Testing Game Theory in the Field: Swedish LUPI Lottery Games*

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Abstract

Game theory is usually difficult to test precisely in the field because predictions typically depend sensitively on features that are not controlled or observed. We conduct one such test using field data from the Swedish lowest unique positive integer (LUPI) game. In the LUPI game, players pick positive integers and whoever chose the lowest unique number wins a fixed prize. Theoretical equilibrium predictions are derived assuming Poisson-distributed uncertainty about the number of players, and tested using both field and laboratory data. The field and lab data show similar patterns. Despite various deviations from equilibrium, there is a surprising degree of convergence toward equilibrium. Initial responses can be rationalized by a cognitive hierarchy model and convergence toward equilibrium by a simple learning-by-imitation model.

JEL CLASSIFICATION: C72, C92, L83, C93.

KEYWORDS: Population uncertainty, Poisson game, QRE, congestion game, guessing game, experimental methods, behavioral game theory, level-k.

1 Introduction

Game theory predictions are challenging to test with field data because those predictions are usually sensitive to details about strategies, information and payoffs which are difficult to observe in the field. As Robert Aumann pointed out: "In applications, when you want to do something on the strategic level, you must have very precise rules...An auction is a beautiful example of this, but it is very special. It rarely happens that you have rules like that (cited in van Damme, 1998, p. 196)."

In this paper we exploit such a happening, using field data from a Swedish lottery game. In this lottery, players simultaneously choose positive integers from 1 to K. The winner is the player who chooses the lowest number that nobody else picked. We call this the LUPI game, because the *l*owest *u*nique *p*ositive *i*nteger wins.[1] Because strategies and payoffs are known, the field setting is unusually well-structured compared to other strategic field data on contracting, pricing, entry, information disclosure, or auctions. The price one pays for clear structure is that the game does not very closely resemble any other familiar economic game. Gaining structure at the expense of generality is similar to the tradeoff faced in using data from game shows and sports to understand general strategic principles.

This paper analyzes LUPI theoretically and reports data from the Swedish field experience and from parallel lab experiments. The paper has several theoretical and empirical parts. The parts have a coherent narrative flow because each part raises some new question which is answered by the next part. The overarching question, which is central to all empirical game theory, is this one: What models of strategic thinking best explain behavior in games?

The first specific question is *Q1: What does an equilibrium model of behavior predict in these games?* To answer this question, we first note that the Nash equilibrium with

[1]The Swedish company called the game Limbo, but we think LUPI is more mnemonic, and more apt because in the typical game of limbo, two players who tie in how low they can slide underneath a bar do not lose.

a fixed number of players is practically impossible to compute numerically. Furthermore, the number of players is not the same each day; and the Nash equilibrium corresponding to the empirically observed distribution of the number of players cannot be solved either. However, we can try to approximate the equilibrium by applying the theory of Poisson games.[2] In Poisson games, the number of players is Poisson-distributed (Myerson, 1998).[3] Remarkably, assuming a variable number of players rather than a fixed number makes computation of equilibrium *simpler* if the number of players is Poisson-distributed.

The number of players in the Swedish LUPI games actually varies too much from day-to-day to match the cross-day variance implicit in the Poisson assumption. However, the Poisson-Nash equilibrium is the only computable equilibrium benchmark. Field tests of theory always violate some of the assumptions of the theory, to some degree; it is an empirical question whether the equilibrium benchmark fits reasonably well despite resting on incorrect assumptions. (We revisit this important issue in the conclusion after all the data are presented.)

After deriving the Poisson equilibrium in order to answer Q1, we compare the Poisson equilibrium to the field data. In our view, the equilibrium is surprisingly close (given its complexity and counterintuitive properties). However, there are clearly large deviations from the equilibrium prediction and some behaviorally interesting fine-grained deviations. These empirical results raises question *Q2: Can non-equilibrium behavioral models explain the deviations when the game is first played?*

The simple LUPI structure allows us to provide tentative answers to Q2 by comparing Poisson-Nash equilibrium predictions with predictions of two parametric models of boundedly rational play: quantal response equilibrium (QRE), and a level-k or cognitive hierarchy (CH) approach. QRE and CH have been compared to Nash predictions in many experimental studies, and they often explain deviations from Nash equilibrium in similar

[2] As Milton Friedman (1953) famously noted, theories with false assumptions could often predict well (and, in economics, often do).

[3] This also distinguishes our paper from contemporaneous research on unique bid auctions by Eichberger and Vinogradov (2008), Houba, van der Laan and Veldhuizen (2008), Raviv and Virag (2009), Rapoport, Otsubo, Kim and Stein (2007) and Gallice (2009) which all assume that the number of players is fixed and commonly known.

ways (e.g., Rogers, Palfrey and Camerer, 2009). However, QRE and CH can be clearly distinguished in LUPI games: QRE predicts too few low-number choices and CH predicts too many low-number choices (compared to the Poisson-Nash). The field data tend to favor the CH prediction.

The answer to Q2 naturally raises a third question *Q3: When there is initial non-equilibrium behavior, do learning forces produce convergence toward equilibrium over time?* About 50,000 numbers were played in LUPI (on average) on 49 consecutive days. The large number of players gives enough statistical power to study the rate of learning across the time series in a game in which the structure does not vary, which most other field studies cannot do. For example, several studies have used field data from tennis and soccer to test mixed-strategy equilibrium predictions (Walker and Wooders, 2001, Chiappori, Levitt and Groseclose, 2002, Palacios-Huerta, 2003 and Hsu, Huang and Tang, 2007). These studies use highly experienced players and the studies on soccer pool data across substantial spans of time to test the mixed equilibrium prediction powerfully. They do not study how players learn to play a mixed equilibrium within their samples.[4]

Note that the LUPI data do not invite a natural application of individual-level learning models like reinforcement, fictitious play, and EWA because we only observe an aggregate set of choices. Therefore, we apply a simple imitation learning model that reflects the basic features of the observed changes in number choices over 49 days.

In this narrative so far, we have discussed a new equilibrium approximation to LUPI (addressing Q1), how well CH and QRE models fits the data (addressing Q2), and that imitation learning tracks some basic properties of the data over time (addressing Q3).

[4]Chiappori, Levitt and Groseclose (2002) provide some suggestive evidence about learning by noting that among the kickers with the most experience in their sample (those with eight or more kicks) only one out of nine fails a randomness test at the 10% level. However, this is a crude test for learning effects compared to our data, which compare a much larger sample of choices with day-by-day comparisons. There is also a field study of randomization in gambling choices that are not strategic, and learning is not measured (Sundali and Croson, 2006). Ockenfels and Roth (2004) discuss an interesting natural experiment measuring learning about prices for a surprising new product. They studied prices for retailer-copied "Iraq most wanted cards" originally produced by the US Department of Defense to help soldiers identify high-value targets during the Iraq war. Retailers quickly copied the cards and offered decks for $5.95; but retailer decks were also traded at a much higher "buy-it-now" (BIN) prices on eBay. They found that BIN prices converged to the retail price in about 30 days.

Because the LUPI game is simple, it is easy to go a step further and run a lab experiment that matches many of the key features of the game played in the field. The lab data enable us to address one more question: *Q4: How well does behavior in a lab experiment designed to closely match features of a field environment parallel behavior in the field?* Q4 is important because of an ongoing debate about lab-field parallelism in economics, rekindled with some skepticism by Levitt and List (2007). We conclude that the basic empirical features of the lab and field behavior are comparable. This close match adds to a small amount of evidence of how well experimental lab data can generalize to a particular field setting when the experiment was specifically intended to do so.

The ability to track decisions by each player in the lab also enables us to answer some minor questions that cannot be answered by field data. For example, it appears that imitation learning is operating at the individual level, sociodemographic variables do not correlate strongly with performance, and there are not strong identifiable differences in skill across players (measured by winning frequency).

Before proceeding, we mention an important caveat. LUPI was not designed by the lottery creators to be an exact model of a particular economic game. However, it combines some strategic features of interesting naturally-occurring games. For example, in games with congestion, a player's payoffs are lower if others choose the same strategy. Examples include choices of traffic routes and research topics, or buyers and sellers choosing among multiple markets. LUPI has the property of an extreme congestion game, in which having even one other player choose the same number reduces one's payoff to zero.[5] Indeed, LUPI is similar to a game in which being first matters (e.g., in a patent race), but if players are tied for first they do not win. One close market analogue to LUPI is the lowest unique bid auction (LUBA; see Eichberger and Vinogradov, 2008, Houba et al., 2008, Raviv and Virag, 2009 , Rapoport et al., 2007, and Gallice, 2009). In these auctions, an object is sold to the lowest bidder whose bid is unique (or in some versions, to the

[5]Note, however, that LUPI is not a congestion game as defined by Rosenthal (1973) since the payoff from choosing a particular number does not only depend on how many other players that picked that number, but also on how many that picked lower numbers.

highest unique bidder). LUPI is simpler than LUBA because winners don't have to pay the amount they bid, and there are no private valuations and beliefs about valuations of others. However, LUPI contains the same essential strategic conflict: between wanting to choose low numbers and wanting to choose unique numbers.

Finally, the scientific value of LUPI games is like the scientific value of data from game shows and professional sports, such as "Deal or No Deal" (e.g. Andersen, Harrison, Lau and Rutström, 2008 and Post, van den Assem, Baltussen and Thaler, 2008). Like the LUPI lottery, game shows and sports are not designed to be replicas of typical economic decisions. Nonetheless, game shows and sports are widely studied in economics because they provide very clear field data about actual economic choices (often for high stakes), and they have simple structures that can be analyzed theoretically. The same is true for LUPI.

The next section provides a theoretical analysis of a simple form of the LUPI game, the Poisson-Nash equilibrium. Section 3 reports the basic field data and compare them to the Poisson-Nash approximate benchmark. It also introduces quantal response equilibrium and cognitive hierarchy behavior models, as well as learning, and asks whether they can explain the field data. Section 4 describes the lab replication. Section 5 concludes the paper.

2 Theory

In the simplest form of LUPI, the number of players, N, has a known distribution, the players choose integers from 1 to K simultaneously, and the lowest unique number wins. The winner earns a payoff of 1, while all others earn 0.[6]

In this section we analyze the game when players are assumed to be fully rational,

[6]In this stylized case, we assume that if there is no lowest unique number then there is no winner. This simplifies the analysis because it means that only the probability of being unique must be computed. In the Swedish game, if there is no unique number then the players who picked the smallest and least-frequently-chosen number share the top prize. This is just one of many small differences between the simplified game analyzed in this section and the game as played in the field, which are discussed further below.

best-responding, and have equilibrium beliefs. We assume that the number of players N is a random variable that has a Poisson distribution.[7] The Poisson assumption proves to be easier to work with than a fixed N. In fact, the Nash equilibrium for arbitrary distributions over N, including fixed N, is extremely difficult to compute for the Swedish LUPI lottery. (Appendix A discusses the fixed-N equilibrium and why it is so much more difficult to compute than the Poisson-Nash equilibrium.) The actual variance of N in the field data is much larger than in the Poisson distribution so the Poisson-Nash equilibrium is only a computable approximation to the correct (but incomputable) equilibrium. Whether it is a good approximation will partly be answered by looking at how well the theory fits the field data. In addition, we implement the Poisson distribution of N exactly in lab experiments.

2.1 Properties of Poisson Games

In this section, we briefly summarize the theory of Poisson games developed by Myerson (1998, 2000). The theory is then used in the next section to characterize the Poisson-Nash equilibrium in the LUPI game.

Games with population uncertainty relax the assumption that the exact number of players is common knowledge. In particular, in a *Poisson game* the number of players N is a random variable that follows a Poisson distribution with mean n. We have

$$N \sim \text{Poisson}(n): \quad N = k \text{ with probability } \frac{e^{-n} n^k}{k!}$$

and, in the case of a Bayesian game, players' types are independently determined according to the probability distribution $r = (r(t))_{t \in T}$ on some type space T.[8] Let a type profile be

[7] Players did not know the number of total bets in both the field and lab versions of the LUPI game. Although players in the field could get information about the current number of bets that had been made so far during the day, players had to place their bets before the game closed for the day and therefore could not know with certainty the total number of players that would participate in that day.

[8] The LUPI game itself is not a Bayesian game. However, in the cognitive hierarchy model (developed in Section 3.3), there are players with different degree of strategic sophistication and we therefore include types in our presentation of Poisson games in this section.

a vector of non-negative integers listing the number of players of each type t in T, and let $Z(T)$ be the set of all such type profiles in the game. Combining N and r can describe the *population uncertainty* with the distribution $y \sim Q(y)$ where $y \in Z(T)$ and $y(t)$ is the number of players of type $t \in T$.

Players have a common finite action space C with at least two alternatives, which generates an *action profile* $Z(C)$ containing the number of players that choose each action. Utility is a bounded function $U : Z(C) \times C \times T \to \mathbb{R}$, where $U(x, b, t)$ is the payoff of a player with type t, choosing action b, and facing an opponent action profile of x. Let $x(c)$ denote the number of other players playing action $c \in C$.

Myerson (1998) shows that the Poisson distribution has two important properties that are relevant for Poisson games and simplify computations dramatically. The first is the *decomposition property*, which in the case of Poisson games imply that the distribution of type profiles for any $y \in Z(T)$ is given by

$$Q(y) = \prod_{t \in T} \frac{e^{-nr(t)}(nr(t))^{y(t)}}{y(t)!}.$$

Hence, $\tilde{Y}(t)$, the random number of players of type $t \in T$, is Poisson with mean $nr(t)$, and is independent of $\tilde{Y}(t')$ for any other $t' \in T$. Moreover, suppose each player independently plays the mixed strategy σ, choosing action $c \in C$ with probability $\sigma(c|t)$ given his type t. Then, by the decomposition property, the number of players of type t that chooses action c, $\overline{Y}(c, t)$, is Poisson with mean $nr(t)\sigma(c|t)$ and is independent of $\overline{Y}(c', t')$ for any other c', t'.

The second property of Poisson distributions is the *aggregation property*, which states that any sum of independent Poisson random variables is Poisson distributed. This property implies that the number of players (across all types) who choose action c, $\tilde{X}(c)$, is Poisson with mean $\sum_{t \in T} nr(t)\sigma(c|t)$, independent of $\tilde{X}(c')$ for any other $c' \in C$. We refer to this property of Poisson games as the *independent actions* (IA) property.

Myerson (1998) also shows that the Poisson game has another useful property: *envi-*

ronmental equivalence (EE). Environmental equivalence means that conditional on being in the game, a type t player would perceive the population uncertainty as an outsider would, i.e., $Q(y)$.[9] If the strategy and type spaces are finite, Poisson games are the only games with population uncertainty that satisfy both IA and EE (Myerson, 1998). EE is a surprising property.

Take a Poisson LUPI game with 27 players on average. In our lab implementation, a large number of players are recruited and are told that the number of players who will be active (i.e. play for real money) in each period varies. Consider a player who is told she is active. On the one hand, she might then act as if she is playing against the number of opponent players that is Poisson-distributed with a mean of 26 (since her active status has lowered the mean of the number of remaining players). On the other hand, the fact that she is active is a clue that the number of players in that period is large, not small. If N is Poisson-distributed the two effects *exactly* cancel out so all active players in all periods act as if they face a Poisson-distributed number of opponents. EE, combined with IA, makes the analysis rather simple.

A *equilibrium* for the Poisson game is defined as a strategy function σ such that every type assigns positive probability only to actions that maximize the expected utility for players of this type; that is, for every action $c \in C$ and every type $t \in T$,

$$\text{if } \sigma(c|t) > 0 \text{ then } \overline{U}(c|t,\sigma) = \max_{b \in C} \overline{U}(b|t,\sigma)$$

for the expected utility

$$\overline{U}(b|s,\sigma) = \sum_{x \in Z(C)} \prod_{c \in C} \left(\frac{e^{-n\tau(c)}(n\tau(c))^{x(c)}}{x(c)!} \right) U(x,b,s)$$

where

$$\tau(c) = \sum_{t \in T} r(t)\sigma(c|t)$$

[9] In particular, for a Poisson game, the number of opponents he faces is also a random variable of Poisson(n).

is the marginal probability that a random sampled player will choose action c under σ. Note that this equilibrium is by definition symmetric; asymmetric equilibrium where players of the same type could play differently are not defined in games with population uncertainty since ex ante we do not know the list of participating players.

Myerson (1998) proves existence of equilibrium under all games of population uncertainty with finite action and type spaces, which includes Poisson games.[10] Note that the equilibria in games with population uncertainty must be symmetric in the sense that each type plays the same strategy. This existence result provides the basis for the following characterization of the Poisson-Nash equilibrium.

2.2 Poisson Equilibrium for the LUPI Game

In the symmetric Poisson equilibrium, all players employ the same mixed strategy $\mathbf{p} = (p_1, p_2, \cdots, p_K)$ where $\sum_{i=1}^{K} p_i = 1$. Let the random variable $X(k)$ be the number of players who pick k in equilibrium. Then, $Pr(X(k) = i)$ is the probability that the number of players who pick k in equilibrium is exactly i. By environmental equivalence (EE), $Pr(X(k) = i)$ is also the probability that i opponents pick k. Hence, the expected payoffs for choosing different numbers are:[11]

$$
\begin{aligned}
\pi(1) &= Pr(X(1) = 0) = e^{-np_1} \\
\pi(2) &= Pr(X(1) \neq 1) \cdot Pr(X(2) = 0) \\
\pi(3) &= Pr(X(1) \neq 1) \cdot Pr(X(2) \neq 1) \cdot Pr(X(3) = 0) \\
&\vdots \\
\pi(k) &= \left(\prod_{i=1}^{k-1} Pr(X(i) \neq 1) \right) \cdot Pr(X(k) = 0) \\
&= \left(\prod_{i=1}^{k-1} \left[1 - np_i e^{-np_i} \right] \right) \cdot e^{-np_k}
\end{aligned}
$$

[10] For infinite types, Myerson (2000) proves existence of equilibrium for Poisson games alone.

[11] Recall that winner's payoff is normalized to 1, and others are 0.

for all $k > 1$. If both k and $k+1$ are chosen with positive probability in equilibrium, then $\pi(k) = \pi(k+1)$. Rearranging this equilibrium condition implies

$$e^{np_{k+1}} = e^{np_k} - np_k. \tag{1}$$

In addition to this condition, the probabilities must sum up to one and the expected payoff from playing numbers not in the support of the equilibrium strategy cannot be higher than the numbers played with positive probability.

The three equilibrium conditions allows us to characterize the equilibrium and show that it is unique.

Proposition 1 *There is a unique mixed equilibrium* $\mathbf{p} = (p_1, p_2, \cdots, p_K)$ *of the Poisson LUPI game that satisfies the following properties:*

1. *Full support:* $p_k > 0$ *for all* k.
2. *Decreasing probabilities:* $p_{k+1} < p_k$ *for all* k.
3. *Convexity/concavity:* $(p_k - p_{k+1})$ *is increasing in* k *for* $p_k < 1/n$ *and decreasing in* k *for* $p_k > 1/n$.
4. *Convergence to uniform play with many players: for any fixed* K, $n \to \infty$ *implies* $p_{k+1} \to p_k$.[12]

Proof. See Appendix B. Q.E.D.

In the Swedish game the average number of players was $n = 53,783$ and number choices were positive integers up to $K = 99,999$. As Figure 1 shows, the equilibrium involves mixing with substantial probability between 1 and 5500, starting from $p_1 = 0.0002025$. The predicted probabilities drop off very sharply at around 5500. Figure 1 shows only

[12]To illustrate the convergence to uniform distribution as $n \to \infty$ numerically, when $K = 100$ and $n = 500$ the mixture probabilities start at $p_1 = 0.0124$ and end with $p_{97} = 0.0043, p_{98} = 0.0038, p_{99} = 0.0031, p_{100} = 0.0023$; so the ratio of highest to lowest probabilities is about six-to-one. When $K = 100$ and $n = 5,000$, all mixture probabilities for numbers 1 to 100 are 0.01 (up to two-decimal precision).

the predicted probabilities for 1 to 10,000, since probabilities for numbers above 10,000 are positive but minuscule.

The central empirical question that will be answered later is how well actual behavior in the field matches the equilibrium prediction in Figure 1. Keep in mind that the simplified game analyzed in this section differs in some potentially important ways from the actual Swedish game. Computing the equilibrium is complicated and its properties are not particularly intuitive. It might therefore be surprising if the actual data matched the equilibrium closely. Because there are 49 days of data, we can also see whether choices move in the direction of the Poisson-Nash benchmark over time.

3 The Field LUPI Game

The field version of LUPI, called Limbo, was introduced by the government-owned Swedish gambling monopoly Svenska Spel on the 29th of January 2007.[13] This section describes its essential elements; additional description is in Appendix D.

In Limbo, players chose an integer between 1 and 99,999. Each number bet costs 10 SEK (approximately 1 EURO). The game was played daily and the winning number was presented on TV in the evening and on the Internet. The winner received 18 percent of the total sum of bets, with the prize guaranteed to be at least 100,000 SEK (approximately 10,000 EURO). If no number was unique the prize was shared evenly among those who chose the smallest and least-frequently chosen number. There were also smaller second and third prizes (1000 SEK and 20 SEK) for being close to the winning number.

During the first three to four weeks, it was only possible to play the game at physical branches of Svenska Spel by filling out a form (Figure A9). The form allowed players to bet on up to six numbers[14], to play the same numbers for up to 7 days in a row, or to let

[13]Stefan Molin at Svenska Spel told us that he invented the game in 2001 after taking a game theory course from the Swedish theorist and experimenter Martin Dufwenberg.

[14]The rule that players could only pick up to six numbers a day was enforced by the requirement that players had to use a "gambler's card" linked to their personal identification number when they played. Colluding in LUPI can conceivably increase the probability of winning but would require a remarkable

a computer choose random numbers for them (a "HuxFlux" option).

Daily data were downloaded for the first seven weeks, ending on the 18th of March 2007. The game was stopped on March 24th, one day after a newspaper article claimed that some players had colluded in the game, but it is unclear whether collusion actually occurred or how it could be detected.

Unfortunately, we have only gained access to aggregate daily frequencies, not to individual-level data. We also do not know how many players used the randomization HuxFlux option. However, because the operators told us how HuxFlux worked, we can estimate that approximately 19 percent of players were randomizing in the first week.[15]

Note that the theoretical analysis of the LUPI game in the previous section differs from the field LUPI game in three ways. First, the theory used a tie-breaking rule in which nobody wins if there is no uniquely chosen number (to simplify expected payoff calculations enormously). In the field game, however, players who tie by choosing the smallest and least-frequently chosen number share the prize. This is a minor difference because the probability that there is no unique number is very small and it never happened during the 49 days for which we have data. A second, more important, difference is that we assume that each player can only pick one number. In the field game, players are allowed to bet on up to six numbers. This does play a role for the theoretical predictions, since it allows players to "knock out" a likely low-number winner by choosing the same number as the winner and then bet on a higher number hoping that the higher number will be unique and win. Finally, we do not take the second and third prizes present in the field version into account, but this is unlikely to make a big difference given the strategic nature of the game.

Nevertheless, these three differences between the payoff structures of the game analyzed theoretically, and the field game as it was played, are a motivation for running laboratory experiments with single bets, no opportunity for direct collusion, and only a

degree of coordination across a large syndicate, and is also risky if others might be colluding in a similar way.

[15]In the first week, the randomizer chose numbers from 1 to 15,000 with equal probability. The drop in numbers just below and above 15,000 suggests the 19 percent figure.

first prize, which match the game analyzed theoretically more closely.

3.1 Descriptive Statistics

Table 1 reports summary statistics for the first 49 days of the game. Two additional columns display the corresponding daily averages for the first and last weeks to see how much learning takes place. The last column displays the corresponding statistics that would result from play according to the Poisson equilibrium.

Overall, the average number of bets N was 53,783, but there was considerable variation over time. There is no apparent time trend in the number of participating players, but there is less participation on Sundays and Mondays (see Figure A11).[16] The variation of the number of bets across days is therefore much higher than what the Poisson distribution predicts (its standard deviation is 232). However, note that larger variance in N means sometimes there are many fewer players (so chosen numbers should be smaller) and sometimes there are many more players (so chosen numbers should be larger). Fixing the mean of N and increasing the variance might therefore have little overall impact on the equilibrium number distribution (and has little effect in the lab data reported later).

Despite some differences between the simplified theory and the way the field lottery game was implemented, the average number chosen overall was 2835, which is close to the equilibrium prediction of 2595.[17] Winning numbers, and the lowest numbers not chosen by anyone, also varied a lot over time. *All* the aggregate statistics using chosen numbers in Table 1 are closer to the equilibrium predictions in the last week than in the first week. Many of the statistics converge rather swiftly and closely. The mean number in the last

[16] The Sunday-Monday average N (std. dev.) is $44,886$ (4001) and the Tuesday-Saturday average is $57,341(5810)$. Dividing the sample in this way does reduce the variance in N by almost half. However, the summary statistics in the two groups are very close together (the means are 2792 and 2941).

[17] To judge the significance of the difference between theory and data we simulated 1000 weekly average numbers from the Poisson-Nash equilibrium. That is, 350,000 i.i.d. draws were drawn from the distribution and the average number was computed. This yields one simulated average. The procedure was then repeated a total of 1000 times to create 1000 simulated averages. The low and high range of 950 of these simulated averages—a 95% confidence interval—is 2590 to 2599. Since the weekly averages in the data lie outside this extremely tight interval, we can conclude that the data are significantly different than those predicted by theory. But note that this is an extremely demanding test because the very large sample sizes mean that the data must lie very close to the theory to *not* reject the theory.

week is 2484, compared to the prediction of 2595. In equilibrium essentially nobody (fewer than .01 percent) should choose a number above 10,000. In the first week 12 percent chose these high numbers, but in the last week only 1 percent did.

	All days				1^{st} week	7^{th} week	Eq.
	Avg.	Std.	Min	Max	Avg.	Avg.	Avg.
# Bets	53783	7782	38933	69830	57017	47907	53783
Average number played	2835	813	2168	6752	4512	2484	2595
Median number played	1674	348	435	2272	1203	1935	2541
Winning number	2095	1266	162	4465	1159	1982	2585
Lowest number not played	3103	929	480	4597	1745	3462	4077
Below 100 (%)	6.08	4.84	2.72	2.97	15.16	3.19	2.02
Below 1000 (%)	32.31	8.14	21.68	63.32	44.91	27.52	20.05
Below 5000 (%)	92.52	6.44	68.26	97.74	78.75	96.44	93.34
Below 10000 (%)	96.63	3.80	80.70	98.94	88.07	98.81	100.00
Even numbers (%)	46.75	0.58	45.05	48.06	45.91	47.45	49.99
Divisible by 10 (%)	8.54	0.466	7.61	9.81	8.43	9.01	9.99
Proportion 1900–2010 (%)	71.61	4.28	67.33	87.01	79.39	68.79	49.78
11, 22,...,99 (%)	12.2	0.82	10.8	14.4	12.4	11.4	9.00
111, 222,...,999 (%)	3.48	0.65	2.48	4.70	4.27	2.78	0.90
1111, 2222,...,9999 (1/1000)	4.52	0.73	2.81	5.80	4.74	3.95	0.74
11111, 22222,...,99999 (1/1000)	0.76	0.84	0.15	5.45	2.26	0.21	0

Proportion of numbers between 1900 and 2010 refers to the proportion relative to numbers between 1844 and 2066. “11, 22,...,99” refers to the proportion relative to numbers below 100, “111,222,...,999” relative to numbers below 1000, and so on. The “eq. avg” predictions refers to the prediction of the Poisson-Nash equilibrium with $n = 53,783$ and $K = 99,999$.

Table 1: Descriptive statistics and Poisson-Nash equilibrium predictions for field LUPI game data

An interesting feature of the data is a tendency to avoid round or focal numbers and choose quirky numbers that are perceived as “anti-focal” (as in hide-and-seek games, see Crawford and Iriberri, 2007*a*). Even numbers were chosen less often than odd ones (46.75% vs. 53.25%). Numbers divisible by 10 are chosen a little less often than predicted. Strings of repeating digits (e.g., 1111) are chosen too often.[18] Players also overchoose

[18]Similar behavior can be found in the federal tax evasion case of Joe Francis, the founder of “Girls Gone Wild.” Mr. Francis was indicted on April 11, 2007 for claiming false business expenses such as $333,333.33 and $1,666,666.67 in insurance, which were too suspicious *not* to attract attention. See `http://consumerist.com/consumer/taxes/girls-gone-wild-tax-indictment-teaches-us-not-to-deduct-funny+looking-numbers-252097.php` for the proposed tax lesson and `http://www.thesmokinggun.com/archive/years/2007/0411072joefrancis1.html` for the original court order.

numbers that represent years in modern time (perhaps their birth years). If players had played according to equilibrium, the fraction of numbers between 1900 and 2010 divided by all numbers between 1844 and 2066 should be 49.78 percent, but the actual fraction was 70 percent.[19]

Figure 2 shows this focality in a histogram of numbers between 1900 and 2010 (aggregating all 49 days). Note that although the numbers around 1950 are most popular, there are noticeable dips at focal years that are divisible by ten.[20] Figure 2 also shows the aggregate distribution of numbers between 1844 and 2066, which clearly shows the popularity of numbers around 1950 and 2007. There are also spikes in the data for special numbers like 2121, 2222 and 2345. Explaining these "focal" numbers with the cognitive hierarchy and quantal response equilibrium models presented below is not easy (unless the 0-step player distribution is defined to include focality), so we will not comment on them further (though see Crawford and Iriberri, 2007*a* for a successful application in simpler hide-and-seek games).

3.2 Results

Do subjects in the field LUPI game play according to the Poisson-Nash equilibrium benchmark? In order to investigate this, we assume that the number of players is Poisson distributed with mean equal to the empirical daily average number of numbers chosen $(53,783)$. As noted previously, this assumption is wrong because the variation in number of bets across days is much higher than what the Poisson distribution predicts. We do not know how close the approximation is because the equilibrium using either the actual distribution of N, or fixed N, cannot be computed. However, computations with small-N games show that fixed-N and Poisson-distributed N equilibria are very close together (see

[19]We compare the number of choices between 1900 and 2010 to the number of choices between 1844 and 2066 since there are twice as many strategies to choose from in the latter range compared to the first. If all players randomized uniformly (an approximation to the equilibrium strategy with large n and K), the proportion of numbers between 1900 and 2010 would be about 50 percent.

[20]Note that it would be unlikely to observe these dips reliably with typical experimental sample sizes. It is only with the large amount of data available from the field, 2.5 million observations, that these dips are visually obvious and different in frequency than neighboring unround numbers.

Appendix A).

Figure 3 shows the average daily frequencies from the first week together with the equilibrium prediction (the dashed line), for all numbers up to 99,999 and for the restricted interval up to 10,000. Recall that in the Poisson-Nash equilibrium, probabilities of choosing higher numbers first decrease slowly, drop quite sharply at around 5500, and asymptotes to zero after $p_{5513} \approx 1/n$ (recall Proposition 1 and Figure 1). Compared to equilibrium, there is overshooting at numbers below 1000 and undershooting at numbers between between 2000 and 5500. It is also noteworthy how spiky the data is compared to the equilibrium prediction, which is a reflection of clustering on special numbers, as described above. Nonetheless, the ability of the very complicated Poisson-Nash equilibrium approximation to capture some of the basic features of the data is surprisingly good.

Figure 4 shows average daily frequencies of choices from the second through the last (7th) week. Behavior in this period is even closer to equilibrium than in the first week. However, when only numbers below 10,000 are plotted, the overshooting of low numbers and undershooting of intermediate numbers is still clear (although the undershooting region shrinks to numbers between 4000 and 5500) and there are still many spikes of clustered choices.

The next question is whether alternative theories can explain both the degree to which the equilibrium prediction is surprisingly accurate *and* the degree to which there is systematic deviation.

3.3 Rationalizing Non-Equilibrium Play

This section describes two potential approaches to rationalizing deviations from the Poisson-Nash approximation: Quantal response equilibrium (QRE) and a cognitive hierarchy (CH) approach. The theories are presented together for coherence. However, for computational reasons, QRE cannot be easily fitted to the field data. So we first describe how well CH can fit the field data. Then the lab experiments are described and all three theories are applied to those data.

3.3.1 Quantal Response Equilibrium

As described in McKelvey and Palfrey (1995) and Chen, Friedman and Thisse (1997), the quantal response equilibrium (QRE) replaces best responses by quantal responses, allowing for either error in actions or uncertainty about payoffs. QRE has been applied to hundreds of experimental data sets and can often account for both behavior close to equilibrium and behavior that deviates from equilibrium (e.g. Goeree and Holt, 2001, Goeree, Holt and Palfrey, 2002, Levine and Palfrey, 2007, and Goeree and Holt, 2005).

As in stochastic consumer choice models, QRE can fit any pattern of data if the error structure is general enough (Haile, Hortaçsu and Kosenok, 2008). Therefore, as is always done in empirical work we use a particular restriction, that choice probabilities are given by normalized power functions of expected payoffs of strategies.[21] In a power QRE, a vector $\mathbf{p} = (p_1, p_2, \cdots, p_K)$ is a symmetric equilibrium if all probabilities satisfy

$$p_k = \frac{\pi(k)^\lambda}{\sum_{j=1}^K \pi(j)^\lambda},$$

where $\lambda > 0$ and $\pi(k) \geq 0$ are expected payoffs given the equilibrium probabilities.

If we assume that the number of players are Poisson distributed, we can use the expression for the payoff from playing the k^{th} number from the previous section. This gives the following symmetric QRE probabilities of the game:

$$p_k = \frac{\left(\prod_{i=1}^{k-1} \left[1 - np_i e^{-np_i}\right] e^{-np_k}\right)^\lambda}{\sum_{j=1}^K \left(\prod_{i=1}^{j-1} \left[1 - np_i e^{-np_i}\right] e^{-np_j}\right)^\lambda}.$$

Note that in a power QRE, as in the Poisson equilibrium, all numbers are played with positive probability and larger numbers are chosen less often ($p_{k+1} \leq p_k$, for $\lambda > 0$).[22]

[21] Most studies of QRE use logit probability functions. But since we use a power distribution in the cognitive hierarchy model (to be presented below), we use that for the QRE as well to maintain comparability.

[22] To see why this is the case, suppose by contradiction that $p_{k+1} > p_k$, i.e., $p_{k+1}/p_k > 1$. From the

Some intuition about how QRE behaves[23] can be obtained from the case implemented in the lab experiments, which has $n = 26.9$ players and number choices from 1 to $K = 99$. Figure 5 shows a 3-dimensional plot of the QRE probability distributions for many values of λ, along with the Poisson-Nash equilibrium. When λ is low, the distribution is approximately uniform. As λ increases more probability is placed on lower numbers 1-12. When λ is high enough the QRE closely approximates the Poisson-Nash equilibrium, which puts roughly linear declining weight on numbers 1 to 15 and infinitesimal weight on higher numbers. We conjecture that power QRE always approaches the Poisson-Nash equilibrium in this way, shifting weight from higher numbers to lower numbers in the transition from random ($\lambda = 0$) to Poisson-Nash ($\lambda \to \infty$) behavior, but have not been able to prove the conjecture.

3.3.2 Cognitive Hierarchy with Quantal Response

A natural way to model limits on strategic thinking is by assuming that different players carry out different numbers of steps of iterated strategic thinking in a cognitive hierarchy (CH). This idea has been developed in behavioral game theory by several authors (e.g., Nagel, 1995, Stahl and Wilson, 1995, Costa-Gomes, Crawford and Broseta, 2001, Camerer, Ho and Chong, 2004 and Costa-Gomes and Crawford, 2006) and applied to many games of different structures (e.g., Crawford, 2003, Camerer, Ho and Chong, 2004 and Crawford

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expression for the ratio p_{k+1}/p_k we know that this implies that

$$\left[\left(1 - np_k e^{-np_k}\right) e^{-np_{k+1}}\right]^\lambda > \left[e^{-np_k}\right]^\lambda .$$

Raising both sides to the power of $1/\lambda$ (which is valid since both sides are positive) and rearranging we get

$$\left(1 - np_k e^{-np_k}\right) e^{np_k} > e^{np_{k+1}}.$$

Taking logarithms

$$\frac{1}{n} \ln\left(1 - np_k e^{-np_k}\right) > p_{k+1} - p_k.$$

Since $p_{k+1} > p_k$, the right hand side is positive. The left hand side, however, is always negative since $1 - np_k e^{-np_k} = P\left(X\left(k\right) \neq 1\right)$ (which is a probability between zero and one). This is a contradiction, and we can therefore conclude that $p_k > p_{k+1}$ whenever $\lambda > 0$.

[23]We have not shown that the symmetric power QRE is unique, but no other symmetric equilibria have emerged during numerical calculations.

and Iriberri, 2007*b*).[24]

These models require a specification of how k-step players behave and the proportions of players for various k. We follow Camerer, Ho and Chong (2004) and assume that the proportion of players that do k thinking steps is Poisson distributed with mean τ, i.e., the proportion of players that think in k steps is given by

$$f(k) = e^{-\tau}\tau^k/k!.$$

We assume that k-step thinkers correctly guess the proportions of players doing 0 to $k-1$ steps.[25] Then the conditional density function for the belief of a k-step thinker about the proportion of $l < k$ step thinkers is

$$g_k(l) = \frac{f(l)}{\sum_{h=0}^{k-1} f(h)}.$$

The IA and EE properties of Poisson games (together with the general type specification described earlier) imply that the number of players that a k-step thinker believes will play strategy i is Poisson distributed with mean

$$nq_i^k = n\sum_{j=0}^{k-1} g_k(j)\, p_i^j.$$

Hence, the expected payoff for a k-step thinker of choosing number i is

$$\pi^k(i) = \prod_{j=1}^{i-1}\left[1 - nq_j^k e^{-nq_j^k}\right]\cdot e^{-nq_i^k}.$$

[24] A precursor to these models was the insight, developed much earlier in the 1980's by researchers studying negotiation, that people often 'ignore the cognitions of others' in asymmetric-information bidding and negotiation games (Bazerman, Curhan, Moore and Valley, 2000).

[25] An alternative approach which often has advantages is that level-k types think all others are level $k-1$. If we start out with L0 types playing random, L1 types should all play 1. If L2 types best respond to only L1 types, then they should play uniformly among 2–K. If L3 types best respond to only L2 types, then they should all play 1 (since they believe nobody is playing 1), and this logic can cycle. Note that this problem typically occurs in games with mixed strategy equilibrium, such as matching pennies—if you start out with L0 playing H, you would have all even types play H and all odd types play T (and if you start out with random, L0 trivially coincides with equilibrium).

To fit the data well, it is necessary to assume that players respond stochastically (as in QRE) rather than always choose best responses (see also Rogers, Palfrey and Camerer, 2009).[26] We assume that level 0 players randomize uniformly across all numbers 1 to K, and higher-step players best respond with probabilities determined by a normalized power function of expected payoffs.[27]

The probability that a k step player plays number i is given by

$$p_i^k = \frac{\left(\prod_{j=1}^{i-1}\left[1 - nq_j^k e^{-nq_j^k}\right] e^{-nq_i^k}\right)^\lambda}{\sum_{l=1}^{K}\left(\prod_{j=1}^{i-1}\left[1 - nq_j^k e^{-nq_j^k}\right] e^{-nq_l}\right)^\lambda},$$

for $\lambda > 0$. Since q_j^k is defined recursively—it only depends of what lower step thinkers do—it is straightforward to compute the predicted choice probabilities numerically for each type of k-step thinker (for given values of τ and λ) using a loop, then aggregating the estimated p_k^k across steps k. Apart from the number of players and the number of strategies, there are two parameters: the average number of thinking steps, τ, and the precision parameter, λ.

Figure 6 shows the prediction of the cognitive hierarchy model for the parameters of the field LUPI game, i.e., when $n = 53,783$ and $K = 99,999$. The dashed line corresponds to the case when players do relatively few steps of reasoning and their responses are very noisy ($\tau = 3$ and $\lambda = 0.008$). The dotted line corresponds to the case when players do more steps of reasoning and respond more precisely ($\tau = 10$ and $\lambda = 0.011$). Increasing τ and λ creates a closer approximation to the Poisson-Nash equilibrium, although even

[26] The CH model with best-response piles up most predicted responses at a very small range of the lowest integers (1-step thinkers choose 1, 2-step thinkers choose 2, and k-step thinkers will never pick a number higher than k). Assuming quantal response smoothes out the predicted choices over a wider number range.

[27] In many previous studies logit choice functions are typically used and they fit comparably to power functions (e.g., Camerer and Ho, 1998 for learning models). Some QRE applications have used power functions and found better fits (e.g., in auctions, Goeree, Holt and Palfrey, 2002). However, in this case a logit choice function fits substantially worse for the field data (with 99,999 numbers to choose from). The reason is that logit choice probabilities are convex in expected payoff. This implies, numerically that probabilities are either substantial for only a small number of the 99,999 possible numbers, or are close to uniform across numbers. The logit CH model simply cannot fit the intermediate case in which thousands of number are chosen with high probability and many other numbers have very low probability (as in the data).

with a high τ there are too many choices of low numbers.

There is a clear contrast between the ways in which QRE and CH models deviate from equilibrium. QRE predicts number choices will be more evenly spread across the entire range than what equilibrium predicts, so it predicts too *few* low numbers compared to equilibrium. CH predicts there will be too *many* low numbers (see Figure 6). This distinction in how the two theories deviate from equilibrium is useful for comparing them because the deviations these two theories predict from equilibrium often coincide (see Rogers, Palfrey and Camerer, 2009).

Unfortunately, the QRE model is not estimated for two reasons: First, it is very computationally challenging to estimate for the large-scale field data and we have not been able to do so.[28] Second, if the QRE approaches the Poisson-Nash equilibrium smoothly from random to Poisson-Nash, then it cannot account for overshooting of low numbers and will not explain the major deviation in the data. Indeed, it is likely that the best-fitting QRE function is very close to Poisson-Nash (i.e., a high value of λ), since most of the choices are below 5000 and there is substantial overshooting in that region which we conjecture (but cannot prove) that QRE can only fit by approximating Poisson-Nash.

Can the cognitive hierarchy model account for the main deviations from equilibrium described in the previous section? Table 2 reports the results from the maximum likelihood estimation of the data using the cognitive hierarchy model.[29] The best-fitting estimates week-by-week, shown in Table 2, suggest that both parameters increase over time. The average number of thinking steps that people carry out, τ, increases from about 3 in the first week—an estimate reasonably close to estimates from 1.0 to 2.5 that typical fit experimental data sets well (Camerer et al., 2004)—to 10 in the last week.

Figure 7 shows the average daily frequencies from the first week together with the CH estimation and the equilibrium prediction. The CH model does a reasonable job of

[28]Keep in mind that the CH model includes a quantal response component as well. However, because the CH model is recursive (level-k behavior is determined by lower-level behavior and λ) it is much easier to estimate.

[29]It is difficult to guarantee that these estimates are global maxima since the likelihood function is not smooth and concave. We also used a relatively coarse grid search, so there may be other parameter values that yield slightly higher likelihoods and different parameter values.

Week	1	2	3	4	5	6	7
τ	2.98	5.83	7.32	7.20	7.82	10.27	10.27
λ	0.0080	0.0094	0.0103	0.0108	0.0110	0.0108	0.0107

Table 2: Maximum likelihood estimation of the cognitive hierarchy model for field data

accounting for the over- and undershooting tendencies at low and intermediate numbers (with the estimated $\hat{\tau} = 2.98$). In later weeks, the week-by-week estimates of τ drift upward a little (and λ increases slightly), which is a reduced-form model of learning as an increase in the mean number of thinking steps (see more details below). In the last week the cognitive hierarchy prediction is much closer to equilibrium (because τ is around 10) but is still consistent with the smaller amounts of over- and undershooting of low and intermediate numbers (see Figure 8).

To get some notion of how close to the data the fitted cognitive hierarchy model is, Table 3 displays two goodness-of-fit statistics. First, the log-likelihoods reveal that the cognitive hierarchy model does better in explaining the data toward the last week and is always much better than Poisson-Nash.[30] Second, in order to compare the CH model with the equilibrium prediction, we calculate the proportion of the empirical density that lies below the predicted density. This measure is one minus the summed "miss rates", the differences between actual and predicted frequencies, for numbers which are chosen more often than predicted. If there is a lot of overshooting this statistic is low and if there is very little overshooting this statistic is close to 1.

The cognitive hierarchy model does better than the equilibrium prediction in all seven weeks based on this statistic. For example, in the first week, 61 percent of players' choices were consistent with the cognitive hierarchy model, whereas only 50 percent were consistent with equilibrium. However, both models improve substantially across the weeks.

[30]Since the computed Poisson-Nash equilibrium probabilities are ϵ for $k > 5518$, the likelihood is always essentially zero for the equilibrium prediction. In Appendix C, however, we compute the log-likelihood for the low numbers only. Based on the Schwarz (1978) information criterion, the cognitive hierarchy model still performs better in all weeks.

Week	1	2	3	4	5	6	7
Log-likelihood CH	-63956	-36390	-23716	-20546	-20255	-19748	-18083
Proportion below CH (%)	61.08	72.50	77.69	79.87	81.86	82.63	81.94
Proportion below equil. (%)	49.56	61.82	67.66	67.70	70.23	76.79	76.61

The proportion below the theoretical prediction refers to the fraction of the empirical density that lies below the theoretical prediction, or one minus the fraction of overshooting.

Table 3: Goodness-of-fit for cognitive hierarchy and equilibrium for field data

3.4 Learning

Explaining rapid convergence in the LUPI game is challenging for traditional models of learning in games. A wide range of learning dynamics are likely to converge to equilibrium in the limit, but it is more difficult to explain how players can learn to play close to equilibrium in only 49 rounds. For example, simple learning based on reinforcement of chosen strategies is far too slow because players win rarely (and hence, their strategies are rarely reinforced). Belief-based models like fictitious play also have a hard time explaining the speed of learning because of the special structure of the game: In LUPI, a belief learner *should not* respond to the winning number, but should instead choose the lowest unchosen number below the winning number. But this number is only reported on the website. Moreover, in the field, the lowest unchosen number is typically *above* the winning number (in 43 of 49 days). Hybrid models like EWA (Camerer and Ho, 1999, Ho, Camerer and Chong, 2007) require the same information as fictitious play and therefore do not fit any better in this information environment.

Explaining learning therefore requires a model that 1) does not rely on best responses to the full empirical distribution, that 2) does not only consider a player's own payoff and 3) is not based on any other information than the structure of the game, a player's own experience, and the winning numbers.[31] An appealing alternative which satisfies these three criteria is a simple imitation-learning model in which all players imitate a window of numbers around the previous winning number. Since players' payoffs are sym-

[31]Moreover, we can only fit models of representative agents since not all players participate in each round (due to population uncertainty), and we do not have individual level data in the field.

metric, imitating winning numbers is psychologically similar to counterfactual "fictive" reinforcement of unchosen numbers. In fact, in explaining learning in weak-link games (Roth, 1995) and proposer competition ultimatum games ("market games", Roth and Erev, 1995), Roth and Erev note that reinforcement according to chosen strategies fits very poorly, so they substitute a differnt a model based on imitating the most successful players.

An imitation model is empirically motivated by the fact that players clearly change strategies in the direction of previous winners (as in "direction learning", see Selten and Buchta, 1998). Figure 11 shows how the relationship between the median number chosen in period t in the field is related to the median of the winning numbers from period 1 until $t-1$.

Let $A_k(t)$ denote the attraction of strategy k in period t. Based on these attractions, players probabilistically pick numbers in the next period using a power function so that the probability of picking number k in the next period is

$$p_k(t+1) = \frac{A_k(t)^{\lambda}}{\sum_{j=1}^{K} A_j(t)^{\lambda}}. \tag{2}$$

Note that $\lambda = 0$ means uniform randomization and $\lambda \to \infty$ means playing only the strategy with the highest attraction.

Any learning model requires an assumption about the choice probabilities in the first period, $p_k(1)$. We use the empirical frequencies to create choice probabilities in the first period ("burning in"). Given these probabilities and λ, we determine $A(1)$ so that equation (2) gives the assumed choice probabilities $p_k(1)$. Since the power choice function is invariant to scaling, we determine the attractions in the first period so that they sum to one, i.e., $\sum_{k=1}^{K} A_k(1) = 1$. From the second period onwards, strategies are reinforced by a factor $r_k(t)$, which depends on the winning number in period $t-1$. For the empirical estimation of the learning model we use the actual winning numbers from the field. Attractions in period $t > 1$ are given by[32]

[32]More complicated variants of this function are possible, e.g., weighting lagged attractions by a "forget-

$$A_k(t) = \frac{A_k(t-1) + r_k(t)}{1 + \sum_{j=1}^{K} r_j(t)}.$$

The reinforcement factors are determined by the winning number in the previous period (if there is no winning number, the same attractions carry over to the next period). However, since the strategy sets are so large, only reinforcing the previous winning number would predict learning that is too slow and much too tightly clustered on previous winners. We therefore follow Sarin and Vahid (2004) by assuming that numbers that are "similar" to the winning number are also reinforced. We use the triangular Bartlett similarity function used by Sarin and Vahid (2004), which puts reinforcement on strategies near the previous winner that declines linearly with distance from that winning number. Let W denote the size of the "similarity window" and $k^*(t-1)$ the winning number in the previous round. Then the reinforcement factors in period t are given by

$$r_k(t) = \frac{\max\{0, 1 - |k - k^*(t-1)|/W\}}{\sum_{j=1}^{K} r_j(t)}.$$

Note that the reinforcement factors are scaled so that they sum to one, just as the first period attractions were scaled to sum to one.[33]

The learning model has two parameters: the size of the similarity window, W, and the precision of the choice function, λ. We estimate the best-fitting values by minimizing the squared deviation between predicted choice densities and empirical densities summed over all numbers and rounds. The estimated values for the field data are $W = 344$ and $\lambda = 0.0085$.

To see how the learning model fits the data, Figure 12 displays the average weekly predicted densities of the learning model for numbers up to 6000 (along with the data and Poisson-Nash equilibrium). The main feature of learning is that the number of low numbers shrinks and the gap between the predicted frequency of numbers between 2000

ting" or decay factor. However, it is difficult to identify parameters in these more complicated functions (e.g., a faster decay factor will look like a wider window) and creating an ideal specification lies beyond the major scope of the paper.

[33]Figure A7 shows an example of the reinforcement factors when $k^*(t-1) = 10$ and $W = 3$.

and 5000 is gradually filled in. Figure A23 shows summary statistics week-by-week in a boxplot.

4 The Laboratory LUPI Game

We conducted a parallel lab experiment for two reasons.

First, the rules of the field LUPI game do not exactly match the theoretical assumptions used to generate the Poisson-Nash equilibrium prediction. In the field data some choices were made by a random number generator, some players might have chosen multiple numbers or colluded, there were multiple prizes, and the variance in N is larger than the Poisson distribution variance.

In the lab, we can more closely implement the assumptions of the theory. If the theory fits poorly in the field and closely in the lab, then that suggests the theory is on the right track when its underlying assumptions are most carefully controlled. If the theory fits closely in both cases, that suggests that the additional factors in the field that are excluded from the theory do not matter.

Second, because the field game is rather simple, it is possible to design a lab experiment which closely matches the field in its key features. How closely the lab and field data match provides some evidence in ongoing debate about how well lab results generalize to comparable field settings (e.g., Levitt and List, 2007).

In designing the laboratory game, we compromise between two goals: to create a simple environment in which theory should apply (theoretical validity), and to recreate the features of the field LUPI game in the lab (specialized external validity). Because we use this opportunity to create an experimental protocol that is closely matched to a particular field setting, we often sacrificed theoretical validity in favor of close field replication.

The first choice is the scale of the game: The number of players (N), possible number choices (K), and stakes. We choose to scale down the number of players and the largest payoff by a factor of 2000. This implies that there were on average 26.9 players and the prize to the winner in each round was \$7. For $K = 99$ the shape of the equilibrium distribution has some of the basic features of the equilibrium distribution for the field data parameters (e.g. most numbers should be below 10 percent of K). Since the field data span 49 days, the experiment also has 49 rounds in each session. (We typically refer to experimental rounds as "days" and seven-"day" intervals as "weeks" for semantic comparability between the lab and field descriptions.)

The number of players in each round was drawn from a distribution with mean 26.9. In three of the four sessions, subjects were told the mean number of players, and that the number varied from round to round, but did not know the distribution (in order to match the field situation in which players were very unlikely to know the total number playing each day). Due to a technical error, in these three sessions, the variance was lower than the Poisson variance (7.2 to 8.6 rather than 26.9). However, this mistake is likely to have little effect on behavior because subjects did not know the total number of players in each round.[34] In the last session, the number of players in each round was drawn from a Poisson distribution with mean 26.9 and the subjects were informed about this. Furthermore, the data from the true Poisson session and the lower-variance sessions look statistically similar so we pool them for all analysis (see below).

Some design choices made the lab setting different from the field setting but closer to the assumptions of the theory. In contrast to the field game, in the lab each player was allowed to choose only one number, they could not use a random number generator, there was only one prize per round, \$7, and if there was no unique number nobody won.

In the field data we do not know how much Swedish players learned each day about the full distribution of numbers that were chosen. The numbers were available online and partially reported on a TV show. To maintain parallelism with the field, only the winning number was announced in the lab.

[34]Note that it is very difficult to draw an exact inference about the underlying distribution of the number of players based only on winning numbers – winning numbers are likely to be lower when there are few players, but the relationship is far from deterministic.

Four laboratory sessions were conducted at the California Social Science Experimental Laboratory (CASSEL) at the University of California Los Angeles on the 22nd and 25th of March 2007, and on the 3rd of March 2009. The experiments were conducted using the Zürich Toolbox for Ready-made Economic Experiments (zTree) developed by Urs Fischbacher, as described in Fischbacher (2007). Within each session, 38 graduate and undergraduate students were recruited, through CASSEL's web-based recruiting system. All subjects knew that their payoff will be determined by their performance. We made no attempt to replicate the demographics of the field data, which we unfortunately know very little about. However, the players in the laboratory are likely to differ in terms of gender, age and ethnicity compared to the Swedish players. In the four sessions, we had slightly more male than female subjects, with the great majority clustered in the age bracket of 18 to 22, and the majority spoke a second language. Half of the subjects had never participated in any form of lottery before. Subjects had various levels of exposure to game theory, but very few had seen or heard of a similar game prior to this experiment.

4.1 Experimental Procedure

At the beginning of each session, the experimenter first explained the rules of the LUPI game. The instructions were based on a version of the lottery form for the field game translated from Swedish to English (see Appendix E). Subjects were then given the option of leaving the experiment, in order to see how much self-selection influences experimental generalizability. None of the recruited subjects chose to leave, which indicates a limited role for self-selection (after recruitment and instruction).

In three of the four sessions, subjects were told that the experiment would end at a predetermined, but non-disclosed time to avoid an end-game effect (also matching the field setting, which ended abruptly and unexpectedly). Subjects were also told that participation was randomly determined at the beginning of each round, with 26.9 subjects participating on average. Subjects in the fourth session were explicitly told there were 49 rounds, and the number of players was drawn from a Poisson distribution. They were also shown in the instructions a graph showing a distribution function for a Poisson distribution with mean 26.9.

In the beginning of each round, subjects were informed whether they would actively participate in the current round (i.e., if they had a chance to win). They were required to submit a number in each round, even if they were not selected to participate. The difference between behavior of selected and non-selected players gives us some information about the effect of marginal incentives on performance (cf. Camerer and Hogarth, 1999).

When all subjects had submitted their chosen numbers, the lowest unique positive integer was determined. If there was a lowest unique positive integer, the winner earned \$7; if no number was unique, no subject won. Each subject was privately informed, immediately after each round, what the winning number was, whether they had won that particular round, and their payoff so far during the experiment. This procedure was repeated 49 times, with no practice rounds (as is the case of the field). After the last round, subjects were asked to complete a short questionnaire which allowed us to build the demographics of our subjects and a classification of strategies used. In two of the sessions, we included the cognitive reflection test as a way to measure cognitive ability (to be described below). All sessions lasted for less than an hour, and subjects received a show-up fee of \$8 or \$13 in addition to earnings from the experiment (which averaged \$8.60). Screenshots from the experiment are shown in Appendix E.

4.2 Lab Descriptive Statistics

We focus only on the choices from incentivized subjects that were selected to actively participate in the remainder of the paper. It is noteworthy, however, that the choices of participating and non-participating subjects did not significantly differ (p-value 0.16, Mann-Whitney). The choices from the session with the announced Poisson distribution and the pooled other three sessions do not significantly differ at the five percent level ($p = 0.59$, t-test with clustered standard errors). In the remainder of the paper we therefore pool all four sessions.

Figure 9 shows the data for the choices of participating players (together with the Poisson-Nash equilibrium prediction). There are very few numbers above 20 so the numbers 1 to 20 are the focus in subsequent graphs. In line with the field data, players have a slight predilection for certain numbers, while others are avoided. Judging from Figure 9, subjects avoid some even numbers, especially 10, while they endorse the odd (and prime) numbers 11, 13 and 17. Interestingly, only one subject played 20, while 19 was played ten times and 21 was played seven times.

	All rounds				R 1-7	R. 43–49	Equil.
	Avg.	Std.dev.	Min	Max	Avg.	Avg.	Avg.
Average number played	6.0	1.4	4.3	12.5	8.6	5.8	5.2
Median number played	4.7	1.0	3	10	6.1	5	5
Below 20 (%)	98.02	2.77	81.98	100.00	93.94	98.42	100.00
Even numbers (%)	45.19	4.47	35.16	53.47	42.11	49.15	46.86
Session 1							
Winning number	6.0	9.4	1	67	13.0	2.5	2.9
Lowest number not played	8.1	2.6	1	12	4.9	8.1	3.3
Session 2							
Winning number	5.1	2.6	1	10	5.8	5.1	2.9
Lowest number not played	7.5	3.0	1	12	6.3	8.4	3.3
Session 3							
Winning number	5.6	3.3	1	14	6.1	5.7	2.9
Lowest number not played	7.5	2.7	2	13	7.4	10.0	3.3
Session 4 (Poisson)							
Winning number	5.8	3.6	1	17	6.7	3.1	2.9
Lowest number not played	7.6	3.4	1	13	5.1	8.1	3.3

Summary statistics are based only on choices of subjects who are selected to participate. The equilibrium column refers to what would result if all players played according to equilibrium ($n = 26.9$ and $K = 99$)

Table 4: Descriptive statistics for laboratory data

Table 4 shows some descriptive statistics for the participating subjects in the lab experiment. As in the field, some players in the first week tend to pick very high numbers (above 20) but the percentage shrinks by the seventh week. The average number chosen in the last week corresponds closely to the equilibrium prediction (5.8 vs. 5.2) and the medians are identical (5.0). The average winning numbers are too high compared to equilibrium play, which is consistent with the observation that players pick very low numbers too much, creating non-uniqueness among those numbers so that unique numbers are unusually high. The tendency to pick odd numbers decreases over time—42 percent of all numbers are even in the first week, whereas 49 percent are even in the last week. As in the field data, the overwhelming impression from Table 4 is that convergence to equilibrium is quite rapid over the 49 periods (despite receiving feedback only about the winning number).

4.3 Aggregate Results

In the Poisson equilibrium with 26.9 average number of players, strictly positive probability is put on numbers 1 to 16, while other numbers have probabilities numerically indistinguishable from zero. Figure 10 shows the average frequencies played in week 1 to 7 together with the equilibrium prediction (dashed line) and the estimated week-by-week results using the cognitive hierarchy model (solid line). These graphs clearly indicates that learning is quicker in the laboratory than in the field. Despite that the only feedback given to players in each round is the winning number, behavior is remarkably close to equilibrium already in the second week. However, we can also observe the same discrepancies between the equilibrium prediction and observed behavior as in the field. The distribution of numbers is too spiky and there is overshooting of low numbers and undershooting at numbers just below the equilibrium cutoff (at number 16).

Figure 10 also displays the estimates from a maximum likelihood estimation of the cognitive hierarchy model presented in the previous section (solid line).[35] The cognitive hierarchy model can account both for the spikes and the over- and undershooting. Table 5 shows the estimated parameters.[36] There is no clear time trend in the two parameters, and in some rounds the average number of thinking steps is unreasonably large compared

[35] To illustrate how the CH model behaves, consider $N = 26.9$ and $K = 99$, with $\tau = 1.5$ and $\lambda = 2$. Figure A22 shows how 0 to 5 step thinkers play LUPI and the predicted aggregate frequency, summing across all thinking steps. In this example, 1-step thinkers put most probability on number 1, 2-step thinkers put most probability on number 5, and 3–step thinkers put most probability on numbers 3 and 7.

[36] The log-likelihood function is neither smooth nor concave, so the estimated parameters may not reflect a global maximum of the likelihood.

to other experiments showing τ around 1.5. Since there are two free parameters with relatively few choice probabilities to estimate, we might be over-fitting by allowing two free parameters. We therefore estimate the precision parameter λ while keeping the average number of thinking steps fixed. We set the average number of thinking steps to 1.5, which has been shown to be a value of τ that predicts experimental data well in a large number of games (Camerer et al., 2004). The estimated precision parameter is considerably lower in the first week, but is then relatively constant.[37]

Table 5 also displays the maximum likelihood estimate of λ for the power QRE. The precision parameter is high from the second week and onwards. Recall from Figure 5 that the QRE prediction for such high λ is very close to the Poisson-Nash equilibrium.

Week	1	2	3	4	5	6	7
τ	8.98	11.80	12.91	13.97	11.85	11.98	7.00
λ	1.31	11.79	16.23	16.50	15.20	18.29	9.55
λ ($\tau = 1.5$)	1.09	2.52	2.57	2.63	2.60	2.31	2.08
λ_{QRE}	1.32	10.99	10.25	8.95	10.75	12.73	6.89

Table 5: Maximum likelihood estimation of the cognitive hierarchy model and QRE for laboratory data

Table 6 provides some goodness-of-fit statistics for the cognitive hierarchy model, QRE and the equilibrium prediction. Based the proportion of the empirical density that lies below the predicted density, the equilibrium prediction does remarkably well. However, the cognitive hierarchy model (with two free parameters) does better than the equilibrium prediction in all but the sixth week. QRE performs better than equilibrium in the first week, but is practically indistinguishable from equilibrium after the first week (due to high λ). The log-likelihood of the cognitive hierarchy model (with two parameters) is higher than the QRE during all weeks.[38]

[37]Figure A4 shows the fitted cognitive hierarchy model when τ is restricted to 1.5. It is clear that the model with $\tau = 1.5$ can account for the undershooting also when the number of thinking steps is fixed, but it has difficulties in explaining the overshooting of low numbers. The main problem is that with $\tau = 1.5$, there are too many zero-step thinkers that play all numbers between 1 and 99 with uniform probability. The log-likelihoods for the CH model with $\tau = 1.5$ range from -241 in week 1 to -212 in week 2, which are much worse than power QRE or unrestricted CH.

[38]In Appendix C we calculate the log-likelihoods using data from numbers 1 to 16, which allows us

Week	1	2	3	4	5	6	7
Log-likelihood CH	-166.9	-84.5	-80.6	-86.1	-86.3	-83.7	-87.7
Log-likelihood power QRE	-167.0	-94.6	-108.5	-108.8	-102.4	-94.5	-112.4
Proportion below CH (%)	87.83	91.55	92.78	93.50	91.65	91.86	92.98
Proportion below power QRE (%)	87.83	88.75	87.23	88.12	88.39	91.73	87.17
Proportion below eq. (%)	82.25	88.55	87.61	88.64	88.64	92.86	87.06

The proportion below the theoretical prediction refers to the fraction of the empirical density that lies below the theoretical prediction.

Table 6: Goodness-of-fit for cognitive hierarchy, QRE and equilibrium for laboratory data

On the aggregate level, behavior in the lab is remarkably close to equilibrium from the second to the last week. The cognitive hierarchy model can rationalize the tendencies that some numbers are played more, as well as the undershooting below the equilibrium cutoff. The value-added of the cognitive hierarchy model is not primarily that it gives a slightly better fit, but that it provides a plausible story for how players manage to play so close to equilibrium. Most likely, few players would be capable of calculating the equilibrium during the course of the experiment, whereas many of them should be able to carry out a few steps of reasoning along the lines of the cognitive hierarchy model.

4.4 Individual Results

An advantage of the lab over the field, in this case, is that the behavior of individual subjects can be tracked over time and we can gather more information about them to link to choices. Appendix E discusses some details of these analyses but we summarize them here only briefly.

In a post-experimental questionnaire, we asked people to state why they played as they did. We coded their responses into four categories (sometimes with multiple categories): "Random", "stick" (with one number), "lucky", and "strategic" (explicitly mentioning response to strategies of others). The four categories were coded 35%, 30%, 11% and 38% of the time. These categories had some relation to actual choices because "stick"

to compare the equilibrium prediction with the other models. Based on Schwarz (1978) information criterion, both QRE and cognitive hierarchy (with two parameters) outperforms equilibrium.

players chose fewer distinct numbers and "lucky" players had number choices with a higher mean and higher variance. The only demographic variable with a significant effect on choices and payoffs was "exposure to game theory"; those subjects chose numbers with less variation across rounds. A measure of "cognitive reflection" (Frederick, 2005), a short-form IQ test, did not correlate with choice measures or with payoffs.

As is often seen in games with mixed equilibria, there is some mild evidence of "purification" since subjects chose only 9.65 different numbers on average (see Appendix E), compared to 10.9 expected in Poisson-Nash equilibrium.

	All periods	Week 1	Week 2	Week 3-7
Round (1-49)	0.001	-0.109	-0.065	0.023
	(0.13)	(-0.42)	(-0.62)	(1.58)
$t-1$ winner	0.178***	0.148**	0.304***	0.059*
	(4.89)	(2.38)	(2.98)	(1.89)
$t-2$ winner	0.133***	0.096	0.242**	0.038*
	(2.98)	(1.18)	(2.40)	(1.68)
$t-3$ winner	0.083***	0.052	-0.050	0.030
	(1.94)	(0.65)	(-0.63)	(1.18)
Fixed effects	Yes	Yes	Yes	Yes
Observations	4360	421	585	3354
R^2	0.03	0.09	0.01	0.00

*=10 percent, **=5 percent and ***=1 percent significance level. The table report results from a linear subject fixed effects panel regression. Only actively participating subjects are included. $t-$statistics based on clustered standard errors are within parentheses.

Table 7: Panel data regressions explaining individual number choices in the laboratory

In the post-experimental questionnaire, several subjects said that they responded to previous winning numbers. To measure the strength of this learning effect we regressed players' choices on the winning number in the three previous periods. Table 7 shows that the winning numbers in previous rounds do affect players' choices early on, but this tendency to respond to previous winning numbers is considerably weaker in later weeks (3 to 7). The small round-specific coefficients in Table 7 also show that there does not appear to be any general trend in players' choices over the 49 rounds.

4.5 Learning Results

The regression analysis reported in Table 7 shows that players' choices in the lab depend on previous winners (at least in early rounds). Hence, this leads naturally to the investigation on learning. Unfortunately, we cannot estimate standard individual learning models since not all players participate in each round.

To maintain comparison with the field, we use the same imitation learning model as discussed in section 3.4. For the laboratory implementation, we divide the estimated window size from the field by 100 and fix $W = 3$. The estimated λ for the laboratory data is 0.31.[39]

As was discussed in the previous section, players in the laboratory seem to learn to play the game much quicker, so there is not so much learning to be explained by the learning model. The learning model can explain some of the ups and downs during the first 14 rounds in the laboratory, as well as the shrinking dispersion of numbers over time, but there is no trend toward higher numbers as seen in the field data. Figure A8 displays box plots for the 14 first rounds in the four sessions. Note that the learning model predicts much more dispersion of numbers in the early rounds in the first session. This is explained by the fact that players played very high numbers in the first round in that session and that a very high number, 67, won in the fourth period. The imitation-based model is substantially affected by that outlying win.

5 Conclusion

It is often difficult to test game theory using field data because equilibrium predictions depend so sensitively on strategies, information and payoffs, which are usually not ob-

[39]Estimating both W and λ for the laboratory data gives $W = 11$ and $\lambda = 1.84$. However, the fit is nearly identical with the smaller window size. For the lab data, W and λ largely play inverse roles. Higher window sizes W combined with higher response sensitivities λ often generate very close squared deviations (since higher W is generating a wider spread of responses and higher λ is tightening the response). The higher W is, the higher is λ, but the overall fit is nearly unchanged as W varies between 3 and 12. See Appendix C for details.

servable in the field. This paper exploits an empirical opportunity to test game theory in a field setting which is simple enough that clear predictions apply (when some simplifying assumptions are made). The game is a LUPI lottery, in which the lowest unique positive integer wins a fixed prize. LUPI is a close relative of auctions in which the lowest unique bid wins.

One contribution of our paper is to characterize the Poisson-Nash equilibrium of the LUPI game and analyze behavior in this game using both a field data set, including more than two million choices, and parallel laboratory experiments which are designed to first permit a clear test of the theory while also matching the field setting. In both the field and lab, players quickly learn to play close to equilibrium, but there are some diagnostic discrepancies between players' behavior and equilibrium predictions.

As noted earlier, the variance in the number of players in the field data is much larger than the variance assumed in the Poisson-Nash equilibrium. So the field data is not an ideal test of this theory, strictly speaking. Therefore, the key issues are how much the theory's predictions vary with changes in $\text{var}(N)$, and how much behavior changes in response to $\text{var}(N)$. If either theory or behavior is insensitive to $\text{var}(N)$, then the Poisson-Nash equilibrium could be a useful approximation to the field data.

As for theory: For the simple examples in which fixed N and Poisson equilibria can be computed, zero variance (fixed N) and Poisson variance equilibria are almost exactly the same (see Appendix A). Keep in mind that increasing $\text{var}(N)$ (holding n constant) implies that sometimes there are a lot of extra players so number choices should be higher, and sometimes there are fewer players so number choices should be lower. These two opposing effects could minimize the effect of variance on mean choices (as the low-K cases in Appendix A suggest they do).

As for behavior: There are two sources of evidence that actual behavior is not too sensitive to $\text{var}(N)$. First, in the field data the Sunday and Monday sessions have lower n and lower standard deviation than all days, but choices are very comparable to data from all days (in which $\text{var}(N)$ about twice as large). Second, in the lab data different sessions with var(N) ≈ 8 and var(N) $= 27$ lead to indistinguishable behavior.

These theoretical and behavioral considerations suggest why the 'wrong' theory (Poisson-Nash) might approximate actual behavior surprisingly well in the field (despite the field var(N) being empirically far from what the theory assumes).

A different way to describe our contribution is this: A LUPI game was actually played in the field, with specific rules. Can we produce *any* kind of theory which fits the data from this game? In this view, it does not matter whether the field setting matches the predictions of a theory exactly. Instead, all that matters is whether the theory fits well, even if its assumptions are wrong.

Here the answer is rather clear: The empirical distribution of choices clearly is moving in the direction of the Poisson-Nash equilibrium over the 49 days (as judged by every number choice statistic) and is numerically close. As a bonus, the CH model improves a little on the Poisson-Nash equilibrium, when optimally parameterized, in the sense that it can explain the key ways in which behavior departs from Poisson-Nash (too many low and very high numbers) in the short run. The estimated number of thinking steps is in the first week 2.98, which is a little higher than estimates from many lab experiments, but within an order of magnitude.

Another contribution is measuring learning week-by-week. Since the subjects have only the winning number to learn from, fictitious play and hybrid EWA models do not apply well. Therefore, we apply a model in which players imitate successful strategies by shifting reinforcement (and hence, choice probability) to strategies in a window around the previous winning number. This model does a reasonable job of explaining the time path of change in the field data. It does a less impressive job in the lab data, largely because choices are so close to the equilibrium in early periods that there is little to learn.

The game is also useful for distinguishing the CH approach and QRE, which often account for deviations from Nash play in the same direction (e.g., Rogers, Palfrey and Camerer, 2009). CH predicts too many low numbers (compared to Poisson-Nash) and QRE predicts too few (in the smaller lab game where QRE can be computed). The field data clearly favor CH over QRE and the lab data also favor CH by a smaller margin.

Note that the point of the learning and cognitive hierarchy models is not simply to fit the data better than Poisson-Nash, but also to show how people with limited computational power might start near, and converge to, such a complex equilibrium.

Finally, because the LUPI field game is simple, it is possible to do a lab experiment that closely replicates the essential features of the field setting (which most experiments are not designed to do). This close lab-field parallelism in design adds evidence to the ongoing debate about when lab findings generalize to parallel field settings (e.g., Levitt and List, 2007). The lab game was described very much like the Swedish lottery (controlling context), experimental subjects were allowed to select out of the experiment after it was described (allowing self-selection), and lab stakes were made equal to the field stakes. Basic lab and field findings are fairly close: In both settings, choices are close to equilibrium, but there are too many large numbers and too few agents choose intermediate numbers at the high end of the equilibrium range. We interpret this as a good example of close lab-field generalization, when the lab environment is designed to be close to a particular field environment.[40]

[40]Of course, it is also conceivable that there is a genuine lab-field behavioral difference but it is approximately cancelled by differences in the design details which have opposite effects. A referee opined that this is "a longshot" but said we should mention it.

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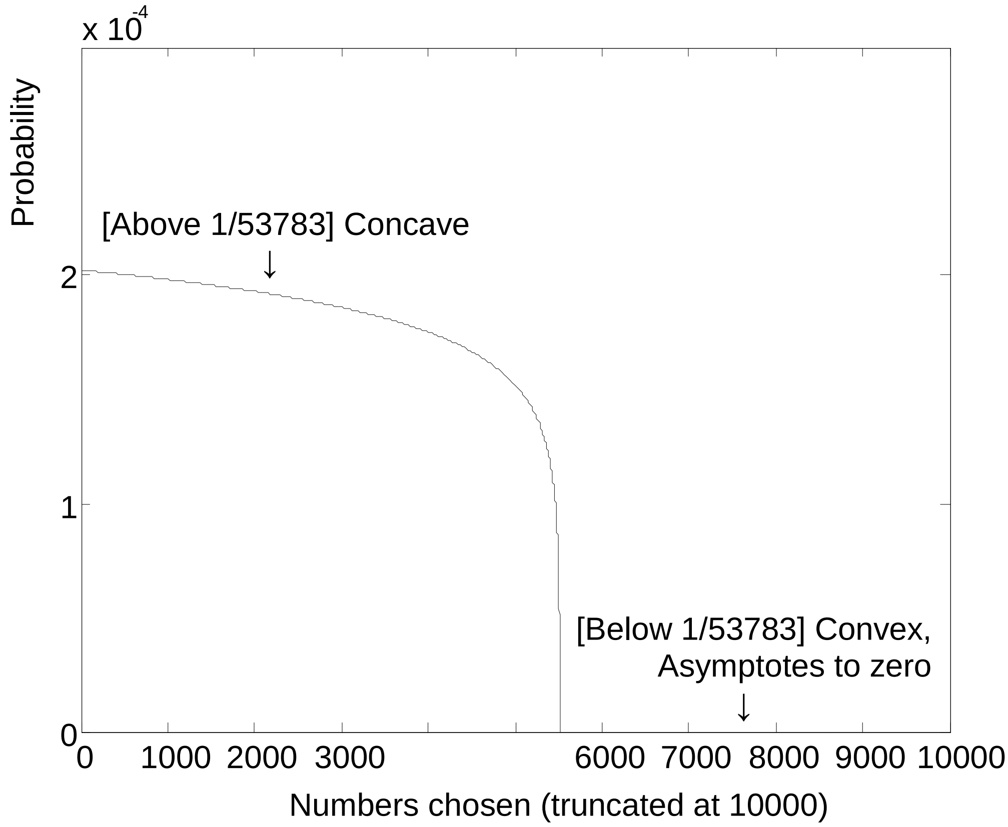


Figure 1. Poisson-Nash equilibrium for the LUPI game (*n*=53783, *K*=99999).

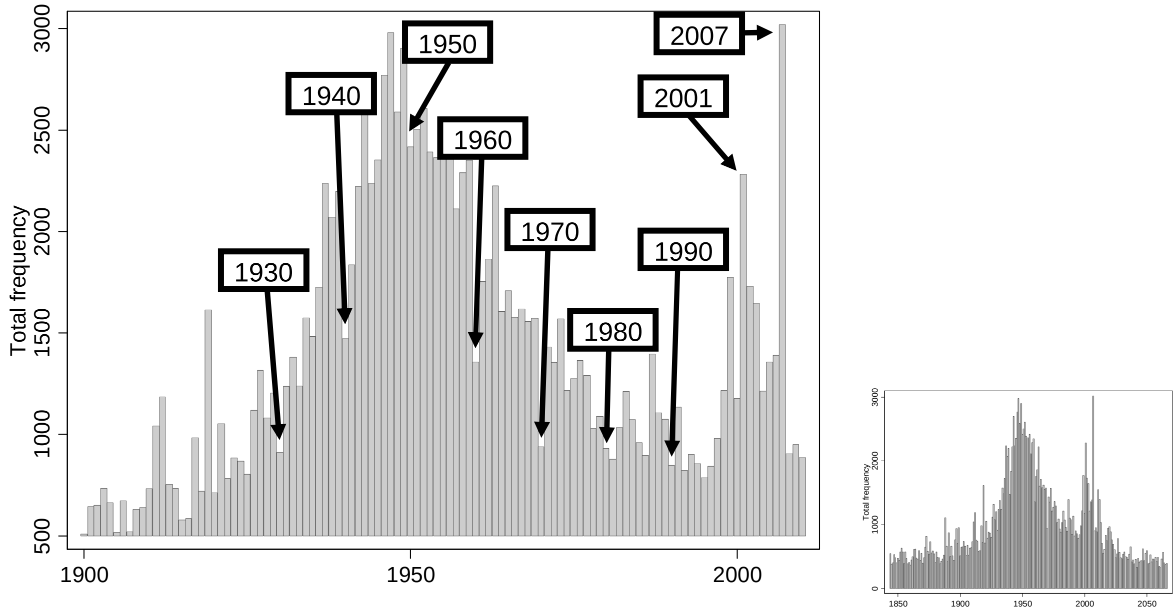


Figure 2. Numbers chosen between 1900 and 2010, and between 1844 and 2066, during all days in the field.

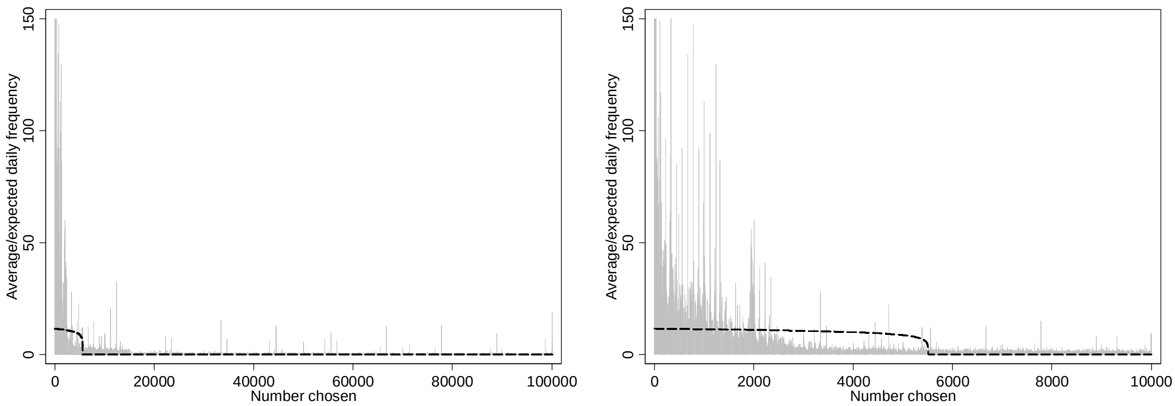


Figure 3. Average daily frequencies and Poisson-Nash equilibrium prediction for the first week in the field (*n*=53783, *K*=99999).

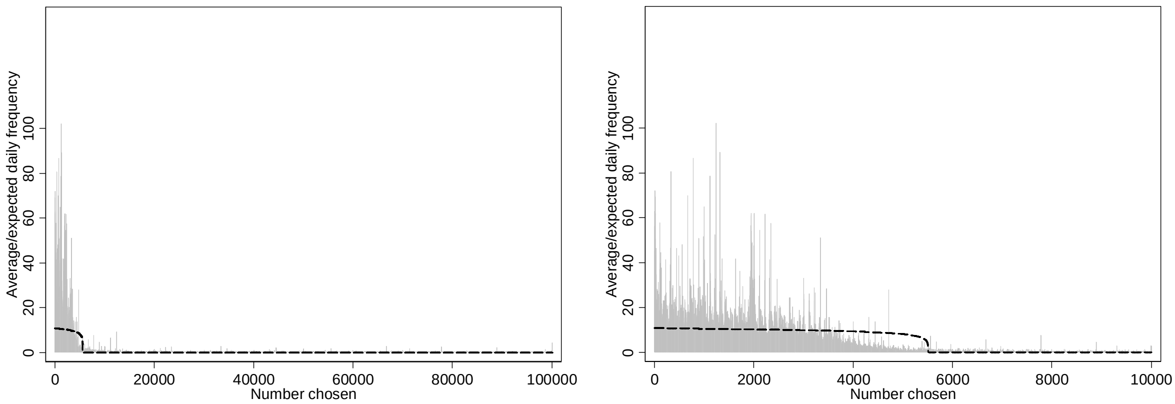


Figure 4. Average daily frequencies and Poisson-Nash equilibrium prediction for week 2-7 in the field (*n*=53783, *K*=99999).

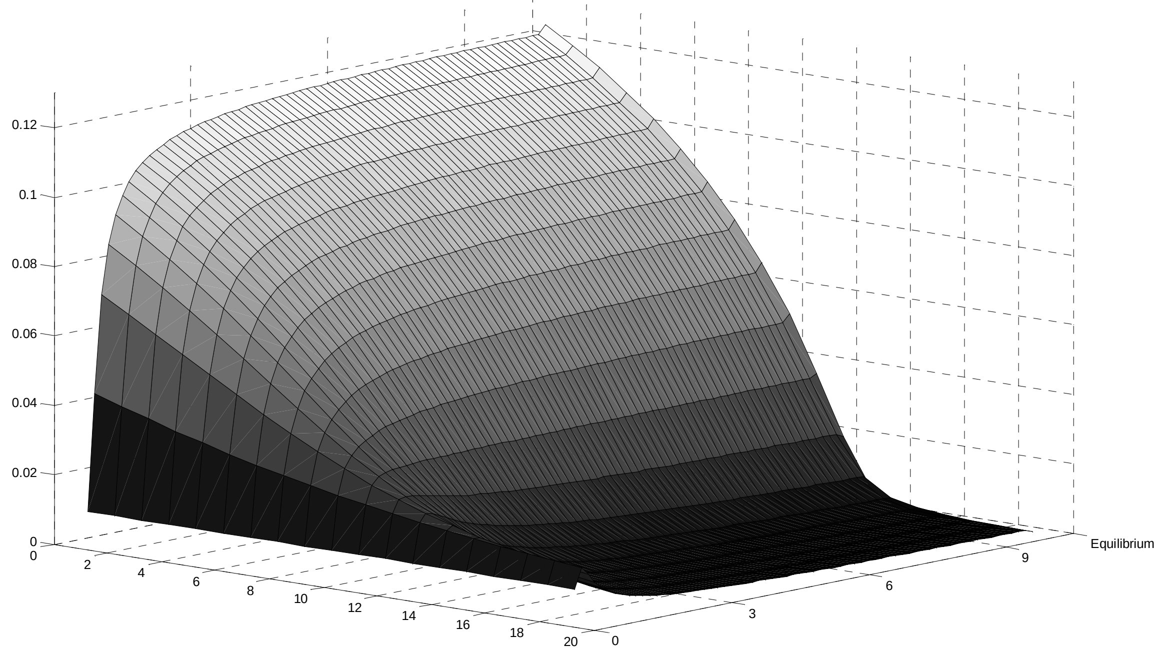


Figure 5. Probability of choosing numbers 1 to 20 in symmetric QRE (*n*=26.9, *K*=99, λ =0.001,...,10) and in the Poisson-Nash equilibrium (*n*=26.9, *K*=99).

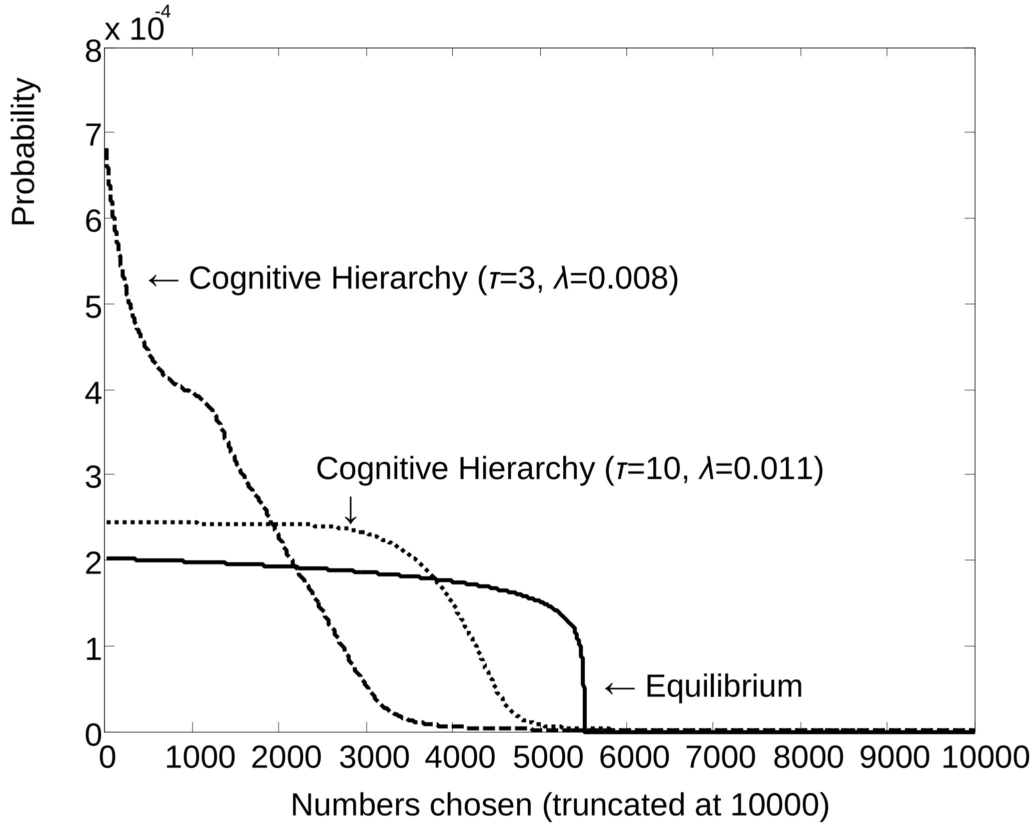


Figure 6. Probability of choosing numbers 1 to 10000 in the Poisson-Nash equilibrium and the cognitive hierarchy model (*n*=53783, *K*=99999).

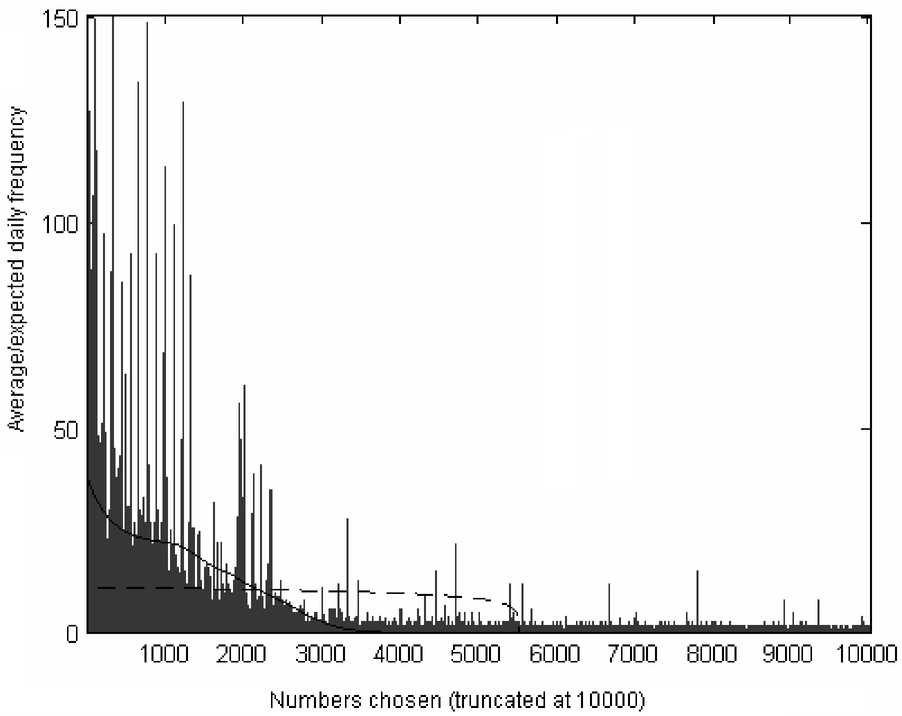


Figure 7. Average daily frequencies, cognitive hierarchy (solid line) and Poisson-Nash equilibrium prediction (dashed line) for the first week in the field (*n*=53783, *K*=99999, τ=2.98, λ=0.008).

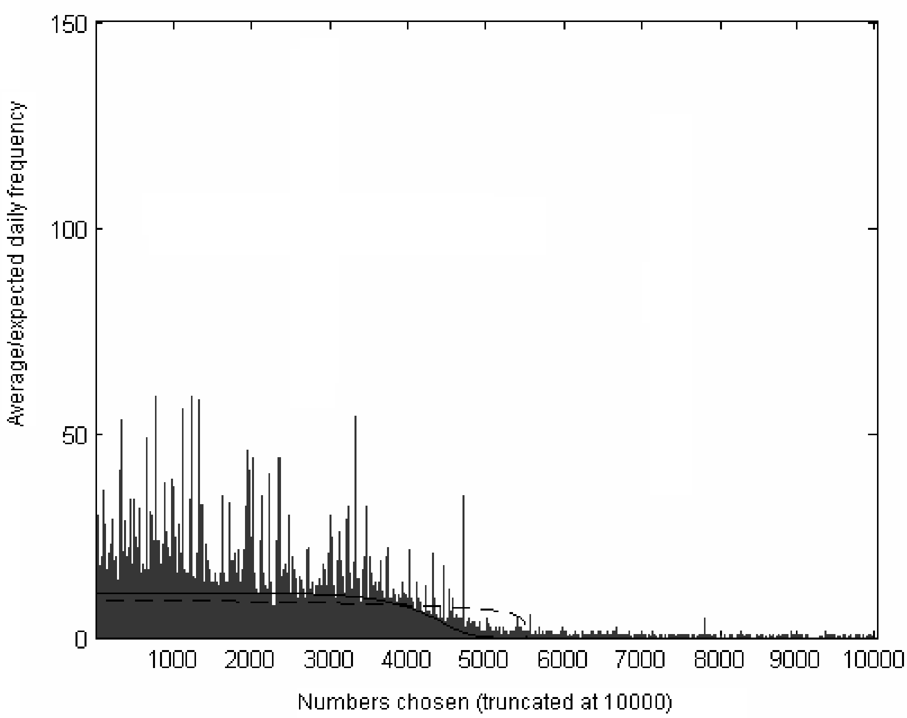


Figure 8. Average daily frequencies, cognitive hierarchy (solid line) and Poisson-Nash equilibrium prediction (dashed line) for the last week in the field (*n*=53783, *K*=99999, τ=10.27, λ=0.0107).

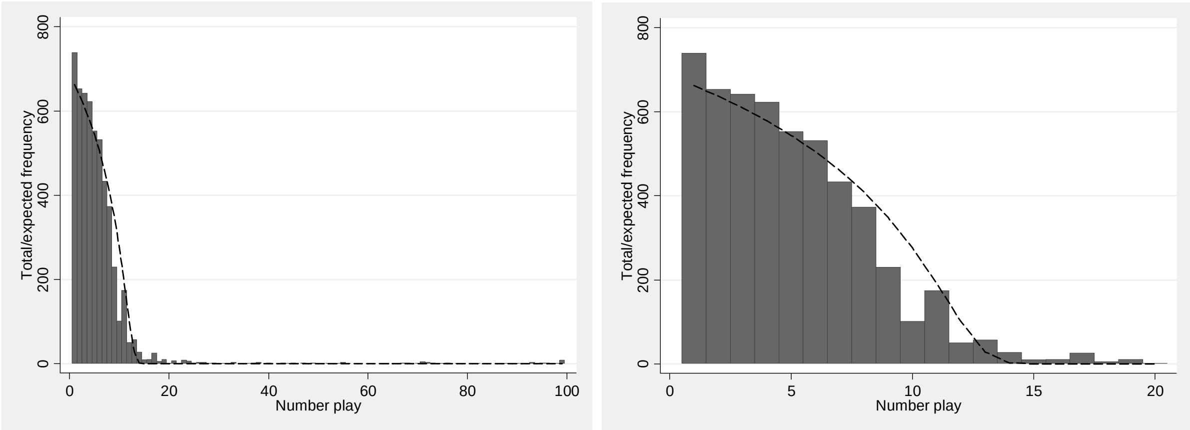


Figure 9. Laboratory total frequencies and Poisson-Nash equilibrium prediction (all sessions, participating players only, *n*=26.9, *K*=99).

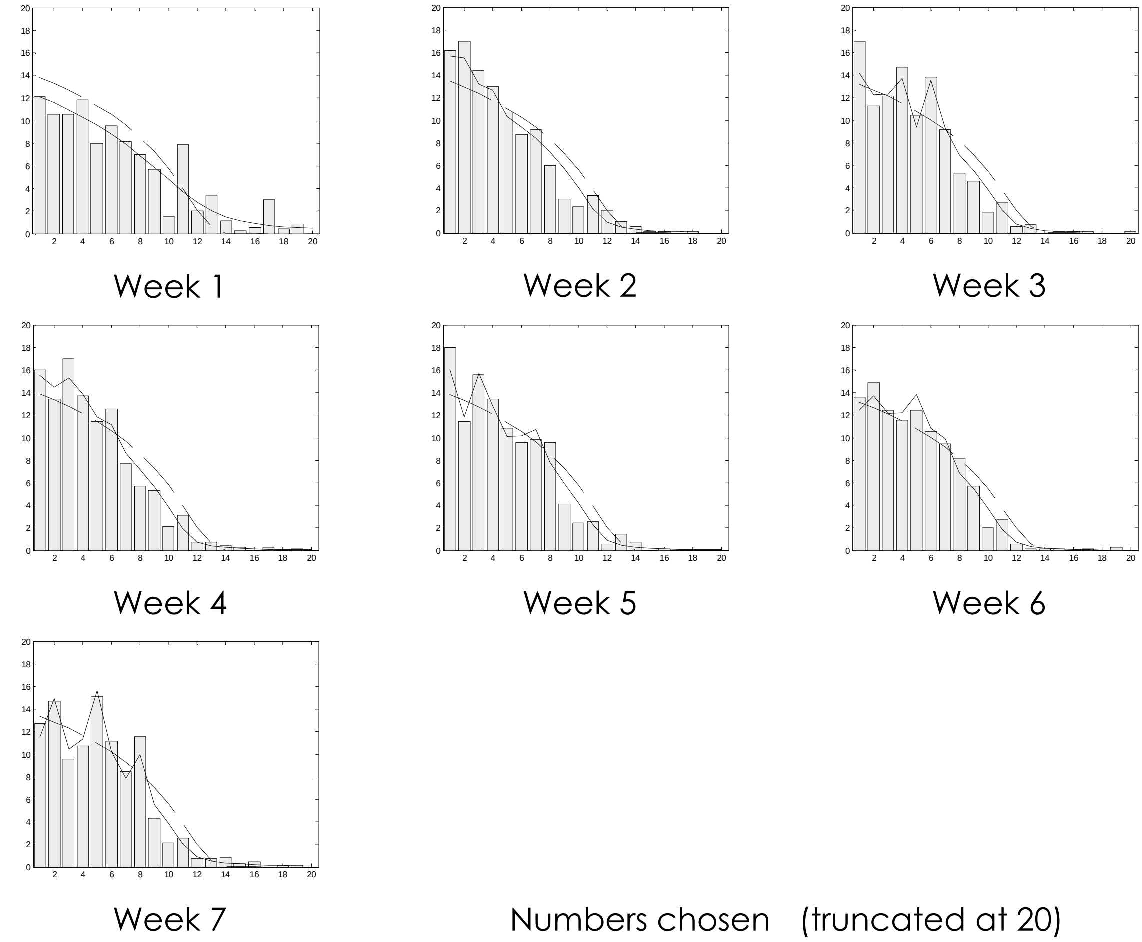


Figure 10. Average daily frequencies in the laboratory, Poisson-Nash equilibrium prediction (dashed lines) and estimated cognitive hierarchy (solid lines), week 1 to 7 (*n*=26.9, *K*=99).

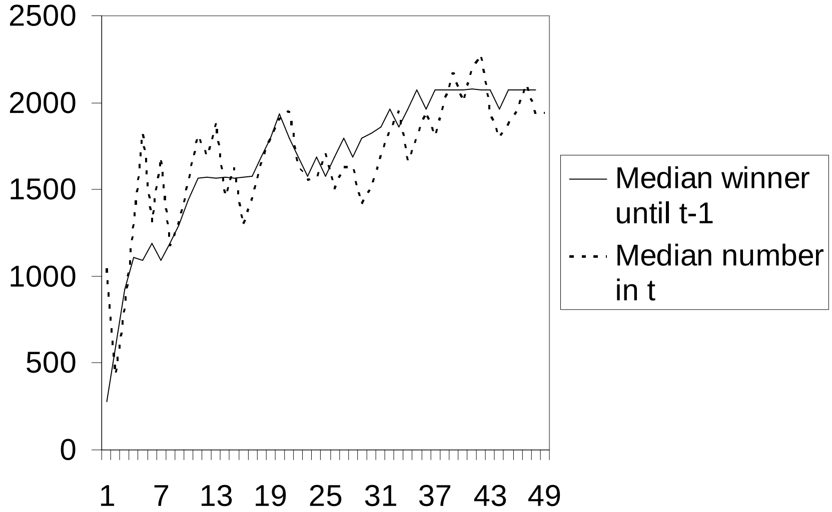


Figure 11. Median winner and median choices in the field

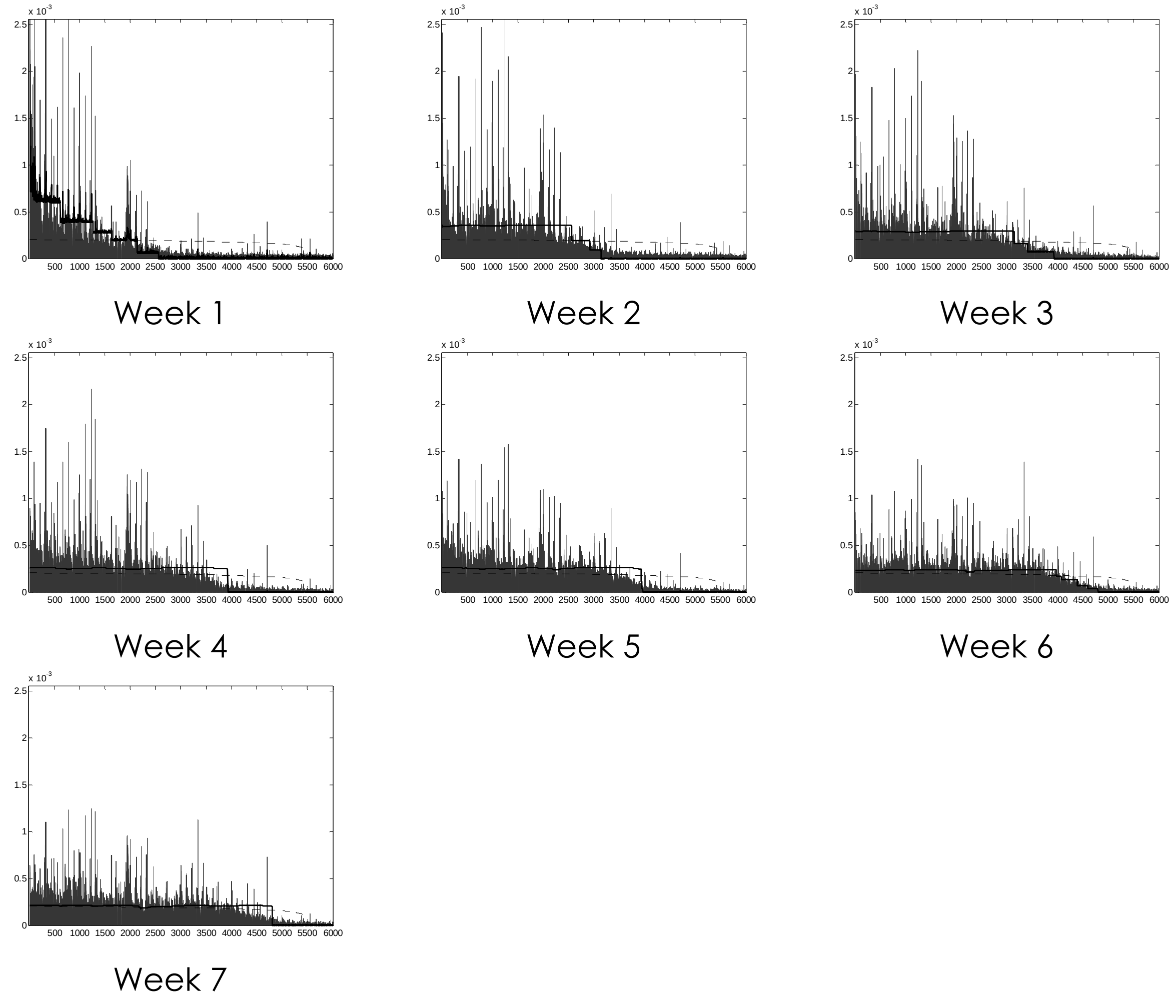


Figure 12. Average weekly empirical densities (bars), estimated learning model (lines) and Poisson-Nash equilibrium (dotted lines) for the field (*W* = 344, λ = 0.0085).

Note that the learning model fits extremely well in week 1 by construction because it was initialized using actual data from week 1.

Appendix [For referees and online availability only]

A. The Symmetric Fixed-n Nash Equilibrium

Let there be a finite number of n players that each pick an integer between 1 and K. If there are numbers that are only chosen by one player, then the player that picks the lowest such number wins a prize, which we normalize to 1, and all other players get zero. If there is no number that only one player chooses, everybody gets zero.

To get some intuition for the equilibrium in the game with many players, we first consider the cases with two and three players. If there are only two players and two numbers to choose from, the game reduces to the following bimatrix game.

	1	2
1	$0,0$	$1,0$
2	$0,1$	$0,0$

This game has three equilibria. There are two asymmetric equilibria in which one player picks 1 and the other player picks 2, and one symmetric equilibrium in which both players pick 1.

Now suppose that there are three players and three numbers to choose from (i.e., $n = K = 3$). In any pure strategy equilibrium it must be the case that at least one player plays the number 1, but not more than two players play the number 1 (if all three play 1, it is optimal to deviate for one player and pick 2). In pure strategy equilibria where only one player plays 1, the other players can play in any combination of the other two numbers. In pure strategy equilibria where two players play 1, the third player plays 2. In total there are 18 pure strategy equilibria. To find the symmetric mixed strategy equilibrium, let p_1 denote the probability with which 1 is played and p_2 the probability with which 2 is played. The expected payoff from playing the pure strategies if the other two players randomize is given by

$$
\begin{aligned}
\pi(1) &= (1-p_1)^2, \\
\pi(2) &= \left[(1-p_1-p_2)^2 + p_1^2\right], \\
\pi(3) &= \left[p_1^2 + p_2^2\right].
\end{aligned}
$$

Setting the payoff from the three pure strategies yields $p_1 = 2\sqrt{3} - 3 = 0.464$ and $p_2 = p_3 = 2 - \sqrt{3} = 0.268$.

In the game with n players, there are numerous asymmetric pure strategy equilibria as in the three-player case. For example, in one type of equilibrium exactly one player picks 1 and the other players pick the other numbers in arbitrary ways. In order to find symmetric mixed strategy equilibria, let p_k denote the probability put on number k.[41] In a symmetric mixed strategy equilibrium, the distribution of guesses will follow the multinomial distribution. The probability of x_1 players guessing 1, x_2 players guessing 2 and so on is given by

$$
f(x_1, ..., x_K; n) = \begin{cases} \frac{n!}{x_1! \cdots x_K!} p_1^{x_1} \cdots p_K^{x_K} & \text{if } \sum_{i=1}^{K} x_i = n, \\ 0 & \text{otherwise,} \end{cases}
$$

where we use the convention that $0^0 = 1$ in case any of the numbers is picked with zero probability. The marginal density function for the k^{th} number is the binomial distribution

$$
f_k(x_k; n) = \frac{n!}{x_k!\,(n-x_k)!} p_k^{x_k} (1-p_k)^{n-x_k}.
$$

Let $g_k(x_1, x_2, ..., x_k; n)$ denote the marginal distribution for the first k numbers. In other

[41] We have not been able to show that there is a unique symmetric equilibrium, but when numerically solving for a symmetric equilibrium we have not found any other equilibria than the ones reported below. Existence of a symmetric equilibrium is guaranteed since players have finite strategy sets. (A straightforward extension of Proposition 1.5 in Weibull, 1995 shows that all symmetric normal form games with finite number of strategies and players have a symmetric equilibrium.)

words, we define g_k for $k < K$ as

$$g_k(x_1, x_2, ..., x_k; n) = \sum_{x_{k+1}+x_{k+2}+\cdots+x_K = n-(x_1+x_2+\cdots+x_k)} \frac{n!}{x_1!x_2!\cdots x_K!} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}.$$

Using the multinomial theorem we can simplify this to[42]

$$g_k(x_1, x_2, ..., x_k; n) = \frac{n!}{x_1!\cdots x_k!} p_1^{x_1} \cdots p_k^{x_k} \frac{(p_{k+1} + p_{k+2} + \cdots + p_K)^{n-(x_1+x_2+\cdots+x_k)}}{(n-(x_1+x_2+\cdots+x_k))!}.$$

If $k = K$, then $g_k(x_1, x_2, ..., x_k; n) = f(x_1, x_2, ..., x_k; n)$. Finally, let $h_k(n)$ denote the probability that nobody guessed k and there is at least one number between 1 to $k-1$ that only one player guessed. This probability is given by (again if $k < K$)

$$h_k(n) = \sum_{\substack{(x_1,...,x_{k-1})\text{: some } x_i=1 \\ \& \ x_1+\cdots+x_{k-1}\leq n}} g_k(x_1, x_2, ..., x_{k-1}, 0; n).$$

If $k = K$, then this probability is given by

$$h_K(n) = \sum_{\substack{(x_1,...,x_{k-1})\text{: some } x_i=1 \\ \& \ x_1+\cdots+x_{k-1}=n}} f(x_1, x_2, ..., x_{K-1}, 0; n).$$

The probability of winning when guessing 1 and all other players follow the symmetric mixed strategy is given by

$$\pi(1) = f_1(0; n-1) = (1-p_1)^{n-1}.$$

[42]The multinomial theorem states that the following holds

$$(p_1 + p_2 + \cdots + p_K)^n = \sum_{x_1+x_2+\cdots+x_K=n} \frac{n!}{x_1!x_2!\cdots x_K!} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K},$$

given that all $x_i \geq 0$.

The probability of winning when playing $1 < k < K$ is given by[43]

$$\pi(k) = f_k(0; n-1) - h_k(n-1),$$
$$= (1-p_k)^{n-1} - h_k(n-1).$$

Similarly, the probability of winning when playing $k = K$ is given by

$$\pi(K) = f_K(0; n-1) - h_K(n-1).$$

In a symmetric mixed strategy equilibrium, the probability of winning from all pure strategies in the support of the equilibrium must be the same. In the special case when $n = K$ and all numbers are played with positive probability, we can simply solve the system of $K-2$ equations where each equation is

$$(1-p_k)^{n-1} - h_k(n-1) = (1-p_1)^{n-1},$$

for all $2 < k < K$ and the Kth equation

$$(1-p_K)^{n-1} - h_K(n-1) = (1-p_1)^{n-1}.$$

[43]The easiest way to see this is to draw a Venn diagram. More formally, let $A = \{$No other player picks $k\}$ and let $B = \{$No number below k is unique$\}$, so that $P(A) = f_k(0; n-1)$ and $P(B) = h_k(n-1)$. We want to determine $P(A \cap B)$, which is equal to

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

To determine $P(A \cup B)$, note that it can be written as the union between two independent events

$$P(A \cup B) = P(B \cup (B' \cap A)).$$

Since B and $B' \cap A$ are independent,

$$P(A \cup B) = P(B) + P(B' \cap A).$$

Combining this with the expression for $P(A \cap B)$ we get

$$P(A \cap B) = P(A) - P(A \cap B').$$

In principle, it is straightforward to solve this system numerically. However, computing the h_k function is computationally explosive because it requires the summation over a large set of vectors of length $k-1$. The number of combinations explodes as n and K gets large and it is non-trivial to solve for equilibrium for more than 8 players. As an illustration, when $n = K = 7$, $h_7(6)$ involves the summation over 391 vectors, and when $n = K = 8$ computing $h_8(7)$ involves 1520 vectors. To understand the magnitude of the complexity, suppose we want to compute $h_K(n-1)$. This involves the summation over all vectors $(x_1, ..., x_{K-1})$ such that some $x_i = 1$ and $x_1 + \cdots + x_{K-1} = n-1$. Only a small subset of all these vectors are the ones where $x_1 = 1$. How many such vectors are there? For those vectors there must be $n-2$ players that play numbers $x_2, ..., x_{K-1}$, i.e., potentially $K-2$ different strategies. The total number of such vectors are

$$\frac{(K+n-5)!}{(n-2)!(K-3)!},$$

where we have used the fact that the number of sequences of n natural numbers that sum to k is $(n+k-1)!/(k!(n-1)!)$. For example, when $n = 27$ and $K = 99$, the number of vectors in which $x_1 = 1$ is larger than 10^{25}. Note that this number is much lower than the actual total number of vectors since we have only counted vectors such that $x_1 = 1$.

Assuming $n = K$, the table below show the equilibrium for up to eight players.[44]

[44] See Appendix C for details about how these probabilities were computed.

	3x3	4x4	5x5	6x6	7x7	8x8
1	0.4641	0.4477	0.3582	0.3266	0.2946	0.2710
2	0.2679	0.4249	0.3156	0.2975	0.2705	0.2512
3	0.2679	0.1257	0.1918	0.2314	0.2248	0.2176
4		0.0017	0.0968	0.1225	0.1407	0.1571
5			0.0376	0.0216	0.0581	0.0822
6				0.0005	0.0110	0.0199
7					0.0004	0.0010
8						0.0000

These probabilities are close to the Poisson-Nash equilibrium probabilities. To see this, the table below shows the Poisson-Nash equilibrium probabilities when n is equal to K for 3 to 8 players. Note that all the fixed-n and Poisson-Nash probabilities for all strategies in the 5x5 game and larger are within 0.02.

	3x3	4x4	5x5	6x6	7x7	8x8
1	0.4773	0.4057	0.3589	0.3244	0.2971	0.2747
2	0.3378	0.3092	0.2881	0.2701	0.2541	0.2397
3	0.1849	0.1980	0.2046	0.2057	0.2030	0.1983
4		0.0870	0.1129	0.1315	0.1430	0.1492
5			0.0355	0.0575	0.0775	0.0931
6				0.0108	0.0234	0.0385
7					0.0020	0.0064
8						0.0002

B. Proof of Proposition 1

We first prove the four properties and then prove that the equilibrium is unique.

1. We prove this property by induction. For $k = 1$, we must have $p_1 > 0$. Otherwise,

deviating from the proposed equilibrium by choosing 1 would guarantee winning for sure. Now suppose that there is some number $k+1$ that is not played in equilibrium, but that k is played with positive probability. We show that $\pi(k+1) > \pi(k)$, implying that this cannot be an equilibrium. To see this, note that the expressions for the expected payoffs allows us to write the ratio $\pi(k+1)/\pi(k)$ as

$$\begin{aligned}\frac{\pi(k+1)}{\pi(k)} &= \frac{\prod_{i=1}^{k} Pr(X(i) \neq 1) \cdot Pr(X(k+1) = 0)}{\prod_{i=1}^{k-1} Pr(X(i) \neq 1) \cdot Pr(X(k) = 0)} \\ &= \frac{Pr(X(k) \neq 1) \cdot Pr(X(k+1) = 0)}{Pr(X(k) = 0)}.\end{aligned}$$

If $k+1$ is not used in equilibrium, $Pr(X(k+1) = 0) = 1$, implying that the ratio is above one. This shows that all integers between 1 and K are played with positive probability in equilibrium.

2. Rewrite equation (1) as

$$e^{np_{k+1}} - e^{np_k} = -np_k.$$

By the first property, both p_k and p_{k+1} are positive, so that the right hand side is negative. Since the exponential is an increasing function, we conclude that $p_k > p_{k+1}$.

3. First rearrange equation (1) as

$$p_{k+1} = p_k + \frac{1}{n} \ln\left(1 - np_k e^{-np_k}\right). \tag{A1}$$

We want to determine $(p_k - p_{k+1})/(p_{k+1} - p_{k+2})$. Using (A1) we can write this ratio as

$$\frac{p_k - p_{k+1}}{p_{k+1} - p_{k+2}} = \frac{\ln(1 - np_k e^{-np_k})}{\ln(1 - np_{k+1} e^{-np_{k+1}})} = \frac{\ln(Pr(X(k) \neq 1))}{\ln(Pr(X(k+1) \neq 1))}.$$

The derivative of $Pr(X(k) \neq 1)$ with respect to p_k is positive if $p_k > 1/n$ and negative if $p_k < 1/n$. We therefore have shown that $(p_k - p_{k+1})$ is increasing in k

when $p_k > 1/n$, whereas the difference is decreasing for $p_k > 1/n$.

4. Taking the limit of (A1) as $n \to \infty$ implies that $p_{k+1} = p_k$.

In order to show that the equilibrium $\mathbf{p} = (p_1, p_2, \cdots, p_K)$ is unique, suppose by contradiction that there is another equilibrium $\mathbf{p}' = (p'_1, p'_2, \cdots, p'_K)$. By the equilibrium condition (1), p_1 uniquely determines all probabilities $p_2, ..., p_K$, while p'_1 uniquely determines $p'_2, ..., p'_K$. Without loss of generality, we assume $p'_1 > p_1$. Since in any equilibrium, p_{k+1} is strictly increasing in p_k by condition (1), it must be the case that all positive probabilities in $\mathbf{p}'$ are higher than in $\mathbf{p}$. However, since $\mathbf{p}$ is an equilibrium, $\sum_{k=1}^{K} p_k = 1$. This means that $\sum_{k=1}^{K} p'_k > 1$, contradicting the assumption that $\mathbf{p}'$ is an equilibrium.

C. Computational and Estimation Issues

This appendix provides details about the numerical computations and estimations that are reported in the paper. We have used MATLAB for all computations and estimations. Both the data and all MATLAB programs that have been used for the paper can be obtained from the authors upon request.

Poisson-Nash Equilibrium

The Poisson-Nash equilibrium was computed in MATLAB through iteration of the equilibrium condition (1). Unfortunately, MATLAB cannot handle the extremely small probabilities that are attached to high numbers in equilibrium, so the estimated probabilities are zero for high numbers (17 and above for the laboratory and 5519 and above for the field).

Fixed-n Equilibrium

To compute the equilibrium when the number of players is fixed and commonly known, we programmed the functions f_k, f_K, h_k and h_K in MATLAB and then solved the system of equations characterizing equilibrium using MATLAB's solver fsolve. However, the h_k function includes the summation of a large number of vectors. For high k and n the number of different vectors involved in the summation grows explosively and we only managed solve for equilibrium for up to 8 players.

Cognitive Hierarchy with Quantal Response

Calculating the cognitive hierarchy prediction for a given τ and λ is straightforward. However, the cognitive hierarchy prediction is non-monotonic in τ and λ, implying that the log-likelihood function isn't generally smooth.

In order to calculate the log-likelihood, we assume that all players play according to the same aggregate cognitive hierarchy prediction, i.e., the log-likelihood function is calculated using the multinomial distribution as if all players played the same strategy. For the field data, we calculated the log-likelihood for the daily average frequency for each week, but the frequency was rounded to integers in order to be able to calculate the log-likelihood. For the lab data, we instead calculated the log-likelihood by summing the frequencies for each week since we didn't want unnecessary estimation errors due to rounding off to integers.

Maximum likelihood estimation for the field data is computationally demanding so we used a relatively coarse two-dimensional grid search. We used a 20x20 grid and restricted τ to be between 0.05 and 12, and restricted λ to be between 0.0001 and 0.05. We tried wider bounds on the parameters as well, but that didn't change the results. The log-likelihood function is shown in Figure A1. The log-likelihood appears relatively smooth, but since we have been forced to use a very coarse grid we might not have found the global maximum.

For the maximum likelihood estimation of the lab data, we used a two-dimensional 300x300 grid search. We tried different bounds on τ and λ, then let both parameters vary between 0.001 and 20. The three-dimensional log-likelihood function is shown in Figure A2. It is clear that the log-likelihood function isn't smooth and that it is very flat with respect to λ when λ is low. There is therefore no guarantee that we have found a global maximum, but we have tried different grid sizes and bounds on the parameters which resulted in the same estimates.

When τ is fixed at 1.5, the maximum likelihood estimation is simpler. We used a grid size of 300 and tried different bounds for λ with unchanged results. The log-likelihood function for $\lambda = 0.001$ to $\lambda = 100$ from the first week is shown in Figure A3. The log-likelihood function is not globally concave, but seems to be concave around the global maximum, so it is likely that we have found a global maximum. Figure A4 shows the cognitive hierarchy prediction week-by-week for the laboratory data when τ is 1.5.

QRE

In order to calculate the QRE for a given level of λ, we used MATLAB's solver fsolve to solve the fixed-point equation that characterizes the QRE. In the ML estimation for the laboratory data we allowed λ between 0.001 and 700. To find the optimal value we used a grid search with a grid size of 100. The log-likelihood function for the first week is shown in Figure A5. The log-likelihood function is smooth and concave, indicating that we have are likely to have found a global maximum.

Learning

To estimate the learning model, we use the actual winning numbers in the field and in each laboratory session. The predicted choice probabilities are evaluated based on the sum of squared distances from the empirical densities, summed over numbers, days and sessions (in the laboratory). For the field data, we estimated λ through a grid search (with a grid size of 15) for window sizes between 100 and 400 and λ between 0.005 and 0.5. The sum of squared deviations with respect to both W and λ appears to be relatively smooth and convex, so it is likely that we have find the best-fitting values. For the laboratory data, we estimated λ through grid search (with a grid size of 1000) for window sizes between 1 and 13 and λ between 0.01 and 2. Figure A6 shows the sum of the squared deviations for the laboratory data. As can be seen from the graph, the fit is relatively flat with respect to both W and λ when both parameters are increased proportionally. We have tried different bounds on the parameters and grid sizes and the estimated parameters appears robust. Figure A7 shows an example of a Bartlett similarity window and Figure A8 shows box plots with the data and learning model for the first 14 rounds in the laboratory.

Model Selection

Since the Poisson-Nash equilibrium probabilities are zero for high numbers, the likelihood of the equilibrium prediction is always zero. However, to be able to compare the equilibrium prediction with the cognitive hierarchy model and QRE, we calculate the log-likelihoods using only data on numbers up to 5518 (field) and 16 (laboratory). These log-likelihoods cannot be directly compared with the log-likelihoods in Table 3 and 6, however, since those are calculated using data on all numbers. For comparison, we therefore compute the log-likelihoods for the cognitive hierarchy model (as well as QRE for the laboratory) in the same way as for the equilibrium prediction. In order for these probabilities to sum up to one, we divide the probabilities by the total probability attach to numbers up to the threshold (5518 or 16). Using the estimated parameters reported in Table 2, Table A1 shows the log-likehoods only based on numbers up to 5518.

Week	1	2	3	4	5	6	7
Log-likelihood eq. (<5519)	-43365	-32073	-28453	-27759	-28087	-21452	-19719
Log-likelihood CH (< 5519)	-25307	-21606	-18630	-16253	-16123	-15829	-15010

Table A1: Log-likelihoods for cognitive hierarchy and equilibrium for field data (up to 5518)

The log-likelihoods are higher for the cognitive hierarchy model in all weeks. The cognitive hierarchy model is estimated with two parameters, while the equilibrium prediction has no free parameters. One way to compare the models is to use Schwarz (1978) information criterion which penalizes a model depending on the number of estimated parameters by subtracting a factor $\log(n) \times m/2$ from the log-likelihood value, where n is the number of observations and m the number of estimated parameters. The log-likelihoods in Table A1 are calculated based on daily averages, so the penalty for the cognitive hierarchy model is approximately $\log(53783) = 10.9$, indicating that the cognitive hierarchy model is the better model in all weeks. Schwarz information criterion penalizes the number of estimated parameters more harshly than for example Aikake's information criterion. However, it should be kept in mind that the two parameters in cognitive hierarchy model are estimated using the data, whereas the equilibrium prediction is not estimated at all, so any comparison based on information criteria is likely to be unfair.

Week	1	2	3	4	5	6	7
Log-likelihood eq. (<17)	-192.9	-95.3	-91.3	-81.4	-93.0	-59.7	-145.2
Log-likelihood CH (<17)	-70.2	-54.8	-45.8	-55.8	-54.2	-48.9	-46.3
Log-likelihood CH $\tau = 1.5$ (<17)	-79.0	-52.9	-63.3	-56.5	-56.9	-65.2	-70.3
Log-likelihood QRE (<17)	-70.0	-65.1	-72.5	-66.2	-67.4	-57.1	-70.8
BIC eq. (<17)	-192.9	-95.3	-91.3	-81.4	-93.0	-59.7	-145.2
BIC CH (<17)	-76.7	-61.4	-53.4	-62.5	-60.8	-55.5	-52.9
BIC CH $\tau = 1.5$ (<17)	-82.3	-56.2	-66.6	-59.9	-60.2	-68.5	-73.6
BIC QRE (<17)	-73.3	-68.5	-75.8	-69.5	-70.7	-60.4	-74.1

Table A2: Log-likelihood and Schwarz information criterion (BIC) for the cognitive hierarchy, QRE and equilibrium models in the laboratory (up to 16)

Table A2 reports the restricted log-likelihoods and the corresponding values of the Schwarz information criterion for the laboratory data. Based on Schwarz information criterion, the cognitive hierarchy model outperforms equilibrium in all weeks, but the equilibrium prediction does better than QRE and the cognitive hierarchy model with $\tau = 1.5$ in the sixth week.

D. Additional Details About the Field LUPI Game

This part of the Appendix provides some additional details about the field game that was not discussed in the main text.

The prize guarantee for the winner of 100,000 SEK was first extended until the 11th of March and then to the 18th of March, so the prize guarantee covered all days for which we have data. The thresholds for the second and third prizes were determined so that the second prizes constituted 11 percent of all bets and the third prizes 17.5 percent. The winner of the first prize also won the possibility to participate in a "final game".[45] The final game ran weekly and had four to seven participants. The "final game" consisted of three rounds where the participants chose two numbers in each round. The rules of this game were very similar to the original game, but what happened in this game did not depend on what number you chose in the main game, so we leave out the details about this game.

The Hux Flux randomization option involved a uniform distribution where the support of the distribution was determined by the play during the 7 previous days.[46] It became possible to play the game on the Internet sometime between the 21st and 26th of February 2007. The web interface for online play is shown in Figure A10. This interface also included the option HuxFlux, but in this case players could see the number that was generated by the computer before deciding whether to place the bet.

We use daily data from the first seven weeks. The reason is that the game was withdrawn from the market on the 24th of March 2007 and we were only able to access data up to the 18th of March 2007.

Figure A11 shows histograms for the total number of daily bets separately for all days and for Sundays and Mondays. Figure A12 shows empirical frequencies together with the Poisson-Nash equilibrium for the last week in the field.

The game was heavily advertised around the days when it was launched and the main message was that this was a new game where you should be alone with the lowest number. The winning numbers (for the first, second, and third prizes) were reported on TV, text-

[45] 3.5 percent of all daily bets were reserved for this "final game".

[46] In the first week HuxFlux randomized numbers uniformly between 1 and 15000. After seven days of play, the computer randomized uniformly between 1 and the average 90th percentile from the previous seven days. However, the only information given to players about HuxFlux was that a computer would choose a number for them.

TV and the Internet every day. In the TV programs they reported not only the winning numbers, but also commented briefly about how people had played previously.

The richest information about the history of play was given on the home page of Svenska Spel. People could display and download the frequencies of all numbers played for all previous days. However, this data was presented in a raw format and therefore not very accessible. The homepage also displayed a histogram of yesterday's guesses which made the data easier to digest. An example of how this histogram looked is shown in Figure A13. The homepage also showed the total number of bets that had been made so far during the day.

The web interface for online play also contained some easily accessible information. Besides links to the data discussed above as well as information about the rules of the game, there were some pieces of statistics that could easily be displayed from the main screen. The default information shown was the first name and home town of yesterday's first prize winner and the number that that person guessed. By clicking on the pull-down menu in the middle, you could also see the seven most popular guesses from yesterday. This information was shown in the way shown in Figure A14. By moving the mouse over the bars you can see how many people guessed that number. In this example, the most popular number was 1234 with 85 guesses! Note that this information was not easily available before online play was possible. From the same pull-down menu, you could also see the total number of distinct numbers people guessed on during the last seven days. Finally, you could display the numbers of the second- and third prize winners of yesterday.

In addition to this information, Svenska Spel also published posters with summary statistics for previous rounds of the game (see Figure A15). The information given on these posters varied slightly, but the one in Figure A15 shows the winning numbers, the number of bets, the size of the first prize and if there was any numbers below the winning number that no other player chose. It also shows the average, lowest and highest winning number, as well as the most frequently played numbers.

E. Additional Details About the Lab Experiment

Screenshots from the input and results screens of the laboratory experiment are shown in Figure A16 and A17. Figure A18 shows screenshots from the post-experimental questionnaire and Figure A19 a screenshot from the CRT.

Figure A20 displays the aggregate data from non-selected and selected subjects' choices. Subjects are slightly more likely to play high numbers above 20 when they are not selected to participate, but overall the pattern looks very similar. This implies that subjects' behavior in a particular round is almost unaffected depending on whether they had marginal monetary incentives or not.

Experimental Instructions

Instructions for the laboratory experiment are as follows (translated directly by one of the authors from the Swedish field instructions, but modified in order to fit the laboratory game):

Instruction for Limbo[47]

Limbo is a game in which you choose to play a number, between 1 and 99, that you think nobody else will play in that round. The lowest number that has been played only once wins.

The total number of rounds will not be announced. At the beginning of each round, the computer will indicate whether you have been selected to participate in that round. The computer selects participating players randomly so that the average number of participating players in each round is 26.9. Please choose a number even if you are not selected to participate in that round.

[**Instructions where the Poisson distribution is explicitly described:**

The game is played in 49 rounds. In the beginning of each round, the computer will indicate whether you have been selected to participate in this round. The computer

[47] In order to mirror the field game as closely as possible, we referred to the LUPI game as "Limbo" in the lab.

selects participating players randomly so that the average number of participating players in each round is 26.9.

Specifically, the number of players in each round is pre-drawn from a so called Poisson distribution. The diagram below shows the Poisson distribution with mean 26.9. The horizontal axis shows different possible numbers of participants, and the vertical axis shows the probability of having that many participants. Notice that in some rounds there are more than 27 players and in other rounds there are fewer than 27 players. You will not know how many players are participating in each round. All you know is the probabilities of what the number of players might be, given by the distribution shown in the diagram.

On the second screen you can indicate which number between 1 and 99 that you want to play in that round.

Note: We also attached Figure A24 at the end of the instructions.]

After all participating players have selected a number, the round is closed and all bets are checked. The lowest unique number that has been received is identified and the person that picked that number is awarded a prize of 7$.

The winning number is reported on the screen and shown to everybody after each round.

Prizes are paid out to you at the end of the experiment.

If you have any questions, raise your hand to get the experimenter's attention. Please be quiet during the experiment and do not talk to anybody except the experimenter.

Individual Lab Results

The regression results in Table 7 mask a considerably degree of heterogeneity between individual subjects. Based on the responses in the post-experimental questionnaire, we coded four variables depending on whether they mentioned each aspect as a motivation for their strategy.

Random All subjects who claimed that they played numbers randomly were coded in this category.[48]

Stick All subjects who stated that they stuck to one number throughout parts of the experiment were included in this category. Many of these subjects explained their choices by arguing that if they stuck with the same number, they would increase the probability of winning.

Lucky This category includes all subjects who claimed that they played a favorite or lucky number.

Strategic This category includes all players who explicitly motivated their strategy by referring to what the other players would do.[49]

Several subjects were coded into more than one category.[50]

How well does the classification based on the self-reported strategies explain behavior? Table A3 reports regressions where the dependent variables are four summary statistics of subjects' behavior—the number of distinct choices, the mean number, the standard deviation of number, and the total payoff. In the first column for each measure of individual play only the four categories above are included as dummy variables. There are few statistically significant relationships. Subjects coded into the "Stick" category did tend to choose fewer and less dispersed numbers, and subjects coded as "Lucky" tend to pick higher and more dispersed numbers. Table A3 also report regressions for the same dependent variables and some demographic variables. The only statistically significant

[48]For example, one subject motivated this strategy choice in a particular sophisticated way: "First I tried logic, one number up or down, how likely was it that someone else would pick that, etc. That wasn't doing any good, as someone else was probably doing the exact same thing. So I started mentally singing scales, and whatever number I was on in my head I typed in. This made it rather random. A couple of times I just threw curveballs from nowhere for the hell of it. I didn't pay any attention to whether or not I was selected to play that round after the first 3 or so."

[49]For example, one subject stated the following: "I tried to pick numbers that I thought other people wouldn't think of—whatever my first intuition was, I went against. Then I went against my second intuition, then picked my number. After awhile, I just used the same # for the entire thing."

[50]For example, the following subject was classified into all but the "Lucky" category: "At first I picked 4 for almost all rounds (stick) because it isn't considered to be a popular number like 3 and 5 (strategic). Afterwards, I realized that it wasn't helping so I picked random numbers (random)."

relationship is that subjects familiar with game theory tend to pick less dispersed numbers (though their payoffs are not higher). Note that the explanatory power is very low and that there are no significant coefficients in the regressions on the total payoff from the experiment. This suggests that it is hard to affect the payoff by using a particular strategy, which is consistent with the fully mixed equilibrium (where payoffs are the same for all strategies).

The questionnaire in two of the sessions also contained the three-question Cognitive Reflection Test (CRT) developed by Frederick (2005).[51] The purpose with collecting subjects' responses to the CRT is to get some measure of cognitive ability. In line with the results reported in Frederick (2005), a majority of the UCLA subjects answered only zero or one questions correctly. Interestingly, there does not appear to any relation between player's behavior or payoff in the LUPI game and the number of correctly answered questions, but the sample size is small ($n = 76$). The number of correctly answered CRT questions is not significant when the four measures in Table A3 are regressed on the CRT score.

Figure A21 shows a histogram of the number of distinct numbers that subjects played during the experiments. Based only on choices when players were selected to participate, subjects played on average 9.65 different numbers, compared to 10.9 expected in Poisson-Nash equilibrium. Figure A21 also shows a simulated distribution of how many distinct numbers players would pick if they played according to the equilibrium distribution.

[51]The CRT consists of three questions, all of which would have an instinctive answer, and a counterintuitive, but correct, answer. See Frederick (2005) or the screenshot in Figure A19 for the questions that we used.

	# Distinct		Mean		Std. dev.		Payoff	
Random	0.77		-0.37		-0.54		-0.26	
	(1.44)		(-0.42)		(-0.61)		(-0.21)	
Stick	-1.48^{***}		-1.12		-1.50^{*}		-0.36	
	(-2.80)		(-1.30)		(-1.70)		(-0.29)	
Lucky	1.24		4.43^{***}		3.73^{***}		-0.39	
	(1.60)		(3.52)		(2.88)		(-0.22)	
Strategic	0.35		-0.65		-0.54		1.42	
	(0.68)		(-0.78)		(-0.63)		(1.21)	
Age		-0.02		-0.00		0.02		0.26
		(-0.18)		(-0.02)		(0.11)		(1.32)
Female		-0.23		-0.92		-1.03		-1.10
		(-0.46)		(-1.12)		(-1.23)		(-0.99)
Income (1-4)		-0.13		-0.35		-0.50		0.38
		(-0.48)		(-0.81)		(-1.17)		(0.67)
Lottery player		0.17		0.59		0.39		-0.13
		(0.34)		(0.70)		(0.47)		(-0.12)
Game theory		0.25		0.23		-1.48^{*}		-0.55
		(0.63)		(-0.28)		(-1.74)		(-0.49)
R^2	0.08	0.01	0.10	0.02	0.08	0.04	0.01	0.02
Obs.	152	152	152	152	152	152	152	152

Only selected choices are included in the calculation of the dependent variables. $t-$statistics within parentheses. Constant included in all regressions. *=10 percent, **=5 percent and ***=1 percent significance level.

Table A3: Linear regressions explaining individual behavior

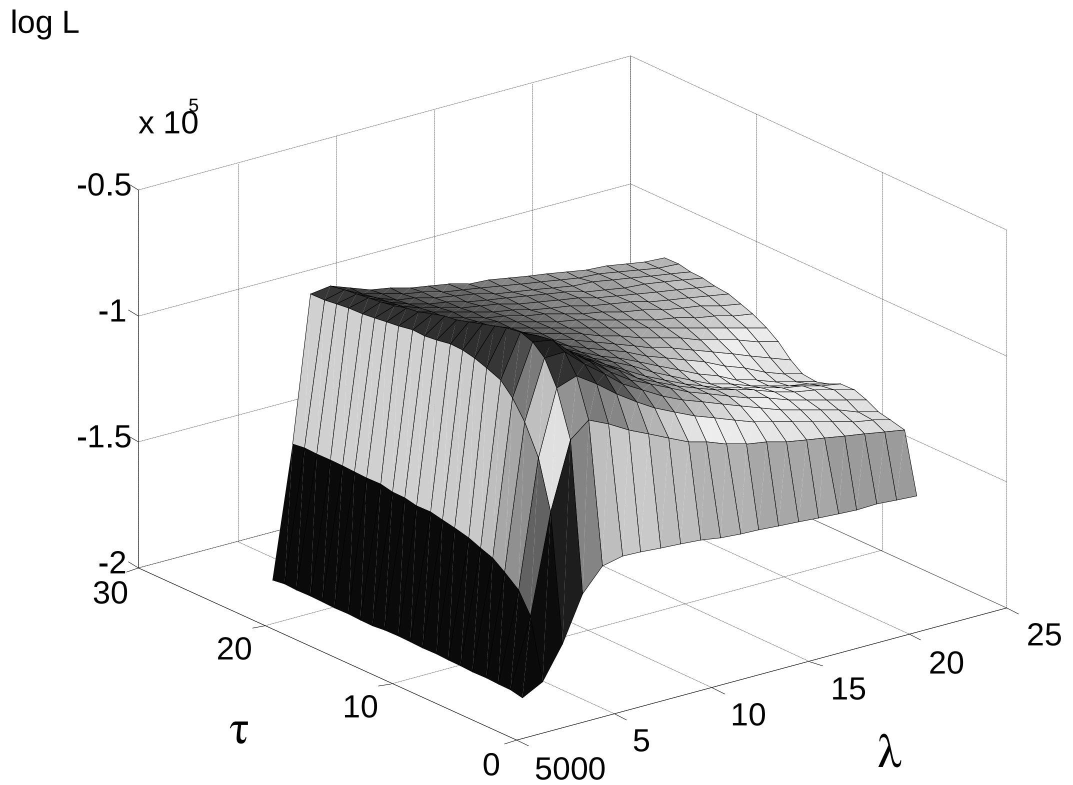


Figure A1. Log-likelihood for cognitive hierarchy in the field (first week).

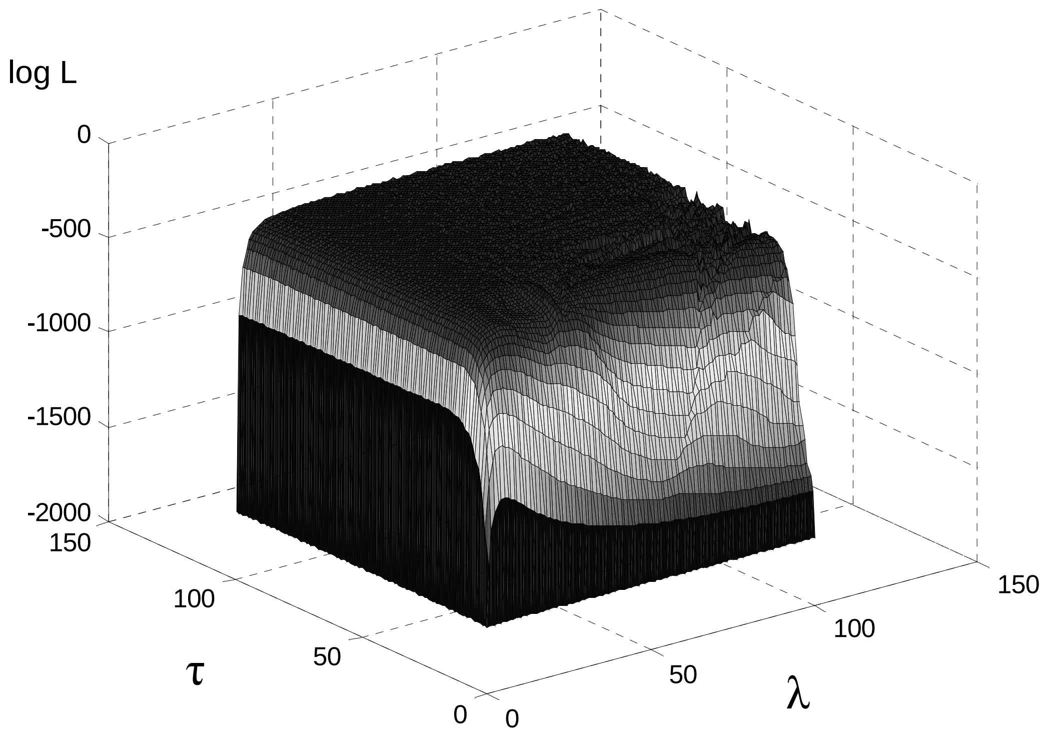


Figure A2. Log-likelihood for cognitive hierarchy in the laboratory (first week).

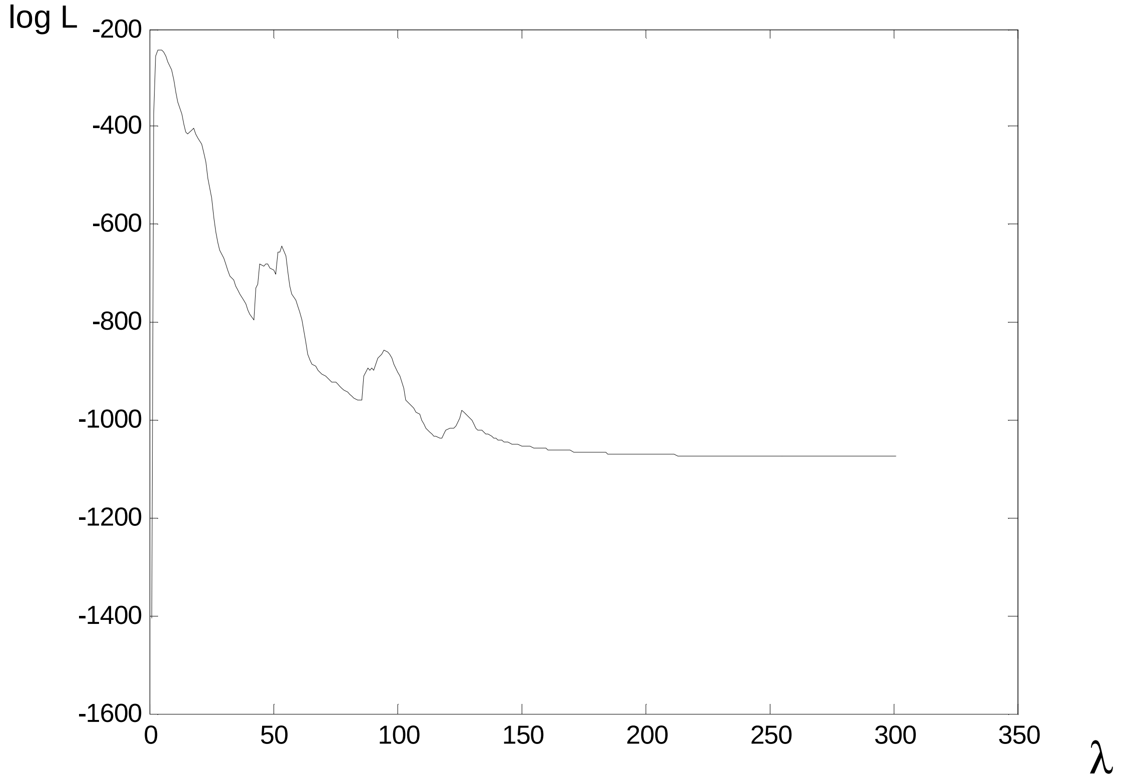


Figure A3. Log-likelihood function for cognitive hierarchy in the laboratory (first week, τ=1.5).

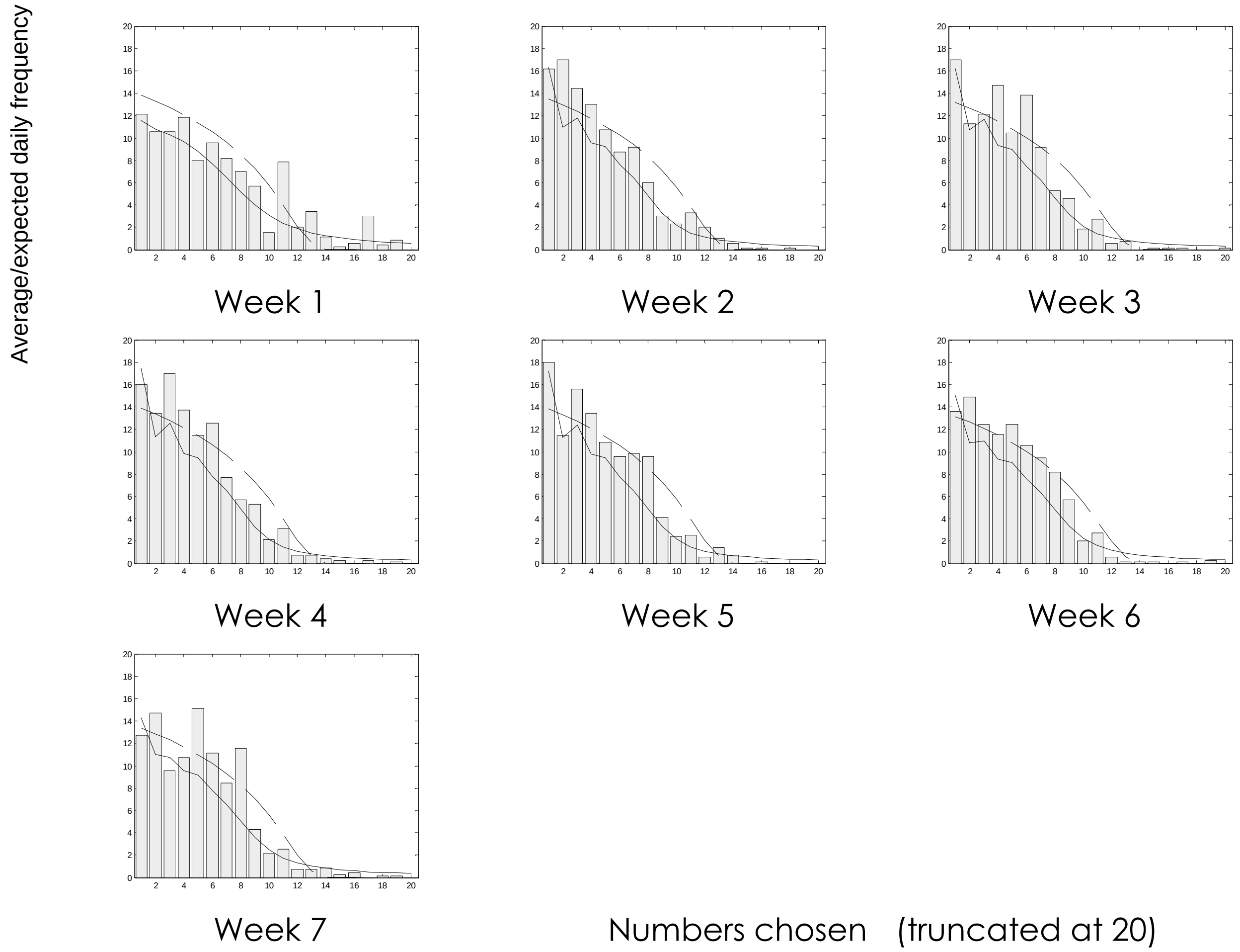


Figure A4. Average daily frequencies in the laboratory, Poisson-Nash equilibrium prediction (dashed lines) and estimated cognitive hierarchy (solid lines) when tau = 1.5 (line), week 1 to 7.

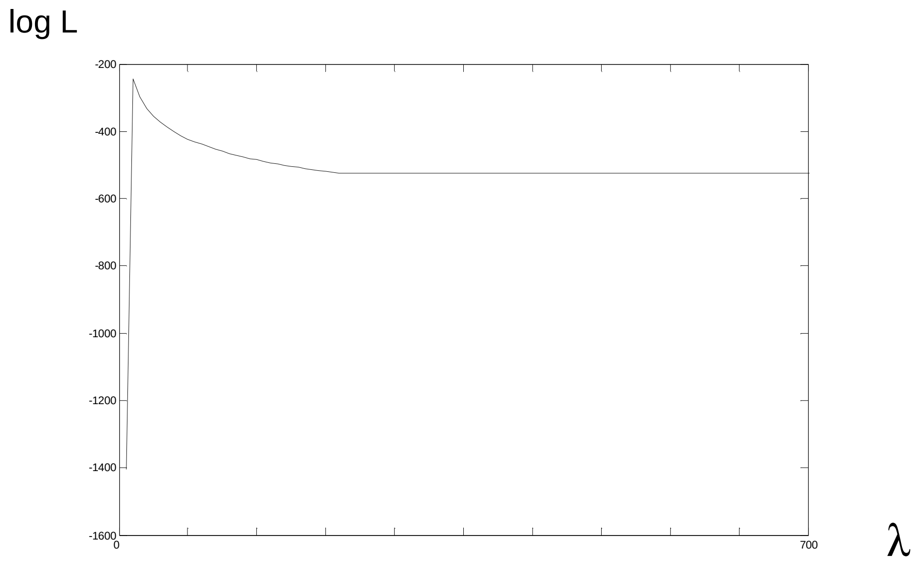


Figure A5. Log-likelihood function for QRE in the laboratory (first week).

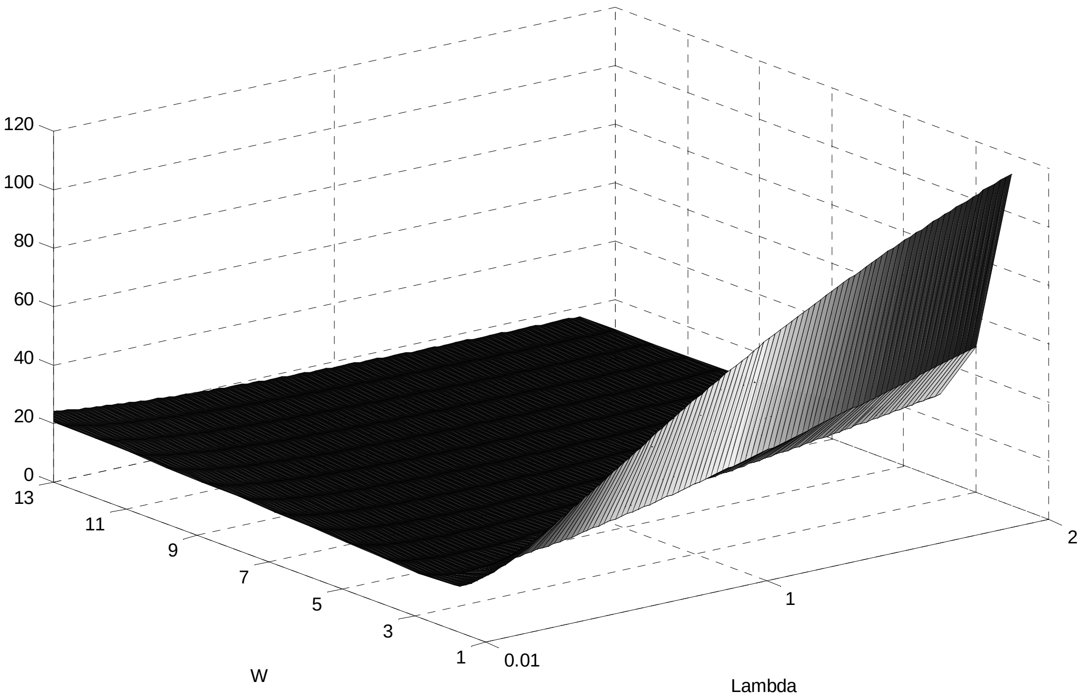


Figure A6. Sum of squared deviation for learning model in the laboratory (*W* = 1,...,13, λ = 0.01,...,2).

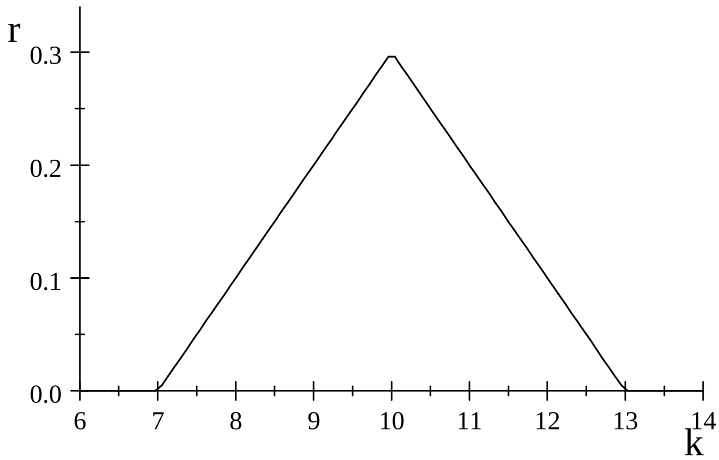
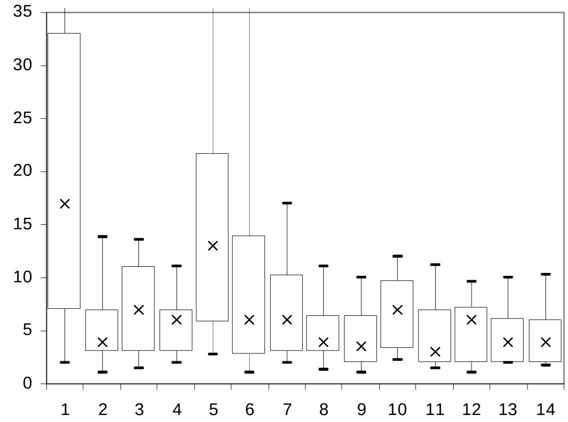
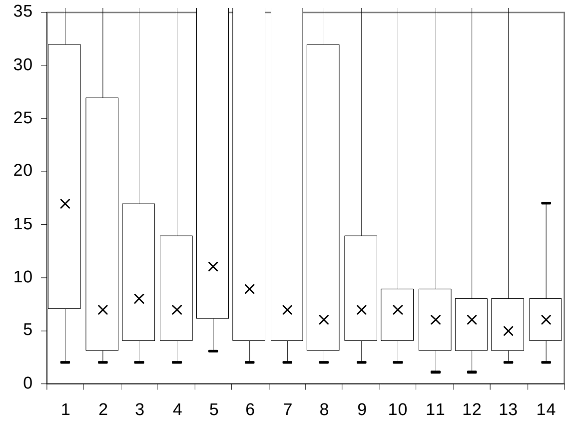


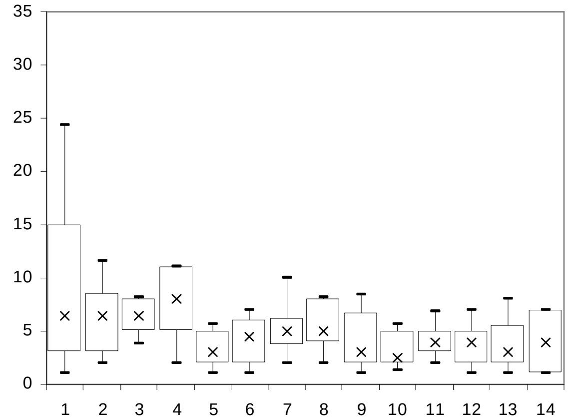
Figure A7. Bartlett similarity window (*k=10, *W*=3).**



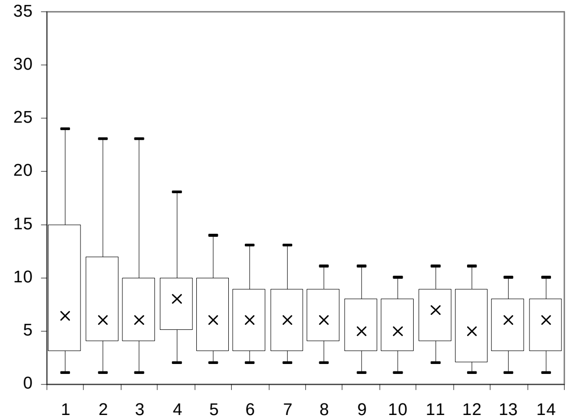
Data (Session 1)



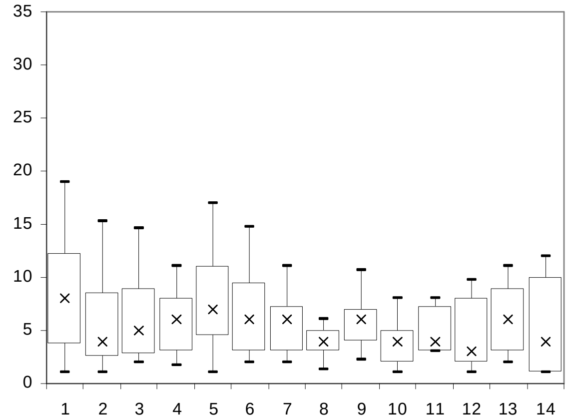
Learning model (Session 1)



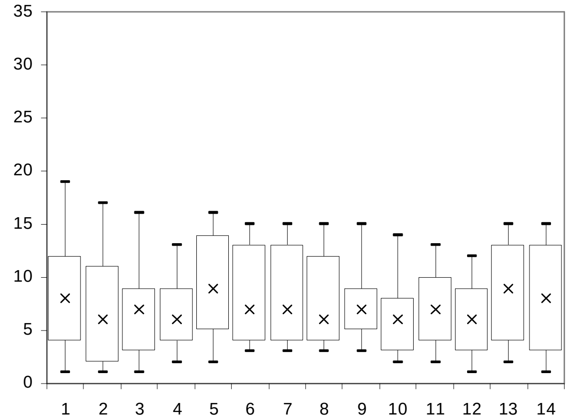
Data (Session 2)



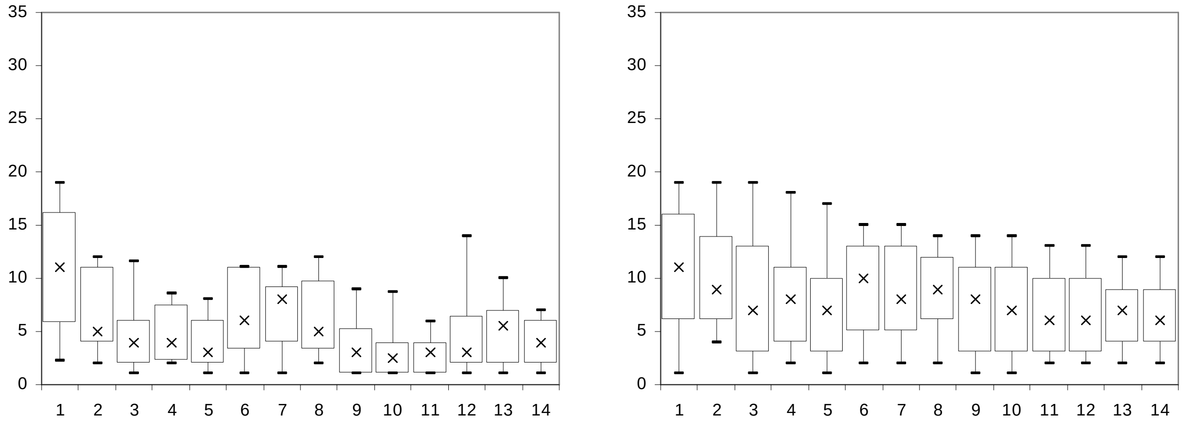
Learning model (Session 2)



Data (Session 3)



Learning model (Session 3)



Data (Session 4) Learning model (Session 4)

Figure A8. Box plots of data (left) and estimated learning model (right) for round 1-14 in the four laboratory sessions (10-25-50-75-90 percentile box plots, W = 3, λ = 0.31).

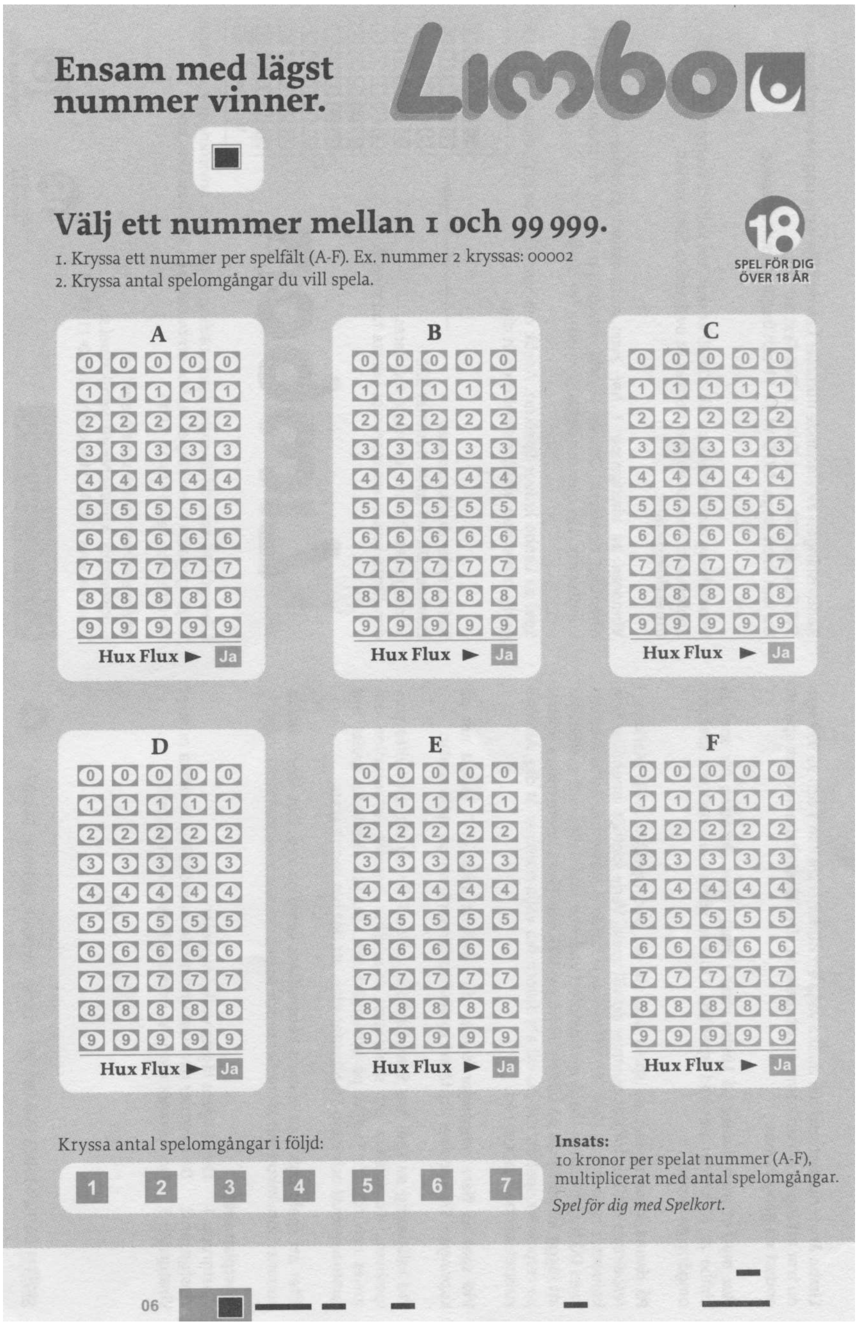


Figure A9. The paper entry form for the Swedish LUPI (Limbo) game.

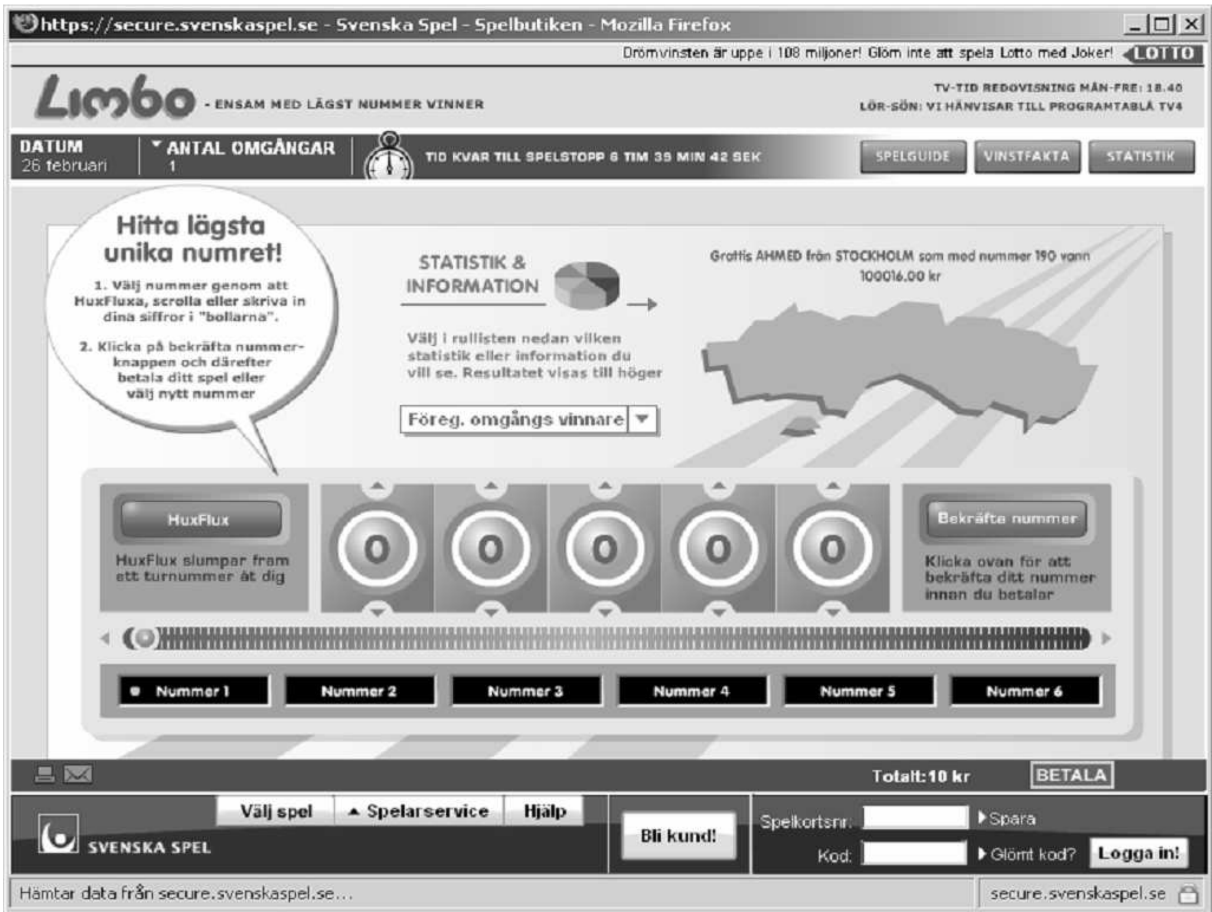


Figure A10. Online entry interface for the Swedish LUPI (Limbo) game.

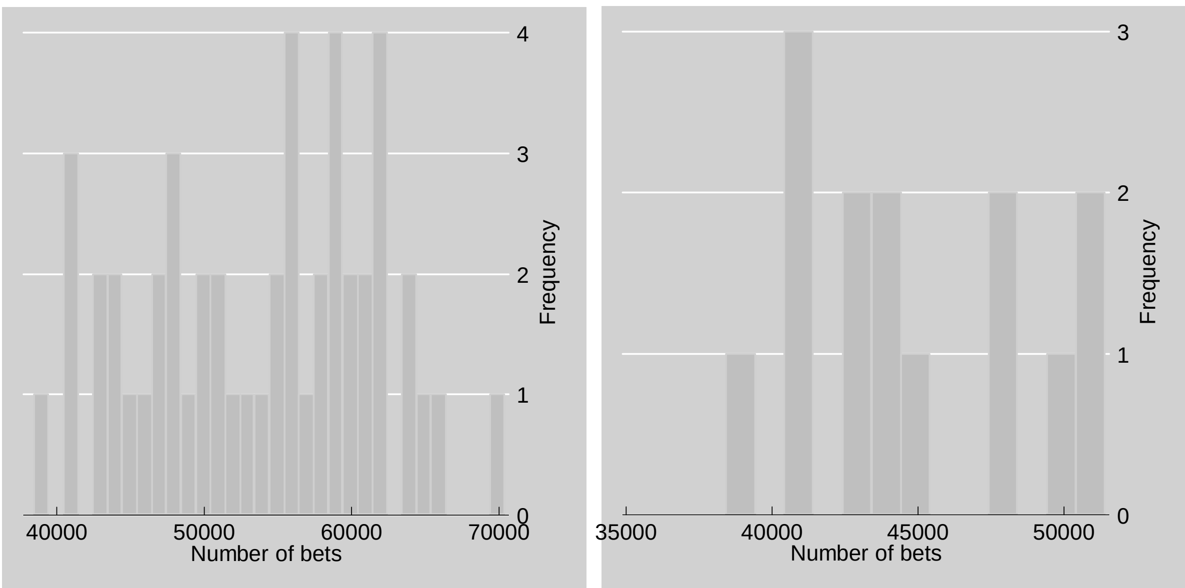


Figure A11. Total number of daily bets on all days (left) and Sundays and Mondays (right).

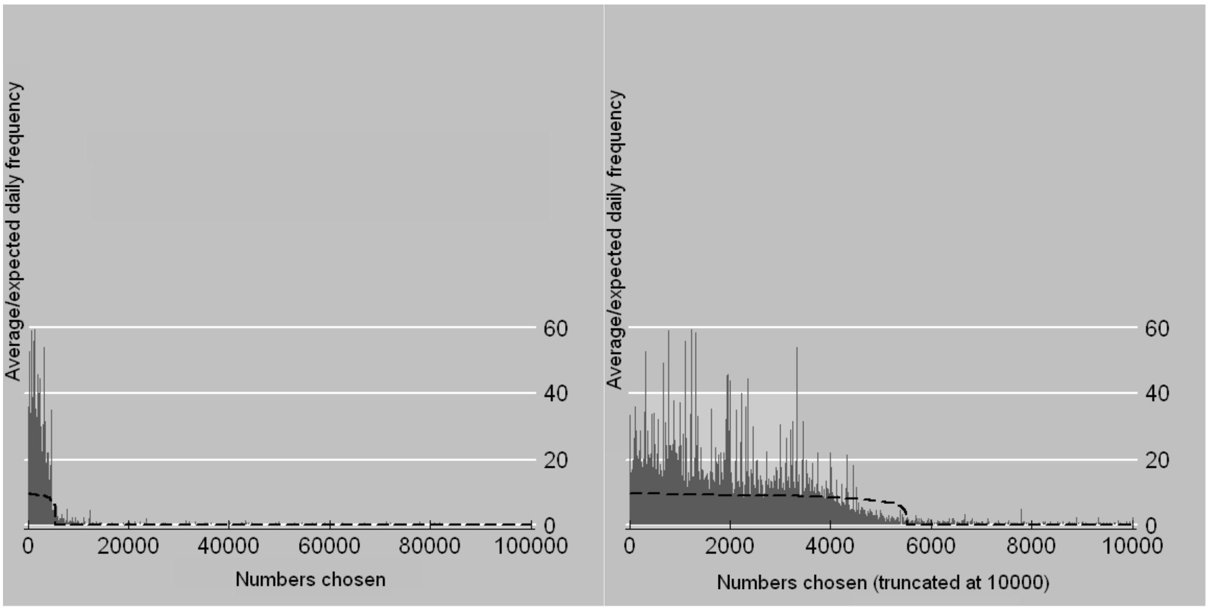


Figure A12. Average daily frequencies and equilibrium prediction for the last week in the field.

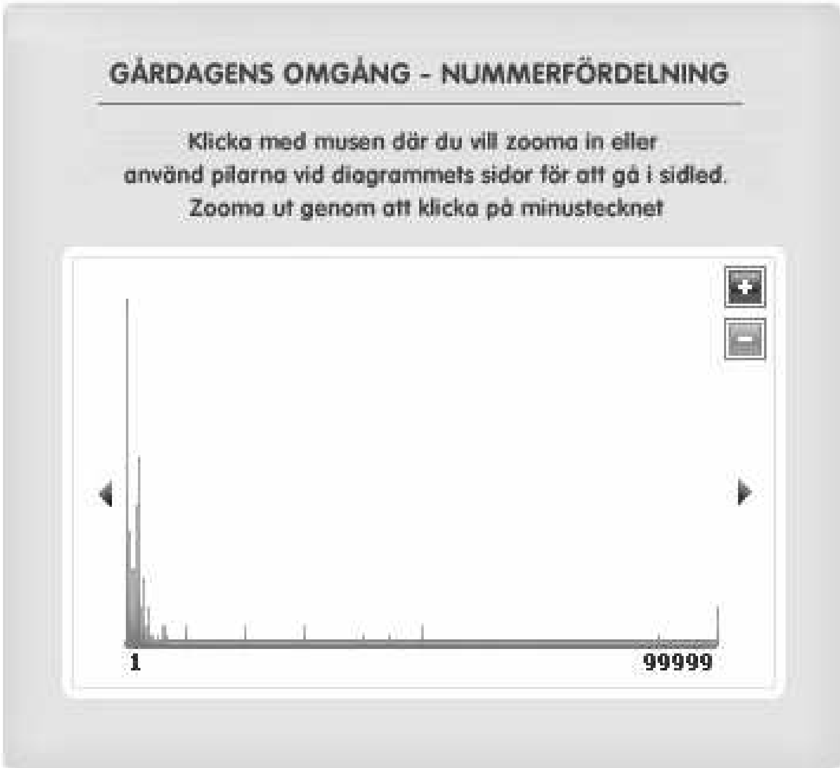


Figure A13. Histogram of yesterday's bets as shown online.

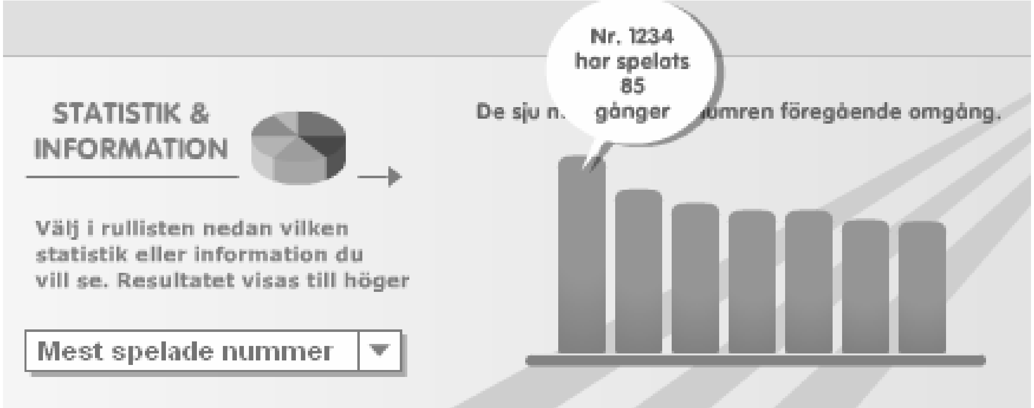


Figure A14. Most popular numbers yesterday as shown online.

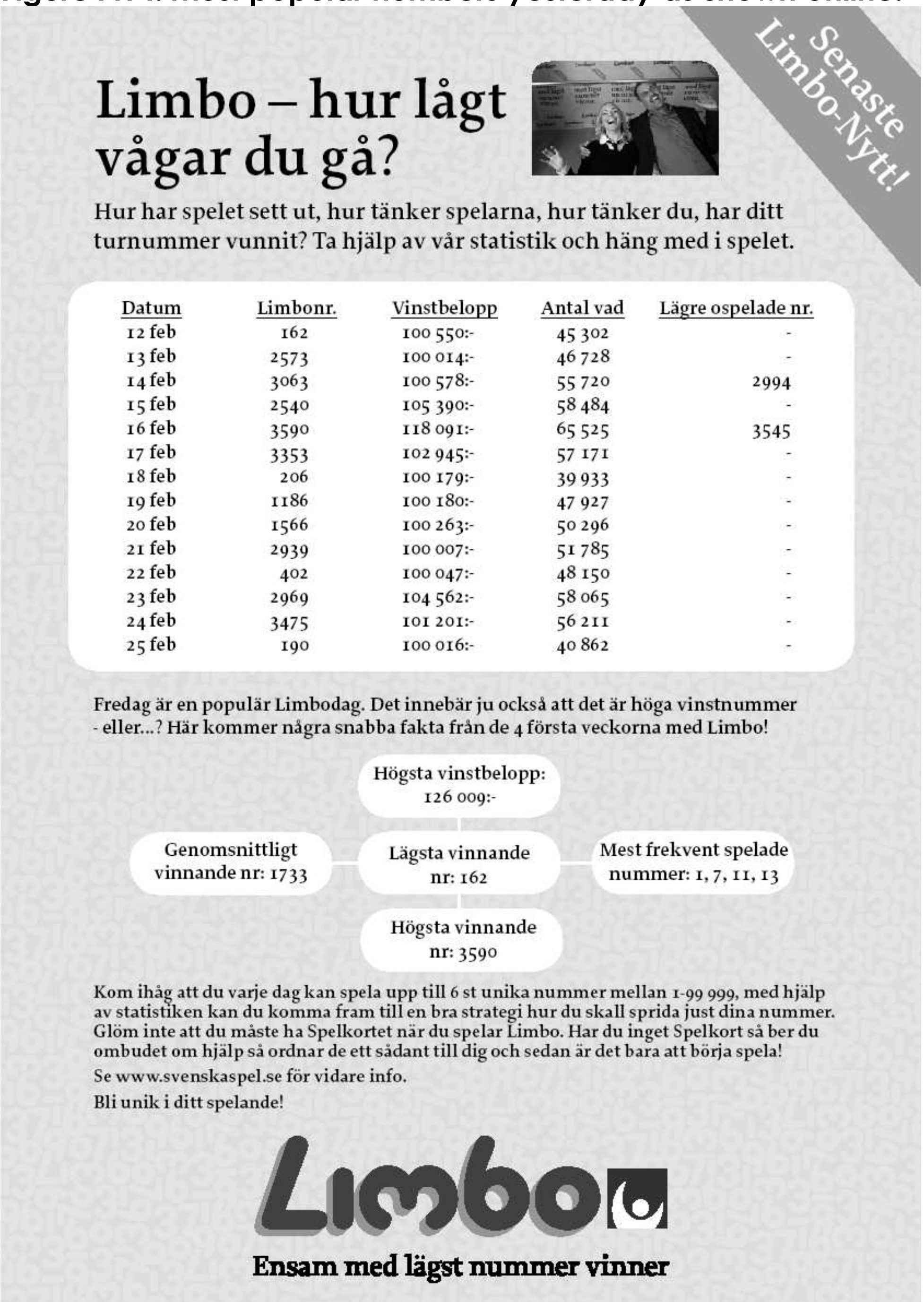


Figure A15. Example of Limbo poster.

Figure A16. Screenshot of input screen in the laboratory experiment.

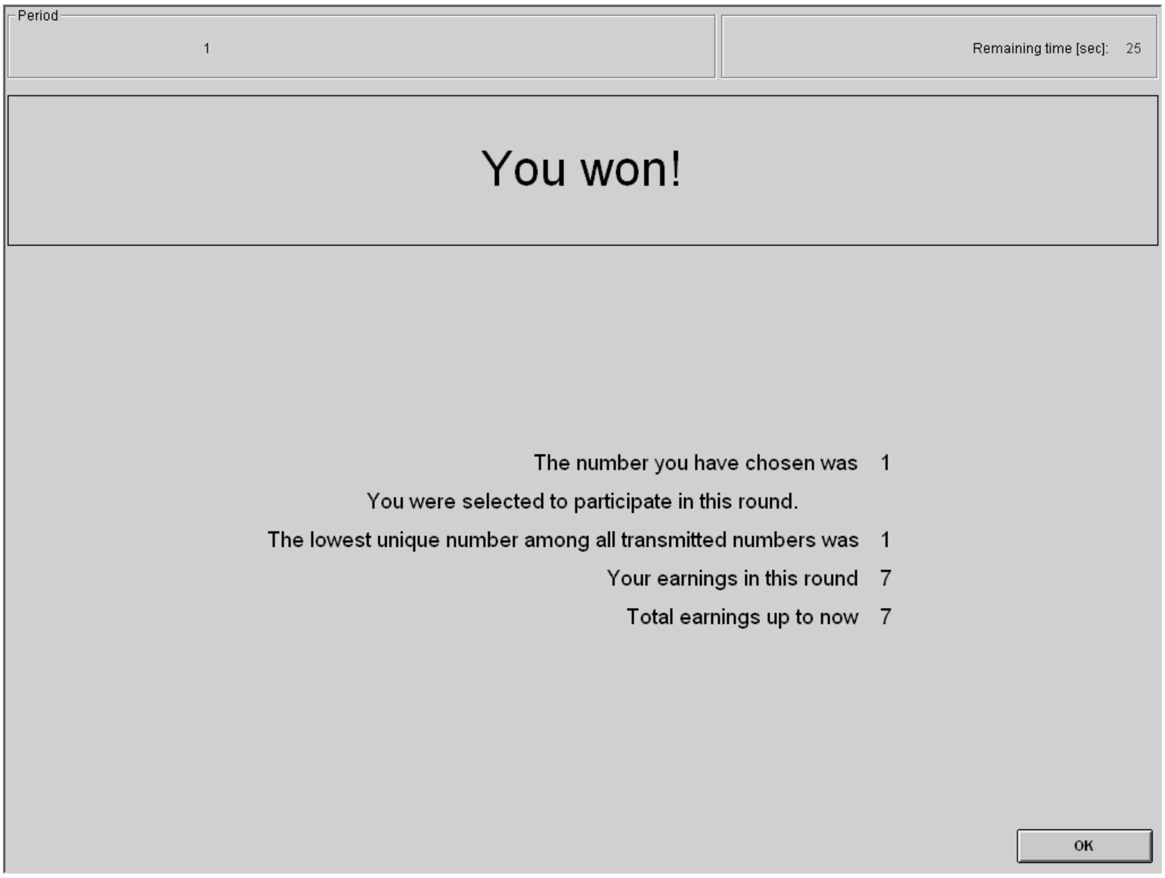
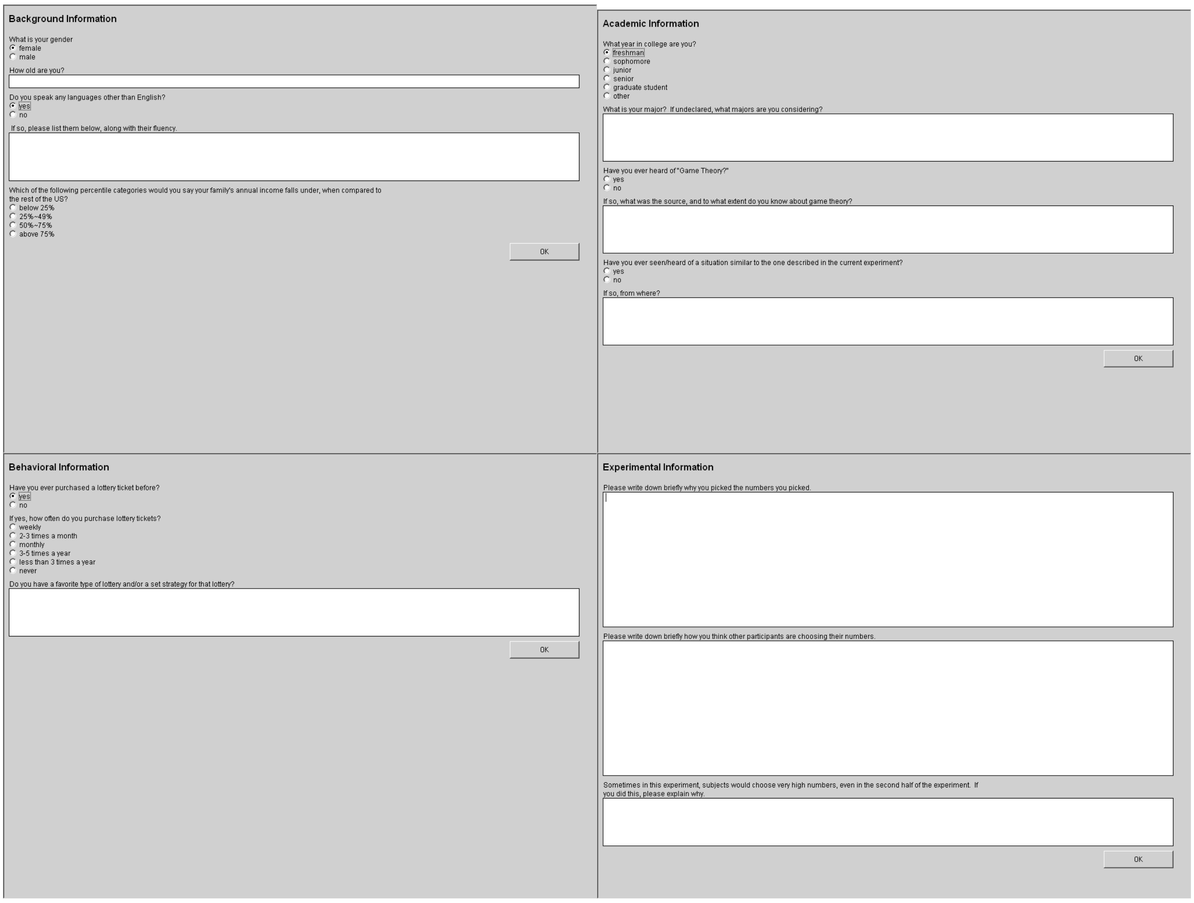


Figure A17. Screenshot of result screen in the laboratory experiment.



Background Information

What is your gender
- female
- male

How old are you?

Do you speak any languages other than English?
- yes
- no

If so, please list them below, along with their fluency.

Which of the following percentile categories would you say your family's annual income falls under, when compared to the rest of the US?
- below 25%
- 25%~49%
- 50%~75%
- above 75%

OK

Academic Information

What year in college are you?
- freshman
- sophomore
- junior
- senior
- graduate student
- other

What is your major? If undeclared, what majors are you considering?

Have you ever heard of "Game Theory?"
- yes
- no

If so, what was the source, and to what extent do you know about game theory?

Have you ever seen/heard of a situation similar to the one described in the current experiment?
- yes
- no

If so, from where?

OK

Behavioral Information

Have you ever purchased a lottery ticket before?
- yes
- no

If yes, how often do you purchase lottery tickets?
- weekly
- 2-3 times a month
- monthly
- 3-5 times a year
- less than 3 times a year
- never

Do you have a favorite type of lottery and/or a set strategy for that lottery?

OK

Experimental Information

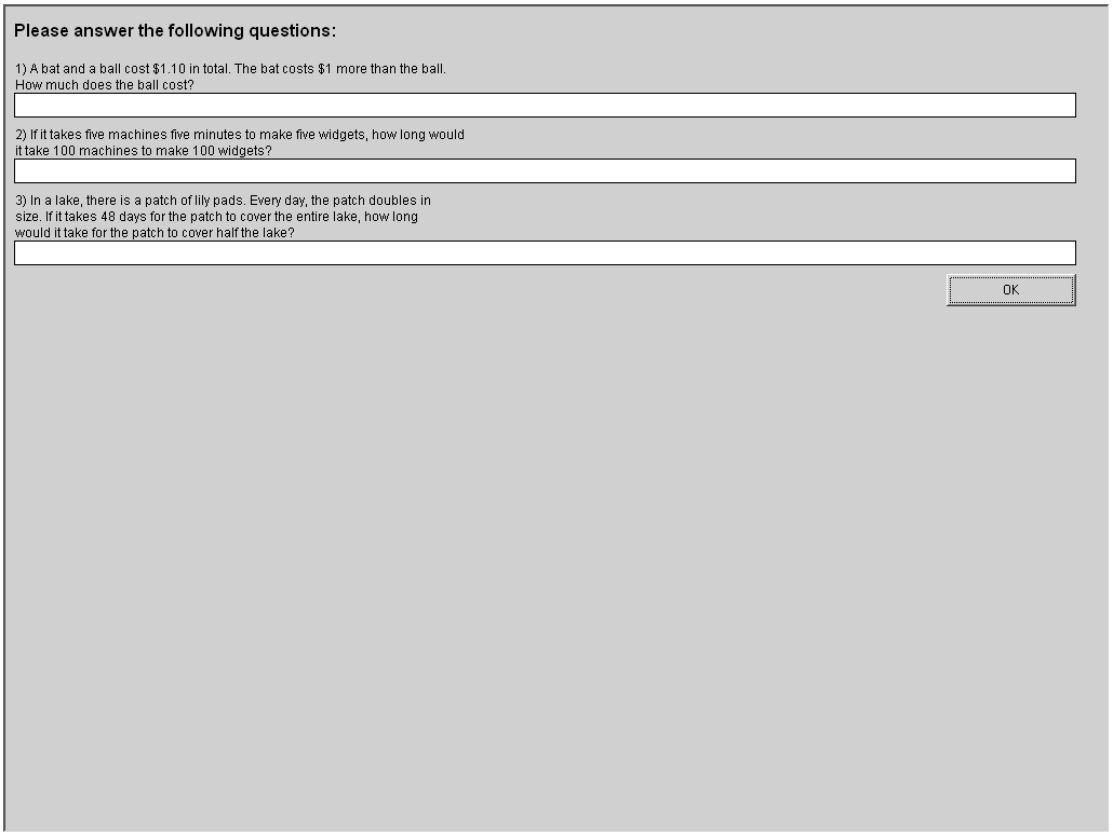
Please write down briefly why you picked the numbers you picked.

Please write down briefly how you think other participants are choosing their numbers.

Sometimes in this experiment, subjects would choose very high numbers, even in the second half of the experiment. If you did this, please explain why.

OK

Figure A18. Screenshots of questionnaire in the laboratory experiment.



Please answer the following questions:

1) A bat and a ball cost $1.10 in total. The bat costs $1 more than the ball.
How much does the ball cost?

2) If it takes five machines five minutes to make five widgets, how long would it take 100 machines to make 100 widgets?

3) In a lake, there is a patch of lily pads. Every day, the patch doubles in size. If it takes 48 days for the patch to cover the entire lake, how long would it take for the patch to cover half the lake?

OK

Figure A19. Screenshot of CRT in the laboratory experiment.

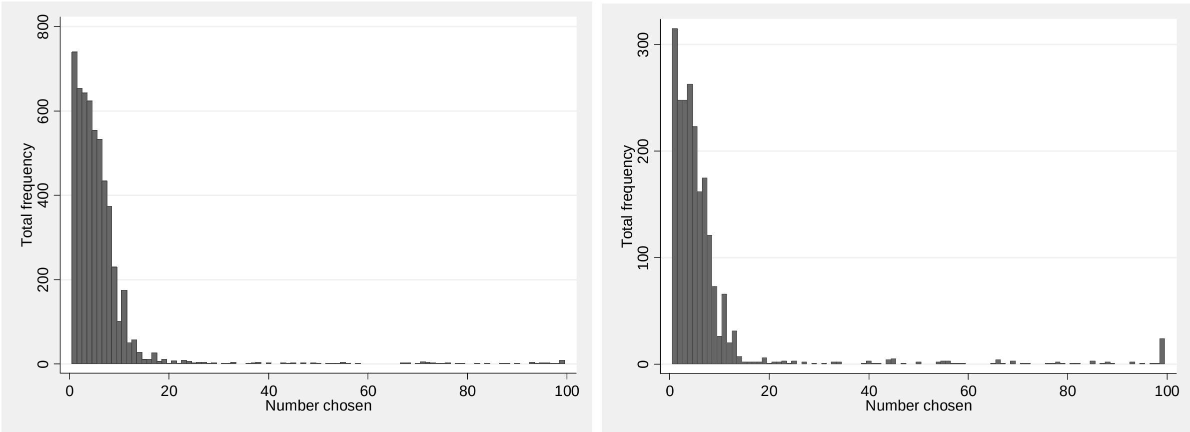


Figure A20. Laboratory total frequencies, selected (left) vs non-selected (right) subjects.

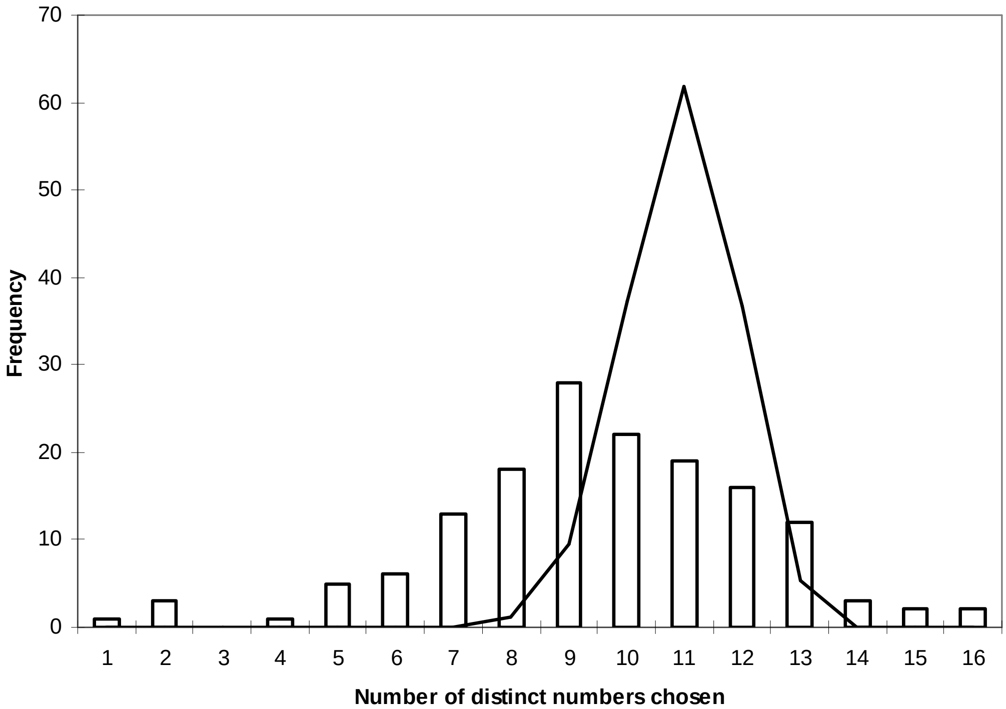


Figure A21. Histogram of the number of distinct numbers chosen by subjects (selected subjects' choices from all sessions, one subject choosing 27 distinct numbers excluded) and the corresponding simulated number of distinct numbers if subjects were playing the Poisson-Nash equilibrium.

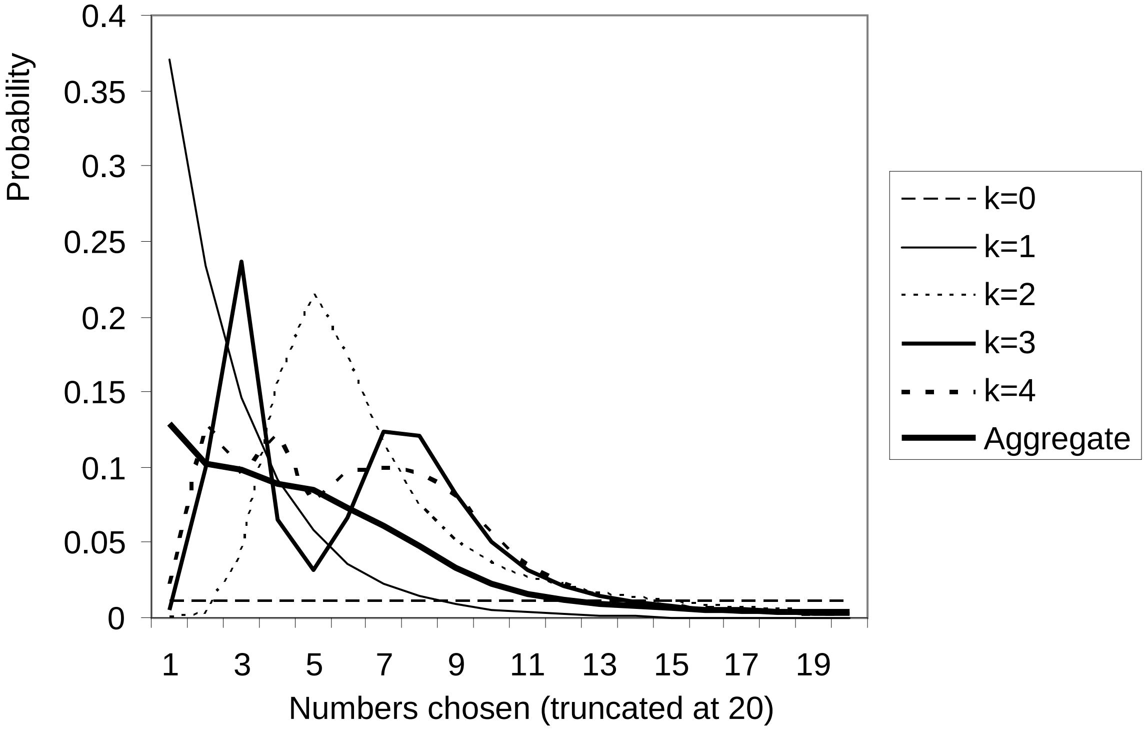


Figure A22. Probability of choosing numbers 1 to 20 in cognitive hierarchy model (*n*=26.9, *K*=99, τ=1.5, λ=2).

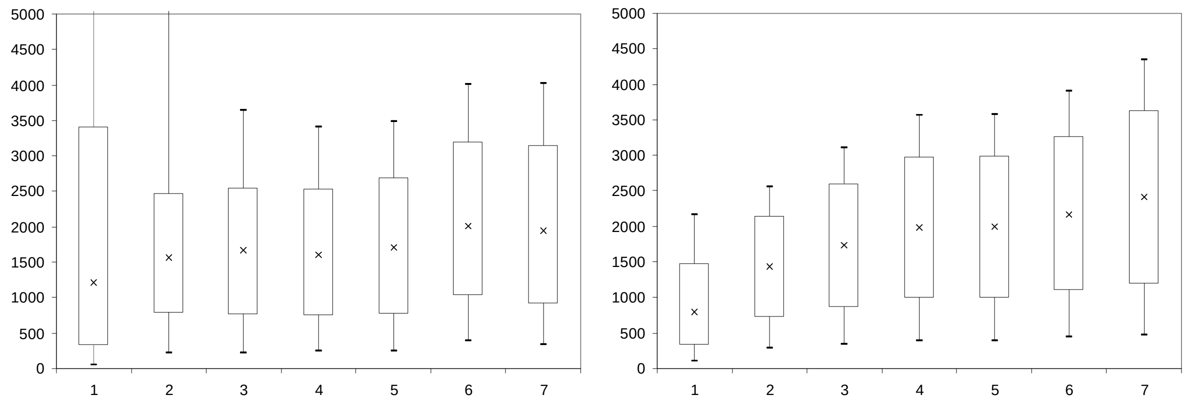


Figure A23. Weekly box plots of data (left) and estimated learning model (right) (10-25-50-75-90 percentile box plots, *W* = 344, λ = 0.0085).

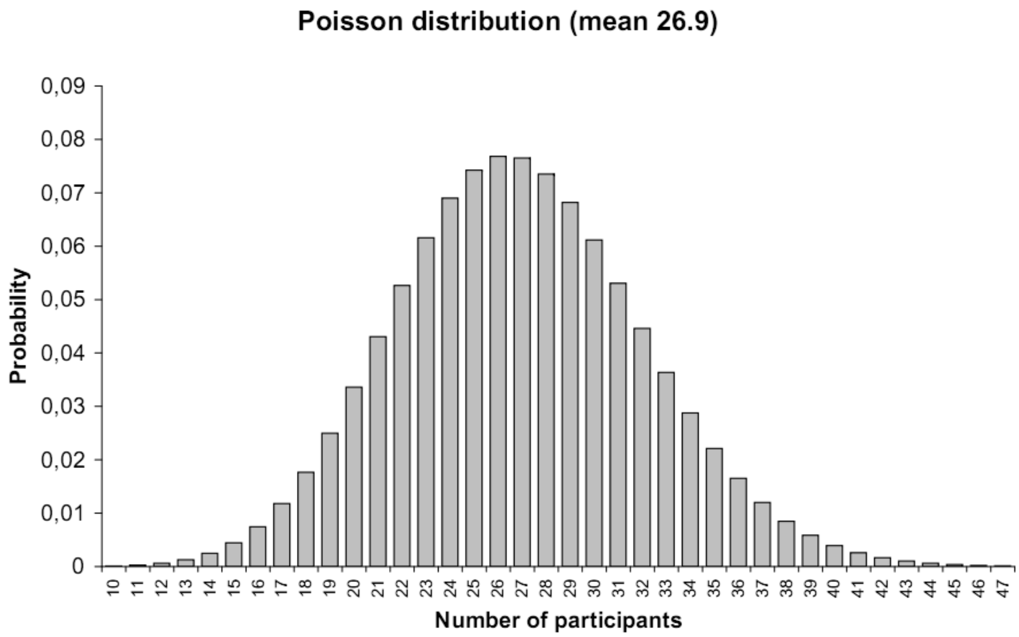


Figure A24. Poisson distribution pdf shown in the instructions of the fourth (Poisson) session.