Reallocating and pricing illiquid capital

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Firm-level and sectoral heterogeneity is pervasive in data. Equilibrium models in macro and finance typically assume a representative firm, as in Cochrane (1991). This representative firm paradigm leaves no role for the distribution of capital. We jointly model capital reallocation and asset pricing in a general equilibrium model with two sectors. Existing multisector equilibrium models assume either that capital is perfectly liquid and can be reallocated frictionlessly, as in Cox, Ingersoll, and Ross (1985, hereafter CIR), or that capital is completely illiquid and fixed, as in Lucas (1978) and multisector versions by Santos and Veronesi (2006) and Cochrane, Longstaff, and Santa Clara (2008, hereafter CLS). When capital is perfectly liquid as in CIR, Tobin's q is one at all times and heterogeneity plays no role in equilibrium. When capital is completely illiquid as in CLS, investment is zero at all times. In our model, investment drives dynamics of both Tobin's q and the distribution of capital; these results fundamentally differ from both CIR and CLS due to the illiquidity of capital.

We use capital adjustment costs to capture illiquidity. We follow Hayashi (1982) in assuming that the production technology is linearly homogeneous, which allows us to focus on the economic impact of the sectoral distribution of capital. Similarly, we assume the two sectors have identical technologies to highlight the effects of endogenous investment and reallocation, rather than relying on *ex ante* heterogeneity. We show that the distribution of capital is the single state variable determining equilibrium reallocation and asset pricing.

We analytically characterize the interdependence between price and quantity variables, including investment, growth, the interest rate, risk premia, and the price of capital (Tobin's q) at the aggregate level, along with the effect of sectoral heterogeneity on these variables. When the two sectors are of equal size, diversification achieves its maximum attainable level, hence precautionary savings demand is lowest, which implies the risk premium is lowest and the interest rate is highest (to encourage sufficient savings to sustain equilibrium). Sectoral investment generates *endogenous* mean reversion which tends to pull the economy towards a more balanced distribution of capital.

At the sectoral level, investment is higher in the smaller sector, not because of higher marginal productivity (we have constant-returns-to-scale in production), but because in equilibrium the required rate of return is lower and sectoral Tobin's q is higher for the sector that is in relatively scarce supply. Because the survival and recovery of a small sector is so valuable, its dividend yield may even be negative in order to capture the valuable diversification option embodied in the small sector.

Our model allows us to consider the impact of the liquidity of capital, unlike previous models which make the extreme assumption that this margin is either frictionless or unavailable. The more liquid is capital, the higher is welfare, but importantly, more liquid capital endogenously increases the growth rate of the economy, which raises the equilibrium interest rate and the aggregate market return. As a result, the value of installed capital (Tobin's q) is lower in both sectors, consistent with Hall's (2001) argument that greater liquidity reduces the rents to installed capital. These results suggest that economies with more liquid capital invest more and grow faster, but have higher interest rates and lower asset values.

I. Model

Consider an infinite-horizon continuous-time production economy. There are two productive sectors in the economy, sector 0 and 1. We consider the case where the two sectors are symmetric.[1] Let K_n, I_n, and Y_n denote the representative firm's capital stock, investment and output processes in sector n where $n = 0, 1$. This firm has the following "AK" production technology:

$$(1) \qquad Y_n(t) = AK_n(t),$$

where $A > 0$ is constant. Capital accumulation is stochastic and is given by

$$(2) \quad dK_n(t) = \Phi(I_n(t), K_n(t))dt + \sigma K_n(t)dB_n,$$

where $\sigma > 0$ is the volatility parameter, and $\Phi(I, K)$ denotes the effectiveness in converting investment goods into installed capital. Assume the correlation between Brownian motions B_0 and B_1 is 0. Shocks appear in capital accumulation dynamics as in CIR and the endogenous growth literature (Jones and Manuelli (2005)). As in Hayashi (1982) and Jermann (1998), we assume that the adjustment technology is homogeneous of degree one in I and K, in that

$$(3) \qquad \Phi(I, K) = \phi(i)K,$$

where $i = I/K$ is the investment-capital ratio. We require $\phi'(i) > 0$ and $\phi''(i) \leq 0$. Our model nests "AK" models (such as CIR and Jones and Manuelli (2005)) as special cases.

A representative consumer has a logarithmic utility given by:

$$(4) \qquad \mathbb{E}\left(\int_0^\infty e^{-\delta t}\delta \ln C(s)\,ds\right),$$

where $\delta > 0$ is the subjective discount rate. The consumer is endowed with financial claims on the aggregate output from both sectors in the economy. Markets are complete.

We now describe the market equilibrium. The representative consumer chooses his consump-

[1] See Eberly and Wang (2009) for the general formulation with heterogeneity between the two sectors and recursive utility, which separates risk aversion from elasticity of intertemporal substitution.

tion process and a complete set of financial claims to maximize (4). The representative firm in both sectors takes the equilibrium stochastic discount factor as given and maximizes firm value. All produced goods are either consumed or invested in either sector, so the goods-market clearing condition holds:

$$(5) \qquad C = Y_0 + Y_1 - I_0 - I_1.$$

In equilibrium, the representative consumer holds his financial claims on aggregate output in both sectors. This is the standard no-trade equilibrium as in Lucas (1978). Using the standard results in complete-markets competitive equilibrium analysis, we may obtain the equilibrium allocation by solving a central planner's problem and then decentralize the allocation using price system. We refer readers to the appendix and relegate additional details to online materials.

II. Model Results and Analysis

One-sector economy. First, we summarize the main results for the one-sector model. The capital stock K is the single state variable in this economy. The equilibrium of the one-sector economy features stochastic growth, where the stochastic growth rates of consumption, investment, and capital, and output are all equal. Moreover, these growth rate are independently and identically distributed. Therefore, after scaling by capital, the consumption-capital ratio $c = C/K$, investment-capital ratio $i = I/K$, and Tobin's q are all constant. For logarithmic utility, the first-order condition (FOC) with respect to consumption gives $c = \delta q$, where q is the firm value-capital ratio, also referred to as average q or Tobin's q. The FOC with respect to investment gives $q = 1/\phi'(i)$. The FOCs for consumption and investment together with goods market clearing condition (investment equals saving, i.e. $A - c = i$) jointly determine the optimal investment-capital ratio as the solution of $(A-i)\phi'(i) = \delta$. The equilibrium interest rate r is given by $r = \delta + \phi(i) - \sigma^2$, the sum of the subjective discount rate δ and the expected growth rate $\phi(i)$, minus the last standard precautionary saving term. The expected return of a financial claim on aggregate output is $\mu^m = \delta + \phi(i)$ implying that the aggregate

risk premium is equal to σ^2. The CIR model is a special case of ours with $q = 1$ because capital is liquid (no adjustment cost, i.e. $\phi'(i) = 1$).

Two-sector economy. Capital stocks in both sectors are the natural state variables. By exploiting the homogeneity properties, we show that the effective state variable is the relative size of capital stocks in the two sectors. Let

$$z \equiv \frac{K_1}{K_0 + K_1}$$

denote the ratio between sector-1 capital K_1 and the aggregate capital $(K_0 + K_1)$. Since physical capital is non-negative, we have $0 \le z \le 1$.

Let i_n denote sector-n investment capital ratio, in that $i_n = I_n/K_n$, where $n = 0, 1$. Let i denote the ratio between aggregate investment $(I_0 + I_1)$ and aggregate capital $(K_0 + K_1)$, in that $i = (I_0 + I_1)/(K_0 + K_1)$. We may write

$$(6) \qquad i(z) = (1 - z) i_0(z) + z i_1(z).$$

Recall that the adjustment cost drives a wedge between gross investment I and $\Phi(I, K)$. Let $g_n(z)$ denote the expected growth rate of capital in sector n. Using (2), we have $g_n(z) = \phi(i_n(z))$, which differs from $i_n(z)$. Let $g(z)$ denote the expected growth rate of aggregate capital $(K_0 + K_1)$. We have

$$(7) \qquad g(z) = (1 - z) g_0(z) + z g_1(z),$$

The concavity of $\phi(i)$ implies $g(z) \le \phi(i(z))$. When $z = 0, 1$, the equality holds. Intuitively, *ceteris paribus,* the expected growth rate $g(z)$ is lower in a two-sector economy than the corresponding one-sector economy, since both sectors incur adjustment costs and hence production is less efficient.

The equilibrium dynamics of z are given by:

$$(8) \quad dz_t = \mu_z(z_t)dt + \sigma z_t(1 - z_t)\left(dB_1 - dB_0\right),$$

where the drift $\mu_z(z)$ is given by
$$(9)$$
$$\mu_z(z) = z(1 - z)\left[g_1(z) - g_0(z) + (1 - 2z)\sigma^2\right].$$

Note that the volatility of dz is a quadratic function in z which attains its highest value at $z = 1/2$ and becomes zero at $z = 0, 1$ (i.e. the one-sector economy is absorbing), as in the two-tree pure exchange model of CLS. More interestingly in our model, the drift μ_z depends on $g_1(z) - g_0(z)$, the wedge between the *endogenous* capital growth rates in the two sectors. The larger this difference, the more capital reallocation will occur in equilibrium. This sectoral growth-wedge-induced component fundamentally differentiates our model from CLS.

We now turn to investment and the valuation of capital. The FOC for $i_1(z)$ is given by:

$$(10) \qquad \frac{\delta}{\phi'(i_1(z))} = c(z)\left(1 + (1 - z)\frac{N'(z)}{N(z)}\right),$$

where $N(z)$, the log of the value function (per unit of aggregate capital), is given in the appendix. A similar FOC holds for $i_0(z)$ and is also in the appendix. Let $Q_n(K_n; z)$ denote the firm value in sector n. Using the homogeneity property, we have

$$(11) \qquad Q_n(K_n; z) = q_n(z)K_n, \quad n = 0, 1,$$

where Tobin's q in sector n is given by

$$(12) \qquad q_n(z) = \frac{1}{\phi'(i_n(z))}.$$

The market value of the aggregate capital is $Q(z) = Q_0(z) + Q_1(z) = q(z)(K_0 + K_1)$, where Tobin's q for the aggregate capital is given by

$$(13) \qquad q(z) = (1 - z)q_0(z) + zq_1(z).$$

In complete-markets models with log utility, the aggregate consumption-wealth ratio $C(z)/Q(z)$ is equal to the discount rate δ, or equivalently $c(z) = \delta q(z)$, as we noted in the one-sector setting. While the aggregate dividend yield (i.e. consumption/wealth) ratio is constant and equal to the discount rate δ, the sector-specific dividend yield dy_n is stochastic and is given by

$$(14) \qquad dy_n(z) = \frac{A - i_n(z)}{q_n(z)}, \quad n = 0, 1.$$

III. Example: Log Adjustment Costs

For analytical convenience, we now specify

$$(15) \qquad \phi(i) = \alpha + \Gamma \ln\left(1 + \frac{i}{\theta}\right),$$

where $\Gamma, \theta > 0$. While z is stochastic, at the aggregate level, the consumption-capital ratio $c(z)$, the investment-capital ratio $i(z)$, and Tobin's q are all constant, in that $c(z) = c^*$, $i(z) = i^*$, and $q(z) = q^*$, where

$$(16) \quad c^* = \delta q^*, \ i^* = \frac{\Gamma A - \delta \theta}{\delta + \Gamma}, \ q^* = \frac{A + \theta}{\delta + \Gamma}.$$

While the aggregate investment-capital ratio $i(z)$ is constant, the aggregate expected growth rate $g(z)$ is not, as we see from (7). The expected return of the market portfolio (aggregate output) is given by

$$(17) \quad \mu^m(z) = \delta + g(z).$$

The equilibrium interest rate is given by

$$(18) \quad r(z) = \delta + g(z) - \sigma_m^2(z),$$

where the last term, the variance of the market portfolio return $\sigma_m^2(z)$, captures the precautionary saving effect and is given by

$$(19) \quad \sigma_m^2(z) = \sigma^2 \left[(1 - z)^2 + z^2 \right].$$

With two symmetric sectors, diversification is highest, and hence precautionary saving demand is lowest, when $z = 1/2$. Note that both $\mu^m(z)$ and the interest rate $r(z)$ increase one-for-one with the aggregate growth rate $g(z)$. Let $rp^m(z) = \mu^m(z) - r(z)$ denote the corresponding aggregate risk premium. We have

$$(20) \quad rp^m(z) = \sigma_m^2(z) = \sigma^2 \left[(1 - z)^2 + z^2 \right].$$

The aggregate risk premium and the variance of the market portfolio are independent of investment and growth, as in CLS.

Now consider sectoral heterogeneity. Recall that the production technologies in two sectors are identical. Despite the identical technologies, the two sectors price investment differently and carry different risk premia because of differences in their capital stocks. The investment-capital ratio $i_1(z)$ is given by

$$(21) \quad i_1(z) = \frac{\Gamma (A + \theta)}{\delta + \Gamma} \left[1 + (1 - z) \frac{N'(z)}{N(z)} \right] - \theta,$$

The expression for $i_0(z)$ is symmetric, (i.e.

$i_0(z) = i_1(1 - z))$ and omitted for brevity.

We choose model parameters to generate sensible aggregate predictions and to highlight the impact of endogenous investment and growth on equilibrium pricing and capital reallocation. The annual subjective discount rate is $\delta = 0.025$. Annual volatility parameter is $\sigma = 0.15$. The annual productivity parameter is $A = 0.12$. Finally, we choose $\Gamma = 0.025$, $\alpha = -0.035$, and $\theta = 0.01$ to generate the following aggregate predictions for the one-sector economy: Tobin's $q = 2.6$, the expected growth rate $\phi(i^*) = 0.012$, and annual risk-free rate $r = 0.014$.

Figure 1 shows $N(z)$, the logarithm of the representative consumer's value function per unit of aggregate capital $(K_0 + K_1)$, as a function of $z = K_1/(K_0 + K_1)$. Intuitively, we expect that $N(z)$ is maximized at $z = 1/2$, where the consumer achieves the maximally attainable level of diversification. The value function is higher in the economy with endogenous investment compared to the equivalent pure exchange "two tree" economy of CLS. For the baseline parameter values, the quantitative difference is substantial.

Figure 2 plots the drift of z, $\mu_z(z)$ given in equation (9). Recall that the economy is not stationary, since one of the sectors may vanish eventually. However, there is a natural tendency for z to move towards the center (i.e. when $z < 1/2$, $\mu_z(z) > 0$ and hence on average z increases towards $1/2$.) This mean reversion effect is also present in CLS, due to the definition of z; this effect is shown by the dotted line labelled "pure exchange" in Figure 2. Unlike CLS, however, the "central tendency" is stronger in our production economy due to endogenous investment and growth. Controlling for size (that is, per unit of capital), the consumer has a "bigger" demand for the smaller sector, and hence invests more per unit of capital, *ceteris paribus*. For example, when $0 < z < 1/2$, sector 1 is the smaller one, so it invests and grows at a faster rate, in that $i_1(z) > i_0(z)$ and $g_1(z) > g_0(z)$. The solid line shows the enhanced "central tendency" of $\mu_z(z)$ due to endogenous growth.

In Figure 3, we plot the aggregate expected return $\mu^m(z)$ and the equilibrium interest rate $r(z)$. Note that $\mu^m(z)$ increases one-for-one with the expected aggregate growth rate $g(z)$ and is larger than $g(z)$ by δ for all z. Interestingly, $g(z)$ features an "W"shape. First, *ex ante* sectoral

homogeneity implies $g(0) = g(1) = \phi(i^*)$. Increasing z away from zero makes sector 1 less scarce and investment in that sector becomes less urgent (large investment in the miniscule sector lowers the probability that the sector vanishes). Therefore, $i_1(z)$ decreases and $i_0(z)$ increases, while keeping $i(z)$ constant (recall that the total investment-capital ratio $i(z) = i^*$ is independent of z.) The concavity of $g(z)$ and $i(z) = i^*$ imply $g(z) \leq \phi(i^*)$ for all z. Because $i(z) = i^*$, a two-sector economy incurs more adjustment costs, and hence grows at a slower rate, *ceteris paribus*. After reaching the minimum level around $z = 0.06$, $g(z)$ then starts to increase when z moves towards the center ($z = 1/2$), reaching the maximum value $g(1/2) = \phi(i^*)$. This follows from the continuity of $g(z)$ and the fact that $g(1/2) = \phi(i^*)$, which relies on both symmetry (i.e. $i_0(1/2) = i_1(1/2)$) and the model's implication that the aggregate investment-capital ratio $i(z)$ is independent of z, i.e. $i(1/2) = i^*$.

While the risk-free rate $r(z)$ also increases one-for-one with $g(z)$, the precautionary saving effect (given by the variance of the market portfolio, $\sigma_m^2(z)$) dominates fluctuations in $g(z)$ and hence effectively determines the level of the interest rate. Note that $r(z)$ reaches its maximum at $z = 1/2$, where diversification achieves the highest possible level and precautionary saving demand is lowest. A high interest rate is necessary to encourage saving and sustain equilibrium when the precautionary saving effect is weak.

The next set of figures shows sectoral values; in each panel we graph results for sector 1 only for brevity since results are symmetric for sector 0. Figure 4 plots the investment-capital ratio $i_1(z)$ and the expected growth rate of capital $g_1(z)$. Both $i_1(z)$ and $g_1(z)$ decrease with z for $z \leq 0.8$. The smaller sector invests more and grows faster, in that $i_1(z) \geq i_0(z)$ and $g_1(z) \geq g_0(z)$ for $z \leq 1/2$. Moreover, symmetry between the two sectors implies $i_0(1/2) = i_1(1/2) = i^*$ and $g_0(1/2) = g_1(1/2) = \phi(i^*)$. Note that $i_1(z)$ and $g_1(z)$ increase with z for sufficiently high z, i.e. $z \geq 0.8$. This is due to the model's prediction that equilibrium aggregate investment is constant $i(z) = i^*$ and must be equal to capital-share-weighted sectoral investments, i.e. $i(z) = (1 - z)i_0(z) + zi_1(z)$. When z is high enough, the vanishing sector's

contribution to total investment is negligible and hence to keep aggregate investment constant at the level of i^*, the investment-capital ratio in the larger sector must rise. The increasing relation for $g_1(z)$ over the region $z \geq 0.8$ follows naturally from $g_1'(z) = \phi'(i_1(z))i_1'(z)$.

Recall that Tobin's q is given by $q_1(z) = 1/\phi'(i_1(z)) = (\theta + i_1(z))/\Gamma$, an affine function of $i_1(z)$. Tobin's q becomes significantly larger as the sector becomes smaller, because the consumer values the smaller sector more for diversification benefits, *ceteris paribus*. The diversification benefits of keeping the small sector "alive" with the potential to grow are very valuable. Upon vanishing, the sector will never be reborn, and the economy (with only the one surviving sector) will be significantly riskier thereafter.

Figure 5 graphs the sectoral risk premium $rp_1(z)$ and dividend yield $dy_1(z)$. The risk premium of a miniscule sector is effectively zero, because this sector carries almost no weight in aggregate consumption, and the correlation between the two shocks is zero. The same intuition applies in the pure-exchange economy (e.g. CLS). Recall that the interest rate is lowest at $z = 0$, therefore, the discount rate for a sector is lowest when it is vanishing. Intuitively, in equilibrium, the preferences for consumption smoothing and risk diversification lower the risk premium and discount rate for the shrinking sector. Since both the physical production technology and investment opportunities remain unchanged, the vanishing sector invests at the highest rate $i_1(0)$ to take advantage of its lowest cost of capital. To finance this high level of investment around $z = 0$ (e.g. $i_1(0) = 0.15$), the firm issues equity and hence the dividend yield is negative for sufficiently small z. That is, it is optimal to invest beyond current earnings in the shrinking sector in order to increase the odds that the sector survives. Tobin's q reaches the maximal level $q_1(0)$ at $z = 0$ despite the negative dividend yield. Note that the high valuation of capital for the vanishing sector is purely driven by the discount rate effect. Unlike the aggregate dividend yield, which is equal to the discount rate δ, the sectoral dividend yield can be negative and varies significantly with z.

We now turn to sectoral risk measures. In Figure 6, we plot the sectoral β and return volatility

$\sigma_1^r(z)$. We show that sectoral β is given by

(22)
$$\beta_1(z) = \frac{z}{z^2 + (1-z)^2} \left[1 + \frac{q_1'(z)}{q_1(z)}(1-z)(2z-1) \right],$$

First, zero risk premium for the disappearing sector implies $\beta_1(0) = 0$. Second, with increases in the share of capital z, more consumption is financed out of sector-1's output and hence $\beta_1(z)$ rises. Third, $\beta(1/2) = 1$, which follows from symmetry between the two sectors and $\beta = 1$ for the market portfolio by definition. When z exceeds $1/2$, β_1 exceeds one, because the other sector becomes smaller and carries smaller β (again by symmetry). Therefore, the bigger sector is riskier, *ceteris paribus*. When the sector becomes sufficiently large (i.e. high enough z), β has to fall as the sector becomes effectively the market portfolio, which has $\beta = 1$ by definition. Indeed, in the limit, when sector 1 comprises the whole economy ($z = 1$), $\beta_1 = 1$.

Now consider return volatility for sector 1, $\sigma_1^r(z)$. We have

(23)
$$\sigma_1^r = \sigma \sqrt{\left(\frac{q_1'(z)}{q_1(z)} z(1-z) \right)^2 + \left(1 + \frac{q_1'(z)}{q_1(z)} z(1-z) \right)^2}$$

While $\sigma_1^r(z)$ also varies non-monotonically with z, its behavior is rather different from $\beta_1(z)$. At $z = 0$, β_1 is zero and hence all return volatility comes from the idiosyncratic component (the sector carries no weight in the aggregate). Since total return volatility is the same as capital stock growth volatility σ for the miniscule sector, we have $\sigma_1^r(0) = \sigma = 0.15$. As z increases, Tobin's q falls (i.e. $q_1'(z) < 0$), which reduces the volatility of returns in equation (23). Note that increasing z raises $\beta_1(z)$, which increases the systematic component of volatility. But the increase in the systematic component is dominated by the decrease in the idiosyncratic component, as we see from equation (23). When z passes 0.04, sector 1 is sufficiently relevant that the systematic component of $\sigma_1^r(z)$ becomes relatively more significant and hence total volatility increases with z. For $z = 1$, sector 1 comprises the whole economy and hence $\sigma_1^r(1) = \sigma = .15$. By continuity, for sufficiently high z (around 0.96), $\sigma_1^r(z)$ must be larger than 0.15 in order for the vanishing sector (sector 0) to have a lower $\sigma_0^r(z)$ (as sector 1 does at small z in equilibrium). This

continuity argument and symmetry between sectors explain the "overshooting" of $\sigma_1^r(z)$ before it converges to σ at $z = 1$.

Finally, consider a comparative static change in the efficiency of reallocating capital, by changing the adjustment parameter Γ, which may be viewed as a measure of the liquidity of capital. In typical equilibrium models, this experiment is not possible, since capital reallocation is either frictionless (CIR) or ruled out in pure-exchange settings (CLS). Comparing the shape of $\mu_z(z)$ when $\Gamma = 0.05$ (the dashed line in Figure 2) with that in our baseline case (when $\Gamma = 0.025$), we see that the more liquid capital (a higher Γ), the stronger central tendency the economy exhibits. The log of the (scaled) value function $N(z)$ increases with Γ, reflecting the higher value of more liquid capital. Investment and saving rise with Γ. Consumption falls relative to aggregate capital and output (while remaining a constant fraction of wealth). The fact that greater liquidity of capital endogenously increases the aggregate growth rate has strong equilibrium implications for pricing. The equilibrium interest rate shifts up for all values of z and Tobin's q falls; there is no effect on aggregate risk premia. Interestingly, while investment rises to reflect the more efficient technology, the general equilibrium effects imply that Tobin's q falls, so capital becomes less valuable in equilibrium. This reflects both the elimination of rents to installed capital (since capital is more liquid) and a higher discount rate.

Future work. These results lead us to consider heterogeneity across sectors to potentially better understand pricing and reallocation. With symmetric sectors, changes such as an increase in liquidity that generate higher growth imply a higher equilibrium interest rate. This general equilibrium effect tends to reduce asset prices. This finding relies, however, on symmetry, or *ex ante* identical sectors. Eberly and Wang (2009) consider asymmetric sectors, and allow for *ex ante*, as well as *ex post*, heterogeneity. Such a structure tends to dilute and potentially mute this general equilibrium impact on pricing.

References

Cochrane, John, "Production-Based Asset Pricing and the Link Between Stock Returns and Economic Fluctuations," Journal of Finance, 46, 1, (March 1991) 209-237.

Cochrane, John, Francis Longstaff and Pedro Santa-Clara, "Two Trees," Review of Financial Studies, 21, 1 (January 2008), 347-385.

Cox, John, J. E. Ingersoll and S. A. Ross, "A Theory of the Term Structure of Interest Rates," Econometrica, 53, 2 (March 1985), 385–407.

Eberly, Janice and Neng Wang, "Investment, Asset Pricing, and the Equilibrium Distribution of Capital," working paper, January 2009.

Hall, Robert E., "The Stock Market and Capital Accumulation," American Economic Review, 91, 5 (December 2001), 1185-1202.

Hayashi, Fumio, "Tobin's marginal Q and average Q : A neoclassical interpretation," Econometrica, 50, 215-224, 1982.

Jermann, Urban J, "Asset Pricing in Production Economies," Journal of Monetary Economics, April 1998, 257-275.

Jones, Larry and Rodolfo E. Manuelli, "Neoclassical Models of Endogenous Growth: The Effects of Fiscal Policy, Innovation and Fluctuations," in The Handbook of Economic Growth, Phillippe Aghion and Steven Durlauf, Eds., Elsevier, 2005.

Lucas, Robert E., Jr., "Asset Prices in an Exchange Economy," Econometrica, 46, 6 (November 1978), 1429-1445.

Santos, Tano and Pietro Veronesi, "Labor Income and Predictable Stock Returns," Review of Financial Studies, 19, 1 (Spring 2006), 1-44.

Mathematical Appendix

We solve the equilibrium by solving the planner's problem. By the standard principle of optimality, we conjecture that the following Hamilton-Jacobi-Bellman (HJB) equation holds:

$$(24) \qquad \delta V = \max_{I_0, I_1} \ \delta \ln C + \phi(i_0) K_0 V_0 + \phi(i_1) K_1 V_1 + \frac{1}{2} \sigma^2 \left(K_0^2 V_{00} + K_1^2 V_{11} \right),$$

where V_n and V_{nn} denote the first and the second derivative of V with respect to K_n, where $n = 0, 1$. Using the homogeneity property, we conjecture

$$(25) \qquad V(K_0, K_1) = \ln \left((K_0 + K_1) N(z) \right),$$

where $N(z)$ is a function to be determined. Substituting (25) into (24), we obtain the following:

$$(26) \ \ 0 = \delta \ln \frac{c(z)}{N(z)} + \phi(i_0(z)) L_0(z) + \phi(i_1(z)) L_1(z) - \frac{\sigma^2}{2} \left(L_0(z)^2 + L_1(z)^2 \right) + \sigma^2 z^2 (1-z)^2 \frac{N''(z)}{N(z)},$$

subject to the boundary conditions:

$$N(0) = N(1) = v,$$

where v is the value function coefficient for the one-sector economy and is given by (32), and

$$(27) \qquad L_0(z) \quad = \quad (1-z) \left[1 - z \frac{N'(z)}{N(z)} \right],$$

$$(28) \qquad L_1(z) \quad = \quad z \left[1 + (1-z) \frac{N'(z)}{N(z)} \right].$$

Note $L_0(z) + L_1(z) = 1$. Optimality and equilibrium imply that $i_0(z)$, $i_1(z)$, and $c(z)$ solve the FOCs:

$$(29) \qquad \frac{\delta}{\phi'(i_0(z))} = c(z)\left(1 - \frac{zN'(z)}{N(z)}\right),$$

$$(30) \qquad \frac{\delta}{\phi'(i_1(z))} = c(z)\left(1 + (1-z)\frac{N'(z)}{N(z)}\right),$$

and the goods-market clearing condition:

$$(31) \qquad c(z) + (1-z)i_0(z) + zi_1(z) = A.$$

The one-sector economy is a special case of the above two-sector economy (i.e. $z = 0$ and $z = 1$ are absorbing barriers in the two-sector optimality problem). We have $N(0) = N(1) = v$, where

$$(32) \qquad v = (A - i)\exp\left[\frac{1}{\delta}\left(\phi(i) - \frac{\sigma^2}{2}\right)\right],$$

and the optimal investment-capital ratio i is the solution of $(A - i)\phi'(i) = \delta$. The transversality conditions are provided in the online appendix.

Let $R_1(z)$ denote the instantaneous cumulative return for sector 1. We have

$$(33) \qquad dR_1(z) = \mu_1^r(z)dt - \sigma\frac{q_1'(z)}{q_1(z)}z(1-z)dB_0(t) + \sigma\left(1 + \frac{q_1'(z)}{q_1(z)}z(1-z)\right)dB_1(t),$$

where the expected rate of return $\mu_1^r(z)$ is given by

$$(34) \quad \mu_1^r(z) = dy_1(z) + \phi(i_1(z)) + z(1-z)\left[\phi(i_1(z)) - \phi(i_0(z)) - 2z\sigma^2\right]\frac{q_1'(z)}{q_1(z)} + \frac{q_1''(z)}{q_1(z)}z^2(1-z)^2\sigma^2.$$

The instantaneous return volatility is thus given by

$$(35) \qquad \sigma_1^r(z) = \sigma\sqrt{\left(\frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2 + \left(1 + \frac{q_1'(z)}{q_1(z)}z(1-z)\right)^2}.$$

The instantaneous correlation $\rho(z)$ between returns in sectors 0 and 1 is given by

$$(36) \qquad \rho(z) = \frac{\sigma^2 z(1-z)}{\sigma_0^r(z)\sigma_1^r(z)}\left[-\frac{q_1'(z)}{q_1(z)}\left(1 - \frac{q_0'(z)}{q_0(z)}z(1-z)\right) + \left(1 + \frac{q_1'(z)}{q_1(z)}z(1-z)\right)\frac{q_0'(z)}{q_0(z)}\right].$$
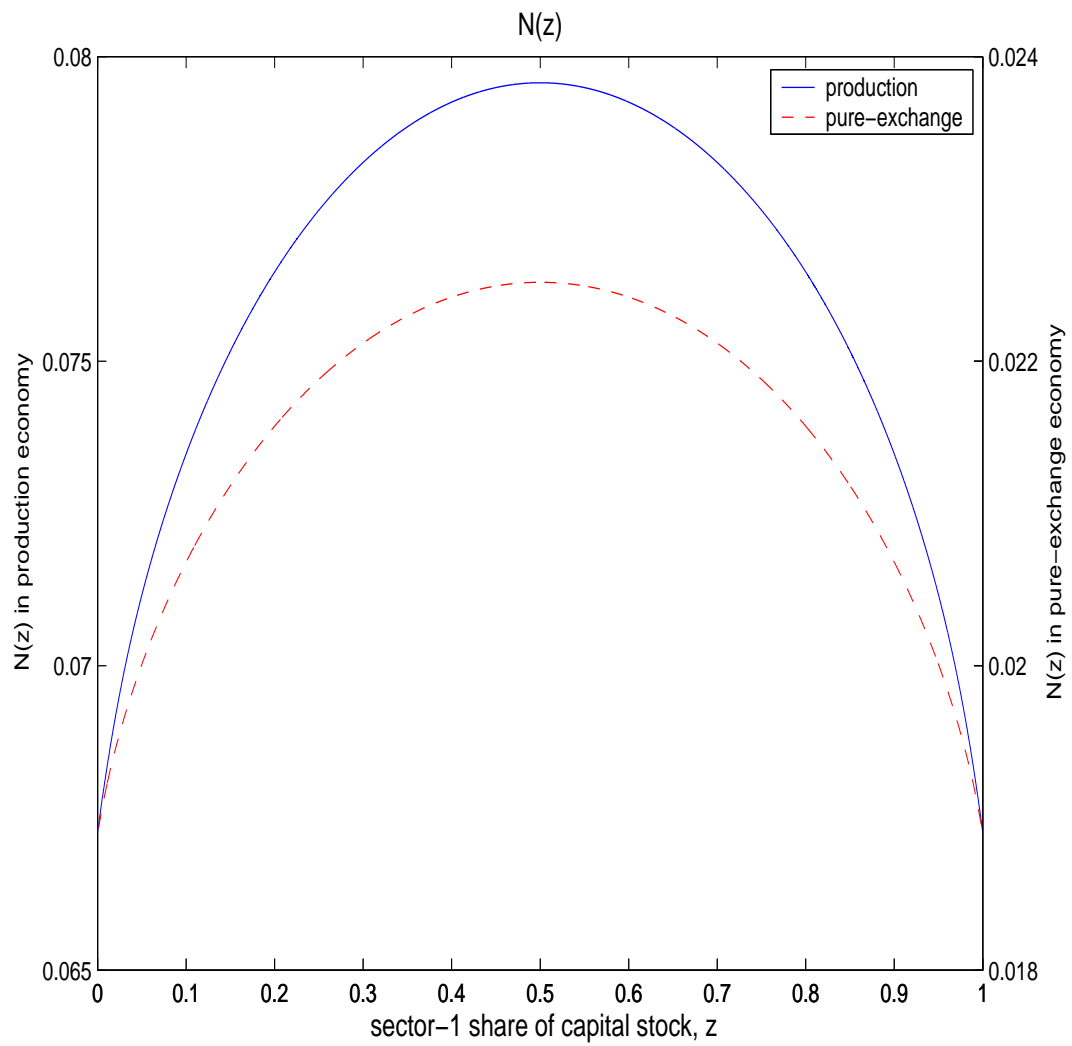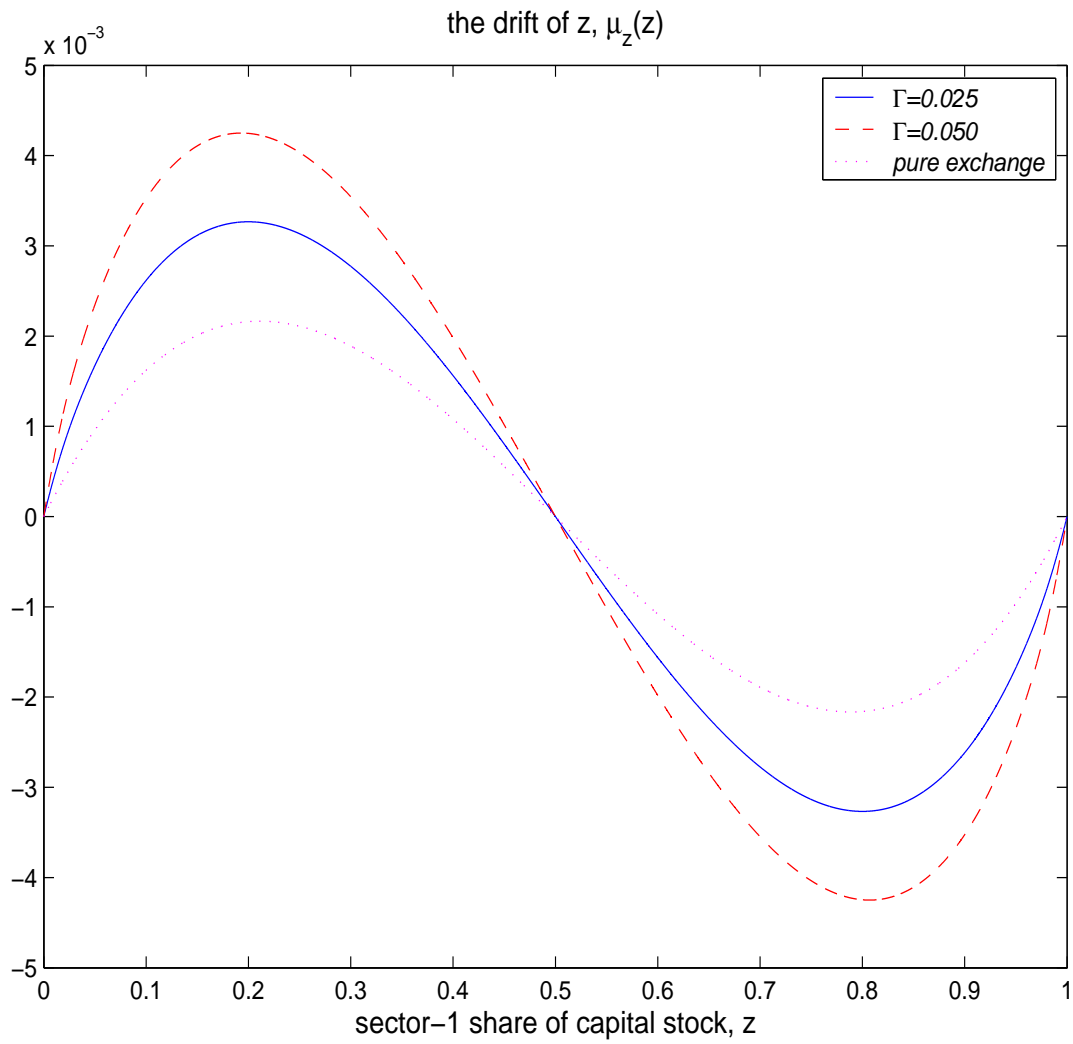
FIGURE 1: **The scaled value function in pure exchange and production economies.**
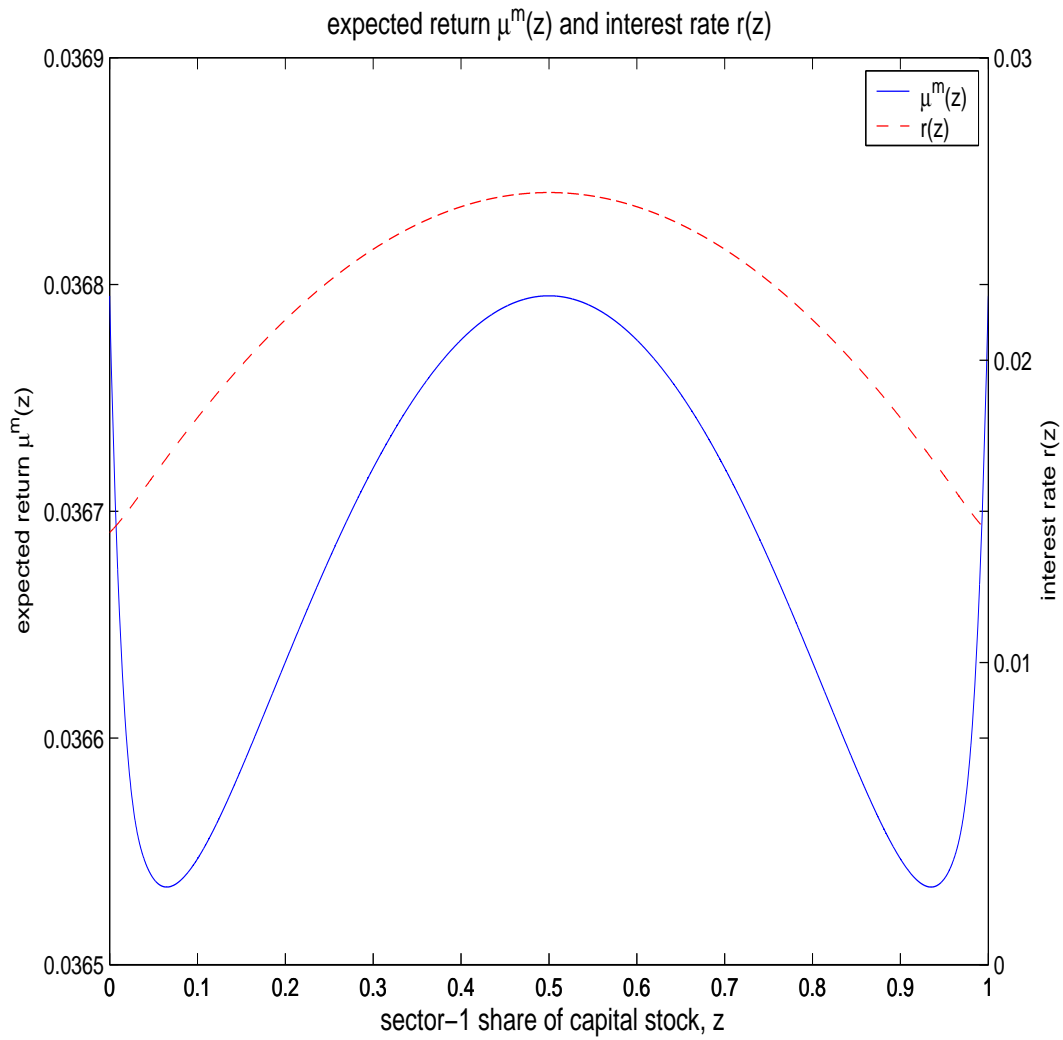
FIGURE 2: **The central tendency in** z.

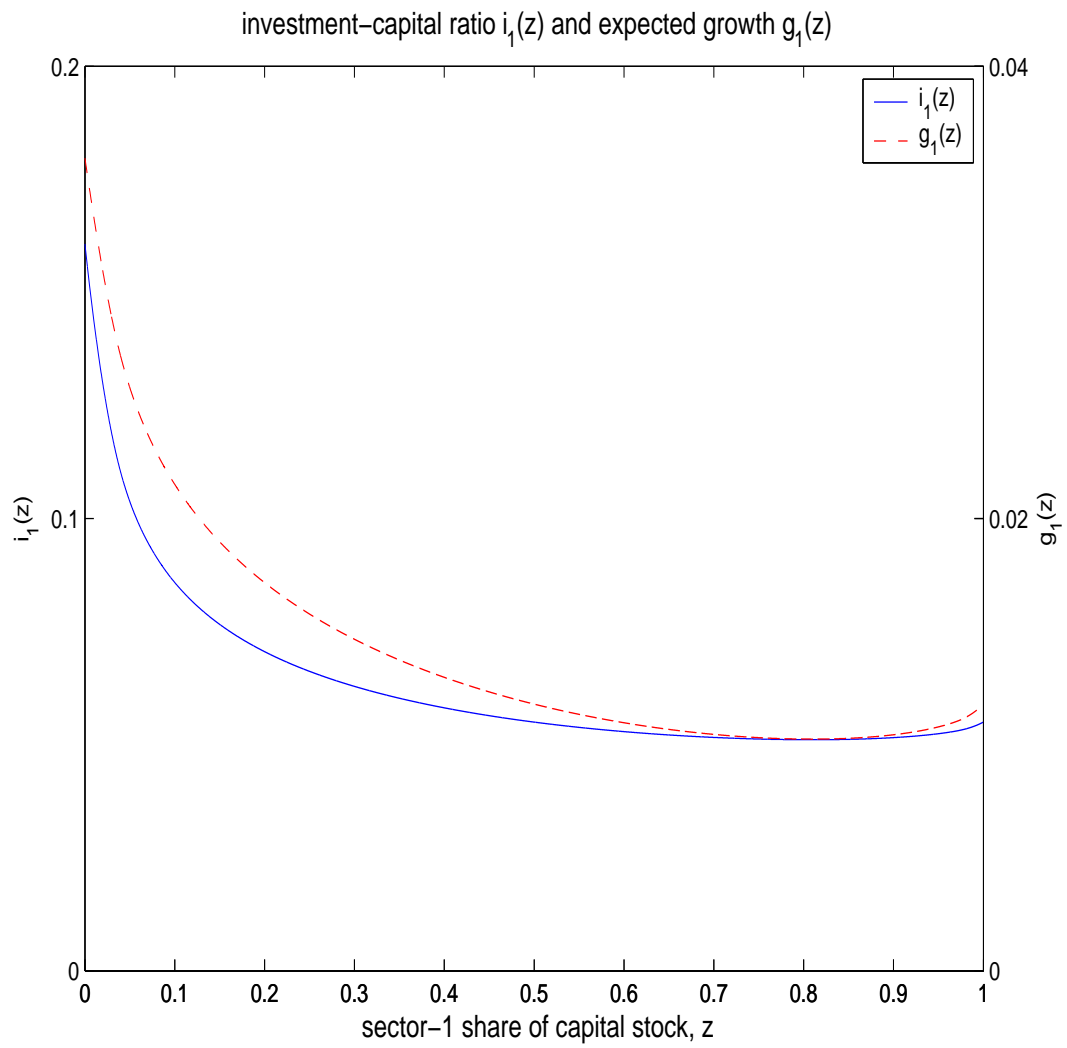FIGURE 3: **The aggregate expected return and interest rate.**

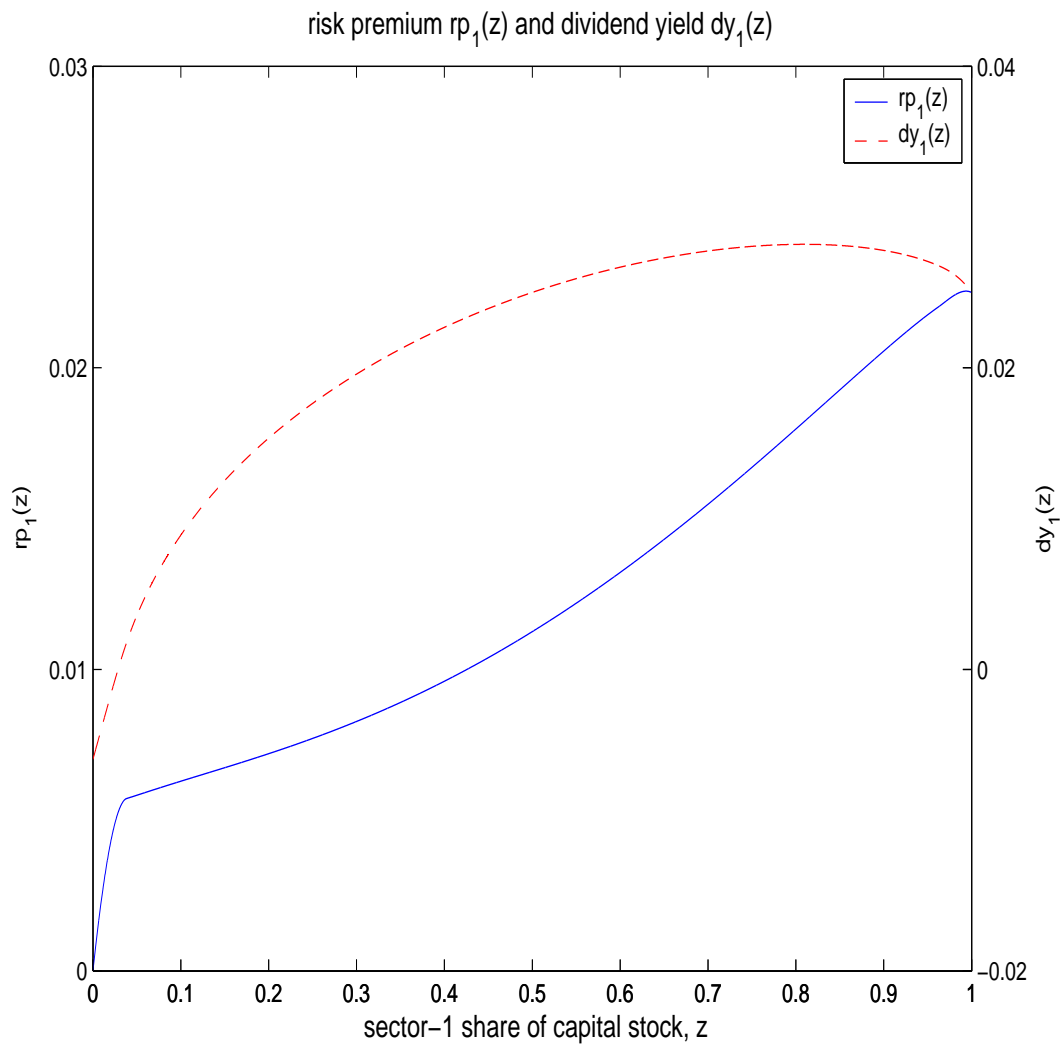FIGURE 4: **Sectoral investment-capital ratio and expected growth rate.**
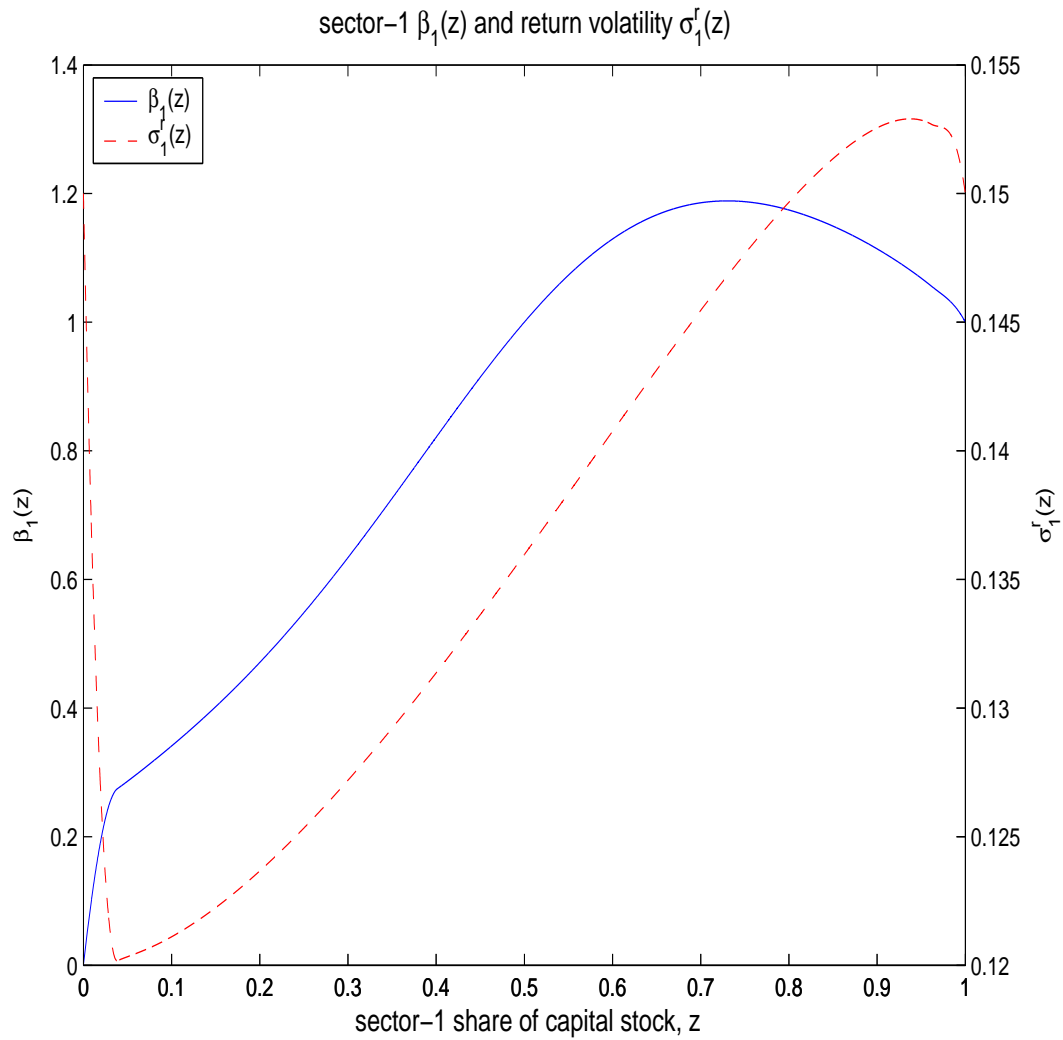
FIGURE 5: **Sectoral risk premium and dividend yield.**

FIGURE 6: **Sectoral beta and return volatility.**