## Terrorism and the Optimal Defense of Networks of Targets

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This paper has benefited from the helpful comments of Kai Konrad and Daniel G. Arce as well as participants in presentations at the 2006 Midwest Economic Theory Meetings, the 2008 Tournaments, Contests and Relative Performance Evaluation Conference held at North Carolina State University, and the 2008 Contests: Theory and Applications Conference held at Stockhom School of Economics. Part of this work was completed while Kovenock was Visiting Professor at the Social Science Research Center Berlin (WZB). Roberson gratefully acknowledges financial support from the Social Science Research Center Berlin (WZB), the Miami University Committee on Faculty Research, and the Farmer School of Business. The authors, of course, remain solely responsible for any errors or omissions.

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**Abstract** This paper examines a game-theoretic model of attack and defense of multiple networks of targets in which there exist intra-network strategic complementarities among targets. The defender's objective is to successfully defend all of the networks and the attacker's objective is to successfully attack at least one network of targets. In this context, our results highlight the importance of modeling asymmetric attack and defense as a conflict between "fully" strategic actors with endogenous entry and force expenditure decisions as well as allowing for general correlation structures for force expenditures within and across the networks of targets.

**JEL Classification:** C7, D74
**Keywords:** Asymmetric Conflict, Attack and Defense, Weakest-Link, Best-Shot

## 1 Introduction

In the literature on optimal defense from intentional attack there has been growing interest in not only the attack and defense of isolated targets[1] but also networks of targets[2] and even complex supra-networks of targets.[3] This move towards increasing network complexity emphasizes the role that strategic complementarities among targets play in creating structural asymmetries between the attack and defense of such combinations of targets. For example in complex infrastructure supra-networks — such as communication systems, electrical power grids, water and sewage systems, oil pipeline systems, transportation systems, and cyber security systems — there often exist particular targets or combinations of targets which if destroyed would be sufficient to either: (a) disable the entire supra-network or (b) create a terrorist ‘spectacular.’

In order to highlight the importance of modeling the asymmetric attack and defense of complex supra-networks as a conflict between ‘fully’ strategic actors with endogenous entry and force expenditure decisions, we examine a contest-theoretic model of the attack and defense of a complex supra-network and allow for the players to use general correlation structures for force expenditures within and across the networks of targets. The supra-network of targets is made up of an arbitrary combination of two simple types of networks which capture the two extreme endpoints of an exposure-redundancy spectrum for network types. The maximal exposure network, which we label a *weakest-link network*, is successfully defended if and only if the defender successfully defends all targets within the network.[4] The maximal redundancy network, which we label a *best-shot network*, is successfully defended if the defender successfully defends at least one target within the network. At each target the conflict is modeled as a deterministic contest in which the player who allocates the higher level of force wins the target with probability one. Given that the loss of a single network may be sufficient to either disable the entire supra-network or create a terrorist ‘spectacular,’ we focus on the case that the attacker’s

[1] See for example Bier et al. (2007), Powell (2007a, b), and Rosendorff and Sandler (2004).

[2] See for example Bier and Abhichandani (2003), Bier et al. (2005), and Clark and Konrad (2007).

[3] See for example Azaiez and Bier (2007), Hausken (2008), and Levitin and Ben-Haim (2008).

[4] See Hirshleifer (1983) who coins the terms best-shot and weakest-link in the context of voluntary provision of public goods.

objective is to successfully attack a single network, and that the defender's objective is to successfully defend all of the networks.

A distinctive feature of this environment is that a mixed strategy is a joint distribution function in which the randomization in the force allocation to each target is represented as a separate dimension. A pair of equilibrium joint distribution functions specifies not only each player's randomization in force expenditures for each target but also the correlation structure of the force expenditures within and across the networks of targets. For all parameter configurations, we completely characterize the unique set of Nash equilibrium univariate marginal distributions for each player as well as the unique equilibrium payoff for each player. Furthermore, in any equilibrium we find that the attacker launches an attack on at most one network of targets, and there exist parameter configurations for which the attacker optimally launches no attack with positive probability. While at most one network is attacked, the attacker randomizes over which network is attacked, and each of the networks is attacked with positive probability. In the event that a weakest-link network is attacked, the attacker optimally launches an attack on only a single target. When a best-shot network is attacked, the attacker optimally attacks every target in that network with a strictly positive force level.

As emphasized in the *National Strategy for Homeland Security*, "terrorists are strategic actors." However, much of the existing literature [e.g. Azaiez and Bier (2007), Bier and Abhichandani (2003), Bier et al. (2005), Bier et al. (2007), Levitin and Ben-Haim (2008), Powell (2007a, b), and Rosendorff and Sandler (2004).] assumes that terrorists (henceforth attackers) are not 'fully' strategic in the sense that the number of attacks (which is usually set to one) is exogenously specified. By endogenizing the attacker's entry and force expenditure decisions, we examine not only the conditions under which the assumption of one attack is likely to hold, but also related issues such as how the defender's actions can decrease the number of terrorist attacks. Furthermore, the few previous models which allow for the attacker to endogenously choose the number of targets to attack [e.g. Clark and Konrad (2007) and Hausken (2008)][5] obtain the result

[5] Utilizing probabilistic contest success functions [Clark and Konrad (2007) utilize the Tullock contest success function, Hausken (2008) utilizes both the Tullock and difference-form contest success functions], Clark and Konrad (2007) and Hausken (2008) examine a single weakest-link network and a supra-network consisting of any arbitrary combination of weakest-link and best-shot networks [as in this paper, a successful attack on any one network is sufficient to disable the entire supra-network], respectively.

that even when the attacker's objective is to disable a single network — and the attacker derives no additional benefit from successfully disabling more than one network — the attacker optimally chooses to attack every target in every network with certainty. By showing that in all equilibria of our model the attacker optimally engages in a from of stochastic guerilla warfare in which they attack at most one network of targets (but with positive probability each network is chosen as the one to be attacked), our results also provide a sharp contrast with existing models of 'fully' strategic attackers.

Section 2 presents the model of attack and defense with networks of targets. Section 3 characterizes a Nash equilibrium and explores properties of the equilibrium distributions of force. Section 4 concludes.

## 2 The Model

Players

The model is formally described as follows. Two players, an attacker, $A$, and a defender, $D$, simultaneously allocate their forces across a finite number, $n \geq 2$, of heterogeneous targets. The players' payoffs depend on the composition of each of the networks of targets in the supra-network. We examine a supra-network consisting of any arbitrary combination of two types of simple networks.

The targets are partitioned into a finite number $k \geq 1$ of disjoint networks, where network $j \in \{1, \ldots, k\}$ consists of a finite number $n_j \geq 1$ of targets with $\sum_{j=1}^{k} n_j = n$. Let $N_j$ denote the set of targets in network $j$. Let $\mathscr{W}$ denote the set of weakest-link networks and $\mathscr{B}$ denote the set of best-shot networks.

In a *best-shot network* the network is successfully defended if the defender allocates at least as high a level of force to at least one target within the network. Conversely, an attack on a best-shot network is successful if the attacker allocates a higher level of force to each target in the network. Let $x_A^i$ ($x_D^i$) denote the level of force allocated by the attacker (defender) to target $i$. Define

$$\iota_j^B = \begin{cases} 1 & \text{if } \forall\ i \in N_j \mid x_A^i > x_D^i \\ 0 & \text{otherwise} \end{cases}.$$

Observe that for each target, the player that allocates the higher level of force wins that target, but in order to win the network the attacker must win all of the targets. In a best-shot network, a tie arises when player A allocates a level of force to each target in the network that is at least as great as player D's allocation, and there exists at least one target in the network to which the players allocate the same level of force. In this case, the defender wins the network.

In the second type of network, which we label a *weakest-link network*, the network is successfully defended if the defender allocates at least as high a level of force to all targets within the network. Conversely, an attack on a weakest-link network is successful if the attacker allocates a higher level of force to any target in the network. Define

$$\iota_j^W = \begin{cases} 1 & \text{if } \exists\ i \in N_j \mid x_A^i > x_D^i \\ 0 & \text{otherwise} \end{cases}.$$

Again, in the case of a tie, the defender is assumed to win the network.

The players are risk neutral and have asymmetric objectives. The attacker's objective is to successfully attack at least one network, and the attacker's payoff for the successful attack of at least one network is $v_A$. The attacker's payoff function is given by

$$\pi_A(\mathbf{x}_A, \mathbf{x}_D) = v_A \max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right) - \sum_{i=1}^{n} x_A^i$$

The defender's objective is to preserve the entire supra-network, and the defender's payoff for successfully defending the supra-network is $v_D$. The defender's payoff function is given by

$$\pi_D(\mathbf{x}_A, \mathbf{x}_D) = v_D\left(1 - \max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right)\right) - \sum_{i=1}^{n} x_D^i.$$

The force allocated to each target must be nonnegative.

It is important to note that our formulation utilizes the all-pay auction contest success function.[6] Within the all-pay auction literature it is well known that the equilibrium of the game in which the players have differing unit costs of resources is equivalent up to a linear scaling of the equilibrium of the game with asymmetric valuations. This result extends directly to the environment examined here, and thus, our focus on asymmetric valuations also covers the case in which the players have differing unit costs of resources.

Also observe that in the formulation described above the supra-network is a weakest-link supra-network. That is if the defender loses a single network then the entire supra-network is inoperable. By interchanging the identities of player A and player D, our

[6] See Baye, Kovenock, and de Vries (1996).

results on weakest-link supra-networks apply directly to the case of best-shot supra-networks (where a best-shot supra-network is a supra-network which is successfully defended if the defender successfully defends at least one network).

Figure 1 provides a representative supra-network consisting of 5 networks (A, B, C, D, and E). Networks A, C, and E are weakest-link (series) networks with two targets each. Networks B and D are best-shot (parallel) networks with five targets each. In order to preserve the entire supra-network player D's objective is to preserve a path across the entire network. If a single target in networks A, C, or E is destroyed then the supra-network is inoperable. Conversely, in networks B and D all of the targets must be destroyed in order to render the supra-network inoperable.

[Insert Figure 1 here]

Strategies

It is clear that there is no pure strategy equilibrium for this class of games. A mixed strategy, which we term a *distribution of force*, for player $i$ is an $n$-variate distribution function $P_i : \mathbb{R}^n_+ \to [0,1]$. The $n$-tuple of player $i$'s allocation of force across the $n$ targets is a random $n$-tuple drawn from the $n$-variate distribution function $P_i$.

Model of Attack and Defense with Networks of Targets

The model of attack and defense with networks of targets, which we label

$$ADN\left\{\left\{N_j\right\}_{j\in\mathscr{B}}, \left\{N_j\right\}_{j\in\mathscr{W}}, v_A, v_D\right\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of force, each target is won by the player that provides the higher allocation of force for that target, ties are resolved as described above, and players' payoffs, $\pi_A$ and $\pi_D$, are specified above.

## 3 Optimal Distributions of Force

It will be useful to introduce a simple summary statistic which captures both the asymmetry in the players' valuations and the structural asymmetries arising in the supra-network.

**Definition 1** Let $\alpha = v_D/(v_A[\sum_{j\in\mathscr{W}} n_j + \sum_{j\in\mathscr{B}} \frac{1}{n_j}])$ denote the *normed relative strength of the defender*.

Several properties of this summary statistic should be noted. First, the normed relative strength of the defender is increasing in the relative valuation of the defender to the attacker ($v_D/v_A$), and is decreasing in the level of exposure arising in the supra-network ($\sum_{j\in\mathscr{W}} n_j + \sum_{j\in\mathscr{B}} \frac{1}{n_j}$). In particular, the defender's exposure is increasing in the number of weakest-link targets ($\sum_{j\in\mathscr{W}} n_j$), and is decreasing in the number of targets within each best-shot network ($\sum_{j\in\mathscr{B}} \frac{1}{n_j}$).

For all parameter ranges, Theorem 1 establishes the uniqueness of: (i) the players' equilibrium expected payoffs and (ii) the players' sets of univariate marginal distributions. Theorem 1 also provides a pair of equilibrium distributions of force for all parameters ranges. Case (1) of Theorem 1 examines the parameter configurations for which the defender has a normed relative strength advantage, i.e. $\alpha \geq 1$. Case (2) of Theorem 1 addresses the parameter configurations for which the defender has a normed relative strength disadvantage, i.e. $\alpha < 1$. It is important to note that the stated equilibrium distributions of force ($n$-variate distributions) are not unique. However, in Propositions 1-3 we characterize properties of optimal attack and defense that hold in all equilibria.

**Theorem 1** *For all feasible parameter figurations of the game ADN*$\{\{N_j\}_{j\in\mathscr{B}}, \{N_j\}_{j\in\mathscr{W}}, v_A, v_D\}$ *(i.e.,* $v_A, v_D > 0$ *and* $N_j \neq \emptyset$ *for all* $j$*) there exists a unique set of Nash equilibrium univariate marginal distributions and a unique equilibrium payoff for each player. One such equilibrium is for each player to allocate his forces according to the following n-variate distribution functions.*

*(1) If* $\alpha \geq 1$*, then for player A and* $\boldsymbol{x} \in \prod_{j\in\mathscr{W}}[0, v_A]^{n_j} \times \prod_{j\in\mathscr{B}}[0, \frac{v_A}{n_j}]^{n_j}$

$$P_A(\boldsymbol{x}) = 1 - \frac{1}{\alpha} + \frac{\sum_{j\in\mathscr{W}} \sum_{i\in N_j} x^i + \sum_{j\in\mathscr{B}} \min_{i\in N_j}\{x^i\}}{v_D}$$

*Similarly for player D and* $\boldsymbol{x} \in \prod_{j\in\mathscr{W}}[0, v_A]^{n_j} \times \prod_{j\in\mathscr{B}}[0, \frac{v_A}{n_j}]^{n_j}$

$$P_D(\boldsymbol{x}) = \min\left(\left\{\frac{\min_{i\in N_j}\{x^i\}}{v_A}\right\}_{j\in\mathscr{W}}, \left\{\frac{\sum_{i\in N_j} x^i}{v_A}\right\}_{j\in\mathscr{B}}\right)$$

*The expected payoff for player A is* $0$*, and the expected payoff for player D is* $v_D(1 - \frac{1}{\alpha})$.

*(2) If $\alpha < 1$, then for player A and $\boldsymbol{x} \in \prod_{j\in\mathscr{W}}[0,\alpha v_A]^{n_j} \times \prod_{j\in\mathscr{B}}[0,\frac{\alpha v_A}{n_j}]^{n_j}$*

$$P_A(\boldsymbol{x}) = \frac{\sum_{j\in\mathscr{W}}\sum_{i\in N_j} x^i + \sum_{j\in\mathscr{B}}\min_{i\in N_j}\{x^i\}}{v_D}$$

*Similarly for player D and $\boldsymbol{x} \in \prod_{j\in\mathscr{W}}[0,\alpha v_A]^{n_j} \times \prod_{j\in\mathscr{B}}[0,\frac{\alpha v_A}{n_j}]^{n_j}$*

$$P_D(\boldsymbol{x}) = 1-\alpha+\min\left(\left\{\frac{\min\{x^i\}_{i\in N_j}}{v_A}\right\}_{j\in\mathscr{W}}, \left\{\frac{\sum_{i\in N_j} x^i}{v_A}\right\}_{j\in\mathscr{B}}\right)$$

*The expected payoff for player D is $0$, and the expected payoff for player A is $v_A(1-\alpha)$.*

*Proof* The proof of the uniqueness of the players' equilibrium expected payoffs and sets of univariate marginal distributions is given in the appendix. This proof establishes that the pair of $n$-variate distribution functions given in case (1) constitute an equilibrium within the case (1) parameter range. The proof of case (2) is analogous. The appendix (see Lemma 5) establishes that in any $n$-tuple drawn from any equilibrium $n$-variate distribution $P_A$ player $A$ allocates a strictly positive level of force to at most one network of targets. If the network which receives the strictly positive level of force is a weakest-link network, then exactly one target in that network receives a strictly positive level of force. While not a necessary condition for equilibrium, the $P_A$ described in Theorem 1 also displays the property that when the network which receives the strictly positive level of force is a best-shot network the force allocated to each target in that network is an almost surely increasing function of the force allocated to any of the other targets in that network. The appendix (see Lemma 5) also establishes that in any $n$-tuple drawn from any equilibrium $n$-variate distribution $P_D$ player $D$ allocates a strictly positive level of force to at most one target in each best-shot network of targets.

We will now show that for each player each point in the support of their equilibrium $n$-variate distribution function $\{P_A, P_D\}$ given in case (1) of Theorem 1 results in the same expected payoff, and then show that there are no profitable deviations from this support.

We begin with the case in which player A attacks a single target in a single weakest-link network. The probability that player $A$ wins target $i$ in network $j \in \mathscr{W}$ is given by the univariate marginal distribution $P_D(x_A^i, \{\{v_A\}_{i'\in N_{j'}|x_A^{i'}=0}\}_{j'\in\mathscr{W}}, \{\{\frac{v_A}{n_{j'}}\}_{i'\in N_{j'}}\}_{j'\in\mathscr{B}})$. Given that player $D$ is using the equilibrium strategy $P_D$ described above, the payoff to player $A$ for any allocation of force $\mathbf{x}_A \in \mathbb{R}_+^n$ which allocates a strictly positive level of force to a single target $i$ in a weakest-link network $j \in \mathscr{W}$ is

$$\pi_A(\mathbf{x}_A, P_D) = v_A P_D^i(x_A^i) - x_A^i.$$

Simplifying,

$$\pi_A(\mathbf{x}_A, P_D) = v_A \left(\frac{x_A^i}{v_A}\right) - x_A^i = 0.$$

Thus the expected payoff to player $A$ from allocating a strictly positive level of force to only one target in any weakest-link network is 0 regardless of which target is attacked.

Next, we examine the case in which player A attacks a single best-shot network. The probability that player $A$ wins every target in network $j \in \mathscr{B}$ is given by the $n_j$-variate marginal distribution $P_D(\{x_A^i\}_{i\in N_j}, \{\{v_A\}_{i'\in N_{j'}}\}_{j'\in\mathscr{W}}, \{\{\frac{v_A}{n_{j'}}\}_{i'\in N_{j'}}\}_{j'\in\mathscr{B}|j'\neq j})$, which we will denote as $P_D^{N_j}(\{x_A^i\}_{i\in N_j})$. Given that player $D$ is using the equilibrium strategy $P_D$ described above, the payoff to player $A$ for any allocation of force $\mathbf{x}_A \in \mathbb{R}_+^n$ which allocates a strictly positive level of force only to the targets in a best-shot network $j \in \mathscr{B}$, and allocates zero forces to every other network is

$$\pi_A(\mathbf{x}_A, P_D) = v_A P_D^{N_j}\left(\{x_A^i\}_{i\in N_j}\right) - \sum_{i\in N_j} x_A^i.$$

Simplifying,

$$\pi_A(\mathbf{x}_A, P_D) = v_A \left(\frac{\sum_{i\in N_j} x_A^i}{v_A}\right) - \sum_{i\in N_j} x_A^i = 0.$$

Thus, the expected payoff to player $A$ from allocating a strictly positive level of force to only one best-shot network is 0 regardless of which best-shot network is attacked.

For player $A$, possible deviations from the support include allocating a strictly positive level of force to: (a) two or more targets in the same weakest-link network, (b) two or more targets in different weakest-link networks, (c) two or more best-shot networks, and (d) any combination of both weakest-link and best-shot networks.

Beginning with (a), the probability that player $A$ wins both targets $i$ and $i'$ in network $j \in \mathscr{W}$ is given by the bivariate marginal distribution $P_D(x_A^i, x_A^{i'}, \{\{v_A\}_{i''\in N_{j'}|i''\neq i,i'}\}_{j'\in\mathscr{W}}, \{\{\frac{v_A}{n_{j'}}\}_{i''\in N_{j'}}\}_{j'\in\mathscr{B}})$, which we will denote as $P_D^{i,i'}(x_A^i, x_A^{i'})$. The payoff to player $A$ for any allocation of force $\mathbf{x}_A \in \mathbb{R}_+^n$ which allocates a strictly positive level of force to two targets $i$, $i'$ in a weakest-link network $j \in \mathscr{W}$ is

$$\pi_A(\mathbf{x}_A, P_D) = v_A P_D^i\left(x_A^i\right) + v_A P_D^{i'}\left(x_A^{i'}\right) - v_A P_D^{i,i'}\left(x_A^i, x_A^{i'}\right) - x_A^i - x_A^{i'}.$$

Simplifying,

$$\pi_A(\mathbf{x}_A, P_D) = v_A \left( \frac{x_A^i}{v_A} + \frac{x_A^{i'}}{v_A} - \frac{\min\left\{x_A^i, x_A^{i'}\right\}}{v_A} \right) - x_A^i - x_A^{i'} < 0.$$

The case of player $A$ allocating a strictly positive level of force to more than two targets in a weakest-link network follows directly. Clearly, in any optimal strategy player $A$ never allocates a strictly positive level of force to more than one target within a weakest-link network.

The proof for type (b) deviations follows along similar lines. Thus, in any optimal strategy player $A$ never allocates a strictly positive level of force to more than one target within a weakest-link network of targets or in different weakest-link networks.

For type (c) deviations, the probability that player $A$ wins all of the targets in both best-shot networks $j, j' \in \mathscr{B}$ is given by the $(n_j + n_{j'})$-variate marginal distribution $P_D(\{x_A^i\}_{i \in N_j \cup N_{j'}}, \{\{v_A\}_{i'' \in N_{j''}}\}_{j'' \in \mathscr{W}}, \{\{\frac{v_A}{n_{j''}}\}_{i'' \in N_{j''}}\}_{j'' \in \mathscr{B} | j'' \neq j, j'})$, which we will denote as $P_D^{N_j, N_{j'}}\left(\{x_A^i\}_{i \in N_j \cup N_{j'}}\right)$. The payoff to player $A$ for any allocation of force $\mathbf{x}_A \in \mathbb{R}_+^n$ which allocates a strictly positive level of force to exactly two best-shot networks $j, j' \in \mathscr{B}$ is

$$\begin{aligned}
&\pi_A(\mathbf{x}_A, P_D) = \\
&\quad v_A P_D^{N_j}\left(\{x_A^i\}_{i \in N_j}\right) + v_A P_D^{N_{j'}}\left(\{x_A^i\}_{i \in N_{j'}}\right) - v_A P_D^{N_j, N_{j'}}\left(\{x_A^i\}_{i \in N_j \cup N_{j'}}\right) - \sum_{i \in N_j \cup N_{j'}} x_A^i.
\end{aligned}$$

Simplifying,

$$\pi_A(\mathbf{x}_A, P_D) = -v_A \min\left\{ \frac{\sum_{i \in N_j} x_A^i}{v_A}, \frac{\sum_{i \in N_{j'}} x_A^i}{v_A} \right\}$$

The case of player $A$ allocating a strictly positive level of force to more than two best-shot networks follows directly. Clearly, in any optimal strategy player $A$ never allocates a strictly positive level of force to more than one best-shot network.

The case of type (d), follows along similar lines. Thus, the expected payoff from each point in the support of the $n$-variate distribution $P_A$ results in the same expected payoff, 0, and there exist no allocations of force which have a higher expected payoff.

The case for player $D$ follows along similar lines. $\square$

While the equilibrium distributions of force stated in Theorem 1 are not unique,[7] it is useful to provide some intuition regarding the existence of this particular equilibrium before moving on to the characterization of properties of optimal attack and defense that hold in all equilibria (Propositions 1-3). The supports of the equilibrium distributions of force stated in Theorem 1 are given in Figure (2). Panels (i) and (ii) of Figure (2) provide the supports for the attacker and defender, respectively, in the case that there is one weakest-link network with two targets ($i = 1,2$). Panels (iii) and (iv) of Figure (2) provide the supports for the attacker and defender, respectively, in the case that there is one best-shot network with two targets ($i = 1,2$) and one weakest-link network with one target ($i = 3$).

[Insert Figure 2]

Across all of the Panels (i)-(iv), if $\alpha = 1$ then each player randomizes continuously over their respective shaded line segments. In the event that the defender has a normed relative strength advantage ($\alpha > 1$), the defender's strategy stays the same, but the attacker now places a mass point of size $1-(1/\alpha)$ at the origin and randomizes continuously over the respective line segments with the remaining probability. Conversely, if the defender does not have a normed relative strength advantage ($\alpha < 1$) then it is the defender who places a mass point (of size $1-\alpha$) at the origin.

Beginning with Panels (i) and (ii), recall that if the attacker successfully attacks a single target in a weakest-link network the entire network is disabled. As shown in Panel (i) the attacker launches an attack on at most one target. To successfully defend a weakest-link network, the defender must win every target within the network. As shown in Panel (ii) the defender's allocation of force to target $i$ is an almost surely strictly increasing function of the force allocated to target $-i$. Note that if the attacker launches an attack on at most one target, then the probability that any single attack is successful depends only on the univariate marginal distributions of the defender's ($n$-variate joint)

[7] For example, in the case (1) parameter range of Theorem 1 another equilibrium strategy for player D is to use the distribution of force

$$P_D(\mathbf{x}) = \min\left(\left\{\frac{\prod_{i\in N_j} x^i}{v_A}\right\}_{j\in\mathscr{W}}, \left\{\frac{\sum_{i\in N_j} x^i}{v_A}\right\}_{j\in\mathscr{B}}\right).$$

distribution of force. In addition, the defender's expected force expenditure depends only on his set of univariate marginal distributions, and, for a given set of univariate marginal distributions, is invariant to the correlation structure.[8] Finally, note that for the given correlation structure in the defender's support [panel (ii)] the probability that the attacker launches at least one successful attack depends only on the maximum of his force allocations across the two targets. That is, given the defender's distribution of force, if there exists any points in the support of the attacker's distribution of force in which $x_A^i > x_A^{-i} > 0$ with positive probability, then the attacker can strictly increase his expected payoff by changing to $x_A^{-i} = 0$ in all such points. In such a deviation, the probability of at least one successful attack is unaffected, but the attacker's expected force expenditure decreases. Thus, at each point in the support of an optimal distribution of force the attacker launches at most one attack.

Panels (iii) and (iv) examine a simple supra-network with one best-shot network and one weakest-link network . In Panel (iii), note that the attacker launches an attack on at most one network. In the event that the best-shot network is attacked, the attacker's allocation of force to target $i$ is an almost surely strictly increasing function of the force allocated to target $-i$. In Panel (iv), note that the defender allocates a strictly positive level of force to at most one of the targets in the best-shot network, and that level of force allocated to the weakest-link network is an almost surely increasing function of the level of force allocated to the best-shot network. Given these correlation structures, the intuition for the attacker launching an attack on at most one network in the supra-network follows along the lines given above for the weakest-link network in which at most one target was attacked.

We now characterize the qualitative features arising in all equilibrium distributions of force. Proposition 1 examines the number of networks that are simultaneously attacked as well as the number of targets within each network that are simultaneously attacked and defended. Propositions 2 and 3 examine the likelihood that the attacker optimally chooses to launch an attack on any given network, and the likelihood that the attacker launches no attack or the defender leaves the supra-network undefended.

**Proposition 1** *In any equilibrium* $\{P_A, P_D\}$*:*

1. *Player A allocates a strictly positive level of force to at most one network.*

[8] More formally, for a given set of univariate marginal distribution functions, the expected force expenditure is invariant to the mapping into a joint distribution function, i.e. the $n$-copula.

2. *If the network to which player A allocates a strictly positive level of force is a weakest-link network, then at most one target in that weakest-link network receives a strictly positive level of force.*
3. *In each best-shot network player D allocates a strictly positive level of force to at most one target in the network.*

The formal proof of Proposition 1 is given in the appendix (see Lemma 5). The intuition for Proposition 1 follows from the fact that the likelihood that player D successfully defends all of the networks (and therefore player D's expected payoff) is weakly decreasing in the number of networks that player A chooses to simultaneously attack. However, player D has the ability to vary the correlation structure of his force allocations while leaving invariant: (i) his network specific multivariate marginal distributions of force, (ii) his univariate marginal distributions of force, and (iii) his expected expenditure. Furthermore, there exist correlation structures for which the likelihood that player D successfully defends all of the networks depends only on player A's force allocation to the one network which receives the highest level of force from player A. Given that player D is using such a correlation structure, player A optimally attacks at most one network at a time. A similar result extends directly to (2) the case of weakest-link networks and to (3) the case of best-shot networks.

**Proposition 2** *If* $\alpha \geq 1$, *then in any equilibrium* $\{P_A, P_D\}$*:*

1. *The probability that any weakest-link network* $j$ *is attacked (i.e., the probability that the attacker allocates a strictly positive level of force to weakest-link network* $j$*) is* $(n_j v_A / v_D)$, *which is increasing in the number of targets in network* $j$ *and the attacker's valuation of success and decreasing in the defender's valuation of successfully defending the entire supra-network.*
2. *The probability that any best-shot network* $j$ *is attacked is* $(v_A)/(n_j v_D)$, *which is increasing in the attacker's valuation of success and is decreasing in both the defender's valuation and the number of targets in network* $j$.
3. *The attacker optimally attacks no network in the supra-network with probability* $1-(1/\alpha)$.

For the attacker's joint distribution, the appendix characterizes the attacker's mass point at the origin (see Lemma 9) as well as his set of univariate marginal distributions. Proposition 2 follows directly. The probability that a network $j$ is attacked is equal to one minus the attacker's mass point at zero in the $n_j$-variate marginal distribution for network $j$, where the $n_j$-variate marginal distribution for network $j$ is given by $P_A^{N_j}(\{x_i\}_{i \in N_j})$.

The likelihood that the attacker optimally chooses to launch no attack is increasing in the defender's valuation of success and decreasing in the attacker's valuation of success.

In the case (1) parameter range, the attacker's valuation is low enough relative to the defender's valuation that the optimal strategy includes not launching an attack with positive probability. As we move to the case (2) parameter range, the attacker optimally launches an attack with certainty. In this case the probability that any given network of targets is attacked depends only on the number of targets in the network and the type of network. The proof of Proposition 3 also follows from the characterization of the equilibrium joint distributions given in the appendix.

**Proposition 3** *If $\alpha < 1$, then in any equilibrium $\{P_A, P_D\}$:*

1. *The probability that any weakest-link network $j$ is attacked (i.e., the probability that the attacker allocates a strictly positive level of force to weakest-link network $j$) is $n_j/([\sum_{j' \in \mathscr{W}} n_{j'} + \sum_{j' \in \mathscr{B}} \frac{1}{n_{j'}}])$, which is increasing in the number of targets in network $j$.*
2. *The probability that any best-shot network $j$ is attacked is $1/(n_j[\sum_{j' \in \mathscr{W}} n_{j'} + \sum_{j' \in \mathscr{B}} \frac{1}{n_{j'}}])$, which is decreasing in the number of targets in network $j$.*
3. *The defender optimally leaves the entire supra-network undefended with probability $1 - \alpha$.*

In the case (1) parameter range, the defender optimally chooses, with certainty, to allocate a strictly positive level of defensive force. However, in the case (2) parameter range, the defender optimally chooses to leave the entire supra-network undefended with positive probability. Furthermore, the likelihood that the defender chooses to leave the entire supra-network undefended is increasing in the attacker's valuation of success and decreasing in the defender's valuation of successfully defending the entire supra-network.

To summarize, the following conditions hold in all equilibria. In the case (1) parameter range the attacker optimally chooses not to launch an attack with positive probability. In both cases (1) and (2), the attacker optimally launches an attack on at most one network. In the event that a weakest-link network is attacked, only one target within the network is attacked. The likelihood that any individual network is attacked depends on the number of targets within the network. In each weakest-link network the likelihood of attack is increasing in the number of targets. In each best-shot network the likelihood of attack is decreasing in the number of targets. In the case (2) parameter range, the defender optimally leaves the entire supra-network undefended. Lastly, in both cases (1) and (2) when the defender chooses to defend the supra-network, within each best-shot network, the defender randomly chooses at most one target to defend.

## 4 Conclusion

This paper examines a game-theoretic model of attack and defense with multiple networks of targets and intra-network strategic complementarities among targets. In equilibrium we find that the correlation structure of the optimal attack and defense strategies depends critically on the composition of the supra-network. In addition, network redundancies, as in best-shot networks, strengthen the defender's strategic position. Conversely, the absence of network redundancies, as in weakest-link networks, weaken the defender's strategic position. In the context of networks of targets with asymmetric attack and defense, our results highlight the importance of allowing for endogenous entry and force expenditure decisions including general correlation structures for force expenditures within and across the networks of targets.

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## Appendix

This appendix characterizes the supports of the equilibrium joint distributions, the unique equilibrium payoffs, and the unique sets of equilibrium univariate marginal distributions. Before proceeding, observe the following notational conventions which will be used throughout the appendix. For points in $\mathbb{R}^n$, will use the vector notation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. For $a_k \leq b_k$ for all $k = 1, 2, \ldots, n$, let $[\mathbf{a}, \mathbf{b}]$ denote the $n$-box $B = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$, the Cartesian product of $n$ closed intervals. The vertices of an $n$-box $B$ are the points $(c_1, c_2, \ldots, c_n)$ where $c_k$ is equal to $a_k$ or $b_k$.

Given that the defender is using the distribution of force $P_D$, let

$$Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) \tag{1}$$

denote the probability that with a force allocation of $\mathbf{x}_A$ the attacker wins at least one network. Thus, the attacker's expected payoff from any pure strategy $\mathbf{x}_A$ is

$$v_A Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) - \sum_i x_A^i. \tag{2}$$

It will also be useful to note that the attacker's expected payoff from any distribution of force $P_A$ is

$$v_A E_{P_A}\left[Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right) = 1 \middle| P_D, \mathbf{x}_A\right)\right] - \sum_i E_{F_A^i}\left[x_A^i\right] \tag{3}$$

where $E_{P_A}$ denotes the expectation with respect to the joint distribution of force $P_A$ and $E_{F_A^i}$ denotes the expectation with respect to the univariate marginal distribution for target $i$, henceforth $F_A^i$, of the joint distribution of force $P_A$.

Similarly, given that the attacker is using the distribution of force $P_A$, let

$$Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right) = 0 \middle| P_A, \mathbf{x}_D\right) \tag{4}$$

denote the probability that with a force allocation of $\mathbf{x}_D$ the defender wins all of the networks in the supra-network. Thus, the defender's expected payoff from any pure strategy $\mathbf{x}_D$ is

$$v_D Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right) = 0 \middle| P_A, \mathbf{x}_D\right) - \sum_i x_D^i. \tag{5}$$

Lastly, the defender's expected payoff from any distribution of force $P_D$ is

$$v_D E_{P_D}\left[Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathscr{B}}, \left\{\iota_j^W\right\}_{j\in\mathscr{W}}\right) = 0 \middle| P_A, \mathbf{x}_D\right)\right] - \sum_i E_{F_D^i}\left[x_D^i\right] \tag{6}$$

where $E_{P_D}$ and $E_{F_D^i}$ denote the expectation with respect to the joint distribution of force $P_D$ and the expectation with respect to the univariate marginal distribution for target $i$, $F_D^i$, respectively.

**Lemma 1** *For each $i$ and $j$ such that $i \in N_j | j \in \mathscr{W}$, $\bar{s}_A^i = \bar{s}_D^i = \bar{s}_W^j$ and $\underline{s}_A^i = \underline{s}_D^i = 0$. For $i$ and $j$ such that $i \in N_j | j \in \mathscr{B}$, $\bar{s}_A^i = \bar{s}_D^i = \bar{s}_B^j$ and $\underline{s}_A^i = \underline{s}_D^i = 0$.*

*Proof* We begin with the proof that $\underline{s}_A^i = \underline{s}_D^i = 0$ for all $i$. By way of contradiction, suppose $\underline{s}_A^i \neq \underline{s}_D^i$. Let $\hat{\underline{s}}^i \equiv \max\{\underline{s}_A^i, \underline{s}_D^i\}$, and let $k$ be the identity of the player attaining $\hat{\underline{s}}^i$ (that is $\hat{\underline{s}}^i = \underline{s}_k^i$ and $\hat{\underline{s}}^i > \underline{s}^i_{-k}$).

If $\underline{s}^i_{-k} > 0$, when player $-k$ allocates $\underline{s}^i_{-k}$ to target $i$ player $-k$ is losing target $i$ with certainty and can strictly increase his payoff by setting $\underline{s}^i_{-k} = 0$. It follows directly, that player $-k$ does not randomize over the open interval $(0, \hat{\underline{s}}^i)$, and thus player $-k$ must have a mass point 0.

In the case that $\underline{s}^i_{-k} = 0$ (where player $-k$ does not randomize over the open interval $(0, \hat{\underline{s}}^i)$ and has a mass point at 0), we know that (i) both players can not have a mass point at $\underline{s}_k^i$ and (ii) player $k$ can strictly increase his payoff by lowering $\underline{s}_k^i$ to a neighborhood above 0.

Thus, we conclude that $\underline{s}_A^i = \underline{s}_D^i = 0$ for all $i$.

Lastly, for the proof that for each $i$ and $j$ such that $i \in N_j | j \in \mathscr{W}$, $\bar{s}_A^i = \bar{s}_D^i = \bar{s}_W^j$, note that for $i, k \in N_j | j \in \mathscr{W}$ it follows that if $\bar{s}_A^i = \bar{s}_D^i < \bar{s}_A^k = \bar{s}_D^k$ then player A would do better by moving mass from $\bar{s}_A^k$ to $\bar{s}_A^i$. The proof that for $i$ and $j$ such that $i \in N_j | j \in \mathscr{B}$, $\bar{s}_A^i = \bar{s}_D^i = \bar{s}_B^j$ follows for the same reasons. $\square$

**Lemma 2** *In any equilibrium $\{P_A, P_D\}$, for each target $j$ neither player's univariate marginal distributions place positive mass on any point except possibly at zero.*

*Proof* If $x_j^i$ is such a point for player $j$, then player $-j$ would either benefit from moving mass from an $\varepsilon$-neighborhood below $x_j^i$ to zero or to a $\delta$-neighborhood above $x_j^i$. $\square$

**Lemma 3** *In any equilibrium $\{P_A, P_D\}$, each player's expected payoff is constant over the support of their joint distribution except possibly at points of discontinuity of the payoff function.*

*Proof* By Lemma 2, for each target $i$ there are no mass points in the half-open interval $(0, \bar{s}^i]$. Thus for each point in the support of player $j$'s joint distribution, player $j$ must make his equilibrium payoff except for possibly at points of discontinuity of the payoff function. $\square$

**Lemma 4** *In any equilibrium $\{P_A, P_D\}$, for each target $i$ each player randomizes continuously over the interval $(0, \bar{s}^i]$.*

*Proof* By way of contradiction, suppose that there exists an equilibrium in which for some target $i$, player $j$'s univariate marginal distribution for target $i$, $F_j^i$, is constant over the interval $[\alpha, \beta) \subset (0, \bar{s}^i]$ and strictly increasing above $\beta$ in its support. For this to be an equilibrium, it must be the case that $F_{-j}^i$ is also constant over the interval $[\alpha, \beta)$. Otherwise, player $-j$ could increase his payoff.

If $F_{-j}^i(\alpha) = F_{-j}^i(\beta)$, then for any $\varepsilon > 0$ spending $\beta + \varepsilon$ in target $i$ cannot be optimal for player $j$. Indeed, by discretely reducing his expenditure from $\beta + \varepsilon$ to $\alpha + \varepsilon$ player $j$'s payoff would strictly increase. Consequently, if $F_j^i$ is constant over $[\alpha, \beta)$ it must also be constant over $[\alpha, \bar{s}^i]$, a contradiction to the definition of $\bar{s}^i$. $\square$

**Lemma 5** *In any equilibrium $\{P_A, P_D\}$:*

*(a) If $\mathbf{x}_A$ is an n-tuple contained in the support of $P_A$, then $\mathbf{x}_A$ allocates a strictly positive level of force to at most one network.*
*(b) If the network to which the n-tuple $\mathbf{x}_A$ (contained in the support of $P_A$) allocates a strictly positive level of force is a weakest-link network, then at most one target in that weakest-link network receives a strictly positive level of force.*
*(c) If $\mathbf{x}_D$ is an n-tuple contained in the support of $P_D$, then within each best-shot network $\mathbf{x}_D$ allocates a strictly positive level of force to at most one target in the network.*

*Proof* Beginning with (a), by way of contradiction suppose that there exists an equilibrium in which player A simultaneously allocates a strictly positive level of force to two or more networks. Without loss of generality, we will also assume that there exists at least one point in the support of an equilibrium strategy $P_A$ for which only networks $j$ and $j'$ simultaneously receive a strictly positive level of force from player A (henceforth networks $j$ and $j'$ are simultaneously "attacked"). Observe that this assumption allows for any number and/or combination of networks to be simultaneously attacked as long as at some point in the support of $P_A$ only networks $j$ and $j'$ are simultaneously attacked. Furthermore, while the focus on the case in which the minimum number of networks being simultaneously attacked is equal to two simplifies the expressions that follow, the case in which the minimum number of networks which are simultaneously attacked is greater than two follows directly.

Since $\max(\{\iota_j^B\}_{j\in\mathscr{B}},\{\iota_j^W\}_{j\in\mathscr{W}})$ is equal to either 0 or 1, the expected payoff for player D may be written as

$$\begin{aligned} v_D - v_D E_{P_A}\Big[Pr\Big(\max\left(\iota_j,\iota_{j'}\right)=1\Big|P_D,\mathbf{x}_A \text{ s.t. only } j \text{ and } j' \text{ attacked}\Big)\Big] \\ - v_D E_{P_A}\Big[Pr\Big(\max\Big(\left\{\iota_j^B\right\}_{j\in\mathscr{B}},\left\{\iota_j^W\right\}_{j\in\mathscr{W}}\Big)=1\Big|P_D,\mathbf{x}_A \text{ s.t. not } j \text{ and } j' \text{ attacked}\Big)\Big] \\ - \sum_i F_D^i\left(x_D^i\right). \quad (7) \end{aligned}$$

The expectation in the first line of (7) is the probability that player A successfully attacks at least one of the networks $j$ or $j'$ given that player A attacks only networks $j$ and $j'$. The expectation in the second line of (7) is the probability that player A successfully attacks at least one network conditional on the attack being on any single network or any combination of networks other than only $j$ and $j'$.

Let $\mathbf{x_A^j}$ denote the restriction of the vector $\mathbf{x_A}$ to the set of targets contained in network $j$, i.e. $\{x_A^i\}_{i\in N_j}$. Note that

$$\begin{aligned} Pr\Big(\max\left(\iota_j,\iota_{j'}\right)=1\Big|P_D,\mathbf{x}_A \text{ s.t. } j \text{ and } j' \text{ attacked}\Big) = Pr\Big(\iota_j=1\Big|P_D^{n_j},\mathbf{x}_A^j\Big) \\ + Pr\Big(\iota_{j'}=1\Big|P_D^{n_{j'}},\mathbf{x}_A^{j'}\Big) - Pr\Big(\iota_j=1 \text{ and } \iota_{j'}=1\Big|P_D^{n_j,n_{j'}},\mathbf{x}_A^j,\mathbf{x}_A^{j'}\Big). \quad (8) \end{aligned}$$

If network $j$ is a best-shot network then $Pr(\iota_j^B=1|P_D^{n_j},\mathbf{x}_A^j)=P_D^{n_j}(\mathbf{x}_A^j)$. If network $j$ is a weakest-link network then the probability that player A wins at least one target, $Pr(\iota_j^W=1|P_D^{n_j},\mathbf{x}_A^j)$, depends critically on both $P_D^{n_j}$ and the number of targets in network $j$ which are attacked. In both cases, it is clear that player D's payoff depends not only on the $n_j$-variate marginal distribution for network $j$, $P_D^{n_j}$, and the $n_{j'}$-variate marginal distribution for network $j'$,$P_D^{n_{j'}}$, but also on the correlation between these two multivariate marginal distributions. There are 3 possible cases to consider: (i) both networks $j$ and $j'$ are best-shot networks, (ii) both networks $j$ and $j'$ are weakest-link networks, and (iii) either network $j$ or $j'$ is a best-shot network and the other is a weakest-link network.

If both networks $j$ and $j'$ are best shot networks, then (8) becomes $P_D^{n_j}(\mathbf{x}_A^j)+P_D^{n_{j'}}(\mathbf{x}_A^{j'})-P_D^{n_j,n_{j'}}(\mathbf{x}_A^j,\mathbf{x}_A^{j'})$. However, if $P_D^{n_j,n_{j'}}(\mathbf{x}_A^j,\mathbf{x}_A^{j'})\neq\min\{P_D^{n_j}(\mathbf{x}_A^j),P_D^{n_{j'}}(\mathbf{x}_A^{j'})\}$ then player D could increase the first line of (7) without affecting the univariate marginals and thus the third line of (7). Furthermore, if $P_D(\mathbf{x})=\min_j\{P_D^{n_j}(\mathbf{x}^j)\}$ then each $n_j$-variate marginal distribution $(P_D^{n_j}(\mathbf{x}^j))$ is preserved, each univariate marginal distribution $(F_D^i(x^i))$ is preserved, and for any set of networks $j\in\mathscr{J}$ the joint marginal distribution for the set $\mathscr{J}$ is $P_D^{\mathscr{J}}(\mathbf{x}^{\mathscr{J}})=\min_{j\in\mathscr{J}}\{P_D^{n_j}(\mathbf{x}^j)\}$. Clearly, if networks $j$ and $j'$ are the only two networks which player A simultaneously attacks, then the deviation to such a correlation strategy strictly increases player D's payoff. A contradiction to the assumption that $\{P_A, P_D\}$ is an equilibrium. Furthermore, if $P_D^{n_j,n_{j'}}(\mathbf{x}_A^j, \mathbf{x}_A^{j'}) = \min\{P_D^{n_j}(\mathbf{x}_A^j), P_D^{n_{j'}}(\mathbf{x}_A^{j'})\}$ then player A could increase his payoff by attacking network $j$ or network $j'$ but not both simultaneously; also a contradiction.

If networks $j$ and $j'$ are *not* the only networks which player A simultaneously attacks but all of the networks in the supra-network are best-shot networks, then the second line of (7) can be broken into components for each of the sets of networks which player A simultaneously attacks and each of the networks which are attacked in isolation. In this case, the proof follows along the lines of the proceeding case. That is if the supra-network is comprised of only best-shot networks, player A attacks at most one network.

For cases (ii) and (iii) as well as the remaining case (i) network configurations, note that the result on a supra-network consisting of only best-shot networks can be modified to show that within each weakest-link network player A attacks at most one target. That is, without loss of generality, assume that there exists at least one point in the support of an equilibrium strategy $P_A$ for which within weakest-link network $j$ only targets $i$ and $i'$ simultaneously receive a strictly positive level of force from player A. In such a case, $Pr(\iota_j^W = 1|P_D^{n_j}, \mathbf{x}_A^j) = F_D^i(x_A^i) + F_D^{i'}(x_A^{i'}) - P_D^{n_j}(x_A^i, x_A^{i'}, \{\bar{s}_W^j\}_{i''\in N_j|i''\neq i,i'})$ where $(x_A^i, x_A^{i'}, \{\bar{s}_W^j\}_{i''\in N_j|i''\neq i,i'})$ denotes the vector formed by replacing each zero in $\mathbf{x}_A^j$ (all targets except $i$ and $i'$) with $\bar{s}_W^j$. Setting $P_D^{n_j}(\mathbf{x}_A^j) = \min_{i\in j}\{F_D^i(x^i)\}$, the result follows directly, as does the result for the case in which the number of targets which are simultaneously attacked is greater than two or involves any arbitrary combinations of weakest-link targets. Thus, player A attacks at most one target in each weakest-link network, and $Pr(\iota_j = 1|P_D^{n_j}, \mathbf{x}_A^j) = F_D^i(x_A^i)$. Inserting this back into (8), the proof for cases (ii) and (iii) as well as the remaining case (i) network configurations follows directly.

The proof for part (c) follows from a symmetric argument. $\square$

**Lemma 6** $\forall\ j, j' \in \mathscr{W}$, $\bar{s}_W^j = \bar{s}_W^{j'} \equiv \bar{s}_W$.

*Proof* Following from Lemmas 2 and 5, in the support of any optimal strategy, when player A allocates $\bar{s}_W^j$ to a single target in network $j$ the force allocated to each of the remaining targets is 0, player A wins network $j$ with certainty, and player A's expected payoff is $v_A - \bar{s}_W^j$.

From Lemma 3, player A's expected payoff is constant across all points in the support of $P_A$. Thus, $\forall\ j, j' \in \mathscr{W}$, $\bar{s}_W^j = \bar{s}_W^{j'} \equiv \bar{s}_W$. $\square$

**Lemma 7** $\forall\ j \in \mathscr{B}$, $\bar{s}_W = n_j \bar{s}_B^j$.

*Proof* From Lemma 5 part (a) in the support of any optimal strategy player A attacks at most one network. In the case that player A attacks best-shot network $j$, from Lemma 3 there exists a $k_A \geq 0$ such that

$$Pr\left(\iota_j^B = 1 \middle| P_D^{n_j}, \mathbf{x}_A\right) \leq \frac{k_A}{v_A} + \frac{\sum_i x_A^i}{v_A} \tag{9}$$

which holds with equality for each $\mathbf{x}_A$ in the support of $P_A$ such that player A attacks best-shot network $j$.

From Lemma 5 part (c) in the support of any optimal strategy player D allocates a strictly positive level of force to at most one target in network $j$, and thus the support of player D's $n_j$-variate marginal distribution for network $j$, $P_D^{n_j}$ is contained on the each of the $n_j$ axes in $\mathbb{R}^{n_j}$. From Lemmas 2 and 4, it follows that equation (1) holds with equality not only for each $\mathbf{x}_A$ in the support of $P_A$ such that player A attacks best-shot network $j$, but — given that network $j$ is the only network attacked — for all $n_j$-tuples $\mathbf{x}^\mathbf{j} \in [0, \bar{s}_B^j]^{n_j}$. That is given that the support of player D's $n_j$-variate marginal distribution for network $j$ is: (i) contained on the each of the $n_j$ axes in $\mathbb{R}^{n_j}$, (ii) has no mass points except possible at the origin in $\mathbb{R}^{n_j}$, and (iii) is continuous on each axis, it follows that for $\mathbf{x}_\mathbf{A}^\mathbf{j} \in [0, \bar{s}_B^j]^{n_j}$, $P_D^{n_j}(\mathbf{x}_A^j) = Pr(\iota_j^B = 1 \big| P_D^{n_j}, \mathbf{x}_A) = \frac{k_A}{v_A} + \frac{\sum_i x_A^i}{v_A}$.

Thus, if player A chooses the $n_j$-tuple with $\bar{s}_B^j$ for each element then from Lemma 2 player A's expected payoff from such an $n_j$-tuple is $v_A - n_j \bar{s}_B^j$.

From Lemmas 3 and 6, $k_A = v_A - \bar{s}_W$. From Lemma 3, player A's expected payoff is constant across all points in the support of $P_A$. Thus, $\forall\ j \in \mathscr{B}$, $\bar{s}_W = n_j \bar{s}_B^j$. $\square$

**Lemma 8** $\bar{s}_W = \min\{v_A, v_D/[\sum_{j\in\mathscr{W}} n_j + \sum_{j\in\mathscr{B}}(1/n_j)]\}$.

*Proof* If player D allocates: (i) $\bar{s}_W$ to each target in each weakest-link network, (ii) $\bar{s}_B^j$ to exactly one target in each best-shot network $j$, and (iii) 0 to each of the remaining targets in the best-shot networks, then player D wins with certainty and has an expected payoff of $v_D - \sum_{j\in\mathscr{W}} n_j \bar{s}_W + \sum_{j\in\mathscr{B}}(\bar{s}_W/n_j) \geq 0$. Thus, in such a case it must be that $\bar{s}_W \leq v_D/[\sum_{j\in\mathscr{W}} n_j + \sum_{j\in\mathscr{B}}(1/n_j)]$. Similarly, player A's expected payoff is $v_A - \bar{s}_W \geq 0$, and thus, $\bar{s}_W \leq v_A$.

Since player A attacks at most one network, and in the case of a weakest-link network only one target, we know that the origin is contained in the support of any equilibrium distribution of force for player A, $P_A$.

By way of contradiction suppose that there exists an equilibrium $\{P_A, P_D\}$ in which the origin is not contained in the support of $P_D$. Thus, there exists an $\varepsilon > 0$ such that for at least two targets, denoted as targets 1 and 2, the intersection of the projection of player D's support onto the $x_1, x_2$-plane with the box $[0,\varepsilon]^2$ is empty. There are five configurations to consider: (i) targets 1 and 2 are in the same weakest-link network, (ii) targets 1 and 2 are in separate weakest-link networks, (iii) targets 1 and 2 are in the same best-shot network, (iv) targets 1 and 2 are in separate best-shot networks, (v) target 1 is in a weakest-link network and target 2 is in a best-shot network. From Lemma 5, it is clear that we can rule out case (iii). In cases (i) and (ii), for any $(x_1,x_2) \in [0,\varepsilon]^2$ player D's bivariate marginal distribution for targets 1 and 2, $P_D^{1,2}$, is equal to zero and player A can strictly increase his payoff by allocating a level of force less than $\varepsilon$ to both targets 1 and 2 a contradiction to Lemma 5 if targets 1 and 2 are in the same weakest-link network or if targets 1 and 2 are in separate weakest-link networks. Following along similar lines, cases (iv) and (v) lead to a similar contradiction to Lemma 5. Thus, the origin is contained in the support of any equilibrium distribution $P_D$ for player D.

Since only one player can have a mass point at the origin, we have that if $v_A - \bar{s}_W > 0$ player A must outbid player D with a probability that is bounded away from zero. Thus, player D places positive mass at the origin, but if player D has a mass point at the origin then it must be the case that $v_D - \sum_{j\in\mathscr{W}} n_j \bar{s}_W + \sum_{j\in\mathscr{B}} (\bar{s}_W/n_j) = 0$. Similarly, if $v_D - \sum_{j\in\mathscr{W}} n_j \bar{s}_W + \sum_{j\in\mathscr{B}} (\bar{s}_W/n_j) > 0$ then player D must outbid player A with a probability that is bounded away from zero. Thus, $v_A - \bar{s}_W = 0$ and player A places positive mass at the origin. $\square$

The next two lemmas follow directly from Lemma 8. Recall that $\alpha = v_D/(v_A[\sum_{j\in\mathscr{W}} n_j + \sum_{j\in\mathscr{B}} \frac{1}{n_j}])$.

**Lemma 9** *If $\alpha \geq 1$, then (i) player A places mass $1-(1/\alpha)$ at the origin, (ii) player A's expected payoff is 0, (iii) player D does not place positive mass at the origin, and (iv) player D's expected payoff is $v_D - (v_D/\alpha)$.*

**Lemma 10** *If $\alpha < 1$, then (i) player D places mass $1-\alpha$ at the origin, (ii) player D's expected payoff is 0, (iii) player A does not place positive mass at the origin, and (iv) player A's expected payoff is $v_A - v_A\alpha$.*

**Lemma 11** *There exists a unique set of equilibrium univariate marginal distributions $\{\{F_A^j\}_{j\in\mathscr{B}\cup\mathscr{W}}, \{F_D^j\}_{j\in\mathscr{B}\cup\mathscr{W}}\}$.*

*Proof* This proof is for the uniqueness of player D's set of univariate marginal distributions. The proof for player A is analogous. For each best-shot network $j \in \mathscr{B}$, from Lemma 7 for $\mathbf{x^j} \in [0, \bar{s}_B^j]^{n_j}$, $P_D^{n_j}(\mathbf{x}^j) = \frac{v_A - \bar{s}_W}{v_A} + \frac{\sum_i x^i}{v_A}$, where $\bar{s}_W = \min\{v_A, v_D/[\sum_{j\in\mathscr{W}} n_j +$

$\sum_{j\in\mathscr{B}}(1/n_j)]\}$ and $\bar{s}_B^j = \frac{\bar{s}_W}{n_j}$. Thus, in each best-shot network $j$ player D's unique univariate marginal distributions follow from player D's unique $n_j$-variate marginal distribution for network $j$.

From Lemma 5 parts (a) and (b), player A attacks at most one target in one weakest-link network. From Lemmas 2, 3, and 4 it follows that for each target $i$ in each weakest-link network $j \in \mathscr{W}$,

$$v_A F_D^i\left(x_A^i\right) - x_A^i = v_A - \bar{s}_W$$

for $x^i \in [0, \bar{s}_W]$. Thus, player D's univariate marginal distributions are uniquely determined in each weakest-link network. $\square$

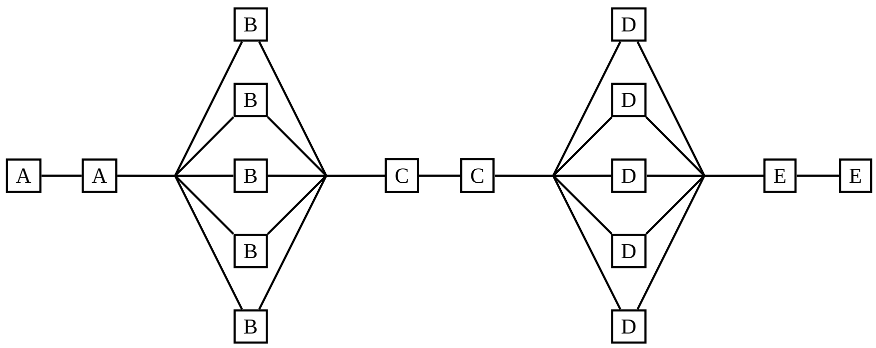


**Fig. 1** Example Supra-Network with Five Networks (A, B, C, D, and E)

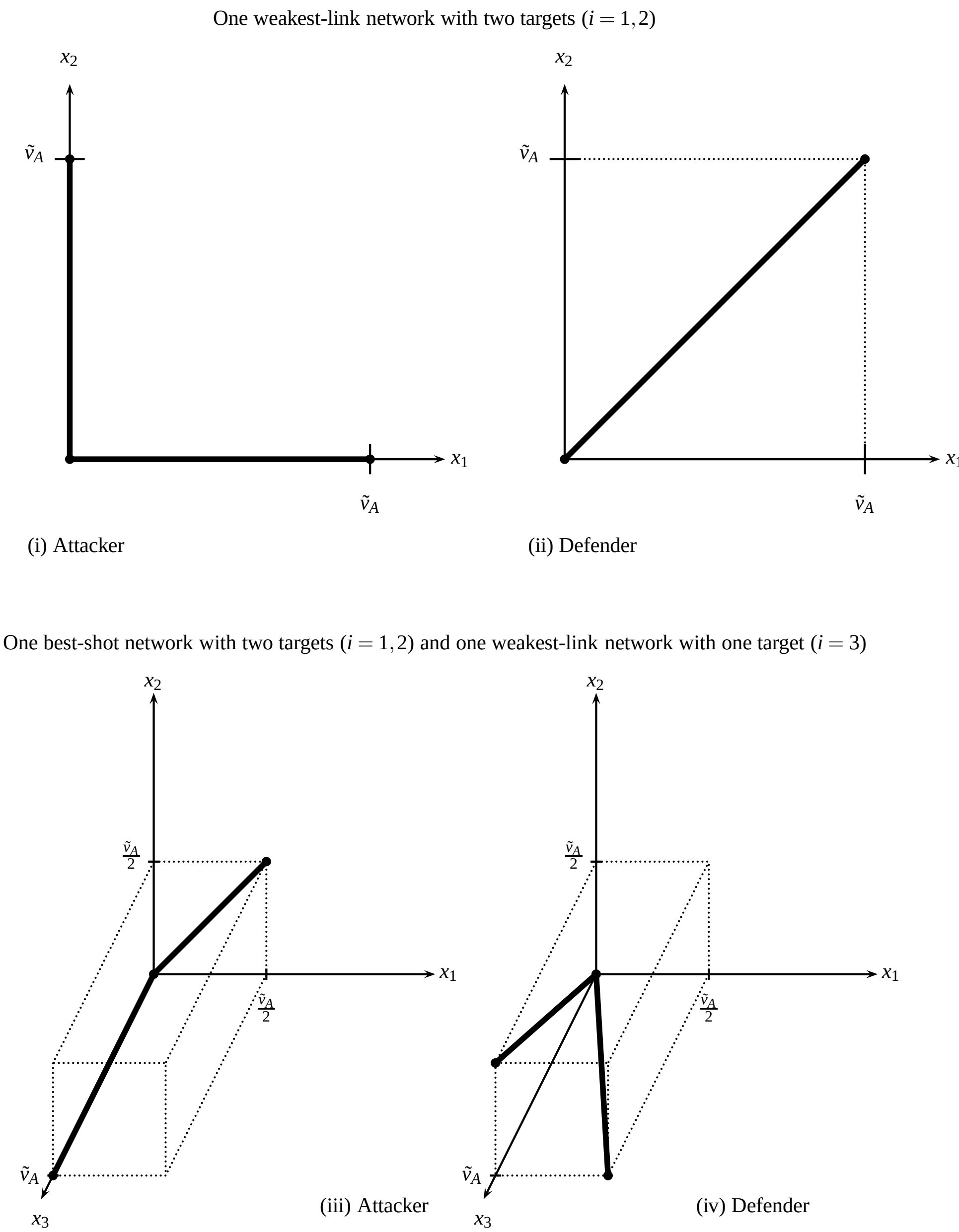


**Fig. 2** Supports of the equilibrium joint distributions stated in Theorem 1 ($\tilde{v}_A = \min\{\alpha v_A, v_A\}$).