Supplemental Appendix

Sharing Model Uncertainty

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A. Ambiguity aversion and revealed beliefs

Denote by $b\left(\Omega_{\mathbf{P}}\right)$ a bet that pays c^* on $\Omega_{\mathbf{P}}$ and c_* off it, and by $b\left(\Omega \setminus \Omega_{\mathbf{P}}\right)$ the bet on the complementary event $\Omega \setminus \Omega_{\mathbf{P}}$. Note that $\mathbf{P}\left(\Omega_{\mathbf{P}}\right) = 1$. Normalize u_i and ϕ_i so that $u_i\left(c_*\right) = 0$ and $\phi_i\left(0\right) = 0$, and write $h = u(c^*)$. Consumer i evaluates these bets as: $U_i(b\left(\Omega_{\mathbf{P}}\right)) = \mu\left(\mathbf{P}\right)\phi_i\left(\mathbf{P}\left(\Omega_{\mathbf{P}}\right)h\right) = \mu\left(\mathbf{P}\right)\phi_i\left(h\right)$ and $U_i(b\left(\Omega \setminus \Omega_{\mathbf{P}}\right)) = \left(1 - \mu\left(\mathbf{P}\right)\right)\phi_i\left(h\right)$. Consider next a lottery ℓ^π which pays c^* with a probability π and c_* with probability $1 - \pi$. Then, $U_i(\ell^\pi) = \phi_i\left(\pi h\right)$. If ϕ_i is strictly concave, then $U_i(b\left(\Omega_{\mathbf{P}}\right)) < U_i(\ell^{\mu(\mathbf{P})})$ and $U_i(b\left(\Omega \setminus \Omega_{\mathbf{P}}\right)) < U_i(\ell^{1-\mu(\mathbf{P})})$. Define $\underline{\pi}, \bar{\pi} \in [0,1]$ so that $U_i(\ell^{\underline{\pi}}) = U_i(b\left(\Omega_{\mathbf{P}}\right))$ and $U_i(\ell^{1-\bar{\pi}}) = U_i(b\left(\Omega \setminus \Omega_{\mathbf{P}}\right))$. Since ϕ is strictly increasing, $\underline{\pi} < \mu\left(\mathbf{P}\right) < \bar{\pi}$. Moreover, $\underline{\pi}$ satisfies

$$\begin{aligned}
\phi_i\left(\underline{\pi}h + (1-\underline{\pi})0\right) &= \mu\left(\mathbf{P}\right)\phi_i\left(h\right) + (1-\mu\left(\mathbf{P}\right))\phi_i\left(0\right) \\
\Leftrightarrow \underline{\pi}h &= \phi_i^{-1}\left(\mu\left(\mathbf{P}\right)\phi_i\left(h\right)\right)
\end{aligned}$$

Applying a quadratic approximation, we get, letting λ_{ϕ_i} be the Arrow-Pratt measure of absolute risk aversion for the function ϕ_i (see Supplemental Appendix B for further detail).

$$\begin{aligned}
\underline{\pi}h &= \mu\left(\mathbf{P}\right)h - \frac{\lambda_{\phi_i}\left(0\right)}{2}\left[\mu\left(\mathbf{P}\right)h^2 - \left(\mu\left(\mathbf{P}\right)h\right)^2\right] + o\left(h^2\right) \\
\Leftrightarrow \underline{\pi} &= \mu\left(\mathbf{P}\right) - \frac{\lambda_{\phi_i}\left(0\right)}{2}\mu\left(\mathbf{P}\right)\left(1-\mu\left(\mathbf{P}\right)\right)h + o\left(h\right)
\end{aligned}$$

Similarly, $\bar{\pi} = \mu\left(\mathbf{P}\right) + \frac{\lambda_{\phi_i}(0)}{2}\mu\left(\mathbf{P}\right)\left(1-\mu\left(\mathbf{P}\right)\right)h + o(h)$. Hence, the "probability matching" interval for $\Omega_{\mathbf{P}}$ is given by $[\underline{\pi}, \bar{\pi}]$. Its length is increasing in λ_{ϕ_i}.

B. Relative ambiguity aversion

We relate the measure of relative ambiguity aversion introduced in Section II.B to ambiguity premiums (see also (Cerreia-Vioglio, Maccheroni and Marinacci 2022)). Let h be a random variable defined on Ω and w be the initial consumption level. Denote by $\mathbf{P}^\mu$ the reduced measure $\int_{\mathcal{P}}\mathbf{Q}\mu(d\mathbf{Q})$, and by λ_u the Arrow-Pratt measure of absolute risk aversion for a Bernoulli utility u. The variance $(\sigma^\mu)^2\left(E^{\cdot}\left(h\right)\right)$ of the function $E^{\cdot}(h) : \mathbf{P} \mapsto E^{\mathbf{P}}(h)$ under μ reflects the uncertainty on the expected values and encapsulates ambiguity.

The certainty equivalent for a proportional ambiguous prospect xh can be approximated as[30]

$$\begin{aligned} C(x+xh) &= x + E^{\mathbf{P}^{\mu}}(xh) - \frac{x^2}{2}\lambda_u(x)\left(\sigma^{\mathbf{P}^{\mu}}\right)^2(h) \\ &\quad - \frac{x^2}{2}\left(\lambda_v(x) - \lambda_u(x)\right)(\sigma^{\mu})^2\left(E^{\mathbf{P}^{\mu}}(h)\right) + o\left(\|h\|^2\right) \end{aligned}$$

Since $\phi = v \circ u^{-1}$, $\lambda_{\phi}(u(x)) = \frac{1}{u'(x)}\left(\lambda_v(x) - \lambda_u(x)\right)$, that is, $\lambda_{\phi}(u(x))u'(x) = \lambda_v(x) - \lambda_u(x)$. The ambiguity premium for xh is obtained by subtracting the risk premium from the overall uncertainty premium and, as a proportion of wealth, equal to

$$\left(\left(\lambda_v(x) - \lambda_u(x)\right)x\right) \times \frac{1}{2}(\sigma^{\mu})^2\left(E^{\mathbf{P}^{\mu}}(h)\right) = \lambda_{\phi}(u(x))u'(x)\,x \times \frac{1}{2}(\sigma^{\mu})^2\left(E^{\mathbf{P}^{\mu}}(h)\right).$$

In our HARA specification, it is convenient to express the ambiguity premium in terms of the effective consumption $x - \zeta$. By differentiating $v = \phi \circ u$, we obtain

$$-\frac{v''(x)}{v'(x)} = -\frac{\phi''(u(x))}{\phi'(u(x))}u'(x) - \frac{u''(x)}{u'(x)}. \tag{B.1}$$

By multiplying both sides by $x - \zeta$, we obtain, under Condition 2:

$$-\frac{\phi''(u(x))}{\phi'(u(x))}u'(x)(x-\zeta) = \gamma - \alpha. \tag{B.2}$$

C. *Proof of Proposition 4.*

For the purpose of this Appendix, denote the value function of (9) by $u(x,\lambda)$. Then, u is the representative consumer's (inner) Bernoulli utility function, where dependence on the vector λ of utility weights is made explicit. Similarly, denote the value function of (3) by $V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)$ and the value function of (2) by $V\left((\bar{X}^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}},\lambda\right)$. Then,

$$V\left((\bar{Y}^{\mathbf{P}})_{\mathbf{P}},\lambda\right) = \sum_{\mathbf{P}} \mu(\mathbf{P}) V^{P}\left(\bar{Y}^{\mathbf{P}},\lambda\right) \tag{C.1}$$

for every $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$.

Denote the solution to (9) by $(f_i(x,\lambda))_i$. Then $(f_i)_i$ is the risk-sharing rule, with the dependence on the vector λ of utility weights made explicit. By the

[30] This is akin to the quadratic approximation of certainty equivalent obtained by (Maccheroni, Marinacci and Ruffino 2013)

envelope theorem,

$$\frac{\partial u}{\partial x}(x,\lambda) = \lambda_i u_i'(f_i(x,\lambda)) \tag{C.2}$$

for every i. Denote the risk tolerances of u_i and u by t_i and t. (Wilson 1968) showed that $t(x,\lambda) = \sum_i t_i(f_i(x,\lambda))$ for every (x,λ). Hence,

$$\nabla_\lambda t(x,\lambda) = \sum_i t_i'(f_i(x,\lambda))\nabla_\lambda f_i(x,\lambda). \tag{C.3}$$

LEMMA 5: $\nabla_\lambda t(x,\lambda) = 0$ *if and only if* $t_1'(f_1(x,\lambda)) = \cdots = t_I'(f_I(x,\lambda))$.

Proof of Lemma 5 Although this lemma is true for an arbitrary I, we give a proof only for $I = 2$ to save space. By (C.2), $\lambda_1 u_1'(f_1(x,\lambda)) = \lambda_2 u_2'(f_2(x,\lambda))$. By differentiating both sides w.r.t. λ_1:

$$u_1'(f_1(x,\lambda)) + \lambda_1 u_1''(f_1(x,\lambda))\frac{\partial f_1}{\partial \lambda_1}(x,\lambda) = \lambda_2 u_2''(f_2(x,\lambda))\frac{\partial f_2}{\partial \lambda_1}(x,\lambda).$$

Since $\sum_i(\partial f_i/\partial\lambda_1)(x,\lambda) = 0$, $u_1'(f_1(x,\lambda)) = -\frac{\partial f_1}{\partial \lambda_1}(x,\lambda)\sum_i \lambda_i u_i''(f_i(x,\lambda))$. Hence, $(\partial f_1/\partial\lambda_1)(x,\lambda) > 0$. Thus,

$$\frac{\partial t}{\partial \lambda_1}(x,\lambda) = \left(t_1'(f_1(x,\lambda)) - t_2'(f_2(x,\lambda))\right)\frac{\partial f_1}{\partial \lambda_1}(x,\lambda) = 0$$

if and only if $t_1'(f_1(x,\lambda)) = t_2'(f_2(x,\lambda))$. □

If $(X_i^{\mathbf{P}})_i$ is a solution to (3), then, by the envelope theorem,

$$\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}(\omega)} = \lambda_i\phi_i'\left(E^{\mathbf{P}}u_i(X_i^{\mathbf{P}})\right)u_i'(X_i^{\mathbf{P}}(\omega))\mathbf{P}(\omega) \text{ for all } i \text{ and } \omega. \tag{C.4}$$

LEMMA 6: *For each* $\mathbf{P} \in \mathcal{P}$, *let* $(X_i^{\mathbf{P}})_i$ *be a solution to* (3). *Write* $\lambda_i^{\mathbf{P}} = \lambda_i\phi_i'\left(E^{\mathbf{P}}u_i(X_i^{\mathbf{P}})\right)$ *and* $\lambda^{\mathbf{P}} = (\lambda_i^{\mathbf{P}})_i$. *Suppose that there is a pair of a differentiable function* $u : \mathbb{X} \to \mathbb{R}$ *and a differentiable function* $\phi : u(\mathbb{X}) \to \mathbb{R}$ *such that* $V^{\mathbf{P}}(\bar{Y},\lambda) = \phi(E^{\mathbf{P}}u(\bar{Y}))$ *for all* $\mathbf{P} \in \mathcal{P}$ *and* $\bar{Y} : \Omega \to \mathbb{X}$. *Then, for all* ω_1 *and* $\omega_2 \in \Omega_{\mathbf{P}}$,

$$\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})}$$

depends only on the values of $\bar{X}^{\mathbf{P}}(\omega_1)$ *and* $\bar{X}^{\mathbf{P}}(\omega_2)$, *that is, if* $\bar{X}^{\mathbf{P}}(\omega_1) = \bar{X}^{\mathbf{Q}}(\omega_3)$ *and* $\bar{X}^{\mathbf{P}}(\omega_2) = \bar{X}^{\mathbf{Q}}(\omega_4)$, *then* $\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})} = \int_{\bar{X}^{\mathbf{Q}}(\omega_3)}^{\bar{X}^{\mathbf{Q}}(\omega_4)} \frac{dx}{t(x,\lambda^{\mathbf{Q}})}$ *for all* $\mathbf{P},\mathbf{Q} \in \mathcal{P}$,

$\omega_1, \omega_2 \in \Omega_{\mathbf{P}}$, *and* $\omega_3, \omega_4 \in \Omega_{\mathbf{Q}}$.

Proof of Lemma 6 First, we prove that

$$\frac{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \exp\left(-\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})}\right).$$

Indeed, by (C.4),

$$\frac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega)}}{\mathbf{P}(\omega)} = \lambda_i^{\mathbf{P}} u_i'(X_i^{\mathbf{P}}(\omega))$$

for every ω. Thus, the right-hand side is independent of i. Hence, the first-order condition for a solution to (9) is met, and $X_i^{\mathbf{P}}(\omega) = f_i(\bar{X}^{\mathbf{P}}(\omega), \lambda^{\mathbf{P}})$ for all i and $\omega \in \Omega$. Thus, by (C.2), $\lambda_i^{\mathbf{P}} u_i'(X_i^{\mathbf{P}}(\omega)) = \frac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega), \lambda^{\mathbf{P}})$. Hence,

$$\frac{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \frac{\dfrac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega_2), \lambda^{\mathbf{P}})}{\dfrac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega_1), \lambda^{\mathbf{P}})} = \exp\left(-\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})}\right). \tag{C.5}$$

On the other hand, by assumption, the chain rule implies that

$$\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega)} = \phi'(E^{\mathbf{P}} u(\bar{X})) u'(\bar{X}^{\mathbf{P}}(\omega)) \mathbf{P}(\omega)$$

for every ω. Thus,

$$\frac{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \frac{u'(\bar{X}^{\mathbf{P}}(\omega_2))}{u'(\bar{X}^{\mathbf{P}}(\omega_1))}$$

for all ω_1 and ω_2. Since the right-hand side depends only on the values of $\bar{X}^{\mathbf{P}}(\omega_1)$ and $\bar{X}^{\mathbf{P}}(\omega_2)$, so is the left-hand side. The lemma follows now from (C.5). $\square$

LEMMA 7: *Suppose that there is a pair of a differentiable function $u : \mathbb{X} \to \mathbb{R}$ and a differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ such that $V((\bar{Y}^{\mathbf{P}})_{\mathbf{P}}, \lambda) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}}))$ for every $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$, where $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$ for every $\mathbf{P}$. Then, $V^{\mathbf{P}}(\bar{Y}, \lambda) = \phi(E^{\mathbf{P}} u(\bar{Y}))$ for all $\mathbf{P}$ and $\bar{Y} : \Omega \to \mathbb{X}$.*

Proof of Lemma 7 Let $\mathbf{Q} \in \mathcal{P}$, and $(\bar{X}^{\mathbf{P}})_{\mathbf{P}}$ and $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$ be two endowments such that $\bar{X}^{\mathbf{P}} = \bar{Y}^{\mathbf{P}}$ for every $\mathbf{P} \in \mathcal{P} \setminus \{\mathbf{Q}\}$. By assumption and (C.1),

$$\phi(E^{\mathbf{Q}} u(\bar{X}^{\mathbf{Q}})) - \phi(E^{\mathbf{Q}} u(\bar{Y}^{\mathbf{Q}})) = V^{\mathbf{Q}}\left(\bar{X}^{\mathbf{Q}}, \lambda\right) - V^{\mathbf{Q}}\left(\bar{Y}^{\mathbf{Q}}, \lambda\right).$$

Therefore, for every $\mathbf{P} \in \mathcal{P}$, there is an $a^{\mathbf{P}} \in \mathbb{R}$ such that $V^{\mathbf{P}}(\bar{Y}^{\mathbf{P}}) = \phi(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})) + a^{\mathbf{P}}$ for every $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$. Hence, $\sum_{\mathbf{P}} \mu(\mathbf{P}) a^{\mathbf{P}} = 0$. Let $\bar{Y} : \Omega \to \mathbb{X}$ be (deterministic) endowments for which there is an $x \in \mathbb{X}$ such that $\bar{Y}(\omega) = x$ for every ω. Then, for every $\mathbf{P}$, the solution $(Y_i^{\mathbf{P}})_i$ to (2) is given by letting $Y_i^{\mathbf{P}}$ be the deterministic consumption x_i such that $\lambda_i \phi_i'(u_i(x_i))u_i'(x_i)$ is independent of i, and $V^{\mathbf{P}}(\bar{Y}) = \sum_i \lambda_i \phi_i(u_i(x_i))$. Thus, whenever $\bar{Y}$ is deterministic, $V^{\mathbf{P}}(\bar{Y})$ is independent of $\mathbf{P}$. Hence, $a^{\mathbf{P}}$ is independent of $\mathbf{P}$. Thus, $a^{\mathbf{P}} = 0$ for every $\mathbf{P}$. Hence, $V^{\mathbf{P}}(\bar{Y}) = \phi(E^{\mathbf{P}} u(\bar{Y}))$ for all $\mathbf{P}$ and $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$. $\square$

PROPOSITION 7: *Assume $|\Omega| \geq 4$. For each i, let u_i be a (inner) Bernoulli utility function with the following property: for each i, there is an $x_i^* \in \mathbb{X}_i$ such that it is not true that $t_1'(x_1^*) = t_2'(x_2^*) = \cdots = t_I'(x_I^*)$. Then, there are: for each i, a Bernoulli utility function ϕ_i over expected utility levels; a (common) second-order belief μ on Ω; endowments $\bar{X} : \Omega \to \mathbb{X}$ whose range is model-independent; and a vector $\lambda^\star$ of utility weights, such that such that if V is defined by (2), then there is no pair of a (inner) Bernoulli utility function u and a Bernoulli utility function ϕ over expected utility levels such that $V\left((\bar{Y}^{\mathbf{P}})_{\mathbf{P}}, \lambda\right) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$.*

Proof of Proposition 7 Suppose that for each i, there is an $x_i^* \in \mathbb{X}_i$ such that it is not true that $t_1'(x_1^*) = t_2'(x_2^*) = \cdots = t_I'(x_I^*)$. For each i, let $\lambda_i^* = (u_i'(x_i^*))^{-1}$, and $\lambda^* = (\lambda_i^*)_i$. Write $x^* = \sum_i x_i^*$. Then, $x_i^* = f_i(x^*, \lambda^*)$ for every i. By Lemma 5, $\nabla_\lambda t(x^*, \lambda^*)$ is a nonzero vector. Thus, there is a $\kappa \in \mathbb{R}^I$ such that $\nabla_\lambda t(x^*, \lambda^*)\kappa > 0$. Note here that

$$D_\lambda u_i(f_i(x^*, \lambda^*)) = u_i'(f_i(x^*, \lambda^*))\nabla_\lambda f_i(x^*, \lambda^*) \in \mathbb{R}^I.$$

Let $\delta > 0$ be so large that $D_\lambda u_i(f_i(x^*, \lambda^*))\kappa + \delta > 0$ for every i, then there is a neighborhood $\mathbb{Y}$ of x^* and a neighborhood Λ of λ^* such that $D_\lambda u_i(f_i(x, \lambda))\kappa + \delta >$

0 and $\nabla_\lambda t(x,\lambda)\kappa > 0$ for all i and $(x,\lambda) \in \mathbb{Y} \times \Lambda$. Then,

$$\frac{d}{d\varepsilon} t(x, \lambda^* + \varepsilon\kappa) = \nabla_\lambda t(x, \lambda^* + \varepsilon\kappa)\kappa > 0$$

for every $x \in \mathbb{Y}$ and every ε sufficiently close to 0. Hence, for every $x \in \mathbb{Y}$, $t(x, \lambda^* + \varepsilon\kappa)$ is a strictly increasing function of ε around 0.

Since $\Omega \geq 4$, there is a partition $(\Xi^1, \Xi^2, \Xi^3, \Xi^4)$ of Ω where each Ξ^n is nonempty. Let $x^1, x^2 \in \mathbb{X}$ be such that $x^1 < x^2$. Define $\bar{X} : \Omega \to \mathbb{X}$ by

$$\bar{X}(\omega) = \begin{cases} x^1 & \text{if } \omega \in \Xi^1 \cup \Xi^3 \\ x^2 & \text{if } \omega \in \Xi^2 \cup \Xi^4 \end{cases}$$

Define $\rho > 0$ so that $(u_i(f_i(x^2,\lambda)) - u_i(f_i(x^1,\lambda)))\rho > \delta$ for all i. Let $\mathbf{P}^0 \in \Delta(\Omega)$ be s. th. $\mathbf{P}^0(\omega) > 0$ for all $\omega \in \Xi^1 \cup \Xi^2$ and $\mathbf{P}^0(\omega) = 0$ for all $\omega \in \Xi^3 \cup \Xi^4$. For each $\varepsilon > 0$ sufficiently close to 0, let $\mathbf{P}^\varepsilon \in \Delta(\Omega)$ be s. th.

$$\mathbf{P}^\varepsilon(\omega) = \begin{cases} \frac{1}{|\Xi^3|}(\mathbf{P}^0(\Xi^1) - \varepsilon\rho) & \text{if } \omega \in \Xi^3, \\ \frac{1}{|\Xi^4|}(\mathbf{P}^0(\Xi^2) + \varepsilon\rho) & \text{if } \omega \in \Xi^4, \\ 0 & \text{if } \omega \in \Xi^1 \cup \Xi^2 \end{cases}$$

Then, $\mathbf{P}^\varepsilon(\Xi^3) = \mathbf{P}^0(\Xi^1) - \varepsilon\rho$, $\mathbf{P}^\varepsilon(\Xi^4) = \mathbf{P}^0(\Xi^2) + \varepsilon\rho$ and $\mathbf{P}^\varepsilon(\Xi^1 \cup \Xi^2) = 0$. Fix a sufficiently small $\varepsilon^\star > 0$ and let $\mathcal{P} = \{\mathbf{P}^0, \mathbf{P}^{\varepsilon^\star}\}$. Then, $\mathcal{P}$ is point-identified with kernel k s.th.

$$k(\omega) = \begin{cases} \mathbf{P}^0 & \text{if } \omega \in \Xi^1 \cup \Xi^2 \\ \mathbf{P}^{\varepsilon^\star} & \text{if } \omega \in \Xi^3 \cup \Xi^4 \end{cases}$$

Moreover, $\Omega_{\mathbf{P}^0} = \Xi^1 \cup \Xi^2$ and $\Omega_{\mathbf{P}^{\varepsilon^\star}} = \Xi^3 \cup \Xi^4$. Thus, the range of $\bar{X}$ is model independent. Let μ be a second-order belief s.th. $\mu(\mathbf{P}^0) > 0$ and $\mu(\mathbf{P}^{\varepsilon^\star}) > 0$. Then, by definition of δ and ρ, $\forall \varepsilon > 0$:

$$\begin{aligned} \frac{d}{d\varepsilon} E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)) &= (u_i(f_i(x^2, \lambda^* + \varepsilon\kappa))) - u_i(f_i(x^1, \lambda^* + \varepsilon\kappa))\rho \\ &\quad + \sum_{\omega\in\Omega} \mathbf{P}^\varepsilon(\omega) D_\lambda u_i(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))\kappa \\ &> \delta + \sum_{\omega\in\Omega} \mathbf{P}^\varepsilon(\omega)(-\delta) = 0. \end{aligned}$$

Thus, by Proposition 10 of (Hara et al. 2022) for each i, there is a twice continuously differentiable ϕ_i with $\phi_i'' \leq 0 < \phi_i'$ such that $\left(\left(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)\right)_i\right)_{\varepsilon=0,\varepsilon^\star}$ is an efficient allocation of the economy $\left((u_i, \phi_i, \mu)_i, \bar{X}\right)$.

Since $\left(\left(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)\right)_i\right)_{\varepsilon=0,\varepsilon^\star}$ is an efficient allocation of the economy $\left((u_i, \phi_i, \mu)_i, \bar{X}\right)$, there is a $\nu \in \mathbb{R}^I_{++}$ such that it is a solution to (2) when λ is replaced by ν. The first-order condition is that for all ε and ω,

$$\nu_i \phi_i' \left(E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)) \right) u_i'(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))$$

is independent of i. Write $\lambda_i^{\mathbf{P}^\varepsilon} = \nu_i \phi_i' \left(E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)) \right)$. By definition,

$$(\lambda_i^* + \varepsilon\kappa_i) u_i'(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))$$

is independent of i. Thus, $\lambda_i^{\mathbf{P}^\varepsilon}/(\lambda_i^* + \varepsilon\kappa_i)$ is independent of i. Denote it by c^ε. Then $\lambda^{\mathbf{P}^\varepsilon} = c^\varepsilon(\lambda^* + \varepsilon\kappa)$. Hence, $u\left(\cdot, \lambda^{\mathbf{P}^\varepsilon}\right) = c^\varepsilon u\left(\cdot, \lambda^* + \varepsilon\kappa\right)$. Thus, $t\left(\cdot, \lambda^{\mathbf{P}^\varepsilon}\right) = t\left(\cdot, \lambda^* + \varepsilon\kappa\right)$. Hence,

$$\int_{x^1}^{x^2} \frac{dx}{t(x, \lambda^{\mathbf{P}^\varepsilon})} = \int_{x^1}^{x^2} \frac{dx}{t(x, \lambda^* + \varepsilon\kappa)}. \tag{C.6}$$

Since $t(x, \lambda^* + \varepsilon\kappa)$ is a strictly increasing function of ε for every x, each side of this equality is a strictly decreasing function of ε. In particular, each side is greater for $\varepsilon = 0$ than for $\varepsilon = \varepsilon^\star$.

Suppose that there is a pair of a twice continuously differentiable function $u : \mathbb{X} \to \mathbb{R}$ satisfying $u'' < 0 < u'$ and a twice continuously differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ satisfying $\phi'' \leq 0 < \phi'$ such that $V(\left(\bar{Y}^{\mathbf{P}}\right)_{\mathbf{P}}) = \sum_{\mathbf{P}} \mu(\mathbf{P}) \phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$, where V is the value function of (2). Then, by Lemma 7, for every $\mathbf{P} \in \mathcal{P}$, $V^{\mathbf{P}}(\bar{Y}^{\mathbf{P}}) = \phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $\bar{Y}^{\mathbf{P}} : \Omega \to \mathbb{X}$, where $V^{\mathbf{P}}$ is the value function of (3). Thus, by Lemma 6, the left-hand side of (C.6) is independent of ε. In particular, it takes the same value for $\varepsilon = 0$ and $\varepsilon = \varepsilon^\star$. This is a contradiction. Hence, there is no pair of a differentiable function $u : \mathbb{X} \to \mathbb{R}$ and a differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ such that $V(\left(\bar{Y}^{\mathbf{P}}\right)_{\mathbf{P}}) = \sum_{\mathbf{P}} \mu(\mathbf{P}) \phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$. $\square$

D. *Constant absolute risk aversion*

We study here an economy where u_i and v_i are HARA with zero marginal risk tolerance.[31]

ASSUMPTION 2: *Assume u_i is CARA with risk aversion $\alpha_i > 0$ and v_i is CARA with risk aversion $\gamma_i \geq \alpha_i$*

Assumption 2 is equivalent to assume u_i and v_i are HARA with CMRT (with parameters $(0, \frac{1}{\alpha_i})$ and $(0, \frac{1}{\gamma_i})$ respectively). Let $\phi_i = v_i \circ u_i^{-1}$, so $\phi_i(t) \propto -(-t^{\gamma_i/\alpha_i})$. Hence, our economy consists of smooth ambiguity-averse consumers with heterogeneous risk aversion and heterogeneous ambiguity aversion, parameterized by

[31] While this class of utility functions is usually not the one considered in the DSGE literature, it admits an easy representation for the efficient allocations and the representative consumer's utility function, while allowing for heterogeneity.

CARA Bernoulli utilities with risk aversion coefficient $\alpha_i > 0$ and by a power function with index $\frac{\gamma_i}{\alpha_i} \geq 1$, respectively.

PROPOSITION 8: *Let $(X_i^{\mathbf{P}})_{\mathbf{P},i}$ be an efficient allocation of an economy that satisfies Assumption 2. Let $\alpha = \left(\sum_i \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_i \gamma_i^{-1}\right)^{-1}$. Then,*

1) *For each P, there are constants $(\tau_i^{\mathbf{P}})_{i=1,\ldots,I}$ s.th. $\sum_i \tau_i^{\mathbf{P}} = 0$ and $X_i^{\mathbf{P}} = (\alpha/\alpha_i)\bar{X} + \tau_i^{\mathbf{P}}$ for every i.*
2) *For every i, there is a function $\tau_i : (-\infty, \infty) \to (-\infty, \infty)$ and constants κ_i such that $\tau_i(c) = \frac{\gamma}{\gamma_i}\left(1 - \frac{\gamma_i/\alpha_i}{\gamma/\alpha}\right)c + \kappa_i$ with $\sum_i \kappa_i = 0$ and*

   $$\tau_i^{\mathbf{P}} = \tau_i(c^{\mathbf{P}}) \tag{D.1}$$

   with $c^{\mathbf{P}} = u^{-1}(E^{\mathbf{P}} u(\bar{X}))$, where u, the representative consumer's utility function, is CARA with absolute risk aversion coefficient α.
3) *In the smooth ambiguity representative consumer's utility $\phi(t) \propto -(-t^{\gamma/\alpha})$ and $v = \phi \circ u$ is CARA with parameter γ.*

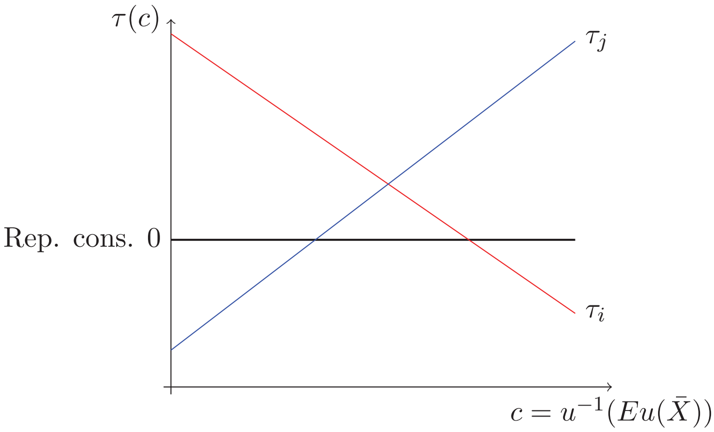

FIGURE A1. CONSTANT RISK TOLERANCE CASE.

Note: The Figure shows the transfers as a function of the certainty equivalents for two consumers, i and j. Consumer i is more ambiguity-averse than, and j is less ambiguity-averse than, the representative consumer.

As P varies, the efficient allocation rule adjusts by varying the intercept term of the linear sharing rule, $\tau_i^{\mathbf{P}}$, a term denoting transfers that sum to zero across all the consumers. The function $\tau_i^{\mathbf{P}}$ is itself linear in the aggregate certainty equivalent. Figure (A1) gives a graphical depiction showing how $\tau_i^{\mathbf{P}}$ varies as a function of the representative consumer's certainty equivalent for two consumers in this economy as established in Proposition 8.

If ambiguity attitudes were homogeneous, i.e., $\gamma_i/\alpha_i = \gamma_j/\alpha_j$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected-utility consumers: for all i, $\tau_i^{\mathbf{P}}$ is independent of $\mathbf{P}$.

E. Non-zero marginal risk tolerance

We provide here a complement to Proposition 5 and give the limit behavior of $\theta_i(.)$ and b.

PROPOSITION 9: *Consider the functions θ_i and RAA_ϕ constructed in Proposition 5. Then,*

1) *$\theta_i(z) \to 0$ as $z \to 0$ if $\gamma_i \neq \max_{i=1,\ldots,I}\gamma_i$ and $\theta_i(z) \to 0$ as $z \to \infty$ if $\gamma_i \neq \min_{i=1,\ldots,I}\gamma_i$.*
2) *$RAA_\phi(z) \to \max_{i=1,\ldots,I}\gamma_i - \alpha$ as $z \to 0$, and $RAA_\phi(z) \to \min_{i=1,\ldots,I}\gamma_i - \alpha$ as $z \to \infty$.*

Proof of Proposition 9 The l.h.s. of (A5) is equal to the derivative of the logarithm of the function $z \mapsto (f_i(z))^{\gamma_i}\, v'(z+\zeta)$. Hence this function is, in fact, constant. Thus, if there were an i s.th. $f_i(z)$ is bounded from above, then $v'(z)$ would be bounded away from zero. Then, $f_i(z)$ would be bounded from above for every i. This would contradict the assumption that $\sum_i f_i(z) = z$ for every $z > 0$. Hence, for every i, $f_i(z) \to \infty$ as $z \to \infty$. We can analogously show that for every i, $f_i(z) \to 0$ as $z \to 0$. This also shows that $v'(x) \to \infty$ as $x \to \zeta$ and $v'(x) \to 0$ as $x \to \infty$.

Denote the constant value of $(f_i(z))^{\gamma_i}\, v'(z+\zeta)$ by κ_i. Then, for every i and j,

$$0 < \theta_i(z) = \frac{f_i(z)}{z} < \frac{f_i(z)}{f_j(z)} = \frac{\left(\dfrac{\kappa_i}{v'(z+\zeta)}\right)^{1/\gamma_i}}{\left(\dfrac{\kappa_j}{v'(z+\zeta)}\right)^{1/\gamma_j}} = \frac{\kappa_i^{1/\gamma_i}}{\kappa_j^{1/\gamma_j}}\left(v'(z+\zeta)\right)^{1/\gamma_j - 1/\gamma_i}.$$

If $\gamma_i < \max_{i=1,\ldots,I}\gamma_i = \gamma_j$, then $1/\gamma_j - 1/\gamma_i < 0$. Since $v'(z+\zeta) \to \infty$ as $z \to 0$, the far right-hand side of the above equality converges to 0 as $z \to 0$. Hence $\theta_i(z) \to 0$ as $z \to 0$. We can analogously show that for every i, if $\gamma_i > \min_{i=1,\ldots,I}\gamma_i$, then $\theta_i(z) \to 0$ as $z \to \infty$. The limiting behavior of RAA_ϕ follows. □

We now explain the qualitative features of the graph of the shares θ_i as a function of the aggregate certainty equivalent.

Part 1(b) of Proposition 5 implies that, as we move from worse to better models, a consumer whose relative ambiguity aversion is greater (smaller) than that of the representative consumer around $c^{\mathbf{P}}$ will see their share decrease (resp. increase) for models with certainty equivalents marginally greater than $c^{\mathbf{P}}$ as shown in Figure A2.

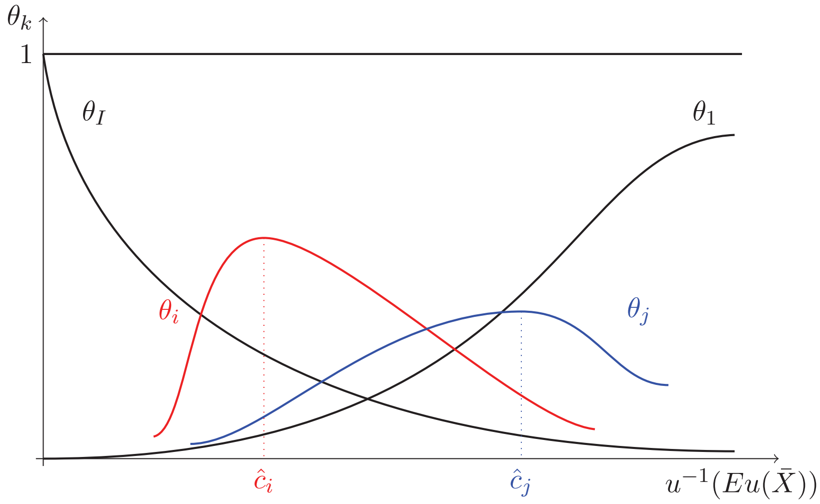

FIGURE A2. COMPARING CONSUMPTION SHARES θ_k UNDER CONDITION 2.

Note: Consumer I (resp. 1) is the most (resp. the least) relatively ambiguity-averse. i is more relatively ambiguity-averse than consumer j.

Consider consumer I with the largest relative ambiguity aversion in the economy. By part 2 of Proposition 5, their relative ambiguity aversion is greater than that of the representative consumer (at all $c^{\mathbf{P}}$). By part 1(b) of Proposition 5, θ_I will be negatively sloped everywhere. Analogously, consumer 1, with the lowest relative ambiguity aversion in the economy, will have a θ_1 that is positively sloped everywhere. From 1 of Proposition 9, the most relatively ambiguity-averse consumers get all of $\bar{X} - \zeta$ at the worst models. Therefore, at these models the representative consumer's relative ambiguity aversion is $\max_{i=1,\ldots,I} \gamma_i - \alpha$. Hence, by part 1(b) of Proposition 5, any consumer i with relative ambiguity aversion less than $\max_{i=1,\ldots,I} \gamma_i - \alpha$ will have their share increasing at least initially. Since the representative consumer has decreasing relative ambiguity aversion, we will reach a model, identified by $\hat{c}_i$ in Figure A2, where the representative consumer's relative ambiguity aversion falls below i's; hence, i's share is decreasing to the right of $\hat{c}_i$. For a consumer j relatively less ambiguity-averse than i, the representative consumer's ambiguity aversion has to decrease further before j's share peaks. Hence, $\hat{c}_j$ is to the right of $\hat{c}_i$. Taken together, the most relatively ambiguity-averse consumers get protected with extra shares at the worst models, the "middling" relative ambiguity-averse consumers get extra shares at the "middling" models and the least relatively ambiguity-averse ones get compensated by extra shares at the best models.

Finally, note that if $\gamma_i - \alpha = \gamma_j - \alpha$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected-utility consumers: for all i, θ_i is a constant function.

F. *Strict log-supermodularity*

In this Appendix, we give a general result on strict log-supermodularity (SLSPM for short) from which part 2 of Proposition 6 can be derived.

Let N be a positive integer. For each $x = (x_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$ and each $y = (y_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$, we write $x \geq y$ when $x_n \geq y_n$ for every n. We also write $x \vee y = (\max\{x_n, y_n\})_{n=1,2,\ldots,N}$ and $x \wedge y = (\min\{x_n, y_n\})_{n=1,2,\ldots,N}$. For each $x = (x_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$, we write $x_{-N} = (x_n)_{n=1,2,\ldots,N-1} \in \mathbb{R}^{N-1}$. By a slight abuse of notation, we use $\geq$, $\leq$, $\vee$, and $\wedge$ for vectors in $\mathbb{R}^{N-1}$ as well.

Let $f : \mathbb{R}^N \to \mathbb{R}_+$. We say that f is *strictly log-supermodular* (SLSPM for short) if

$$f(x)f(y) < f(x \vee y)f(x \wedge y)$$

for every $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ unless $x \leq y$ or $x \geq y$. That is, the strict log-supermodularity is a stronger property than the log-supermodularity (LSPM) in that the left-hand side is strictly smaller than the right-hand side. If $x \leq y$ or $x \geq y$, then $\{x, y\} = \{x \vee y, x \wedge y\}$ and the left- and right-hand sides would necessarily be equal. The constraint that neither should hold is needed to exclude this case. If $f(x) > 0$ for every $x \in \mathbb{R}^N$, then f is SLSPM if and only if $\ln f$ is strictly supermodular in the sense of Topkis (1998, Section 2.6.1).

Throughout this Appendix, we assume, for every $f : \mathbb{R}^N \to \mathbb{R}_+$ under consideration, that f is differentiable and $f(x) > 0$ for every $x \in \mathbb{R}^N$.

The first part of the following result is stated in Topkis (1998, Section 2.6.1). The second part can be proved in an analogues manner. The proof is omitted.

LEMMA 8: *1) f is LSPM if and only if, for all n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a nondecreasing function of x_m.*

2) f is SLSPM if, for every n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a strictly increasing function of x_m.

PROPOSITION 10: *Suppose that for all $m < N$ and n, $\partial \ln f(x)/\partial x_m$ is nondecreasing in x_n, and strictly increasing in x_n if $n = N$. Define $g : \mathbb{R}^{N-1} \to \mathbb{R}_{++}$ by $g(x_{-N}) = \int_{\mathbb{R}} f(x_{-N}, x_N)\,\mathrm{d}x_N$ for every $x_{-N} \in \mathbb{R}^{N-1}$. Then g is SLSPM.*

The assumptions of this proposition imply that f is LSPM but not that f is SLSPM. In fact, they can be met even when f is not SLSPM. The proposition, thus, implies that g can be SLSPM even when f is not. For a twice continuously differentiable f, they are satisfied if, for every $x \in \mathbb{R}^N$, $\frac{\partial^2}{\partial x_m \partial x_N} \ln f(x) > 0$ for every $m < N$, and $\frac{\partial^2}{\partial x_m \partial x_n} \ln f(x) \geq 0$for all $m < N$ and $n \neq m$.

The following proof method is essentially due to Karlin and Rinott (1980, Theorem 2.1). We only need to take special care of preserving strict inequalities under integration.

Proof of Proposition 10 By Fubini's theorem,

$$\begin{aligned}
&g(x_{-N})g(y_{-N})\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}} f(x_{-N},z)f(y_{-N},w)\,\mathrm{d}w\mathrm{d}z=\int_{\mathbb{R}\times\mathbb{R}} f(x_{-N},z)f(y_{-N},w)\,\mathrm{d}(z,w)\\
&=\int_{\{(z,w)\in\mathbb{R}\times\mathbb{R}\mid z=w\}} f(x_{-N},z)f(y_{-N},w)\,\mathrm{d}(z,w)\\
&\quad+\int_{\{(z,w)\in\mathbb{R}\times\mathbb{R}\mid z<w\}} \left(f(x_{-N},z)f(y_{-N},w)+f(y_{-N},w)f(x_{-N},z)\right)\mathrm{d}(z,w).
\end{aligned}\tag{F.1}$$

We can similarly show that

$$\begin{aligned}
&g(x_{-N}\vee y_{-N})g(x_{-N}\wedge y_{-N})\\
&=\int_{\{(z,w)\in\mathbb{R}\times\mathbb{R}\mid z=w\}} f(x_{-N}\vee y_{-N},z)f(x_{-N}\wedge y_{-N},w)\,\mathrm{d}(z,w)\\
&\quad+\int_{\{(z,w)\in\mathbb{R}\times\mathbb{R}\mid z<w\}} \big(f(x_{-N}\vee y_{-N},z)f(y_{-N}\wedge y_{-N},w)\\
&\qquad\qquad+f(x_{-N}\vee y_{-N},w)f(x_{-N}\wedge y_{-N},z)\big)\,\mathrm{d}(z,w).
\end{aligned}\tag{F.2}$$

When $z=w$, $(x_{-N},z)\vee(y_{-N},w)=(x_{-N}\vee y_{-N},z)$ and $(x_{-N},z)\wedge(y_{-N},w)=(x_{-N}\wedge y_{-N},w)$. Since f is LSPM,

$$f(x_{-N},z)f(y_{-N},w)\le f(x_{-N}\vee y_{-N},z)f(x_{-N}\wedge y_{-N},w).$$

Thus, the first term of the right-hand side of (F.1) is less than or equal to that of (F.2). To compare the second terms, assume that $z<w$ and that it is false that $x_{-N}\le y_{-N}$. Write

$$\begin{aligned}
A(z,w)&=f(x_{-N},z)f(y_{-N},w), &\quad C(z,w)&=f(x_{-N}\vee y_{-N},z)f(y_{-N}\wedge y_{-N},w),\\
B(z,w)&=f(x_{-N},w)f(y_{-N},z), &\quad D(z,w)&=f(x_{-N}\vee y_{-N},w)f(x_{-N}\wedge y_{-N},z).
\end{aligned}$$

Note first that

$$\begin{aligned}
A(z,w)B(z,w)&=\left(f(x_{-N},z)f(y_{-N},z)\right)\left(f(x_{-N},w)f(y_{-N},w)\right)\\
&\le\left(f(x_{-N}\vee y_{-N},z)f(x_{-N}\wedge y_{-N},z)\right)\left(f(x_{-N}\vee y_{-N},w)f(y_{-N}\wedge y_{-N},w)\right)\\
&=C(z,w)D(z,w).
\end{aligned}$$

Next, without loss of generality, we can assume that there is an M with $1 \leq M < N$ s.th. $x_n > y_n$ if and only if $n \leq M$. Then,

$$\begin{aligned} x_{-N} \vee y_{-N} &= (x_1, \dots, x_M, y_{M+1}, \dots, y_{N-1}), \\ x_{-N} \wedge y_{-N} &= (y_1, \dots, y_M, x_{M+1}, \dots, x_{N-1}). \end{aligned}$$

Moreover,

$$x_{-N} - x_{-N} \wedge y_{-N} = x_{-N} \vee y_{-N} - y_{-N} = (x_1 - y_1, \dots, x_M - y_M, 0, \dots, 0).$$

Denote this by v. For each $m \leq M$, write $v^m = (x_1 - y_1, \dots, x_m - y_m, 0, \dots, 0)$. Then $v^M = v$, $v^0 = 0$, and $v^m - v^{m-1} = (0, \dots, 0, x_m - y_m, 0, \dots, 0)$. Write $h = \ln f$. Then, for every $m \leq M$

$$\begin{aligned} &h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z) \\ =& \int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, x_{M+1} \dots, x_{N-1}, z)\, \mathrm{d}r, \\ &h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w) \\ =& \int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, y_{M+1} \dots, y_{N-1}, w)\, \mathrm{d}r. \end{aligned}$$

Since $\partial h / \partial x_m$ is nondecreasing in x_n with $n = M+1, \dots, N-1$ and strictly increasing in x_N,

$$\begin{aligned} &\frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, x_{M+1} \dots, x_{N-1}, z) \\ <& \frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, y_{M+1} \dots, y_{N-1}, w) \end{aligned}$$

for every r. Thus,

$$h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z) < h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w).$$

Since $x_{-N} \wedge y_{-N} + v^M = x_{-N}$ and $y_{-N} + v^M = x_{-N} \vee y_{-N}$, by taking the summation of each side over $m \leq M$, we obtain

$$h(x_{-N}, z) - h(x_{-N} \wedge y_{-N}, z) < h(x_{-N} \vee y_{-N}, w) - h(y_{-N}, w).$$

That is, $A(z, w) < D(z, w)$. By swapping the roles of x_{-N} and y_{-N} (while maintaining that $z < w$), we can show that $B(z, w) < D(z, w)$.

Since $A(z, w)B(z, w) \leq C(z, w)D(z, w)$, $A(z, w) < D(z, w)$, $B(z, w) < D(z, w)$, and

$$\begin{aligned}&(C(z,w)+D(z,w))-(A(z,w)+B(z,w))\\&=\frac{1}{D(z,w)}\left((C(z,w)D(z,w)-A(z,w)B(z,w))+(D(z,w)-A(z,w))(D(z,w)-B(z,w))\right),\end{aligned}$$

we have $A(z,w)+B(z,w) < C(z,w)+D(z,w)$. Since the second term of the right-hand side of (F.1) is nothing but the integral of $A(z,w)+B(z,w)$ on $\{(z,w) \in \mathbb{R}\times\mathbb{R} \mid z < w\}$ and that of (F.2) is nothing but the integral of $C(z,w)+D(z,w)$ on the same domain, this completes the proof. □

This proposition can be extended to the case in which the domain of the function is $X_1 \times X_2 \times \cdots \times X_N$, where X_n is an interval in $\mathbb{R}$ for every n.

G. *Comparing kernels*

PROPOSITION 11: *For each $n = 1, 2$, let $\pi_n : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be differentiable and suppose that $\pi_n' < 0$. Suppose, moreover, that $\varepsilon(s;\pi_1)$ is independent of s, $\varepsilon(s;\pi_2)$ is strictly decreasing in s, and the value of the former is contained in the range of the latter. Suppose, furthermore, that there is a non-degenerate probability P on $\mathbb{R}_{++}$ s.th. $\int \pi_1(x)P(dx) = \int \pi_2(x)P(dx)$. Then, there are x_* and x^* in $\mathbb{R}_{++}$ with $x_* < x^*$ s.th. $\pi_1(x) < \pi_2(x)$ if $x < x_*$ or $x > x^*$; $\pi_1(x) > \pi_2(x)$ if $x_* < x < x^*$; and $\pi_1(x) = \pi_2(x)$ if $x = x_*$ or $x = x^*$.*

Proof of Proposition 11 Define $g : \mathbb{R} \to \mathbb{R}$ by $g(z) = \ln \pi_2(\exp z) - \ln \pi_1(\exp z)$. Then,

$$g'(z) = \frac{\pi_2'(\exp z)\exp z}{\pi_2(\exp z)} - \frac{\pi_1'(\exp z)\exp z}{\pi_1(\exp z)}.$$

Thus, g' is strictly increasing, and there are $\underline{z}$ and $\overline{z}$ s.th. $g'(\underline{z}) < 0 < g'(\overline{z})$. Then, $g'(z) \le g'(\underline{z})$ for every $z \le \underline{z}$ and $g'(z) \ge g'(\overline{z})$ for every $z \ge \overline{z}$. By applying the mean-value theorem to g on the interval $[z, \underline{z}]$ and the strict increasingness of g', we obtain $g(\underline{z}) \le g'(\underline{z})(\underline{z} - z) + g(z)$, that is, $g(z) \ge -g'(\underline{z})(\underline{z} - z) + g(\underline{z})$ for every $z < \underline{z}$. As $z \to -\infty$, the right-hand side diverges to ∞. Similarly, $g(z) \ge g'(\overline{z})(z - \overline{z}) + g(\overline{z})$ for every $z > \overline{z}$. As $z \to \infty$, the right-hand side diverges to ∞. Thus, g attains its minimum (over the entire $\mathbb{R}$). Denote by $\hat{z}$ a point at which the minimum is attained. Then, $g'(\hat{z}) = 0$ by the first-order condition. Since g' is strictly increasing, $g'(z) < 0$ for every $z < \hat{z}$, and $g'(z) > 0$ for every $z > \hat{z}$. Thus, g is strictly decreasing on $(-\infty, \hat{z})$ and strictly increasing on $(\hat{z}, -\infty)$.

If $g(\hat{z}) \ge 0$, then $g(z) \ge 0$ for every z, with a strict inequality possibly except at $z = \hat{z}$. Thus, $\pi_2(x) \ge \pi_1(x)$ for every x, with a strict inequality possibly except for $x = \exp \hat{z}$, and the integral assumption is violated. Thus, $g(\hat{z}) < 0$. By the intermediate value theorem, there is a unique $z_* < \hat{z}$ s.th. $g(z_*) = 0$; and there is a unique $z^* > \hat{z}$ s.th. $g(z^*) = 0$. Let $x_* = \exp z_*$ and $x^* = \exp z^*$, to complete the proof. $\square$

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