{\partial}{\partial p} \phi(v; p, k)$ and $\phi_k(v; p, k) := \frac{\partial}{\partial k} \phi(v; p, k)$. After some algebra, one can show that

$$
\phi_p(v; p, k) = \frac{1 - \phi(v; p, k)}{(1 - 2p) \cdot \left(p + e^{\phi(v; p, k)} \cdot (1 + \phi(v; p, k)) \frac{1 - 2p}{e^{v/k - 1}} \right)} \ge 0, \tag{16}
$$

$$\phi_k(v;p,k) = -\frac{v}{k^2} \cdot \frac{\phi(v;p,k)e^{\phi(v;p,k)}}{\frac{p}{1-2p}e^{v/k-1} + e^{\phi(v;p,k)}(1+\phi(v;p,k))} \leq 0. \tag{17}$$

Note that, as $k \uparrow k^t$ and, therefore, $p \to 0$, we have $\phi \to 0$. Therefore, $\phi_p \to e^{v/k^t-1}$ and $\phi_k \to 0$ as $k \uparrow k^t$.

Next, we totally differentiate the equation $F_k^C(p^C(k)) = 1$ with respect to $k > k^e$. One can show that

$$\frac{\partial}{\partial k} p^C(k) = -\frac{A_C}{B_C} \tag{18}$$

where

$$A_C = \mathbb{E}_\lambda \left[\frac{1}{D_C^2} \cdot \left((1-2p^C(k)) e^{\phi(v;p^C(k),k)+1-v/k} \cdot \left(\frac{v}{k^2} + \phi_k(v;p^C(k),k) \right) \right) \right],$$

$$B_C = \mathbb{E}_\lambda \left[\frac{1}{D_C^2} \cdot \left(2(1-e^{\phi(v;p^C(k),k)+1-v/k}) + (1-2p^C(k)) \cdot e^{\phi(v;p^C(k),k)+1-v/k} \cdot \phi_p(v;p^C(k),k) \right) \right],$$

and

$$D_C = 2p^C(k) + (1-2p^C(k)) \cdot e^{\phi(v;p^C(k),k)+1-v/k}.$$

Letting $k \uparrow k^t$, we conclude that

$$\frac{\partial}{\partial k} p^C(k) \to -\mathbb{E}_\lambda \left[\frac{v}{(k^t)^2} \cdot e^{v/k^t-1} \right] / \mathbb{E}_\lambda \left[\frac{2(1-e^{1-v/k^t})+1}{e^{2\cdot(1-v/k^t)}} \right]$$

as required. □

Concluding the proof of Theorem 2. We use *L'Hopital's rule* to show that as $k \uparrow k^t$, the ratio $p^B(k)/p^M(k)$ is bounded above 1/2 *strictly*. Formally:

Lemma 9. *There exists* $\Theta > 0$ *such that*

$$\lim_{k \uparrow k^t} \frac{p^C(k)}{p^M(k)} > \frac{1}{2} + \Theta.$$

Proof. Note that $\lim_{k\uparrow k^t} \frac{\partial}{\partial k} p^M(k)$ exists and is different from 0. Therefore, by

L'Hopital's rule

$$\lim_{k\uparrow k^t}\frac{p^C(k)}{p^M(k)}=\lim_{k\uparrow k^t}\frac{\frac{\partial}{\partial k}p^C(k)}{\frac{\partial}{\partial k}p^M(k)}=\frac{\mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^t}}{e^{2\cdot(1-v/k^t)}}\right]}{\mathbb{E}_\lambda\left[\frac{2(1-e^{1-v/k^t})+1}{e^{2\cdot(1-v/k^t)}}\right]}=\frac{1}{2-\frac{\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right]}{2\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right]-1}}.$$

Since

$$2\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right]-1>\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right]-1\geq 0$$

because of Jensen inequality, the conclusion of the lemma follows. □

As the last step, note that as $k\uparrow k^t$, $p^M(k),p^C(k)\to 0$. It follows that $x_k^M(v),x_k^C(v)\to k^t$ for all $v\in V$. Now, fix $\varepsilon>0$ so small that $1+2(\Theta-\varepsilon)>(k^t+\varepsilon)/(k^t-\varepsilon)$, and let $\hat{k}\in(k^e,k^t)$ be such that $p^C(k)/p^M(k)>1/2+\Theta-\varepsilon$ and $x_k^m(v)\in(k^t-\varepsilon,k^t+\varepsilon)$ for all $k>\hat{k}$, $v\in V$, and $m\in\{C,M\}$. For all $k>0$, we have that $2p^C(k)(k^t-\varepsilon)>p^M(k)(k^t+\varepsilon)$ if and only if

$$2\cdot\frac{p^C(k)}{p^M(k)}>\frac{k^t+\varepsilon}{k^t-\varepsilon}. \tag{19}$$

Notice that (19) holds by assumption as long as $k\in(\hat{k},k^t)$. Since by construction we have $\Pi^C(k)\geq p^C(k)(k^t-\varepsilon)$ and $p^M(k)(k+\varepsilon)\geq\Pi^M(k)$, we conclude that $2\Pi^C(k)>\Pi^M(k)$ for all $k\in(\hat{k},k^t)$ as required. *Q.E.D.*

Proof of Lemma 4

Proof. The proof of Lemma 4 relies on the following lemma.

Lemma 10. *There exists threshold $v^*>0$ such that $x^M(v)>x^C(v)$ iff $v>v^*$.*

Proof of Lemma 10. Let π^M be the overall equilibrium engagement level of the consumer when the firms collude, and $2\pi^C$ be the overall engagement level of the consumer in the competitive trading equilibrium. For every $v\in V$, let $W^M(v)=W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)$ and $\phi^C(v)=\phi(v;\pi^C)$ solving (5). Since $x^M(v)=k(1+W^M(v))$ and $x^C(v)=k(1+\phi^C(v))$, it follows that $x^M(v)>x^C(v)$ if and only if $W^M(v)>\phi^C(v)$. By the definition of the Lambert function, $W^M(v)e^{W^M(v)}=\frac{\pi^M}{1-\pi^M}e^{v/k-1}$.

Moreover, $x \mapsto xe^x$ is a strictly increasing function of $x > 0$. Therefore, $W^M(v) > \phi^C(v)$ if and only if

$$\frac{\pi^M}{1-\pi^M} e^{v/k-1} > \phi^C(v) e^{\phi^C(v)}. \tag{20}$$

From equation (5), we know that $\phi^C(v)e^{\phi^C(v)} = \frac{1-\phi^C(v)}{2} \cdot \frac{2\pi^C}{1-2\pi^C} e^{v/k-1}$. Therefore, (20) is equivalent to

$$\frac{1-\phi^C(v)}{2} < \frac{\pi^M}{1-\pi^M} \cdot \frac{1-2\pi^C}{2\pi^C}. \tag{21}$$

Because $\phi^C(v)$ is strictly increasing in v, the conclusion of the lemma follows. $\square$

Known results in rational inattention show that, in equilibrium,[44]

$$\mathbb{E}[U^M] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^M} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right]$$

$$\mathbb{E}[U^C] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right].$$

Consider the random variables Y^C and Y^M defined by $Y^C(v) := v - x^C(v)$ and $Y^M(v) = v - x^M(v)$. Let G^C and G^M be the CDF of Y^C and Y^M respectively, and define $\omega := \mathbb{E}_\lambda[x^M(v)] - \mathbb{E}_\lambda[x^C(v)]$. By assumption, $\omega \geq 0$. Finally, denote with u_1 and u_0 the maximal and minimal element in the support of Y^C respectively. From Lemma 10, we know that $u_0 \leq Y^M \leq u_1$ with probability 1. Furthermore, one can verify that $\omega \geq 0$ together with Lemma 10 imply

$$\int_u^{\bar{u}} G^C(y)dy \leq \int_u^{\bar{u}} G^M(y)dy \ \text{ for all } u \in [u_0, u_1].$$

This means that any expected utility maximizer with an increasing and convex Bernoulli utility function $w : [u_0, u_1] \to \mathbb{R}$ would prefer the lottery Y^C over Y^M (see Theorem 4 in Meyer (1977)). Observe that for every $\pi \in [0, 1/2]$, we have

$$k \cdot \mathbb{E}_{\mu^M} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right] = k \cdot \mathbb{E} \left[\ln \left(2\pi \cdot e^{\frac{Y^M}{k}} + 1 - 2\pi \right) \right],$$

[44] See, e.g., Lemma 2 in Matějka and McKay (2015). See also Denti, Marinacci, and Montrucchio (2020).

$$k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = k \cdot \mathbb{E}\left[\ln\left(2\pi \cdot e^{\frac{Y^C}{k}} + 1 - 2\pi\right)\right].$$

Furthermore, the function $y \in (0, +\infty) \mapsto \ln\left(2\pi \cdot e^{y/k} + 1 - 2\pi\right)$ is strictly increasing and strictly convex in $y > 0$ whenever $\pi \in (0, 1/2)$. Therefore,

$$\begin{aligned}
\mathbb{E}[U^M] &= \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^M}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] \\
&= k \cdot \mathbb{E}_{\mu^M}\left[\ln\left(\pi^M \cdot e^{\frac{v-x}{k}} + 1 - \pi^M\right)\right] = k \cdot \mathbb{E}\left[\ln\left(\pi^M \cdot e^{\frac{Y^M}{k}} + 1 - \pi^M\right)\right] \\
&\leq k \cdot \mathbb{E}\left[\ln\left(\pi^M \cdot e^{\frac{Y^C}{k}} + 1 - \pi^M\right)\right] = k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(\pi^M \cdot e^{\frac{v-x}{k}} + 1 - \pi^M\right)\right] \\
&< \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C}\left[\ln\left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi\right)\right] = \mathbb{E}[U^C].
\end{aligned}$$

where the first inequality is implied by $\pi^M \in (0, 1)$, while the last strict inequality is implied by the fact that the overall engagement level π^M is not a best response to μ^C. We conclude that $\mathbb{E}[U^C] > \mathbb{E}[U^M]$ as required. □

Proof of Proposition 4

Proof. First, we show that the threshold $\bar{k} \geq k^e$ exists. To this goal, we use inequality (21) introduced earlier. Specifically, the proof of Theorem 2 shows that while both π^C and π^M converge to 0 as $k \uparrow k^t$, we have $\lim_{k \uparrow k^t} \frac{\pi^C(k)}{\pi^M(k)} = \frac{1}{2 - \frac{\mathbb{E}_\lambda\left[e^{2(v/k^t - 1)}\right]}{2\mathbb{E}_\lambda\left[e^{2(v/k^t - 1)}\right] - 1}}$. Such limit is strictly less than 1 because $\mathbb{E}_\lambda\left[e^{2(v/k^t-1)}\right] > \left(\mathbb{E}_\lambda\left[e^{v/k^t - 1}\right]\right)^2$ due to Jensen inequality. (Recall that $\mathbb{E}_\lambda\left[e^{v/k^t-1}\right] = 1$ by definition.) Therefore, while the LHS of inequality (21) converges to $\frac{1}{2}$ because $\phi^C(v) \downarrow 0$ as $k \uparrow k^t$, the RHS of (21) is converging to a limit strictly greater than 1/2. As a result, inequality (21) is satisfied eventually (i.e., as k approaches k^t from below) for all $v \in V$. The existence of the threshold $\bar{k}$ follows immediately from this observation.

We now show that $\mathbb{E}_\lambda[x^M(v)] \geq \mathbb{E}_\lambda[x^C(v)]$ for all $k \in (0, k^e]$. To this goal, we begin by showing that the aggregate engagement level under collusion is weakly larger than the engagement level each competitive firm experience in the efficient equilibrium. Formally:

Lemma 11. *Suppose $k \leq k^e$. Then, $\pi^M \geq 1/2$.*

Proof. Ravid (2020) shows that the function $F^M(\cdot)$ defined in the proof of Theorem 2 is strictly convex when $k < k^t$, and satisfies $F^M(0) > 1 > F^M(1^-)$. This implies that, if $\pi^M \in (0,1)$ is the unique solution to $F^M(\pi^M) = 1$, we have $F^M(\pi) \geq 1$ if and only if $\pi^M \geq \pi$. Therefore, it is sufficient to show that:[45]

$$\frac{1}{2}F^M(1/2) = \mathbb{E}_\lambda\left[\frac{W(1/2,v)}{W(1/2,v)+1}\right] \geq 1/2 \tag{22}$$

whenever

$$\mathbb{E}_\lambda\left[e^{1-v/k}\right] \leq 1/e, \tag{23}$$

where $W(\pi, v) = W\left(\frac{\pi}{1-\pi}e^{v/k-1}\right)$. Observe that we can interpret (22) as an objective function and (23) as a constraint set on the distribution over quality levels $\lambda \in \Delta(\mathbb{R}_+)$. To simplify the problem, change variable from v to $y = e^{1-v/k}$. That is, let $\mathcal{F} := \{F \in \Delta(\mathbb{R}_+) : F \text{ is finitely supported}\}$. We need to show that $V^* \geq 1/2$, where

$$V^* := \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[\frac{W(1/y)}{W(1/y)+1}\right] \quad \text{subject to} \quad \mathbb{E}_F[y] \leq 1/e.$$

The function $y \mapsto H(y) := \frac{W(1/y)}{W(1/y)+1}$ is strictly decreasing and strictly convex in $y \geq 0$. Therefore, V^* is achieved by the degenerate distribution $F = \delta_{1/e}$. Plugging in $y = 1/e$ in $H(\cdot)$, we get $H(1/e) = 1/2$, that is (22) holds. This shows that $\pi^M \geq 1/2$, as required. □

Given equations (4), the monopolist's equilibrium pricing strategy, and the fact that $\phi^C(v) = 1$ for all $v \in V$ when competitive trade is efficient, to complete the proof of Proposition 4, it is sufficient to show that $\mathbb{E}_\lambda[W(\pi^M, v)] \geq 1$, for all $k \leq k^e$. To this goal, we use the optimization approach introduced earlier once again. Formally, define $\mathcal{F} := \{F \in \Delta(\mathbb{R}_+) : F \text{ is finitely supported}\}$. Since $\pi^M \geq 1/2$ (Lemma 11), it is enough to argue that $V^{**} \geq 1$, where

$$V^{**} := \inf_{F\in\mathcal{F}} \mathbb{E}_F[W(1/y)] \quad \text{subject to} \quad \mathbb{E}_F[y] \leq 1/e.$$

[45] Observe that $k \leq k^e$ if and only if (23) holds.

The function $y \mapsto G(y) := W(1/y)$ is strictly decreasing and strictly convex. Therefore, V^{**} is achieved at $F = \delta_{1/e}$. Since, $G(1/e) = 1$, we are done. $\square$

Proof of Lemma 5 (sketch)

Proof. Suppose (μ, σ, β) is a competitive trading equilibrium with $N \geq 2$ firms. That such equilibrium assessment must be symmetric follows from the same arguments used for the duopoly model. From Milgrom and Roberts (1990), we know that for every $v \in V$, each firm $i \in I$ uses a pure strategy $\sigma_i(\cdot|v) = \delta_{x(v,N)}$ that solves

$$\max_{x_i \geq 0} \frac{\pi e^{\frac{v-x_i}{k}}}{\pi\left(e^{\frac{v-x_i}{k}} + \sum_{j\neq i} e^{\frac{v-x(v,N)}{k}}\right) + 1 - N\pi} \cdot x_i$$

The result follows from a re-arranging of the FOCs. See the proof of Lemma 2. $\square$

Proof of Proposition 5 (sketch)

Proof. **Part** (i)**:** Follows from a simple extension of the proof of Theorem 1.

Part (ii)**:** Follows from a straightforward extension of the proof of Proposition 2. Here, we show how the instrumental first step is extended.

Lemma 12 (Pricing effect for an arbitrary number of firms)**.** *Fix* $N_1 > N_2 \geq 2$ *arbitrarily, and for every* $j \in \{1, 2\}$*, let* $\phi_j > 0$ *be the unique solution to*

$$\phi_j \left(N_j - 1 + e^{\phi_j} \frac{1 - N_j \pi^j}{\pi^j e^{v/k-1}}\right) = 1, \tag{24}$$

where $\pi^j \in (0, 1/N_j]$*. If* $N_1\pi^1 = N_2\pi^2$*, then* $\phi_2 > \phi_1$*.*

Proof. Equation (24) can be equivalently re-written as

$$\pi^j e^{v/k-1} = \phi_j(N_j - 1)\pi^j e^{v/k-1} + (1 - N_j\pi^j)\phi_j e^{\phi_j}.$$

Suppose by way of contradiction that $\phi_2 \leq \phi_1$. Then, $\phi_2 e^{\phi_2} \leq \phi_1 e^{\phi_1}$ which, given our assumption $N_1\pi^1 = N_2\pi^2$, implies $\pi_2(1 - \phi_2(N_2 - 1)) \leq \pi^1(1 - \phi_1(N_1 - 1))$. This is equivalent to $\frac{N_1}{N_2}(1 - \phi_2(N_2 - 1)) \leq 1 - \phi_1(N_1 - 1)$ which, in turn, implies $\phi_2 \geq \frac{N_2(N_1-1)}{N_1(N_2-1)}\phi_1 + \frac{N_1-N_2}{N_1(N_2-1)} > \phi_1$, a contradiction since $\frac{N_1-N_2}{N_1(N_2-1)} > 0$ and $\frac{N_2(N_1-1)}{N_1(N_2-1)} > 1$. □

Part (iii)**:** With $N \geq 2$ firms, the maximal price that can be sustained in a symmetric equilibrium is $x(v, N) = k \cdot \frac{N}{N-1}$.[46] Therefore, an efficient equilibrium exists if and only if $k \leq k^e(N)$, where $k^e(N)$ is the unique solution to $\mathbb{E}_\lambda \left[e^{\frac{N}{N-1} - v/k}\right] = 1$.

Part (iv)**:** Fix $N \geq 2$ arbitrarily and let $k \in (k^e(N), k^t)$. For each $p \in [0, 1/N)$, define $F_k^{C,N}(p)$ as $F_k^{C,N}(p) := \mathbb{E}_\lambda \left[\frac{1}{Np+(1-Np)\cdot e^{\phi(p,v,k,N)+1-v/k}}\right]$ where, for $p > 0$, we let $\phi(p, v, k, N)$ be defined as the unique solution to equation (6), and we set $\phi(0, v, k, N) := 0$ for all $v \in V$ and $k \in (k^e(N), k^t)$. For fixed k, the consumer trade engagement level with each firm in the competitive trading equilibrium is given by the unique solution to $F_k^{C,N}(p) = 1$. Denote with $p^{C,N}(k)$ such solution. Because as $k \uparrow k^t$, offers are converging to k^t irrespective of N, the crucial step to prove (iv) is to show that, for $N_1 > N_2 \geq 2$,

$$\lim_{k \uparrow k^t} \frac{p^{C,N_1}(k)}{p^{C,N_2}(k)} > \frac{N_2}{N_1} + \Theta, \tag{25}$$

for some $\Theta > 0$. Using the same arguments as in the proof of Theorem 2, one obtains

$$\lim_{k \uparrow k^t} \frac{p^{C,N_1}(k)}{p^{C,N_2}(k)} = \frac{\mathbb{E}_\lambda \left[\frac{N_2\left(1-e^{1-v/k^t}\right)+1}{e^{2(1-v/k^t)}}\right]}{\mathbb{E}_\lambda \left[\frac{N_1\left(1-e^{1-v/k^t}\right)+1}{e^{2(1-v/k^t)}}\right]},$$

which implies that (25) is satisfied. □

[46] This price corresponds to $\pi = 1/N$, which generalizes the case $\pi = 1/2$ of the duopoly setting.