The Venture Capital Cycle*

Rafael Silviera Randall Wright

November 30, 2007

Abstract

We develop an equilibrium model of venture capital markets, characterized by the following cycle: (i) capitalists raise funds; (ii) capitalists match with entrepreneurs; (iii) once matched, capitalists and entrepreneurs both take active roles in implementation; (iv) when the venture matures, the capitalist exits to start the cycle anew. We determine the durations of each phases in the cycle, the amount of funds that flow into the market, and the returns to both entrepreneurs and capitalists.

*Both authors are affiliated with The Department of Economics, University of Pennsylvania. For their input on work related to this project, we thank Ken Burdett, Boyan Jovanovic, Philipp Kircher, Christian Helwig, and Iuorii Manovskii.

1 Introduction

The venture capital (private equity) market works like this: Some entepreneurs with viable ideas for projects, or ventures, have trouble getting financing from banks or other conventional sources, typically because their ideas are technically difficult for non-experts to evaluate and/or because they lack sufficient collateral. Venture capitalists, or VC's, are agents with access to funds that specialize in locating, evaluating, and selecting high-risk but potentially high-return projects. They also have expertise in implementation that is a key input into the project; more than just lending money, VC's enter into partnerships with entrepreneurs, taking an active role in development, management, monitoring, etc. As projects mature, especially once they can be more easily evaluated and operated by others, VC's exit to raise funds for new projects. This is called *the venture capital cycle*.[1]

The economics of this market seems worth investigation. There have been rapid increases in the size of the venture capital industry over the past two decades: private equity increased from under \$5 billion in 1980 to over \$300 billion in 2004, and in the past 25 years more than \$1 trillion has passed through these funds (Lerner 2003; Lerner et al. 2004). Despite the 2001-2003 downturn, commitments to US venture capital funds increased from \$8.7 billion in 2003 to \$16-17 billion in 2004. Although it still is only a fraction of the total financial sector, this market is considered an engine for economic growth. For example, a significant fraction of blue-chip firms, including Apple, Compaq, FedEx, Intel, Microsoft and Cisco Systems,

[1] We're not making this up. In the words of the experts, "To understand the venture capital industry, one must understand the whole 'venture cycle.' The venture capital cycle starts with raising a venture fund; proceeds through the investment in, monitoring of, and adding value to firms; continues as the venture capital firm exits successful deals and returns capital to its investors; and renews itself with the venture capitalist raising additional funds." (Gompers and Lerner 2001).

received venture capital in their early stages, and generally venture capital-backed firms perform quite well, by a variety of criteria.[2]

There is much research on particular aspects of this market, surveyed e.g. in Gompers and Lerner (1999, 2001) and Kaplan and Stromberg (2001). But we think there is much more to be done. Two questions that we think especially interesting are, what determines the duration of each phase of venture cycle, and what determines the size of the fund? The former (timing) is a major concern; as Cochrane (2004) puts it, e.g., "The risk facing a venture capital investor is as much when his or her return will occur as it is how much that return will be." The latter (fund size) is a decision affected by various factors discussed in the empirical literature; but scholars still seem uncertain about the determinants of the optimal level of investment in the industry, or, at the individual level, optimal fund size. To address these issues, we develop an equilibrium model that makes explicit various frictions and imperfections in the market, using some tools from search theory.[3]

The idea that frictions may be relevant here is not new. According to Gompers and Lerner (2000), e.g., there is a positive relation between the valuation of venture capital investment and liquidity inflows in the market not driven by improvements in potential prospects, as would be predicted

[2] A study by NVCAA (2002) found that during 1970-2000 "venture capital-backed companies had approximately twice the sales, paid almost three times the federal taxes, generated almost twice the exports, and invested almost three times as much in R&D as the average non-venture capital-backed public company, per each $1,000 of assets." Kortum and Lerner (2000) estimated that although such firms account for only about 3 percent of all R&D spending, they generate about 14 percent of the innovation.

[3] Search has proved a rigorous yet tractable way to study many markets with frictions; see Rogerson et al. (2005) for a recent survey of applications to labor markets, and the references therein for applications to the theory of marriage, monetary economics, and industrial organization. Recent applications in finance inclue Duffie, Garleanu and Pedersen (2002), Weill (2004), and Lagos (2006). Formally, although different on several dimensions, our framework shares some features with the macro-labor model in Pissarides (1990) or Mortensen and Pissarides (1994) and the monetary model in Lagos and Wright (2005).

by classic asset pricing theory, and interpret this as evidence against the assumption of a perfect and frictionless market. Gompers and Lerner (1999) and Lerner (2002) also describe recurrent imbalances between supply and demand for funds. Our model helps clarify these observations by explaining capital flows into this market, the decision to enter (or not) into a venture, and how we can simultaneously have entrepreneurs in search of funding and capitalists with idle funds. The model also determines the lengths of the different phases of the venture cycle, and how these depend on various factors.

There is a considerable literature in financial economics that explores how VC's screen, select, finance, monitor and advise their companies.[4] At the rsik of generalizing, it is mainly partial equilibrium, and focuses on optimal contracts between VC's and entrepreneurs in the presence of adverse selection and moral hazard; there is little consideration of other market frictions. Inderst and Muller (2004), Michelacci and Suarez (2004), and Keuschnigg (2003) do consider search-based models, but our approach and focus are very different. For instance, while they assume that every match between an entrepreneur and capitalist is the same, we take seriously the notion of heterogeneity, which seems critical for analyzing project selection. We also model explicitly liquidity in a way that is realistic and has not been done before, and is critical for analyzing the duration of the differen phases of the cycle and optimal fund size.[5]

[4] In addition to research summarized in the surveys cited above, we mention Aghion and Tirole (1994), Bascha and Walz (2001), Bergemann and Hege (1998), Casamatta (2002), Cumming and MacIntosh (2002), Hellmann (1998), Hellmann and Puri (2000,2002), Kaplan and Strömberg (2000,2001), Schmidt (2002), and Babarino and Jovanovic (2006).

[5] In Silviera and Wright (2006) we also endogenized liquidity in the market for "ideas" in a way that is similar to the approach here, but that paper was not about the venture capital market and had nothing to say about the venture cycle.

Related work includes Jovanovic and B. Szentes (2006) and Pinheiro (2007), both of which consider the initial decision of an entrpreneur try to get venture financing in the first place (instead of alternative like, say, saving up the funds himself), and worry about the timing of the IPO. In particular, an issue discussed in the literature is whether the VC has an extra incentive to go public early, since he wants to renter the market in search of new projects, while the entrepreneur is typically not expected to renter the market. A popular explanation since Gompers (1996) known as granstanding says that the VC is trying to signal his ability to the rest of the market by going public early, even if this means underpricing. Pinheiro (2007) proposes an alternative view, which is simply that the VC has to finish up with one project before he starts another due to limited capacity to be involved in several projects at the same time. This is very similar to the line in Michelacci and Suarez (2004), and very much in the spirit of our model.

2 A Simple Model

There are two types of risk neutral agents: entrepreneurs labeled e, and capitalists labeled . They need each other to start a venture. The notion is that e has an idea and technical ability, but lacks either the funds or the expertise to implement it, while k is a specialist in finding and evaluating potentialy profitable ideas who can potentially provide funds, expertise and other inputs.[6] For simplicity we assume that k wants to enter into one and only one

[6] As we said in the Introduction, e cannot easily access banks or other conventional sources of funds because he lacks sugnificant tangible assets and because his idea may be hard for non-experts to evaluate. We abstract from the question of whether e should, instead of entering the venture capital market, try to save up and finance the project himself (Babarino and Jovanovic 2006); even if e has access to his own funds, however, he may still value the expertise of k in terms of implementation.

project at a time, and symmetrically e needs exactly one VC. Let ℓ_e and ℓ_k denote inputs, or investments, by the two parties during the implementation stage; for now we take these as fixed, but they are endogenized below. The marginal cost of investments for each agent can be normalized to 1 for each agent without loss of generality. Agents meet bilaterally in continuous time according to a standard Poisson matching process, with arrival rates α_e and α_k that they take as given, but are endogenized below.

In general, when e and k meet, they draw (R, C) from some distribution $F(R, C) = pr(\tilde{R} \leq R, \tilde{C} \leq C)$, where C is the start-up cost and R the return on the project when it comes to fruition in the future. Although one could precoeed differently, we assume that agents are ex ante homogenous, but (R, C) differs across matches ex post, say because k may have more or less expertise in the area corresponding to the idea of a particular e. We abstract from private information, not because we think it is uninteresting, but because we want to focus on different issues; thus, when they meet, e and k both know the realization (R, C). In this section we begin with the simplest case where $C = 0$ and $R > 0$ are constant across matches. With $C = 0$, funding is not an issue, but k still brings expertise to the table. In any event, we will soon allow random C as well as R.

If they agree to start a venture, a payment p from e to k is negotiated, due when the project comes to fruition. We can trivially reinterpret k as receiving an equity share μ, as is typically the case in reality, simply by setting $\mu = p/R$; even if returns are random, it does not matter if they bargain over a fixed payment p or an equity share μ, since agents are risk neutral. We assume that the agreement is binding – e.g. an enforcable contract, not subject to opportunistic renegotiation. The venture comes to fruition at some random date after implementation begins. Let σ be the

Possion rate at which this happens as a function of the investments made during implementation, $\sigma = \sigma(\ell_e, \ell_k)$. When the venture comes to fruition, k receives p and returns to the market, while e keeps the residual $R-p$, exits, and is replaced by a clone of himself in order to maintain steady state.[7]

In our framework there are four recurring stages in the life of a VC, corresponding to the cycle described in the Introduction: (i) fund raising; (ii) search for a partner/project; (iii) implementation; and (iv) exit upon maturation. For now we ignore the first stage, since $C = 0$, but fund raising will be a crucial part of the general model discussed below. The second stage is random in duration because search takes time, and depends on market tightness, we discuss below. The third stage is also random because the rate at which projects come to fruition depends on investment inputs, as we also discuss below. The exit stage can also be random, but since this is the one phase that has been analyzed in a related model, by Michelacci and Suarez (2004), we assume for now that k can exit and reap his return costlessly once the project matures.

Let the value function for type i in state j be V_i^j, where $j = 0$ means that i is in the partner search stage and $j = 1$ means that i in the implementation stage. Let $\bar{p}$ be the value of p prevailing in the market. Given the discount rate r, the value functions for e satisfy[8]

$$\begin{aligned} rV_e^0 &= \alpha_e(V_e^1 - V_e^0) && (1) \\ rV_e^1 &= \sigma(R - \bar{p} - V_e^1) - \ell_e, && (2) \end{aligned}$$

[7] In reality, the parties may actually sell the venture – i.e. *go public* – but again this does not matter because of risk neutrality, so we assume e simply pays off k.

[8] These standard programming equations have simple interpretations: e.g. the per period value of the venture to e is rV_e^1, which equals the rate at which it comes to fruition σ, times the capital gain $R-\bar{p}-V_e^1$, minus the flow cost ℓ_e. Notice that, as we said, k goes back to the market when he exits a project while e does not; this seems fairly realistic, although the model works fine with althernative assumptions – e.g. we can allow e to go back to the market with a new idea.

and those for k satisfy

$$rV_k^0 = \alpha_k(V_k^1 - V_k^0) \tag{3}$$

$$rV_k^1 = \sigma(\bar{p} + V_k^0 - V_e^1) - \ell_k. \tag{4}$$

It is simple to solve these (linear) equations for

$$V_e^0 = \frac{\alpha_e\left[\sigma(R-\bar{p}) - \ell_e\right]}{(r+\sigma)(r+\alpha_e)} \text{ and } V_e^1 = \frac{\sigma(R-\bar{p}) - \ell_e}{r+\sigma} \tag{5}$$

$$V_k^0 = \frac{\alpha_k\left(\sigma\bar{p} - \ell_k\right)}{r(r+\sigma+\alpha_k)} \text{ and } V_k^1 = \frac{(r+\alpha_k)\left(\sigma\bar{p} - \ell_k\right)}{r(r+\sigma+\alpha_k)}. \tag{6}$$

When e and k meet, they negotiate over p for their particular project, taking as given the market $\bar{p}$. We adopt the generalized Nash bargaining solution

$$\max_p S_k^\theta S_e^{1-\theta} \tag{7}$$

where θ is the bargaining power of k, while $S_k = V_k^1 - V_k^0$ and $S_e = V_e^1 - V_e^0$ are the surpluses. It is convenient rearrange (4) as $V_k^1 = \frac{\sigma(p+V_k^0)-\ell_k}{r+\sigma}$ and (2) as $V_e^1 = \frac{\sigma(R-p)-\ell_e}{r+\sigma}$, to express

$$S_k = \frac{\sigma p - rV_k^0 - \ell_k}{r+\sigma} \tag{8}$$

$$S_e = \frac{\sigma(R-p) - (r+\sigma)V_e^0 - \ell_e}{r+\sigma} \tag{9}$$

as functions of the p in the current venture, as well as the outside options V_k^0 and V_e^0, which are themselves functions of $\bar{p}$. For future reference, write the total surplus $S = S_k + S_e$ as

$$S = \frac{\sigma R - (r+\sigma)V_e^0 - rV_k^0 - \ell_e - \ell_k}{r+\sigma}. \tag{10}$$

Inserting (8) and (9) into (7), taking the first order condition, and simplifying, we find the p for a particular venture in terms of its expected return and costs, plus the outside options:

$$\sigma p = \theta\sigma R - \theta\ell_e - (1-\theta)\ell_k - \theta(r+\sigma)V_e^0 + (1-\theta)\,rV_k^0 \tag{11}$$

Inserting (5)-(6), setting $\bar{p} = p = p^*$, and simplifying, we get

$$\sigma p^* = \frac{\theta r\,(r+\sigma+\alpha_k)\,(\sigma R-\ell_e)+(1-\theta)(r+\sigma)(r+\alpha_e)\ell_k}{\theta r\,(r+\sigma+\alpha_k)+(1-\theta)\,(r+\sigma)\,(r+\alpha_e)}. \tag{12}$$

Equilibrium thus yields p^* as a weighted average of $\sigma R - \ell_e$ and ℓ_k. There are, however, two participation constraints to check: $S_k \geq 0$ and $S_e \geq 0$. It turns out that these both hold iff $S \geq 0$, which simplifies to $\sigma R - \ell_k - \ell_e \geq 0$. Hence, as long as we satisfy this obviously neccesary condition for a venture to be worthwhile, the constraints are not binding.[9]

Even this basic model makes interesting predictions about the returns p^* and $R - p^*$ going to the two parties. For example, $\partial p^*/\partial\theta > 0$, $\theta = 1 \implies p^* = R$, and $\theta = 0 \implies p^* = 0$, naturally. Also, $\partial p/\partial\alpha_k > 0$ and $\partial p/\partial\alpha_e < 0$, so higher arrival rates and hence better outside options for the capitalist (entrepreneur) increase (decrease) p^*. Also, $\partial p^*/\partial R > 0$ and $\partial p^*/\partial\sigma < 0$, so more lucarative projects generate higher payments to k, while less time-consuming ones generate lower payments. Finally, $\partial p/\partial\ell_k > 0$ and $\partial p/\partial\ell_e < 0$, so that when the project requires greater investment by the capitalist (entrepreneur) p^* goes up (down) in compensation. All of these predictions seem reasonable, relevant, and potentially testable.[10]

revision

[9] This is standard in search-and-bargaining models with transferable utility, and implies that one side wants to form a relationship iff the other side wants to form a relationship (see e.g. Burdett and Wright 1998). Of course, for now all relationships are the same, but the same result holds with (R, C) random.

[10] If we interpret k as receiving an equity share $\mu = p/R$ in the venture, comparative statics on μ are all obvious, except $\partial\mu/\partial r$, which is poisitive iff $\theta r(r+\sigma+\alpha_k)\ell_e > (1-\theta)(r+\sigma)\,(r+\alpha_e)\,\ell_k$. One can also use alternative bargaining solutions and get similar results. Suppose e.g. the threat point of agent j is 0, rather than V_j^0, which is equally valid (either can be derived from an underlying stategic bargaining model; see e.g. Osborne and Rubinstein 1990). In this case, letting $\ell_k = \ell_e = 0$ to reduce notation,

$$\sigma p = \frac{\theta r(r+\sigma+\alpha_k)\,(\sigma R-\ell_e)+(1-\theta)\,(r+\sigma)(r+\alpha_k)\ell_k}{r(r+\sigma+\alpha_k)+(1-\theta)\alpha_k\sigma}.$$

Predictions are similar, if not exactly the same.

Here we discuss arrival rates, without free entry. Let the measure of type k be n and the measure of type e be (normalized to) 1. Let the number of ventures in the implementation stage be m. Since each venture takes one e and one k off the market, there are $1-m$ and $n-m$ of each type in the partner-search phase at any point in time. We have the identity $\alpha_k(n-m) = \alpha_e(1-m)$, both sides giving the flow of new ventures. In steady state, this must equal σm, and therefore

$$m = \frac{n\alpha_k}{\alpha_k + \sigma}.$$

The measures of e and k in the partner-search phase therefore at any point in time are then $1-m = \frac{(1-n)\alpha_k+\sigma}{\alpha_k+\sigma}$ and $n-m = \frac{n\sigma}{\alpha_k+\sigma}$, and market tightness is $\tau = \frac{n\sigma}{(1-n)\alpha_k+\sigma}$. The total number of meetings is given by the constant returns technology $\mu(1-m, n-m)$, and so

$$\alpha_k = \frac{\mu(1-m, n-m)}{n-m} = \mu\left[\frac{(1-n)\alpha_k + \sigma}{n\sigma}, 1\right].$$

The RHS is a function of α_k with intercept $\mu[1/n, 1] > 0$, and is increasing if $n > 1$, or decreasing if $n < 1$. In the latter case there clearly exists a unique α_k satifying this condition, and in the former case the same is true as long as we make the standard assumption that μ is increasing and concave with $\mu_1(\infty, 1) = 0$. In the border line case where $n = 1$, α_k is simply given by the intercept $\mu[1/n, 1]$. Given α_k, we recover α_e from the identity

$$\alpha_e = \alpha_k \frac{n-m}{1-m} = \mu\left[1, \frac{n\sigma}{(1-n)\alpha_k + \sigma}\right].$$

Obviously, increasing n lowers α_k and raises α_e.

revision

So far, the model determines returns, but the length of the cycle is exogenous, with hazard rate α_e between the search and implementation stages and hazard rate σ between implementation and exit. We now endogenize these. First, following Pissarides' (1990) labor-market model, we consider entry. Let the measure of entepreneurs and capitalists in the market be n_e and n_k. If we assume a standard CRS matching technology $m(n_e, n_k)$, the arrival rates in the partner-search stage satisfy

$$\alpha_e = m(n_e, n_k)/n_e = m(1, \tau) \tag{13}$$

$$\alpha_k = m(n_e, n_k)/n_k = m(1, \tau)/\tau, \tag{14}$$

where $\tau = n_e/n_k$ denotes market tightness. Hence, we write $\alpha_k = \alpha_k(\tau)$ and $\alpha_e = \alpha_e(\tau)$.

If n_e and n_k are fixed then so are the arrival rates, as long as we satisfy the identity $n_e\alpha_e = n_k\alpha_k$, which is how one interprets the previous simple model. Now we fix $n_e = 1$, but assume a perfectly elastic supply of potential VC's who can enter the market as long as they pay some fixed cost κ (say, a cost to acquiring expertise; later we introduce the cost of raising funds). Then equilibrium requires $\kappa = V_k^0$, which by (6) is equivalent to

$$\kappa = \frac{\alpha_k (\sigma p - \ell_k)}{r (r + \sigma + \alpha_k)}. \tag{15}$$

Using (12) to eliminate p, then inserting $\alpha_j = \alpha_j(\tau)$, we arrive at

$$\kappa = \frac{\alpha_k(\tau)\theta (\sigma R - \ell_k - \ell_e)}{\theta r [r + \sigma + \alpha_k(\tau)] + (1 - \theta) (r + \sigma) [r + \alpha_e(\tau)]}. \tag{16}$$

Equilibrium with entry is a pair (τ^*, p^*) solving (16) and (12). Conveniently, equilibrium is recursive: we first look for a τ^* solving (16); then, given $\alpha_j(\tau^*)$, we get p^* exactly as in the model with no entry. We make the usual assumptions that the matching technology $m(\cdot)$ implies $\alpha_k(0) = \infty$, $\alpha_k(\infty) = 0$, $\alpha_k'(\tau) < 0$, $\alpha_e(0) = 0$, $\alpha_e(\infty) = \infty$, and $\alpha_e'(\tau) < 0$. Then it is easy to check, that as long as $\theta > 0$, the RHS of (16) is decreasing in τ, approaches 0 as $\tau \to \infty$, and approaches $(\sigma R - \ell_k - \ell_e)/r$ as $\tau \to 0$. Hence, given $\sigma R - \ell_k - \ell_e \geq 0$, which we obviously need for ventures to be profitable in the first place, there exists a unique equilibrium, with $n_e = \tau^* > 0$ iff $\kappa < \bar{\kappa} = (\sigma R - \ell_k - \ell_e)/r$. Given $\kappa < \bar{\kappa}$, it is easy to verify $\partial\tau^*/\partial\kappa < 0$, $\partial\tau^*/\partial R > 0$, $\partial\tau^*/\partial\sigma > 0$, $\partial\tau^*/\partial\ell_k < 0$, $\partial\tau^*/\partial\ell_e < 0$, $\partial\tau^*/\partial\theta > 0$ and $\partial\tau^*/\partial r < 0$.

Therefore, once we add entry, the framework implies that the number of VC's will increase when the cost of entry falls, when ventures are more lucrative, and when ventures are less costly in terms of either time or other investments. Their number will also increase when VC's have more bargaining power, and when they are more patient. By affecting market tightness, entry affects the arrival rates α_e and α_k, and hence the duration of search for a partner on both sides of the market. In general equilibrium, all of these changes also affect p in ways that can be easily computed. Hence, again the model can be used to make predictions that seem reasonable, relevant, and testable empirically.

Having determined the duration of search, we now endogenize the length of the implemtation stage, by letting agents choose the investments (ℓ_e, ℓ_k). We assume that more of either input increases the rate at which the project comes to fruition $\sigma = \sigma(\ell_e, \ell_k)$, where $\sigma(\cdot)$ is strictly concave, and without any real loss of generality restrict $(\ell_e, \ell_k) \in \mathcal{L} = [0, 1]^2$. When e and k meet, they bargain over p as well as (ℓ_e, ℓ_k). We can find the outcome by inserting (8)-(9) into (7) and maximizing with respect to all three variables. However, as is standard, an easier route to the same end is to first maximize the total surplus S (which does not depend on p) with respect to (ℓ_e, ℓ_k), and then split S by maximizing the Nash product with respect to p.

The problem of maximizing the surplus is:

$$\max_{(\ell_e,\ell_k)\in\mathcal{L}} S = \frac{\sigma(\ell_e,\ell_k)R - [r+\sigma(\ell_e,\ell_k)]\,V_e^0 - rV_k^0 - \ell_e - \ell_k}{r+\sigma(\ell_e,\ell_k)} \tag{17}$$

Because S is continuous and $\mathcal{L} = [0,1]^2$ a solution exists. Standard curvature conditions can be imposed to guaranntee it is interior, and hence satisfies the first-order conditions

$$\ell_e \quad : \quad [rR + rV_k^0 + \ell_e + \ell_k]\sigma_e - (r+\sigma) = 0 \tag{18}$$

$$\ell_k \quad : \quad [rR + rV_k^0 + \ell_e + \ell_k]\sigma_k - (r+\sigma) = 0. \tag{19}$$

It is easy to check that S is strictly concave at any point where the first-order conditions hold, and so there is a unique choice (ℓ_e, ℓ_k) that solves (17). From (18)-(19) we immediately see that $\sigma_e = \sigma_k$, which is natural, since ℓ_e and ℓ_k have the same marginal cost. Also, this choice of (ℓ_e, ℓ_k) takes all the market values V_i^j as given, but actually depends only on V_k^0.

In equilibrium, of course, V_k^0 is given by (6) as a function of the $\bar{p}$ prevailing in the market, which is in turn given by (12). Inserting these results and simplifying, we get

$$\sigma_e = \sigma_k = \frac{\theta r(r+\sigma+\alpha_k) + (1-\theta)\,(r+\sigma)\,(r+\alpha_e)}{[r+(1-\theta)\,\alpha_e+\alpha_k]\,rR + [r+(1-\theta)\,\alpha_e]\,(\ell_e+\ell_k)}. \tag{20}$$

The two equalities in (20) determine (ℓ_e^*, ℓ_k^*), and then we get p^* from (12) as in the model with ℓ_j fixed.[11] Assuming for the moment that α_e and α_k are fixed, we claim that an equilibrium always exists. Thus, consider an arbitray pair $(\bar{\ell}_e, \bar{\ell}_k) \in \mathcal{L}$ prevailing in the market. Clearly the values of σ and V_j^i prevailing in the market are continous in $(\bar{\ell}_e, \bar{\ell}_k)$. Given this, the unique solution $(\ell_e, \ell_k) \in \mathcal{L}$ to (17) discussed in the previous paragraph is

[11] Notice the solution to the bargaining problem (17) does not depend on θ, since agents at this stage are trying simply to maximize the joint surplus; in equilibrium, however, (20) indicates that (ℓ_e^*, ℓ_k^*) does depend on θ, since it influences the continuation value V_k^0.

continuous in $(\bar{\ell}_e, \bar{\ell}_k)$ by the Theorem of the Maximum. Since $\mathcal{L} = [0,1]^2$ there exists a fixed point (ℓ_e^*, ℓ_k^*).

Thus we endogenize the implementation phase. To illustrate the essential workings of the model, consider for simplicity the limiting case $\theta \to 1$. Then (20) reduces to

$$\sigma_e = \sigma_k = \frac{r + \sigma + \alpha_k}{(r + \alpha_k) R + \ell_e + \ell_k}. \tag{21}$$

It is easy to check $\partial \ell_e / \partial R \simeq \sigma_{ek} - \sigma_{kk}$, where $a \simeq b$ means a and b have the same sign; hence, $\partial \ell_e / \partial R > 0$ iff ℓ_e is a normal input.[12] Similarly, $\partial \ell_e / \partial r > 0$ and $\partial \ell_e / \partial \alpha_k > 0$ under the same conditions. Symmetrically, $\partial \ell_k / \partial R > 0$, $\partial \ell_k / \partial r > 0$ and $\partial \ell_k / \partial \alpha_k > 0$ iff ℓ_k is a normal input. The intuition is clear: increasing R raises the value of the venture once it comes to fruition, while increasing r or α_k makes k more impatient for this to happen, and in either case it is desirable to increase σ. This means either ℓ_e or ℓ_k must increase, and both increase if they are normal inputs.

One can of course combine the models that endogenize the partner-search phase and the implementation phase. To ease the presentation, suppose $\ell_k = \bar{\ell}_k$ is given – e.g. the VC contributes some fixed factor, but does not contribute to σ at the margin – so that we can define equilibrium in terms of (ℓ_e, τ) and represent the outcome using a graph in $\mathbb{R}^2$, and again we let $\theta \to 1$. This has the following extreme feature: with $\theta = 1$, e gets no surplus since he is compensated only for the cost of ℓ_e; and with free entry, k gets no ex ante profit (although there is positive expected profit ex post once k meets e and implemetation begins). Hence, if taken literally, this case implies all ex ante gains from trade go to whoever is selling expertise to e at price κ – say, economics professors. But we do not need to take it literally,

[12] In standard price theory, in general, saying ℓ_e is a normal input means that in the problem $\min\{p_k \ell_k + p_e \ell_e\}$ s.t. $\sigma(\ell_e, \ell_k) = \bar{\sigma}$, the solution satisfies $\partial \ell_e / \partial \bar{\sigma} = \sigma_k \sigma_{ek} - \sigma_e \sigma_{kk} > 0$. In our problem, since $\sigma_k = \sigma_e$, this reduces to $\sigma_{ek} > \sigma_{kk}$.

since by continuity, all substantive conclusions also hold for $\theta \approx 1$.

Given $\bar{\ell}_k$ is fixed, equilibrium is characterized by the first order condition (20) for the optimal investment ℓ_e, and the free entry condition (16) for τ. When $\theta = 1$, after algebraic manipulation to isolate $\alpha_k(\tau)$ on the LHS, these can be written

$$\alpha_k(\tau) = \frac{\sigma_e(\ell_e)\left(rR + \ell_e + \bar{\ell}_k\right) - r - \sigma(\ell_e)}{1 - \sigma_e(\ell_e) rR} \tag{22}$$

$$\alpha_k(\tau) = \frac{\kappa r\left[r + \sigma(\ell_e)\right]}{\sigma(\ell_e)R - \ell_e - \bar{\ell}_k - \kappa r}. \tag{23}$$

We prove in the Appendix the following:

Claim 1 *There exists a unique equilibrium; if $\kappa \geq \bar{\kappa}$ where $\bar{\kappa}$ is defined in terms of parameters, then the equilibrium entails $n_e = 0$ and the market shuts down; if $\kappa < \bar{\kappa}$ then $n_e = \tau^* > 0$ and $\ell_e = \ell_e^*$, where ℓ_e^* is the value of ℓ_e that maximizes τ subject to free entry.*

These results are illustrated in Figure 1 in (ℓ_e, α_k) space, since it is obviously equivalent to solve for either τ or $\alpha_k(\tau)$. The FE curve is the locus of points satisfying the free entry condition (23) and the OI curve is the locus of points satisfying the optimal investment condition (22), drawn assuming $\kappa < \bar{\kappa}$. These curves intersect uniquely in the positive quadrant, at the minimum of the FE curve over the interval (ℓ_1, ℓ_2), where ℓ_1 and ℓ_2 are the two solutions to $\sigma(\ell_e)R - \ell_e - \bar{\ell}_k - r\kappa = 0$.[13] These curves make it relatively easy to analyze the full equilibrium. For example, as κ increases, OI does not change while FE shifts up, increasing ℓ^* and $\alpha_k(\tau^*)$ (i.e., decreasing entry). As κ increases, however, the interval (ℓ_1, ℓ_2) gets smaller and eventually vanishes, so we lose equilibrium.

[13] Thus, ℓ_1 and ℓ_2 imply 0 profit given the VC gets a project without having to search. Note that for analytic completeness the Figure shows what happens when $\alpha_k < 0$, although obviously this cannot be an equilibrium.

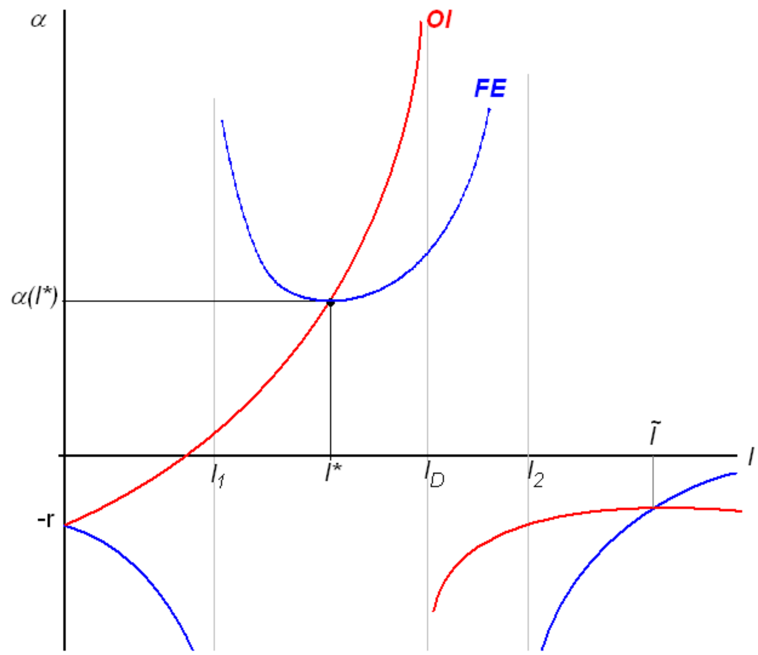


Figure 1: Results in Claim 1

Similarly, one can ask about a change in R. This shifts FE down as entry rises, reducing α_k and ℓ along a given OI curve; but the IO curve also shifts down, causing ℓ to move back up and further reducing α_k since we know the new equilibrium still lies at the minimum of FE. The net effect is a fall in α_k due to increased entry, and apparently an ambiguous effect on ℓ. *But* we can use the result in Claim 1 that the intersection of FE and OI occurs at the minium of the FE curve to deduce that in fact ℓ^* actually must rise after an increase in R, because it is easy to see that the minimum point on FE shifts right. A similar argument can be used to establish that ℓ^* and $\alpha_k(\tau^*)$ must rise after an increase in r. See Figure 2 for parametric examples.

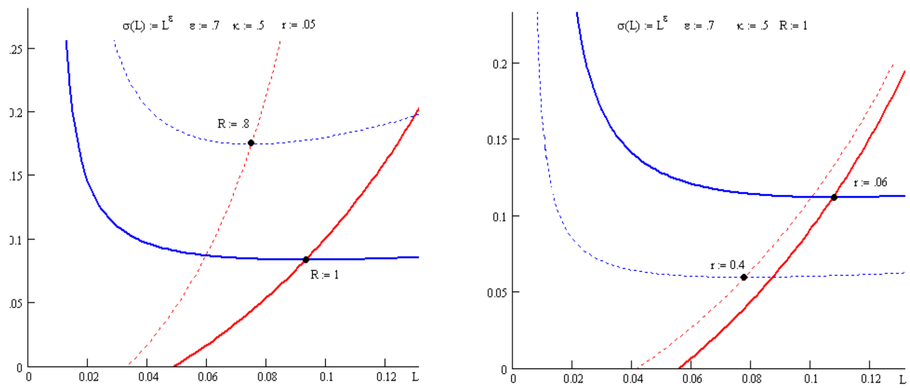


Figure 2: Changes in R and r in examples

3 Random Return and Cost

The models in the previous section generate as an equilibrium outcome the return p, as well as the duration of the partner-search and implementation stages of the VC cycle. The duration of the partner-search phase is made endogenous through entry by VC's, which affects the arrival rates α_e and α_k through market tightness. Another way to make this stage endogenous is to let the quality of the match between e and k project be random. To be precise, a non-negative (R, C) is observed when agents meet, but it can differ across meetings, even though all agents are ex ante homogeneous. We can summarize a general joint distribution of (R, C) by the marginal $F_C(C)$ and the conditional $F_R(R|C)$. However, we start with $C = 0$ in all matches and $F_R(R|C = 0)$ simple as $F_R(R|0)$. We take ℓ_e and ℓ_k as fixed for now.

The value functions for implementation satisfy the same conditions as above, (2) and (4), except $V_e^1 = V_e^1(R)$ and $V_k^1 = V_k^1(R)$ can now vary across matches. The value functions for partner search are slightly more complicated. For an entrepreneur, e.g.,

$$rV_e^0 = \alpha_e \int_0^{\bar{R}} \max\{V_e^1(\tilde{R}) - V_e^0, 0\} dF_R(\tilde{R}) = \alpha_e \int_{R^*}^{\bar{R}} \left[V_e^1(\tilde{R}) - V_e^0\right] dF_R(\tilde{R}) \tag{24}$$

where, to clarify notation, $\tilde{R}$ is the random variable (a dummy variable for integration), $\bar{R}$ is the upper bound of its support, and R^* is the endogenous reservation return below which partnerships do not form. The reservation return obviously satisfies $S(R^*) = S_e(R^*) = S_k(R^*) = 0$; in other words, it generates 0 surplus, which from (10) means

$$\sigma R^* = rV_k^0 + (r + \sigma)V_e^0 + \ell_e + \ell_k. \tag{25}$$

Inserting (9) into (24), we get

$$rV_e^0 = \frac{\alpha_e}{r + \sigma} \int_{R^*}^{\bar{R}} \left[\sigma(\tilde{R} - p) - (r + \sigma)V_e^0 - \ell_e\right] dF_R(\tilde{R}). \tag{26}$$

The barganing problem is the same as in the simpler model, and yields the same solution (11), which we can insert into (26) to get

$$\begin{aligned} rV_e^0 &= \frac{\alpha_e(1 - \theta)}{r + \sigma} \int_{R^*}^{\bar{R}} \left[\sigma\tilde{R} - rV_k^0 - (r + \sigma)V_e^0 - \ell_e - \ell_k\right] dF_R(\tilde{R}) \\ &= \frac{\alpha_e(1 - \theta)\sigma}{r + \sigma} \int_{R^*}^{\bar{R}} \left(\tilde{R} - R^*\right) dF_R(\tilde{R}), \end{aligned}$$

after using (25). Integrating by parts yields

$$rV_e^0 = \frac{\alpha_e(1 - \theta)\sigma}{r + \sigma} \int_{R^*}^{\bar{R}} \left[1 - F_R(\tilde{R})\right] d\tilde{R}. \tag{27}$$

Similarly,

$$rV_k^0 = \frac{\alpha_k \theta\sigma}{r + \sigma} \int_{R^*}^{\bar{R}} \left[1 - F_R(\tilde{R})\right] d\tilde{R}. \tag{28}$$

Inserting (27)-(28) into (25), we arrive at

$$\sigma R^* = \Pi \int_{R^*}^{\bar{R}} \left[1 - F_R(\tilde{R})\right] d\tilde{R} + \ell_e + \ell_k, \tag{29}$$

where to save space we introduce

$$\Pi = \Pi(\sigma, r, \theta, \alpha_e \alpha_k) \equiv \frac{\sigma\left[\alpha_e(1-\theta)(r+\sigma) + \alpha_k \theta r\right]}{r\,(r+\sigma)} \tag{30}$$

Given fixed n_e and n_k, and fixed ℓ_e and ℓ_k, equilibrium is characterized simply as a solution R^* to (29), since given this we can compute $p^* = p^*(R)$ for any venture with $R \geq R^*$ from (12) (notice that p^* is an increasing function). It is easy to verify that a unique equilibrium exists. The duration of partner search for k is endogenous, with hazard rate $H_k \equiv \alpha_k[1 - F_R(R^*)]$, because the parties agree to start a venture iff $R \geq R^*$. The expected duration of partner search for the VC is $1/H_k$. It is straightforward to show $\partial R^*/\partial r < 0$ and $\partial R^*/\partial \sigma < 0$; when impatience increases or projects becomes less time consuming, agents get less picky about R, and the expected duration of search falls. Also, $\partial R^*/\partial\left(\ell_e + \ell_k\right) > 0$, since agents get more picky when ventures are more costly. We cannot sign $\partial R^*/\partial \theta$, in general, as a change in θ raises the outside option for one side and lowers it for the other.

One can show $\partial R^*/\partial \alpha_j > 0$; more frequent meetings make agents more picky. However, $\partial H_k/\partial_k$ is ambigous, precisely because agents get more picky. One can show $\partial H_k/\partial \alpha_k > 0$ if the density is log concave.[14] Using some other tricks from job search theory, one can study changes in the distribution. For instance, a translation of F_R (increase every R to $R + \varepsilon$)

[14] That is, $\log F'(R)$ is concave in R, which holds for many but not all common distributions, and has been a common tool in search since Burdett (1981). See Sec. 2.5 of the notes at http://www.ssc.upenn.edu/~rwright/courses/oss.pdf for details concerning results discussed in this paragraph.

raises R^*, but by less than the amount of the translation as long as $\ell_e + \ell_k > 0$, which means that H_k increases. Perhaps surprisingly, the expected return conditional on implementation $E[R|R \geq R^*]$ can fall after a translation, but we can rule this out again by log concavity. Similar results apply to a scale transformation of F_R (increase every R to $R + R\varepsilon$). One can also show that an increase in risk (a mean-preserving spread of F_R) increases R^*; intuitively, more risk makes search more attractive due to the option to pass on poor ventures.

As in the previous section, instead of assuming fixed n_e and n_k we can add free entry, which in this model reduces to

$$r\kappa = \frac{\alpha_k(\tau)\theta\sigma}{r+\sigma} \int_{R^*}^{\bar{R}} \left[1 - F_R(\tilde{R})\right] d\tilde{R}. \tag{31}$$

Given this, we can simplify (29) using $\alpha_e = \alpha_k/\tau$ to

$$\theta\sigma R^* = [\theta r + (1-\theta)(r+\theta)\tau]\,\kappa + \theta\,(\ell_e + \ell_k)\,. \tag{32}$$

Equilibrium is now a pair (τ, R^*) solving (31)-(32), two curves labeled FE and RR (for Free Entry and Reservation Return) in Figure 3. It is clear that FE is decreasing and approaches the axes under the usual assumptions on the $m(\cdot)$, and that RR gives R^* as a linearly increasing funciton of τ.

Existence and uniqueness results are now rountine. It is also easy to perform comparative static exercises. For example, if κ increases, both curves shift left, which reduces τ by choking off entry, but has an ambiguious effect on R^*. The same thing happens if r increases; for a fixed τ we earlier showed that R^* unambiguiously decreases with r, but now the impact of changing entry on the arrival rates clouds the issue. The same thing happens if $\ell_e + \ell_k$ increases. If θ or σ increases, both curves shift right, which increases entry and τ and has an ambiguious effect on R^*. All of this says that there

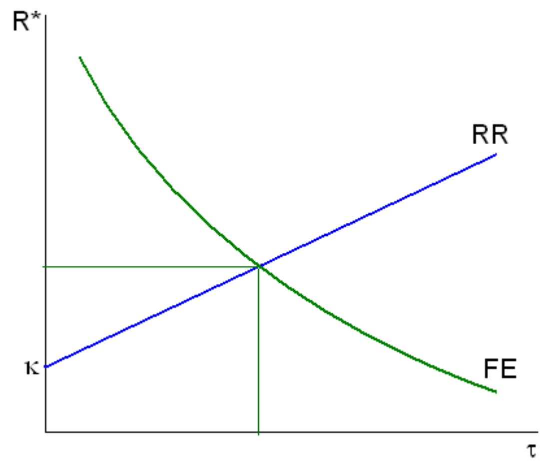


Figure 3: Equilibrium with R random and free entry

can be interesting general equilibrium effects on the duration of the VC cycle.

The case where R and C are both random is not much more difficult. In general, $V_j^1 = V_j^1(R, C)$ now depends on the joint realization. Thus, e.g., we have

$$rV_k^0 = \alpha_k \int_0^{\bar{C}} \int_0^{\bar{R}} \max\{V_k^1(\tilde{R}, \tilde{C}) - \tilde{C} - V_k^0, 0\} dF_R(\tilde{R}|\tilde{C}) dF_C(\tilde{C}). \tag{33}$$

Notice k has to pay C up front. The reservation return $R^*(C)$ now depends on the realized value of C, and satisfies a generalized version of (25)

$$\sigma R^*(C) = rV_k^0 + (r+\sigma)V_e^0 + (r+\sigma)C + \ell_e + \ell_k. \tag{34}$$

The bargaining problem yields a generalized version of (11),

$$\sigma p = \theta\sigma R - \theta\ell_e + (1-\theta)\ell_k + (1-\theta)(r+\sigma)C - \theta(r+\sigma)V_e^0 + (1-\theta) rV_k^0 \tag{35}$$

where we note that p is increasing in the realization of C as long as $\theta = 1$. This leads to a generalization of (28), and similarly, of (27), given by

$$rV_k^0 = \frac{\alpha_k \theta \sigma}{r+\sigma} \int_0^{\bar{C}} \int_{R^*(C)}^{\bar{R}} \left[1 - F_R(\tilde{R}|\tilde{C})\right] d\tilde{R} dF_C(\tilde{C}) \tag{36}$$

$$rV_e^0 = \frac{\alpha_e (1-\theta)\sigma}{r+\sigma} \int_0^{\bar{C}} \int_{R^*(\tilde{C})}^{\bar{R}} \left[1 - F_R(\tilde{R}|\tilde{C})\right] d\tilde{R} dF_C(\tilde{C}). \tag{37}$$

Inserting (36)-(37) into (34), we get the analog of (29),

$$\sigma R^*(C) = (r+\sigma)\, C + \Pi \int_0^{\bar{C}} \int_{R^*(\tilde{C})}^{\bar{R}} \left[1 - F_R(\tilde{R}|\tilde{C})\right] d\tilde{R} dF_C(\tilde{C}) + \ell_e + \ell_k, \tag{38}$$

where $\Pi = \Pi(\sigma, r, \theta, \alpha_e \alpha_k)$ was define in (30). Thus $R^*(C)$ is linear in C, with slope $(r+\sigma)/\sigma$ and intercept R_0^*. This is technically useful because we can describe the outcome in terms of a number R_0^* instead of the function $R^*(C)$. With n_e and n_k fixed, then, we define equilibrium simply in terms of the number R_0^*. Or, with free entry, we get generalizations of (31)-(32),

$$r\kappa = \frac{\alpha_k(\tau)\theta\sigma}{r+\sigma} \int_0^{\bar{C}} \int_{R_0^*+C(r+\sigma)/\sigma}^{\bar{R}} \left[1 - F_R(R|C)\right] dR dF_C(C) \tag{39}$$

$$\theta\sigma R_0^* = \left[\theta r + (1-\theta)(r+\sigma)\tau\right]\kappa + \theta\left(\ell_e + \ell_k\right), \tag{40}$$

which pin down (τ, R_0^*). Comparing these to (31)-(32), we see that making C random does not change the basic model much – but this will not be true in the next section when we introduce liquidity considerations.

4 Fund Raising and Liquidity

In this section we start to take seriously the fund raising stage of the VC cycle. The essential idea is not that fund raising takes time, but rather that it inolves some cost, and therefore a VC will not enter the next stage with an arbitrarily large amount of money. Moreover, we will assume that once he enters the next stage he cannot easily go back and raise additional funds. To capture this succintly, we take the position that he simply cannot spend more than the amount he brings to partner search. This assumption is quite realistic, based on what we read in the institutional literature, and not hard to model rigorously based on modern monetary theory. What is important is the idea that one may have to strike when the iron is hot: a VC who says, "that's a good idea, let me see if I can raise some money" may lose the deal. This is what the notion of liqudity is all about.[15]

To make this interesting we need C is random – otherwise the VC would bring in exactly enough to cover the start-up cost. Given C is random, we can assume R is constant without much sacrifice. We also assume $F_C(C)$ is continuously differentiable when we need it. We also set $\ell_e = \ell_k = 0$ here to reduce notation. To reiterate, the key part of the specification is that k needs to choose a fund size m before the partner-search stage; any match with $C > m$ is out of reach and cannot be implemented. While this is obviously extreme, it captures the idea that VC's do commit to a fund size ex ante and typically have difficulty extending themselves beyond this commitment ex post. One can think of as also choosing a reservation cost C^*, but it adds nothing to the outcome, since m is the binding constraint: as long as raising funds is costly there is no sense choosing $m > C^*$. Hence we do not need to worry about C^*, and we can focus on the choice of m.

[15] In Silviera and Wright (2005) we introduced into a different but related model the idea that agents can *try* to raise additional funds after they meet someone, but with some probability δ the deal falls through. To make the essential point here we assume $\delta = 1$, but the results hold more generally.

The value function for k at the fundraising stage is given by

$$W_k(m_{-1}) = \max_m \left\{ m_{-1} - m + V_k^0(m) \right\}, \tag{41}$$

where m_{-1} denotes funds he starts with, typically given by his returns from exiting the previous cycle, and m denotes what he takes to the next cycle. If $m_{-1} > m$ he consumes the difference; else he has to raise funds. We assume he can raise m instantly – although this would be easy to relax – at unit cost, and we restrict $m \in [0, R]$, obviously without loss in generality. One interpretation is that he can borrow at the same real interest rate that he uses to discount, which means that the cost of borrowing m between t and t' in present value terms is simply m.[16] The important point is that m is taken into the next stage as a state variable, which is why we write $V_k^0(m)$. The envelope condition is $\partial W_k / \partial m_{-1} = 1$. The first-order condition from the maximization problem is $\partial V_k^0 / \partial m = 1$, although as we show below, this condition may or may not characterize the solution.

Given the fund size m, the value to partner search is

$$rV_k^0(m) = \alpha_k \int_0^m \left[V_k^1(\tilde{C}) + m - \tilde{C} - V_k^0(m) \right] dF_C(\tilde{C}). \tag{42}$$

Note that if the venture has startup cost $C < m$ then k cashes in the difference immediately, and, again, if $C > m$ he simply cannot afford it. Note also that the implementation value $V_k^1(C)$ may depend on C, even though it is a sunk cost at the implementation stage, because the payment $p = p(C, m)$ in general can depend on C. Emulating the analysis of bargaining in the

[16] Because agents are risk neutral, there is a perfectly elastic supply of funds available at rate r. Alternatively, we can say that if $m > m_{-1}$ then k takes the difference out of an endowment – it does not matter with risk neutrality. In any case, with R deterministic $m > m_{-1}$ it does not arise in steady state – no one is going to take funds to the market in excess of his share of the value of the venture – although, of course, a new entrant VC has to either borrow or work to raise the initial fund.

previous sections, we get the generalization of (35):

$$\sigma p = \theta\sigma R - \theta(r+\sigma)V_e^0 + (1-\theta)\, rV_k^0 + (1-\theta)\,(r+\sigma)C - (1-\theta)\, rm \quad (43)$$

As long as $\theta < 1$, p is again increasing in C, and now it is also decreasing in m if we hold V_k^0 constant at some given level; but if the first-order condition for m holds then $\partial V_k^0/\partial m = 1$, and hence $\partial p/\partial m = 0$.[17]

Further emulating the analysis in the previous sections, we get the generalized versions of (27)-(28)

$$\begin{aligned} rV_e^0 &= \frac{\alpha_e(1-\theta)}{r+\sigma}\int_0^M \left[\sigma R + rM - (r+\sigma)\tilde{C} - rV_k^0 - (r+\sigma)V_e^0\right] dF_C(\tilde{C}) \\ rV_k^0 &= \frac{\alpha_k\theta}{r+\sigma}\int_0^m \left[\sigma R + rm - (r+\sigma)\tilde{C} - rV_k^0 - (r+\sigma)V_e^0\right] dF_C(\tilde{C}). \quad (44) \end{aligned}$$

Note that we write V_e^0 in terms of the value M that e expects the representative k to hold in equilibrium, while V_k^0 depends directly on the m that k himself chooses. In equilibrium, of course, we can set $m = M$, and solve these (linear) equations for

$$V_e^0 = \frac{\alpha_e(1-\theta)F(m)\left[\sigma R + rm - (r+\sigma)E\left(C|C \le m\right)\right]}{r\,(r+\sigma) + \alpha_k\theta F(m)r + \alpha_e(1-\theta)F(m)(r+\sigma)} \quad (45)$$

$$V_k^0 = \frac{\alpha_k\theta F(m)\left[\sigma R + rm - (r+\sigma)E\left(C|C \le m\right)\right]}{r\,(r+\sigma) + \alpha_k\theta F(m)r + \alpha_e(1-\theta)F(m)(r+\sigma)}, \quad (46)$$

where $E\left(C|C \le m\right) = \int_0^m \frac{CdF_C(C)}{F_C(m)}$.

Now consider the fund-raising decision in (41). Differentiation of both sides of (44) (which gives V_k^0 *before* we impose the equilibrium condition

[17] The key point is that the realization of C applies only to the current match, while the value of m is carried forward in the outside option V_k^0.

$m = M$) leads to

$$r\frac{\partial V_k^0}{\partial m} = \frac{\alpha_k\theta}{r+\sigma}\int_0^m \left(r - r\frac{\partial V_k^0}{\partial m}\right) dF_C(\tilde{C}) + \frac{\alpha_k\theta}{r+\sigma}\left[\sigma\left(R-m\right) - rV_k^0(\hat{m}) - (r+\sigma)V_e^0\right] F_C'(m),$$

using Leibnez's rule. By virtue of the first-order condition $\partial V_k^0/\partial m = 1$, this reduces to

$$r = \frac{\alpha_k F_C'(m)\theta\left[\sigma(R-m) - rV_k^0 - (r+\sigma)V_e^0\right]}{r+\sigma}. \tag{47}$$

This equates the marginal cost of funds, r, to the marginal benefit, which is given by the probability per unit time that an additional dollar just allows some venture to be funded, $\alpha_k F_C'(m)$, times the VC's share of the surplus, appropriately discounted.

This marginal calculation takes V_k^0 and V_e^0 as given. To describe equilibrium, we can insert (45)-(46) into (47) and reduce the system to

$$T(m) = T_1(m)T_2(m) - T_3(m)T_4(m) = 0, \tag{48}$$

where

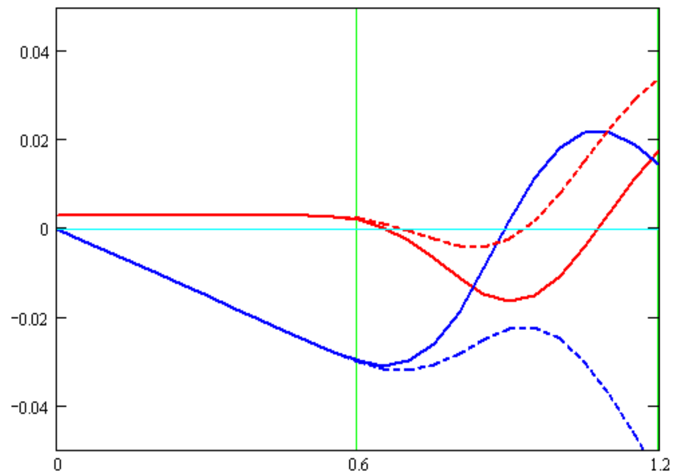
$$\begin{aligned}
T_1(m) &= \alpha_k\theta F_C(m)F_C'(m)\left[r\alpha_k\theta + (r+\sigma)\alpha_e(1-\theta)\right] \\
T_2(m) &= \left[\sigma R + rm - (r+\sigma)E\left(C|C \le m\right)\right] \\
T_3(m) &= \left[r(r+\sigma) + \alpha_k\theta F_C(m)r + \alpha_e(1-\theta)F_C(m)(r+\sigma)\right] \\
T_4(m) &= \left[\alpha_k\theta\sigma(R-m)F_C'(m) - r(r+\sigma)\right].
\end{aligned}$$

This is *almost* our final answer. Since $T(m)$ equals (in sign) marginal cost minus marginal benefit, $T(m) = 0$ is the first-order condition for an interior solution for fund size in equilibirum; but we also must check the second-order condition and the participation condition $V_k^0(m) \ge m$. One can show the second-order condition holds iff $T'(m) > 0$ (which simply says marginal cost is increasing faster than marginal benefit). The participation condition turns out to hold iff R is above some threshold, as we now discuss.

The first thing to do is to graph $T(m)$ (see Figure 4 below). The relevant range is $m \in [0, R]$, and notice that $T(R) > 0$. Suppose $T(m) \geq 0$ $\forall m \in [0, R]$, which must be true if R is small enough; then $m = 0$ is an equilibrium, since the marginal cost exceeds the benefit $\forall m \in [0, R]$. So suppose $T(m) < 0$ for some $m \in [0, R)$. Then two things are clear: there is at least one solution to $T(m) = 0$ in $(0, R)$ with $T'(m) \geq 0$, and hence there is a value of m that satisfies first- and second-order conditions; and if there are multiple solutions, one of them $\bar{m}$ acheives the maximum over $[0, R]$ of $w(m) = V_k^0(m) - m$, where $V_k^0(m)$ is given by (**??**).[18] If $w(\bar{m}) \geq 0$ then $\bar{m}$ is an equilibrium, since it satisfies the first- and second-order conditions, plus participation. If $w(\bar{m}) < 0$ then no solution to the first- and second-order conditions satisfies participation, and hence $m = 0$ is the unique equilibrium. In either case we have existence of an equilibrium.

In Figure 4, an example with F_c log-normal shows $T(m)$ in red, with the dashed curve corresponding to $R = 1$ and the solid curve corresponding to $R = 1.2$. The dashed and solid blue curves show $w(m)$ for the same two values of R. For each case, there are two solutions to $T(m) = 0$ (two places where each red curve cuts the horizontal axis), and hence two values of m that satisfy the first-order condition; but only the higher value of m satisfies the second-order condition, which requires $T'(m) > 0$. With the lower $R = 1$ shown as the dashed curve, the relevant solution violates participation since $w(m) < 0$ and hence the only equilibrium is $m = 0$. With the higher $R = 1.2$ shown as the solid curve, the relevant solution satisfies participation since

[18] We assume that $T(m)$ is continuous, and that there there are at most a finite number of solutions to $T(m) = 0$, which can be guaranteed with primitive assumptions on $F_C(\cdot)$.



$w(m) > 0$ and hence it is an equilibrium.

The key point is that equilibrium always exists, but if R is sufficiently small then the only equilibrium is $m = 0$ – i.e. the venture capital market shuts down – and if R is sufficiently small then equilibrium exists with $m > 0$. Since an increase in R shifts $T(m)$ up, it is also clear that this will increase the equilibrium value of m, assuming equilibrium with $m > 0$ exists, since $T'(m) > 0$ in equilibrium by the second-order condition.

5 Appendix

We prove Claim 1. First, we can work with α_k, instead of τ, since $\alpha_k = \alpha_k(\tau)$. Second, given $\bar{\ell}_k$ is fixed, we can normalize it to 0. Then equilibrium is a pair (ℓ_e, α_k) satisfying

$$\begin{aligned} \alpha_k &= \alpha_k^F(\ell_e) \equiv \frac{r\kappa\left[r + \sigma(\ell_e)\right]}{\sigma(\ell_e)R - \ell_e - r\kappa} \\ \alpha_k &= \alpha_k^I(\ell_e) \equiv \frac{\sigma_e(\ell_e)(rR + \ell_e) - \left[r + \sigma(\ell_e)\right]}{1 - \sigma_e(\ell_e)R}. \end{aligned}$$

As $\ell_e \to 0$, $\sigma(\ell_e) \to 0$ and $\sigma_e(\ell_e) \to \infty$ by assumption, and therefore l'Hopital's Rule implies $\alpha_k^I(\ell_e) \to -r$. Also, clearly, $\alpha_k^F(0) = -r$.

One can check that $\alpha_k^F(\ell_e) > 0$ iff $\ell_e \in (\ell_1, \ell_2)$, where ℓ_1 and ℓ_2 are the two solutions to $\sigma(\ell_e)R - \ell_e = r\kappa$, which means (ℓ_1, ℓ_2) is nonempty iff $\kappa < \bar{\kappa} = \max\{\sigma(\ell_e)R - \ell_e\}$. One can check that α_k^F is strictly convex in (ℓ_1, ℓ_2). It is also easy to check that α_k^I is strictly increasing iff $\ell_e \in (0, \tilde{\ell})$, with $\tilde{\ell} > \ell_2$, and $\alpha_k^I \to \pm\infty$ as $\ell_e \to \ell_D$ from the left or right, where ℓ_D is the zero of the denominator of α_k^I. Notice that as long as $\kappa < \bar{\kappa}$, we have $\ell_D \in (\ell_1, \ell_2)$, because $\ell_D = \arg\max\{\sigma(\ell_e)R - \ell_e - r\kappa\}$ while (ℓ_1, ℓ_2) are the zeros of $\sigma(\ell_e)R - \ell_e - r\kappa$.

Hence the situation is as depicted in Figure 1 in the text. Indeed we can prove the following: α_k^I meets α_k^F at $\ell_e = \ell^*$ where ℓ^* minimizes α_k^F over (ℓ_1, ℓ_2). To verify this, notice that since α_k^F is strictlty convex over this interval, a necessary and sufficient condition for the minimization is $\partial\alpha_k^F/\partial\ell_e = 0$, which can be rearranged as

$$\sigma_e(\ell^*) = \frac{r + \sigma(\ell^*)}{rR + \ell_e + r\kappa}.$$

Minor algebra reveals $\alpha_k^I(\ell^*) = \alpha_k^F(\ell^*)$. This establishes the claim: equilibrium exists iff $\kappa < \bar{\kappa}$, since this is required for (ℓ_1, ℓ_2) to be nonempty, and when it exists it is uniquely determined by $\ell_e = \ell^*$ and $\alpha_k = \alpha_k^F(\ell^*)$. ■

Next we consider the model discussed in Claim 1 for the case of a general value of $\theta \in [0, 1]$. To simplify the notation slightly we set $\ell_k = 0$ and write $\ell_e = \ell$. The free entry and optimal investment conditions are given by

$$\begin{aligned}
\kappa &= \frac{\alpha_k(\tau)\theta\left[\sigma\left(\ell\right)R - \ell\right]}{\theta r\left[r + \sigma\left(\ell\right) + \alpha_k(\tau)\right] + (1-\theta)\left[r + \sigma\left(\ell\right)\right]\left[r + \alpha_e(\tau)\right]} \\
\sigma_e &= \frac{\theta r\left[r + \sigma\left(\ell\right) + \alpha_k\right] + (1-\theta)\left[r + \sigma\left(\ell\right)\right]\left(r + \alpha_e\right)}{\left[r + (1-\theta)\,\alpha_e + \alpha_k\right]rR + \left[r + (1-\theta)\,\alpha_e\right]\ell}.
\end{aligned}$$

We can solve each of these equations for α_k

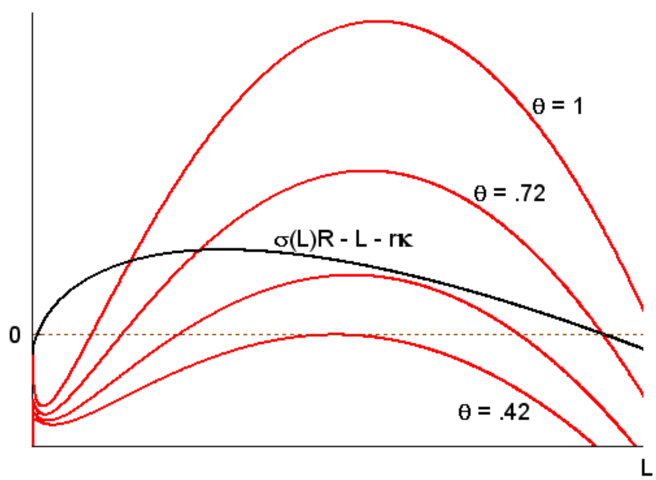
$$
\begin{aligned}
\alpha_k(\tau) &= \frac{\kappa\left\{\theta r\left[r+\sigma(\ell)\right]+(1-\theta)\left[r+\sigma(\ell)\right]\left[r+\alpha_e(\tau)\right]\right\}}{\theta\left[\sigma(\ell)R-\ell-r\kappa\right]} \\
\alpha_k(\tau) &= \frac{\left[r+\sigma(\ell)\right]\left\{\theta r+(1-\theta)\left[r+\alpha_e(\tau)\right]\right\}}{r\left[\sigma_e(\ell)R-\theta\right]} \\
&\quad -\frac{\left[r+(1-\theta)\alpha_e(\tau)\right](rR+\ell)\,\sigma_e(\ell)}{r\left[\sigma_e(\ell)R-\theta\right]},
\end{aligned}
$$

although these are not that useful, since $\alpha_e(\tau)$ appears on the RHS. But if we equate the two expressions, after some minor algebra, a minor miracle occurs and $\alpha_e(\tau)$ vanishes, leaving $T(\ell)=0$, where

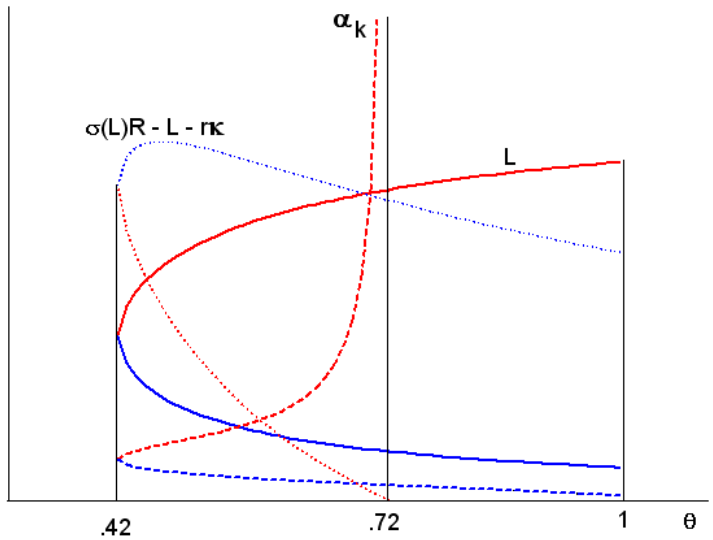
$$
\begin{aligned}
T(\ell) &= \theta\left[r+\sigma(\ell)-(rR+\ell)\,\sigma_e(\ell)\right]\left[\sigma(\ell)R-\ell\right] \\
&\quad -r\kappa\left[(1-\theta)rR+\sigma(\ell)R-\theta\ell\right]\sigma_e(\ell).
\end{aligned}
$$

This is nice because an equilibrium with $\ell>0$ has to satisfy at least $T(\ell)=0$, which one equation in ℓ, independent of any other endogenous variable. Now, if $\theta=0$ there are obviously no solutions to $T(\ell)=0$, and if $\theta=1$ from Claim 1 we already know there exists a solution iff κ is not too big. Hence, given κ not too big, there is a solution to $T(\ell)=0$ iff θ is not too small. Suppose there is a solution ℓ^* to $T(\ell^*)=0$; then ℓ^* is a *candidate* equilibrium, but we still have to check that no side conditions are violated. In particular, the expression for $\alpha_k^E(\tau)$ derived above indicates that $\alpha_k(\tau)\geq 0$ at the candidate solution iff $\sigma(\ell^*)R-\ell^*\geq r\kappa$, which is a participation constraint conditional on having found a partner; one can also interpret it as a non-negative profit condition in th limiting case where $\alpha_k(\tau)\to\infty$, since $\sigma(\ell^*)R-\ell^*$ is the expected flow payoff during implementation and $r\kappa$ is the capitalized entry cost. It is clear that $\sigma(\ell)R-\ell\geq r\kappa$ iff $\ell\in(\ell_1,\ell_2)$, where $\ell_1>0$, and $\ell_2>\ell_1$ iff $\kappa<\bar{\kappa}=\max_\ell\{\sigma(\ell)R-\ell\}$.

Notice that at $\ell = \ell_1$ or $\ell = \ell_2$, we have $\sigma(\ell) R - \ell = r\kappa$, and $T(\ell) = r\kappa [r + \sigma(\ell)] [\theta - \sigma_e(\ell) R]$. For small θ, $T(\ell) < 0$ $\forall \ell \in (\ell_1, \ell_2)$, so $T(\ell) = 0$ has no solution in (ℓ_1, ℓ_2); for $\theta = 1$ we already know $T(\ell) = 0$ has a solution iff κ is not too big; therefore, in general, there is a solution iff θ is not too small, given that κ is not too big. As shown in the Figure, again given that κ is not too big, there are two solutions to $T(\ell) = 0$ when $\theta < \hat{\theta}$ and no solutions when $\theta > \hat{\theta}$. We now check whether these solutions satisfy the participation constraint, which holds for an arbitrary ℓ iff $\sigma(\ell) R - \ell - r\kappa \geq 0$, as shown in the Figure. From the analysis of the case with $\theta = 1$, and continuity, for any $\theta \approx 1$ we know the participation condition holds at the lower but not the upper solution; i.e., we know $T(\ell_1) < 0$ and $T(\ell_2) > 0$, or in other wordes, the lower solution is in (ℓ_1, ℓ_2) and the upper solution is not. As we lower θ, $T(\ell)$ unambiguously shifts down, until we reach some $\tilde{\theta}$ such that $T(\ell_2) = 0$. As we continue to lower θ from $\tilde{\theta}$ to $\hat{\theta}$ both solutions are in (ℓ_1, ℓ_2) and hence satisfy the participation constraint. And, again, once we lower θ past $\hat{\theta}$ there are no solutions to $T(\ell) = 0$.



The above argument, based on the parametric example in the Figure, proceeds as if T were concave in the relevant interval (ℓ_1, ℓ_2), which we have not been able to establish, in general, since it depends on the third derivative of σ. This however does not affect the logic, except that there could potentially be more than two solutions to $T(\ell) = 0$ for some θ; the generalized version of the discussion would say the lowest solution (rather than the lower solution) satisfies participation and the highest one (rather than the higher one) does not satisfy participation for big θ, while all of them (rather than both of them) satisfy participation for intermediate θ. Taking the case where there are exactly two solutions, the next figure shows the candidate equilibrium values of ℓ – i.e. the upper and lower solutions of $T(\ell) = 0$ – when they exist as we vary θ. Also shown are the values of α_k and $\sigma(\ell) R - \ell - r\kappa$ implied by the candidate equilibrium values of ℓ. As can be seen, when $\theta > \tilde{\theta} \approx 0.72$, the upper solution implies $\sigma(\ell) R - \ell - r\kappa < 0$, or equivalently, there is no α_k consistent with free entry because $\alpha_k \to \infty$ as $\theta \to \tilde{\theta}$. For $\theta \in (\tilde{\theta}, \hat{\theta}) \approx (0.44, 0.72)$ both solutions satisfy participation, although notice that the upper solution imples a much higher α_k and lower profits as measured by $\sigma(\ell) R - \ell - r\kappa$. At $\theta = \hat{\theta} \approx 0.44$ the upper and lower solutions coalesce; below $\hat{\theta}$ there is no solution, so the only equilibrium implies the market shuts down.



REFERENCES

Aghion, Philippe and Jean Tirole (1994), The Management of Innovation, *Quarterly Journal of Economics* 109, 1185-1209.

Bascha, Andreas and Uwe Walz (2001), Convertible Securities and Optimal Exit Decisions in Venture Capital Finance, *Journal of Corporate Finance* 7, 285-306.

Bergemann, Dirk and Ulrich Hege (1998), Venture Capital Financing, Moral Harzard and Learning, *Journal of Banking and Finance* 104, 703-735.

Casamatta, Catherine (2002), Financing and Advising: Optimal Financial Contracts With Venture Capitalists, CEPR DP 3475.

Cumming, Douglas and Jeffrey G.MacIntosh (2002), A Cross Country Comparison of Full and Partial Venture Capital Exits, *Journal of Banking and Finance*, forthcoming.

Gompers, Paul A. (1996) "Grandstanding in the Venture Capital Industry, *Journal of Financial Economics* 42, 133-156.

Gompers, Paul A., and Josh Lerner, 1999, *The Venture Capital Cycle* (MIT Press, Cambridge, MA).

Gompers, Paul A., and J. Lerner (2000), Money Chasing Deals? The Impact of Fund Inflows on the Valuation of Private Equity Investments, *Journal of Financial Economics* 55 (2), pp. 281-325.

Gompers, Paul A., and Josh Lerner (2001), *The Money of Invention: How Venture Capital Creates New Wealth*. Cambridge: Harvard Business School Press.

Gompers, Paul A., and Josh Lerner (2001), The Venture Capital Revolution, *Journal of Economic Perspectives*, 15 (2), pp. 145–168.

Gompers, P.A., and J. Lerner (2002), *Short-Term America Revisited? Boom and Bust in the Venture Capital Industry and the Impact on Innovation*, Innovation Policy and the Economy, 3 pp. 1-27.

Hellmann, Thomas (1998), The Allocation of Control Rights in Venture Capital Contracts, *Rand Journal of Economics* 29, 57-76.

Hellmann, Thomas and Manju Puri (2000), The Interaction Between ProductMarket and Financing Strategy: The Role of Venture Capital, *Review of Financial Studies* 13, 995-984.

Hellmann, Thomas and Manju Puri (2002), Venture Capital and the Professionalization of Start-Up Firms, *Journal of Finance* 57, 169-197.

Inderst, R. and H. Muller 2004. The Effect of Capital Market Characteristics on the Value of Start-Up Firms, *Journal of Financial Economics* 72: 319–56.

Jovanovic, B. and B. Szentes (2006) On the Return to Vencture Capital

Jovanovic, B. and B. Szentes (2007) IPO Underpricing: Auctions vs. Book Building.

Kaplan, Steven N. and Per Strömberg (2001), Venture Capitalists as Principals: Contracting, Screening, and Monitoring, *American Economic Review* 91, 426-430.

Kaplan, Steven N. and Per Strömberg (2000), Financial Contracting Theory Meets the Real World: An Empirical Analysis of Venture Capital Contracts, NBER working paper #7660.

Keuschnigg, C. 2003. Optimal Public Policy for Venture Capital Backed Innovation, CEPR Discussion Paper No. 3850.

Kortum, S. and J. Lerner. 2000. "Does Venture Capital Spur Innovation?' *Rand Journal of Economics*. 31, pp. 674–92.

Schmidt, Klaus M. (2002), Convertible Securities and Venture Capital Finance, CEPR DP 2317, forthcoming in *Journal of Finance*.

Lerner, J., 2003. Venture Capital and Private Equity, *International Journal of Entrepreneurship Education*, 1 (3), pp. 359-384.

Lerner, J., F. Hardymon, and A. Leamon, 2004, *Venture Capital and Private Equity: A Casebook, 3rd edition*, John Wiley & Sons.

Michelacci, C. and Suarez, J (2004). Business Creation and the Stock Market, *Review of Economic Studies* 71, 459–481

NVCA, 2002. *DRI-WEFA Study Identifies Venture Capital as a Key Factor Powering U.S. Economic Growth*. National Venture Capital Association Press Release, June 26, Washington, DC.

Osborne, Martin J. and Ariel Rubinstein, *Bargaining and Markets,* London: Academic Press, 1990.

Pinheiro, Roberto, Venture Capital and Underpricing: Capacity Constraints and Early Sales, mimeo, U. of Penn.

Pissarides, Christopher A., *Equilibrium Unemployment Theory*, Oxford: Basil Blackwell, 1990.

References to add: Rogerson et al. (2005); Duffie, Garleanu and Pedersen (2002); Weill (2004); Lagos (2006); Mortensen-Pissarides (1994); Lagos-Wright (2005); Burdett (1981).

Burdett, Kenneth and Randall Wright, Two-Sided Search with Nontransferable Utility, *Rev. Econ. Dynamics* 1 (1998), 220-45.