Research Cycles

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Abstract: This paper studies the dynamics of fundamental research. We develop a simple model where researchers allocate their effort between improving existing fields and inventing new ones. A key assumption is that scientists derive utility from recognition from other scientists. We show that the economy can be either in a regime where new fields are constantly invented, and then converges to a steady state, or in a cyclical regime where periods of innovation alternate with periods of exploitation. We characterize the cyclical dynamics of the economy, show that indeterminacy may appear, and establish some comparative statics and welfare implications.

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> "my love of natural science (...) has been much aided by the ambition to be esteemed by my fellow naturalists." Darwin (1958).

1 Introduction

This paper studies the dynamics of fundamental research. We observe that periods of intense innovations are followed by periods of exploitation of existing fields. We want to understand these dynamics and be able to study whether they are efficient from the point of view of social welfare.

A key aspect we are interested in is the credential one. Scientists derive utility from recognition from other scientists, which often takes the form of citations. In our model, the value derived by a scientist from a paper he has written is the sum of an "intrinsic" value of the paper, which depends on the field in which it is written and its order of appearance in that field, and a "citation premium" which depends on the number of subsequent papers written in that field.

We show that our model yields a rich set of results, both with respect to the cyclical dynamics of the allocation of the research effort and in terms of the comparative statics around the steady state, when it exists.

More specifically, we show that the economy can be either in a regime where new fields are constantly invented, and then converges to a steady state, or in a cyclical regime where periods of innovation alternate with periods where one only exploits existing fields. Furthermore, these cycles are very irregular and the duration of a cycle is "unpredictable" from the duration of the previous cycle – i.e. related to it by a very nonlinear function.

Furthermore, we are able to precisely understand when cycles emerge and show that a (i) higher citation premium makes cycles less likely, (ii) a mean-preserving spread in the distribution of the value of new fields makes cycles more likely if the citation premium is not too large and (iii) a larger citation premium alleviates (and potentially reverses) that effect.

We introduce the distinction between extensive and intensive research to the study of scientific progress. Studies of technological changes have long stressed the difference between improvements of known processes and innovations leading to new products (e.g. Rosenberg (1972)). Similarly, it seems that some scientific contributions are pioneering and open up new avenues for future research, while others mainly refine or extend previous work. This distinction lies at the core of Kuhn (1970)'s influential theory of scientific evolution. In his view, science alternates between periods of normal science and scientific revolutions. Under normal science, progress is gradual and builds up on past achievements. In contrast, scientific revolutions correspond to paradigm shifts during which scientists qualitatively change their focus and assumptions. Without necessarily adhering to Kuhn's view, other observers have noted the importance of fads and fashions in science. Bronfenbrenner (1966) gives an early account of fads in economics. Stephan and Levin (1991) discuss how scientific fashions might affect a scientist's career. Sunstein (2001) relates academic fads to informational cascades.

This literature remains relatively undeveloped.[2] We develop the first formal model of the evolution of science that gives rise to innovation cycles and scientific fashions. We also explicitly account for the unique reward structure of science, by assuming that scientists care for recognition by their peers through citations of their work.[3]

In the literature on growth,[4] several papers look at innovation cycles.

[2] Exceptions include Levin and Stephan (1991), Brock and Durlauf (1999), Goyal et al. (2006), see also Stephan (1996) and the references therein.

[3] Scientists could also care for citations because of the financial gains they generate, see e.g. Diamond (1986).

[4] Our paper is also related to the literature on directed innovation in growth models, which studies the incentive to innovate in one sector vs. another (See Acemoglu, 1998). The determinants of innovation in existing vs. new fields which we discuss here, however,

Jovanovic and Rob (1990) and Jovanovic and Nyarko (1996) build learning models. In Jovanovic and Rob (1990), an agent can explore a known or an unknown dimension of the technological space. Innovation cycles emerge for an intermediate range of the parameters. In Jovanovic and Nyarko (1996), the agent chooses between a known and a better, but unknown, technology. Permanent upgrading and growth can coexist with technological lock-in and stagnation. Matsuyama (1999) shows how innovation cycles can emerge in an endogenous growth model. Phases where investment is concentrated on old technologies alternate with phases with innovation. Innovation cycles emerge because of the temporary monopoly power enjoyed by innovators.

In contrast, the mechanism that drives our cycles is due to a multiplier effect, namely the fact that one unit of effort devoted to creating a new field today induces more than one unit of effort to exploit this field tomorrow.

A main novelty of our specification, relative to these papers, is the citation premium.[5] Through citations, payoffs from a scientist's current choice depend on the future evolution of science. This introduces a dynamic linkage between current and future actions. This linkage dramatically complicates the proof of existence of an equilibrium. Especially, it prevents the use of standard fixed point techniques. We develop appropriate reformulations of our equilibrium conditions to overcome this difficulty. Our methodology is original, and may have applicability in other dynamic settings where classical results do not hold.

Note that our results could also potentially be applied to the analysis of commercial R & D, with our citation premium being reinterpreted as

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are substantially different from the ones studied in that literature. The two approaches could be brought together, however, by assuming that exploiting existing fields uses different factors of production than invention.

[5] Another difference is that the agent can make one search per period in Jovanovic and Rob (1990) and Jovanovic and Nyarko (1996), while in our model research effort is continuous and allocated among different alternatives in every period.

the income derived by an innovator from the royalties paid by subsequent innovators building on his or her invention. The level of the citation premium can then be interpreted as the level of intellectual property.[6]

The rest of the paper is organized as follows. We introduce the model in section 2. We present our main result in section 3 and interpret it in section 4. We derive comparative statics in section 5 and comparative dynamics in section 6. In section 7, we analyze the local transitional dynamics to the steady state in regime II, when it is stable. A key result is that for large enough citation premia, sunspots may arise; that captures the idea of fads and fashions in academic research. Section 8 introduces some welfare analysis, by assuming that the only market failure is that researchers do not internalize the fact that their papers will benefit future generations. We then show that absent a citation premium the value of a new field in the equilibrium steady state is lower than at the optimum steady state, and that an optimal "pigovian" citation premium can be introduced so as to induce the socially optimal level of fundamental. We provide a formula for computing this citation premium. Section 9 contains concluding comments.

2 The model

We consider an infinite horizon model with discrete time. At each date t there is a continuum of existing fields of research, which we index by i. Each field is characterized by a stock of contributions (or 'papers') $n_t(i)$ at the end of period t. We also think of this stock as a continuum. $n_t(i)$ is the advancement level of field i at date t. Creating a new field puts it at a fixed, initial advancement level $\bar{n}$.

Papers are produced by researchers. Researchers live for two periods,

[6] Or, more generally, as the degree of appropriability of the returns to innovation through their embodiment in physical goods, as in Boldrin and Levine (2005).

hence we have an overlapping generation structure. In the first period of their life, researchers produce contributions. In the second period of their life, they enjoy the returns from their scientific "reputation", which defines their utility function. A researcher's scientific reputation is the sum of the contribution of each individual paper he or she has written. An (infinitesimal) individual paper written at date t in existing field i yields the following contribution to its author's reputation:

$$v_t(i) = \omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)).$$

This reputation is the sum of two terms. The first term, $\omega(i)-\beta(\ln n_t(i)-\ln \bar{n})$, defines the *intrinsic* value of the paper. $\omega(i)$ is a field-specific constant which represents the field's value (or initial research potential) as a whole. The term $\beta(\ln n_t(i) - \ln \bar{n})$, where β and $\bar{n}$ are positive parameters, captures the fact that there are decreasing returns to research: the larger the stock of knowledge in field i, the smaller the intrinsic value of additional contributions. The second term, $\theta(\ln n_{t+1}(i) - \ln n_t(i))$, is the *citation premium.* It tells us that the reputation obtained from papers written at t is greater, the greater the flow of further advances in the relevant field at $t+1$. Underlying this formulation is the idea that papers come in a given order, and that new papers cite previous papers, thus enhancing their author's reputation. Note that contemporaneous papers do not cite each other, so that what matters for citations is the log difference between the stock of papers written at the end of $t+1$ and that at the end of t.

The total mass of researchers per generation is normalized to 1. Each researcher is endowed with υ units of time. He allocates his time optimally between writing papers in different fields. In addition to that, one may create new fields.[7]

[7] Note that this distinction between fundamental and secondary innovation is different from the one used by Aghion and Howitt (1996, 2000 ch. 6), who assume that secondary

When one writes the first paper in a new field, its potential $\omega(i)$ is drawn from some distribution, with pdf $f(.)$, such that all moments exist. The realization of $\omega(i)$ is unknown when one decides to write the paper. At the end of the period when the new field is created, its advancement level is set at the initial value $\bar{n}$. Therefore, one must wait one period before making further contributions to a new field.

We assume that one unit of time produces either 1 paper in an existing field or γ papers in a new field.

We make two technical assumptions that we need to be able to solve the model:

Assumption A1 – If at date t, there is a strictly positive measure of new fields invented, then all fields invented before date t can no longer be researched from date $t+1$ on.

This assumption is a useful simplification that avoids having to keep track of all the fields ever invented at any date t.[8] Only the fields invented in the last wave of innovation can be exploited at a given date.[9]

Assumption A2 – $\gamma < 1$.

This assumption states that inventing a new field requires more labor than writing a paper in an existingf field. It is a plausible, but merely technical

innovation results from learning by doing only.

[8]This assumption is consistent with Kuhn's theory: "When it repudiates a paradigm, a scientific community simultaneously renounces most of the books and articles in which that paradigm had been embodied", Kuhn (1970). See also theorem 2 in Jovanonvic and Rob (1990) and the "no-recall" assumption of Jovanovic and Nyarko (1996).

[9]It is not necessary to make this assumption in the special case where $\theta = 0$. In such a case the value of inventing a new field is $V_N = \gamma\bar{\omega} = \gamma E(\omega)$, which is also the lower bound of the value of working on an existing field, since one could always produce new fields instead. Consequently, when new fields are invented, all previous fields reach their maximum advancement level, such that the value of the marginal paper is equal to V_N; they will not be exploited thereafter.

assumption, required to prove the existence of an equilibrium for $\theta > 0$.[10]

3 Equilibrium

In this section, we show the existence of an equilibrium, and the conditions under which it is cyclical as opposed to converging to a steady state. We provide a result for uniqueness in the case where $\theta = 0$. We first discuss the equilibrium conditions of the model in the two regimes of interest. We then state the paper's main result, whose proof is relegated to the Appendix. In the next section, we discuss its economic interpretation using a graphical illustration, confining ourselves to the $\theta = 0$ case. We then work out numerical examples. Finally, we give a sketch of the proof when the citation premium is positive.

3.1 Equilibrium conditions

At any point in time, the economy may be in one of two regimes:

In Regime I, all the research input is allocated to improving existing fields. There exists a shadow value of time λ_t; a field is exploited if and only if the first paper written in the current period has a value greater than λ_t, that is:[11]

$$\omega - \beta(\ln n_{t-1} - \ln \bar{n}) + \theta(\ln n_{t+1} - \ln n_{t-1}) > \lambda_t. \qquad (1)$$

The number of papers written in such a field, at t, must satisfy

$$\omega - \beta(\ln n_t - \ln \bar{n}) + \theta(\ln n_{t+1} - \ln n_t) = \lambda_t. \qquad (2)$$

The equilibrium value of λ_t must adjust so that the total mass of papers being written is equal to υ. Call s the last period where invention took place, and μ_s the mass of new fields invented at s.

[10] It is again not needed for $\theta = 0$.

[11] We drop the index i as two fields with the same ω will behave identically.

Using (1) and (2), the full employment condition can be written as

$$\mu_s \int_{\omega > (\beta+\theta)\ln n_{t-1}(\omega) - \beta \ln \bar{n} - \theta \ln n_{t+1}(\omega) + \lambda_t} (n_t(\omega) - n_{t-1}(\omega)) f(\omega) d\omega = \upsilon. \quad (3)$$

Finally, the value of writing a paper in a new field, denoted by V_t, must be lower than that of working on an existing field:

$$V_t < \lambda_t.$$

In Regime II, people exploit existing fields, and work on new fields as well. They must be indifferent between the two activities, so that one must have $\lambda_t = V_t$. Conditions (1) and (2) remain valid with λ_t replaced with V_t.

Because of Assumption (A1), the existing fields will disappear at $t+1$ and be replaced by the mass μ_t of new fields, which will start with advancement level $\bar{n}$ at $t+1$.Therefore, $n_{t+1}(\omega) = n_t(\omega)$, since existing fields at t are no longer exploited at $t+1$. Substituting into (2), the final advancement level is:

$$n_t(\omega) = \bar{n} e^{\frac{\omega - V_t}{\beta}}, \quad (4)$$

while (1) can be rewritten in this regime as

$$\omega > V_t + \beta(\ln n_{t-1}(\omega) - \ln \bar{n}).$$

Note that this condition collapses to

$$\omega > V_t \quad (5)$$

if the economy was also in regime II at date $t-1$, since the field must then have been invented at that date.

In regime II, the resource constraint states that total time devoted to existing fields cannot exceed υ :

$$\mu_s \int_{\omega > \beta(\ln n_{t-1}(\omega) - \ln \bar{n}) + V_t} (n_t(\omega) - n_{t-1}(\omega)) f(\omega) d\omega \leq \upsilon.$$

The remaining time endowment must be devoted to new fields; this determines the mass of new fields invented at t :

$$\mu_t = \gamma \left[\upsilon - \mu_s \int_{\omega > \beta(\ln n_{t-1}(\omega) - \ln \bar{n}) + V_t} (n_t(\omega) - n_{t-1}(\omega)) f(\omega) d\omega \right]. \quad (6)$$

Finally, in both regimes, the value of working in a new field V_t is determined as follows. Consider a researcher writing a paper in a new field with value ω. Then, $n_t(\omega) = \bar{n}$. If (1) holds at $t+1$, which is equivalent to

$$\omega > \theta \ln \bar{n} - \theta \ln n_{t+2}(\omega) + \lambda_{t+1},$$

then the field will be active, and the inventor will benefit from citations. The value to the inventor is then given by

$$v_t(\omega) = \omega + \theta(\ln n_{t+1}(\omega) - \bar{n})$$

where $n_{t+1}(\omega) = \bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+2}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega - \lambda_{t+1}}{\beta+\theta}}$.

Otherwise, the field will not be active at $t+1$, and the inventor just gets the intrinsic value of the first paper:

$$v_t(\omega) = \omega.$$

Thus, the value of working on a new field at t is given by:

$$\begin{aligned} V_t &= \gamma E v_t(\omega) \\ &= \gamma \left[\bar{\omega} + \frac{\theta}{\beta+\theta} \int_{\omega > \lambda_{t+1} - \theta(\ln n_{t+2}(\omega) - \ln \bar{n})} (\omega - \lambda_{t+1} + \theta(\ln n_{t+2}(\omega) - \ln \bar{n})) f(\omega) d\omega \right]. \end{aligned}$$

3.2 Existence, uniqueness, and cycles

We now state the central results of the paper. To do so, we need to introduce the following two functions:

$$\Phi(y) = \gamma \left[\bar{\omega} + \frac{\theta}{\beta} \int_y^{+\infty} (\omega - y) f(\omega) d\omega \right], \tag{7}$$

and

$$I^*(y) = \bar{n} \int_y^{+\infty} (e^{\frac{\omega - y}{\beta}} - 1) f(\omega) d\omega.$$

The Φ function captures how the value of invention evolves during regime II, as a function of the value of invention next period, denoted by y. It consists of two terms: the average intrinsic value of the first paper in the field, $\bar{\omega}$, and the contribution to the inventor's welfare of future citations, $\frac{\theta}{\beta} \int_y^{+\infty} (\omega - y) f(\omega) d\omega$. That contribution falls with y, since a greater value of invention tomorrow reduces the number of papers written in my field and thus my citations.

As for I^*, it is a measure of the mass of researchers who devote themselves to existing virgin fields, as a function of the current opportunity cost of writing a paper. The greater that value y, the smaller the equilibrium labor input in existing fields. For example, if the economy is in regime II in periods t and $t+1$, and if a unit mass of new fields is invented at t, exploiting those with field value greater than or equal to y at $t+1$ requires $I^*(y)$ research input.[12]

Both functions are continuous and decreasing. Since $\Phi(0) \geq \gamma\bar{\omega}$ and $\Phi(+\infty) = \gamma\bar{\omega}$, Φ has a fixed point $\bar{V}$:

$$\Phi(\bar{V}) = \bar{V}. \tag{8}$$

The paper's main result can be stated as follows. (All proofs are given in Appendix).

[12] As another example, if $\theta = 0$ and fields are exploited for the first time, then the LHS of (3) is equal to $\mu_s I^*(\lambda_t)$.

PROPOSITION 1 – Assume that the economy starts at $t = 0$ *with an initial mass of fields* μ_{-1}*, whose intrinsic value is distributed with* $f()$*, and whose initial advancement level is given by* $\bar{n}$*. Then:*

(i) There exists an equilibrium path.

(ii) *If*

$$\gamma I^*(\bar{V}) > 1, \tag{9}$$

then any equilibrium is cyclical, i.e. periods in regime I alternate with periods in regime II. During periods in regime II, the mass of invented fields follows explosive oscillations, until the economy reverts to regime I. During periods in regime I, the set of exploited fields grows. The duration of a period in regime I cannot exceed $\gamma I^*(\gamma\bar{\omega})$*.*

(iii) If

$$\gamma I^*(\bar{V}) < 1,$$

then there exists an equilibrium such that

-the economy is in regime II from $t = 0$ *on.*

-the value of working in a new field is equal to $\bar{V}$ *at all dates.*

-the mass of invented fields converges to its steady state value, given by

$$\bar{\mu} = \frac{\gamma\upsilon}{1 + \gamma I^*(\bar{V})}, \tag{10}$$

by dampened oscillations.

(iv) If $\theta = 0$*, equilibrium is unique.*[13]

We interpret the key condition (9). Consider a regime II episode where the value of invention is constant and equal to $\bar{V}$. By definition of I^*, a unit increase in the mass of new fields invented at t triggers an increase in the research input exploiting these fields at $t + 1$ equal to $I^*(\bar{V})$. Since total

[13] We conjecture that the equilibrium is unique for θ small enough, but cannot prove it.

labor input is fixed, this represents a decrease in the research input devoted to invention at $t+1$. Recall, one unit of time produces γ papers in a new field. Therefore, a unit increase in new fields at t leads to a decrease in new fields at $t+1$ equal to $\gamma I^*(\bar{V})$. If $\gamma I^*(\bar{V}) > 1$, the initial effect is amplified and regime II dynamics are unstable. The economy eventually reverts to regime I, and cycles emerge. In contrast, the initial effect is attenuated if $\gamma I^*(\bar{V}) < 1$. In that case, stable regime II dynamics lead to a steady-state equilibrium. The quantity $I^*(\bar{V})$ represents the attractiveness of existing fields in regime II. We describe this process more formally, and explain how our proof is constructed, in the next section. In section 5, we study how the main parameters of the model affect the emergence of cycles.

4 Interpretation

To analyze the reason behind cycles, let us focus on the simpler case where $\theta = 0$. In the absence of a citation premium, inventors of new fields just get the intrinsic value of the field, ω, as a reward. Consequently, the value of a new field is pinned down and equal to $V = \gamma\bar{\omega}$ in any period.

Figure 1 plots the value of working in an existing field at any date t, λ_t, as a function of the total input in existing fields; that defines the LL schedule. This curve is downward-sloping, because of decreasing returns, captured by the $-\beta(\ln n_t(i) - \ln \bar{n})$ term in the utility function. For the same reason, its position is lower, the higher the initial advancement level of those fields, $n_{t-1}(i)$. Finally, given that level, its position is higher, the greater the mass of available fields μ_s, since the same total research input is now associated with a lower advancement level $n_t(i)$ in each field.

If, as is the case in Figure 1, that schedule intersects the horizontal line VV at $\lambda = V$, then the economy is in regime II. The horizontal distance AB determines the labor input into new fields, and hence the mass of fields being invented.

If that is not the case, then the economy must be in regime I, and equilibrium determination is illustrated in Figure 2. At date t, all researchers work in existing fields. Advancement in these fields generate a downward shift in LL, and the intercept of the LL schedule for the next period must be equal to λ_t – which simply means that the value of the first marginal paper at $t+1$ in a given field is equal to the value of the last paper written in that field at t. The process continues until the LL schedule cuts the VV schedule, in which case one is back to regime II (at $t+2$ in the case of Figure 2). This must happen in finite time, otherwise decreasing returns would eventually drive VV below the $\lambda = 0$ line. Note that the λ_ts fall during the regime I period. That is the reason why the set of fields being exploited grows during that phase.[14]

What happens, next, in regime II? At each date, a given mass of fields is invented. The greater that mass, the greater the value of exploiting these fields next period (i.e. LL shifts up). On average, one field invented at date t, with a quality distribution $f(\omega)$, triggers an amount $I^*(V)$ of research input devoted to exploiting that field at date $t+1$. That reduces the amount of time devoted to innovation: the greater the mass of fields invented today, the lower the mass of fields invented tomorrow. The evolution of μ_t, the mass of fields invented at t, evolves according to

$$\mu_t = \gamma(\upsilon - \mu_{t-1} I^*(V)). \tag{11}$$

If these dynamics are stable ($\gamma I^*(V) < 1$, Figure 3), then the economy converges to a steady state. Otherwise, ($\gamma I^*(V) > 1$, Figure 4), the economy

[14] Because of assumption A1, a field exploited during that phase must have been invented in the last period in regime II before the regime I phase. It enters regime I with an initial advancement level equal to $\bar{n}$. Using (1) with $\theta = 0$, it will therefore be exploited as soon as $\omega > \lambda_t$. Because the λs are falling, it will continue to be exploited until the economy reverts to regime II, when it becomes obsolete.

cannot remain in regime II forever: it will revert to regime I. As regime I itself cannot last forever, the two regimes must prevail alternatively.

The instability condition $\gamma I^*(V) > 1$ simply means that a unit of labor employed in inventing a new field today attracts more than one unit of labor into exploiting that field tomorrow. That in turn reduces the amount of labor inventing new fields tomorrow more than one for one, thus generating the explosive oscillatory dynamics and the subsequent exit from regime II. The greater the quantity $I^*(V)$, the more existing fields are attractive, and the more likely it is that cycles arise.

From these considerations, it is in easy to show that an equilibrium exists in the $\theta = 0$ case by using the following iterative procedure.

We start from a given inherited measure of invented fields at date s, μ_s (thus the economy was in regime II at date s). At date $t = s + 1$, we allocate all research to improving these fields, in an optimal way, i.e. so that the resulting advancement level of each field satisfies (2) – the marginal value of an extra contribution is common across all fields and equal to λ_t. Integrating the number of papers written across all active fields and using the resource constraints (3), this allows to solve for λ_t.

If the result is such that $\lambda_t < V$, it can be checked that we can construct a new period in regime II by simply applying (11) between dates s and $s+1$; and we can restart the procedure from date $s + 1$.

If the result is such that $\lambda_t > V$, we have constructed one period in regime I, and we can repeat the procedure for $t = s + 2$. That leads to a decreasing sequence of values of λ_t. The procedure is stopped when we get $\lambda_t < V$, in which case we are back to regime II at t and the number of invented fields is set according to (6). We then just apply (11) until the economy exits regime II, in which case we restart the procedure.

The basic principle behind the proof of Proposition 1 is to extend that strategy to the case where $\theta > 0$, it involves two essential ingredients: First, the λ_t's that intervene in the construction of regime I are substituted by a pseudo-shadow cost $\hat{\lambda}_t$ which reflects future citations. Second, the value of working in a new field V_t is no longer constant and a sophisticated continuity argument must be elaborated so as to prove that there exists an initial value of V_t which matches the equilibrium condition for the transition from regime I to regime II.

4.1 Numerical illustration

In this section we provide some simulations in order to get a better idea of the irregular nature of the innovation cycles. We assume that the quality of a field ω is drawn from a uniform distribution over $[0, \omega_u]$, implying $\bar{\omega} = \omega_u/2$. We stick to the $\theta = 0$ case.

Figures 5 to 10 report the simulation results for the following set of parameters: $\bar{n} = 2$; $\omega_u = 1; \beta = 0.3$; $\gamma = 0.7; \upsilon = 1$. The initial measure of existing fields was taken as $\mu_c = 1$.

It is easy to show that (9) holds in this case, so that the equilibrium must be cyclical. The simulation shows that the economy follows cycles that are irregular, both in the duration spent in regime I and the duration spent in regime II. The time spent in regime I oscillates between 1 and 2 periods (Fig. 5), while time spent in regime II oscillates between 1 and up to 6 periods (Fig. 8)[15]. There are also chaotic oscillations in the stock of new fields available for exploration at the beginning of each regime I phase (Figure 6). Furthermore, and that can be proved analytically[16], there is a tight positive connection between that initial stock and the length of the time spent in period 1 (Fig. 7); the regime I cycle lasts for 2 periods if the initial stock of knowledge is $>\approx 0.6$, and for 1 period otherwise.

[15] These figures report the 70 first cycles after the initial one.

[16] See equation (25) in the Appendix.

Figure 9 reports the average rate of innovation during the time spent in regime II. We see that it exhibits irregular fluctuations. We also see (Figure 10), that cycles where a longer time is spent in regime II, have a lower rate of innovation. Intuitively, if a large number of researchers produce new fields, it is more likely that the economy reverts to regime I in the following period in order to exploit the potential of these new fields[17].

Relative to that benchmark simulation, we can perform some exercices. Figures 11 and 12 report the structure of cycles when we reduce the decreasing returns parameter from $\beta = 0.3$ to $\beta = 0.2$.[18] We see that overall, the economy spends more time in regime I and less time in regime II. In a cycle, regime I last between 1 and 5 periods, although that is quite often just 1 period, and regime II typically does not exceed 2 periods, although there are very rare occurences of cycles where the economy spends 3 periods in regime II.

In fact, while there is a maximum duration for the regime I phase, if the dynamics are truly chaotic one will have (very rare) regime II phases of arbitrary length. The reason is that the initial values of μ will span all the $[0, \gamma v]$ interval, becoming sometimes arbitrarily close to the unstable steady state value $\bar{\mu}$.

We are now in a position to analyze how the parameters of interest affect the equilibrium.

[17] Another interesting property of that simulation, is that cycles where regime I lasts for two periods, are such that the economy only spends 1 period in regime II. The explanation could be as follows: at the end of such cycles, fields are quite exhausted, and the value of working in new fields in regime II is quite high. Thus a large mass of innovation will take place during a short period of time, after which people revert to exploiting the new fields.

However, this explanation is incomplete, since longer cycles are also those with a higher total initial potential. And that regularity is not robust to parameter changes.

[18] Given the richer results, simulation are reported over 140 cycles rather than 70.

5 Comparative statics and dynamics

In this section, we study how the equilibrium is affected by the main parameters of the model:

- The citation premium θ,
- The distribution of field quality $f(\omega)$; in particular: how does the riskiness of invention, measured by the variance of $f(.)$, affects the likelihood to obtain a cyclical equilibrium,
- The strength of decreasing returns β.

To do so, we study how these parameters affect the quantity $I^*(\bar{V})$. As discussed above, $I^*(\bar{V})$ measures the relative attractiveness of exploiting existing fields during regime II episodes. Proposition 1 shows that cycles are more likely to emerge when $I^*(\bar{V})$ is higher. Therefore, an increase in $I^*(\bar{V})$ means that cycles are more likely, and also that the steady state measure of invented fields ($\bar{\mu}$ in (10)) in regime II becomes smaller. Our results thus relate to both the likelihood of cycles and the equilibrium invention rate in steady state, when it exists.

5.1 The effect of the citation premium

Equation (7) implies that $\bar{V}$ is an increasing function of θ. Furthermore, one can check that $dI^*/d\bar{V} < 0$. Consequently,

PROPOSITION 2 – Cycles are less likely to emerge, the higher θ. Furthermore, $\bar{\mu}$ increases with θ.

This result is not totally obvious. In principle, the citation premium increases incentives to work both in new fields and in existing fields. However, in regime II, existing fields are only exploited during one period; thus one earns no citation premium on them. An increase in θ clearly decreases the value of working on existing fields in regime II, hence the chance of ever reaching regime I, as well as the steady state measure of invented fields.

5.2 The role of research uncertainty

Next, we look at the role of uncertainty in research; we want to know how the variance of ω – or any mean-preserving spread parameter denoted by σ – affects the arbitrage between working in new fields vs. existing fields in regime II. As we shall see, option values intervene in two conflicting ways.

We first note that $I^*(V)$ can be written as $\bar{n}E(z(\omega))$, where $z(\omega) = \max(e^{\frac{\omega(i)-V}{\beta}} - 1, 0)$ is a convex function of $\omega(i)$. By Jensen's inequality, a mean-preserving spread in the distribution of ω raises $I^*(V)$ for any given V. If $\bar{V}$ were to remain unchanged, or move by only a little, $I^*(\bar{V})$ would actually increase: more research uncertainty reduces the incentives to work in new fields.

If $\theta = 0$, it is actually true that $\bar{V}$ does not change in response to a mean-preserving change in the distribution of ω, for it is equal to $\gamma\bar{\omega}$. Similarly, by continuity, eq. (7) implies that for θ arbitrarily small the change in $\bar{V}$ can be made arbitrarily small. Thus we also expect a mean-preserving spread in ω to increase $I^*(V)$ as long as θ is not too large.

PROPOSITION 3 – Assume θ small enough, i.e.

$$\theta \leq \frac{\beta}{\gamma \int_0^{+\infty} (e^{\omega/\beta} - 1) f(\omega) d\omega}, \tag{12}$$

then

$$\frac{\partial I^*(\bar{V})}{\partial \sigma} > 0.$$

Therefore, a mean preserving spread in the distribution of ω reduces the equilibrium value of $\bar{\mu}$ and makes cycles more likely.

Uncertainty increases the value of existing fields because one can select those of them with the highest potential. A greater variance of ω means that it is more valuable to work in the top field, while the value of working in the bottom fields is unchanged, because these fields are abandoned anyway. In contrast, the value of writing the first paper in an unknown field is increased if the field turns out to be good, but reduced if it turns out to be bad – if θ is small, then that value will roughly equal $\omega(i)$, regardless of the fate of the field after its invention. Hence greater uncertainty increases the value to work in known fields relative to unknown, new fields in regime II.

Against that logic, runs the fact that uncertainty increases the value of new fields, because of the citation premium. That is apparent from (7): a mean-preserving spread in ω increases its RHS. The option value of working in an existing field only if it is good enough also affects the value of working in new fields through the citation premium. When uncertainty goes up, researchers gain from their good ideas being cited more, but do not lose from their bad, uncited ideas, being cited less. In other words, the higher the citation premium, the less risk-averse the researchers. If the condition in Proposition 3 holds, then that effect is smaller than the direct effect of a spread in ω. But if θ is larger, we can work out examples of mean-preserving spreads that raise the incentives to produce new fields, thus making cycles less likely and raising the equilibrium invention rate in steady state (see Appendix).

5.3 The role of decreasing returns

A similar trade-off appears regarding the effect of β. It is easy to see that $I^*(V)$ is a decreasing function of β. Given the value of writing a new paper V, existing fields lose their value more quickly. Researchers thus devote more time to invention, which makes the emergence of cycles less likely. As with uncertainty, an opposite effect comes into play through the citation premium.

Note that $\bar{V}$ is decreasing in β when $\theta > 0$. A higher β means that existing fields will be less exploited. This decreases the value of writing a paper in a new field (they will be cited less), which tends to counteract the first effect. We can again show that if the same condition (12) holds, then the first effect dominates:

PROPOSITION 4 – *Assume* (12) *holds. Then*

$$\frac{\partial I^*(\bar{V})}{\partial \beta} < 0.$$

Therefore, a rise in β raises the equilibrium value of $\bar{\mu}$ and makes cycles less likely.

5.4 Comparative dynamics on cycles

Given the highly nonlinear nature of our cycles, it is difficult to establish comparative dynamics properties. We provide two results, here. First, Proposition 5 gives some information on the impact of the citation premium on the structure of cycles. Typically, the casual idea that a greater citation premium makes "fads" more important and therefore cycles more likely, is not supported by the model. The reason is that the value of new fields goes up with the citation premium, which reduces the attractivity of existing fields, thus making it less likely that instability arises in (11). As the next section shows, however, a larger citation premium makes fads more likely in an indeterminacy sense.

PROPOSITION 5 – Conditional on the initial mass of invented fields, the economy spends less time in the regime I phase for $\theta > 0$ than for $\theta = 0$. Furthermore, if the amount of time spent in regime I is the same, then more invention takes place at the beginning of the subsequent regime II phase, if $\theta > 0$.

Second, we describe an example of a regular cycle. We saw in section 4 that cycles are usually irregular. Nonetheless, we can build regular cycles by specifying appropriate initial conditions. We can notably show that cycles of length 1 always exist if $\theta > 0$. Let V_1 be the solution to the following equation:

$$V_1 = \Phi(I^{*-1}(\frac{1}{2}(\frac{1}{\gamma} + I^*(V_1))) \tag{13}$$

Properties of I^* and Φ imply that this equation has a unique solution.

PROPOSITION 6 – Suppose that $\theta > 0$ and that $\gamma I^*(\bar{V}) > 1$. An equilibrium where a single period in regime II alternates with a single period in regime I always exists. In this equilibrium, the value of invention is constant and equal to V_1, the unique solution of (13). In addition, $V_1 < \bar{V}$. The mass of new fields invented in regime II is equal to

$$\mu = \frac{2\gamma\upsilon}{1 + \gamma I^*(V_1)}$$

In this short regular cycle, the value of invention is lower than at the steady-state. Interestingly, however, both equilibria possess similar comparative statics. An increase in θ leads to an increase in V_1 and to an increase in the mass of new fields invented. If θ is small enough, a mean-preserving spread in the distribution of ω or a decrease in β reduce innovation effort. Overall, these results confirm the idea that the citation premium θ has a positive impact on innovation.

6 Indeterminacy and "sunspots"

The greater θ, the more expectations about future citations have a strong effect on the decision to work on a given field. By analogy with the literature on indeterminacy, we can speculate that there are multiple equilibria for θ large enough. That is actually the case. The following result shows that there is local indeterminacy around the regime II steady state for large enough values of θ.

PROPOSITION 7 – Assume

$$\frac{\gamma\theta}{\beta}(1 - F(\bar{V})) > 1 \tag{14}$$

and

$$\gamma I^*(\bar{V}) < 1.$$

Then there exists a continuum of equilibria indexed by any initial value $V_0 = \bar{V} + v_t$, *for* v_t *sufficiently small.*

Condition (14) implies that the conditions for saddle-path stability in the dynamics of V_t in regime II are violated locally, so that the dynamical system $V_t = \Phi(V_{t+1})$ no longer has the steady state $\bar{V}$ as its unique non-explosive solution. As always, that is because the current valuation of invention is "too sensitive" to expectations about the future. Condition (14) reveals that that will be the case if research in new fields is productive (γ high), if the citation premium θ is high, if decreasing returns are not strong (β small), and if the fraction of new fields that are exploited next period is large ($1 - F(\bar{V})$ large).

If scientists think that opening new fields brings a higher payoff, they devote more effort to invention. The mass of papers in new fields is higher. This increases subsequent research in the best of these new fields. The citation premium originally associated to the new fields is thus effectively larger, which confirms the original expectation. In short, expecting invention to bring a high payoff can be a self-fulfilling prophecy.

7 Some welfare results

Due to the complexity of our model, it is not easy to make a thorough comparison between the equilibrium and the social optimum. However, it is possible to compare the steady state in regime II to its equivalent for the social planner. That is what we do in this section.

In order to perform a welfare analysis, a criterion is needed. There are many options since our model only specifies the value of innovation to researchers. An ample literature discusses the appropriability problems associated with research. Here we want to use our model to focus on only one market failure, which is that the stock of knowledge created by researchers is durable and will benefit future generations beyond their lifetime. We then show that absent a citation premium the value of a new field in the equilibrium steady state is lower than at the optimum steady state, and that an optimal "pigovian" citation premium can be introduced so as to induce the socially optimal level of fundamental. We provide a formula for computing this citation premium.

The social welfare function we use is as follows. At each date t there is a stock of knowledge K_t, which grows because of the introduction of new fields and because of improvements in existing fields. We assume that the increase in the stock of knowledge is equal to the intrinsic value of all papers written at date t. Thus, the intrinsic value perceived by each researcher captures well their contribution to the knowledge stock. Researchers only fail to internalize the fact that their contribution increases the stock of knowledge forever. They get rewards from the flow of ideas they produce while society gets rewards from the stock of ideas.

We capture that with an intertemporal social welfare function given by

$$SW = \sum_{t=0}^{+\infty} \frac{K_t}{(1+\phi)^t},$$

where K_t is the stock of knowledge at t and ϕ the social discount rate, which can conveniently be interpreted as an inverse measure of the weight put on future generations. The evolution of the knowledge stock is then given by, in regime II,

$$K_t = K_{t-1}+\mu_t \int \omega f(\omega)d\omega+\mu_{t-1} \int_\omega \left(\int_{\bar{n}}^{n_t(\omega)} (\omega - \beta(\ln z - \ln \bar{n}))dz \right) f(\omega)d\omega.$$

The first integral is the initial value of the fields invented at date t. The second integral is the contribution of the improvements made during t to the fields invented at $t-1$. Note that we integrate the marginal contribution of all papers ranked between $\bar{n}$ and n_t. This guarantees that researchers internalize the congestion externality they exert upon others by moving, through their contribution, the state of the field down the marginal value curve. In other words, the intrinsic value of writing a paper in a field with potential ω is equal to its marginal effect on K, $(\omega - \beta(\ln n_t - \ln \bar{n}))$.

This equation may be rewritten

$$K_t = K_{t-1} + \mu_t\bar{\omega} + \mu_{t-1} \int_\omega ((\omega + \beta)(n_t(\omega) - \bar{n}) - \beta n_t (\ln n_t - \ln \bar{n}))f(\omega)d\omega. \tag{15}$$

It can easily be shown that, as in the equilibrium, given the fraction of researchers who work in new fields, it is optimal to allocate the others so as to equate their intrinsic marginal value across active fields. Otherwise, one could reallocate the research effort across existing fields to get a higher value of the last term in (15). Consequently, at each date there exists a critical field ω_t^* such that $n_t(\omega) = \bar{n}$ for $\omega < \omega_t^*$ and $n_t(\omega) = \bar{n}e^{\frac{\omega-\omega_t^*}{\beta}}$ for $\omega \geq \omega_t^*$. In steady state, ω_t^* will be constant through time. Using this property, the evolution equation for knowledge can be rewritten as

$$K_t = K_{t-1} + \mu_t\bar{\omega} + \mu_{t-1}\Gamma(\omega_t^*),$$

with

$$\Gamma(\omega^*) = \bar{n} \int_{\omega^*}^{+\infty} \left[(\beta + \omega^*)\, e^{\frac{\omega-\omega^*}{\beta}} - (\beta + \omega) \right] f(\omega)d\omega.$$

The social planner's problem can be rewritten recursively by introducing the value function

$$V(\mu_{t-1}, K_{t-1}) = \max(K_t + \frac{1}{1+\phi} V(\mu_t, K_t)).$$

Maximization takes place with respect to x_t, the fraction of research allocated to new fields. We thus have

$$\mu_t = \gamma \upsilon x_t, \tag{16}$$

while the resource constraint allows to compute ω_t^* as a function of x. Aggregating the number of papers written in existing fields, we get

$$\upsilon(1-x) = \mu_{t-1} I^*(\omega_t^*). \tag{17}$$

PROPOSITION 8 – The steady-state, welfare maximizing value of ω_t^* is determined by the following equation:

$$\omega^* = \Psi(\omega^*),$$

where $\Psi(.)$ is a decreasing function defined by

$$\Psi(\omega^*) = \gamma\bar{\omega} + \frac{\bar{n}\gamma}{1+\phi} \int_{\omega^*}^{+\infty} \Delta(\omega - \omega^*) f(\omega) d\omega, \tag{18}$$

and where $\Delta(.)$ is a positive, increasing, convex function defined by $\Delta(x) = \beta(e^{x/\beta} - 1) - x$.

The critical level ω^* is the social opportunity cost of working in an existing field rather than a new field. Its equivalent in the analysis of the equilibrium is V_t, which is equal to $\bar{V}$, the fixed point of Φ in the equilibrium. Furthermore, (4) and (5) show that a market economy will allocate employment across existing fields in exactly the same way as the social optimum if $\bar{V} = \omega^*$. Since the resource constraints (16) and (17) are the same in the equilibrium case and the optimum case, all that is needed to compare the equilibrium with the optimum is to compare the fixed point of Φ with that of Ψ. If they coincide, then the equilibrium steady state is identical to the social optimum steady state. Confronting (7) with (18) we then get that the two fixed points coincide provided the citation premium is equal to

$$\theta^* = \frac{\bar{n}\beta}{1+\phi} \frac{\int_{\omega^*}^{+\infty} \Delta(\omega - \omega^*) f(\omega) d\omega}{\int_{\omega^*}^{+\infty} (\omega - \omega^*) f(\omega) d\omega}.$$

This citation premium goes down with ϕ, which means that it must be higher when the social planner cares more about future generations.[19] That is because the social planner puts more weight on subsequent improvements of a new field, the lower ϕ. The value of these subsequent improvements– which raises the value of a new field beyond its contemporaneous effect $\bar{\omega}$– is internalized by the inventor only through the citation premium. Thus it must go up when ϕ goes down.

8 Conclusion

This paper has developed a simple model of the allocation of effort between fundamental research, which invents new fields, and applied research, which improves existing fields. Despite the model's simplicity, our results are quite rich.

We were able to characterize the cyclical dynamics of the economy and derive a necessary and sufficient condition for cycles to arise. We have shown that indeterminacy may also appear, and that the citation premium makes the equilibrium less cyclical, but at the same time makes indeterminacy more likely.

[19] To see this, simply rewrite (8) as $\bar{V} = \Phi(\bar{V};\theta)$, $\Phi_1' < 0$, $\Phi_2' > 0$, and (18) as $\omega^* = \Psi(\omega^*, \phi)$, $\Psi_1' < 0, \Psi_2' < 0$. The welfare maximizing value of θ, θ^*, is the unique solution to $\omega^* = \Phi(\omega^*;\theta)$. Hence, $\partial\theta^*/\partial\omega^* > 0$. Since ω^* falls with ϕ, so does θ^*.

We have also established some comparative statics for a steady-state in regime II, and to compare this steady state to the welfare optimum. We were able to highlight the role of the option value in determining the optimal and equilibrium allocation of research between the two activities.

9 Appendix:

9.1 Proof of proposition 1: roadmap

We define λ_t as the shadow cost of a paper at t. The optimality conditions imply that at date t, a field i is exploited if and only if

$$\omega(i) - \beta(\ln n_{t-1}(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_{t-1}(i)) < \lambda_t,$$

in which case n_t is determined by

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t.$$

The proof then follows the following steps. The details are precisely described in a separate Appendix available from the authors upon request. Here we focus on the essence of the reasoning.

A. First, one proves that one cannot remain forever in regime I. That is based on the following idea: to only exploit new fields forever, despite that each field has decreasing returns, one must be compensated by a "citation bubble": that is, despite that the intrinsic value of a contribution falls without bounds, one is compensated by future citations, because more papers will be written in the future. But then, the speed at which new papers are written in any field must accelerate, since the intrinsic value keeps falling. In the end, the economy resource constraint is violated, which is a contradiction.

B. Next, we are able to characterize the dynamics when the economy is in regime I. We can construct a sequence $\hat{\lambda}_t$ of pseudo-shadow costs, which reflect both the shadow cost λ_t and the value of future citations. The optimality condition can then be expressed by comparing the pseudo-shadow cost with the intrinsic value of the paper, as summarized by property P1:

PROPERTY P1 – A field is active iff $\omega(i) - \beta(\ln n_{t-1} - \ln \bar{n}) > \hat{\lambda}_t$, in which case $\omega(i) - \beta(\ln n_t - \ln \bar{n})) = \hat{\lambda}_t$.

Denoting by T the date at which regime I ends, T_0 the date at which it starts, we show that the $\hat{\lambda}_t$'s can be constructed recursively:

$$\begin{aligned}\hat{\lambda}_T &= V_T; \\ \hat{\lambda}_t &= \min(\frac{\theta\hat{\lambda}_{t+1} + \beta\lambda_t}{\theta + \beta}, \lambda_t).\end{aligned}$$

Property P1 allows to compute, as a function of $\hat{\lambda}_t$, the set of fields that are researched at t, as well as n_t for each of those fields. The lower $\hat{\lambda}_t$, the larger that set and the larger n_t, so that total research in existing field is a decreasing function of $\hat{\lambda}_t$. Since one is in regime I, all researchers work in existing fields. There is a unique value of $\hat{\lambda}_t$ which satisfies that full-employment condition. This determines the evolution of $\hat{\lambda}_t$ during regime I. Denoting by μ_{T_0-1} the measure of exploitable fields, we get:

$$\mu_{T_0-1}I^*(\hat{\lambda}_{T_0}) = \upsilon. \tag{19}$$

$$\mu_{T_0-1}(I^*(\hat{\lambda}_{t+1}) - I^*(\hat{\lambda}_t)) = \upsilon. \tag{20}$$

Finally, one is in regime II as long as the value of working on a new field is lower than the shadow cost of a paper. We can show that that is equivalent to

$$\Phi(\hat{\lambda}_{t+1}) < \lambda_t. \tag{21}$$

$\hat{\lambda}_{t+1}$ appears in that condition because the lower it is, the more likely the seminal paper written today will be cited tomorrow.[20]

[20] As the new field is infinitesimal, it does not make existing fields obsolete, and the researcher who considers working on a new field assumes that his invention has no impact on the regime prevailing at $t+1$ and on $\hat{\lambda}_{t+1}$.

C. In regime II, people produce both new fields and papers in existing fields. V_t is both the shadow cost of an additional paper in an existing field and the expected value of starting a new field. The value of working in a new field depends on its expected intrinsic value, plus the contribution of next period's citations. The latter is larger, the larger the number of papers that will be written in existing fields at $t + 1$, which is a decreasing function of V_{t+1}. This allows to derive the following recursive relationship:

$$V_t = \Phi(V_{t+1}).$$

Using similar computations as in regime I, we can get the fraction of researchers in existing fields:

$$\upsilon_{At+1} = I^*(V_{t+1})\mu_t.$$

We then subtract it to get the labor input into new fields, which gives us the measure of new fields next period:

$$\mu_{t+1} = \gamma(\upsilon - I^*(V_{t+1})\mu_t). \tag{22}$$

One can show that if $\bar{V}$ is the fixed point of Φ, and if

$$I^*(\bar{V})\gamma > 1, \tag{23}$$

then the dynamics of μ_t in regime II cannot be stable, so that one must eventually leave regime II.

D.E.F. We can now work out the transitions between the two regimes. If we know the terminal value V_T and the initial measure μ_{T_0-1}, we can construct a decreasing sequence $\hat{\lambda}_t$ by applying (19) and (20). It is easy to see that the transition takes place at T such that $\hat{\lambda}_{T-1} > V_T > \hat{\lambda}_T$. That in turn allows to compute the allocation of researchers between existing fields and new fields at T, and thus the measure of new fields invented at T :

$$\mu_T = \gamma(\upsilon - \mu_{T_0-1}(I^*(V_T) - I^*(\hat{\lambda}_{T-1}))). \tag{24}$$

As a corollary, the duration of the phase in regime I is given by[21]

$$T - T_0 = INT(\frac{\mu_{T_0-1}}{\upsilon} I^*(V_T)). \tag{25}$$

As for the transition from II to I, we get an extra condition by noting that the value of a new field at $T_0 - 1$ is equal to $\Phi(\hat{\lambda}_{T_0})$, while (20) must hold. This gives us

$$V_{T_0-1} = \Phi(I^{*-1}\left(\frac{\upsilon}{\mu_{T_0-1}}\right)). \tag{26}$$

G. We can finally construct an equilibrium which matches the conditions we have derived above. The first step consists in constructing the $m(.,.)$ function, which allows to compute μ_t in regime II as a function of μ_s, the inherited measure of fields from the last period in regime II, and V_t, the current value of a job. If $t = s+1$, then $m(.,.)$ is defined as the RHS of (22). Otherwise, there is a period in regime I between s and t, and we can use the steps in D.E.F. to get[22]

$$\mu_t = \gamma\upsilon(1 - DEC\left(\frac{\mu_s}{\upsilon} I^*(V_t)\right)) = m(\mu_s, V_t), \tag{27}$$

of which (22) is a special case.

The next step is to determine the equilibrium value of V_t. It must be such that the terminal condition (26) at the end of the regime II (i.e. with T instead of $T_0 - 1$ in the formula) holds. We do that using a continuity-type argument, which must be worked out carefully as the final measure μ_T is not a continuous function of V_t. If (23) holds, we can construct the last date in the current regime T, which satisfies (26).

[21] $INT(x)$ is the largest integer number y such that $y \leq x$.

[22] $DEC()$ is the decimal part of a number $DEC(x) = x - INT(x)$.

Therefore, given an inherited measure μ_s, we can construct a full cycle in regime II, which may or may not be preceded by a period in regime I. The procedure can be repeated at the end of that cycle, using the new measure of fields μ_T. The only equilibrium condition that remains to be checked is (21) in Regime I, which we do.

H. Finally, if (23) does not hold, we can construct an infinitely lived path in regime II by picking $V = \bar{V}$ throughout and let the economy evolve according to (22).

I. In the case where $\theta = 0$, in regime II one must always have $V_t = \bar{V} = \gamma\omega$, which allows to prove uniqueness.

9.2 Proof of Proposition 3

Consider a change in the distribution $f()$ denoted by $\Delta f()$. The implied shifts in $\bar{V}$ and $I^*(\bar{V})$ satisfy

$$\Delta I^* = -\frac{\bar{n}\Delta\bar{V}}{\beta}\int_{\bar{V}}^{+\infty} e^{\frac{\omega-\bar{V}}{\beta}} f(\omega)d\omega + \bar{n}\int_{\bar{V}}^{+\infty}\left(e^{\frac{\omega-\bar{V}}{\beta}}-1\right)\Delta f(\omega)d\omega, \qquad (28)$$

and

$$\Delta\bar{V} = -\frac{\gamma\theta}{\beta}\Delta\bar{V}(1-F(\bar{V})) + \frac{\gamma\theta}{\beta}\int_{\bar{V}}^{+\infty}\left(\omega-\bar{V}\right)\Delta f(\omega)d\omega. \qquad (29)$$

By Jensen's inequality, it must be that if $\Delta f(\omega)$ is a mean-preserving spread, then

$$\int_{\bar{V}}^{+\infty}\left(e^{\frac{\omega-\bar{V}}{\beta}}-1\right)\Delta f(\omega)d\omega > \int_{\bar{V}}^{+\infty}\frac{\omega-\bar{V}}{\beta}\Delta f(\omega)d\omega > 0,$$

since the functions $g_1(\omega) = \max(\frac{\omega-\bar{V}}{\beta}, 0)$ and $\exp(g_1(\omega)-1) - g_1(\omega)$ both are convex. Furthermore, eliminating $\Delta\bar{V}$ between (28) and (29) we see that $\Delta I^* > 0$ if and only if

$$\frac{\gamma\theta\int_{\bar{V}}^{+\infty} e^{\frac{\omega-\bar{V}}{\beta}} f(\omega)d\omega}{\beta+\gamma\theta(1-F(\bar{V}))} < \frac{\beta\int_{\bar{V}}^{+\infty}\left(e^{\frac{\omega-\bar{V}}{\beta}}-1\right)\Delta f(\omega)d\omega}{\int_{\bar{V}}^{+\infty}\left(\omega-\bar{V}\right)\Delta f(\omega)d\omega}. \qquad (30)$$

Since the RHS is greater than 1, that inequality will be satisfied if the LHS is lower than 1, which is equivalent to

$$\int_{\bar{V}}^{+\infty}\left(e^{\frac{\omega-\bar{V}}{\beta}}-1\right)f(\omega)d\omega<\frac{\beta}{\gamma\theta}.$$

Since the LHS decreases with $\bar{V}$, it reaches its maximum at $\bar{V}=0$, and that inequality will therefore always hold if (12) holds. Q.E.D.

9.2.1 Counter-example

We now construct a counter-example where $\Delta I^*<0$. We assume ω is uniformly distributed over [0,2], $\gamma=1$, and $\theta=8\beta$. That implies $\bar{\omega}=1$ and it can be checked that $\bar{V}=3/2$. The LHS of (30) is then equal to $\frac{4}{3}(e^{1/(2\beta)}-1)$. We consider a specific mean-preserving spread whose only action above $\bar{V}$ is to add a finite (Dirac) mass at some $\tilde{\omega}>\bar{V}$. Condition (30) will then be violated iff

$$\frac{4}{3}(e^{\frac{1}{2\beta}}-1)>\frac{\beta(e^{\frac{\tilde{\omega}-3/2}{\beta}}-1)}{\tilde{\omega}-3/2}. \tag{31}$$

The RHS rises from 1 to $2\beta(e^{\frac{1}{2\beta}}-1)$ as $\tilde{\omega}$ increases from 3/2 to 2. As long as $\beta<(2\ln\frac{7}{4})^{-1}\approx 0.893$, the LHS is greater than 1 and the condition holds for $\tilde{\omega}$ not too above 3/2. Furthermore, if $\beta<2/3$, then (31) holds for $\tilde{\omega}=2$ and therefore for any $\tilde{\omega}\in[3/2,2]$.

9.3 Proof of Proposition 4

We take the same steps as in Prop. 3, but with respect to a change in β. We now get

$$dI^*=-\frac{\bar{n}d\bar{V}}{\beta}\int_{\bar{V}}^{+\infty}e^{\frac{\omega-\bar{V}}{\beta}}f(\omega)d\omega-\frac{\bar{n}}{\beta^2}\int_{\bar{V}}^{+\infty}e^{\frac{\omega-\bar{V}}{\beta}}\left(\omega-\bar{V}\right)f(\omega)d\omega.d\beta,$$

and

$$d\bar{V}=-\frac{\gamma\theta}{\beta}d\bar{V}(1-F(\bar{V}))-\frac{\gamma\theta}{\beta^2}\int_{\bar{V}}^{+\infty}\left(\omega-\bar{V}\right)f(\omega)d\omega.d\beta.$$

We get that $dI^* < 0$ iff

$$\frac{\gamma\theta \int_{\bar{V}}^{+\infty} e^{\frac{\omega-\bar{V}}{\beta}} f(\omega)d\omega}{\beta + \gamma\theta(1-F(\bar{V}))} < \frac{\int_{\bar{V}}^{+\infty} e^{\frac{\omega-\bar{V}}{\beta}} \left(\omega - \bar{V}\right) f(\omega)d\omega}{\int_{\bar{V}}^{+\infty} \left(\omega - \bar{V}\right) f(\omega)d\omega}.$$

The LHS is the same as in (30), while the RHS is always greater than 1. Therefore, that inequality again holds if (12) holds. Q.E.D.

9.4 Proof of Proposition 5

(i) Recall, $\bar{V}$ is increasing in θ and I^* is decreasing. Thus, if the condition $I^*(\bar{V}) < \frac{1}{\gamma}$ is satisfied for θ, it is also satisfied for $\theta' > \theta$.

(ii) By virtue of (25), the duration of the first regime I phase is $INT(I^*(\gamma\bar{\omega})\frac{\mu_{-1}}{\upsilon})$ when $\theta = 0$ and $INT(I^*(V_T)\frac{\mu_{-1}}{\upsilon})$ when $\theta > 0$. Since $V_T \geq \gamma\bar{\omega}$, the first regime I phase is necessarily longer when $\theta > 0$. Examining the equation for μ gives the second result. Q.E.D.

9.5 Proof of Proposition 6

Consider a profile where a single regime I period alternates with a single regime II period, and where the mass of new fields invented is constant and equal to μ while the value of invention is constant and equal to V. This is an equilibrium if and only if the four following conditions are satisfied

$$\begin{aligned} 1 &< \frac{\mu}{\upsilon}I^*(V) < 2 \\ \mu &= \gamma\upsilon(1 - DEC(\frac{\mu}{\upsilon}I^*(V))) \\ V &< I^{*-1}(\frac{\upsilon}{\mu}) \\ V &= \phi(I^{*-1}(\frac{\upsilon}{\mu})) \end{aligned}$$

Observe that the first condition implies the third. Since $INT(\frac{\mu}{\upsilon}I^*(V)) = 1$, the second condition becomes $\mu = 2\gamma\upsilon/[1+\gamma I^*(V)]$. Substituting into the fourth condition yields

$$V = \Phi(I^{*-1}(\frac{1}{2}(\frac{1}{\gamma} + I^*(V)))$$

Recall, Φ takes value in $\left[\gamma\bar{\omega}, \gamma\bar{\omega}(1+\frac{\theta}{\beta})\right]$. Since Φ and I^* are decreasing, this equation has a unique solution. Rearranging, the first condition becomes $1 < 2\frac{\gamma I^*(V)}{1+\gamma I^*(V)} < 2$, which is equivalent to $I^*(V) > 1/\gamma$.

Next, recall $I^*(\bar{V}) > 1/\gamma$. This means that $I^*(\bar{V}) > \frac{1}{2}(\frac{1}{\gamma} + I^*(\bar{V}))$. Thus, $\bar{V} < I^{*-1}(\frac{1}{2}(\frac{1}{\gamma} + I^*(\bar{V})))$ and $\Phi(\bar{V}) = \bar{V} > \Phi(I^{*-1}(\frac{1}{2}(\frac{1}{\gamma} + I^*(\bar{V}))))$. This implies that $V < \bar{V}$ and $I^*(V) > I^*(\bar{V}) > 1/\gamma$. Q.E.D.

9.6 Proof of Proposition 7

To prove Proposition 7, just differentiate the dynamics of V_t in regime II, $V_t = \Phi(V_{t+1})$, around the fixed point $\bar{V}$. Denoting by $v_t = V_t - \bar{V}$, we get

$$v_t = -\frac{\gamma\theta}{\beta}(1 - F(\bar{V}))v_{t+1}.$$

If $\frac{\gamma\theta}{\beta}(1 - F(\bar{V})) > 1$, then we can construct an equilibrium for any initial value of v_t. Q.E.D.

9.7 Proof of Proposition 8

The first order condition for maximization of the value function with respect to x is

$$\begin{aligned} 0 &= \left(\bar{\omega}\frac{\partial \mu_t}{\partial x} + \mu_{t-1}\Gamma'(\omega^*)\frac{\partial \omega^*}{\partial x}\right)\left(1 + \frac{1}{1+\phi}\frac{\partial V(\mu_t, K_t)}{\partial K}\right) \\ &\quad + \frac{1}{1+\phi}\frac{\partial \mu_t}{\partial x}\frac{\partial V(\mu_t, K_t)}{\partial \mu_t}. \end{aligned} \tag{33}$$

The resource constraints allow us to compute the following derivatives:

- $\frac{\partial \mu_t}{\partial x} = \gamma\upsilon$

- $\frac{\partial \omega^*}{\partial x} = -\frac{\upsilon}{I^{*\prime}(\omega^*)\mu_{t-1}\bar{n}}$

  $\frac{\partial \omega^*}{\partial \mu_{t-1}} = -\frac{I^*(\omega^*)}{I^{*\prime}(\omega^*)\mu_{t-1}}.$

Differentiating the value function while ignoring the changes in x because of the envelope theorem allows to compute the following:

- $\frac{\partial V}{\partial K} = \frac{1+\phi}{\phi}.$
- $\frac{\partial V(\mu_{t-1},K_{t-1})}{\partial \mu_{t-1}} = \left(\Gamma(\omega^*) + \mu_{t-1}\Gamma'(\omega^*)\frac{\partial \omega^*}{\partial \mu_{t-1}}\right)\frac{1+\phi}{\phi}.$

Substituting these formulas into (33) while making use of the steady state assumption, we get

$$\begin{aligned} 0 \quad = \quad & \left(\bar{\omega}\gamma\upsilon - \Gamma'(\omega^*)\frac{\upsilon}{I^{*\prime}(\omega^*)\bar{n}}\right) \\ & + \frac{1}{1+\phi}\gamma\upsilon\left(\Gamma(\omega^*) - \Gamma'(\omega^*)\frac{I^*(\omega^*)}{I^{*\prime}(\omega^*)}\right). \end{aligned} \tag{34}$$

To get to (18), compute the derivatives of Γ and I^* :

- $\Gamma'(\omega^*) = -\bar{n}\frac{\omega^*}{\beta}\int_{\omega^*}^{+\infty} e^{\frac{\omega-\omega^*}{\beta}} f(\omega)d\omega;$
- $I^{*\prime}(\omega^*) = -\frac{1}{\beta}\int_{\omega^*}^{+\infty} e^{\frac{\omega-\omega^*}{\beta}} f(\omega)d\omega.$

Replace all the terms in $\Gamma'(\omega^*)/I^{*\prime}(\omega^*)$ in (34) by ω^*, and replace the term in $\Gamma(\omega^*)$ by the following expression (it can be checked that it is indeed equal to $\Gamma(\omega^*)$):

$$(\beta + \omega^*)I^*(\omega^*) - \bar{n}\int_{\omega^*}^{+\infty} (\omega - \omega^*) f(\omega)d\omega.$$

These operations yield equation (18). Q.E.D.

10 References

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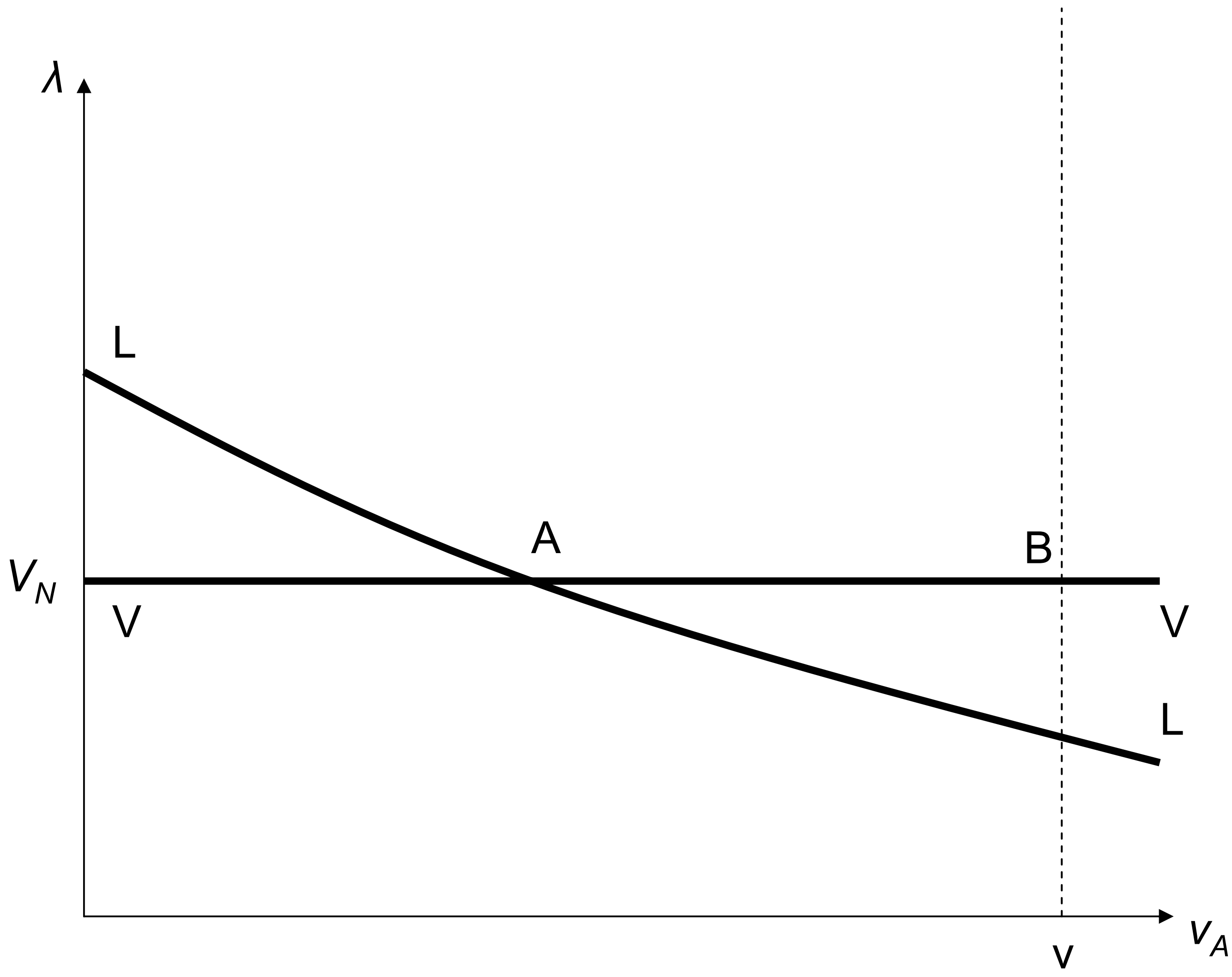


Figure 1 – Equilibrium determination in regime II

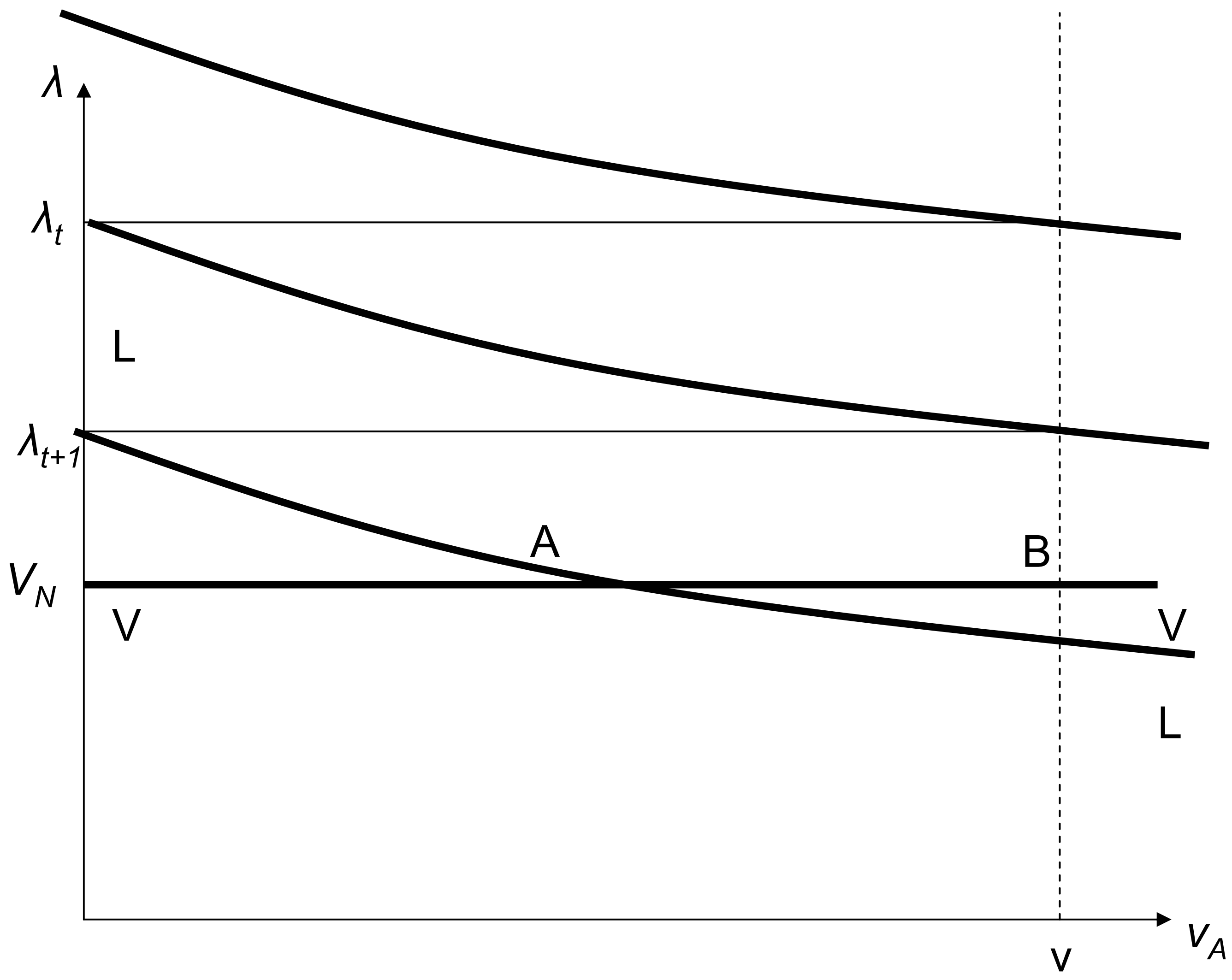


Figure 2 – Equilibrium determination in regime I

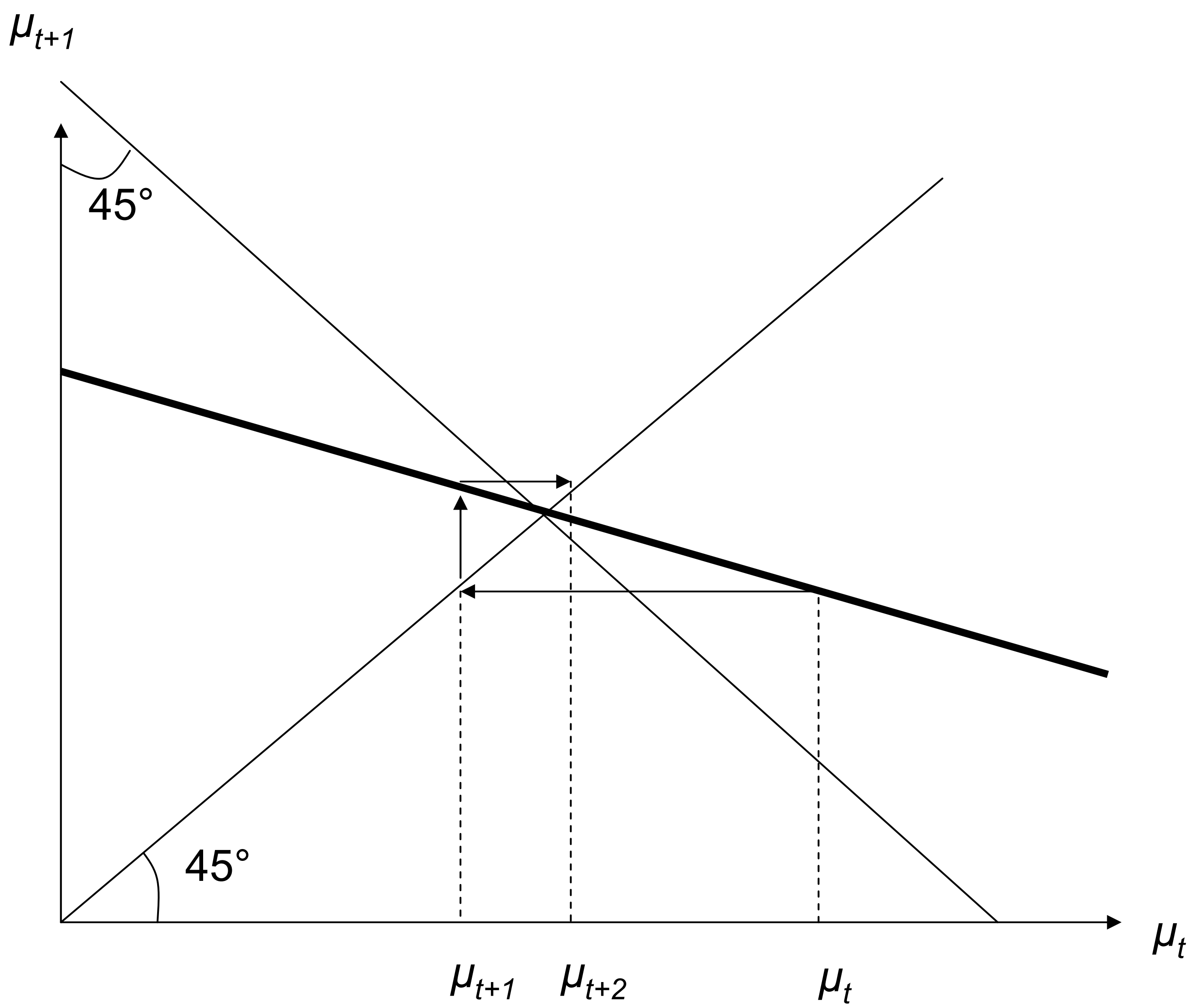


Figure 3 – Convergence to the regime II steady state

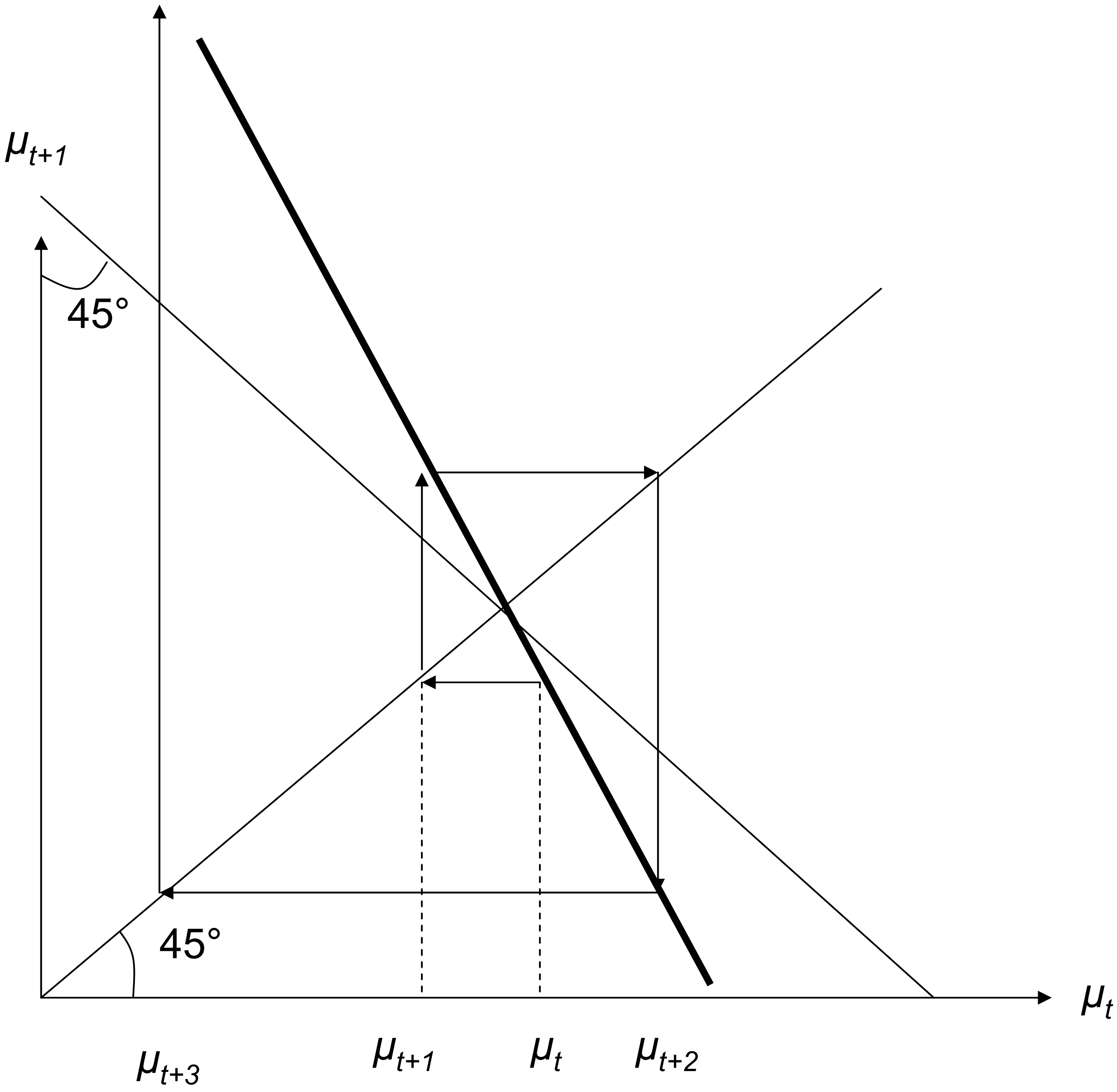


Figure 4 – The economy eventually leaves regime II

Fig. 5: time in I per cycle

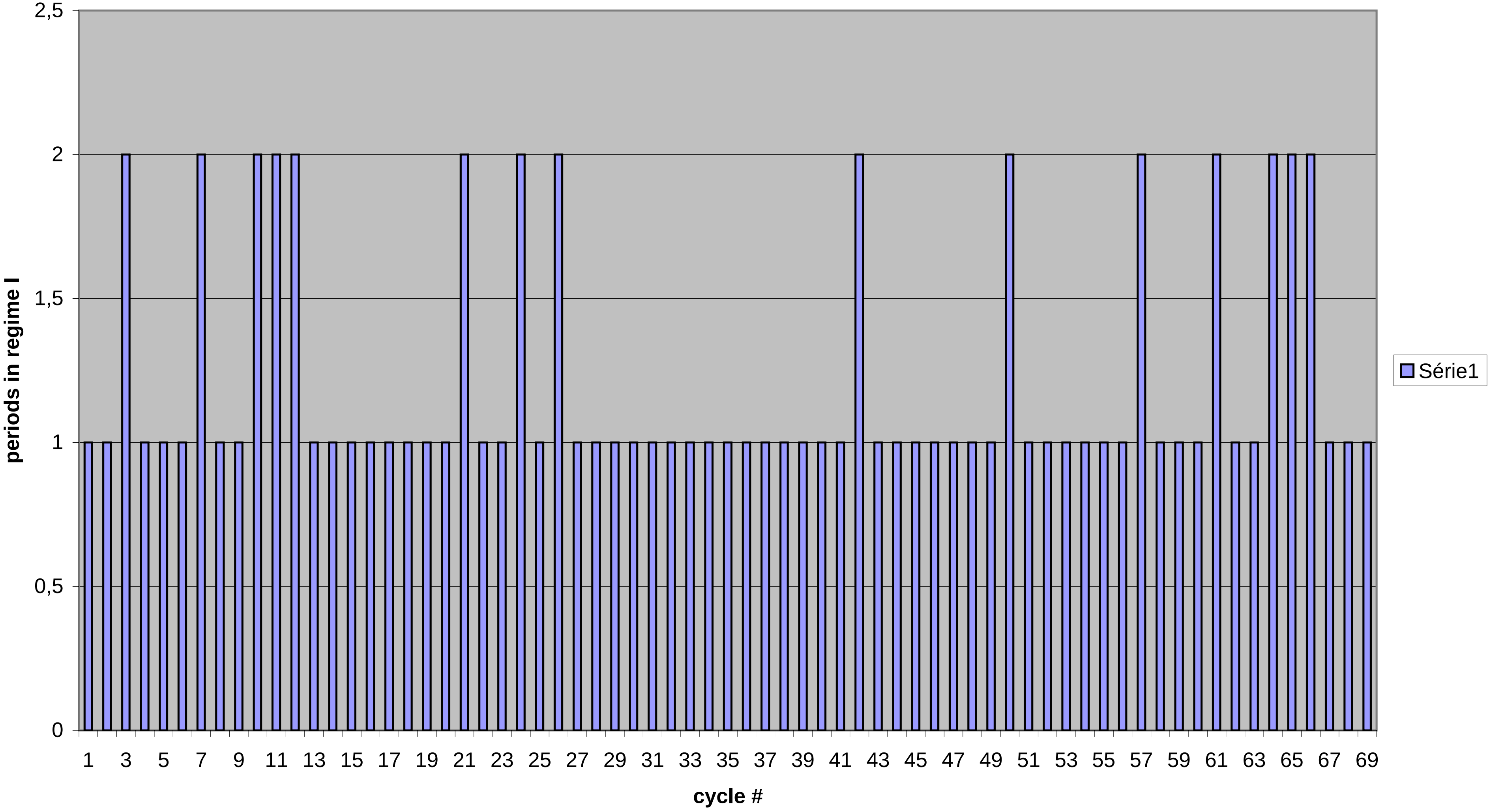


Fig 6: mass of new fields per cycle

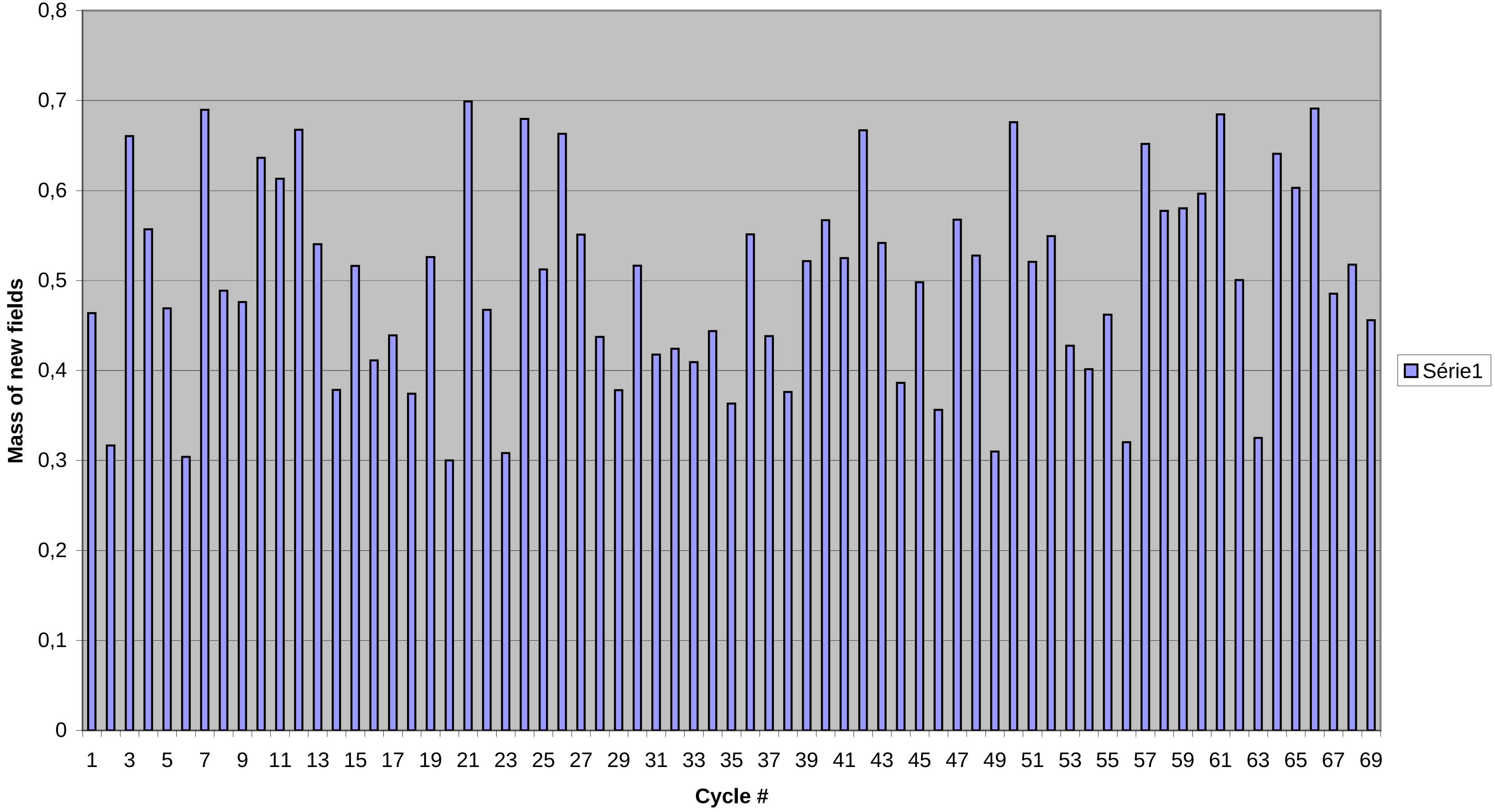


Fig. 7 cycle length and mass of new fields

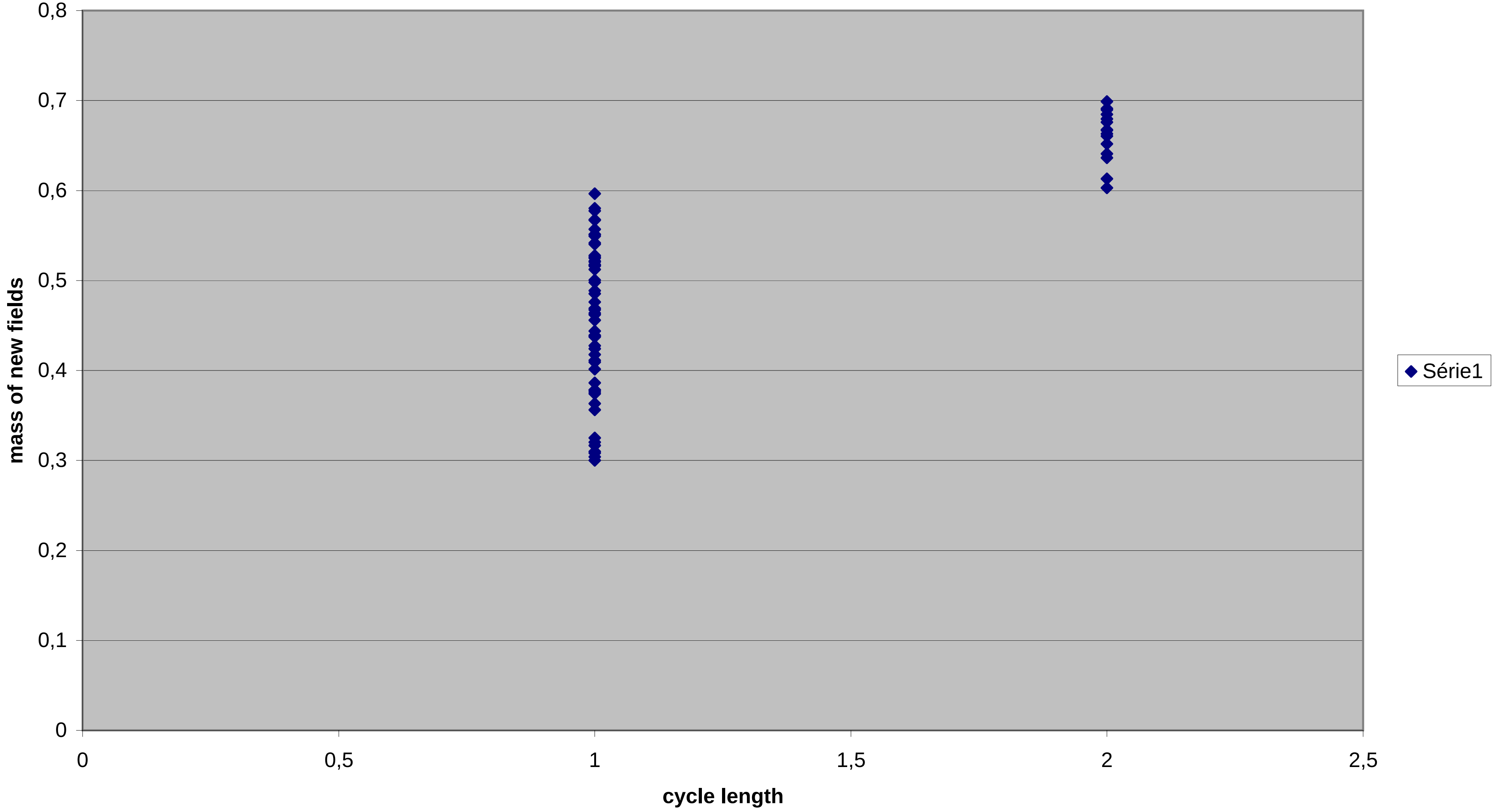


Fig. 8 time spent in regime II

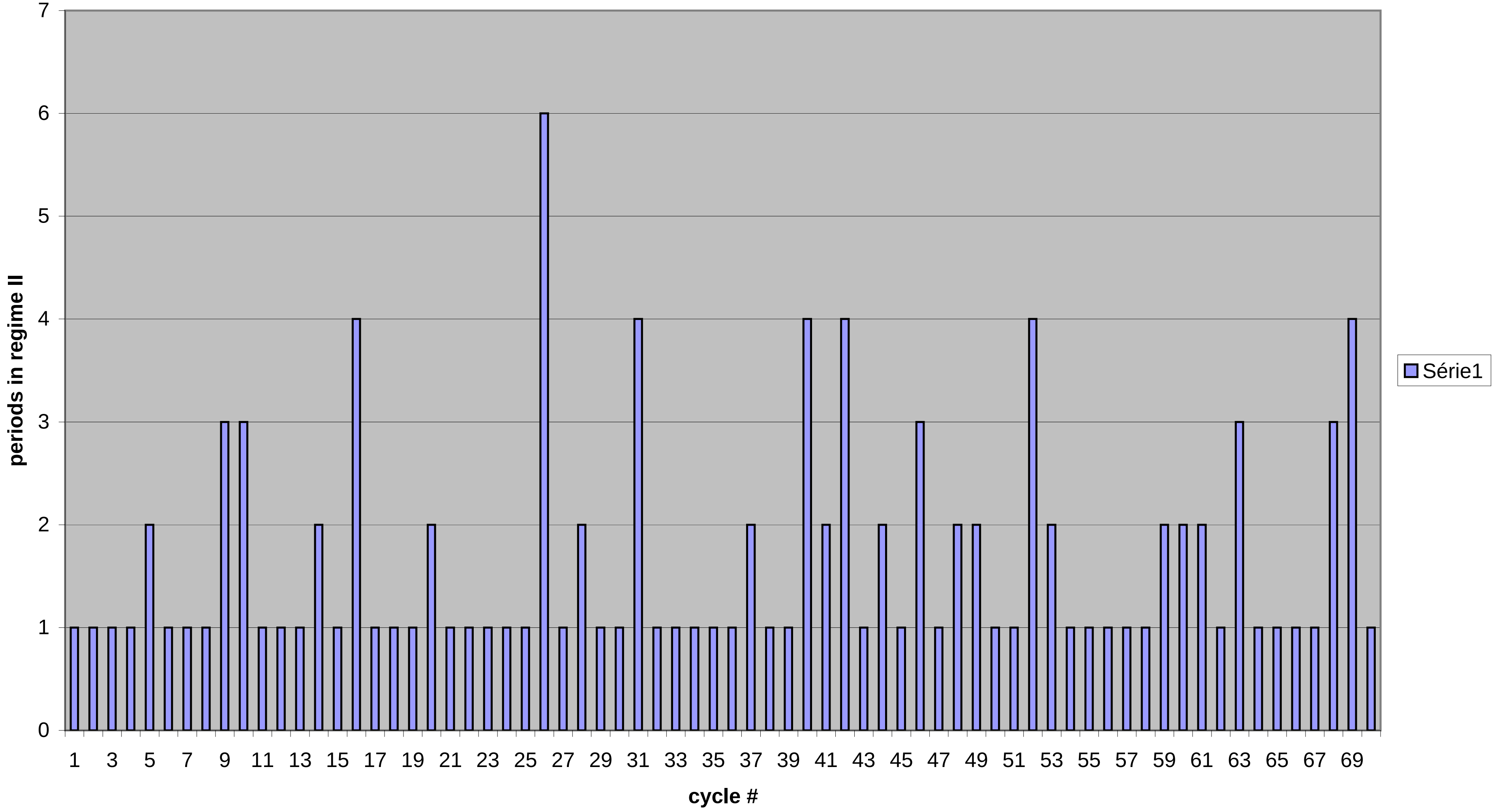


Fig. 9: Average production of new fields in regime II per cycle

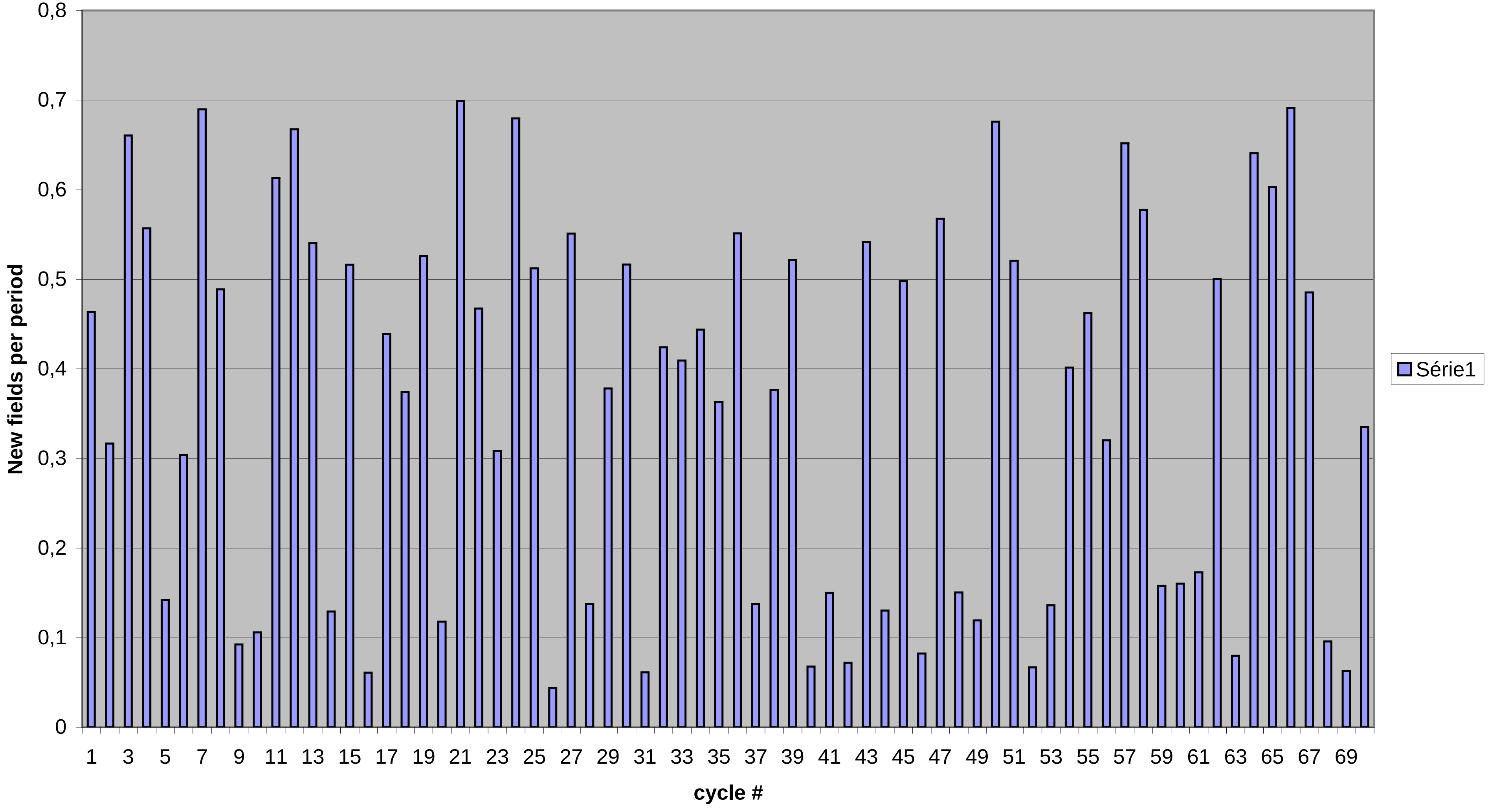


Fig. 10: Time in regime II and average innovation

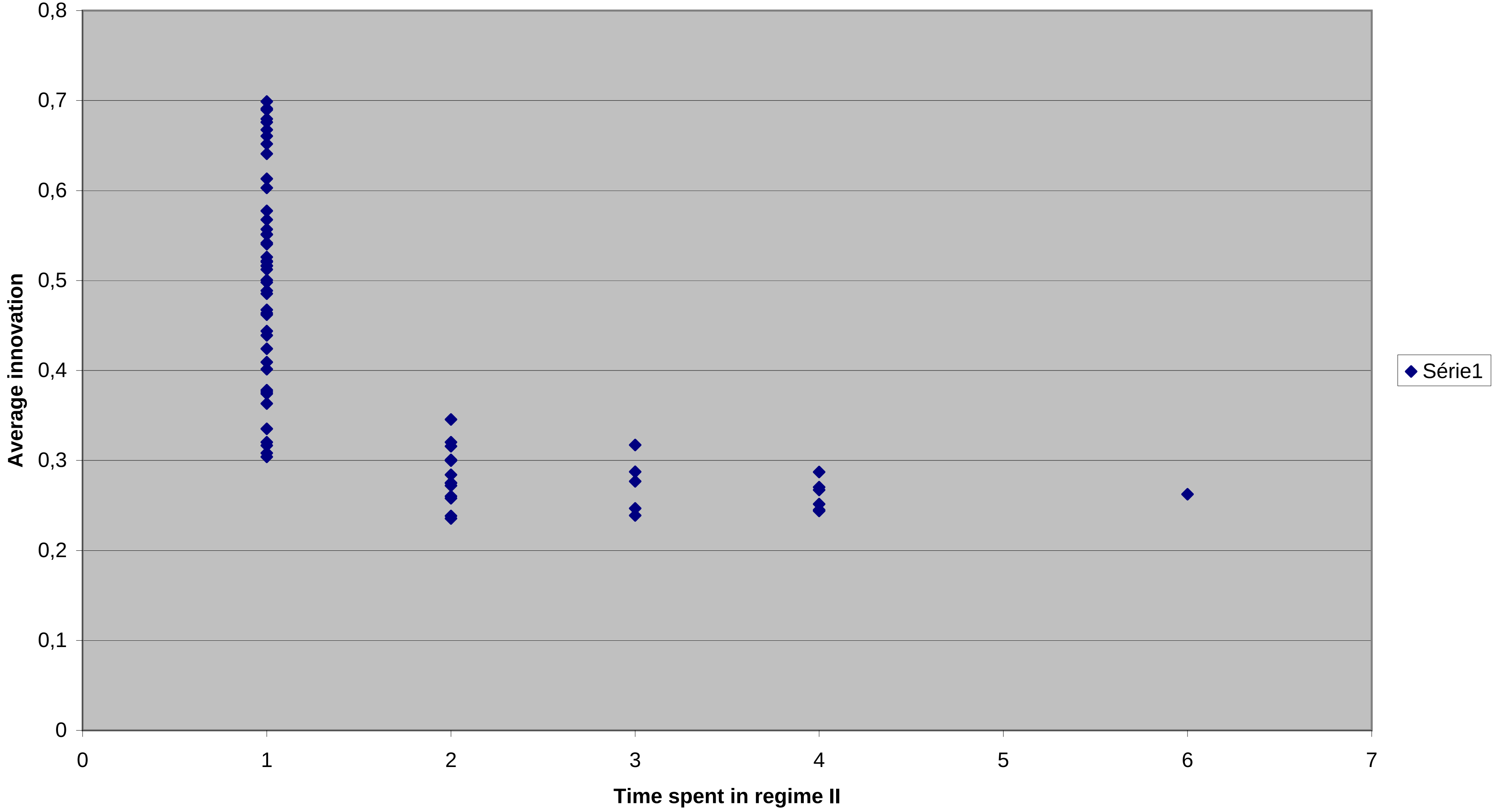
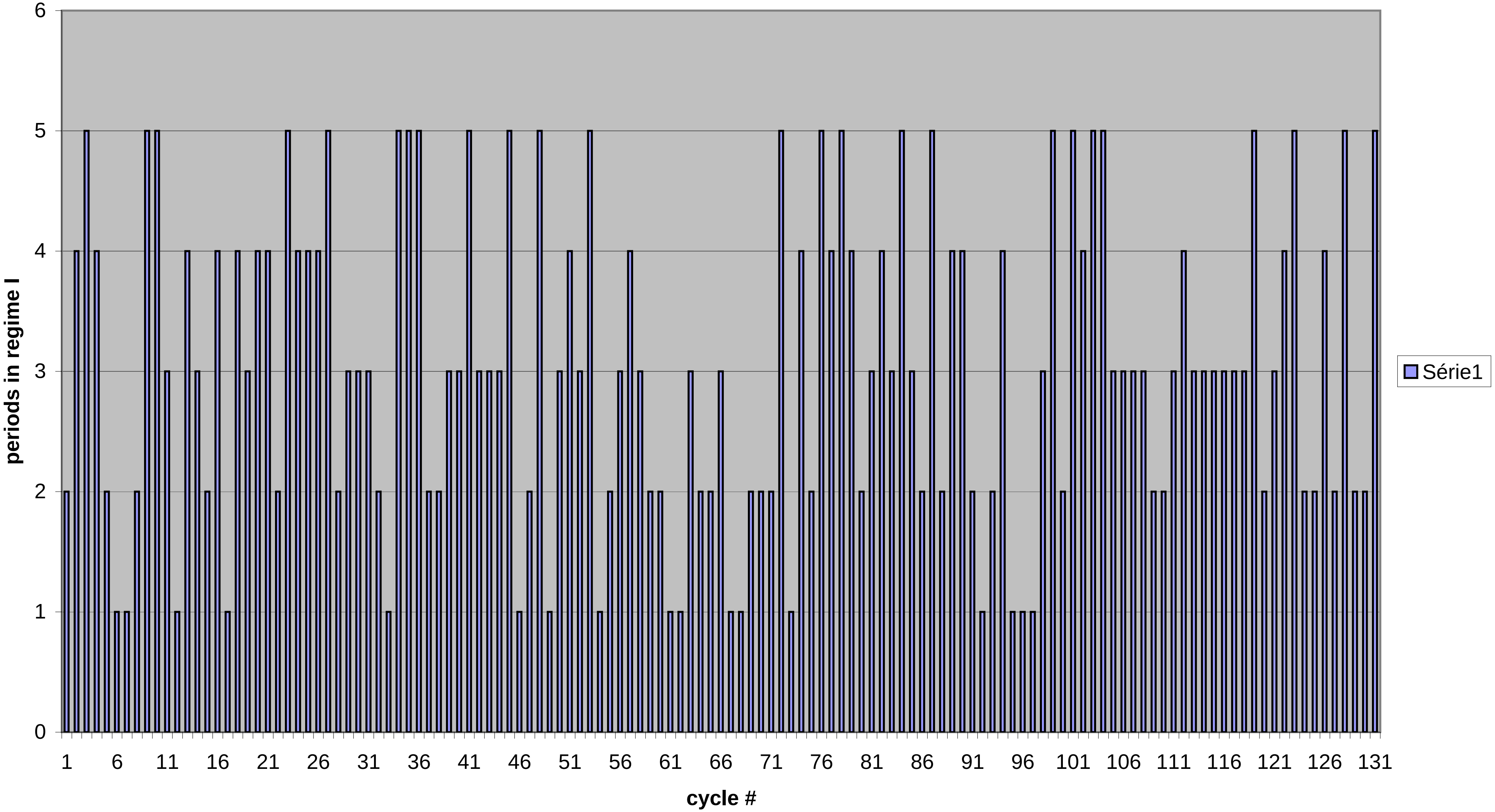


Fig. 11: cycle duration in reg. I, beta = 0.2



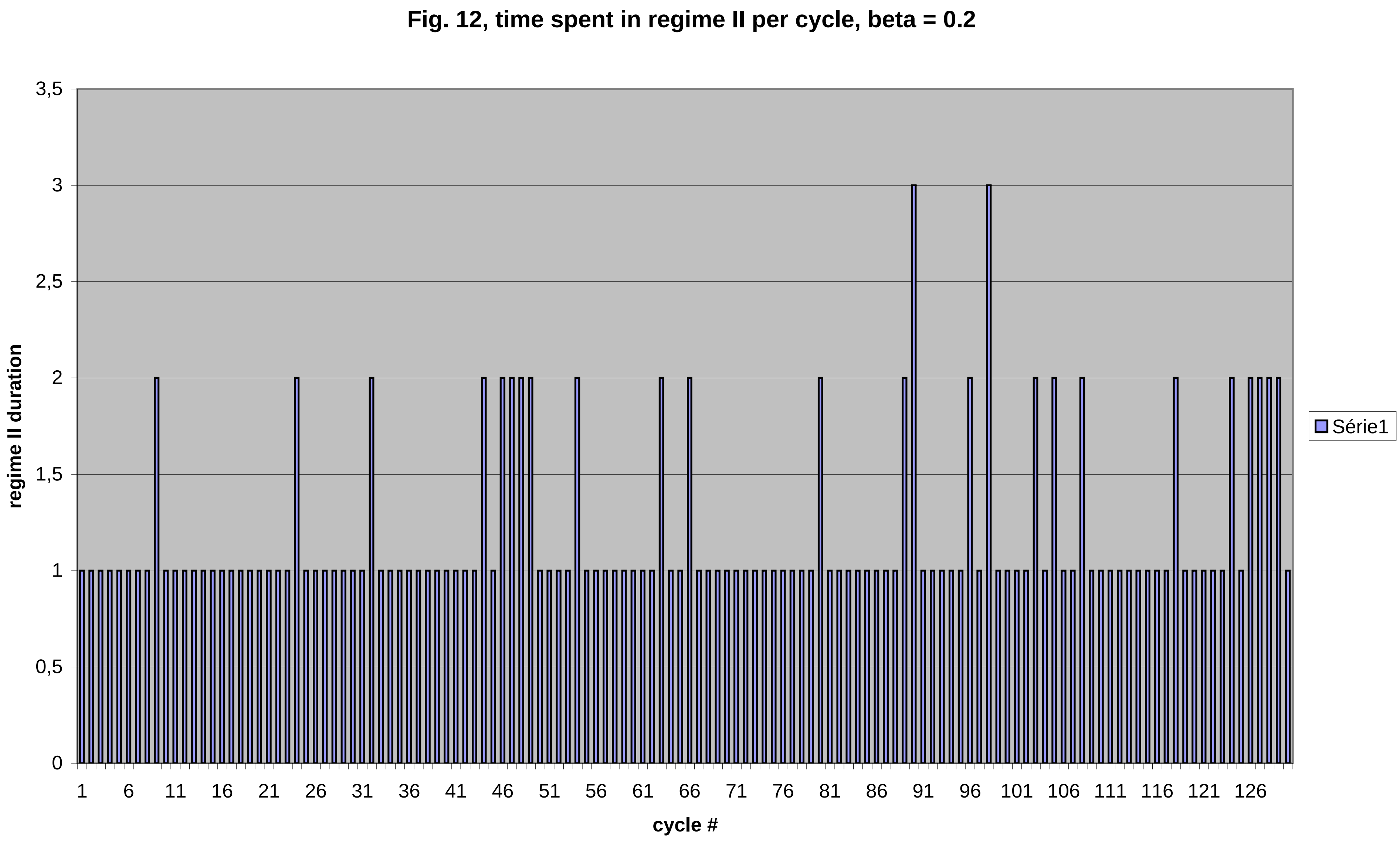


Fig. 12, time spent in regime II per cycle, beta = 0.2
3,5
3
2,5
2
1,5
1
0,5
0
regime II duration
1
6
11
16
21
26
31
36
41
46
51
56
61
66
71
76
81
86
91
96
101
106
111
116
121
126
cycle #
Série1