# Supplemental Appendix to
# "Monetary Policy without Commitment"

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# A Continuous-Time Limit

In this appendix, we solve the discrete-time model for an arbitrary time step of length $\mathrm{d}t$ and derive the continuous-time limit as $\mathrm{d}t \to 0$. For completeness, we first reiterate the derivations of the discrete-time model for a given $\mathrm{d}t$, where $\mathrm{d}t = 1$ corresponds to the derivations in the main text.

Time now runs at increments of $\mathrm{d}t$, so that $t \in T_{\mathrm{d}t} \equiv \{0, \mathrm{d}t, 2\mathrm{d}t, \ldots\}$. Let $\rho \equiv -\log(\beta)$. For a given $\mathrm{d}t$, the household's problem (see (1)) can be written as

$$\max_{C_t, L_t, B_t, (s_{j,t}, C_{j,t})_{j\in[0,1]}} \sum_{t \in T_{\mathrm{d}t}} e^{-\rho t} \left( \log(C_t) - \frac{L_t^{1+\psi}}{1+\psi} \right) \mathrm{d}t$$

subject to

$$\int_0^1 P_{j,t} C_{j,t} \mathrm{d}j \mathrm{d}t + B_t \leq W_t L_t \mathrm{d}t + (1 + i_{t-\mathrm{d}t} \mathrm{d}t) B_{t-\mathrm{d}t} + \int_0^1 s_{j,t} X_{j,t} \mathrm{d}j \mathrm{d}t + \int_0^1 (s_{j,t-\mathrm{d}t} - s_{j,t}) P_{j,t}^S \mathrm{d}j - T_t \mathrm{d}t,$$

where $C_t = \left( \int_0^1 C_{j,t}^{1-\sigma^{-1}} \mathrm{d}j \right)^{\frac{1}{1-\sigma^{-1}}}$. Note that this formulation of the problem redefines $C_{j,t}$, $C_t$, $L_t$, $X_{j,t}$ and $T_t$ as *rates* of consumption, labor supply, profits, and lump-sum taxes per $\mathrm{d}t$.

The implied demand for each variety $j \in [0, 1]$, the definition of the aggregate price $P_t$, the price dispersion measure $D_t$, and the intratemporal labor supply condition are all identical to those in the main text because they follow from static decisions that are not affected by the time step $\mathrm{d}t$. To reiterate these, we have

$$C_{j,t} = C_t \left( \frac{P_{j,t}}{P_t} \right)^{1-\sigma}, \quad P_t = \left( \int_0^1 P_{j,t}^{1-\sigma} \mathrm{d}j \right)^{\frac{1}{1-\sigma}}, \quad D_t = \int_0^1 \left( \frac{P_{j,t}}{P_t} \right)^{-\sigma} \mathrm{d}j, \quad \frac{W_t}{P_t} = C_t L_t^{\psi}.$$

As in the main text, we can use the labor market clearing conditions to derive the aggregate production function of the economy as follows:

$$L_t = \int_0^1 L_{j,t} \mathrm{d}j = \int_0^1 C_{j,t} \mathrm{d}j = C_t \int_0^1 \left( \frac{P_{j,t}}{P_t} \right)^{-\sigma} \mathrm{d}j = C_t D_t \implies C_t = \frac{L_t}{D_t}.$$

The Euler equations for nominal bonds and stocks for a given d$t$ are

$$\frac{1}{P_tC_t} = e^{-\rho \mathrm{d}t}(1 + i_t\mathrm{d}t)\frac{1}{P_{t+\mathrm{d}t}C_{t+\mathrm{d}t}},$$
$$P^S_{j,t} = X_{j,t}\mathrm{d}t + \frac{1}{1 + i_t\mathrm{d}t}\mathbb{E}^j_t[P^S_{j,t+\mathrm{d}t}],$$

for all $j \in [0, 1]$. Iterating the Euler equation for stocks forward, using the Euler equation for nominal bonds, and assuming no bubbles gives us the household's valuation of firms at time $t$ as:

$$P^S_{j,t} = \sum_{h\in T_{\mathrm{d}t}} e^{-\rho h}\frac{P_tC_t}{P_{t+h}C_{t+h}}\mathbb{E}^j_t[X_{j,t+h}]\mathrm{d}t$$

We will use this valuation to rewrite the optimization problem of a flexible-price firm. Before we do so, we have to adjust the frequency of price changes so that the probability with which a firm can adjust its price is independent of the choice of d$t$. To this end, let $\theta^{\mathrm{d}t}$ be the probability of not having the opportunity to adjust prices at an interval of length d$t$. This defines a consistent distribution of the price adjustment frequency for different values of d$t$ such that, for any interval length $T$, the probability of not adjusting prices is $\theta^T$, independent of d$t$. With $T = 1$, this corresponds to the model in the main text where d$t = 1$. With d$t \to 0$, it corresponds to a Poisson process, where the arrival rate of price adjustment opportunities is $\lambda \equiv -\log(\theta)$. We obtain a well-defined limit: under the Poisson arrival rate of $\lambda$, the implied distribution of time between price changes is exponential with scale $\lambda$. Accordingly, the probability of not adjusting the price in a period of length $T$ is $e^{-\lambda T} = e^{\log(\theta)T} = \theta^T$.

For a given d$t$, the problem of a flexible-price firm (see (9)) is

$$\max_{P^*_t} \sum_{h\in T_{\mathrm{d}t}} e^{-(\rho+\lambda)h}\frac{P_tC_t}{P_{t+h}C_{t+h}}[P^*_t - (1+\tau)W_{t+h}]C_{t+h}\left(\frac{P^*_t}{P_{t+h}}\right)^{-\sigma}\mathrm{d}t.$$

The first-order condition for $P^*_t$ is

$$\sum_{h\in T_{\mathrm{d}t}} e^{-(\rho+\lambda)h}P^{\sigma-1}_{t+h}\left[P^*_t - \frac{\sigma(1+\tau)}{\sigma-1}W_{t+h}\right]\mathrm{d}t = 0,$$

which, following the main text, can be simplified and rewritten as

$$\frac{P_t^*}{P_t} = \frac{\sigma(1+\tau)}{\sigma-1} \frac{\sum_{h\in T_{\mathrm{dt}}} e^{-(\rho+\lambda)h} \left(\frac{P_{t+h}}{P_t}\right)^{\sigma} \frac{W_{t+h}}{P_{t+h}} \mathrm{d}t}{\sum_{h\in T_{\mathrm{dt}}} e^{-(\rho+\lambda)h} \left(\frac{P_{t+h}}{P_t}\right)^{\sigma-1} \mathrm{d}t}. \tag{A.1}$$

The auxiliary variable $\delta_t$ is defined as the inverse of the denominator in (A.1), and can be written recursively as

$$\delta_t^{-1} \equiv \sum_{h\in T_{\mathrm{dt}}} e^{-(\rho+\lambda)h} \left(\frac{P_{t+h}}{P_t}\right)^{\sigma-1} \mathrm{d}t = \mathrm{d}t + e^{-(\rho+\lambda)\mathrm{d}t} \left(\frac{P_{t+\mathrm{d}t}}{P_t}\right)^{\sigma-1} \delta_{t+\mathrm{d}t}^{-1}. \tag{A.2}$$

Similarly, we can write (A.1) recursively as

$$\begin{aligned} \frac{P_t^*}{P_t} &= \frac{\sigma(1+\tau)}{\sigma-1} \frac{W_t}{P_t} \delta_t \mathrm{d}t + e^{-(\rho+\lambda)\mathrm{d}t} \left(\frac{P_{t+\mathrm{d}t}}{P_t}\right)^{\sigma} \frac{\delta_t}{\delta_{t+\mathrm{d}t}} \frac{P_{t+\mathrm{d}t}^*}{P_{t+\mathrm{d}t}} \\ &= \frac{\sigma(1+\tau)}{\sigma-1} \frac{W_t}{P_t} \delta_t \mathrm{d}t + (1-\delta_t \mathrm{d}t) \frac{P_{t+\mathrm{d}t}}{P_t} \frac{P_{t+\mathrm{d}t}^*}{P_{t+\mathrm{d}t}}, \end{aligned} \tag{A.3}$$

where the second line follows from substituting (A.2) in (A.3).

Next, we can derive the aggregate price as

$$P_t^{1-\sigma} = \int_0^1 P_{i,t}^{1-\sigma} \mathrm{d}j = (1-e^{-\lambda \mathrm{d}t})(P_t^*)^{1-\sigma} + e^{-\lambda \mathrm{d}t} P_{t-\mathrm{d}t}^{1-\sigma},$$

where we have used the fact that the set of sticky-price firms is a random sample of the population at each instant. This equation implies the following relationship between relative reset price and gross inflation rate:

$$1 = (1-e^{-\lambda \mathrm{d}t}) \left(\frac{P_t^*}{P_t}\right)^{1-\sigma} + e^{-\lambda \mathrm{d}t} \left(\frac{P_t}{P_{t-\mathrm{d}t}}\right)^{\sigma-1}.$$

Defining $\pi_t \equiv \frac{1}{\mathrm{d}t} \log(P_t/P_{t-\mathrm{d}t})$ as the rate of inflation at time $t$, we can rewrite the above equation as

$$\frac{P_t^*}{P_t} = \left[\frac{1-e^{[(\sigma-1)\pi_t - \lambda]\mathrm{d}t}}{1-e^{-\lambda \mathrm{d}t}}\right]^{\frac{1}{1-\sigma}},$$

which is the analog of (17) in the main text. Moreover, using this equation, combined with the intratemporal labor supply condition and the aggregate production function $C_t = Y_t = L_t/D_t$, equations (A.2) and (A.3) become

$$\delta_t^{-1} = \mathrm{d}t + e^{[(\sigma-1)\pi_{t+\mathrm{d}t}-(\rho+\lambda)]\mathrm{d}t}\delta_{t+\mathrm{d}t}^{-1}, \tag{A.4}$$

$$\left[\frac{1-e^{[(\sigma-1)\pi_t-\lambda]\mathrm{d}t}}{1-e^{-\lambda\mathrm{d}t}}\right]^{\frac{1}{1-\sigma}} = \frac{\sigma(1+\tau)}{\sigma-1}Y_t^{1+\psi}D_t^{\psi}\delta_t\mathrm{d}t + (1-\delta_t\mathrm{d}t)e^{\pi_{t+\mathrm{d}t}\mathrm{d}t}\left[\frac{1-e^{[(\sigma-1)\pi_{t+\mathrm{d}t}-\lambda]\mathrm{d}t}}{1-e^{-\lambda\mathrm{d}t}}\right]^{\frac{1}{1-\sigma}}, \tag{A.5}$$

which are the analogs of (19) and (20), respectively.

We next write the equation for the price dispersion dynamics to obtain the analog of (18). By random selection of price-setters at any given $t$, we have

$$\begin{aligned} D_t = \int_0^1 \left(\frac{P_{j,t}}{P_t}\right)^{-\sigma}\mathrm{d}j &= (1-e^{-\lambda\mathrm{d}t})\left(\frac{P_t^*}{P_t}\right)^{-\sigma} + e^{-\lambda\mathrm{d}t}\left(\frac{P_t}{P_{t-\mathrm{d}t}}\right)^{\sigma}\int_0^1\left(\frac{P_{j,t-\mathrm{d}t}}{P_{t-\mathrm{d}t}}\right)^{-\sigma}\mathrm{d}j \\ &= (1-e^{-\lambda\mathrm{d}t})\left[\frac{1-e^{[(\sigma-1)\pi_t-\lambda]\mathrm{d}t}}{1-e^{-\lambda\mathrm{d}t}}\right]^{\frac{\sigma}{\sigma-1}} + e^{\sigma\pi_t\mathrm{d}t-\lambda\mathrm{d}t}D_{t-\mathrm{d}t}. \end{aligned} \tag{A.6}$$

Finally, we consider the central bank's problem under lack of commitment. Analogous to (24), the central bank's objective with a general time step can be written as

$$V(D_{t-\mathrm{d}t}) = \left(\log(Y_t) - \frac{(D_tY_t)^{1+\psi}}{1+\psi}\right)\mathrm{d}t + e^{-\rho\mathrm{d}t}V(D_t).$$

The central bank's problem yields the same optimal policy as in the main text, $Y_t = 1/D_t$. This policy implies that the real wage from the intratemporal labor supply condition is given by

$$\frac{W_t}{P_t} = Y_tL_t^{\psi} = Y_t^{1+\psi}D_t^{\psi} = \frac{1}{D_t}.$$

Plugging this optimal policy into equation (A.5) and taking the limit as $\mathrm{d}t \to 0$ in equations (A.4)-(A.6), we obtain the continuous-time analogs of the equations that characterize $D_t$, $\pi_t$, and $\delta_t$, as presented in the main text:

$$\begin{aligned}
\dot{D}_t &= \lambda\left(1-\frac{\sigma-1}{\lambda}\pi_t\right)^{\frac{\sigma}{\sigma-1}} + (\sigma\pi_t-\lambda)D_t, \\
\dot{\pi}_t &= -\lambda\frac{\sigma(1+\tau)}{\sigma-1}\left(1-\frac{\sigma-1}{\lambda}\pi_t\right)^{\frac{\sigma}{\sigma-1}}\frac{\delta_t}{D_t} + (\delta_t-\pi_t)[\lambda-(\sigma-1)\pi_t], \\
\dot{\delta}_t &= \delta_t^2 + [(\sigma-1)\pi_t-(\rho+\lambda)]\delta_t.
\end{aligned}$$

# B Proofs

This appendix contains the proofs of all formal results stated in the paper.

## B.1 Proof of Lemma 1

Take an initial price distribution $(P_{j,-1})_{j\in[0,1]}$ and a sequence of policies $(i_t)_{t=0}^{\infty}$. The arguments in the text show that if a sequence of allocations and prices $(L_t, Y_t, D_t, \delta_t, \Pi_t)_{t=0}^{\infty}$ is supported by a competitive equilibrium, then it satisfies (12), (13), (18), (19), (20), and (21). This proves the necessity claim.

To prove the sufficiency claim, suppose that a sequence $(L_t, Y_t, D_t, \delta_t, \Pi_t)_{t=0}^{\infty}$ satisfies (12), (13), (18), (19), (20), and (21) given $(P_{j,-1})_{j\in[0,1]}$ and $(i_t)_{t=0}^{\infty}$. The set $(P_{j,-1})_{j\in[0,1]}$ defines $P_{-1}$, and we can define $P_t = \Pi_{t-1}P_{t-1}$ recursively. Let $P_{j,t} = P_{j,t-1}$ if firm $j$ cannot change prices at $t$, and $P_{j,t} = P_t^*$ if the firm can change prices at $t$, where $P_t^*$ is given by (17). Define $W_t$ according to (15) and let $B_t = 0$ at all dates with $T_t$ chosen to satisfy (11). Letting $C_t = Y_t$, define $C_{j,t}$ according to (2), and let $Y_{j,t} = L_{j,t} = C_{j,t}$. Additionally, let

$$X_{j,t} = [P_{j,t}-(1+\tau)W_t]C_t\left(\frac{P_{j,t}}{P_t}\right)^{-\sigma},$$

define $P_{j,t}^S$ according to (6), and let $s_{j,t} = 1$ so that the representative household holds a share of every firm $j \in [0,1]$. The household's problem (1) is concave and yields a unique solution. It can be verified that the values of $(C_t, L_t, B_t, (s_{j,t}, C_{j,t})_{j\in[0,1]})_{t=0}^{\infty}$ satisfy all optimality conditions of the household's problem, with the transversality condition being verified below. The firm's problem (9) is concave and yields a unique solution. It can be verified that the values of $(P_t^*, Y_{j,t}, L_{j,t})_{t=0}^{\infty}$ satisfy all optimality conditions of the firm's problem. Therefore, we conclude that the sequence $(L_t, Y_t, D_t, \delta_t, \Pi_t)_{t=0}^{\infty}$ supports a competitive equilibrium.

We next verify the transversality condition. Consider the date-$t$ price of an Arrow-Debreu security that pays a coupon equal to firm $j$'s profits at date $t+h$ for $h > 0$. There are three cases to consider. First, suppose the firm's price has always been sticky. Then the probability of arriving at such a history at $t+h$ from the perspective of date $t$ is $\theta^h$, and the price that the firm is charging at $t+h$ is $P_{j,-1}$. Appealing to the intertemporal condition, we can write the limiting price of the Arrow-Debreu security at date $t$ as $h \to \infty$ as

$$\lim_{h\to\infty} \beta^h \theta^h \frac{P_t C_t}{P_{t+h} C_{t+h}} \left[P_{j,-1} - (1+\tau) W_{t+h}\right] C_{t+h} \left(\frac{P_{j,-1}}{P_{t+h}}\right)^{-\sigma} = 0, \tag{B.1}$$

where transversality requires that this price go to zero.

Second, suppose the firm's price has been sticky since date $\ell$ for $0 \leq \ell \leq t$. Then the probability of arriving at such a history at $t+h$ from the perspective of date $t$ is $\theta^h$, and the price that the firm is charging at $t+h$ is $P_\ell^*$. The transversality condition in this case is

$$\lim_{h\to\infty} \beta^h \theta^h \frac{P_t C_t}{P_{t+h} C_{t+h}} \left[P_\ell^* - (1+\tau) W_{t+h}\right] C_{t+h} \left(\frac{P_\ell^*}{P_{t+h}}\right)^{-\sigma} = 0. \tag{B.2}$$

Finally, suppose the firm's price has been sticky since date $\ell > t$. Then the probability of arriving at such a history at $t+h$ from the perspective of date $t$ is $(1-\theta)\,\theta^{t+h-\ell}$, and the price that the firm is charging at $t+h$ is $P_\ell^*$. The transversality condition in this case is

$$\lim_{h\to\infty} \beta^h (1-\theta)\, \theta^{t+h-\ell} \frac{P_t C_t}{P_{t+h} C_{t+h}} \left[P_\ell^* - (1+\tau) W_{t+h}\right] C_{t+h} \left(\frac{P_l^*}{P_{t+h}}\right)^{-\sigma} = 0. \tag{B.3}$$

To verify that (B.2) and (B.3) are satisfied, note that we can multiply (B.2) by $\beta^{-\ell}\theta^{-\ell} P_\ell C_\ell / P_t C_t$ without changing its limit as $h \to \infty$, which means that satisfaction of (B.2) is equivalent to

$$\lim_{h\to\infty}\beta^{h-\ell}\theta^{h-\ell}\frac{P_\ell C_\ell}{P_{t+h}C_{t+h}}\left[P_\ell^*-(1+\tau)\,W_{t+h}\right]C_{t+h}\left(\frac{P_\ell^*}{P_{t+h}}\right)^{-\sigma}=0. \tag{B.4}$$

Similarly, we can multiply (B.3) by $(1-\theta)^{-1}\theta^{-t}P_\ell C_\ell/P_tC_t$ without changing its limit as $h\to\infty$, which means that satisfaction of (B.3) is also equivalent to (B.4). Moreover, observe that given (14), (15), and (17), and noting that $P_tC_t\left(\frac{1-\theta\Pi_t^{\sigma-1}}{1-\theta}\right)^{-\sigma}>0$, it follows that satisfaction of (21) implies satisfaction of (B.4). Hence, (B.2) and (B.3) are both satisfied.

We are left to verify that (B.1) is also satisfied. We can multiply (B.1) by $P_{j,-1}^{\sigma}/P_tC_t$ without changing its limit as $h\to\infty$, which means that satisfaction of (B.1) is equivalent to

$$\lim_{h\to\infty}\beta^h\theta^hP_h^\sigma\left[\left(\frac{P_{j,-1}}{P_{-1}}\right)\frac{P_{-1}}{P_h}-(1+\tau)\frac{W_h}{P_h}\right]=0.$$

Under the constructed equilibrium, this limit can be rewritten as

$$\lim_{h\to\infty}\left[\beta\theta\left(\prod_{\ell=0}^{h}\Pi_\ell\right)^{\frac{\sigma}{h}}\right]^h\left[\left(\frac{P_{j,-1}}{P_{-1}}\right)\frac{1}{\prod_{\ell=0}^{h}\Pi_\ell}-(1+\tau)D_h^\psi Y_h^{1+\psi}\right]=0. \tag{B.5}$$

There are two possible cases. Suppose first that $\lim_{h\to\infty}\left[\beta\theta\left(\prod_{\ell=0}^{h}\Pi_\ell\right)^{\frac{\sigma}{h}}\right]^h=0$. Then note that by (B.4) for $\ell=0$, the second bracket stays finite as $h\to\infty$. Hence, in this case, (B.5) and thus (B.1) are satisfied.

Suppose next that $\lim_{h\to\infty}\left[\beta\theta\left(\prod_{\ell=0}^{h}\Pi_\ell\right)^{\frac{\sigma}{h}}\right]^h\neq 0$. Then satisfaction of (B.4) (setting $\ell=0$ in that equation) implies

$$\lim_{h\to\infty}\left[\frac{P_0^*}{P_0}\frac{1}{\prod_{\ell=1}^{h}\Pi_\ell}-(1+\tau)D_h^\psi Y_h^{1+\psi}\right]=0.$$

It follows that if (B.5) is not satisfied, then we must have

$$\lim_{h\to\infty}\left[\frac{P_0^*}{P_0}\frac{1}{\prod_{\ell=1}^{h}\Pi_\ell}-(1+\tau)D_h^{\psi}Y_h^{1+\psi}-\left(\frac{P_{j,-1}}{P_{-1}}\right)\frac{1}{\prod_{\ell=0}^{h}\Pi_\ell}+(1+\tau)D_h^{\psi}Y_h^{1+\psi}\right]\neq 0,$$

or, equivalently,

$$\lim_{h\to\infty}\left\{\left[\frac{P_0^*}{P_0}\frac{P_0}{P_{-1}}-\left(\frac{P_{j,-1}}{P_{-1}}\right)\right]\frac{1}{\prod_{\ell=0}^{h}\Pi_\ell}\right\}\neq 0.$$

But this means that $\frac{1}{\prod_{\ell=0}^{h}\Pi_\ell}$ does not approach zero as $h\to\infty$, which contradicts the assumption that $\lim_{h\to\infty}\left[\beta\theta\left(\prod_{\ell=0}^{h}\Pi_\ell\right)^{\frac{\sigma}{h}}\right]^{h}\neq 0$. Hence, (B.5) and thus (B.1) are satisfied.

## B.2 Proof of Lemma 2

Consider first price dispersion $D$. Equation (22) defines $D$ as a function of $\Pi$ in the steady state. Differentiating this equation yields

$$\begin{aligned}\frac{\partial}{\partial\Pi}D &= \theta\sigma D\Pi^{\sigma-2}\left(-\frac{1}{1-\theta\Pi^{\sigma-1}}+\frac{\Pi}{1-\theta\Pi^{\sigma}}\right)\\ &= \theta\sigma D\Pi^{\sigma-2}\frac{\Pi-1}{(1-\theta\Pi^{\sigma-1})(1-\theta\Pi^{\sigma})}.\end{aligned}$$

This expression is strictly positive for $\Pi\in(1,\theta^{-1/\sigma})$, including $D$ itself (which is a function of $\Pi$ per equation (22)). Thus, $D$ is strictly increasing in $\Pi$ for $\Pi\in[1,\theta^{-1/\sigma})$.

Consider next the labor share $\mu$. Raising equation (22) to the power of $1+\psi$ and substituting in equation (23) yields

$$\begin{aligned}\mu &= \frac{\sigma-1}{\sigma(1+\tau)}\frac{1-\theta\Pi^{\sigma-1}}{1-\theta\Pi^{\sigma}}\frac{1-\beta\theta\Pi^{\sigma}}{1-\beta\theta\Pi^{\sigma-1}}\\ &= \frac{\sigma-1}{\sigma(1+\tau)}\left[1+\frac{(1-\beta)\theta\Pi^{\sigma-1}(\Pi-1)}{(1-\theta\Pi^{\sigma})(1-\beta\theta\Pi^{\sigma-1})}\right].\end{aligned}$$

Note that the fraction inside the brackets is strictly positive for $\Pi\in(1,\theta^{-1/\sigma})$

and is equal to zero for $\Pi = 1$. Thus, $\mu \geq (\sigma - 1)/[\sigma(1+\tau)]$, with equality only when $\Pi = 1$. Differentiating this equation yields

$$\frac{\partial}{\partial \Pi}\mu = \left[\mu - \frac{\sigma - 1}{\sigma(1+\tau)}\right]\left[\frac{\sigma - 1}{\Pi} + \frac{1}{\Pi - 1} + \frac{\sigma\theta\Pi^{\sigma-1}}{1-\theta\Pi^{\sigma}} + \frac{(\sigma-1)\beta\theta\Pi^{\sigma-2}}{1-\beta\theta\Pi^{\sigma-1}}\right].$$

This expression is strictly positive for $\Pi \in (1, \theta^{-1/\sigma})$. Thus, $\mu$ is strictly increasing in $\Pi$ for $\Pi \in [1, \theta^{-1/\sigma})$.

## B.3 Proof of Proposition 1

Below, we first restate the central bank's commitment problem presented in the main text. We then show that any steady state that satisfies the first-order conditions and Envelope conditions of this problem must feature $\Pi = 1$.

**Statement of the Problem.** Given an initial value for dispersion $D_{-1}$, and substituting with $Y_t = L_t/D_t$ and $L_t^{1+\psi} = \mu_t$, the central bank's commitment problem is

$$\max_{(\mu_t, D_t, \Pi_t, \delta_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(-\log(D_t) + \frac{\log(\mu_t) - \mu_t}{1+\psi}\right)$$

subject to

$$D_t = (1-\theta)\left(\frac{1-\theta\Pi_t^{\sigma-1}}{1-\theta}\right)^{\frac{\sigma}{\sigma-1}} + \theta\Pi_t^{\sigma}D_{t-1}, \qquad (\beta^t\zeta_t)$$

$$\left(\frac{1-\theta\Pi_t^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}} = \gamma\delta_t\mu_t D_t^{-1} + (1-\delta_t)\Pi_{t+1}\left(\frac{1-\theta\Pi_{t+1}^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}}, \qquad (\beta^t\xi_t)$$

$$\delta_t^{-1} = 1 + \beta\theta\Pi_{t+1}^{\sigma-1}\delta_{t+1}^{-1}, \qquad (\beta^t\chi_t)$$

where $\gamma \equiv \frac{(1+\tau)\sigma}{\sigma-1}$, and $\zeta$, $\xi$, and $\chi$ are the assigned Lagrange multipliers to each of the corresponding constraints.

**First-Order Conditions.** We write below the first-order conditions of the central bank's problem above for any $t \geq 1$. (Our results will not make use of the first-order conditions for $t = 0$).

$$
\begin{aligned}
\mu_t:&\quad \frac{\mu_t^{-1}-1}{1+\psi}+\gamma\xi_t\delta_t D_t^{-1}=0 \iff \gamma\xi_t\delta_t D_t^{-1}=\frac{1-\mu_t^{-1}}{1+\psi},\\
D_t:&\quad -D_t^{-1}-\zeta_t+\beta\theta\Pi_{t+1}^{\sigma}\zeta_{t+1}-\gamma\delta_t\mu_t D_t^{-2}\xi_t=0,\\
\Pi_t:&\quad \sigma\theta\Pi_t^{\sigma-2}\beta\zeta_t\left[-\left(\frac{1-\theta\Pi_t^{\sigma-1}}{1-\theta}\right)^{\frac{1}{\sigma-1}}+D_{t-1}\Pi_t\right]\\
&\quad +\beta\theta(\sigma-1)\Pi_t^{\sigma-2}\delta_t^{-1}\chi_{t-1}-\beta\xi_t\left(\frac{\theta\Pi_t^{\sigma-2}}{1-\theta\Pi_t^{\sigma-1}}\right)\left(\frac{1-\theta\Pi_t^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}}\\
&\quad +\xi_{t-1}(1-\delta_{t-1})\frac{1}{1-\theta\Pi_t^{\sigma-1}}\left(\frac{1-\theta\Pi_t^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}}=0,\\
\delta_t:&\quad -\theta\Pi_t^{\sigma-1}\delta_t^{-2}\chi_{t-1}+\delta_t^{-2}\chi_t+\left[\gamma\mu_t D_t^{-1}-\Pi_{t+1}\left(\frac{1-\theta\Pi_{t+1}^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}}\right]\xi_t=0.
\end{aligned}
$$

**Steady State.** In the steady state, the constraints become

$$
\begin{aligned}
&D=\frac{1-\theta}{1-\theta\Pi^{\sigma}}\left(\frac{1-\theta\Pi^{\sigma-1}}{1-\theta}\right)^{\frac{\sigma}{\sigma-1}},\\
&\left(\frac{1-\theta\Pi^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}}=\frac{1-\beta\theta\Pi^{\sigma-1}}{1-\beta\theta\Pi^{\sigma}}\gamma\mu D^{-1},\\
&\delta=1-\beta\theta\Pi^{\sigma-1},
\end{aligned}
$$

where the second equation can be rewritten as

$$
(\gamma\mu)^{-1}=\frac{1-\beta\theta\Pi^{\sigma-1}}{1-\beta\theta\Pi^{\sigma}}\frac{1-\theta\Pi^{\sigma}}{1-\theta\Pi^{\sigma-1}}.
$$

For the first-order conditions, recall that we have defined a steady state as finite and constant values for the endogenous variables under a constant policy. This necessarily requires that the multipliers on the constraints of the central bank's problem also be finite and constant in the steady state. Hence, the first-order conditions become

$$
\mu:\quad \gamma\xi\delta D^{-1}=\frac{1-\mu^{-1}}{1+\psi},
$$

$$
\begin{aligned}
D:&\quad \zeta D = -\frac{1+\gamma\delta\mu D^{-1}\xi}{1-\beta\theta\Pi^{\sigma}},\\
\Pi:&\quad \sigma\theta\Pi^{\sigma-2}\beta\zeta\left[-\left(\frac{1-\theta\Pi^{\sigma-1}}{1-\theta}\right)^{\frac{1}{\sigma-1}}+D\Pi\right]+\beta\theta(\sigma-1)\Pi^{\sigma-2}\delta^{-1}\chi\\
&\quad +\left(\frac{1-\delta-\beta\theta\Pi^{\sigma-2}}{1-\theta\Pi^{\sigma-1}}\right)\left(\frac{1-\theta\Pi^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}}\xi=0,\\
\delta:&\quad \delta^{-2}\chi=\frac{1}{1-\theta\Pi^{\sigma-1}}\left[\Pi\left(\frac{1-\theta\Pi^{\sigma-1}}{1-\theta}\right)^{\frac{1}{1-\sigma}}-\gamma\mu D^{-1}\right]\xi.
\end{aligned}
$$

**Step by Step Characterization.** To prove the proposition, we proceed in steps as follows:

1. We use the first-order conditions for $\delta$ and $\mu$ in the steady state to obtain

$$
\begin{aligned}
\delta^{-2}\chi &= \frac{1}{1-\theta\Pi^{\sigma-1}}\left(\frac{\Pi-1}{1-\beta\theta\Pi^{\sigma}}\right)\gamma\mu D^{-1}\xi\\
\implies \delta^{-1}\chi &= \frac{1}{1-\theta\Pi^{\sigma-1}}\left(\frac{\Pi-1}{1-\beta\theta\Pi^{\sigma}}\right)\frac{\mu-1}{1+\psi}.
\end{aligned}
$$

2. We use the first-order conditions for $D$ and $\mu$ in the steady state to obtain

$$
\zeta D = -\frac{1}{1-\beta\theta\Pi^{\sigma}}\frac{\mu+\psi}{1+\psi}.
$$

3. Consider the first-order condition for $\Pi$. Substituting with the expressions from steps 1 and 2 as well as the steady-state values of the constraints, we obtain

$$
\begin{aligned}
&-\sigma\beta\theta\Pi^{\sigma-2}\frac{1}{1-\beta\theta\Pi^{\sigma}}\frac{\mu+\psi}{1+\psi}\left(\frac{\Pi-1}{1-\theta\Pi^{\sigma-1}}\right)\\
&+\beta\theta(\sigma-1)\Pi^{\sigma-2}\left(\frac{\Pi-1}{1-\theta\Pi^{\sigma-1}}\right)\frac{1}{1-\beta\theta\Pi^{\sigma}}\frac{\mu-1}{1+\psi}\\
&+\beta\theta\Pi^{\sigma-2}\left(\frac{\Pi-1}{1-\theta\Pi^{\sigma-1}}\right)\frac{1}{1-\beta\theta\Pi^{\sigma}}\frac{\mu-1}{1+\psi}=0.
\end{aligned}
$$

4. Factoring out the common terms, the equation from step 3 yields

$$\sigma \frac{\beta\theta\Pi^{\sigma-2}}{1-\beta\theta\Pi^{\sigma}} \left( \frac{\Pi - 1}{1-\theta\Pi^{\sigma-1}} \right) = 0.$$

It follows that the only possible steady-state value for inflation that respects the positivity of prices is $\Pi = 1$.

## B.4 Proof of Proposition 2

**Uniqueness.** In the steady-state, $\dot{D}_t = \dot{\pi}_t = \dot{\delta}_t = 0$. Setting these to zero, dropping the time subscript, and recalling that $\delta > 0$, we obtain the following system of equations:

$$(\delta - \pi)[\lambda - (\sigma - 1)\pi] = \lambda \frac{\sigma(1+\tau)}{\sigma - 1} \left(1 - \frac{\sigma - 1}{\lambda}\pi\right)^{\frac{\sigma}{\sigma-1}} \frac{\delta}{D}, \tag{B.6}$$

$$(\lambda - \sigma\pi)D = \lambda \left(1 - \frac{\sigma - 1}{\lambda}\pi\right)^{\frac{\sigma}{\sigma-1}}, \tag{B.7}$$

$$\delta = \rho + \lambda - (\sigma - 1)\pi. \tag{B.8}$$

Substituting the last two equations into the first one gives

$$(\rho + \lambda - \sigma\pi)[\lambda - (\sigma - 1)\pi] = \frac{\sigma(1+\tau)}{\sigma - 1}(\lambda - \sigma\pi)[\rho + \lambda - (\sigma - 1)\pi],$$

which can be rearranged to yield

$$\frac{\rho(\sigma - 1)}{1 + \sigma\tau}\pi = (\lambda - \sigma\pi)[\rho + \lambda - (\sigma - 1)\pi]. \tag{B.9}$$

Since this is a quadratic equation, there are at most two steady-state values of $\pi$ that solve it. Rather than solving for these roots explicitly, observe that the left-hand side of the equation is a linear increasing function of $\pi$, while the right-hand side has two zeros, one at $\pi = \frac{\lambda}{\sigma}$ and another at $\pi = \frac{\rho+\lambda}{\sigma-1}$. Since $\frac{\lambda}{\sigma} < \frac{\rho+\lambda}{\sigma-1}$, we need to consider three regions:

1. $\pi < \frac{\lambda}{\sigma}$: In this region, the right-hand side of (B.9) is positive. The two sides intersect at a point where both are positive, so the quadratic has at

least one root $\pi \in (0, \frac{\lambda}{\sigma})$.

2. $\frac{\lambda}{\sigma} \leq \pi \leq \frac{\rho+\lambda}{\sigma-1}$: In this region, the right-hand side of (B.9) is negative while the left-hand side is strictly positive. Thus, there cannot be a solution here.
3. $\pi > \frac{\rho+\lambda}{\sigma-1}$: In this region, the right-hand side of (B.9) is positive and grows quadratically from 0, whereas the left-hand side grows linearly from a positive number. The two sides intersect at a point where both are positive, so the quadratic has at least one root $\pi \in (\frac{\rho+\lambda}{\sigma-1}, \infty)$.

Since a quadratic cannot have more than two roots, we conclude that the roots found in the first and third regions above are unique within their regions.

Finally, note that the root $\pi > \frac{\rho+\lambda}{\sigma-1}$ violates the natural bound on inflation implied by sticky prices $\pi < \frac{\lambda}{\sigma-1}$ and thus cannot be a steady state. Therefore, the unique steady state is the one found in the first region, $\pi \in (0, \frac{\lambda}{\sigma})$.

**Comparative Statics.** It follows from the proof of uniqueness above that steady-state inflation $\pi_{ss}(\tau, \sigma)$ solves

$$\frac{\rho(\sigma-1)}{1+\sigma\tau}\pi_{ss}(\tau,\sigma) = (\lambda - \sigma\pi_{ss}(\tau,\sigma))[\rho + \lambda - (\sigma-1)\pi_{ss}(\tau,\sigma)], \tag{B.10}$$

where the value of $\pi_{ss}(\tau, \sigma)$ is the root of this quadratic equation in the interval $(0, \frac{\lambda}{\sigma})$. Given this value, we can then derive steady-state price dispersion $D_{ss}(\tau, \sigma)$ using equation (B.7):

$$D_{ss}(\tau,\sigma) = \frac{\lambda}{\lambda - \sigma\pi_{ss}(\tau,\sigma)}\left(1 - \frac{\sigma-1}{\lambda}\pi_{ss}(\tau,\sigma)\right)^{\frac{\sigma}{\sigma-1}}. \tag{B.11}$$

<u>Part 1.</u> Consider first $\pi_{ss}(\tau, \sigma)$. Differentiating (B.10) with respect to $\tau$ yields

$$\left[\frac{\sigma}{\lambda - \sigma\pi_{ss}(\tau,\sigma)} + \frac{\sigma-1}{\rho+\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)} + \frac{1}{\pi_{ss}(\tau,\sigma)}\right]\frac{\partial}{\partial\tau}\pi_{ss}(\tau,\sigma) = \frac{\sigma}{1+\sigma\tau}.$$

All the terms in the bracket on the left-hand side are positive given $\pi_{ss}(\tau, \sigma) \in (0, \frac{\lambda}{\sigma})$. The right-hand side is also positive by Assumption 1. Thus, $\frac{\partial}{\partial\tau}\pi_{ss}(\tau,\sigma) > 0$ and $\pi_{ss}(\tau, \sigma)$ is strictly increasing in $\tau$.

Consider next $D_{ss}(\tau, \sigma)$. From (B.11), we see that $D_{ss}(\tau, \sigma)$ depends on $\tau$

only through $\pi_{ss}(\tau,\sigma)$. Thus,

$$\begin{aligned}\frac{\partial}{\partial\tau}D_{ss}(\tau,\sigma) &= \frac{\partial}{\partial\pi_{ss}(\tau,\sigma)}D_{ss}(\tau,\sigma)\times\frac{\partial}{\partial\tau}\pi_{ss}(\tau,\sigma)\\ &= \frac{\sigma D_{ss}(\tau,\sigma)\pi_{ss}(\tau,\sigma)}{(\lambda-\sigma\pi_{ss}(\tau,\sigma))[\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)]}\frac{\partial}{\partial\tau}\pi_{ss}(\tau,\sigma).\end{aligned}$$

All the terms involved are positive given $\pi_{ss}(\tau,\sigma)\in(0,\frac{\lambda}{\sigma})$. Thus, $\frac{\partial}{\partial\tau}D_{ss}(\tau,\sigma)>0$ and $D_{ss}(\tau,\sigma)$ is strictly increasing in $\tau$.

Part 2. Consider first $\pi_{ss}(\tau,\sigma)$. Differentiating (B.10) with respect to $\sigma$ yields

$$\begin{aligned}&\left[\frac{\sigma-1}{\rho+\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)}+\frac{\sigma}{\lambda-\sigma\pi_{ss}(\tau,\sigma)}+\frac{1}{\pi_{ss}(\tau,\sigma)}\right]\frac{\partial}{\partial\sigma}\pi_{ss}(\tau,\sigma) \quad \text{(B.12)}\\ &=-\left[\frac{1+\tau}{(\sigma-1)(1+\sigma\tau)}+\frac{\pi_{ss}(\tau,\sigma)}{\lambda-\sigma\pi_{ss}(\tau,\sigma)}+\frac{\pi_{ss}(\tau,\sigma)}{\rho+\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)}\right].\end{aligned}$$

Using $\pi_{ss}(\tau,\sigma)\in(0,\frac{\lambda}{\sigma})$ and Assumption 1, we can conclude that all the terms inside the brackets on both sides are positive. Thus, by the negative sign on the right-hand side, $\frac{\partial}{\partial\sigma}\pi_{ss}(\tau,\sigma)<0$ and $\pi_{ss}(\tau,\sigma)$ is strictly decreasing in $\sigma$.

Consider next $D_{ss}(\tau,\sigma)$. Observe that $D_{ss}(\tau,\sigma)$ depends on $\sigma$ both directly through aggregation, and indirectly through $\pi_{ss}(\tau,\sigma)$ as the central bank's optimal policy changes $\pi_{ss}(\sigma,\tau)$ when $\sigma$ varies. Accordingly, we will investigate the total derivative of $D_{ss}(\tau,\sigma)$ by decomposing it into these direct and indirect effects of $\sigma$:

$$\frac{\partial}{\partial\sigma}D_{ss}(\tau,\sigma)=\frac{\partial}{\partial\sigma}D_{ss}(\tau,\sigma)\big|_{\pi_{ss}(\tau,\sigma)}+\frac{\partial}{\partial\pi_{ss}(\tau,\sigma)}D_{ss}(\tau,\sigma)\big|_{\sigma}\times\frac{\partial}{\partial\sigma}\pi_{ss}(\tau,\sigma). \tag{B.13}$$

To derive the first term on the right-hand side, we use (B.11) to obtain

$$\begin{aligned}\frac{\partial}{\partial\sigma}D_{ss}(\tau,\sigma)\big|_{\pi_{ss}(\tau,\sigma)}=&\frac{D_{ss}(\tau,\sigma)}{(\sigma-1)^2}\left(1-\frac{1}{1-\frac{\sigma-1}{\lambda}\pi_{ss}(\tau,\sigma)}-\log\left(1-\frac{\sigma-1}{\lambda}\pi_{ss}(\tau,\sigma)\right)\right)\\ &+D_{ss}(\tau,\sigma)\left[\frac{\pi_{ss}(\tau,\sigma)}{\lambda-\sigma\pi_{ss}(\tau,\sigma)}-\frac{\pi_{ss}(\tau,\sigma)}{\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)}\right].\end{aligned}$$

As for the partial derivative of $D_{ss}(\tau,\sigma)$ with respect to $\pi_{ss}(\tau,\sigma)$, holding $\sigma$

fixed, we use (B.11) to obtain

$$\frac{\partial}{\partial \pi_{ss}(\tau,\sigma)} D_{ss}(\tau,\sigma)\Big|_{\sigma} = D_{ss}(\tau,\sigma)\left[\frac{\sigma}{\lambda-\sigma\pi_{ss}(\tau,\sigma)} - \frac{\sigma}{\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)}\right].$$

Substituting these into (B.13) yields

$$\begin{aligned}\frac{\partial}{\partial \sigma} D_{ss}(\tau,\sigma) =& \frac{D_{ss}(\tau,\sigma)}{(\sigma-1)^2}\overbrace{\left(1-\frac{1}{1-\frac{\sigma-1}{\lambda}\pi_{ss}(\tau,\sigma)} - \log\left(1-\frac{\sigma-1}{\lambda}\pi_{ss}(\tau,\sigma)\right)\right)}^{\textcircled{1}<0} \\ &+ D_{ss}(\tau,\sigma)\underbrace{\left(\sigma\frac{\partial}{\partial\sigma}\pi_{ss}(\tau,\sigma)+\pi_{ss}(\tau,\sigma)\right)}_{\textcircled{2}}\underbrace{\left[\frac{1}{\lambda-\sigma\pi_{ss}(\tau,\sigma)} - \frac{1}{\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)}\right]}_{\textcircled{3}>0}.\end{aligned}$$

It is straightforward to show that ① is strictly negative for $\pi_{ss}(\tau,\sigma) \in (0,\frac{\lambda}{\sigma})$.[1] Moreover, ③ is strictly positive for $\pi_{ss}(\tau,\sigma) \in (0,\frac{\lambda}{\sigma})$. Thus, a sufficient condition for $\frac{\partial}{\partial\sigma}D_{ss}(\tau,\sigma)$ to be strictly negative is that ② is negative. We next show that this holds under $\tau < \bar{\tau}(\sigma)$. Using (B.12), we have

$$\begin{aligned}\textcircled{2} &= -\frac{\frac{\sigma(1+\tau)}{(\sigma-1)(1+\sigma\tau)} + \frac{\sigma\pi_{ss}(\tau,\sigma)}{\lambda-\sigma\pi_{ss}(\tau,\sigma)} + \frac{\sigma\pi_{ss}(\tau,\sigma)}{\rho+\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)}}{\frac{\sigma-1}{\rho+\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)} + \frac{\sigma}{\lambda-\sigma\pi_{ss}(\tau,\sigma)} + \frac{1}{\pi_{ss}(\tau,\sigma)}} + \pi_{ss}(\tau,\sigma) \\ &= \frac{-\frac{\sigma(1+\tau)}{(\sigma-1)(1+\sigma\tau)} - \frac{\pi_{ss}(\tau,\sigma)}{\rho+\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)} + 1}{\frac{\sigma-1}{\rho+\lambda-(\sigma-1)\pi_{ss}(\tau,\sigma)} + \frac{\sigma}{\lambda-\sigma\pi_{ss}(\tau,\sigma)} + \frac{1}{\pi_{ss}(\tau,\sigma)}}.\end{aligned}$$

The denominator is positive for $\pi_{ss}(\tau,\sigma)$ in $(0,\frac{\lambda}{\sigma})$. We show that the numerator is negative for $\tau < \bar{\tau}(\sigma)$. To see this, note that the fraction involving $\pi_{ss}(\tau,\sigma)$

---

[1] To see this, note that $\pi_{ss}(\tau,\sigma) \in (0,\frac{\lambda}{\sigma})$ implies that $1-\frac{\sigma-1}{\lambda}\pi_{ss}(\tau,\sigma) \in (\frac{1}{\sigma},1)$. Moreover, note that the function $f(x) \equiv 1-1/x-\log(x)$ is strictly increasing in $x \in (0,1)$ (as $f'(x) = 1/x^2 - 1/x > 0, x \in (0,1)$), so that $\forall x \in (\frac{1}{\sigma},1) : f(x) < f(1) = 0$.

is negative, so it is sufficient to show that

$$-\frac{\sigma(1+\tau)}{(\sigma-1)(1+\sigma\tau)}+1<0 \iff (\sigma-2)\sigma\tau<1.$$

Now note that under $\tau<\bar{\tau}(\sigma)$ and Assumption 1, we have

$$\begin{aligned}
&1<\sigma<2 \implies (\sigma-2)\sigma\tau<(2-\sigma)<1,\\
&\sigma\geq 2 \implies (\sigma-2)\sigma\tau<(\sigma-2)\sigma\bar{\tau}(\sigma)=(\sigma-2)\sigma\frac{1}{\sigma(\sigma-2)}=1.
\end{aligned}$$

Hence, given $\tau<\bar{\tau}(\sigma)$ and $\sigma>1$, we obtain ② $<0$. It follows that $\frac{\partial}{\partial\sigma}D_{ss}(\tau,\sigma)<0$ and $D_{ss}(\tau,\sigma)$ is strictly decreasing in $\sigma$ for all $\tau<\bar{\tau}(\sigma)$.

## B.5 Proof of Proposition 3

To prove this proposition, we will rely on the Stable Manifold and the Hartman-Grobman theorems (Perko, 2001, pages 107 and 120, respectively). These two theorems relate the dynamics of a non-linear dynamical system to its local linearized dynamics around a fixed point (in our case, the unique steady state). To make use of their predictions, we rewrite our dynamical system involving the variables $\pi_t, D_t$ and $\delta_t$ in the following form. Let $\Omega_t\equiv(\pi_t,D_t,\delta_t)$. Then the non-linear dynamical system implied by the model can be characterized by a function $f:\mathbb{R}^3\to\mathbb{R}^3$ defined as

$$\dot{\Omega}_t=f(\Omega_t)\equiv\begin{bmatrix}-\lambda\tfrac{\sigma(1+\tau)}{\sigma-1}\left(1-\tfrac{\sigma-1}{\lambda}\pi_t\right)^{\frac{\sigma}{\sigma-1}}\tfrac{\delta_t}{D_t}+(\delta_t-\pi_t)[\lambda-(\sigma-1)\pi_t]\\ \lambda\left(1-\tfrac{\sigma-1}{\lambda}\pi_t\right)^{\frac{\sigma}{\sigma-1}}+(\sigma\pi_t-\lambda)D_t\\ \delta_t^2+[(\sigma-1)\pi_t-(\rho+\lambda)]\delta_t\end{bmatrix},$$

where the unique steady state that we characterized is a fixed point of this system.

Note that $f(\cdot)$ is a smooth function; importantly, it is continuously differentiable, which implies that the flows of the system are also continuous. In order to understand the dynamics of the system and how the transition to a new steady state happens, we need to first characterize the nature of the unique steady state for the above system. To do this, we can apply the

Hartman-Grobman theorem, which states that if the eigenvalues of the Jacobian of the function $f$ evaluated at the fixed point have non-zero real parts, then there exists a neighborhood $N$ around the fixed point of the system where the flows of the non-linear system are topologically conjugate to the flows of the linearized system. We will apply this theorem in the following way. First, we will show that the fixed point is a saddle point of the linearized system. Then verifying the assumptions of the Hartman-Grobman theorem, we will conclude from topological conjugacy that the steady state is also a saddle point of the non-linear system.

To show that the steady state is a saddle point of the linearized system, we first need to compute the Jacobian of $f$ at the steady state. Letting $\Omega_{ss} = (\pi_{ss}, D_{ss}, \delta_{ss})$ denote the steady state under a certain set of parameters, note that

$$0 = \dot{\Omega}_{ss} = f(\Omega_{ss}) \implies \begin{cases} \frac{\rho(\sigma-1)}{1+\sigma\tau}\pi_{ss} = (\lambda - \sigma\pi_{ss})[\rho + \lambda - (\sigma-1)\pi_{ss}] \\ D_{ss} = \frac{\lambda}{\lambda - \sigma\pi_{ss}}\left(1 - \frac{\sigma-1}{\lambda}\pi_{ss}\right)^{\frac{\sigma}{\sigma-1}} \\ \delta_{ss} = \rho + \lambda - (\sigma-1)\pi_{ss} \end{cases}, \tag{B.14}$$

and, letting $\mathbf{D}f$ denote the Jacobian of $f$ evaluated at $\Omega_{ss}$, we have

$$\mathbf{D}f = \begin{bmatrix} \frac{\partial}{\partial\pi}f_1 & \frac{\partial}{\partial D}f_1 & \frac{\partial}{\partial\delta}f_1 \\ \frac{\partial}{\partial\pi}f_2 & \frac{\partial}{\partial D}f_2 & \frac{\partial}{\partial\delta}f_2 \\ \frac{\partial}{\partial\pi}f_3 & \frac{\partial}{\partial D}f_3 & \frac{\partial}{\partial\delta}f_3 \end{bmatrix},$$

where all the partial derivatives are evaluated at $\Omega_{ss}$ and are given by

$$\begin{aligned} \frac{\partial}{\partial\pi}f_1 &= \frac{\sigma^2(1+\tau)}{\sigma-1}\left(1 - \frac{\sigma-1}{\lambda}\pi_{ss}\right)^{\frac{1}{\sigma-1}}\frac{\delta_{ss}}{D_{ss}} - [\lambda - (\sigma-1)\pi_{ss}] - (\sigma-1)(\delta_{ss} - \pi_{ss}) \\ &= \rho - \pi_{ss}, \qquad \text{(using equation (B.14))} \\ \frac{\partial}{\partial D}f_1 &= \lambda\frac{\sigma(1+\tau)}{\sigma-1}\left(1 - \frac{\sigma-1}{\lambda}\pi_{ss}\right)^{\frac{\sigma}{\sigma-1}}\frac{\delta_{ss}}{D_{ss}^2} = \frac{\rho\pi_{ss} + (\lambda - \sigma\pi_{ss})\delta_{ss}}{D_{ss}}, \\ \frac{\partial}{\partial\delta}f_1 &= -\frac{1+\sigma\tau}{\sigma-1}(\lambda - \sigma\pi_{ss}) + \pi_{ss} = \frac{\pi_{ss}}{\delta_{ss}}[\lambda - (\sigma-1)\pi_{ss}], \\ \frac{\partial}{\partial\pi}f_2 &= -\sigma\left(1 - \frac{\sigma-1}{\lambda}\pi_{ss}\right)^{\frac{1}{\sigma-1}} + \sigma D_{ss} = \sigma D_{ss}\frac{\pi_{ss}}{\lambda - (\sigma-1)\pi_{ss}}, \end{aligned}$$

$$\begin{aligned}
&\tfrac{\partial}{\partial D} f_2 = \sigma\pi_{ss} - \lambda,\\
&\tfrac{\partial}{\partial \delta} f_2 = 0,\\
&\tfrac{\partial}{\partial \pi} f_3 = (\sigma - 1)\delta_{ss},\\
&\tfrac{\partial}{\partial D} f_3 = 0,\\
&\tfrac{\partial}{\partial \delta} f_3 = 2\delta_{ss} + (\sigma - 1)\pi_{ss} - (\rho + \lambda) = \delta_{ss}.
\end{aligned}$$

To show that the Hartman-Grobman theorem applies, we need to show that $\Omega_{ss}$ is a hyperbolic fixed point—i.e., all the eigenvalues of $\mathbf{D}f$ have non-zero real parts. To calculate the eigenvalues of $\mathbf{D}f$, we need to compute the roots of its characteristic polynomial:

$$\det\left(\mathbf{D}f - \eta\mathbf{I}\right) = 0,$$

where any $\eta$ that solves this polynomial is an eigenvalue of the Jacobian. The characteristic polynomial is given by

$$\begin{aligned}
&\det\left(\mathbf{D}f - \eta\mathbf{I}\right) =\\
&(\tfrac{\partial}{\partial \pi} f_1 - \eta)(\tfrac{\partial}{\partial D} f_2 - \eta)(\tfrac{\partial}{\partial \delta} f_3 - \eta) - \tfrac{\partial}{\partial D} f_1 \tfrac{\partial}{\partial \pi} f_2 \left(\tfrac{\partial}{\partial \delta} f_3 - \eta\right) - \tfrac{\partial}{\partial \delta} f_1 \tfrac{\partial}{\partial \pi} f_3 \left(\tfrac{\partial}{\partial D} f_2 - \eta\right),
\end{aligned}$$

where we have used $\frac{\partial}{\partial \delta} f_2 = \frac{\partial}{\partial D} f_3 = 0$. Plugging in the derived values for other partial derivatives, we obtain the following cubic polynomial:

$$\begin{aligned}
&\det\left(\mathbf{D}f - \eta\mathbf{I}\right)\\
&= (\rho - \pi_{ss} - \eta)(\sigma\pi_{ss} - \lambda - \eta)(\delta_{ss} - \eta) - \sigma\pi_{ss}(\rho + \lambda - \sigma\pi_{ss})(\delta_{ss} - \eta)\\
&- (\sigma - 1)\pi_{ss}[\lambda - (\sigma - 1)\pi_{ss}](\sigma\pi_{ss} - \lambda - \eta).
\end{aligned}$$

We now need to compute the roots of this cubic equation. One could use the general formula for roots of a cubic but that requires some tedious algebra. An easier path is to guess and verify that one of the roots is $\rho$.[2] To verify this,

---

[2] There is an economic intuition for this guess. We know that at $\rho = 0$, the Phillips curve of the economy is fully vertical, which implies that $\rho = 0$ is a bifurcation point for the system. So the behavior of the system should switch at $\rho = 0$, making it reasonable to guess that $\rho$ is one of its eigenvalues.

observe that at $\eta = \rho$,

$$\det\left(\mathbf{D}f - \rho\mathbf{I}\right) = (\sigma-1)\pi_{ss}(\sigma\pi_{ss} - \lambda - \rho)(\delta_{ss} - \rho) - (\sigma-1)\pi_{ss}(\delta_{ss} - \rho)(\sigma\pi_{ss} - \lambda - \rho) = 0.$$

Thus, the characteristic polynomial is divisible by $\rho - \eta$. Using this fact, we can factorize the characteristic polynomial as

$$\det(\mathbf{D}f - \eta\mathbf{I}) = (\rho - \eta)\left[\eta^2 - \rho\eta - (\rho+\lambda)\lambda + \sigma(\sigma-1)\pi_{ss}^2\right],$$

where the rest of the eigenvalues are the roots of the quadratic equation $\eta^2 - \rho\eta - (\rho+\lambda)\lambda + \sigma(\sigma-1)\pi_{ss}^2 = 0$. Therefore, the eigenvalues of the Jacobian at the steady state are

$$\eta = \begin{cases} \eta_1 \equiv \rho \\ \eta_2 \equiv \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}+\lambda\right)^2 - \sigma(\sigma-1)\pi_{ss}^2} \\ \eta_3 \equiv \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}+\lambda\right)^2 - \sigma(\sigma-1)\pi_{ss}^2} \end{cases}.$$

We can make the following observations about these eigenvalues. First, all of them are real. To see this, we just need to confirm that the term inside the square root is always positive. This follows from $\rho > 0$ and the fact that $\pi_{ss} \in (0, \lambda/\sigma)$ under Assumption 1:

$$\left(\frac{\rho}{2}+\lambda\right)^2 - \sigma(\sigma-1)\pi_{ss}^2 > \lambda^2 - \sigma^2\pi_{ss}^2 = (\lambda - \sigma\pi_{ss})(\lambda + \sigma\pi_{ss}) > 0.$$

A second observation is that the first two eigenvalues are strictly positive (which is straightforward to confirm from the observation above) and the third one is negative. To verify the latter, note that

$$\begin{aligned} \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}+\lambda\right)^2 - \sigma(\sigma-1)\pi_{ss}^2} < 0 &\iff \left(\frac{\rho}{2}\right)^2 < \left(\frac{\rho}{2}+\lambda\right)^2 - \sigma(\sigma-1)\pi_{ss}^2 \\ &\iff 0 < \lambda^2 + \rho\lambda - \sigma(\sigma-1)\pi_{ss}^2, \end{aligned}$$

and the last inequality holds since

$$\lambda^2 + \rho\lambda - \sigma(\sigma - 1)\pi_{ss}^2 > \lambda^2 - \sigma^2\pi_{ss}^2 = (\lambda - \sigma\pi_{ss})(\lambda + \sigma\pi_{ss}) > 0.$$

Therefore, the Jacobian $\mathbf{D}f$ has two strictly positive eigenvalues and one strictly negative eigenvalue. This implies that the fixed point $\Omega_{ss}$ is a hyperbolic fixed point and is a saddle point for the linearized dynamical system. Thus, the Hartman-Grobman theorem applies and we can conclude that the fixed point is also a saddle point for the non-linear system.

Since all eigenvalues are distinct, the three eigenvectors associated with them are linearly independent and span $\mathbb{R}^3$. Thus, these eigenvalues imply that the dynamics of the linearized system are stable along the eigenspace spanned by the negative eigenvalue (which is one-dimensional as we show below) and unstable along the eigenspace associated with the two positive eigenvalues. Now, to study the convergence of the non-linear dynamics, we appeal to the Stable Manifold Theorem. When applied to our setting, this theorem states that in an open neighborhood around the fixed point $\Omega_{ss}$ where the function $f$ is continuously differentiable (which is the case for our system), there exists a one-dimensional differentiable manifold $S$ tangent to the stable subspace of the linear system such that for all $t \geq 0$, $\Omega \in S$,

$$\lim_{t \to \infty} \phi_t(\Omega) = \Omega_{ss},$$

where $\phi_t(\Omega)$ denotes the flow of the non-linear system starting from $\Omega$ at time $t = 0$ (i.e., $\phi_0(\Omega) = \Omega$) and evolves according to the non-linear dynamics. Therefore, we have established that in an open neighborhood $N$ of the fixed point $\Omega_{ss}$, the non-linear dynamics converge to the fixed point $\Omega_{ss}$ along a stable manifold $S$ that is one-dimensional and tangent to the one-dimensional eigenspace of the linearized system at the fixed point. It then suffices to characterize the direction of convergence along the stable eigenspace of the linearized system. To this end, consider the linear dynamics around the fixed point $\Omega_{ss}$:

$$\dot{\Omega}_t = \mathbf{D}f\left(\Omega_t - \Omega_{ss}\right).$$

Let $\Lambda_t(\Omega)$ denote the flow of this linearized system starting from some $\Omega \in \mathbb{R}^3$. Since the eigenvectors of $\mathbf{D}f$ are linearly independent, we can write this flow as

$$\Lambda_t(\Omega) = \alpha_{1,\Omega}(t)\mathbf{v}_1 + \alpha_{2,\Omega}(t)\mathbf{v}_2 + \alpha_{3,\Omega}(t)\mathbf{v}_3,$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of $\mathbf{D}f$ that correspond to eigenvalues $\eta_1$, $\eta_2$, and $\eta_3$ respectively. Furthermore, since $\Lambda_0(\Omega) = \Omega$, $\alpha_{i,\Omega}(0)$ for $i = 1, 2, 3$ are given by the coordinates of $\Omega$ in the eigenvector basis of $\mathbf{D}f$. Also, note that since $\Lambda_t(\Omega_{ss}) = \Omega_{ss}$, $\alpha_{i,\Omega_{ss}}(t)$ is constant over time, and we use $\bar{\alpha}_i$ to refer to it. Plugging this decomposition into the linearized system yields

$$\sum_{i=1}^{3} \dot{\alpha}_{i,\Omega}(t)\mathbf{v}_i = \mathbf{D}f \sum_{i=1}^{3} (\alpha_{i,\Omega}(t) - \bar{\alpha}_i)\mathbf{v}_i = \sum_{i=1}^{3} \eta_i(\alpha_{i,\Omega}(t) - \bar{\alpha}_i)\mathbf{v}_i.$$

Therefore, for $i = 1, 2, 3$,

$$\dot{\alpha}_{i,\Omega}(t) = \eta_i(\alpha_{i,\Omega}(t) - \bar{\alpha}_i) \implies \alpha_{i,\Omega}(t) - \bar{\alpha}_i = (\alpha_{i,\Omega}(0) - \bar{\alpha}_i)e^{\eta_i t},$$

which implies

$$\Lambda_t(\Omega) = \Omega_{ss} + \sum_{i=1}^{3} (\alpha_{i,\Omega}(0) - \bar{\alpha}_i)e^{\eta_i t}\mathbf{v}_i.$$

Note that since the $\mathbf{v}_i$'s are linearly independent, $\Lambda_t(\Omega)$ is convergent if and only if $\alpha_{1,\Omega}(0) - \bar{\alpha}_1 = \alpha_{2,\Omega}(0) - \bar{\alpha}_2 = 0$ (since $\eta_1 > 0$ and $\eta_2 > 0$). This identifies the stable eigenspace of the linearized system as the span of $\mathbf{v}_3$ shifted to cross $\Omega_{ss}$; that is,

$$\begin{aligned} \lim_{t\to\infty} \Lambda_t(\Omega) = \Omega_{ss} &\iff \Omega \in \Omega_{ss} + \operatorname{span}(\mathbf{v}_3) \\ &\iff \Lambda_t(\Omega) - \Omega_{ss} = k e^{\eta_3 t}\mathbf{v}_3 \quad \text{for some } k \in \mathbb{R}. \end{aligned}$$

Given that $\mathbf{v}_3 = (v_{3,1}, v_{3,2}, v_{3,3})$ is an eigenvector associated with the negative eigenvalue $\eta_3$, and normalizing $v_{3,1} = 1$, we have

$$\frac{\partial}{\partial \pi} f_2 + \left(\frac{\partial}{\partial D} f_2 - \eta_3\right) v_{3,2} = 0 \implies v_{3,2} = \frac{\frac{\partial}{\partial \pi} f_2}{\eta_3 - \frac{\partial}{\partial D} f_2},$$
$$\frac{\partial}{\partial \pi} f_3 + \left(\frac{\partial}{\partial \delta} f_3 - \eta_3\right) v_{3,3} = 0 \implies v_{3,3} = \frac{\frac{\partial}{\partial \pi} f_3}{\eta_3 - \frac{\partial}{\partial \delta} f_3}.$$

For a given $k \in \mathbb{R}$, let $\Lambda_t(\Omega) - \Omega_{ss} = (\pi_t^L - \pi_{ss}, D_t^L - D_{ss}, \delta_t^L - \delta_{ss})$ denote the flow of the linearized system towards the steady state. We show that along the transition path, if $D_t^L$ converges to $D_{ss}$ from below, then $\pi_t^L$ converges to $\pi_{ss}$ from above and vice versa. To see this, note that

$$\frac{\pi_t^L - \pi_{ss}}{D_t^L - D_{ss}} = \frac{v_{3,1}}{v_{3,2}} = \frac{\eta_3 - \frac{\partial}{\partial D} f_2}{\frac{\partial}{\partial \pi} f_2} = \frac{\eta_3 - \sigma \pi_{ss} + \lambda}{\sigma D_{ss} \pi_{ss}} [\lambda - (\sigma - 1)\pi_{ss}].$$

In the expression above, $\sigma D_{ss}\pi_{ss} > 0$ and $\lambda - (\sigma - 1)\pi_{ss} > 0$ as $\pi_{ss} \in (0, \lambda/\sigma)$. Thus, to conclude that the ratio has a negative sign, we need to show that $\eta_3 - \sigma\pi_{ss} + \lambda < 0$. To see that this is indeed the case, note that

$$\begin{aligned} \eta_3 + \lambda - \sigma\pi_{ss} < 0 &\iff \lambda - \sigma\pi_{ss} + \frac{\rho}{2} < \sqrt{\left(\frac{\rho}{2} + \lambda\right)^2 - \sigma(\sigma - 1)\pi_{ss}^2} \\ &\iff \left(\frac{\rho}{2} + \lambda\right)^2 + \sigma^2\pi_{ss}^2 - (2\lambda + \rho)\sigma\pi_{ss} < \left(\frac{\rho}{2} + \lambda\right)^2 - \sigma(\sigma - 1)\pi_{ss}^2 \\ &\iff 2\sigma\pi_{ss} - 2\lambda - \rho - \pi_{ss} < 0, \end{aligned}$$

and the last inequality holds since $\pi_{ss} \in (0, \lambda/\sigma)$. Hence, linearized dynamics are such that

$$\kappa \equiv \frac{\pi_t^L - \pi_{ss}}{D_t^L - D_{ss}} < 0.$$

Finally, let $\phi_t(\Omega) - \Omega_{ss} = (\pi_t - \pi_{ss}, D_t - D_{ss}, \delta_t - \delta_{ss})$ denote the flow of the non-linear system starting from an $\Omega$ on the one-dimensional stable manifold so that $\lim_{t\to\infty} \phi_t(\Omega) = \Omega_{ss}$. Since the stable manifold is tangent to the stable subspace of the linearized system, for sufficiently small $\varepsilon > 0$ such that $\varepsilon + \kappa < 0$, there exists $\bar{t} \geq 0$ such that for all $t > \bar{t}$,

$$\frac{\pi_t - \pi_{ss}}{D_t - D_{ss}} \in (\kappa - \varepsilon, \kappa + \varepsilon) \implies \frac{\pi_t - \pi_{ss}}{D_t - D_{ss}} < 0.$$

Hence, there exists $\bar{t} \geq 0$ such that, after time $\bar{t}$, if $D_t$ of the non-linear system converges to $D_{ss}$ from below, then $\pi_t$ of the non-linear system converges to $\pi_{ss}$ from above and vice versa.

To conclude the proof of Proposition 3, consider a change in the parameters of the model that leads to an increase in $D_{ss}$, as is the case in both parts 1 and 2 of the proposition. First note that since our non-linear system is continuously differentiable, $D_t$ (along with $\pi_t$ and $\delta_t$) have continuous paths along the transition. Moreover, since $D_t$ is backward-looking, it is also continuous at $t = 0$ (i.e., $\lim_{t \to 0} D_t = D_0$, unlike $\pi_t$ and $\delta_t$ which jump to the stable manifold to accommodate convergence to the steady state). Thus, it has to be that conditional on converging to the new steady state, $D_t$ is a continuous function of time with $D_0 < D_{ss} = \lim_{t \to \infty} D_t$.

If along the transition path $D_t$ never crosses $D_{ss}$, then $D_t - D_{ss} < 0$ for all $t$. This means that there exists $\bar{t} \geq 0$ such that $\pi_t - \pi_{ss} > 0$ for all $t > \bar{t}$.

Suppose instead that $D_t$ crosses $D_{ss}$ along the transition path to possibly converge to $D_{ss}$ from above. If this was possible, then there would be two paths for convergence starting from $D_{ss}$: one that increases and then converges back to $D_{ss}$ from above, and another that starts at $D_{ss}$ and stays at $D_{ss}$ forever. However, in this case, the equilibrium cannot be Markov. Therefore, the only possibility of convergence in a Markov equilibrium is that $D_t$ converges to $D_{ss}$ from below, and thus $\pi_t$ converges to $\pi_{ss}$ from above.

# C Derivations for the Limit Setting with $\sigma \to 1$

In subsection IVC, we considered a special case of our model that takes a limit value for the elasticity of substitution, $\sigma \to 1$, with the labor wedge $\tau$ adjusting so as to keep monopoly power $\gamma \equiv \frac{\sigma(1+\tau)}{\sigma-1}$ constant. Below, we provide the derivations for this limit setting.

Recall from equations (30)-(32) that in the continuous-time limit of our model, the dynamics of the system are given by

$$\dot{D}_t = \lambda \left(1 - \frac{\sigma - 1}{\lambda}\pi_t\right)^{\frac{\sigma}{\sigma-1}} + (\sigma\pi_t - \lambda)D_t, \tag{C.1}$$

$$\dot{\pi}_t = -\lambda\frac{\sigma(1+\tau)}{\sigma - 1}\left(1 - \frac{\sigma - 1}{\lambda}\pi_t\right)^{\frac{\sigma}{\sigma-1}}\frac{\delta_t}{D_t} + (\delta_t - \pi_t)[\lambda - (\sigma - 1)\pi_t], \tag{C.2}$$

$$\dot{\delta}_t = \delta_t^2 + [(\sigma - 1)\pi_t - (\rho + \lambda)]\delta_t. \tag{C.3}$$

Taking the limit of this system as $\sigma \to 1$ while keeping $\gamma$ fixed, we arrive at

$$\dot{D}_t = \lambda e^{-\frac{\pi_t}{\lambda}} + (\pi_t - \lambda)D_t, \tag{C.4}$$

$$\dot{\pi}_t = -\lambda\gamma e^{-\frac{\pi_t}{\lambda}}\frac{\delta_t}{D_t} + (\delta_t - \pi_t)\lambda, \tag{C.5}$$

$$\dot{\delta}_t = \delta_t^2 - (\rho + \lambda)\delta_t. \tag{C.6}$$

Importantly, we observe from equation (C.6) that, in this limit, the dynamics of $\delta_t$ are decoupled from the rest of the system and given by a differential equation that involves only $\delta_t$ itself. The dynamics of $\delta_t$ for an arbitrary flow are thus given by the general solution to this differential equation:

$$\delta_t = \frac{\rho + \lambda}{1 + Ke^{(\rho+\lambda)t}}, \tag{C.7}$$

where $K$ is a constant that indexes the flow of the system. We note that the only value of $K$ that is consistent with converging to the steady state of the system ($\delta_{ss} = \rho + \lambda$) is $K = 0$, implying that $\delta_t = \rho + \lambda$ along the whole transition path. That is, $\delta_t$ jumps to its steady-state value immediately at $t = 0$ and stays there until the rest of the system converges. Plugging this into equations (C.4) and (C.5) yields

$$\dot{D}_t = \lambda e^{-\frac{\pi_t}{\lambda}} + (\pi_t - \lambda)\, D_t,$$

$$\dot{\pi}_t = -\lambda\,(\rho + \lambda)\,\gamma e^{-\frac{\pi_t}{\lambda}}\frac{1}{D_t} + (\rho + \lambda - \pi_t)\,\lambda.$$

This is a system of two non-linear differential equations, but one that can be solved in closed form conditional on converging to the steady state. To see this, consider the following change of variables: let $d_t \equiv \log D_t$ and $x_t \equiv \frac{\pi_t}{\lambda} + d_t$. We then have

$$\begin{aligned} \dot{d}_t &= \lambda e^{-x_t} + (\pi_t - \lambda), \\ \frac{\dot{\pi}_t}{\lambda} &= -(\rho + \lambda)\gamma e^{-x_t} + (\rho + \lambda - \pi_t). \end{aligned}$$

Summing these two equations, we obtain

$$\dot{x}_t = -\left[\lambda(\gamma - 1) + \rho\gamma\right] e^{-x_t} + \rho,$$

where $\lambda(\gamma - 1) + \rho\gamma > 0$ under Assumption 1.[3] This is a univariate differential equation in terms of $x_t$ that can be rearranged as

$$\begin{aligned} (\dot{x}_t - \rho)e^{x_t} &= -\left[\lambda(\gamma - 1) + \rho\gamma\right] \\ \iff \frac{d}{dt}\left(e^{x_t - \rho t}\right) &= -\left[\lambda(\gamma - 1) + \rho\gamma\right] e^{-\rho t}, \end{aligned}$$

yielding the general solution

$$e^{x_t} = \left[\gamma + (\gamma - 1)\frac{\lambda}{\rho}\right] + K' e^{\rho t}$$

for some constant $K'$. We note that conditional on converging to the steady state, where $\dot{\pi}_t = \dot{d}_t = 0$, we must have $\dot{x}_t = 0$. Thus, the only flow for $x_t$ that is consistent with converging to a steady state is when $K' = 0$, giving us the solution

$$x_t = \log\left(\gamma + (\gamma - 1)\frac{\lambda}{\rho}\right), \forall t \geq 0$$

along the transition path. Substituting this into the original system (and using the definition of $x_t$), we observe

$$\dot{d}_t = -\lambda\left[d_t + \frac{(\gamma - 1)(\rho + \lambda)}{\rho\gamma + (\gamma - 1)\lambda} - \log\left(\gamma + (\gamma - 1)\frac{\lambda}{\rho}\right)\right],$$

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[3] Recall that for a given $\sigma$, Assumption 1 requires that $\tau > -1/\sigma$, which can be rewritten as $\gamma = \sigma(1 + \tau)/(\sigma - 1) > 1$. Since we took the limit of $\sigma \to 1$ holding $\gamma$ fixed, this statement of Assumption 1 remains invariable to $\sigma$.

$$\dot{\pi}_t = -\lambda \left[ \pi_t - \frac{(\rho+\lambda)(\gamma-1)\lambda}{\rho\gamma + (\gamma-1)\lambda} \right].$$

These equations are decoupled linear differential equations with steady-state values

$$\begin{aligned}
\pi_{ss} &= \frac{(\rho+\lambda)(\gamma-1)\lambda}{\rho\gamma + (\gamma-1)\lambda}, \\
d_{ss} &= -\frac{(\gamma-1)(\rho+\lambda)}{\rho\gamma + (\gamma-1)\lambda} + \log\left( \gamma + (\gamma-1)\frac{\lambda}{\rho} \right),
\end{aligned}$$

and the following solutions:

$$\begin{aligned}
d_t - d_{ss} &= (d_0 - d_{ss})e^{-\lambda t}, \\
\pi_t - \pi_{ss} &= (\pi_0 - \pi_{ss})e^{-\lambda t},
\end{aligned}$$

where $\pi_0$ is given by

$$\pi_0 = \lambda(\bar{x} - d_0) = \lambda \log\left( \gamma + (\gamma-1)\frac{\lambda}{\rho} \right) - \lambda d_0.$$

Thus, we obtain the following exact solution for the non-linear system under no commitment in the limit setting with $\sigma \to 1$:

$$\begin{aligned}
\log D_t &= \log D_{ss} - \log\left( \frac{D_{ss}}{D_0} \right) e^{-\lambda t}, \\
\pi_t &= \pi_{ss} + \lambda \log\left( \frac{D_{ss}}{D_0} \right) e^{-\lambda t}.
\end{aligned}$$

This solution implies that the saddle path has a negative slope and has the following exact form for all possible values of $D$:

$$\pi(D) - \pi_{ss} = -\lambda \left( \log D - \log D_{ss} \right).$$

# References

Perko, L. (2001): *Differential Equations and Dynamical Systems*. Springer New York, NY.