Supplemental Appendix for "The Effect of Omitted Variables on the Sign of Regression Coefficients"

Matthew A. Masten* Alexandre Poirier†

June 1, 2026

This document contains various supplementary appendices. Appendix A contains our survey of regression sensitivity analysis in practice. Appendix B has further details on our main empirical application. Appendix C describes three additional detailed empirical applications. Appendix D contains additional results for our two meta-analyses. Appendix E describes our formal identification results. Appendix F discusses some common pitfalls with current empirical practice. Appendix G discusses estimation and inference. Appendix H gives all proofs. Appendix I lists all papers included in our empirical survey of Appendix A.

We will often refer to Assumption 1 (or 2, 3, ...) as A1 for brevity in these supplementary appendices.

A A Survey of Regression Sensitivity Analysis in Practice

To understand how empirical researchers are using Oster (2019) in practice, we performed a survey of the empirical economics literature. We constructed the sample by looking at all papers published in the three-year period from 2019 to 2021, which cite Oster (2019), and which were published in either *AER*, *JPE*, *QJE*, *REStud*, or *ECMA*. This yields 35 papers. One of these is an econometric theory paper which cited Oster (2019) as a related method only, and so we drop it from the sample. Three of these papers used the methods in Oster (2019), but only in working paper drafts, not in the final published version. We drop these three from the sample as well. This leaves 31 papers remaining. These papers are listed in Appendix I.

This sample size is similar to the survey of Blandhol, Bonney, Mogstad, and Torgovitsky (2025), who found 112 papers that use 2SLS in the same five journals, over the 19-year span from January 2000 to October 2018. In fact, since we are only looking at 3 years of data, Oster's (2019) method is more popular than 2SLS, on average, since 112 papers over 19 years is only about 18 papers every three years.[1] This is perhaps not surprising since Oster's results can be applied to regressions based on many different identification strategies, including unconfoundedness, instrumental variables, and difference-in-differences.

There are three different ways in which researchers apply the results of Oster (2019): (1) Report the explain away breakdown point, (2) Report a bias adjusted estimand, and (3) Informally discuss coefficient stability and cite Oster (2019) as justification. Table S1 summarizes the distribution of these three methods across the 31 papers in our dataset. About 80% of the papers use and report the results of one of the two formal sensitivity analysis methods. Among these, roughly half only report breakdown points, and about half report bias adjusted estimands. Only a few papers do both. The remaining roughly 20% of papers only informally discuss coefficient stability and refer to Oster (2019) in their discussion. Some of those discussions suggest that they may have used one of the formal methods, but if so they did not report the findings.

Consider the papers that report the explain away breakdown point. Very few also vary R^2_{long}. Most use either $1.3\widehat{R}^2_{\text{med}}$ or 1. One paper also used $2\widehat{R}^2_{\text{med}}$. One paper presented the function $\widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$ as R^2_{long} varied from $\widehat{R}^2_{\text{med}}$ to 1. When discussing their results, researchers used a variety of phrases to

*Department of Economics, Duke University, matt.masten@duke.edu

†Department of Economics, Georgetown University, alexandre.poirier@georgetown.edu

[1] Blandhol et al. (2025) gathered their data from Web of Science, whereas we use Google Scholar. Using our search criteria, Web of Science finds 28 papers. So this difference in counts is not due to differences in the original data sources.

Table S1: How Empirical Researchers Use Oster (2019).

	Proportion	Count
Total	100%	31
1. Compute Explain Away Breakdown Points	45%	14
And vary R^2_{long}	21% (conditional)	3
2. Compute Bias Adjusted Estimates	42%	13
And vary δ	15% (conditional)	2
And vary R^2_{long}	46% (conditional)	6
3. Discuss Informally Only	19%	6
Both 1 and 2	6%	2

Note: Sample contains 31 papers published in *AER*, *QJE*, *JPE*, *ECMA*, or *REStud* between 2019–2021 and which use Oster (2019).

describe their estimates of $\delta^{\text{bp,explain away}}(R^2_{\text{long}})$: It is the amount of selection on unobservables relative to observables needed...

> "to fully explain away the coefficient on the explanatory variable of interest" (Arbatlı, Ashraf, Galor, and Klemp 2020, *ECMA*, page 747)
>
> "to drive the results observed" (Valencia Caicedo 2019, *QJE*, online appendix page 6)
>
> "to reverse the sign of the results" (Hoffman and Tadelis 2021, *JPE*, page 272)

to give a few examples from our survey. As we have shown in this paper, the last quote is incorrect, because the sign change and explain away breakdown points are not equivalent. The first quote correctly describes it as an explain away breakdown point. The middle quote's interpretation depends on what it means "to drive the results"; other authors often use similar language like making estimates "spurious" or "go away." If these statements refer to the conclusion that the effect is not exactly zero (but might be positive or negative), then these descriptions are correct. If, however, these statements refer to the sign of the main finding, then these descriptions are incorrect. Our paper clarifies that empirical researchers must specify whether they are interested in robustness to sign changes, or merely to an exact zero.

Next consider the papers that report bias adjusted estimands. For any given specification, all of these papers report a scalar bias adjusted estimand. Most papers explicitly say that this is because they use Oster's Proposition 1 (see our Proposition S3) and assume $\delta = 1$. Others do not clearly explain which result they are using to obtain their bias correction. None of the papers that use Oster's Proposition 1 discuss the validity of Assumption 7. Eleven of the thirteen papers use just the single value $\delta = 1$. One of those papers considers $\delta = 0.5$ only (and no other values), but also uses the incorrect equation (S11) as described in Appendix F.1. Two papers consider multiple δ values, but one of them also uses the incorrect equation (S11). The other paper uses a variation of Oster's results, which we have not formally analyzed, although we conjecture that the concerns we raise in our paper will apply to that extension as well. About half of the papers (7 of 13) only consider one value of R^2_{long}, always either $1.3\widehat{R}^2_{\text{med}}$ or 1. In the other half (6 of 13), five papers also consider the second value $2\widehat{R}^2_{\text{med}}$, while one paper considers up to 4 different values.

B Additional Details for Empirical Application of Section 6

B.1 Data

Section 3 of Satyanath, Voigtländer, and Voth (2017) gives a full description of the data. Here we focus on the aspects relevant to our replication. Satyanath et al. (2017) use three different measures of social capital. All three are based on "association density," the number of social clubs and associations per 1,000 city inhabitants. There are 22,127 different associations in the sample. The authors classify these into three types:

Table S2: Baseline results for Satyanath et al. (2017).

Dependent Variable: Nazi Party Entry, 1925–January 1933			
	(1)	(2)	(3)
A. Treatment = All Associations			
	0.160	0.172	0.087
	(0.054)	(0.052)	(0.050)
B. Treatment = Civic Associations			
	0.429	0.441	0.284
	(0.132)	(0.125)	(0.108)
C. Treatment = Military Associations			
	0.829	0.853	0.613
	(0.268)	(0.276)	(0.357)
Controls:			
Baseline	X	X	X
Socioeconomic		X	X
Political		X	X
State Fixed Effects			X

Note: These estimates replicate part of Table 3 in Satyanath et al. (2017): Column (1) replicates Panel A, columns (4)–(6). Column (3) replicates Panel B, columns (4)–(6). Panel A, column (2) replicates Panel B, column (3). Panels B and C, column (2) were not presented in Table 3 of Satyanath et al. (2017).

1. *Military* associations: Stahlhelm ("steel helmet"), veterans associations.
2. *Civic* associations (have "a clearly nonmilitaristic/nationalist outlook"): Animal breeding, music, chess, hiking, women's, citizens', and homeland (Heimat) clubs, and some others (predominantly civic clubs, many with an artistic or creative pursuit such as gardening, theatre, or photography).
3. *Other*: Sports, choirs, gymnastics, shooting, students/fraternities, lodges, youth, oldfellows, hunting, gentlemen, corps.

X is then defined to be the density of all associations, of civic associations only, or of military associations only.

Next consider the control variables. Recall from Section 2.2 that we distinguish between two kinds of observed control variables: Those which we use for comparison in the sensitivity analysis (W_1) and those which we do not (W_0). The baseline controls W_0 are population, share Catholic, and share blue-collar, all defined for 1925. The comparison covariates W_1 are

- Socioeconomic controls: share of Jews (1925), share unemployed (1933), welfare recipients per 1000 (1933), war participants per 1000 (1933), social insurance pensioners per 1000 (1933), log average income tax payment (1933), log average property tax payment (1933).
- Political controls: Hitler speeches per 1000 (1932), average DNVP votes 1920–1928, average DVP votes 1920–1928, average SPD votes 1920–1928, average KPD votes 1920–1928.

In their Appendix G, Satyanath et al. (2017) also consider using the variables in W_0 for calibration; we omit this analysis for brevity. In their baseline analysis, they also include state fixed effects as covariates. They do not include these in their sensitivity analysis, however.

B.2 Baseline Model Results

Table S2 shows the baseline regression results. Column (1) shows the estimated coefficient $\widehat{\beta}_{\text{short}}$ on X from OLS of Y on $(1, X, W_0)$, along with its standard error. Panels A–C show this coefficient for the three different definitions of the treatment variable X. First consider Panel A. This is their most broad definition of treatment, accounting for all kinds of associations. We see that the estimated coefficient is positive and statistically significantly different from zero at the 1% level. Thus, adjusting for the baseline controls W_0, cities with more social capital, as measured by density of all associations, had higher rates of new membership in the Nazi party.

Does this result reflect a broad causal effect of social capital on entry into the Nazi party? One counter-argument is that it may merely reflect the effect of a specific kind of social capital—membership in military-specific associations. Indeed, in Panel C we see that, when restricting attention to military associations, the estimated coefficient is again positive and statistically significantly different from zero at the 1% level. However, suppose we restrict attention to civic associations, which are defined to be non-militaristic and non-nationalistic. Panel B shows these estimates. There we continue to see a positive point estimate that is statistically significantly different from zero at the 1% level. This finding suggests that the effect is not driven solely by military-specific social capital.

Are these results driven by omitted variable bias? The authors consider a variety of alternative methods to answer this question. We focus on just two: (i) Using additional controls and (ii) Using formal econometric methods for sensitivity analysis. Column (2) of Table S2 includes the additional controls W_1 in the regressions. Specifically, it shows the estimated coefficient $\widehat{\beta}_{\text{med}}$ on X from OLS of Y on $(1, X, W_0, W_1)$, along with its standard error. Panels A–C again show this coefficient for the three different versions of treatment. We again see that all of the point estimates are positive and statistically significantly different from zero at the 1% level. Column (3) adds state fixed effects. The point estimates are smaller but still positive and statistically significantly different from zero at the 10% level. On page 500, the authors explain that this drop in precision arises from the fact that some of the historical policies which drove variation in association density often varied at the state level.

B.3 Plots of Identified Sets for Fixed δ

Here we plot estimated identified sets $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$ as a function of δ for the three choices of treatment variables and two choices of R^2_{long}. Figure S1 shows these plots. These plots show several of the features we discussed earlier in the paper. For example, at $\delta = 0$, the identified set is the singleton containing $\widehat{\beta}_{\text{med}}$. This is displayed as the vertical intercept. The horizontal intercept, in contrast, is the explain away breakdown point, as reported in column (2) of Table 1. Only two of these intercepts are visible given the range of the plots, however.

As δ varies, the identified set can contain up to three elements. For example, in the top left graph, there are two feasible values of β_{long} at $\delta = 1$. We can also see the vertical asymptote at $\delta = 1$, which implies that for δ's very close to 1, the identified set will contain either very large positive values, very large negative values, or both. This explains why the identified set under the restriction $|\delta| \leq \bar{\delta}$ is bounded and finite for $\bar{\delta} < 1$ but unbounded for $\bar{\delta} > 1$. It also explains why the bias adjustments in Table 2 are so sensitive to the exact choice of δ.

B.4 Empirical Conclusions

By only examining the explain away breakdown point, Satyanath et al. (2017) conclude that

> "selection on unobservables would have to be substantially stronger than selection on observables for our main result to be overturned." (online appendix page 44)

However, the language "for our main result to be overturned" suggests that they might also be interested in the robustness of their point estimates to sign changes. Indeed, if the true coefficient β_{long} was negative, then higher social capital would lead to a *decrease* in Nazi party membership, which would support the opposite of their conclusion that social capital supported the Nazi party and therefore helped to undermine an existing democracy.

When examining the relative robustness of their results for the three versions of the treatment variable, a second concern arises. Based on the explain away breakdown point, they say:

> "Note also that for the logic of our argument, the coefficient for military associations is not the most important—what matters is that the civic clubs and associations have an important effect

Figure S1: Estimated identified sets $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$ for assessing the robustness of Satyanath et al. (2017) results to omitted variables. Left column: $R^2_{\text{long}} = 1$. Right column: $R^2_{\text{long}} = 1.3\widehat{R}^2_{\text{med}}$. Top row: Treatment variable includes all associations. Middle row: Treatment variable includes civic associations only. Bottom row: Treatment variable includes military associations only.

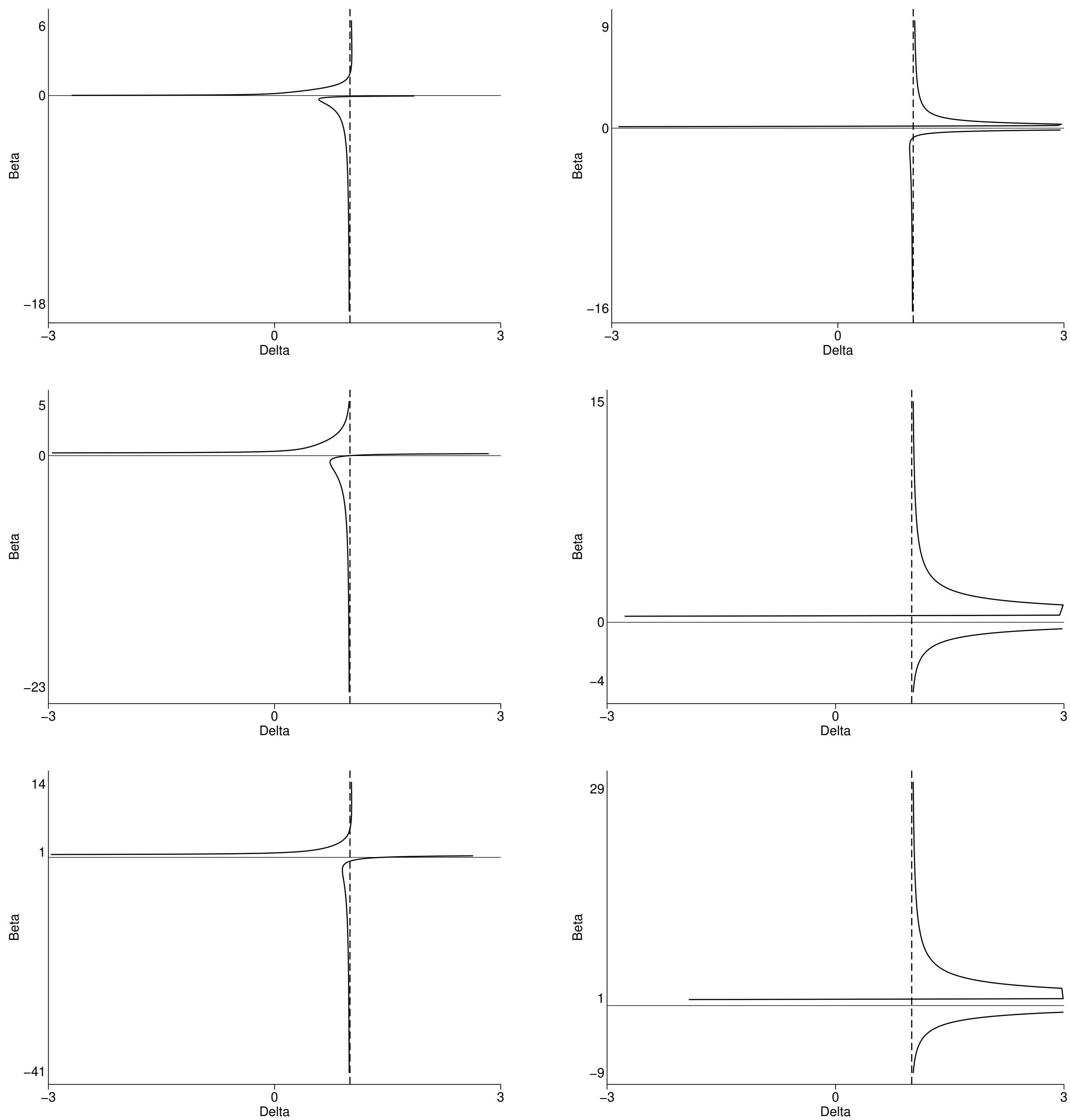

Table S3: Different Types of Breakdown Points for Assessing the Robustness of Squicciarini (2020) Results to Omitted Variables.

	Explain Away	Sign Change with M equal to			
		$+\infty$	$10\lvert\widehat{\beta}_{\text{med}}\rvert$	$3\lvert\widehat{\beta}_{\text{med}}\rvert$	$2\lvert\widehat{\beta}_{\text{med}}\rvert$
Outcome variable	(1)	(2)	(3)	(4)	(5)
Share Ind. Workers	9.0950	0.9982	1.3683	3.4494	4.6986
Machines	4.0699	0.9968	1.0744	1.8306	2.4460

Note: See the original paper for variables included in W_1. $R^2_{\text{long}} = 1$ for all results in columns (1)–(5). Column (1) shows $\widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$. Column (2) shows the sign change breakdown point $\widehat{\delta}^{\text{bp,sign}}(R^2_{\text{long}}, +\infty) = \widehat{\delta}^{\text{bp,sign}}(R^2_{\text{long}})$. Columns (3)–(5) show $\widehat{\delta}^{\text{bp,sign}}(R^2_{\text{long}}, M)$.

on Nazi Party entry. This is never in doubt in the Altonji-Elder-Taber/Oster exercise." (online appendix page 44)

When examining the sign change breakdown points, however, we see that the coefficient for military associations is the most robust, among the three versions of treatment. The coefficient for civic clubs and associations, in contrast, is not so obviously robust to omitted variables. Thus the sign change breakdown points suggest a more tentative conclusion about robustness than the definitive one drawn by the authors.

C Additional Empirical Applications

Beyond our empirical application in Section 6, we also considered all six papers published in the *American Economic Review* from our Appendix A survey which formally implemented the Oster (2019) method. Among these papers, three had complete, publicly available replication data and code. In this section, we apply our methods to the data from these three papers. For brevity we only present the results; see the original papers for a discussion of the full empirical context, analysis, and data. Two of the three papers only presented breakdown point estimates, while the third paper only presented bias adjustments.

1. Squicciarini (2020, *AER*)

In this paper, the author estimates explain away breakdown points for two different outcome variables, and reports them in Table A.11 of the online appendix. Based on these estimates, the author concludes

> "that selection on unobservables would have to be at least 4.1 times stronger than selection on observables to explain away the relationship between the [treatment variable] and the outcome variables, making it unlikely that unobserved factors are confounding the results." (page 20)

In Table S3, we replicate and extend this analysis. Column (1) shows the explain away breakdown point estimates as reported and correctly computed by the original author. Column (2) shows the estimated sign change breakdown points. Consistent with our Theorem 2, these are less than one. However, they are quite close to one. In columns (3)–(5) we add magnitude restrictions. Column (3) shows that even with a fairly loose magnitude restriction we are able to obtain sign change breakdown points above one. And, especially for the first outcome variable, we can substantially increase the breakdown point with further magnitude restrictions.

Overall, our analysis here shows that although the explain away breakdown points are substantially larger than the sign change breakdown points, if one uses the conventional cutoffs for robustness and imposes some magnitude restrictions on the OVB, the author's conclusion that their results are robust to omitted variables continues to hold.

Table S4: Different Types of Breakdown Points for Assessing the Robustness of Dippel and Heblich (2021) Results to Omitted Variables.

Sample	W_1	Explain Away	Sign Change with M equal to			
			$+\infty$	$10\lvert\widehat{\beta}_{\text{med}}\rvert$	$3\lvert\widehat{\beta}_{\text{med}}\rvert$	$2\lvert\widehat{\beta}_{\text{med}}\rvert$
		(1)	(2)	(3)	(4)	(5)
	Core controls	0.5170	0.5170	0.5170	0.5170	0.5170
Full	+ other Vselect	0.5187	0.5187	0.5187	0.5187	0.5187
	All controls	0.5156	0.5156	0.5156	0.5156	0.5156
	Core controls	1.7073	1.0000	1.7073	1.7073	1.7073
Matched	+ other Vselect	1.9654	1.0000	1.9654	1.9654	1.9654
	All controls	1.9554	1.0000	1.9554	1.9554	1.9554

Note: See original paper for variables included in each choice of W_1. $R^2_{\text{long}} = 1$ for all results in columns (1)–(5). Column (1) shows $\widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$. Column (2) shows the sign change breakdown point $\widehat{\delta}^{\text{bp,sign}}(R^2_{\text{long}}, +\infty) = \widehat{\delta}^{\text{bp,sign}}(R^2_{\text{long}})$. Columns (3)–(5) show $\widehat{\delta}^{\text{bp,sign}}(R^2_{\text{long}}, M)$.

2. Dippel and Heblich (2021, *AER*)

In Table 2 on page 491, the authors report estimates of the explain away breakdown point. On that page, they say

> "To get a sense for potential selection on unobservables in the full and matched sample, we report Oster's δ in the bottom of Table 2. Oster's δ measures how large the bias from unobservables would have to be relative to bias from observable *to imply a true value of* $\beta = 0$ in equation (1). In the full sample, δ is consistently around 0.5. In the matched sample by contrast, δ ranges between 1.7 and 1.9. This implies a strong sense of robustness in the matched sample, where selection on unobservables would have to be almost twice as strong as selection on observables to *make our core estimate go away*." [emphasis added]

Our extended analysis, shown in Table S4, is qualitatively similar: The results appear robust for the matched sample but not the full sample. Specifically, first consider the full sample. Consider the first line, the core controls; the results are similar for the other two choices of controls. In column (1) we see that the estimated explain away breakdown point is 0.5170. This is also the sign change breakdown point, as shown in column (2). Given that this value is substantially less than 1, where the asymptote in the identified set appears, imposing magnitude constraints (columns (3)–(5)) does not affect the value of the sign change breakdown point. All of these results together suggest that the results for the full sample are not robust.

Next consider the matched sample. Consider the first line, the core controls; the results are similar for the other two choices of controls. Here we see the estimated explain away breakdown point is 1.7073. This is not the sign change breakdown point, however. In this case, the sign change breakdown point is exactly 1. Nonetheless, with even a loose magnitude constraint—assuming the OVB is bounded above by $10|\widehat{\beta}_{\text{med}}|$—the sign change and explain away breakdown points coincide exactly. Further magnitude restrictions are not helpful because these restrictions only matter insofar as they bring the sign change and explain away breakdown points closer together. Overall, these results suggest that the results for the matched sample are robust.

3. Gregg (2020, *AER*)

This paper uses Oster's results in tables 3, 5, and 7 of the main paper. For brevity we only consider the results in Table 3. The author considers three different outcome variables, and two choices of sample and calibration covariates W_1 for each outcome variable. Unlike the two papers we considered above, Gregg does not estimate breakdown points. Instead, they only present bias adjusted estimates. Specifically, following

Table S5: The Sensitivity of Bias Adjustments in Gregg (2020).

Method	Assumptions	β_{long} Values Specification 1	Specification 2
I. Outcome variable: log(R/L)			
A. Current Practice			
Baseline Estimate	$\delta = 0$	0.495	0.451
Oster's Prop 1 (our Prop S3)	$\delta = 1$, A7	0.434	0.367
B. Sensitivity to δ Assumptions			
Oster's Prop 2 (our Thm S1), $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 1$	$\{0.412, 197\}$	$\{0.331, 152\}$
Oster's Prop 2, $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 0.9999$	$\{0.412, 210, 3200\}$	$\{0.331, 158, 3817\}$
Oster's Prop 2, $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 1.01$	$\{-99.8, 0.411, 67.1\}$	$\{-99.0, 0.329, 60.7\}$
Corollary 2, $\widehat{\bar{\mathcal{B}}}_I(\bar{\delta}, R^2_{\text{long}})$	$\lvert\delta\rvert \leq \bar{\delta} = 1$	$[0.412, 0.550] \cup [197, \infty)$	$[0.331, 0.526] \cup [152, \infty)$
Corollary 2, $\widehat{\bar{\mathcal{B}}}_I(\bar{\delta}, R^2_{\text{long}})$	$\lvert\delta\rvert \leq \bar{\delta} = 1.05$	$(-\infty, -39.2] \cup [0.407, 0.553] \cup [33.8, \infty)$	$(-\infty, -37.9] \cup [0.323, 0.529] \cup [31.4, \infty)$
II. Outcome variable: log(HP/L)			
A. Current Practice			
Baseline Estimate	$\delta = 0$	0.250	0.216
Oster's Prop 1 (our Prop S3)	$\delta = 1$, A7	0.804	0.782
B. Sensitivity to δ Assumptions			
Oster's Prop 2 (our Thm S1), $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 1$	$\{-25.8, 0.9532\}$	$\{-39.4, 0.9525\}$
Oster's Prop 2, $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 0.99$	$\{-367, -27.4, 0.9449\}$	$\{-393, -43.4, 0.9440\}$
Oster's Prop 2, $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 1.01$	$\{-24.4, 0.9615, 411\}$	$\{-36.6, 0.961, 464\}$
Corollary 2, $\widehat{\bar{\mathcal{B}}}_I(\bar{\delta}, R^2_{\text{long}})$	$\lvert\delta\rvert \leq \bar{\delta} = 1$	$(-\infty, -25.8] \cup [-0.301, 0.9532]$	$(-\infty, -39.4] \cup [-0.358, 0.9525]$
Corollary 2, $\widehat{\bar{\mathcal{B}}}_I(\bar{\delta}, R^2_{\text{long}})$	$\lvert\delta\rvert \leq \bar{\delta} = 1.05$	$(-\infty, -21.0] \cup [-0.326, 0.995] \cup [95.0, \infty)$	$(-\infty, -30.2] \cup [-0.383, 0.996] \cup [112, \infty)$
III. Outcome variable: TFP			
A. Current Practice			
Baseline Estimate	$\delta = 0$	0.102	0.015
Oster's Prop 1 (our Prop S3)	$\delta = 1$, A7	0.132	-0.141
B. Sensitivity to δ Assumptions			
Oster's Prop 2 (our Thm S1), $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 1$	$\{-212, 0.163\}$	$\{-0.876, 24.3\}$
Oster's Prop 2, $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 0.99999$	$\{-7090, -218, 0.163\}$	$\{-0.876, 24.3, 22452\}$
Oster's Prop 2, $\widehat{\mathcal{B}}_I(\delta, R^2_{\text{long}})$	$\delta = 1.01$	$\{-35.6, 0.164, 43.0\}$	$\{-35.9, -0.922, 14.6\}$
Corollary 2, $\widehat{\bar{\mathcal{B}}}_I(\bar{\delta}, R^2_{\text{long}})$	$\lvert\delta\rvert \leq \bar{\delta} = 1$	$(-\infty, -212] \cup [0.079, 0.163]$	$[-0.876, 0.115] \cup [24.3, \infty)$
Corollary 2, $\widehat{\bar{\mathcal{B}}}_I(\bar{\delta}, R^2_{\text{long}})$	$\lvert\delta\rvert \leq \bar{\delta} = 1.05$	$(-\infty, -16.5] \cup [.079, 0.168] \cup [18.0, \infty)$	$(-\infty, -11.0] \cup [-1.16, 0.118] \cup [7.93, \infty)$

Note: All sensitivity methods use $R^2_{\text{long}} = 1$. Panel I: Specification 1 corresponds to column (1) of Panel B in Table 3 of Gregg (2020) while specification 2 corresponds to column (4). Panel II: Specification 1 corresponds to column (2) of Panel B in Table 3 of Gregg (2020) while specification 2 corresponds to column (5). Panel III: Specification 1 corresponds to column (3) of Panel B in Table 3 of Gregg (2020) while specification 2 corresponds to column (6). See Section E.2 for a discussion and definition of A7.

Oster's (2019) recommendation and the defaults in the accompanying Stata package `psacalc`, Panel B of Table 3 of Gregg (2020) reports the single element of the estimated identified set obtained from Oster's Proposition 2 (our Theorem S1) which is closest to $\widehat{\beta}_{\text{med}}$ (this is an estimate of the value $\beta^*(\delta, R^2_{\text{long}})$ defined in Section F.2). Using these estimates, Gregg provides the following analysis:

> Some suggestive evidence on the nature of selection is provided by the Oster (2017) bound listed for each regression in panel B of Table 3. Here I report the estimate of the coefficient on corporation in the case where the selection on unobservables is equal to the selection on observables (when δ equals 1). The observables in this case are the province, industry, and year controls. These coefficients suggest some positive selection in revenue per worker, but possibly some negative selection in machine power per worker and TFP. Columns 4–6 introduce controls for the factory's age, which makes little difference for the revenue per worker or power per worker regressions. However, factory age is highly correlated with TFP, since age proxies for survival. Furthermore, with the age control included, the estimated Oster bound indicates positive selection in the TFP regression, likely since selection on the newly included observable, factory age, was

substantial." (pages 414–415)

In this analysis Gregg uses Oster's method to assess the direction of the selection bias, by comparing the bias adjusted estimates with the baseline estimates. Overall, Gregg concludes that there is positive selection bias, meaning that the baseline estimates are too large. In our analysis, we show that Gregg's conclusion about the direction of selection bias no longer holds once we account for the sensitivity of the bias adjusted estimands obtained using Oster's method.

To see this, we replicate and extend her estimates for each of the three outcome variables in Table S5. First consider Panel I, for the log(R/L) outcome variable. Consider specification 1. Gregg compared the element of the set $\{0.412, 197\}$ that is closest to the baseline estimate 0.495 with that baseline estimate. This gives $0.412 < 0.495$, suggesting that the baseline estimate is too large. Similarly, in specification 2 we see that $0.331 < 0.451$, suggesting again that the baseline estimate is too large. There are two main concerns with this comparison, however. First, it ignores the other element of the estimated Proposition 2 identified set. For example, in specification 1 the baseline estimate 0.495 is smaller than 197, the other element of the estimated identified set. This comparison suggests that the baseline estimate is *too small*.

Now, as in our discussion of magnitude restrictions in Section 3.2, one might be willing to eliminate this second element by imposing an additional assumption that restrictions the magnitude of the coefficient. This does not salvage the argument, however, because there is a second issue: In the fourth line of Panel B we show the estimated cumulative identified set under the assumption that $|\delta| \leq 1$. This set is a union of two intervals. Even if we ignore the upper interval, this set is $[0.412, 0.550]$. This set contains values that are smaller *and larger* than the baseline estimate of 0.495. Thus if we allow for δ to take any value so long as its magnitude is not larger than 1, rather than assuming that δ is exactly equal to 1, then we can no longer conclude anything about the direction of the selection bias. A similar analysis holds for the second specification, and the other two outcome variables.

D Additional Results for the Meta-Analyses of Section 7

This section presents an extended version of Table 4 in Section 7. Table 4 presented the distribution of $\widehat{p}_i^{\text{bp}}$, the largest magnitude of OVB (defined as a proportion of the baseline estimate) such that the estimated sign change breakdown point is at least as large as the estimated explain away breakdown point, across both meta-analysis samples and two values of R^2_{long}.

Here we first generalize the definition of $\widehat{p}_i^{\text{bp}}$ to allow us to compare the estimated sign change breakdown point with δ values other than the estimated explain away breakdown point. Specifically, for the ith regression, let $\widehat{p}_i^{\text{bp}}(\delta_{\text{hypo}})$ be the largest magnitude of OVB (defined as a proportion of the baseline estimate) such that the estimated sign change breakdown point is at least as large as $|\delta_{\text{hypo}}|$, where δ_{hypo} is a hypothesized value that is allowed to vary.[2] For example, setting $\delta_{\text{hypo}} = \widehat{\delta}_i^{\text{bp,explain away}}(R^2_{\text{long}})$ yields $\widehat{p}_i^{\text{bp}}(\delta_{\text{hypo}}) = \widehat{p}_i^{\text{bp}}$, the values whose distribution is reported in Table 4. These values answer the question: When can the explain away breakdown point be interpreted as a sign change breakdown point? Table S6 again shows these values, in the rows that set $\delta_{\text{hypo}} = \widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$. Table S6 also includes three new values, however: δ_{hypo} equal to 2, 1.05, and 1.

We can use these additional values $\widehat{p}_i^{\text{bp}}(\delta_{\text{hypo}})$ to answer several questions beyond our primary question of when explain away breakdown points can be interpreted as sign change breakdown points. For example, what is the weakest possible restriction on the magnitude of OVB to ensure that the sign change breakdown point is at least 1? $\widehat{p}_i(\delta_{\text{hypo}})$ with $\delta_{\text{hypo}} = 1$ answers this question. The bottom row of all sub-panels shows the distribution of our estimates of these magnitudes. Here we see that even for this very weak conclusion, at least 10% of regressions still require substantial magnitude restrictions. Most papers, however, require a fairly weak, or essentially no magnitude restriction in order to obtain a sign change breakdown point equal to or larger than 1. However, because of the asymptote at $\delta = 1$, these $\widehat{p}_i^{\text{bp}}$ values may not be informative about values for δ cutoffs slightly above 1. So the middle rows of each sub-panel show estimates of $\widehat{p}_i^{\text{bp}}(\delta_{\text{hypo}})$ for values of δ_{hypo} close to, but slightly above 1. In these middle rows we see that obtaining sign change breakdown points which are meaningfully larger than 1 often requires substantial magnitude restrictions.

[2] Formally, we define $\widehat{p}_i^{\text{bp}}(\delta_{\text{hypo}}) \coloneqq \sup\{p \geq 1 : |\widehat{\delta}_i^{\text{bp,sign}}(R^2_{\text{long}}, p \cdot |\widehat{\beta}_{\text{med},i}|)| \geq |\delta_{\text{hypo}}|\}$.

Table S6: Meta-Analysis: Distribution of $\widehat{p}^{\text{bp}}$, the largest magnitude of OVB (relative to $\widehat{\beta}_{\text{med}}$) we can allow for to guarantee that the sign change breakdown point is at least as large as δ_{hypo}.

		Percentiles				
R^2_{long}	δ_{hypo}	10%	25%	50%	75%	90%
Panel A: Our Sample of Top 5 Papers						
$\min\{1.3\widehat{R}^2_{\text{med}}, 1\}$	$\widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$	1.0	1.0	1.0	1.3	67.2
	2	1.0	1.4	3.4	6.9	67.2
	1.05	2.0	9.2	25.2	54.0	198.9
	1	2.1	709.8	∞	∞	∞
1	$\widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$	1.0	1.0	1.0	13.8	21.3
	2	1.0	1.0	1.3	13.8	21.3
	1.05	1.2	1.2	4.5	53.2	147.8
	1	1.2	1.2	4.9	∞	∞
Panel B: Oster's Sample of Top 5 Papers + *AEJ:Applied*						
$\min\{1.3\widehat{R}^2_{\text{med}}, 1\}$	$\widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$	1.0	1.0	1.0	2.7	11.4
	2	1.0	3.3	12.8	58.6	116.0
	1.05	3.7	16.8	70.7	255.0	771.6
	1	5.3	59.6	1978.6	∞	∞
1	$\widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})$	1.0	1.0	1.0	3.9	61.6
	2	1.0	2.1	10.8	45.8	211.8
	1.05	1.5	16.3	50.5	192.3	760.0
	1	1.6	32.1	216.4	∞	∞

Note: Both samples only include results considered robust according to standard practice (the explain away breakdown point is weakly larger than 1, where we use the value of R^2_{long} specified in the corresponding panel). Panel A: Top: $N = 29$ regressions from 11 papers. Bottom: $N = 11$ regressions from 6 papers. Panel B: Top: $N = 111$ regressions from 47 papers. Bottom: $N = 58$ regressions from 31 papers. The value ∞ means our optimizer to compute $\widehat{p}_i^{\text{bp}}$ stopped at the preset upper bound of 1 million, which can be interpreted as effectively infinite. For $\delta_{\text{hypo}} = 2$ and 1.05, we actually use $\min\{2, \widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})\}$ and $\min\{1.05, \widehat{\delta}^{\text{bp,explain away}}(R^2_{\text{long}})\}$ to ensure that the distributions in each sub-panel are stochastically ranked from top to bottom.

E Formal Identification Results

In this section, we provide additional details on the paper's formal analysis.

Notation Remark

For random vectors A and B, define $A^{\perp B} := A - [\text{var}(B)^{-1}\,\text{cov}(B, A)]'B$. This is the sum of the residual from a linear projection of A onto $(1, B)$ and the intercept in that projection.

E.1 Generalization of Oster (2019) Proposition 2

The following result formally states our theorem deriving the identified set of β_{long}, a generalization of Oster's (2019) Proposition 2.

Theorem S1. Suppose the joint distribution of (Y, X, W_1) is known. Suppose A1–A4 hold. Suppose $R^2_{\text{long}} \in (R^2_{\text{med}}, 1]$ is known. Then the identified set for β_{long} is

$$\mathcal{B}_I(\delta, R^2_{\text{long}}) = \{b \in \mathbb{R} : f(\beta_{\text{med}} - b, \delta, R^2_{\text{long}}) = 0\} \setminus \mathcal{B}_{\text{A3fail}} \tag{5}$$

where

$$\mathcal{B}_{\text{A3fail}} := \{b \in \mathbb{R} : \gamma_{1,\text{med}} + (\beta_{\text{med}} - b)\pi_1 = 0\}$$

and $f(B, \delta, R^2_{\text{long}}) \coloneqq f_0(B) + \delta \cdot f_1(B, R^2_{\text{long}})$ with

$$f_0(B) \coloneqq -B \operatorname{var}(X^{\perp W_1}) \left(\operatorname{var}(\gamma_{1,\text{med}}' W_1) + 2B \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) + B^2 \operatorname{var}(\pi_1' W_1)\right)$$

$$\begin{aligned} f_1(B, R^2_{\text{long}}) \coloneqq{} & (R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) + B(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \operatorname{var}(\pi_1' W_1) \\ & + B^2 \operatorname{var}(X^{\perp W_1}) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) + B^3 \operatorname{var}(X^{\perp W_1}) \operatorname{var}(\pi_1' W_1). \end{aligned}$$

There are several differences between Theorem S1 and Oster's (2019) Proposition 2. First, we do not impose the normalization that $\operatorname{var}(W_1)$ is diagonal. Second, Oster's exogenous controls assumption is stated as $\operatorname{cov}(W_1, \gamma_{2,\text{long}}' W_2) = 0$, which is equivalent to our A4 given that we are assuming for simplicity W_2 is scalar throughout the analysis, and since $\gamma_{2,\text{long}} \neq 0$ by A3.

Third, and most importantly, we show the identified set equals the solutions to the cubic equation except those that lie in the set $\mathcal{B}_{\text{A3fail}}$, which was not present in Oster (2019). Here we explain how to interpret $\mathcal{B}_{\text{A3fail}}$ and how to characterize when it is empty. By Lemma S1 in Appendix H,

$$\gamma_{1,\text{long}} = \gamma_{1,\text{med}} + (\beta_{\text{med}} - \beta_{\text{long}})\pi_1. \tag{S1}$$

So $\mathcal{B}_{\text{A3fail}}$ is the set of β_{long} values which would imply $\gamma_{1,\text{long}} = 0$, which violates A3; recall that δ is not defined when A3 fails. This explains why such values must be removed. Next, notice that $\mathcal{B}_{\text{A3fail}}$ is nonempty if and only if there exists a $C \in \mathbb{R}$ such that $\gamma_{1,\text{med}} = C \cdot \pi_1$: see Proposition S2 below. That is, if and only if π_1 and $\gamma_{1,\text{med}}$ are proportional to each other. When W_1 is a vector, as is usually the case in empirical practice, this proportionality condition will typically not hold, and hence $\mathcal{B}_{\text{A3fail}}$ will be empty. When W_1 is a scalar, however, π_1 and $\gamma_{1,\text{med}}$ are scalars and hence are always proportional when $\pi_1 \neq 0$. In this case, the singleton $\mathcal{B}_{\text{A3fail}} = \{\beta_{\text{med}} + \gamma_{1,\text{med}}/\pi_1\}$ is not in the identified set. While the appearance of the set $\mathcal{B}_{\text{A3fail}}$ in Theorem S1 is mostly technical, it is nonetheless important, for example, when proving that the identified set is a singleton under the stronger assumptions in Proposition S3 below.

Theorem S1 assumed $R^2_{\text{long}} > R^2_{\text{med}}$. The following result explains why: When $R^2_{\text{long}} = R^2_{\text{med}}$ the omitted variable bias is zero, as shown in the following proposition.

Proposition S1. Suppose A1 holds. Suppose $R^2_{\text{long}} = R^2_{\text{med}}$. Then $\beta_{\text{long}} = \beta_{\text{med}}$.

Finally, the analysis simplifies to the baseline case when $\delta = 0$.

Corollary S1. Suppose A1–A4 hold. Suppose $\delta = 0$ and $\gamma_{1,\text{med}} \neq 0$. Then $\beta_{\text{long}} = \beta_{\text{med}}$.

E.2 Adding Assumptions to Point Identify β_{long}

Equation (5) generally does not point identify β_{long}, which means there is not a unique way to compute a bias adjusted estimand. However, as Oster shows, the identified set simplifies to a singleton under two additional assumptions. The first assumption is $\delta = 1$. The second assumption is the following, which is equivalent to Oster's Assumption 2 (page 192).[3]

Assumption 7. There is a constant $C \in \mathbb{R}$ such that $\gamma_{1,\text{long}} = C \cdot \pi_1$.

Recall that π_1 denotes the coefficient on W_1 in the regression of X on $(1, W_1)$. A7 holds immediately when W_1 is a scalar and $\pi_1 \neq 0$, but otherwise is a restriction. In fact, the following result implies that A7 is refutable and confirmable (as defined in Breusch 1986) under exogenous controls. Hence the data alone tells us whether it holds.

Proposition S2. Suppose A1 and A4 hold. Then A7 holds if and only if there is a constant $C_{\text{med}} \in \mathbb{R}$ such that $\gamma_{1,\text{med}} = C_{\text{med}} \cdot \pi_1$.

[3] It is unclear to us whether Oster's (2019) Assumption 2 is meant to be an assumption relating $\gamma_{1,\text{med}}$ to π_1 or relating $\gamma_{1,\text{long}}$ to π_1 (using our notation), since that paper uses the same notation Ψ to denote both $\gamma_{1,\text{long}}$ and $\gamma_{1,\text{med}}$ in different places (for example, compare her equation (1) with the definition of ψ_i in the statement of her Assumption 2). In general, $\gamma_{1,\text{med}} \neq \gamma_{1,\text{long}}$. Regardless, our Proposition S2 shows that the two forms of this assumption are equivalent under the exogenous controls assumption A4. Also note that Oster (2019) does not discuss the testability of this assumption.

For vector W_1, Oster (2019) said A7 "is very unlikely to hold except in pathological cases" (page 192). Proposition S2 shows that researchers do not need to guess about this assumption—it can be directly tested in the data. Such a hypothesis test can be constructed using standard minimum distance methods, by comparing $\widehat{\gamma}_{1,\text{med}}$ with multiples of $\widehat{\pi}_1$; we omit a full development for brevity.

Under these two additional assumptions, $\delta = 1$ and A7, we can now show that β_{long} is point identified.

Proposition S3. Suppose the joint distribution of (Y, X, W_1) is known, A1, A3, A4, and A7 hold, A2 holds with $\delta = 1$, $R^2_{\text{long}} \in (R^2_{\text{med}}, 1]$ is known, and $\beta_{\text{short}} \neq \beta_{\text{med}}$. Then β_{long} is point identified and

$$\beta_{\text{long}} = \beta_{\text{med}} + (\beta_{\text{med}} - \beta_{\text{short}})\frac{R^2_{\text{long}} - R^2_{\text{med}}}{R^2_{\text{med}} - R^2_{\text{short}}}. \tag{7}$$

This result is a version of Oster's Proposition 1. Oster's (2019) proof is slightly incomplete, however, since it only shows that the value in equation (7) is consistent with the data. It does not show that this is the only value that is consistent with the data. Our proof of Proposition S3 shows this last step. In this proof we show how the result follows directly from our Theorem S1. The cubic function $f(\beta_{\text{med}} - b, \delta, R^2_{\text{long}})$ is quadratic with two solutions when $\delta = 1$ and one of the two solutions is eliminated by subtracting $\mathcal{B}_{\text{A3fail}}$. The remaining solution is equation (7). Finally, in this result we assume $\beta_{\text{short}} \neq \beta_{\text{med}}$ to ensure that $R^2_{\text{short}} \neq R^2_{\text{med}}$ and hence that the right hand side of (7) is well defined.

An Alternative Assumption

Proposition S3 showed that β_{long} takes on a single value when $\delta = 1$ and A7 holds. As mentioned earlier, Oster says A7 is "unlikely to hold". Without A7, there are usually two values of β_{long} consistent when $\delta = 1$. Oster briefly suggests an alternative to A7 that also implies a single value of β_{long} is consistent with $\delta = 1$. This alternative, her Assumption 3 on page 194, can be stated as follows.

Assumption 8. $\text{cov}(X, \gamma'_{1,\text{med}} W_1)$ and $\text{cov}(X, \gamma'_{1,\text{long}} W_1)$ have the same sign.

In words, the omitted variable bias does not change the sign of the covariance between the treatment and the index of the observables across the medium and long regressions. Using equations (S13) and (S15) below, we can rewrite A8 as

$$\text{sign}(\beta_{\text{short}} - \beta_{\text{med}}) = \text{sign}\left(\frac{\beta_{\text{short}} - \beta_{\text{med}}}{R^2_{X\sim W_1}} + \beta_{\text{med}} - \beta_{\text{long}}\right).$$

This restriction is therefore a simple inequality assumption on the omitted variable bias:

$$\begin{aligned}
\beta_{\text{med}} - \beta_{\text{long}} &\geq \frac{\beta_{\text{med}} - \beta_{\text{short}}}{R^2_{X\sim W_1}} \qquad \text{when} \quad \beta_{\text{med}} - \beta_{\text{short}} \leq 0,\\
\beta_{\text{med}} - \beta_{\text{long}} &\leq \frac{\beta_{\text{med}} - \beta_{\text{short}}}{R^2_{X\sim W_1}} \qquad \text{when} \quad \beta_{\text{med}} - \beta_{\text{short}} \geq 0.
\end{aligned}$$

Thus, A8 together with an assumed value for δ (not necessarily 1), will rule out certain values for the omitted variable bias. Consequently, the identified set for β_{long} will generally have fewer elements when A8 is imposed. Since $(\beta_{\text{short}}, \beta_{\text{med}}, R^2_{X\sim W_1})$ are all point identified, this assumption can also be easily implemented in practice.

E.3 Properties of R-squared

Here we state without proof five properties of the R-squared and partial R-squared that we use in Section E.4, but which are not necessarily widely known. Let A be a random variable and let B and C be random vectors, all with finite second moments. Let $\text{var}(A, B, C)$ be positive definite.

$$R^2_{A\sim B\bullet C} = \frac{R^2_{A\sim B,C} - R^2_{A\sim C}}{1 - R^2_{A\sim C}} \tag{S2}$$

$$\text{var}(A^{\perp C}) = \text{var}(A)(1 - R^2_{A\sim C}) \tag{S3}$$
$$R^2_{A\sim B,C} = R^2_{A\sim B^{\perp C}} + R^2_{A\sim C} \tag{S4}$$
$$R^2_{A\sim B,C} = R^2_{A\sim B} + R^2_{A\sim C} \text{ when } \text{cov}(B,C) = 0 \tag{S5}$$
$$\text{var}(A)(R^2_{A\sim B,C} - R^2_{A\sim C}) = \text{var}(\gamma_B' B^{\perp C}), \tag{S6}$$

where γ_B is the coefficient on B in the linear regression of A on $(1, B, C)$.

E.4 Formal Results on the Asymptote at $\delta = 1$

We now provide a formal justification for our intuitive explanation behind the presence of the asymptote at $\delta = 1$ in Section 3.1. We show that, for fixed $R^2_{\text{long}} \in (R^2_{\text{med}}, 1]$, the following two conditions are *equivalent*:

1. (X, W_1, W_2) are nearly multicollinear.
2. $|\beta_{\text{long}}|$ is arbitrarily large and δ is arbitrarily close to 1.

To formalize this statement, we consider a sequence of data generating processes for (Y, X, W_1, W_2) indexed by $j \in \{1, 2, \ldots\}$. Since our analysis only depends on the covariance matrix of these variables, we will work with the indexed covariance matrix $\Sigma_j \coloneqq \text{var}_j(Y, X, W_1, W_2)$. Assume that dgps along this sequence are consistent with $\text{var}(Y, X, W_1)$, the known variance matrix of observed variables. We index any aspect of the joint distribution of (Y, X, W_1, W_2) that depends on W_2 with j to indicate that their value depends on Σ_j. For example, we let

$$\delta_j \coloneqq \frac{\text{cov}_j(X, \gamma_{2,\text{long},j} W_2)}{\text{var}_j(\gamma_{2,\text{long},j} W_2)} \Bigg/ \frac{\text{cov}(X, \gamma'_{1,\text{long},j} W_1)}{\text{var}(\gamma'_{1,\text{long},j} W_1)}.$$

Note that the covariance and variance terms which do not involve W_2 are not indexed by j since they are taken over the distribution of $\text{var}(Y, X, W_1)$, which we assume does not vary with j.

We formalize the idea of "near multicollinearity" between X, W_1, and W_2 as $R^2_{X\sim W_1,W_2,j} \to 1$ as $j \to \infty$. This means that a linear combination of W_1 and W_2 approaches X in a mean-square sense as $j \to \infty$. The notions of "$|\beta_{\text{long}}|$ is arbitrarily large" and "δ is arbitrarily close to 1" are formalized as $(|\beta_{\text{long},j}|, \delta_j) \to (+\infty, 1)$ as $j \to \infty$.

Proposition S4. Fix $\text{var}(Y, X, W_1)$ and $R^2_{\text{long}} \in (R^2_{\text{med}}, 1]$. For $j \in \{1, 2, \ldots\}$, let $\Sigma_j \coloneqq \text{var}_j(Y, X, W_1, W_2)$ be a sequence of variance matrices consistent with $\text{var}(Y, X, W_1)$, assumptions A1, A3, and A4, and with $R^2_{Y\sim X,W_1,W_2,j} = R^2_{\text{long}}$. Then, as $j \to \infty$,

$$R^2_{X\sim W_1,W_2,j} \to 1 \quad \text{if and only if} \quad (|\beta_{\text{long},j}|, \delta_j) \to (+\infty, 1).$$

Proof of Proposition S4. ($\Rightarrow$). Suppose $R^2_{X\sim W_1,W_2,j} \to 1$. By equation (S3), this is equivalent to $\text{var}_j(X^{\perp W_1,W_2}) \to 0$. Then

$$\begin{aligned} X^{\perp W_1,W_2} &= X - \pi'_{1,j} W_1 - \pi_{2,j} W_2 \\ &= X - \pi'_1 W_1 - \pi_{2,j} W_2. \end{aligned}$$

The second equality follows from $\pi_1 = \text{var}(W_1)^{-1}\text{cov}(W_1, X)$ and A4. So

$$\text{var}_j(X^{\perp W_1,W_2}) = \text{var}(X) - \text{var}(\pi'_1 W_1) - \text{var}_j(\pi_{2,j} W_2).$$

Hence $\text{var}_j(X^{\perp W_1,W_2}) \to 0$ implies $\text{var}_j(\pi_{2,j} W_2) \to \text{var}(X) - \text{var}(\pi'_1 W_1) = \text{var}(X^{\perp W_1})$, a fixed positive quantity by A1. Next,

$$\begin{aligned} \text{var}(Y)(R^2_{\text{long}} - R^2_{\text{med}}) &= \text{var}_j(\gamma_{2,\text{long},j} W_2^{\perp X,W_1}) \\ &= \text{var}_j(\gamma_{2,\text{long},j} W_2)(1 - R^2_{W_2\sim X,W_1,j}) \\ &= \text{var}_j(\gamma_{2,\text{long},j} W_2)\left(1 - R^2_{X\sim W_2\bullet W_1,j}\right) \end{aligned}$$

$$= \mathrm{var}_j(\gamma_{2,\mathrm{long},j} W_2) \left(1 - \frac{\mathrm{var}_j(\pi_{2,j} W_2)}{\mathrm{var}(X^{\perp W_1})}\right). \tag{S7}$$

The first line follows by (S6). The second by (S3). The third follows from

$$\begin{aligned} R^2_{W_2 \sim X, W_1, j} &= R^2_{W_2 \sim X \bullet W_1, j}(1 - R^2_{W_2 \sim W_1, j}) + R^2_{W_2 \sim W_1, j} \\ &= R^2_{W_2 \sim X \bullet W_1, j} \\ &= R^2_{X \sim W_2 \bullet W_1, j} \end{aligned}$$

where we used (S2) in the first equality and $\mathrm{cov}_j(W_1, W_2) = 0$ in the second. The third equality here uses $R^2_{A \sim B \bullet C} = R^2_{B \sim A \bullet C}$ for scalar A and B, which we can apply since W_2 is a scalar. The last equality in (S7) follows from

$$R^2_{X \sim W_2 \bullet W_1, j} = \frac{R^2_{X \sim W_1, W_2, j} - R^2_{X \sim W_1}}{1 - R^2_{X \sim W_1}} = \frac{R^2_{X \sim W_2, j}}{1 - R^2_{X \sim W_1}} = \frac{\mathrm{var}_j(\pi_{2,j} W_2)}{\mathrm{var}(X^{\perp W_1})}.$$

where we used (S2), (S5) combined with A4, and then (S4) combined with the definition of R-squared.

Equation (S7) can then be used to show $\mathrm{var}_j(\gamma_{2,\mathrm{long},j} W_2) \to \infty$. This is the case because

$$\mathrm{var}_j(\gamma_{2,\mathrm{long},j} W_2) = \frac{\mathrm{var}(Y)(R^2_{\mathrm{long}} - R^2_{\mathrm{med}})}{1 - \mathrm{var}_j(\pi_{2,j} W_2)/\,\mathrm{var}(X^{\perp W_1})}$$

and $\mathrm{var}(Y)(R^2_{\mathrm{long}} - R^2_{\mathrm{med}}) > 0$ is fixed (with respect to j), while $\mathrm{var}_j(\pi_{2,j} W_2) \to \mathrm{var}(X^{\perp W_1})$, implying that $1 - \mathrm{var}_j(\pi_{2,j} W_2)/\,\mathrm{var}(X^{\perp W_1}) \to 0$.

By (S14) we can write the squared omitted variable bias as

$$(\beta_{\mathrm{long},j} - \beta_{\mathrm{med}})^2 = \frac{\mathrm{var}_j(\pi_{2,j} W_2)\mathrm{var}_j(\gamma_{2,\mathrm{long},j} W_2)}{\mathrm{var}(X^{\perp W_1})^2}.$$

Since $\mathrm{var}_j(\pi_{2,j} W_2) \to \mathrm{var}(X^{\perp W_1}) > 0$ and $\mathrm{var}_j(\gamma_{2,\mathrm{long},j} W_2) \to \infty$, we have that $(\beta_{\mathrm{long},j} - \beta_{\mathrm{med}})^2 \to \infty$. Since β_{med} is fixed, we must have $|\beta_{\mathrm{long},j}| \to +\infty$.

We now show $\delta_j \to 1$. We can write

$$\begin{aligned} \delta_j &= \frac{\mathrm{cov}_j(X, \gamma_{2,\mathrm{long},j} W_2)}{\mathrm{var}_j(\gamma_{2,\mathrm{long},j} W_2)} \Bigg/ \frac{\mathrm{cov}(X, \gamma'_{1,\mathrm{long},j} W_1)}{\mathrm{var}(\gamma'_{1,\mathrm{long},j} W_1)} \\ &= \frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}} \frac{\mathrm{var}((\gamma_{1,\mathrm{med}} + (\beta_{\mathrm{med}} - \beta_{\mathrm{long},j})\pi_1)' W_1)}{\mathrm{cov}(\pi_1' W_1, (\gamma_{1,\mathrm{med}} + (\beta_{\mathrm{med}} - \beta_{\mathrm{long},j})\pi_1)' W_1)} \qquad \text{(S8)} \\ &= \frac{\mathrm{var}\left(\left(\frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}}\gamma_{1,\mathrm{med}} + \frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}}(\beta_{\mathrm{med}} - \beta_{\mathrm{long},j})\pi_1\right)' W_1\right)}{\mathrm{cov}\left(\pi_1' W_1, \left(\frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}}\gamma_{1,\mathrm{med}} + \frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}}(\beta_{\mathrm{med}} - \beta_{\mathrm{long},j})\pi_1\right)' W_1\right)}, \qquad \text{(S9)} \end{aligned}$$

where we used A4 and equation (S13) to obtain (S8). The ratio $\pi_{2,j}/\gamma_{2,\mathrm{long},j} \to 0$ as $j \to \infty$ because

$$\left|\frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}}\right| = \left(\frac{\mathrm{var}_j(\pi_{2,j} W_2)}{\mathrm{var}_j(\gamma_{2,\mathrm{long},j} W_2)}\right)^{1/2} \to \left(\frac{\mathrm{var}(X^{\perp W_1})}{\infty}\right)^{1/2} = 0.$$

The product of $\pi_{2,j}/\gamma_{2,\mathrm{long},j}$ and $(\beta_{\mathrm{med}} - \beta_{\mathrm{long},j})$ equals

$$\frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}}(\beta_{\mathrm{med}} - \beta_{\mathrm{long},j}) = \frac{\pi_{2,j}}{\gamma_{2,\mathrm{long},j}} \frac{\pi_{2,j}\gamma_{2,\mathrm{long},j}\mathrm{var}_j(W_2)}{\mathrm{var}(X^{\perp W_1})} = \frac{\mathrm{var}_j(\pi_{2,j} W_2)}{\mathrm{var}(X^{\perp W_1})}, \tag{S10}$$

where equation (S14) was used for the first equality. As derived earlier, $\mathrm{var}_j(\pi_{2,j} W_2)/\,\mathrm{var}(X^{\perp W_1}) \to 1$ as $j \to \infty$, so the ratio in (S10) converges to 1. Examining the terms in (S9) yields that $\delta_j \to 1$.

($\Leftarrow$) Suppose $|\beta_{\text{long},j}| \to +\infty$ and

$$\delta_j = \frac{\text{cov}_j(X, \gamma_{2,\text{long},j} W_2)}{\text{var}_j(\gamma_{2,\text{long},j} W_2)} \Big/ \frac{\text{cov}(X, \gamma_{1,\text{long},j}' W_1)}{\text{var}(\gamma_{1,\text{long},j}' W_1)} \to 1$$

as $j \to \infty$. By equation (S13), we have that $\gamma_{1,\text{long},j} = \gamma_{1,\text{med}} + (\beta_{\text{med}} - \beta_{\text{long},j})\pi_1$. By equation (S7) above and (S14), we have

$$\begin{aligned}
\text{var}_j(\gamma_{2,\text{long},j} W_2) &= \text{var}(Y)(R^2_{\text{long}} - R^2_{\text{med}}) + \frac{\text{var}_j(\pi_{2,j} W_2)\text{var}_j(\gamma_{2,\text{long},j} W_2)}{\text{var}(X^{\perp W_1})^2} \text{var}(X^{\perp W_1}) \\
&= \text{var}(Y)(R^2_{\text{long}} - R^2_{\text{med}}) + (\beta_{\text{long},j} - \beta_{\text{med}})^2 \text{var}(X^{\perp W_1}).
\end{aligned}$$

Since $(\beta_{\text{long},j} - \beta_{\text{med}})^2 \to \infty$ and $\text{var}(X^{\perp W_1}) > 0$, we have that $\text{var}_j(\gamma_{2,\text{long},j} W_2) \to \infty$. Again by equation (S7), we have that

$$1 - \frac{\text{var}_j(\pi_{2,j} W_2)}{\text{var}(X^{\perp W_1})} = \frac{\text{var}(Y)(R^2_{\text{long}} - R^2_{\text{med}})}{\text{var}_j(\gamma_{2,\text{long},j} W_2)} \to 0$$

and therefore $\text{var}_j(\pi_{2,j} W_2) \to \text{var}(X^{\perp W_1})$. This implies that

$$\text{var}_j(X^{\perp W_1, W_2}) = \text{var}(X^{\perp W_1}) - \text{var}_j(\pi_{2,j} W_2) \to 0,$$

as desired. $\square$

E.5 OLS with nearly multicollinear covariates

In this section we analyze the behavior of OLS estimands as the distribution of the covariates approaches perfect multicollinearity. The main result is that any given coefficient can converge to any value in $\mathbb{R} \cup \{-\infty, +\infty\}$.

Let (Y, X_1, X_2) be scalar random variables with distribution indexed by $j \in \{1, 2, \ldots\}$. Assume

$$\text{var}(X_1, X_2) = \begin{pmatrix} 1 & \rho_j \\ \rho_j & 1 \end{pmatrix}$$

where $-1 \leq \rho_j \leq 1$ is the correlation. Let $\sigma_{1,j} \coloneqq \text{cov}(Y, X_1)$ and $\sigma_{2,j} \coloneqq \text{cov}(Y, X_2)$. We will consider sequences with $\rho_j \nearrow 1$ so that both of these covariances converge to a common value. Thus, in the limit, the univariate OLS estimands $\text{cov}(Y, X_1)/\text{var}(X_1)$ and $\text{cov}(Y, X_2)/\text{var}(X_2)$ are well defined and equal to each other.

When $|\rho_j| < 1$, the projection coefficients in the linear projections of Y on $(1, X_1, X_2)$ are well defined and equal

$$\begin{pmatrix} \beta_{1,j} \\ \beta_{2,j} \end{pmatrix} \coloneqq \begin{pmatrix} 1 & \rho_j \\ \rho_j & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{1,j} \\ \sigma_{2,j} \end{pmatrix} = \frac{1}{1 - \rho_j^2} \begin{pmatrix} 1 & -\rho_j \\ -\rho_j & 1 \end{pmatrix} \begin{pmatrix} \sigma_{1,j} \\ \sigma_{2,j} \end{pmatrix} = \begin{pmatrix} \dfrac{\sigma_{1,j} - \rho_j \sigma_{2,j}}{1 - \rho_j^2} \\ \\ \dfrac{\sigma_{2,j} - \rho_j \sigma_{1,j}}{1 - \rho_j^2} \end{pmatrix}.$$

For $\sigma \in \mathbb{R}$, let

$$\mathcal{B}(\sigma) \coloneqq \{(b_1, b_2) \in \mathbb{R}^2 : b_1 + b_2 = \sigma\} \cup \{(-\infty, +\infty), (+\infty, -\infty)\}.$$

Proposition S5. For all $\sigma \in \mathbb{R}$, for each $(\beta_1, \beta_2) \in \mathcal{B}(\sigma)$ there exist sequences $\rho_j \nearrow 1$, $\sigma_{1,j} \to \sigma$, $\sigma_{2,j} \to \sigma$ as $j \to \infty$ such that $(\beta_{1,j}, \beta_{2,j}) \to (\beta_1, \beta_2)$.

E.6 Remark on the definition of a sign change

The sign of zero is often not defined. Our definition of the sign change breakdown point $\delta^{\text{bp,sign}}(R^2_{\text{long}})$ in Section 3.1 makes the convention that a change from a nonzero value to 0 is considered a sign change. We could therefore more precisely call $\delta^{\text{bp,sign}}(R^2_{\text{long}})$ the *weak* sign change breakdown point. Under this convention, $\delta^{\text{bp,sign}}(R^2_{\text{long}})$ will always be weakly smaller than the explain away breakdown point $\delta^{\text{bp,explain away}}(R^2_{\text{long}})$.

We could alternatively define the *strict* sign change breakdown point as requiring the coefficient to actually cross the axis, rather than just touch it. Except in knife-edge cases, the strict and weak sign change breakdown points for δ will be the same. This can be seen from the equation for the explain away breakdown point in Theorem 1, which is the ratio of two cubic polynomials in β_{hypo}. This ratio will generally be strictly monotonic in a neighborhood of $\beta_{\text{hypo}} = 0$, implying that the strict sign change breakdown point cannot be larger than the explain away breakdown point.

Finally, note that for other types of sensitivity parameters besides δ, it is in principle possible for the strict and weak sign change breakdown points to be different. See Section 2 of our older draft of this paper (Masten and Poirier 2022) for a discussion of sign change breakdown points in general models.

F Some Common Pitfalls

In this section we discuss a common mistake some researchers make when computing explain away breakdown points, along with a remark about the "bounding set" defined in Oster (2019).

F.1 A Common Mistake when Computing Explain Away Breakdown Points

As pointed out in Footnote 1 in Diegert, Masten, and Poirier (2025), several papers compute the explain away breakdown point $\delta^{\text{bp,explain away}}(R^2_{\text{long}})$ incorrectly (the correct value is given by Theorem 1). In this section we describe the source of this mistake. It can be avoided by using software based on the correct results, such as Oster's package `psacalc` or our package `regsensitivity` in Stata.

Proposition S3 shows how to point identify β_{long} if $\delta = 1$, R^2_{long} is known, and A7 is assumed. The identification result in Theorem S1 is strictly more general since it allows for any value of δ and does not use A7. Because this general theorem holds for any δ, it should be used for breakdown analysis, as we did earlier. However, on page 193 Oster (2019) states the following equation:

$$\beta_{\text{long}} \approx \beta_{\text{med}} + \delta(\beta_{\text{med}} - \beta_{\text{short}}) \frac{R^2_{\text{long}} - R^2_{\text{med}}}{R^2_{\text{med}} - R^2_{\text{short}}} \tag{S11}$$

where δ is no longer necessarily equal to one. We are not aware of any formal results that justify this equation—Oster (2019) does not provide any and does not explain the nature of the approximation $\approx$ in this equation. To the contrary, as we explained earlier, Theorem S1 implies that β_{long} can be arbitrarily far from its value at $\delta = 1$ even for δ values arbitrarily close to 1. Moreover, even at $\delta = 1$, β_{long} is not necessarily point identified when A7 is not assumed. Consequently, equation (S11) should not be used to perform sensitivity analysis (except when δ is exactly equal to 1 and under the other conditions of Proposition S3).

Several researchers, however, have apparently used equation (S11) to compute breakdown points. This includes Satyanath et al. (2017, *JPE*) and Bazzi, Fiszbein, and Gebresilasse (2020, *ECMA*). Specifically, they fix R^2_{long} and solve

$$\beta_{\text{med}} + \delta(\beta_{\text{med}} - \beta_{\text{short}}) \frac{R^2_{\text{long}} - R^2_{\text{med}}}{R^2_{\text{med}} - R^2_{\text{short}}} = 0$$

for δ. This solution does not equal the explain away breakdown point $\delta^{\text{bp,explain away}}(R^2_{\text{long}})$. Note that this is true even if A7 holds—Proposition S3 is only about the case $\delta = 1$ and does not describe the bias when $\delta \neq 1$.

Equation (S11) also explains why some papers do not report the magnitude of their computed $\delta^{\text{bp,explain away}}$ when it is negative. For example, Satyanath et al. (2017) report "[< 0]" in their Table A.27 results rather than providing the exact magnitude of negative estimates. In the table footnote, they say

"The entry [<0] indicates that the respective Altonji et al. ratio is negative... suggesting a downward bias for our OLS estimates due to unobservables". (online appendix page 44)

If equation (S11) were correct for all δ, then the sign of the omitted variable bias would be determined by the sign of δ *and* $\beta_{\text{med}} - \beta_{\text{short}}$. So it is incorrect to conclude that the true bias $\beta_{\text{med}} - \beta_{\text{long}}$ is negative whenever this breakdown point is negative. Regardless, this line of reasoning is all based on equation (S11) which, as we explained above, generally does not hold except when δ is exactly equal to 1 and under the other conditions of Proposition S3.

F.2 Remark on Oster's "Bounding Set"

Based on the discussion on pages 194–196, Oster (2019) defines

$$\beta^*(\delta, R^2_{\text{long}}) = \underset{b \in \mathcal{B}_I(\delta, R^2_{\text{long}})}{\text{argmin}} |\beta_{\text{med}} - b|.$$

Using this, she defines the set $[\beta_{\text{med}}, \beta^*(\delta, R^2_{\text{long}})]$ (or the reverse if β_{med} is the larger value). She calls this a "bounding set" on page 196, but elsewhere calls it the "identified set" (for example, see her Table 3). The discussion on page 196 suggests that this interval should be interpreted as the identified set for β_{long} under the assumption that $0 \leq \delta \leq 1$. As we have shown, this is not the correct interpretation. This follows because the identified set $\mathcal{B}_I(\delta, R^2_{\text{long}})$ for a known δ is not monotonically increasing (in the set inclusion order) in δ. Note that this is generally true even if we impose the magnitude constraint A5. The correct identified set under the assumption that $0 \leq \delta \leq 1$ is given by $\overline{\mathcal{B}}_I(\bar{\delta}, R^2_{\text{long}})$ from Corollary 2 with $\bar{\delta} = 1$. The left plot in Figure 2 shows an example of this set; the right plot shows its convex hull. We recommend that researchers present this set rather than the "bounding set" described above.

Finally, note that some authors (for example, Mian and Sufi 2014, online appendix page 2) define $\beta^*(\delta, R^2_{\text{long}})$ as the value from equation (7), from Oster's Proposition 1. This is equivalent to the definition above when $\delta = 1$ and A7 holds, but otherwise they are different. Regardless, this approach also leads to an interval that does not have the correct interpretation as an identified set.

G Discussion of Estimation and Inference

Our theoretical analysis in sections 2–5 focuses on population level identification results. In practice, we only have finite sample data. The identified set (5) only depends on the data through seven features of the data-generating process: β_{med}, R^2_{med}, β_{short}, R^2_{short}, $\text{var}(Y)$, $\text{var}(X)$, and $R^2_{X \sim W_1}$. In our empirical analyses of Section 6 and Appendix B, as well as our meta-analysis in Section 7 and Appendix D, we estimate all breakdown points via plug in estimators by taking the sample analog of these seven parameters.

A natural next question is to consider inference on these breakdown points (e.g., as in Masten and Poirier 2020). Our identification results, and the presence of the asymptote at $\delta = 1$ in particular, suggest that there may be substantial sampling uncertainty. For brevity we do not fully study this inferential question here, but keep in mind that all of our empirical analyses and our meta-analyses are based on point estimates. Given that the sign change breakdown points (including those that impose a magnitude restriction) are usually much closer to the $\delta = 1$ asymptote than the explain away breakdown points, we conjecture that adjusting for sampling uncertainty would further exacerbate the difference between the two types of breakdown points.

H Proofs

Let $\mathbb{L}(A \mid B)$ denote the linear projection of the scalar random variable A on the random column vector B, defined as $\mathbb{L}(A \mid B) = \mathbb{E}(AB')\mathbb{E}(BB')^{-1}B$.

Section 2

It is useful to also consider the linear projection of X onto $(1, W_1, W_2)$:

$$X = \pi'_{1,\text{long}} W_1 + \pi_{2,\text{long}} W_2 + X^{\perp W}. \tag{S12}$$

Consider π_1, the coefficient on W_1 in the linear projection of X on $(1, W_1)$. Then,

$$\begin{aligned}
\pi_1 &= \text{var}(W_1)^{-1}\,\text{cov}(W_1, X) \\
&= \text{var}(W_1)^{-1}\,\text{cov}(W_1, W_1'\pi_{1,\text{long}} + W_2\pi_{2,\text{long}} + X^{\perp W}) \\
&= \text{var}(W_1)^{-1}(\text{var}(W_1)\pi_{1,\text{long}} + \text{cov}(W_1, W_2)\pi_{2,\text{long}} + \text{cov}(W_1, X^{\perp W})) \\
&= \pi_{1,\text{long}}.
\end{aligned}$$

The third line follows by the exogenous controls A4. Similarly, if we let π_2 denote the coefficient on W_2 from the linear projection of X onto $(1, W_2)$ then $\pi_2 = \pi_{2,\text{long}}$. Given these results, we use (π_1, π_2) to represent the common values of the regression coefficients throughout the proofs.

We denote the omitted variable bias by $B := \beta_{\text{med}} - \beta_{\text{long}}$. We begin by showing the following identities that we will use throughout our proofs.

Lemma S1 (Regression Algebra)**.** Suppose A1 and A4 hold. Then the following hold:

$$\gamma_{1,\text{long}} = \gamma_{1,\text{med}} + B\pi_1 \tag{S13}$$

$$\pi_2\gamma_{2,\text{long}}\,\text{var}(W_2) = B\,\text{var}(X^{\perp W_1}) \tag{S14}$$

$$\text{cov}(X, \gamma_{1,\text{med}}'W_1) = (\beta_{\text{short}} - \beta_{\text{med}})\,\text{var}(X) \tag{S15}$$

$$(R^2_{\text{med}} - R^2_{\text{short}})\,\text{var}(Y) = \text{var}(\gamma_{1,\text{med}}'W_1) - \text{var}(X)(\beta_{\text{short}} - \beta_{\text{med}})^2. \tag{S16}$$

Proof of Lemma S1. **Proof of** (S13). We have that

$$\begin{aligned}
&\gamma_{1,\text{med}} + (\beta_{\text{med}} - \beta_{\text{long}})\pi_1 \\
&= \text{var}(W_1)^{-1}(\beta_{\text{med}}\,\text{cov}(W_1, X) + \text{var}(W_1)\gamma_{1,\text{med}} - \beta_{\text{med}}\,\text{cov}(W_1, X)) + B\,\text{var}(W_1)^{-1}\,\text{cov}(W_1, X) \\
&= \text{var}(W_1)^{-1}(\text{cov}(W_1, Y) - \beta_{\text{med}}\,\text{cov}(W_1, X)) + B\,\text{var}(W_1)^{-1}\,\text{cov}(W_1, X) \\
&= \text{var}(W_1)^{-1}(\text{cov}(W_1, Y) - \beta_{\text{long}}\,\text{cov}(W_1, X)) \\
&= \text{var}(W_1)^{-1}(\text{cov}(W_1, \beta_{\text{long}}X + \gamma_{1,\text{long}}'W_1 + \gamma_{2,\text{long}}W_2 + Y^{\perp X,W}) - \beta_{\text{long}}\,\text{cov}(W_1, X)) \\
&= \gamma_{1,\text{long}}.
\end{aligned}$$

The second equality follows from

$$\begin{aligned}
\text{cov}(W_1, Y) &= \text{cov}(W_1, \mathbb{L}(Y \mid 1, X, W_1) + Y^{\perp X, W_1}) \\
&= \text{cov}(W_1, \beta_{\text{med}}X + \gamma_{1,\text{med}}'W_1) \\
&= \beta_{\text{med}}\,\text{cov}(W_1, X) + \text{var}(W_1)\gamma_{1,\text{med}},
\end{aligned}$$

while the final equality follows from A4.

Proof of (S14). By the definition of β_{med} and the FWL Theorem,

$$\begin{aligned}
\beta_{\text{med}}\,\text{var}(X^{\perp W_1}) &= \text{cov}(Y, X^{\perp W_1}) \\
&= \text{cov}(\beta_{\text{long}}X + \gamma_{1,\text{long}}'W_1 + \gamma_{2,\text{long}}W_2 + Y^{\perp X,W}, X^{\perp W_1}) \\
&= \beta_{\text{long}}\,\text{var}(X^{\perp W_1}) + \gamma_{2,\text{long}}\,\text{cov}(W_2, X^{\perp W_1}).
\end{aligned}$$

To finish the proof, we note that $\text{cov}(W_2, X^{\perp W_1}) = \text{cov}(W_2, X - \pi_1'W_1) = \text{cov}(W_2, \pi_2 W_2 + X^{\perp W}) = \pi_2\,\text{var}(W_2)$ by A4.

Proof of (S15). By the definition of β_{short},

$$\begin{aligned}
\beta_{\text{short}}\,\text{var}(X) &= \text{cov}(Y, X) \\
&= \text{cov}(\beta_{\text{med}}X + \gamma_{1,\text{med}}'W_1 + Y^{\perp X, W_1}, X) \\
&= \beta_{\text{med}}\,\text{var}(X) + \text{cov}(X, \gamma_{1,\text{med}}'W_1).
\end{aligned}$$

Proof of (S16). By the definitions of R^2_{med} and R^2_{short},

$$\begin{aligned}
&(R^2_{\text{med}} - R^2_{\text{short}}) \operatorname{var}(Y) + \operatorname{var}(X)(\beta_{\text{short}} - \beta_{\text{med}})^2 \\
&= \operatorname{var}(X\beta_{\text{med}} + \gamma'_{1,\text{med}}W_1) - \operatorname{var}(X\beta_{\text{short}}) + \operatorname{var}(X)(\beta_{\text{short}} - \beta_{\text{med}})^2 \\
&= \operatorname{var}(X)(\beta^2_{\text{med}} - \beta^2_{\text{short}}) + 2\beta_{\text{med}} \operatorname{cov}(X, \gamma'_{1,\text{med}}W_1) + \operatorname{var}(\gamma'_{1,\text{med}}W_1) + \operatorname{var}(X)(\beta_{\text{short}} - \beta_{\text{med}})^2 \\
&= \operatorname{var}(X)(\beta^2_{\text{med}} - \beta^2_{\text{short}} + 2\beta_{\text{med}}(\beta_{\text{short}} - \beta_{\text{med}}) + (\beta_{\text{short}} - \beta_{\text{med}})^2) + \operatorname{var}(\gamma'_{1,\text{med}}W_1) \\
&= \operatorname{var}(\gamma'_{1,\text{med}}W_1).
\end{aligned}$$

The third equality follows from equation (S15). □

We also use the following lemma to prove our main results.

Lemma S2 (Oster's 3rd Equation)**.** Suppose A1–A4 hold. Then

$$(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y)\delta \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) = B \operatorname{var}(X^{\perp W_1}) \left(\operatorname{var}(\gamma'_{1,\text{long}}W_1) - B\delta \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1)\right).$$

Proof of Lemma S2. By definition,

$$\begin{aligned}
&R^2_{\text{long}} \operatorname{var}(Y) \\
&= \operatorname{var}(\beta_{\text{long}}X + \gamma'_{1,\text{long}}W_1 + \gamma_{2,\text{long}}W_2) \\
&= \beta^2_{\text{long}} \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}}W_1) + \operatorname{var}(\gamma_{2,\text{long}}W_2) \\
&\quad + 2 \operatorname{cov}(\beta_{\text{long}}X, \gamma'_{1,\text{long}}W_1) + 2 \operatorname{cov}(\beta_{\text{long}}X, \gamma_{2,\text{long}}W_2) + 2 \operatorname{cov}(\gamma'_{1,\text{long}}W_1, \gamma_{2,\text{long}}W_2) \\
&= \beta^2_{\text{long}} \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}}W_1) + \operatorname{var}(\gamma_{2,\text{long}}W_2) \\
&\quad + 2\beta_{\text{long}} \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) + 2\beta_{\text{long}} \operatorname{cov}(X, \gamma_{2,\text{long}}W_2) \\
&= \beta^2_{\text{long}} \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}}W_1) + \operatorname{var}(\gamma_{2,\text{long}}W_2) + 2\beta_{\text{long}} \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) + 2\beta_{\text{long}}B \operatorname{var}(X^{\perp W_1}).
\end{aligned}$$

The third equality follows from A4. The fourth follows from equation (S14). We can show that

$$\begin{aligned}
&R^2_{\text{med}} \operatorname{var}(Y) \\
&= \operatorname{var}(\beta_{\text{med}}X + \gamma'_{1,\text{med}}W_1) \\
&= \beta^2_{\text{med}} \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{med}}W_1) + 2\beta_{\text{med}} \operatorname{cov}(X, \gamma'_{1,\text{med}}W_1) \\
&= (\beta_{\text{long}} + B)^2 \operatorname{var}(X) + \operatorname{var}((\gamma_{1,\text{long}} - B\pi_1)'W_1) + 2(\beta_{\text{long}} + B) \operatorname{cov}(X, (\gamma_{1,\text{long}} - B\pi_1)'W_1) \\
&= (\beta_{\text{long}} + B)^2 \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}}W_1) + B^2\big(\operatorname{var}(X) - \operatorname{var}(X^{\perp W_1})\big) \\
&\qquad - 2B \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) + 2(\beta_{\text{long}} + B)\Big[\operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) + B\big(\operatorname{var}(X^{\perp W_1}) - \operatorname{var}(X)\big)\Big] \\
&= \beta^2_{\text{long}} \operatorname{var}(X) + B^2 \operatorname{var}(X) + 2\beta_{\text{long}}B \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}}W_1) + B^2\big(\operatorname{var}(X) - \operatorname{var}(X^{\perp W_1})\big) \\
&\qquad - 2B \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) + 2(\beta_{\text{long}} + B)\Big[\operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) + B\big(\operatorname{var}(X^{\perp W_1}) - \operatorname{var}(X)\big)\Big] \\
&= \beta^2_{\text{long}} \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}}W_1) + (2\beta_{\text{long}}B + B^2) \operatorname{var}(X^{\perp W_1}) + 2\beta_{\text{long}} \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1).
\end{aligned}$$

The third equality follows from equation (S13). Subtracting this from the R-squared long equation gives

$$\begin{aligned}
&(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \\
&= \beta^2_{\text{long}} \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}}W_1) + \operatorname{var}(\gamma_{2,\text{long}}W_2) + 2\beta_{\text{long}} \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) + 2\beta_{\text{long}}B \operatorname{var}(X^{\perp W_1}) \\
&\qquad - \beta^2_{\text{long}} \operatorname{var}(X) - \operatorname{var}(\gamma'_{1,\text{long}}W_1) - (2\beta_{\text{long}}B + B^2) \operatorname{var}(X^{\perp W_1}) - 2\beta_{\text{long}} \operatorname{cov}(X, \gamma'_{1,\text{long}}W_1) \\
&= \operatorname{var}(\gamma_{2,\text{long}}W_2) - B^2 \operatorname{var}(X^{\perp W_1}).
\end{aligned}$$

Then, multiply both sides of this equation by $\delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1)$:

$$
\begin{aligned}
&(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1) \\
&= \delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1) \operatorname{var}(\gamma_{2,\text{long}} W_2) - B^2 \operatorname{var}(X^{\perp W_1}) \delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1) \\
&= \operatorname{cov}(X, \gamma_{2,\text{long}} W_2) \operatorname{var}(\gamma_{1,\text{long}}' W_1) - B^2 \operatorname{var}(X^{\perp W_1}) \delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1) \\
&= \pi_2 \gamma_{2,\text{long}} \operatorname{var}(W_2) \operatorname{var}(\gamma_{1,\text{long}}' W_1) - B^2 \operatorname{var}(X^{\perp W_1}) \delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1) \\
&= B \operatorname{var}(X^{\perp W_1}) \operatorname{var}(\gamma_{1,\text{long}}' W_1) - B^2 \operatorname{var}(X^{\perp W_1}) \delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1).
\end{aligned}
$$

The second equality follows from A2, the third from A4, and the fourth from equation (S14). □

Proof of Theorem S1. We split this proof into two parts.

Part 1: First we show that $\beta_{\text{long}} \in \mathcal{B}_I(\delta, R^2_{\text{long}})$. To prove this, it suffices to show that $B = \beta_{\text{med}} - \beta_{\text{long}}$ satisfies $f(B, \delta, R^2_{\text{long}}) = 0$ and that $\beta_{\text{long}} \notin \mathcal{B}_{\text{A3fail}}$. By Lemma S2 we can write

$$
(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1) = B \operatorname{var}(X^{\perp W_1}) \left(\operatorname{var}(\gamma_{1,\text{long}}' W_1) - B\delta \operatorname{cov}(X, \gamma_{1,\text{long}}' W_1) \right).
$$

Substituting $\gamma_{1,\text{long}} = \gamma_{1,\text{med}} + B\pi_1$, which was shown in Lemma S1, we obtain the equation

$$
\begin{aligned}
&\delta (R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \left(\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) + B \operatorname{var}(\pi_1' W_1) \right) \\
&= B \operatorname{var}(X^{\perp W_1}) \Big[\operatorname{var}(\gamma_{1,\text{med}}' W_1) + 2B \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) \\
&\quad + B^2 \operatorname{var}(\pi_1' W_1) - B\delta \left(\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) + B \operatorname{var}(\pi_1' W_1) \right) \Big]
\end{aligned}
$$

where we used that $\operatorname{cov}(X, \pi_1' W_1) = \operatorname{var}(\pi_1' W_1)$, which follows from equation (S12) and A4.

Collecting terms, this equation is equivalent to $f(B, \delta, R^2_{\text{long}}) = 0$ as defined above. Recall that $\gamma_{1,\text{long}} = \gamma_{1,\text{med}} + B\pi_1$. By A3, $\gamma_{1,\text{long}} \neq 0$ and thus $\beta_{\text{long}} = \beta_{\text{med}} - B \notin \mathcal{B}_{\text{A3fail}}$. Therefore, $\beta_{\text{long}} \in \mathcal{B}_I(\delta, R^2_{\text{long}})$.

Part 2: Next we show that $\mathcal{B}_I(\delta, R^2_{\text{long}})$ is sharp. Let $b \in \mathcal{B}_I(\delta, R^2_{\text{long}})$ and $B = \beta_{\text{med}} - b$. This implies that $f(B, \delta, R^2_{\text{long}}) = 0$ and $b \notin \mathcal{B}_{\text{A3fail}}$. Fix the distribution of the observables (Y, X, W_1) and assume that $\operatorname{var}(Y, X, W_1)$ is positive definite.

Let

$$
\begin{aligned}
g_1 &\coloneqq \gamma_{1,\text{med}} + B\pi_1 \\
g_2 &\coloneqq \left((R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) + B^2 \operatorname{var}(X^{\perp W_1}) \right)^{1/2} \\
p_1 &\coloneqq \operatorname{var}(W_1)^{-1} \operatorname{cov}(W_1, X) \\
p_2 &\coloneqq \frac{B \operatorname{var}(X^{\perp W_1})}{\left((R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) + B^2 \operatorname{var}(X^{\perp W_1}) \right)^{1/2}}.
\end{aligned}
$$

Note that g_2 is well defined and strictly positive by $R^2_{\text{long}} > R^2_{\text{med}}$ and $\operatorname{var}(Y) > 0$, which follow from A1. p_2 is well defined by $g_2 > 0$. $g_1 \neq 0$ by $b \notin \mathcal{B}_{\text{A3fail}}$. Also note that $p_1 = \pi_1$ by our assumptions.

Under these choices, we have

$$
\begin{aligned}
b \operatorname{var}(X) + g_1' \operatorname{var}(W_1) p_1 + p_2 g_2 &= (\beta_{\text{med}} - B) \operatorname{var}(X) + (\gamma_{1,\text{med}} + B\pi_1)' \operatorname{var}(W_1) \pi_1 + B \operatorname{var}(X^{\perp W_1}) \\
&= \beta_{\text{med}} \operatorname{var}(X) + \gamma_{1,\text{med}}' \operatorname{var}(W_1) \pi_1 \\
&= \beta_{\text{med}} \operatorname{var}(X) + (\operatorname{cov}(Y, W_1) - \beta_{\text{med}} \operatorname{cov}(X, W_1)) \operatorname{var}(W_1)^{-1} \operatorname{var}(W_1) \pi_1 \\
&= \beta_{\text{med}} \operatorname{var}(X^{\perp W_1}) + \operatorname{cov}(Y, W_1)' \pi_1 \\
&= \operatorname{cov}(Y, X) \qquad \text{(S17)}
\end{aligned}
$$

and

$$\begin{aligned}
bp_1' \operatorname{var}(W_1) + g_1' \operatorname{var}(W_1) &= (\beta_{\text{med}} - B)\pi_1' \operatorname{var}(W_1) + (\gamma_{1,\text{med}} + B\pi_1)' \operatorname{var}(W_1) \\
&= \beta_{\text{med}}\pi_1' \operatorname{var}(W_1) + (\operatorname{cov}(Y, W_1) - \beta_{\text{med}} \operatorname{cov}(X, W_1)) \operatorname{var}(W_1)^{-1} \operatorname{var}(W_1) \\
&= \operatorname{cov}(Y, W_1). \qquad \text{(S18)}
\end{aligned}$$

Our goal is to construct a joint distribution for (Y, X, W_1, W_2) that is consistent with the observed distribution of (Y, X, W_1), that is such that $\operatorname{var}(Y, X, W_1, W_2)$ is finite, $\operatorname{var}(X, W_1, W_2)$ is positive definite, A2–A4 hold, that is consistent with the value of R^2_{long}, and such that projection equations (1) and (S12) hold.

To help define the joint distribution of (Y, X, W_1, W_2), let

$$\Sigma := \begin{pmatrix} \operatorname{var}(Y) & \operatorname{cov}(Y, X) & \operatorname{cov}(Y, W_1) & bp_2 + g_2 \\ \operatorname{cov}(Y, X) & \operatorname{var}(X) & \operatorname{cov}(X, W_1) & p_2 \\ \operatorname{cov}(W_1, Y) & \operatorname{cov}(W_1, X) & \operatorname{var}(W_1) & 0 \\ bp_2 + g_2 & p_2 & 0 & 1 \end{pmatrix}.$$

The top left $(d_1 + 2) \times (d_1 + 2)$ block is positive definite by assumption. By Sylvester's criterion, we can then verify that this matrix is positive semidefinite by showing its determinant is nonnegative. Before doing so, we first verify that $\operatorname{var}(X^{\perp W_1}) > p_2^2$:

$$\begin{aligned}
\operatorname{var}(X^{\perp W_1}) - p_2^2 &= \operatorname{var}(X^{\perp W_1}) - \frac{B^2 \operatorname{var}(X^{\perp W_1})^2}{(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) + B^2 \operatorname{var}(X^{\perp W_1})} \\
&= \operatorname{var}(X^{\perp W_1}) \frac{(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y)}{(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) + B^2 \operatorname{var}(X^{\perp W_1})} \\
&> 0.
\end{aligned}$$

The inequality follows from the positive-definiteness of $\operatorname{var}(Y, X, W_1)$, $R^2_{\text{long}} > R^2_{\text{med}}$, and by $\operatorname{var}(Y) > 0$.

This inequality allows us to show that the matrix

$$\Sigma' := \begin{pmatrix} \operatorname{var}(X) & \operatorname{cov}(X, W_1) & p_2 \\ \operatorname{cov}(W_1, X) & \operatorname{var}(W_1) & 0 \\ p_2 & 0 & 1 \end{pmatrix}$$

is invertible via the partitioned inverse matrix formula:

$$\begin{aligned}
&\begin{pmatrix} \operatorname{var}(X) & \operatorname{cov}(X, W_1) & p_2 \\ \operatorname{cov}(W_1, X) & \operatorname{var}(W_1) & 0 \\ p_2 & 0 & 1 \end{pmatrix}^{-1} \\
&= \left(\operatorname{var}(X) - \operatorname{cov}(X, W_1) \operatorname{var}(W_1)^{-1} \operatorname{cov}(W_1, X) - p_2^2\right)^{-1} \\
&\cdot \begin{pmatrix} 1 & -\begin{pmatrix} p_1' & p_2 \end{pmatrix} \\ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} & \left(\operatorname{var}(X) - \operatorname{cov}(X, W_1) \operatorname{var}(W_1)^{-1} \operatorname{cov}(W_1, X) - p_2^2\right) \begin{pmatrix} \operatorname{var}(W_1)^{-1} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \begin{pmatrix} p_1' & p_2 \end{pmatrix} \end{pmatrix} \\
&= \left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right)^{-1} \begin{pmatrix} 1 & -p_1' & -p_2 \\ -p_1 & \operatorname{var}(W_1)^{-1} \left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right) + p_1 p_1' & p_1 p_2 \\ -p_2 & p_2 p_1' & \operatorname{var}(X^{\perp W_1}) \end{pmatrix}.
\end{aligned}$$

Since Σ' is invertible, we can use the partitioned determinant formula to compute the determinant of Σ.

$\det(\Sigma)$

$$
= \det \begin{pmatrix} \operatorname{var}(Y) & \operatorname{cov}(Y,X) & \operatorname{cov}(Y,W_1) & bp_2+g_2 \\ \operatorname{cov}(Y,X) & \operatorname{var}(X) & \operatorname{cov}(X,W_1) & p_2 \\ \operatorname{cov}(W_1,Y) & \operatorname{cov}(W_1,X) & \operatorname{var}(W_1) & 0 \\ bp_2+g_2 & p_2 & 0 & 1 \end{pmatrix}
$$

$$
= 1 \cdot \det \left(\operatorname{var}(Y) - \begin{pmatrix} \operatorname{cov}(Y,X) & \operatorname{cov}(Y,W_1) & bp_2+g_2 \end{pmatrix} \begin{pmatrix} \operatorname{var}(X) & \operatorname{cov}(X,W_1) & p_2 \\ \operatorname{cov}(W_1,X) & \operatorname{var}(W_1) & 0 \\ p_2 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \operatorname{cov}(Y,X) \\ \operatorname{cov}(W_1,Y) \\ bp_2+g_2 \end{pmatrix} \right)
$$

$$
= \operatorname{var}(Y) - \begin{pmatrix} \operatorname{cov}(Y,X) & \operatorname{cov}(Y,W_1) & bp_2+g_2 \end{pmatrix} \left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right)^{-1}
$$

$$
\cdot \begin{pmatrix} 1 & -p_1' & -p_2 \\ -p_1 & \operatorname{var}(W_1)^{-1}\left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right) + p_1 p_1' & p_1 p_2 \\ -p_2 & p_2 p_1' & \operatorname{var}(X^{\perp W_1}) \end{pmatrix} \begin{pmatrix} \operatorname{cov}(Y,X) \\ \operatorname{cov}(W_1,Y) \\ bp_2+g_2 \end{pmatrix}.
$$

Using equations (S17)–(S18), we can write

$$
\begin{pmatrix} 1 & -p_1' & -p_2 \\ -p_1 & \operatorname{var}(W_1)^{-1}\left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right) + p_1 p_1' & p_1 p_2 \\ -p_2 & p_2 p_1' & \operatorname{var}(X^{\perp W_1}) \end{pmatrix} \begin{pmatrix} \operatorname{cov}(Y,X) \\ \operatorname{cov}(W_1,Y) \\ bp_2+g_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & -p_1' & -p_2 \\ -p_1 & \operatorname{var}(W_1)^{-1}\left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right) + p_1 p_1' & p_1 p_2 \\ -p_2 & p_2 p_1' & \operatorname{var}(X^{\perp W_1}) \end{pmatrix} \begin{pmatrix} b\operatorname{var}(X) + g_1' \operatorname{var}(W_1) p_1 + p_2 g_2 \\ b \operatorname{var}(W_1) p_1 + \operatorname{var}(W_1) g_1 \\ bp_2+g_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} b\operatorname{var}(X) + g_1'\operatorname{var}(W_1)p_1 + p_2 g_2 - b\operatorname{var}(p_1'W_1) - p_1'\operatorname{var}(W_1)g_1 - bp_2^2 - g_2 p_2 \\ \\ -p_1(b\operatorname{var}(X) + g_1'\operatorname{var}(W_1)p_1 + p_2 g_2) + (\operatorname{var}(X^{\perp W_1}) - p_2^2)(bp_1 + g_1) \\ +bp_1\operatorname{var}(p_1'W_1) + p_1 p_1'\operatorname{var}(W_1)g_1 + p_1 p_2(bp_2+g_2) \\ \\ -p_2(b\operatorname{var}(X) + g_1'\operatorname{var}(W_1)p_1 + p_2 g_2) + p_2 b\operatorname{var}(p_1'W_1) + p_2 p_1'\operatorname{var}(W_1)g_1 + \operatorname{var}(X^{\perp W_1})(bp_2+g_2) \end{pmatrix}
$$

$$
= \begin{pmatrix} b(\operatorname{var}(X^{\perp W_1}) - p_2^2) \\ (\operatorname{var}(X^{\perp W_1}) - p_2^2) g_1 \\ -p_2(p_2 g_2) + \operatorname{var}(X^{\perp W_1})(g_2) \end{pmatrix}
$$

$$
= (\operatorname{var}(X^{\perp W_1}) - p_2^2) \begin{pmatrix} b \\ g_1 \\ g_2 \end{pmatrix}.
$$

Therefore,

$$
\operatorname{var}(Y) - \begin{pmatrix} \operatorname{cov}(Y,X) & \operatorname{cov}(Y,W_1) & bp_2+g_2 \end{pmatrix} \left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right)^{-1}
$$

$$
\cdot \begin{pmatrix} 1 & -p_1' & -p_2 \\ -p_1 & \operatorname{var}(W_1)^{-1}\left(\operatorname{var}(X^{\perp W_1}) - p_2^2\right) + p_1 p_1' & p_1 p_2 \\ -p_2 & p_2 p_1' & \operatorname{var}(X^{\perp W_1}) \end{pmatrix} \begin{pmatrix} \operatorname{cov}(Y,X) \\ \operatorname{cov}(Y,W_1) \\ bp_2+g_2 \end{pmatrix}
$$

$$
= \operatorname{var}(Y) - \begin{pmatrix} b\operatorname{var}(X) + g_1'\operatorname{var}(W_1)p_1 + p_2 g_2 & b\operatorname{var}(W_1)p_1 + \operatorname{var}(W_1)g_1 & bp_2 + g_2 \end{pmatrix} \begin{pmatrix} b \\ g_1 \\ g_2 \end{pmatrix}
$$

$$
= \operatorname{var}(Y) - b^2\operatorname{var}(X) - 2bg_1'\operatorname{var}(W_1)p_1 - 2bp_2 g_2 - \operatorname{var}(g_1'W_1) - g_2^2.
$$

We now substitute the choices for (b, g_1, p_1, p_2, g_2) and obtain

$$
\det(\Sigma)
$$

$$
= \operatorname{var}(Y) - (\beta_{\text{med}} - B)^2\operatorname{var}(X) - 2(\beta_{\text{med}} - B)(\gamma_{1,\text{med}} + B\pi_1)'\operatorname{var}(W_1)\pi_1 - 2(\beta_{\text{med}} - B)B\operatorname{var}(X^{\perp W_1})
$$

$$
\begin{aligned}
&\quad - \operatorname{var}((\gamma_{1,\text{med}} + B\pi_1)'W_1) - (R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) - B^2 \operatorname{var}(X^{\perp W_1}) \\
&= \operatorname{var}(Y)(1 - R^2_{\text{long}}) - \beta^2_{\text{med}} \operatorname{var}(X) + 2\beta_{\text{med}} B \operatorname{var}(X) - B^2 \operatorname{var}(X) - 2\beta_{\text{med}}\gamma'_{1,\text{med}} \operatorname{var}(W_1)\pi_1 \\
&\quad - 2\beta_{\text{med}} B \operatorname{var}(\pi_1' W_1) + 2B\gamma'_{1,\text{med}} \operatorname{var}(W_1)\pi_1 + 2B^2 \operatorname{var}(W_1'\pi_1) - 2\beta_{\text{med}} B \operatorname{var}(X^{\perp W_1}) + 2B^2 \operatorname{var}(X^{\perp W_1}) \\
&\quad - \operatorname{var}(\gamma'_{1,\text{med}} W_1) - 2B\pi_1' \operatorname{var}(W_1)\gamma_{1,\text{med}} - B^2 \operatorname{var}(\pi_1' W_1) + R^2_{\text{med}} \operatorname{var}(Y) - B^2 \operatorname{var}(X^{\perp W_1}) \\
&= \operatorname{var}(Y)(1 - R^2_{\text{long}}) - \Big(\beta^2_{\text{med}} \operatorname{var}(X) + 2\beta_{\text{med}}\gamma'_{1,\text{med}} \operatorname{var}(W_1)\pi_1 + \operatorname{var}(\gamma'_{1,\text{med}} W_1)\Big) + R^2_{\text{med}} \operatorname{var}(Y) \\
&= \operatorname{var}(Y)(1 - R^2_{\text{long}}) - R^2_{\text{med}} \operatorname{var}(Y) + R^2_{\text{med}} \operatorname{var}(Y) \\
&= \operatorname{var}(Y)(1 - R^2_{\text{long}}) \\
&\geq 0
\end{aligned}
$$

by $R^2_{\text{long}} \leq 1$. From this, we conclude that Σ is positive semidefinite.

To construct W_2, let

$$
v \coloneqq 1 - \begin{pmatrix} bp_2 + g_2 & p_2 & 0 \end{pmatrix} \operatorname{var}(Y, X, W_1)^{-1} \begin{pmatrix} bp_2 + g_2 \\ p_2 \\ 0 \end{pmatrix}.
$$

Since Σ is positive semidefinite and from $\operatorname{var}(Y, X, W_1)$ being positive definite, we have that $v \geq 0$. Let $\widetilde{W}_2$ denote a unit-variance random variable that is uncorrelated with (Y, X, W_1) and let

$$
W_2 \coloneqq \begin{pmatrix} bp_2 + g_2 & p_2 & 0 \end{pmatrix} \operatorname{var}(Y, X, W_1)^{-1} \begin{pmatrix} Y \\ X \\ W_1 \end{pmatrix} + v^{1/2}\widetilde{W}_2.
$$

We can verify that $\operatorname{var}(Y, X, W_1, W_2) = \Sigma$. Note that we arbitrarily selected 1 as the value of $\operatorname{var}(W_2)$. Thus we have constructed a joint distribution for (Y, X, W_1, W_2) that is consistent with the observed distribution of (Y, X, W_1) and has a finite variance matrix equal to Σ. $\operatorname{var}(X, W_1, W_2) = \Sigma'$ is positive definite since it is a positive semidefinite matrix satisfying $\det(\Sigma') > 0$. Therefore, A1 holds.

We now show this joint distribution and (b, g_1, g_2, p_1, p_2) are consistent with A3. In what follows, let $(\beta, \gamma_{1,\text{long}}, \gamma_{2,\text{long}}, \pi_1, \pi_2) = (b, g_1, g_2, p_1, p_2)$. Then $\gamma_{2,\text{long}} \neq 0$ since g_2 was shown to be strictly positive. Similarly, $\gamma_{1,\text{long}} = \gamma_{1,\text{med}} + B\pi_1 \neq 0$ by $\beta_{\text{med}} - B \notin \mathcal{B}_{\text{A3fail}}$. Therefore, A3 is satisfied.

A4 holds since the value of $\operatorname{cov}(W_1, W_2)$ implied by matrix Σ is 0.

We now show this joint distribution and $(\beta, \gamma_{1,\text{long}}, \gamma_{2,\text{long}}, \pi_1, \pi_2)$ are consistent with the definition of R^2_{long} and A2.

For R^2_{long}, we have that

$$
\begin{aligned}
&\beta^2 \operatorname{var}(X) + \operatorname{var}(\gamma'_{1,\text{long}} W_1) + \operatorname{var}(\gamma_{2,\text{long}} W_2) + 2\beta \operatorname{cov}(X, \gamma'_{1,\text{long}} W_1) \\
&\qquad + 2\beta \operatorname{cov}(X, \gamma_{2,\text{long}} W_2) + 2 \operatorname{cov}(\gamma'_{1,\text{long}} W_1, \gamma_{2,\text{long}} W_2) \\
&= \beta^2 \operatorname{var}(X) + 2\beta\gamma'_{1,\text{long}} \operatorname{var}(W_1)\pi_1 + 2\beta\pi_2\gamma_{2,\text{long}} + \operatorname{var}(\gamma'_{1,\text{long}} W_1) + \gamma^2_{2,\text{long}} \\
&= R^2_{\text{long}} \operatorname{var}(Y)
\end{aligned}
$$

as shown above in the proof of positive semidefiniteness of Σ.

For δ, using A3 we can write

$$
\begin{aligned}
&\operatorname{var}(\gamma'_{1,\text{long}} W_1) \operatorname{var}(\gamma_{2,\text{long}} W_2) \left(\frac{\operatorname{cov}(X, \gamma_{2,\text{long}} W_2)}{\operatorname{var}(\gamma_{2,\text{long}} W_2)} - \delta \cdot \frac{\operatorname{cov}(X, \gamma'_{1,\text{long}} W_1)}{\operatorname{var}(\gamma'_{1,\text{long}} W_1)} \right) \\
&= \pi_2 \gamma_{2,\text{long}} \operatorname{var}(\gamma'_{1,\text{long}} W_1) - \delta \operatorname{cov}(X, \gamma'_{1,\text{long}} W_1) \gamma^2_{2,\text{long}} \\
&= B \operatorname{var}(X^{\perp W_1}) \operatorname{var}((\gamma_{1,\text{med}} + B\pi_1)' W_1) - \delta \operatorname{cov}(X, (\gamma_{1,\text{med}} + B\pi_1)' W_1)(\gamma^2_{2,\text{long}})
\end{aligned}
$$

$$
\begin{aligned}
&= B\,\text{var}(X^{\perp W_1})\left(\text{var}(\gamma'_{1,\text{med}}W_1) + 2B\,\text{cov}(X, \gamma'_{1,\text{med}}W_1) + B^2\,\text{var}(\pi'_1 W_1)\right) \\
&\quad - \delta((R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y) + B^2\,\text{var}(X^{\perp W_1}))(\text{cov}(X, \gamma'_{1,\text{med}}W_1) + B\,\text{var}(\pi'_1 W_1)) \\
&= -f_0(B) - \delta f_1(B, R^2_{\text{long}}).
\end{aligned}
$$

By $f(B, \delta, R^2_{\text{long}}) = f_0(B) + \delta f_1(B, R^2_{\text{long}}) = 0$ we have that

$$
\delta \cdot \frac{\text{cov}(X, \gamma'_{1,\text{long}}W_1)}{\text{var}(\gamma'_{1,\text{long}}W_1)} = \frac{\text{cov}(X, \gamma_{2,\text{long}}W_2)}{\text{var}(\gamma_{2,\text{long}}W_2)}
$$

and A2 holds.

To finish the sharpness proof, we verify that this joint distribution is consistent with equations (1) and (S12).

To verify equation (1), we show that

$$
\text{cov}\left(Y - bX - g'_1 W_1 - g_2 W_2, \begin{pmatrix} X \\ W_1 \\ W_2 \end{pmatrix}\right) = 0.
$$

To see this, note that

$$
\begin{aligned}
\text{cov}(Y - bX - g'_1 W_1 - g_2 W_2, X) &= \text{cov}(Y, X) - b\,\text{var}(X) - g'_1\,\text{var}(W_1)p_1 - g_2 p_2 \\
&= 0 \\
\text{cov}(Y - bX - g'_1 W_1 - g_2 W_2, W_1) &= \text{cov}(Y, W_1) - b\,\text{cov}(X, W_1) - g'_1\,\text{var}(W_1) \\
&= 0
\end{aligned}
$$

from equations (S17) and (S18) above. We can directly show that

$$
\begin{aligned}
\text{cov}(Y - bX - g'_1 W_1 - g_2 W_2, W_2) &= bp_2 + g_2 - bp_2 - g_2\,\text{var}(W_2) \\
&= 0
\end{aligned}
$$

since $\text{var}(W_2) = 1$ by construction.

To verify equation (S12), we see that

$$
\begin{aligned}
\text{cov}\left(X - p'_1 W_1 - p_2 W_2, \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}\right) &= \begin{pmatrix} \text{cov}(X, W_1) - p'_1\,\text{var}(W_1) \\ \text{cov}(X, W_2) - p_2\,\text{var}(W_2) \end{pmatrix} \\
&= \begin{pmatrix} p'_1\,\text{var}(W_1) - p'_1\,\text{var}(W_1) \\ p_2 - p_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

□

Proof of Proposition S1. Let $A := \mathbb{L}(Y \mid 1, X, W_1, W_2)$. Note that $R^2_{\text{long}}\,\text{var}(Y) = \text{var}(A)$ and that $R^2_{\text{med}}\,\text{var}(Y) = \text{var}(\mathbb{L}(Y \mid 1, X, W_1))$. By iterated projections, $\mathbb{L}(A \mid 1, X, W_1) = \mathbb{L}(Y \mid 1, X, W_1)$. Therefore,

$$
\begin{aligned}
&\text{var}(A - \mathbb{L}(Y \mid 1, X, W_1)) \\
&= \text{var}(A) - 2\,\text{cov}(A, \mathbb{L}(Y \mid 1, X, W_1)) + \text{var}(\mathbb{L}(Y \mid 1, X, W_1)) \\
&= \text{var}(A) - 2\,\text{cov}(\mathbb{L}(A \mid 1, X, W_1) + A^{\perp X, W_1}, \mathbb{L}(Y \mid 1, X, W_1)) + \text{var}(\mathbb{L}(Y \mid 1, X, W_1)) \\
&= \text{var}(A) - 2\,\text{var}(\mathbb{L}(Y \mid 1, X, W_1)) + \text{var}(\mathbb{L}(Y \mid 1, X, W_1)) \\
&= \text{var}(A) - \text{var}(\mathbb{L}(Y \mid 1, X, W_1)) \\
&= (R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y).
\end{aligned}
$$

Therefore, $R^2_{\text{med}} = R^2_{\text{long}}$ implies that $A = \mathbb{L}(Y \mid 1, X, W_1)$ with probability 1. Rearranging this yields

$$(\beta_{\text{long}} - \beta_{\text{med}})X + (\gamma_{1,\text{long}} - \gamma_{1,\text{med}})'W_1 + \gamma_{2,\text{long}}W_2 = 0$$

with probability 1. By A1, (X, W_1, W_2) are not perfectly collinear and therefore $\beta_{\text{long}} = \beta_{\text{med}}$. □

Proof of Corollary S1. When $\delta = 0$,

$$f(B, \delta, R^2_{\text{long}}) = f_0(B) = -B\,\text{var}(X^{\perp W_1})\,\text{var}((\gamma_{1,\text{med}} + B\pi_1)'W_1),$$

a cubic polynomial. It has up to three roots because $\text{var}(X^{\perp W_1}) > 0$ by A1. $B = 0$ is a solution to the cubic equation $f_0(B) = 0$. Other solutions must satisfy $\text{var}((\gamma_{1,\text{med}} + B\pi_1)'W_1) = 0$, or $\gamma_{1,\text{med}} + B\pi_1 = 0$ by $\text{var}(W_1)$ being positive definite. Under assumptions A1–A4 and by Theorem S1, $\beta_{\text{long}} \notin \mathcal{B}_{\text{A3fail}}$ so these other solutions do not correspond to elements of $\mathcal{B}_I(0, R^2_{\text{long}})$. The solution $B = 0$, corresponding to $\beta_{\text{long}} = \beta_{\text{med}}$, is part of the identified set provided that $\beta_{\text{med}} \notin \mathcal{B}_{\text{A3fail}}$, or that $\gamma_{1,\text{med}} + (\beta_{\text{med}} - \beta_{\text{med}})\pi_1 \neq 0$. This is directly assumed in the statement of the corollary, so $\mathcal{B}_I(0, R^2_{\text{long}}) = \{\beta_{\text{med}}\}$. □

Proof of Theorem 1. If $\beta_{\text{long}} = \beta_{\text{hypo}}$, then $\beta_{\text{hypo}} \in \mathcal{B}_I(\delta, R^2_{\text{long}})$. Then we must have

$$f_0(\beta_{\text{med}} - \beta_{\text{hypo}}) + \delta f_1(\beta_{\text{med}} - \beta_{\text{hypo}}, R^2_{\text{long}}) = 0.$$

Because of this, $f_1(\beta_{\text{med}} - \beta_{\text{hypo}}, R^2_{\text{long}}) = 0$ only if $f_0(\beta_{\text{med}} - \beta_{\text{hypo}}) = 0$. As shown in the proof of Corollary S1, this is the case only when $\beta_{\text{hypo}} = \beta_{\text{med}}$, in which case

$$f_1(0, R^2_{\text{long}}) = (R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{cov}(X, \gamma'_{1,\text{med}}W_1) \neq 0$$

since $\beta_{\text{short}} \neq \beta_{\text{med}}$. Therefore, $f_1(\beta_{\text{med}} - \beta_{\text{hypo}}, R^2_{\text{long}}) \neq 0$, and

$$\delta = \frac{-f_0(\beta_{\text{med}} - \beta_{\text{hypo}})}{f_1(\beta_{\text{med}} - \beta_{\text{hypo}}, R^2_{\text{long}})}$$

is identified.

Finally, by definition, $\{\delta \in \mathbb{R} : \beta_{\text{hypo}} \in \mathcal{B}_I(\delta, R^2_{\text{long}})\}$ is a singleton equal to

$$\frac{-f_0(\beta_{\text{med}} - \beta_{\text{hypo}})}{f_1(\beta_{\text{med}} - \beta_{\text{hypo}}, R^2_{\text{long}})}.$$

Hence the exact value breakdown point equals the absolute value of this singleton. □

Section 3

Proof of Theorem 2. We start this proof by noting that

$$\text{var}(\pi_1'W_1) = \pi_1'\,\text{var}(W_1)\pi_1 = \text{cov}(X, W_1)\,\text{var}(W_1)^{-1}\,\text{cov}(W_1, X) > 0$$

because $\text{var}(W_1)^{-1}$ is positive definite by A1, and because $\text{cov}(X, W_1) \neq 0$ by $\text{cov}(X, \gamma'_{1,\text{med}}W_1) \neq 0$.

We also note that $\pi_1 \neq 0$ implies the set $\mathcal{B}_{\text{A3fail}}$ is a singleton or empty.

Let $B_n \uparrow +\infty$ and

$$\delta_n := \frac{-f_0(B_n)}{f_1(B_n, R^2_{\text{long}})}.$$

By construction, $\beta_{\text{med}} - B_n \in \mathcal{B}_I(\delta_n, R^2_{\text{long}}) \cup \mathcal{B}_{\text{A3fail}}$. Since $\mathcal{B}_{\text{A3fail}}$ contains at most one element, we let $\beta_{\text{med}} - B_n \in \mathcal{B}_I(\delta_n, R^2_{\text{long}})$ without loss of generality by removing at most one element from this sequence.

Then

$$\delta_n = \left(B_n^3\,\text{var}(\pi_1'W_1)\,\text{var}(X^{\perp W_1}) + 2B_n^2\,\text{var}(X^{\perp W_1})\,\text{cov}(X, \gamma'_{1,\text{med}}W_1) + B_n\,\text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}}W_1)\right)$$

$$\Bigg/\Bigg(B_n^3 \operatorname{var}(X^{\perp W_1}) \operatorname{var}(\pi_1' W_1) + B_n^2 \operatorname{var}(X^{\perp W_1}) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)$$
$$+ B_n (R_{\text{long}}^2 - R_{\text{med}}^2) \operatorname{var}(Y) \operatorname{var}(\pi_1' W_1) + (R_{\text{long}}^2 - R_{\text{med}}^2) \operatorname{cov}(X, W_1' \gamma_{1,\text{med}}) \operatorname{var}(Y)\Bigg)$$
$$\to 1$$

as $n \to \infty$ since $\operatorname{var}(X^{\perp W_1}) \operatorname{var}(\pi_1' W_1) > 0$ multiplies the cubic term in both the numerator and denominator. Note that $\operatorname{var}(X^{\perp W_1}) > 0$ by A1. Similarly, we have that

$$\tilde{\delta}_n := \frac{-f_0(-B_n)}{f_1(-B_n, R_{\text{long}}^2)} \to 1$$

as $n \to \infty$. Note that $\beta_{\text{med}} + B_n \in \mathcal{B}_I(\tilde{\delta}_n, R_{\text{long}}^2)$ by construction and by the same argument used to show $\beta_{\text{med}} - B_n \notin \mathcal{B}_{\text{A3fail}}$.

Moreover, for n sufficiently large we have that

$$\operatorname{sign}(\delta_n - 1) = \operatorname{sign}(\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)).$$

We can see this from

$$\begin{aligned}
\delta_n - 1 &= f_1(B_n, R_{\text{long}}^2)^{-1}(-f_0(B_n) - f_1(B_n, R_{\text{long}}^2)) \\
&= f_1(B_n, R_{\text{long}}^2)^{-1}\Big(B_n^3 \operatorname{var}(X^{\perp W_1}) \operatorname{var}(\pi_1' W_1) + 2B_n^2 \operatorname{var}(X^{\perp W_1}) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) \\
&\qquad + B_n \operatorname{var}(X^{\perp W_1}) \operatorname{var}(\gamma_{1,\text{med}}' W_1)\Big) \\
&\quad - f_1(B_n, R_{\text{long}}^2)^{-1}\Big(B_n^3 \operatorname{var}(X^{\perp W_1}) \operatorname{var}(\pi_1' W_1) + B_n^2 \operatorname{var}(X^{\perp W_1}) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) \\
&\qquad + B_n (R_{\text{long}}^2 - R_{\text{med}}^2) \operatorname{var}(Y) \operatorname{var}(\pi_1' W_1) + (R_{\text{long}}^2 - R_{\text{med}}^2) \operatorname{cov}(X, W_1' \gamma_{1,\text{med}}) \operatorname{var}(Y)\Big) \\
&= f_1(B_n, R_{\text{long}}^2)^{-1} B_n^2 \left(\operatorname{var}(X^{\perp W_1}) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) + O(B_n^{-1})\right),
\end{aligned}$$

and from $\operatorname{var}(X^{\perp W_1}) > 0$, $B_n^2 > 0$, $B_n^{-1} \to 0$, and $f_1(B_n, R_{\text{long}}^2) > 0$ as $n \to \infty$.

Likewise, for n sufficiently large we have that

$$\operatorname{sign}(\tilde{\delta}_n - 1) = -\operatorname{sign}(\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1))$$

since

$$-(f_0(-B_n) + f_1(-B_n, R_{\text{long}}^2)) = B_n^2(\operatorname{var}(X^{\perp W_1}) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) + O(B_n^{-1})),$$

and since $f_1(-B_n, R_{\text{long}}^2) < 0$ as $n \to \infty$.

Therefore, if $\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) > 0$, for sufficiently large n the sequence δ_n is such that $\delta_n > 1$, $\delta_n \to 1$, and

$$\min \mathcal{B}_I(\delta_n, R_{\text{long}}^2) \leq \beta_{\text{med}} - B_n \to -\infty$$

as $n \to \infty$. Also let N be such that $\beta_{\text{med}} - B_n < 0$ and $\delta_n > 1$ for all $n \geq N$.

Likewise, for sufficiently large n the sequence $\tilde{\delta}_n$ is such that $\tilde{\delta}_n < 1$, $\tilde{\delta}_n \to 1$, and

$$\max \mathcal{B}_I(\tilde{\delta}_n, R_{\text{long}}^2) \geq \beta_{\text{med}} + B_n \to +\infty$$

as $n \to \infty$. Also let $\widetilde{N}$ be such that $\beta_{\text{med}} + B_n > 0$ and $\tilde{\delta}_n \leq 1$ for all $n \geq \widetilde{N}$.

If $\beta_{\text{med}} > 0$, then

$$\begin{aligned}
\delta^{\text{bp,sign}}(R_{\text{long}}^2) &= \delta^{\text{bp},>}(R_{\text{long}}^2) \\
&= \inf\{|\delta| : \delta \in \mathbb{R}, b \leq 0 \text{ for some } b \in \mathcal{B}_I(\delta, R_{\text{long}}^2)\}
\end{aligned}$$

$$\leq \inf\{|\delta_n| : n \geq N\}.$$

The inequality follows from $\mathcal{B}_I(\delta_n, R^2_{\text{long}})$ containing a negative number for all $n \geq N$. Since $\delta_n \to 1$ and $\delta_n > 1$ for $n \geq N$, $\inf\{|\delta_n| : n \geq N\} = 1$ thus $\delta^{\text{bp,sign}}(R^2_{\text{long}}) \leq 1$.

If $\beta_{\text{med}} < 0$, then

$$\begin{aligned}
\delta^{\text{bp,sign}}(R^2_{\text{long}}) &= \delta^{\text{bp},<}(R^2_{\text{long}}) \\
&= \inf\{|\delta| : \delta \in \mathbb{R}, b \geq 0 \text{ for some } b \in \mathcal{B}_I(\delta, R^2_{\text{long}})\} \\
&\leq \inf\{|\tilde{\delta}_n| : n \geq \widetilde{N}\} \\
&\leq 1.
\end{aligned}$$

The first inequality follows from $\mathcal{B}_I(\tilde{\delta}_n, R^2_{\text{long}})$ containing a positive value for $n \geq \widetilde{N}$. The second inequality follows from $\tilde{\delta}_n \leq 1$ for all $n \geq \widetilde{N}$. An analogous argument shows that this conclusion holds if $\text{cov}(X, \gamma'_{1,\text{med}} W_1) < 0$ as well. □

Proof of Corollary 1. By Theorem S1, $\beta_{\text{long}} \in \mathcal{B}_I(\delta, R^2_{\text{long}})$. By A5, $\beta_{\text{long}} \in [\beta_{\text{med}} - M, \beta_{\text{med}} + M]$. Combining these yields that β_{long} lies in the intersection of these sets. Sharpness follows from the sharpness of $\mathcal{B}_I(\delta, R^2_{\text{long}})$, which was established in Theorem S1, and from $\beta_{\text{long}} \in [\beta_{\text{med}} - M, \beta_{\text{med}} + M]$, which implies A5. □

Proof of Corollary 2. This follows directly from Theorem S1 and Corollary 1. □

Appendix E

Proof of Proposition S2.

1. ($\Rightarrow$) As shown in Lemma S1, $\gamma_{1,\text{long}} = \gamma_{1,\text{med}} + (\beta_{\text{med}} - \beta_{\text{long}})\pi_1$ under assumptions A1 and A4. Therefore, if A7 holds, then $\gamma_{1,\text{med}} = \gamma_{1,\text{long}} - (\beta_{\text{med}} - \beta_{\text{long}})\pi_1 = (C - (\beta_{\text{med}} - \beta_{\text{long}}))\pi_1 = C_{\text{med}} \cdot \pi_1$ with $C_{\text{med}} \coloneqq C - (\beta_{\text{med}} - \beta_{\text{long}})$.
2. ($\Leftarrow$) If there is a constant $C_{\text{med}} \in \mathbb{R}$ such that $\gamma_{1,\text{med}} = C_{\text{med}} \cdot \pi_1$, then we immediately see that $\gamma_{1,\text{long}} = (C_{\text{med}} + (\beta_{\text{med}} - \beta_{\text{long}}))\pi_1$ so A7 holds with $C \coloneqq C_{\text{med}} + (\beta_{\text{med}} - \beta_{\text{long}})$.

□

Proof of Proposition S3. From the expression for $f(B, \delta, R^2_{\text{long}})$, we note that $f(B, 1, R^2_{\text{long}})$ is quadratic in B and is equal to

$$\begin{aligned}
&f(B, 1, R^2_{\text{long}}) \\
&= f_0(B) + f_1(B, R^2_{\text{long}}) \\
&= -B^2 \,\text{var}(X^{\perp W_1}) \,\text{cov}(X, \gamma'_{1,\text{med}} W_1) + B \left((R^2_{\text{long}} - R^2_{\text{med}}) \,\text{var}(Y) \,\text{var}(\pi'_1 W_1) - \text{var}(X^{\perp W_1}) \,\text{var}(\gamma'_{1,\text{med}} W_1)\right) \\
&\quad + (R^2_{\text{long}} - R^2_{\text{med}}) \,\text{cov}(X, W'_1 \gamma_{1,\text{med}}) \,\text{var}(Y).
\end{aligned}$$

The discriminant of this quadratic polynomial is

$$\begin{aligned}
\Delta &\coloneqq \left((R^2_{\text{long}} - R^2_{\text{med}}) \,\text{var}(Y) \,\text{var}(\pi'_1 W_1) - \text{var}(X^{\perp W_1}) \,\text{var}(\gamma'_{1,\text{med}} W_1)\right)^2 \\
&\quad + 4 \,\text{var}(X^{\perp W_1}) \,\text{cov}(X, \gamma'_{1,\text{med}} W_1)^2 (R^2_{\text{long}} - R^2_{\text{med}}) \,\text{var}(Y) \\
&= \left((R^2_{\text{long}} - R^2_{\text{med}}) \,\text{var}(Y) \,\text{var}(\pi'_1 W_1)\right)^2 + \left(\text{var}(X^{\perp W_1}) \,\text{var}(\gamma'_{1,\text{med}} W_1)\right)^2 \\
&\quad - 2(R^2_{\text{long}} - R^2_{\text{med}}) \,\text{var}(Y) \,\text{var}(\pi'_1 W_1) \,\text{var}(X^{\perp W_1}) \,\text{var}(\gamma'_{1,\text{med}} W_1) \\
&\quad + 4 \,\text{var}(X^{\perp W_1}) \,\text{cov}(X, \gamma'_{1,\text{med}} W_1)^2 (R^2_{\text{long}} - R^2_{\text{med}}) \,\text{var}(Y).
\end{aligned}$$

To simplify this expression, we show that $\text{cov}(X, \gamma'_{1,\text{med}} W_1)^2 = \text{var}(\pi'_1 W_1)\,\text{var}(\gamma'_{1,\text{med}} W_1)$. To see this, note that A7 and Proposition S2 implies that $\gamma_{1,\text{med}} = C_{\text{med}}\pi_1$. Therefore

$$\begin{aligned}
\text{cov}(X, \gamma'_{1,\text{med}} W_1)^2 - \text{var}(\pi'_1 W_1)\,\text{var}(\gamma'_{1,\text{med}} W_1) &= C^2_{\text{med}}\,\text{cov}(X, \pi'_1 W_1)^2 - C^2_{\text{med}}\,\text{var}(\pi'_1 W_1)\,\text{var}(\pi'_1 W_1) \\
&= C^2_{\text{med}}\left(\text{cov}(\pi'_1 W_1 + \pi_2 W_2 + X^{\perp W}, \pi'_1 W_1)^2 - \text{var}(\pi'_1 W_1)^2\right) \\
&= C^2_{\text{med}}\left(\text{var}(\pi'_1 W_1)^2 - \text{var}(\pi'_1 W_1)^2\right) \\
&= 0.
\end{aligned}$$

The fourth equality follows from A4. Therefore,

$$\text{cov}(X, \gamma'_{1,\text{med}} W_1)^2 = \text{var}(\pi'_1 W_1)\,\text{var}(\gamma'_{1,\text{med}} W_1) \tag{S19}$$

which implies that the discriminant can be written as

$$\begin{aligned}
\Delta &= \left((R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1)\right)^2 + \left(\text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}} W_1)\right)^2 \\
&\quad + 2(R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1)\,\text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}} W_1) \\
&= \left((R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1) + \text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}} W_1)\right)^2.
\end{aligned}$$

Therefore, the quadratic equation $f(B, 1, R^2_{\text{long}}) = 0$ has the following two solutions

$$\begin{aligned}
&\Bigg((R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1) - \text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}} W_1) \\
&\quad \pm \sqrt{\left((R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1) + \text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}} W_1)\right)^2}\Bigg) \Bigg/ 2\,\text{var}(X^{\perp W_1})\,\text{cov}(X, \gamma'_{1,\text{med}} W_1) \\
&= \Bigg((R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1) - \text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}} W_1) \\
&\quad \pm \left((R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1) + \text{var}(X^{\perp W_1})\,\text{var}(\gamma'_{1,\text{med}} W_1)\right)\Bigg) \Bigg/ 2\,\text{var}(X^{\perp W_1})\,\text{cov}(X, \gamma'_{1,\text{med}} W_1) \\
&= \{B_1, B_2\}
\end{aligned}$$

where

$$B_1 = \frac{(R^2_{\text{long}} - R^2_{\text{med}})\,\text{var}(Y)\,\text{var}(\pi'_1 W_1)}{\text{var}(X^{\perp W_1})\,\text{cov}(X, \gamma'_{1,\text{med}} W_1)} \qquad \text{and} \qquad B_2 = -\frac{\text{var}(\gamma'_{1,\text{med}} W_1)}{\text{cov}(X, \gamma'_{1,\text{med}} W_1)}.$$

These are well defined by $\text{cov}(X, \gamma'_{1,\text{med}} W_1) = \text{var}(X)(\beta_{\text{short}} - \beta_{\text{med}}) \neq 0$. They are distinct by $\Delta > 0$. We now simplify the expressions for B_1 and B_2. Combining equations (S15), (S16), and (S19), we can write

$$\begin{aligned}
\text{var}(\gamma'_{1,\text{med}} W_1) &= (R^2_{\text{med}} - R^2_{\text{short}})\,\text{var}(Y) + \text{var}(X)(\beta_{\text{short}} - \beta_{\text{med}})^2 \\
&= (R^2_{\text{med}} - R^2_{\text{short}})\,\text{var}(Y) + \frac{\text{cov}(X, \gamma'_{1,\text{med}} W_1)^2}{\text{var}(X)} \\
&= (R^2_{\text{med}} - R^2_{\text{short}})\,\text{var}(Y) + \frac{\text{var}(\pi'_1 W_1)\,\text{var}(\gamma'_{1,\text{med}} W_1)}{\text{var}(X)}.
\end{aligned}$$

Solving the previous equation for $\text{var}(\gamma'_{1,\text{med}} W_1)$ yields

$$\text{var}(\gamma'_{1,\text{med}} W_1) = (R^2_{\text{med}} - R^2_{\text{short}})\,\text{var}(Y)\left(1 - \frac{\text{var}(\pi'_1 W_1)}{\text{var}(X)}\right)^{-1}$$

$$= (R^2_{\text{med}} - R^2_{\text{short}}) \operatorname{var}(Y) \frac{\operatorname{var}(X)}{\operatorname{var}(X) - \operatorname{var}(\pi_1' W_1)}$$
$$= (R^2_{\text{med}} - R^2_{\text{short}}) \operatorname{var}(Y) \frac{\operatorname{var}(X)}{\operatorname{var}(X^{\perp W_1})}.$$

This equality can be used to simplify B_1 as follows:

$$\begin{aligned}
B_1 &= \frac{(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \operatorname{var}(\pi_1' W_1)}{\operatorname{var}(X^{\perp W_1}) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)} \\
&= \frac{(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y) \operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)}{\operatorname{var}(X^{\perp W_1}) \operatorname{var}(\gamma_{1,\text{med}}' W_1)} \\
&= \frac{(R^2_{\text{long}} - R^2_{\text{med}}) \operatorname{var}(Y)(\beta_{\text{short}} - \beta_{\text{med}}) \operatorname{var}(X)}{\operatorname{var}(X^{\perp W_1})(R^2_{\text{med}} - R^2_{\text{short}}) \operatorname{var}(Y) \frac{\operatorname{var}(X)}{\operatorname{var}(X^{\perp W_1})}} \\
&= (\beta_{\text{short}} - \beta_{\text{med}}) \frac{R^2_{\text{long}} - R^2_{\text{med}}}{R^2_{\text{med}} - R^2_{\text{short}}}.
\end{aligned}$$

The second equality follows from equation (S19), and the third follows from our expression for $\operatorname{var}(\gamma_{1,\text{med}}' W_1)$.

Meanwhile, we can show that $\beta_{\text{med}} - B_2 \in \mathcal{B}_{\text{A3fail}}$ and thus $\beta_{\text{med}} - B_2 \notin \mathcal{B}_I(1, R^2_{\text{long}})$. To see this, note that

$$\begin{aligned}
&\operatorname{var}\left((\gamma_{1,\text{med}} + B_2 \pi_1)' W_1\right) \\
&= \operatorname{var}\left(\gamma_{1,\text{med}}' W_1 - \frac{\operatorname{var}(\gamma_{1,\text{med}}' W_1)}{\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)} \pi_1' W_1\right) \\
&= \operatorname{var}(\gamma_{1,\text{med}}' W_1) - 2 \frac{\operatorname{var}(\gamma_{1,\text{med}}' W_1)}{\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)} \operatorname{cov}(\gamma_{1,\text{med}}' W_1, \pi_1' W_1) + \frac{\operatorname{var}(\gamma_{1,\text{med}}' W_1)^2}{\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)^2} \operatorname{var}(\pi_1' W_1) \\
&= \operatorname{var}(\gamma_{1,\text{med}}' W_1) - 2 \frac{\operatorname{var}(\gamma_{1,\text{med}}' W_1)}{\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)} \operatorname{cov}(\gamma_{1,\text{med}}' W_1, X) + \frac{\operatorname{var}(\gamma_{1,\text{med}}' W_1)^2}{\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1)^2} \operatorname{var}(\pi_1' W_1) \\
&= \operatorname{var}(\gamma_{1,\text{med}}' W_1) - 2 \operatorname{var}(\gamma_{1,\text{med}}' W_1) + \frac{\operatorname{var}(\gamma_{1,\text{med}}' W_1)^2 \operatorname{var}(\pi_1' W_1)}{\operatorname{var}(\gamma_{1,\text{med}}' W_1) \operatorname{var}(\pi_1' W_1)} \\
&= 0.
\end{aligned}$$

The fourth equality follows from equation (S19) and from $\operatorname{cov}(X, \gamma_{1,\text{med}}' W_1) \neq 0$. Therefore, $\gamma_{1,\text{med}} + B_2 \pi_1 = 0$, which implies that $\beta_{\text{med}} - B_2 \notin \mathcal{B}_I(1, R^2_{\text{long}})$. Since $\pi_1 \neq 0$ and $\gamma_{1,\text{med}} \neq 0$, which follows from $\beta_{\text{short}} \neq \beta_{\text{med}}$, the set $\mathcal{B}_{\text{A3fail}}$ is a singleton. Since B_1 and B_2 are distinct, $\beta_{\text{med}} - B_1 \notin \mathcal{B}_{\text{A3fail}} = \{\beta_{\text{med}} - B_2\}$. By Theorem S1, we conclude that $\beta_{\text{med}} - B_1$ is the unique element of the identified set. □

Proof of Proposition S5. First, consider the case where $|\beta_1| \neq \infty$ and $\beta_1 + \beta_2 = \sigma$. Let $c := \sigma - 2\beta_1$, and define

$$(\sigma_{1,j}, \sigma_{2,j}, \rho_j) = (\sigma, \sigma + c/j, 1 - 1/j).$$

Then,

$$\beta_{1,j} = \frac{\sigma_{1,j} - \rho_j \sigma_{2,j}}{1 - \rho_j^2} = \frac{\sigma - (1 - 1/j)(\sigma + c/j)}{1 - (1 - 1/j)^2} = \frac{\sigma - c + c/j}{2 - 1/j} \to \frac{\sigma - c}{2} = \beta_1$$

and

$$\beta_{2,j} = \frac{\sigma_{2,j} - \rho_j \sigma_{1,j}}{1 - \rho_j^2} = \frac{\sigma + c/j - (1 - 1/j)\sigma}{1 - (1 - 1/j)^2} = \frac{\sigma + c}{2 - 1/j} \to \frac{\sigma + c}{2} = \sigma - \beta_1 = \beta_2.$$

To show a sequence converging to $(-\infty, +\infty)$ let

$$(\sigma_{1,j}, \sigma_{2,j}, \rho_j) = (\sigma, \sigma + 1/j, 1 - 1/j^2).$$

Then,

$$\beta_{1,j} = \frac{\sigma - (1 - 1/j^2)(\sigma + 1/j)}{1 - (1 - 1/j^2)^2} = \frac{\sigma/j^2 - (1 - 1/j^2)1/j}{2/j^2 - 1/j^4} = \frac{\sigma}{2 + o(1)} - \frac{1 - 1/j^2}{2/j - 1/j^3} \to -\infty$$

and

$$\beta_{2,j} = \frac{\sigma + 1/j - (1 - 1/j^2)\sigma}{1 - (1 - 1/j^2)^2} = \frac{1/j + \sigma/j^2}{2/j^2 - 1/j^4} = \frac{j + \sigma}{2 - 1/j^2} \to +\infty.$$

Finally, for a sequence converging to $(+\infty, -\infty)$, let

$$(\sigma_{1,j}, \sigma_{2,j}, \rho_j) = (\sigma, \sigma - 1/j, 1 - 1/j^2).$$

□

I List of Papers Used in Appendix A Survey

Abaluck, J., M. Caceres Bravo, P. Hull, and A. Starc (2021): "Mortality effects and choice across private health insurance plans," *Quarterly Journal of Economics*, 136, 1557–1610.

Abdulkadiroglu, A., P. A. Pathak, J. Schellenberg, and C. R. Walters (2020): "Do parents value school effectiveness?" *American Economic Review*, 110, 1502–39.

Allcott, H., R. Diamond, J.-P. Dubé, J. Handbury, I. Rahkovsky, and M. Schnell (2019a): "Food deserts and the causes of nutritional inequality," *Quarterly Journal of Economics*, 134, 1793–1844.

Allcott, H., B. B. Lockwood, and D. Taubinsky (2019b): "Regressive sin taxes, with an application to the optimal soda tax," *Quarterly Journal of Economics*, 134, 1557–1626.

Arbatli, C. E., Q. H. Ashraf, O. Galor, and M. Klemp (2020): "Diversity and conflict," *Econometrica*, 88, 727–797.

Bazzi, S., M. Fiszbein, and M. Gebresilasse (2020): "Frontier culture: The roots and persistence of "rugged individualism" in the United States," *Econometrica*, 88, 2329–2368.

Bertrand, M., R. Burgess, A. Chawla, and G. Xu (2020): "The glittering prizes: Career incentives and bureaucrat performance," *Review of Economic Studies*, 87, 626–655.

Blader, S., C. Gartenberg, and A. Prat (2020): "The contingent effect of management practices," *Review of Economic Studies*, 87, 721–749.

Blouin, A. and R. Macchiavello (2019): "Strategic default in the international coffee market," *Quarterly Journal of Economics*, 134, 895–951.

Chen, Y. (2021): "Team-specific human capital and team performance: Evidence from doctors," *American Economic Review*, 111, 3923–62.

Couttenier, M., V. Petrencu, D. Rohner, and M. Thoenig (2019): "The violent legacy of conflict: evidence on asylum seekers, crime, and public policy in Switzerland," *American Economic Review*, 109, 4378–4425.

Dippel, C. and S. Heblich (2021): "Leadership in Social Movements: Evidence from the" Forty-Eighters" in the Civil War," *American Economic Review*, 111, 472–505.

Enke, B. (2019): "Kinship, cooperation, and the evolution of moral systems," *Quarterly Journal of Economics*, 134, 953–1019.

——— (2020): "Moral values and voting," *Journal of Political Economy*, 128, 3679–3729.

Eugster, B. and R. Parchet (2019): "Culture and taxes," *Journal of Political Economy*, 127, 296–337.

Finkelstein, A., M. Gentzkow, and H. Williams (2021): "Place-based drivers of mortality: Evidence from migration," *American Economic Review*, 111, 2697–2735.

Gavazza, A., M. Nardotto, and T. Valletti (2019): "Internet and politics: Evidence from UK local elections and local government policies," *Review of Economic Studies*, 86, 2092–2135.

Goldsmith-Pinkham, P., I. Sorkin, and H. Swift (2020): "Bartik instruments: What, when, why, and how," *American Economic Review*, 110, 2586–2624.

Gregg, A. G. (2020): "Factory productivity and the concession system of incorporation in late imperial Russia, 1894–1908," *American Economic Review*, 110, 401–27.

Grosfeld, I., S. O. Sakalli, and E. Zhuravskaya (2020): "Middleman minorities and ethnic violence: anti-Jewish pogroms in the Russian empire," *Review of Economic Studies*, 87, 289–342.

Grosjean, P. and R. Khattar (2019): "It's raining men! Hallelujah? The long-run consequences of male-biased sex ratios," *Review of Economic Studies*, 86, 723–754.

Guriev, S., N. Melnikov, and E. Zhuravskaya (2021): "3G internet and confidence in government," *Quarterly Journal of Economics*, 136, 2533–2613.

Heldring, L. (2021): "The origins of violence in Rwanda," *Review of Economic Studies*, 88, 730–763.

Hoffman, M. and S. Tadelis (2021): "People management skills, employee attrition, and manager rewards: An empirical analysis," *Journal of Political Economy*, 129, 243–285.

Murphy, R. and F. Weinhardt (2020): "Top of the class: The importance of ordinal rank," *Review of Economic Studies*, 87, 2777–2826.

Schindler, D. and M. Westcott (2021): "Shocking racial attitudes: Black GIs in Europe," *Review of Economic Studies*, 88, 489–520.

Squicciarini, M. P. (2020): "Devotion and development: Religiosity, education, and economic progress in nineteenth-century France," *American Economic Review*, 110, 3454–91.

Tabellini, M. (2020): "Gifts of the immigrants, woes of the natives: Lessons from the age of mass migration," *Review of Economic Studies*, 87, 454–486.

Valencia Caicedo, F. (2019): "The mission: Human capital transmission, economic persistence, and culture in South America," *Quarterly Journal of Economics*, 134, 507–556.

Verner, E. and G. Gyöngyösi (2020): "Household debt revaluation and the real economy: Evidence from a foreign currency debt crisis," *American Economic Review*, 110, 2667–2702.

Weigel, J. L. (2020): "The participation dividend of taxation: How citizens in Congo engage more with the state when it tries to tax them," *Quarterly Journal of Economics*, 135, 1849–1903.

References

Bazzi, S., M. Fiszbein, and M. Gebresilasse (2020): "Frontier culture: The roots and persistence of "rugged individualism" in the United States," *Econometrica*, 88, 2329–2368.

Blandhol, C., J. Bonney, M. Mogstad, and A. Torgovitsky (2025): "When is TSLS actually late?" *Review of Economic Studies (forthcoming).*

Breusch, T. S. (1986): "Hypothesis testing in unidentified models," *Review of Economic Studies*, 53, 635–651.

Diegert, P., M. A. Masten, and A. Poirier (2025): "Assessing omitted variable bias when the controls are endogenous," *arXiv preprint arXiv:2206.02303v5.*

Dippel, C. and S. Heblich (2021): "Leadership in Social Movements: Evidence from the "Forty-Eighters" in the Civil War," *American Economic Review*, 111, 472–505.

Gregg, A. G. (2020): "Factory productivity and the concession system of incorporation in late imperial Russia, 1894–1908," *American Economic Review*, 110, 401–27.

Masten, M. A. and A. Poirier (2020): "Inference on breakdown frontiers," *Quantitative Economics*, 11, 41–111.

——— (2022): "The effect of omitted variables on the sign of regression coefficients," *arXiv preprint arXiv:2208.00552v1.*

Mian, A. and A. Sufi (2014): "What explains the 2007–2009 drop in employment?" *Econometrica*, 82, 2197–2223.

Oster, E. (2019): "Unobservable selection and coefficient stability: Theory and evidence," *Journal of Business & Economic Statistics*, 37, 187–204.

Satyanath, S., N. Voigtländer, and H.-J. Voth (2017): "Bowling for fascism: Social capital and the rise of the Nazi Party," *Journal of Political Economy*, 125, 478–526.

Squicciarini, M. P. (2020): "Devotion and development: Religiosity, education, and economic progress in nineteenth-century France," *American Economic Review*, 110, 3454–91.