Investor and Central Bank Uncertainty and Fear Measures Embedded in Index Options*

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Abstract

We provide a structural Bayesian equilibrium learning model that captures the interaction of the uncertainty about fundamentals of investors and the central bank. In our model central bank policy is able to affect fundamental state transitions and investors can learn about future fundamental states from observing policy variables. We show that investors' fear measures — implied volatility (ATMIV) and put-call implied volatility ratios (P/C) — lead industrial capacity utilization, which the central bank reacts to so the fear measures can be used to predict interest rates. The model endogenously generates several of the time series properties of option prices including the counter (pro) cyclicality of ATMIV (P/C), the V-shape (inverse V-shape) relation between ATMIV (P/C) and monetary policy variables, the positive relation between the level and absolute changes in ATMIV, the negative beta of volatility as a priced systematic risk factor, and an economically significant amount of time variation in the volatility premium.

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Since the classic work of Breeden and Litzenberger (1978) it has been clear that option prices contain valuable information on investors' forward looking state price density function. It has been less clear, however, if this information is any way linked to macro economic fundamentals that should affect agents' state prices. While it is intuitive that corporate stock and options prices react to corporate news, there is also substantive evidence that they react to monetary policy shocks. For example, Bernanke and Kuttner (2005) report that monetary policy surprises affect the stock market, while Rigobon and Sack (2003) show that the monetary policy responds to stock returns with a greater reaction during times of higher volatility, and more recently Bekaert, Hoerova, and Duca (2010) find a significant reaction of options prices to lead and lag measures of monetary policy. This compelling empirical evidence though spurs a fundamental question: What is the relation between option prices, corporate fundamentals, and the action of the Central Bank? In this paper we provide a simple dynamic equilibrium model that links options prices to fundamentals and monetary policy, and provide additional empirical evidence of the complex relation between these variables.

Before moving on, it is useful to first present some empirical relations between options and monetary policy. In particular, our analysis focuses on two popularly quoted market wide 'fear' indexes constructed from options prices. The first, the implied volatility of at-the-money (ATMIV) options, which was originally created by Whaley (1993) and trades on the Chicago Board Options Exchange under the ticker VIX, is described by the exchange as:

> One of the most interesting features of VIX, and the reason it has been called the "investor fear guage," is that, historically, VIX hits its highest levels during times of financial turmoil and investor fear. [CBOE Bulletin on VIX, 2003].

The second is the ratio of implied volatilities of out-of-the-money put to call options (P/C), which is a direct market assessment of downside relative to upside risk. This measure has also been studied extensively since the work of Bates (1991) to imply, in particular, investors' fears about a market crash. Quarterly time series plots of these variables for options with three months to maturity are shown in Figure 1. Comparing to the top panel gives the surprising stylized fact that the two fear measures ATMIV and the P/C are negatively related, with the ATMIV (P/C) being generally counter (pro) cyclical. While it is intuitive that a fear index such as ATMIV is high during downturns, it is less obvious why the downside-risk fear index P/C is high during booms and low during recessions.

Indeed, even more interestingly, how does the monetary policy relates to such "fear indices"? By way of motivation, we estimate pairwise Vector Auto Regressions (VAR) with the fear indices and monetary policy variables. The left panels of Figure 2 report resulting impulse responses for the historical series over the options subsample of 1986:Q2 – 2008. The results are striking and all in one direction: fear measures lead to sustained impacts on future monetary policy, while we do not find that monetary policy measures have any sustained impacts on the fear measures (not shown). Indeed, the first panel shows that 3-month Treasury rate decreases for up to six to eight quarters in response to a shock to ATMIV.[1] Even more interestingly, a shock to the P/C downside-risk fear measure induces the 3-month Treasury rate to increase for up to six to eight quarters in the future. The latter result implies that when investors have a higher fear of a stock market decline, future short-term rates increase. To link this discussion to a policy-relevant fundamental variable, the bottom two panels show that indeed we obtain the the same sign impulses for fear measure shocks on capacity utilization (CU), an important determinant in the Federal Reserve monetary policy rule. The impulse responses of the policy measures on the fear indices are statistically insignificant, and are not shown in the figure. Taken together, these results suggest that the central bank responds to market fears, but actions of the central bank do not cause future fears.

This paper provides a dynamic equilibrium model of learning that links options to investors' and central banks' uncertainty about fundamentals, and provides an economic explanation for these relations. More specifically, our model builds on the recent literature that extracts macroeconomic information from the term structure of interest rates [see, e.g. Estrella and Mishkin (1998), Ang and Piazzesi (2003) and Ang, Piazzesi, and Wei (2006).] Indeed, we also exploit the Taylor-rule, which directly links macroeconomic variables to target interest rates set by the Federal Reserve. [2] However, we add to this literature by introducing two additional key features, which we provide evidence for. First, we specify a learning-based Taylor rule, where neither the central bank nor investors observe the true trend growths of nominal as well as real variables. Both sets of agents continuously learn about the hidden trends. In our model, the central bank's has the same information as the investors, and thus its interest

[1]In our empirical analysis, we use the 3-month T-bill rate as our short-term rate, rather than the Federal Funds rate, as the latter is affected by banks' default premium, which is absent in our model. The 3-month T-bill rate and the Fed Funds rate are very highly correlated.

[2]The New Keynesian Economics approach shows the optimality of such rules in settings where price stickiness implies deviations from short run full employment and capacity utilization [see, e.g. Woodford (2003)]. Gallmeyer, Hollifield, and Zin (2005) and Bekaert, Cho, and Moreno (2010) build term-structure models using policy variables.

rate decisions are explicitly determined by a global uncertainty about fundamentals. The interest rate rule is then directly built into investors' pricing kernel to provide pricing implications for the stock index, Treasury bonds, and for the stock index options. Second, to extend the current understanding of the effects of monetary policy on the stock market we follow the suggestions in Lucas (2007)[3] to allow money growth to affects transitions between fundamental drift states. The observations of money growth affect agents beliefs about future fundamental states, which in turn affect stocks, bonds, and, important, options. The econometrician can exploit this additional variable to make inference about investors' state uncertainty.

Our model is based around the role of inflation in signaling future real activity. Essentially, corporate earnings growth is stable at moderate levels of inflation but becomes unstable when inflation is either too high or too low. This "Goldilocks" relationship between inflation and growth implies that the sign of the conditional reaction of the stock market to CPI fluctuations can vary over time and is a key mechanism for understanding several of the time-varying phenomenon we see in the options market, and their relation to monetary policy. Indeed, we show that the central bank's efforts to stabilize growth and inflation implies that the key policy variables, industrial capacity utilization and money growth, exhibit a V-shape relation with the fear measures. In particular, high ATMIV occurs both when capacity utilization is very low or very high. For instance, high ATMIV can lower capacity utilization as firms slow down their activity in the face of high uncertainty, which lowers inflation in the economy below regular levels. Similarly, high ATMIV also occurs when capacity utilization is very high, as investors now fear a future slowdown of the economy. Similarly, the growth of money supply and the ATMIV also have a V-shape relation with the fear measures, as high money growth can result from

[3]Lucas (2007) complains about the lack of use of monetary aggregates in recent models of monetary policy and recommends their use in information extraction:

> One source of this concern is the increasing reliance of central bank research on New-Keynesian modeling. New-Keynesian models define monetary policy in terms of a choice of money market rate and so make direct contact with central banking practice. Money supply measures play no role in their estimation, testing or policy simulation. A role for money in the long run is sometimes verbally acknowledged, but the models themselves are formulated in terms of deviation from trends that are themselves somewhere off stage. It seems likely that these models could be reformulated to give a unified account of trends, including trends in monetary aggregates, and deviations about trend but so far they have not been. This remains an unresolved issue on the frontier of monetary theory. Until it is resolved, monetary information should continue to be used as a kind of add-on or cross-check, just as it is in the ECB policy formulation today.

Coenen, Levin, and Wieland (2005) and Beck and Wieland (2008) show that money growth can help predict real activity when the real output and real money are economically linked but the central bank, which partially controls money growth, receives noisy information on the former.

either efforts by the central bank to stimulate the economy, or from high money demand in times of rising inflationary expectations.

Our model sheds light on the compelling dynamic relation between the fear indices and monetary policy, discussed earlier in Figure 2. The strong evidence that the fear measures lead the policy variables, but not the reverse, is not driven by differences in information between investors and the central bank, as our model assumes they observe the same data and have the same information. Instead, our analysis points to a compelling *real effect* of uncertainty. Consistent with the real options literature, we find that a higher ATMIV (uncertainty) predicts declines in future capacity utilization in the second half of our sample (when options data are available), which feeds into monetary policy through the interest rate rule. Therefore, investors' uncertainty embedded in the fear measures can lead to slowdowns in real activity, which the central bank reacts to by lowering the cost of capital. Similarly, investors' Bayesian learning implies that in good times, investors downwardly revise their beliefs in response to bad news by a larger amount, and thus the endogenous learning dynamics creates a conditionally strongly negatively skewed return distribution. The negatively skewed distribution raises the price of puts relative to calls (P/C). However, it is exactly following such times that the Federal Reserves moves to a tightening of monetary policy, implying an increase in future rates.

All these effects also explains why ATMIV and the P/C are negatively correlated (see Figure 1). As mentioned, in good times the properties of Bayesian learning imply a faster reaction to bad news and thus a negatively skewed return distribution, which increases P/C. In good times, however, their overall belief volatility is low, so that the ATMIV is low. Consequently, the relations between the P/C and the policy variables are inverse V-shaped. We find support for these nonmonotonic relations between policy variables and the fear measures in the data, although it is useful to note that for the post-1986 subsample of our data for which options prices are available, there were few periods of very high capacity utilization.

These nonlinearities are central to our analysis because as we report, most macro fundamental variables explain low amounts of variation in the fear indices in linear regressions. Instead, our model which builds on this nominal-real interaction explains about 50 percent of the variation in the two fear indices and explains their negative relation.

A novel feature of our methodology is to specify a joint unobserved regime switching model of macroeconomic and policy fundamental variables. Agents in the economy (investors and the central bank) are econometricians in the sense of Hansen (2007), that is, they attempt to learn about the drift states of fundamentals from the fundamental realizations. We fit the parameters of our structural model with an overidentified Simulated Method of Moments (SMM) procedure, which uses the likelihood of observing the fundamentals as well as stock, Treasury bond, and options prices to extract investor's beliefs. It is important to note that when analyzing pricing relationships we consider a *single* process of uncertainty over time to guage the success of our model in explaining the time-series variation in fundamentals and asset prices so that investors' beliefs are extracted from asset prices in a dynamic and time consistent way. This distinguishes our work from related work on options with learning[4] that resets the model uncertainty in each period to some proxy of uncertainty in the data and focuses on conditional reactions in options prices.

Our model has numerous additional implications for the behavior of option prices that fit well with the data, which provide additional support for our modeling choice. Indeed, while our estimation method explicitly uses the two fear indices as overidentifying moments, the uncertainty process of the model has additional implications for the volatility of stock market volatility and the dynamics of the implied volatility risk premium, which are not directly fitted by our estimation procedure. The first, as noted by Jones (2003), is that during periods of high ATMIV, ATMIV also fluctuated rapidly, that is, there is a positive relationship between volatility and absolute changes in volatility. Jones (2003) points out that this stylized fact is incompatible with the Heston (1993) stochastic volatility model, and instead requires an explosive volatility process (one which violates certain regularity conditions). We show that our model price instead is nonexplosive, but still displays the correlation of similar magnitude to that in the historical series. We further show that the absolute changes in ATMIV are driven by the models' volatility of volatility series. The intuition on why the model provides the correct correlation rests on the in-built dynamics underlying Bayes' law. In periods of greater uncertainty (and hence volatility), investors also update their beliefs faster, creating the positive relation between volatility and its absolute changes.

[4] For example Guidolin and Timmermann (2005) and Buraschi and Jiltsov (2006) study option prices and volume in models with learning about fundamentals. Dubinsky and Johannes (2006) study the reaction of options prices on individual stocks to news about earnings. Benzoni, Collin-Dufresne, and Goldstein (2005) show that the increase in investors' perception about the average jump size of stock prices led to a steepening of the implied volatility smirk after the stock market crash of 1987, but do not study its time variation in subsequent years. In a paper related to ours, Shaliastovich (2009) models investors' non-Bayesian (behavioral) learning about the long run drift of consumption to generate the smile, but does not study its time series fit to data series.

The second statistic that we take our model implications to is the implied volatility premium, which is the difference between implied volatility and the expectation of volatility under the objective measure. The volatility premium is currently one of the most actively researched statistics in empirical option pricing, and we show that our model volatility premium explains economically significant amounts of variation in two estimates of the historical volatility premium. We break up the implied volatility premium in the model into its components arising from a risk premium for bearing volatility risk and from the fact that the return distribution in our model is non-Gaussian. Each of these components implies that implied volatility is higher than the volatility forecast. We show that the risk premium component, is very highly correlated with the volatility of volatility, which is consistent with volatility being a systematic factor with a negative beta as has been noted in previous research [see e.g. Buraschi and Jackwerth (2001) and Bakshi and Kapadia (2003)]. The non-Gaussian component has a smaller correlation, which results mainly due to the fatness in tails of the return distribution created by the continuous variation in investors' beliefs.

Related Literature

Besides the literature on option prices with learning in footnote 4, this paper contributes to a small set of papers that provides economic explanations of the implied volatility curve for options.[5] Bollen and Whaley (2004) and Garleanu, Pederson, and Poteshman (2008) find that net buying pressure affects the prices of options for several days as market makers fail to provide options at no-arbitrage prices, but charge for the residual risk due to the limits to arbitrage. In addition to focus on lower frequency data and explaining the entire time series of options prices, we do not depart from the no-arbitrage framework. Among theoretical explanations for smirks, Liu, Pan, and Wang (2005) study the implications for ambiguity about rare event risk that raise the prices of puts relative to calls. Drechsler (2008) and Du (2010) provide calibrated models with time-varying ambiguity and with jumps with habit formation

[5] There is also a large literature that explains the volatility smile by assuming exogenous processes for stock prices, volatilities, and jumps. Indeed, since the classic work of Black and Scholes (1977) the major innovations have been the addition of stochastic volatility [see, e.g., Hull and White (1987) and Heston (1993)], jumps in prices [see e.g. Bates (1996) and Bates (2000), and Pan (2002)], and jumps in volatility [see, e.g. Eraker, Johannes, and Polson (2003)]. A tremendous amount of empirical work has been done on these extensions of the BS formula that has enriched our understanding of stock price dynamics, and of options returns. Bakshi, Cao, and Chen (1997) provides a specification analysis of some of these models. Among more recent innovations, Christoffersen, Jacobs, Ornthanalai, and Wang (2008) build multi-factor stochastic volatility models, and somewhat related to our paper, Polson, Johannes, and Stroud (2008) price options when exogenously specified volatility follows an unobserved process that investors learn about. Constantinides, Jackwerth, and Perrakis (2008) find that several exogenously specified volatility models, such as GARCH, can be rejected as possible data generating processes for S&P 500 index options.

preferences, respectively, to generate the left skewed implied volatility smile, but neither paper studies the time series properties of the smile, nor their interaction with monetary policy.

Our work also complements the papers constructing structural models of options prices to understand the volatility premium. Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2010) and Eraker (2008) construct equilibrium models with 'long term risks' in the consumption process to understand the size of the IVP and some of its unconditional moments. In work related to ours, Bekaert and Engstrom (2010) model the time variation in higher order moments of fundamentals to generate a volatility premium. This paper uses habit preferences to generate time variations in the price of risk. Unlike these papers, we shut off both the fundamental heteroskedasticity and time varying risk aversion channels in generating the volatility risk premium. As we will show, all the time variation in the volatility premium in our model arises from variations in the uncertainty of fundamentals (both corporate and policy) which affects the speed of revision of beliefs and the volatility of stock market volatility.

The layout of the paper is as follows. In section 1, we provide the structure of the model and derive some key pricing results. In section 2 we estimate the parameters of our model using an overidentified simulated method of moments procedure. In section 3, we study the ability of our model to explain the two fear indices, and in section 4, we study its ability to understand the volatility of stock market volatility and the volatility premium. Section 5 concludes. Two technical appendices provides proofs of technical results and the estimation methodology, respectively.

1 Structure of the Model

Our main assumption throughout the paper is that the drift rates of the fundamental processes are driven by a $N-$state, continuous time, hidden Markov chain process. It is useful to describe this process first. We denote by s_t the state at time t, where $s_t \in \{s^1, .., s^N\}$, and we let Λ denote the Markov chain infinitesimal generator matrix. That is, over the infinitesimal time interval of length dt

$$\lambda_{ij} dt \ = \ \text{prob}\left(s_{t+dt} \ = \ s^j | s_t = s^i\right), \quad \text{for} \quad i \neq j, \qquad \lambda_{ii} = -\sum_{j\neq i} \lambda_{ij}.$$

We assume that all agents in our economy, both investors and the central bank, do not observe the realizations of s_t but learn about it from the observation of numerous signals, including realized fundamental variables. Given an information filtration $\{\mathcal{F}_t\}$ generated by such signals, we denote the agents' common beliefs at time t about state s^i as

$$\pi_{it} = \text{prob}(s_t = s^i|\mathcal{F}_t), \quad i = 1, ..., N \tag{1}$$

Lemma 1 below characterizes the dynamics of the vector $\pi_t = \{\pi_{1t}, ..., \pi_{Nt}\}$, but before we introduce the learning result, we need to introduce the rest of the model.

There is a single homogeneous good in the economy whose price, Q_t, follows:

$$\frac{dQ_t}{Q_t} \;=\; \beta(s_t)\,dt \;+\; \sigma_Q\,dW_t, \tag{2}$$

where $W_t = (W_{1t}, W_{2t}, W_{3t}, W_{4t}, W_{5t})'$ is a 5-dimensional vector of independent Weiner processes, inflation volatilities are summarized in 1×5 constant vector $\sigma_Q = (\sigma_{Q,1}, 0, 0, 0, 0)$, and the drift rate $\beta(s_t)$ depends on the realization of the (hidden) state s_t.

The main real corporate fundamental in the economy is the process of real earnings, E_t, which follows the jump-diffusion process

$$\frac{dE_t}{E_t} \;=\; (\theta(s_t) - \kappa\,\xi_1)\,dt \;+\; \sigma_E\,dW_t, + (e^{Y_{1t}} - 1)\,dL_t \tag{3}$$

where fundamental volatilities, $\sigma_E = (0, \sigma_{E,2}, 0, 0, 0)$, are constant over time, the drift rate $\theta(s_t)$ depends on the realization of the state s_t, L_t is the counter of a Poisson process with constant intensity κ, i.e. Prob$(dL_t = 1) = \kappa\,dt$, earnings growth conditional on a jump has a distribution Y_{1t} which is i.i.d. normal with mean μ_1 and volatility σ_1, and $\xi_1 = e^{\mu_1 + 0.5\sigma_1^2} - 1$. The state process, s_t, the Brownian Motions, W_t, and the jump process L_t are all independent of each other. We notice that under the assumption of continuous observation of fundamentals, and hence their quadratic variation processes, investors can perfectly observe jumps. Therefore, as we will formalize below, the inclusion of jumps in the fundamental process specification has no impact of investors' learning about the state s_t.

Given their lack of impact on investors' learning process, it is important to understand the reasons for including jumps in our analysis. Jumps to earnings play three important roles: First, their inclusion permits a better estimation of the earnings process, which we will see has some large negative outcomes in our sample. Second, negative mean jumps will be shown to increase the average put-call implied volatility ratio (P/C), and, third, they increase the average volatility premium priced in options in our sample. It is important to note however that we model i.i.d. jump sizes and constant jump intensity so that the modeled jumps in themselves are unable to explain the time series variation in either the P/C or the volatility premium.

The next important fundamental in the economy is de-meaned industrial capacity utilization (CU), K_t which follows the process

$$dK_t \;=\; \rho(s_t)\,dt \,+\, \sigma_K\,dW_t, \tag{4}$$

where $\sigma_K = (\sigma_{K,1}, 0, 0, \sigma_{K,4}, 0)$, is assumed known by investors and the drift $\rho(s_t)$ depends on the realization of of the state s_t. Note that unlike the other state variables, CU is stated in levels, and hence can become negative. The use of CU improves the term structure fit of our model. We will comment on the nonzero instantaneous correlation between CU and inflation in Section 2.

The final state variable in our model is aggregate real money in the economy, H_t, which follows

$$\frac{dH_t}{H_t} \;=\; \omega(s_t)\,dt \,+\, \sigma_H\,dW_t, \tag{5}$$

where $\sigma_H = (0, 0, 0, 0, \sigma_{H,5})$ and the drift $\omega(s_t)$ depends on the state s_t. We emphasize that H_t is the equilibrium quantity of real money in the economy determined both by its demand and supply. It is also useful to note that while ours is not a full structural model in which the quantity of money is endogenously determined, the statistical properties of dH_t/H_t affect agents' beliefs' dynamics, and thus the equilibrium prices.[6]

1.1 The Central Bank Policy Rule

We assume that all agents, investors and Central Bank, observe the same data and thus have the same information about the state of the economy. Thus, the state probabilities π_{it} defined in (1) are common across all agents. We assume that the Central Bank sets the real rate of the economy $\bar{\phi}_t$ by using a forward looking Taylor rule, namely

$$\bar{\phi}_t = \alpha_0 \,+ \alpha_\beta\,\mathbb{E}\left[\frac{dQ_t}{Q_t}|\mathcal{F}_t\right] + \alpha_\rho\,\mathbb{E}\left[dK_t|\mathcal{F}_t\right]. \tag{6}$$

[6] We allow for a generalization the Taylor rule to let interest rates directly be impacted by money growth but do not estimate a significant effect.

where the expectations are taken with respect to all of the information available at time t, $\mathcal{F}_t$. The second and third terms of the real rate capture the essential elements of the Taylor rule, which posits that the central bank increases rates in response to increases in expected inflation and the expected real slack in the economy [see Taylor (1993)]. Our policy rule is hence 'forward-looking' in the sense of Clarida, Gali, and Gertler (2000), who suggested replacing current and/or lagged values of inflation and the output gap by their forward-looking conditional expectations. A significant contribution of our analysis is to jointly estimate the expectations from corporate earnings as well as regular macroeconomic variables, so that there is interaction between uncertainty in the corporate sector and Central Bank policy. In addition, following the assumption in Rudebusch and Wu (2008) we use the industrial capacity utilization series obtained from the Federal Reserve Board rather than the output gap, in the original Taylor Rule.

We finally note that in standard Taylor rules, the central bank sets the nominal interest rate. In our model, we will show that the inflation risk premium is constant, so the policy rule can equivalently be written as the setting of the nominal rate by adding expected inflation and the inflation risk premium on both sides of equation (6).

1.2 No Arbitrage Pricing

To build the policy rule of the Central Bank into a no-arbitrage framework, we follow Ang and Piazzesi (2003) and Piazzesi (2005) in specifying a state price density to price all cash flows in our model. Let M_t be the state price density at date t. As in the modern classic asset pricing theory (see, e.g. Cochrane (2001)), a generic random real cash flow $\{D_t\}$ is priced as

$$M_t P_t \;=\; \mathbb{E}\left[\int_t^\infty M_s\, D_s\, ds|\mathcal{F}_t\right]. \tag{7}$$

It is convenient to first write the process of the state price density in terms of the original hidden Markov process s_t and Brownian motions W_t. We specify M_t taking the form

$$\frac{dM_t}{M_t} \;=\; (-\phi(s_t) - \kappa\,\xi_2)dt \;-\; \sigma_M\, dW_t \;+\; (e^{Y_{2t}} - 1)\, dL_t, \tag{8}$$

where $\phi(s_t)$ denotes the real rate *conditional* on observing the state (see discussion below), $\sigma_M = (\sigma_{M,1}, \sigma_{M,2}, \sigma_{M,3}, \sigma_{M,4}, \sigma_{M,5})$ is a 1×5 constant vector of the market prices of risk, L_t is the same

Poisson counter as in the earnings process in (3), Y_{2t} has an i.i.d. normal distribution with mean μ_2 and volatility σ_2 and perfectly correlated with Y_{2t}, and $\xi_2 = e^{\mu_2 + 0.5\sigma_2^2} - 1$. Note that the jumps in earnings are systematic since they are correlated with the marginal utility of the representative investor in the economy.

To ensure no-arbitrage, the expected drift rate of the state price density must equal the real rate $\bar{\phi}_t$ in (6), so that we impose

$$\mathbb{E}\left[\frac{dM_t}{M_t}|\mathcal{F}_t\right] = -\bar{\phi}_t dt$$

Because of symmetric information, this no arbitrage restriction is naturally obtained by requiring that state by state:[7]

$$\phi(s_t) = \alpha_0 + \alpha_\beta\, \beta(s_t) + \alpha_\rho\, \rho(s_t).$$

We note that constant prices of risk also arise in a simple Lucas (1978) economy with no government where the representative agent has constant relative risk aversion, and where the fundamental volatility of consumption (dividends) is constant. This assumption along with the homoskedasticity of fundamentals ensure that all fluctuations in volatilities and premiums in our model arise endogenously and not from either time variation in risk aversion or built-in fundamental heteroskedasticity. Finally, we assume that the kernel is observed by agents in the economy but not by the econometrician.

1.3 Learning Dynamics

For notational convenience, we stack the fundamental processes (2), (3), (4), and (5) that are observed by the econometrician as signals in a vector $dY_t = (\frac{dQ_t}{Q_t}, \frac{dE_t}{E_t}, dK_t, \frac{dH_t}{H_t})'$, so that

$$dY_t \;=\; \varrho(s_t)\, dt \;+\; \Sigma_4\, dW_t \;+\; J_{4t}\, dL_t, \tag{9}$$

where the drift vector process is $\varrho(s_t) = (\beta(s_t), \theta(s_t) - \kappa\xi_1, \rho(s_t), \omega(s_t))'$, the volatility matrix is $\Sigma_4 = (\sigma_Q', \sigma_E', \sigma_K', \sigma_H')'$, and the vector of jump sizes is $J_{4t} = (0, e^{Y_{1t}} - 1, 0, 0,)$. Similarly, the full set of signals that are observed by the investors in the economy is $dZ_t = (dY_t', \frac{dM_t}{M_t})'$, which has the drift vector $\nu(s_t) = (\varrho(s_t)', -\phi(s_t) - \kappa\, \xi_2)'$, volatility matrix $\Sigma = (\Sigma_4', \sigma_M')'$, and jump size of $J_t = (J_{4t}', e^{Y_{2t}} - 1)'$.

[7] Indeed, from (8) $-\mathbb{E}\left[\frac{dM_t}{M_t}|\mathcal{F}_t\right] = \mathbb{E}\left[\phi(s_t)|\mathcal{F}_t\right] = \alpha_0 + \alpha_\beta\, \mathbb{E}\left[\beta(s_t)|\mathcal{F}_t\right] + \alpha_\rho\, \mathbb{E}\left[\rho(s_t)|\mathcal{F}_t\right]$, which yields (6).

The following Lemma characterizes the dynamics of beliefs $\pi_{it} = \text{prob}(s_t = s^i|\mathcal{F}_t)$. For notational convenience, we denote the drift of the signal vector dZ_t in state i by $\nu^i = \nu(s^i)$.

Lemma 1. *Given an initial condition $\pi_0 = \hat{\pi}$ with $\sum_{i=1}^{N} \hat{\pi}_i = 1$ and $0 \leq \hat{\pi}_i \leq 1$ for all i, the probabilities π_{it} satisfy the N-dimensional system of stochastic differential equations:*

$$d\pi_{it} = \mu_i(\pi_t)dt + \sigma_i(\pi_t)d\tilde{W}_t, \tag{10}$$

$$\textit{in which} \quad \mu_i(\pi_t) = [\pi_t\Lambda]_i, \quad \sigma_i(\pi_t) = \pi_{it}\,[\,\nu^i - \overline{\nu}(\pi_t)]'\,\Sigma'^{-1}, \tag{11}$$

$$\overline{\nu}(\pi_t) = \sum_{i=1}^{N} \pi_{it}\,\nu^i = E_t\left(dZ_t|\mathcal{F}_t\right), \quad \textit{and}$$

$$d\tilde{W}_t = \Sigma^{-1}[dZ_t - J_t dL_t - \overline{\nu}(\pi_t)] = \Sigma^{-1}\left(\nu_t - \overline{\nu}(\pi_t)\right)dt \;+\; dW_t. \tag{12}$$

Moreover, for every $t > 0$, $\sum_{i=1}^{N} \pi_t = 1$.

This filtering result is a straightforward extension of the Wonham filter (see Wonham (1964)), which characterizes the Bayesian learning about the hidden drift with Brownian noise. In the setup here, the observed fundamental vector process has observable jumps in some elements, which do not affect investors' beliefs about the hidden drift. In particular, note that the high frequency variation in investors' beliefs is driven by investors' inferred shocks, $d\tilde{W}$, in equation (12) as opposed to the true shocks, dW, which affect fundamentals. It is also possible to write the fundamental process vector $dZ_t = \nu_t dt + \Sigma dW + J_t dL_t = \nu_t dt + \Sigma[d\tilde{W} - \Sigma^{-1}\left(\nu_t - \overline{\nu}(\pi_t)\right)dt] + J_t dL_t = \bar{\nu}(\pi_t)dt + \Sigma d\tilde{W} + J_t dL_t$. The right hand side of (12) also reveals that the inferred shocks process $d\tilde{W}$, does not depend on the jump parameters, since investors are able to observe jumps which thus do not affect their inference about s_t.

The first application of the Wonham filter in financial economics, as well as several properties of the filtering process, are derived in David (1997). We find it useful to recall that a main advantage of this modeling strategy as opposed to the more commonly used Kalman filter is that investors uncertainty (conditional variance of expectations about the drift terms) fluctuates forever, while in the Kalman filter, this uncertainty converges to a constant. The fluctuating confidence (inverse of the conditional variance) is the driver of the fear indices that we seek to explain in this paper.

1.4 Stock Prices and the Term Structure of Interest Rates

The following proposition provides expressions for the price-earnings (henceforth P/E) ratio and the nominal bond price:

Proposition 1.

(a) The P/E ratio at time t is

$$\frac{P_t}{E_t}(\pi_t) \; = \; \sum_{j=1}^{N} C_j \, \pi_{jt} \; \equiv \; C \cdot \pi_t, \tag{13}$$

where the vector $C = (C_1, .., C_N)$ satisfies $C \; = \; A^{-1} \cdot \mathbf{1}_N$,

$$A = \mathit{Diag}(\phi^1 - \theta^1 + \sigma_M \, \sigma_E' - \kappa(\xi_3 - \xi_1 - \xi_2), \cdots, \phi^N - \theta^N + \sigma_M \, \sigma_E' - \kappa(\xi_3 - \xi_1 - \xi_2)) \; - \; \Lambda. \tag{14}$$

(b) The price of a nominal zero-coupon bond at time t with maturity τ is

$$B_t(\pi_t, \tau) = \sum_{i=1}^{N} \pi_{it} \, B_i(\tau), \tag{15}$$

where the $N \times 1$ vector valued function $B(\tau)$ with element $B_i(\tau) = E\left(\frac{M_{t+\tau}}{M_t} \cdot \frac{Q_t}{Q_{t+\tau}} | \nu_t = \nu^i\right)$ is given by

$$B(\tau) = \Omega \, e^{\omega \, \tau} \, \Omega^{-1} \, 1_N. \tag{16}$$

In (16), Ω and ω denote the matrix of eigenvectors and the vector of eigenvalues, respectively, of the matrix $\hat{\Lambda} = \Lambda - \mathit{Diag}(r^1, r^2, \cdots, r^n)$, where each $r^i = k^i + \beta^i - \sigma_M \sigma_Q' - \sigma_Q \sigma_Q'$, is the nominal rate that would obtain in the i^{th} state, were the states observable. In addition, $e^{\omega\tau}$ denotes the diagonal matrix with $e^{\omega_i \tau}$ in its (i, i) position.

The proof for stocks is in the appendix. The proof for bonds follows from a simple extension of the proof in a similar setting in David and Veronesi (2009). The stock price formula has a similar form to that developed in the pure diffusion setup of David and Veronesi (2009) and further intuition on the formula is provided there. The major difference here is the jump risk in earnings and kernel, which is priced, and adds to the equity risk premium. The constant C_i is the P/E as in the Gordon growth model. In contrast to stocks, bond prices do not jump since the belief processes are continuous and the main bond fundamental, inflation, is continuous. It is useful to note that the actual dynamics of stock and bond prices here are quite different from those in David and Veronesi (2009) since they are in part determined by policy variables, not in their paper, and in addition, stock prices can jump.

Let $P_t^n = P_t \cdot Q_t$ be the nominal value of stock, where P_t is the real value of stocks in Proposition 1. Using the dynamics of the inflation and earnings processes under the observed filtration, we now formulate the nominal return processes for stocks and bonds.

Proposition 2.

(a) The nominal stock return process under the investor's filtration is given by

$$\frac{dP_t^n}{P_t^n}(\pi_t) \;=\; \left(\mu^n(\pi_t) - \delta(\pi_t)\right) dt \;+\; \sigma^n(\pi_t)\, d\tilde{W}_t \;+\; \left(e^{Y_{1t}} - 1\right) dL_t,$$

where $\delta(\pi) = 1/(C \cdot \pi_t)$ *is the earnings yield, and the nominal stock price volatility is*

$$\sigma^n(\pi_t) \;=\; \sigma_E \;+\; \sigma_Q \;+\; \frac{\sum_{i=1}^N C_i\, \pi_{it}\, (\nu_i - \overline{\nu}(\pi_t))'(\Sigma')^{-1}}{\sum_{i=1}^N C_i\, \pi_{it}}. \tag{17}$$

The proof follows from a simple adaptation of the proof in Veronesi (2000) and an application of Ito's formula for jump-diffusions. Asset volatilities have exogenous as well as learning-based components, which depends on the volatility of each state probability π_i. We will discuss these further in the empirical sections of this paper.

1.5 Return Volatility and its Dynamic Properties

As discussed in the introduction, the key variable for understanding a number of features of options prices is the volatility of stock variance. We develop its properties here.

We start by introducing the following notation. Let

$$\pi_i^\circ = \frac{\pi_i C_i}{\Sigma_{j=1}^N \pi_j\, C_j} \tag{18}$$

As in Veronesi (2000), we call $\pi^\circ = (\pi_1^\circ, ..., \pi_n^\circ)$ the *value-weighted probabilities* (notice that $\pi_i^\circ \geq 0$ for each i and $\sum_{i=1}^N \pi_i^\circ = 1$,). From now on, a "$\circ$" denotes a quantity computed with respect to the distribution π°. For example, $\overline{\theta}^\circ$ denotes the mean of the drift vector θ computed using the distribution

π° (whereas e.g. $\bar{\theta}$ denotes the mean drift vector computed using the original distribution π_t), and

$$\sigma_{\theta\beta} = \sum_{i=1}^{N} \pi_i(\theta_i - \overline{\theta})(\beta_i - \overline{\beta}); \text{and} \quad \sigma^\circ_{\theta\beta} = \sum_{i=1}^{N} \pi^\circ_i(\theta_i - \overline{\theta}^\circ)(\beta_i - \overline{\beta}^\circ) \tag{19}$$

are the covariances of the drift vectors θ and β computed using π and π°, respectively. In addition we denote $\sigma_{\theta\nu}$ and $\sigma^\circ_{\theta\nu}$ to be the vectors of covariances of θ with each element of the vector ν using the two sets of probabilities respectively. We then have:

Proposition 3 *(a) Stock return variance is given by*

$$V = \sigma^n(\pi_t)\sigma^{n'}(\pi_t) = (\sigma_E+\sigma_Q)(\sigma_E+\sigma_Q)'+(\bar{\nu}^\circ-\bar{\nu})'(\Sigma\Sigma')^{-1}(\bar{\nu}^\circ-\bar{\nu})+2\left[(\overline{\theta}^\circ-\overline{\theta})+(\overline{\beta}^\circ-\overline{\beta})\right] \tag{20}$$

(b) Return variance V follows the process $dV = \mu_V dt + \sigma_V\, d\tilde{W}$, where $\sigma_V =$

$$2\left[\sum_i \left([\pi^\circ_i\,(\nu_i - \overline{\nu}^\circ) - \pi_i\nu_i]'(\Sigma\Sigma')^{-1}(\bar{\nu}^\circ - \bar{\nu})(\nu_i - \bar{\nu})'\right) + (\sigma^\circ_{\theta_\nu} - \sigma_{\theta_\nu})' + (\sigma^\circ_{\beta_\nu} - \sigma_{\beta_\nu})'\right]\Sigma'^{-1} \tag{21}$$

(c) The volatility of stock volatility is

$$\sigma_\sigma = 0.5\,\frac{\sigma_V}{\sqrt{V}} \tag{22}$$

The proposition implies that return variance is stochastic and so is the covariance between return and variance, given by

$$\text{Cov}\left(dV, \frac{dS}{S}\right) \equiv \sigma_V\sigma_S', \tag{23}$$

where stock volatility is in (17) and and the volatility of variance in (22). We will see below in Section 3 that for our calibrated model this covariance can change sign and magnitude leading to changes in the slope of the implied volatility curve for options prices.

We finally show that the stock price process in our model satisfies important regularity conditions, which guarantee the solutions to the to the option pricing partial differential equation as well as estimation of the likelihood function. These conditions will be useful to compare the properties of our model with standard option pricing models in Section 4.

Proposition 4 *The stock price process, P_t^n, in Proposition 2 satisfies global Lipschitz and growth conditions.*

1.6 Option Prices

To formulate options prices we need to provide the process for the stock index under the risk-neutral measure, which we provide below:

Proposition 5 *The stock price under the risk-neutral measure follows:*

$$\begin{aligned}
\frac{dP_t^{n*}}{P_t^{n*}}(\pi_t^*) &= (\mu^n(\pi_t^*) - \delta(\pi_t^*))\,dt \;+\; \sigma^n(\pi_t^*)\,d\tilde{W}_t^* \;+\; (e^{Y_{1t}^*} - 1)\,dL_t^*, \\
d\pi_t^* &= (\mu(\pi_t^*) - \vartheta(\pi_t^*))\;dt + \sigma(\pi_t^*)\,d\tilde{W}_t^*,
\end{aligned}$$

where $d\tilde{W}_t^* = d\tilde{W}_t + (\sigma_M + \sigma_Q)dt$, L_t^* is the counter of a Poisson process with intensity $\kappa^* = \kappa \cdot e^{\mu_2 + .5\sigma_2^2}$, and Y_{1t}^* is distributed $N(\mu_1 + \sigma_1\sigma_2, \sigma_1^2)$. Finally the market price of risk of the belief of state i, which is the covariance of π_i with the nominal pricing kernel is given by

$$\vartheta_i(\pi_t^*) = \pi_{it}^*\left((\beta_i - \bar{\beta}(\pi_t^*)) - (\phi_i - \bar{\phi}(\pi_t^*)\right). \tag{24}$$

The proof is in Appendix 1.

We appeal to the Feynman-Kac formula to use Monte-Carlo simulations to evaluate the expectation

$$f(t, \pi_t, P_t^n) = \mathbb{E}^Q\left[\exp\left(-\int_{s=t}^{T} r(\pi_s)ds\right) g(P_T^n, \pi_T)\right]. \tag{25}$$

We use some variance reduction techniques for efficiency. The advantage of the simulation methods is that they do not suffer from the curse of dimensionality as the projection methods above. Details of the simulation procedure are provided in Appendix 2.

2 Estimation

Ours is a regime switching model in which the regime s_t affects the drift rates of four different fundamental series, so that shifts in their means are jointly determined by s_t. This feature of our model is important as it introduces an important low-frequency comovement of fundamental variables in addition to the high frequency Brownian shocks. For example, we will see that that earnings growth is more stable in period of moderately low inflation and is unstable when the inflation drift is either too high or too low. In addition, the persistence and transition between states is partly determined by the central bank's efforts to stabilize the economy. We build this feature into our model by including the policy variables, industrial capacity utilization and money growth, in the joint specification of macroeconomic and policy states. An important feature of our methodology is that asset prices in the model are functions of both macro and policy variables and are are used by the econometrician to back out investors' beliefs about these states.

2.1 Estimation Methodology

It is important for the goal of this paper to extract investors' beliefs in a in a dynamic, forward-looking, and time-consistent way. This is done by using an overidentified Simulated Method of Moments (SMM) of the learning model, in which the structural parameters are constant over time, but the arrival of new information leads to investors updating their beliefs about the joint states of macro and policy variables. We (the econometricians) estimate the model by using information in fundamentals (macro and policy), stock and bond prices, and options prices. Fundamentals are included since investors' information sets clearly contain the history of all fundamental data. However, since investors' information sets are likely to be considerably richer than the history of fundamentals, we attempt to extract their forward-looking beliefs embedded in asset prices at discrete (quarterly) points of time. It is important to note that the SMM likelihood function of observing the fundamentals is exactly identified by all the structural parameters. It follows that all asset prices used in the procedure are overidentifying restrictions on the model, and lead to an omnibus test of the model. In fact, we use the objective function to guide us on the number and specification of the composite regimes of fundamentals and essentially use a stopping rule when the model is no longer rejected. Details of the SMM procedure are available in Appendix 2.

2.2 Data Description

We start with the description of the data series used. Our data sample runs from 1967 to 2008. The definitions of the fundamental series are as follows. Aggregate earnings for the economy are approximated as the operating earnings of S&P 500 firms, and these data are obtained from Standard and Poor's. Dividends for these firms, also obtained from Standard and Poor's, are used with the prices to compute returns. The other three fundamentals, the Consumer Price Index (CPI), Industrial CU and money (M1) are obtained from the Federal Reserve Board.

Stock prices are obtained from S&P and P/E ratio is estimated as the equity value of these firms divided by their operating earnings. The time series of zero-coupon yields and returns on Treasury bonds of different maturities are obtained from the Fama-Bliss data set available at the University of Chicago. Options data are obtained from two sources. We obtain transactions data on S&P 500 index options from 1986:Q2 to 1996:Q1 from the CBOE. These data are no longer available from 1996:Q1, and therefore, we use data on these same options from Option Metrics from 1996:Q2 to 2008:Q3. It is important to note that Option Metrics provide the average of bid and ask prices at the end of each trading day, and not prices based on actual transactions. Prices at the beginning of each quarter are fitted with fundamental data available at the end of the previous quarter. Since the well known VIX index hit its record in the fourth quarter of 2008, while Options Metrics stopped supplying its data in the third quarter, we approximated the 2008:Q4 ATMIV by using the VIX for this quarter. For our sample the VIX and our ATMIV measure had a correlation of over 90 percent, so the approximation error here should be small.

2.3 Estimation Results for the Regime Switching Model

In this subsection, we briefly describe the results of the estimation of our model. The procedure in Appendix 2 finally settles on $N = 8$ states. To limit the number of possible drift rates that each fundamental series can take overall, we impose that the functions $\beta(s)$, $\theta(s)$, $\rho(s)$ and $\omega(s)$ can take at most four values each. The fundamental composite states that we estimate are provided below, and investors' conditional probabilities of these states are in Figure 3.

State s^1: $(\beta = 1.5\%, \theta = 6.1\%, \rho = 1\%, \omega = 1\%)$. This is the "regular boom" state of the economy. Inflation is low, earnings growth is strong, and the policy variables are just above their average levels, which are all conditions for stability in future fundamentals. Investors believed on *average* that this is the most likely to be in the sample period, and in particular during non-recessionary periods classified by the NBER.

State s^2: $(\beta = 6.51\%, \theta = -5.21\%, \rho = 1\%, \omega = 5.31\%)$. This is "regular recession" state. Inflation is at a medium level, earnings are shrinking, CU is above average, but money growth is very strong. The strong money growth is consistent with stimulative efforts by the central bank, but also with high demand for money that is pushing up goods prices. The filtered probability of this state was at its maximum in the 1982 recession, at about 50 percent. In the past three recessions, the probability of this state has been small.

State s^3: $(\beta = 6.51\%, \theta = 6.11\%, \rho = 8.71\%, \omega = 1.51\%)$. This is the "over-heating" state. In this state, earnings growth is still strong, while the other fundamentals warn of impending trouble. Inflation hits a medium level, CU is unusually tight, although money growth remains mild, likely as the central bank has not decided to intervene yet. The filtered probabilities suggest that this has been the second most likely state in the sample, and fears of it have it have sporadically increased in most boom periods in the sample.

State s^4: $(\beta = 9.11\%, \theta = -5.21\%, \rho = -2.51\%, \omega = -5.71\%)$. This is the "stagflation" state. In this state, fundamentals are at about their worst shape, with high inflation, low profit growth, low CU and very low money growth. The low money growth is consistent with attempts by the central bank to rein in inflation. Investors' filtered probability of this state peaked around the 1981 recession, and did not fully subside until the end of the following recession in 1983. Notably, the belief of this state increased to nearly 10 percent, before deflation expectations set in the 2008 financial crisis.

State s^5: $(\beta = 1.5\%, \theta = 7.71\%, \rho = 1\%, \omega = -3.1\%)$. This is the "new economy" state. Earnings growth is at its most rapid, inflation and capacity utilization are low, but money has tightened likely a reflection of the central bank's efforts to moderate growth. Investors' probability of this state peaked at about 50% in the late 1990s, but crashed during the 2001 recession as investors' hopes of the new economy tanked. There was a mild increase of this probability in the boom period in the 2000s, which again tanked in the 2008 recession.

State s^6: ($\beta = -0.2\%, \theta = -5.7\%, \rho = -6.6\%, \omega = 5.3\%$). This is the "deflation state of the economy, in which earnings shrink at their most rapid rate in the cycle. CU is 6.6% below its historical average, and money growth is very rapid as the central bank attempts to stimulate growth. Investors' deflation expectations have spiked in the recessions of the current millennium, but were also high after the 1982 recession after the Fed's efforts to tame strong inflation expectations.

State s^7: ($\beta = 6.5\%, \theta = -5.2\%, \rho = -6.6\%, \omega = -3.1\%$). This is a "deep recession" state, in which inflation and earnings growth are similar to those in the mild recession (state 2), but CU is very low and money shrinks, which likely results as monetary policy is no longer effective in stimulating the economy. Investors' filtered probability of this state was at its highest after the oil price induced recession in 1973, but has also been as high as 30% in the current recession. Combined with their high deflation probability in this period, investors' inflation uncertainty has been very in this recession.

State s^8: ($\beta = 1.5\%, \theta = 6.1\%, \rho = -6.6\%, \omega = 5.3\%$). We call this the "low capacity boom" state of the economy in which inflation and earnings growth are as in the regular boom state (state 1), however, the growth seems shaky since CU is very low and money growth is very strong likely as a result of very proactive stimulative efforts by the central bank. Investors' probability of this state hit close to 30 percent in the recovery periods following recessions in the early 1970s and mid 1980s, and was even higher in 2000s prior to the most recent recession. The high money growth in this period is consistent with the easy credit regime that is oft cited as the cause of the increase in stock and house prices in this decade.

The uncertainty of investors is generated in large part by their estimations of transitions between states, which we show in the top and middle panels of Table 2. In the top panels we see how inflation interacts with earnings stability. In the boom states with low inflation (states 1,5 and 8), investors' estimate only a 1.2 percent chance of a recession in the next year, while in state 3, when earnings are still booming, but inflation heats up to a medium level, the transition to a recession state in the following quarter rises to about 8.5 percent. The role of low CU in affecting transitions can be seen by comparing the annual transitions in the regular (state 2) and deep (state 7) recession states to the deflation state. Indeed, the risk of entering deflation rises from about 0.5 from the regular recession, to about 7.7 percent from a deep recession, when CU is extremely weak, which explains the spike in deflation fears in the two recessions of the current millennium. The middle panel of the table, shows the 5-year transitions between states, which show the medium-term risks to fundamentals. Notable among these transitions, is the large persistence of the new-economy growth state, which suggests that even after five years, investors expect to remain in that state with a probability exceeding 92 percent. It is also relevant to point out that the estimated persistence of the deflation state is the lowest among all states. This estimate likely arises from the fairly rapid recovery of industrial CU from its troughs, which we have seen in our sample that began in 1967.

2.4 Estimation Results for the Taylor Rule and State Price Density

We next turn to the parameter estimates determining the pricing kernel. As shown in Table 1, the interest rate rule parameters suggest that the real rate in the economy depends positively on both the hidden state of inflation $\alpha_\beta = 0.362$ and $\alpha_\rho = 0.257$. We note that these estimates are similar to estimates of the Taylor rule in many other papers and indeed Taylor's own work suggested values of each parameter of 0.5. It is useful to remember that in our model we use industrial CU rather than the output gap used by Taylor, and the rates depend on the expected drifts of the variables rather than the variable realizations themselves.

The next line in Table 1 shows the prices of risk. Most notably, the prices or risk of the earnings shock, the CU shock, and and kernel shock itself are all positive and large (around 0.3), however, the prices of risk of the inflation and money growth shocks are small. The inflation shock has a positive price of risk, while the money growth shock has a negative price of risk. As we see in our plot of the fundamental series, money growth is strongly countercyclical, and investors estimate that their marginal utility increases in periods with high money growth.

2.5 Model Fit to the Data

Using the time series of investors' state probabilities in Figure 3 and the estimated parameters, we generate time series of model-implied expected fundamental growth and stock and Treasury bond prices in Figure 4. We start by making a few comments about the trends in fundamentals. As noted by several authors, inflation has largely trended downwards since the early 1980s, even though it picked up before the 1991 and 2008 recessions. Very notably, the CPI fell by nearly 2 percent in the recent recession, fanning deflation fears. Earnings is the most volatile of the fundamental series, and has witnessed massive drops of 15 and 25 percent in the past two recessions, and has generally fallen by smaller amounts in all past recessions in the sample. The other real fundamental, de-meaned industrial CU, is also highly procyclical. It is smoother than the other series, and our model fits it very closely throughout the sample, expect perhaps at the end of the 1982 recession, where it fell far more than predicted by the model. As is perhaps evident from comparing the first and third panels, CU and inflation are positively correlated (correlation of of nearly 35 percent). Indeed for much of the 1970s, there was high CU accompanied by high inflation. The demise of inflation in the 1980s resulting from high interest rates also choked off CU. More recently in the 2000s, deflation fears resulting from low or falling CPI also accompanied very low CU. The final fundamental in the bottom panel, real money growth shows some interesting comovements with the business cycle, which our model captures correctly. In the 1970s, money growth was tightened during recessions, while in the recessions of the 2000s, money growth grew very rapidly. The different policy response in these recessions is likely in part determined by the different trends in CU in these recessions noted above and highlights the interdependence between the different fundamentals.

As noted, the model expected growth rate of the fundamentals seem to track the major trends in realized fundamentals quite closely. The fits of the model are reported in Table 3. The model expected growth rate explains 62, 16, 75, and 37 percent of the variation in the realized fundamental for inflation, earnings growth, capacity utilization, and real money growth, respectively. We note that our SMM procedure, which maximizes the likelihood of investors observing the historical fundamental processes, does not have an explicit prediction on the fitted *actual* fundamentals in each period, but instead characterizes expected fundamental growth. Therefore, these fits reflect not simply the accuracy or our model, but in addition, investors' estimates on the fraction of variation in fundamental growth that is related to shifts in trend growth rates as opposed to purely idiosyncratic variation. So, e.g., earnings growth is the most volatile fundamental, and the model explains the lowest amount of its variation, while CU is the smoothest, and the model explains most of its variation. Also note that the β coefficients in the expectations regressions are between 1.3 and 2.2 so that actual fundamentals are more volatile than their expectations.

The ability of the model to understand fluctuations in stock and Treasury bond prices is displayed in Figure 5 and the fit statistics are reported in Table 3. The model explains most fluctuations in the S&P500 P/E ratio, in particular the single digit P/Es in the late 1970s early 1980s, the return to high teens levels in the 1980s, the rapid rise to over 25 in the late 1990s. and the decline in the 2000s again, overall explaining about 60 percent of the historical variation. The middle and bottom panel shows that the model is also quite successful in explaining most of the variation in the short rate and the slope of the term structure of Treasuries. The use of a Taylor-type rule is mostly instrumental in explaining the sharp dips in short rates following most of the recessions, although notably, the model short rate recovered more rapidly than the historical series in the early 1990s and the 2000s. In addition, the historical slope was higher in these two episodes than our model can explain. Overall, or model explains more than 50 percent of the variation in the short rate and slope. It is worth noting that our model does not rely on 'unobservable' factors that are used in the exponential affine term structure literature [see e.g. Dai and Singleton (2002)] to explain the fluctuations in these variables.

The final components of our SMM error term are the moments based on option prices that we discuss separately in Section 3. Using the scores of the likelihood function and the errors of the price and volatility variables, we evaluate the SMM objective function, which serves as an omnibus test statistic. The overall SMM objective function value, which has a chi-squared distribution with five degrees of freedom, is 10.47, implying a p-value larger than 5%, so we fail to reject our model.

3 Option Implied Fear Indices and Monetary Policy

This section contains our main results: Before discussing the relation between options and monetary policy, we first show that the model fits well the fear indices ATMIV and P/C, and elaborate on their relation with several macro-economic control variables that have been used in the literature. We then turn to the model implications for options and monetary variables.

3.1 Explaining Time-Variation in ATM Implied Volatility

To guage the impact of fundamentals on ATMIV, the bottom panel of Table 2 computes the model's implied ATMIV when investors are 80 percent sure of being in each of the eight states respectively.[8] It is immediately evident from the table that the model implies that implied volatility is generally countercyclical, being higher in states with negative earnings growth (states 2,4, 6, and 7). Its highest

[8] An important caveat to note is that the model ATMIV is not a linear function of beliefs but instead is more directly associated with the uncertainty about the states. Therefore, the implications for intermediate beliefs should not be approximated by interpolating the ATMIV at the state beliefs in the table.

levels occur during stagflation periods (state 4) when investors' beliefs are the most reactive to inflation news. However, implied volatility can be high in strong states of the economy as well. In particular in the new economy state (state 5), ATMIV is close to that in some recession states. In this state, strong economic growth raises investors' future earnings uncertainty causing high volatility. This observation is made in David and Veronesi (2009) for explaining the conditional positive relation between P/E ratios and realized volatility in the late 1990s.

The interaction between the key policy variable, CU and the ATMIV is a compelling one, linked by investors uncertainty about states. Looking again at Table 2 we see that of the two recession states – regular (state 2) and deep (state 7), ATMIV is higher in the former, when the transition to stagflation is a significant possibility. In the latter, CU in the economy is low, and as seen in Table 2, the transition probability to the high volatility stagflation state in the following quarter is close to 0. Low CU and the resulting smaller threat of transition to stagflation also implies that the ATMIV is lower in the deflation state. These observations might suggest that CU and implied volatility are positively related. However, as we will see below, the relation between these variables is actually V-shaped. The apparent contradiction is resolved by noticing in Figure 3 that in periods of extreme CU (either high or low) investors have been more uncertain about the states, which push up ATMIV (see footnote 8). It is also important to note that in the subsample of our data where we have options data (post 1986) CU has never been too high and we will return to this point later in this section.

The historical and model-fitted ATMIV series are shown in the top panel of Figure 1 and some regressions examining the fits are in Table 4. Our historical time series spans a long period of nearly 23 years that covers the recessions of 1991, 2001, and 2008, as well unusual events such as the stock market crash of 1987, the collapse of LTCM in 1998, and the bursting of the technology bubble in 2000. As seen in the figure, during each of these events implied volatility increased above 30 percent, while its average over the sample is 18.5 percent. Our model implied volatility, which only builds in the impact of macroeconomic uncertainty, follows closely the increase in data implied volatility during the three recessions, and remains high in the post technology bubble period, but is unable to explain the surge in volatility during the crash or the LTCM episode. Media commentary at or around these episodes confirms that macro events were not the cause of these crises.[9]

[9] Microstructure issues have been attributed to each of these two crises. Trading problems arose due to the breakdown in market mechanisms by the large trades of portfolio insurers in 1987, while the shutdown of several markets simultaneously led to the severe liquidity problems in 1998.

It is also noteworthy that the model captures well the post-recession decline in implied volatility from 1991 to 1996 and from 2001 to 2006 despite the fairly different macroeconomic conditions in these recessions. While earnings growth rebounded after each recession, the 1991 recession had tight CU and weak money growth, while the opposite conditions prevailed in the 2001 recession. In the former recession, rising inflation was a concern, while in the latter, investors were concerned about deflation. The unwinding of these conditions was therefore quite different in the two periods, but at the end of these cathartic periods, investors beliefs of state 1 (the regular boom) increased to over 60 percent (Figure 3) and implied volatility hit lows in the 10-13 percent range. The model is also quite successful in explaining the spike in implied volatility in the current recession, which started with the fear of an increase in inflation to a medium level and an increase investors' probability of the economy overheating (state 3) and deep recession (state 7), followed by the collapse of inflation and increase in the fear of deflation in the second half of 2008. The model ATMIV hit about 50 percent at the end of 2008, its highest level in the 23 year sample, but was lower than the nearly 70 percent in the data. Undoubtedly, the selling pressure from the collapse of Lehman Brothers and troubles at other financial institutions worldwide had an impact of volatility in this period more than can be accounted for by our model. However, these failures were endogenous, and our model does suggest that there were greater fundamental stresses in this period than any other period in our sample.

Table 4 contains formal statistics of the fit our model for the data ATMIV series. Line 1 shows that the $\bar{R}^2$ of the simple regression of the data ATMIV on the model ATMIV for the full sample is 53 percent, and the beta coefficient is 0.91, which is very close to 1. Excluding the fourth quarter of 1987 from the regression increases the $\bar{R}^2$ by another 5 percentage points (not reported). As well known in the GARCH literature, most measures of volatility are persistent. Line 2 shows that the regression $\bar{R}^2$ of 32 percent of the data ATMIV series on its own lag. In line 3, we include both the model and the lag, and find an $\bar{R}^2$ of 58 percent, or an increase of about 5 percentage points over our model. We will discuss in further detail below the economic interpretation of this lagged information, but before doing that we will look further at the macroeconomic determinants of the ATMIV.

We provide results for the five macroeconomic variables that we found significant individually in lines 4 through 8. These results show point at ATMIV being a procyclical variable, and the fact that none of them is able to explain much of the variation of ATMIV. Indeed, line 9 shows that together all these controls explain 36 percent of the variation in the ATMIV, significantly below that of our model. However, our model suggests, that while these variables are important determinants of implied volatility, they affect it jointly and in a nonlinear way, which our model captures with a formula which combines this information essentially using Bayes' formula and its implication for asset prices. In addition, some variables, like inflation and money growth, are not significant in a linear regression, but their effect is embedded in our model ATMIV, through the joint regime specification with real fundamentals.

We next include the controls and the model ATMIV in a joint specification in line 10. The $\bar{R}^2$ increases to 61 percent, so that the controls contribute jointly to an 8 percentage point increase in explanatory power over our model. Among the controls, only CU and the lagged return retain significance. This increase in explanatory power is likely the result of the highly parsimonious nature of our model, which nonetheless does explain most of the variation in the joint specification. Alternatively stated, our model contributes a 25 percentage point increase in explanatory over over the linear specification of control variables.

We next consider the effects of combing the lags and the controls with our model ATMIV. We note that the practice of including lags in volatility equations has a long tradition starting with the seminal work of Engle (1982) and extended with various specifications in the huge GARCH volatility literature. Line 11 shows that the model and lagged ATMIV are both highly significant, while among the controls, only the capacity utilization and the lagged return retain significance. The overall $\bar{R}^2$ increases to 66 percent, which is a 5 percentage point increase over line 10. The improvement in the explanatory power of the regression tells us that there are persistent economic forces that explain incremental amounts of variation in ATMIV, which are not in our model or controls. The information in the lag likely includes trading disruptions, which as mentioned above were particularly important at the time of the stock market crash and the LTCM failure. It remains a challenge to include such information into a model that already builds in the macroeconomic effects as in our model specification, which as noted explain about 80% of the predictable variation.

3.2 Explaining Time-Variation in the Put-Call Ratio

In this section we study the model's ability to explain the put-call ratio (P/C), which recall we define to be the ratio of implied volatilities of 5 percent out-of-the money put and call options.[10]

To understand what forces affect the P/C ratio, it is useful to first look at Figure 6 that shows the densities of stock returns when investors are 80 percent certain of being in each of the eight states (the remaining states each have equal probability). The top (bottom) panel shows the densities for the boom (recession) states, which are states with expected positive (negative) earnings growth. As seen, the densities are negatively (positively) skewed for the boom (recession) states. These shapes naturally imply that the P/C is great (smaller) than one for the boom (recession) states.[11] The conditional values of the put call ratio and higher moments of the stock return distribution under the risk-neutral measure are given in the bottom panel of Table 2. The table also shows that the densities at these eight beliefs are all highly non-Gaussian with skewness coefficients of between -3.3 and 1.4, and kurtosis coefficient of between 4.3 and 21.5. The non-Gaussianity partly stems from the jumps in returns due to jumps in earnings, and partly from the continuous shifting of the instantaneous moments of the return distribution. Indeed, we note that the jump intensity in earnings is constant, so *all* the time variation in the densities arises from the shifting moments. Its is also interesting to note that the sign of the skewness in each of the states can be calculated quite easily by looking at the sign of covariance between stock returns and stock variance in each state from its closed-form expression in (23). As seen in Table 2, the sign of the conditional correlation matches the sign of conditional skewness in each state. The intuition for the sign is as follows: By Bayes' Law, in the boom states, investors revise their beliefs more rapidly when they receive negative shocks to earnings than positive shocks, leading to more volatility with negative outcomes, which is a negative correlation between returns and variance. The higher volatility in down states give a negative sloped implied volatility curve or a positive put call ratio The opposite holds true for the low earnings growth rate states.

[10]To ensure intertemporal consistency of put-call ratios we follow Bates (1991) and set $K_{\text{put}} = S_t e^{(r-\delta)\tau}/1.05$ and $K_{\text{call}} = S_t e^{(r-\delta)\tau} \times 1.05$.

[11]Notice from the filtered probabilities, that investors were never 80 percent certain of any of the states in our sample, so that the model P/Cs were never as extreme as reported for these states. In particular the model P/C was almost always positive in our sample. It is also important to note, that the P/C is not linear in the beliefs, and values for intermediate beliefs will not be well approximated by interpolation. In particular we find that that the P/C at intermediate states is outside of the range in Table 2.

The bottom panel of Figure 1 shows the data and model P/Cs. As can be seen, both series are almost always greater than one (each is less than one only once in the sample) and the model P/C tracks the data P/C quite closely. Somewhat surprisingly, the figure shows that the P/C is procyclical, falling in each NBER-dated recession. Comparing to the top panel, gives the surprising stylized fact that the ATMIV and the P/C are negatively related. In particular, in 1995 when the ATMIV hovered around lows of near 10 percent, the P/C hovered above 1.6. Similarly, around 2006, when the ATMIV was again around 10 percent, the P/C ratio was again above 1.4. In and after the three recessions in the sample, when the ATMIV rose above 30 percent, the P/C fell below 1.2. We return to the issue of negative correlation between the two fear indices below.

Line 1 of Table 5 provides the simple regressions of the data P/C on the model P/C. As seen, the fit is very solid with a statistically insignificant alpha coefficient, a beta of 0.72 (not different statistically from 1) and $\bar{R}^2$ of 45.4 percent. Line 2 reports that the lagged P/C explains a similar $\bar{R}^2$ of 45.8 percent, but of course does not provide us intuition on the underlying economic forces driving the P/C. When both model and lag are included, each of the variables is statistically significant, and the explanatory power increases by 8 percentage points, implying that the lag has some information over and above that of our model.

Line 4 to 8 provide regression results for the five the five macroeconomic variables that we found significant for the ATMIV, and are inputs to our model. In addition, we also report in Line 9 and 10 results for two market sentiment measures advanced by Han (2008): The first, a"trader sentiment," is the net long position of large speculators on S&P 500 index futures obtained from the *Commodity Futures Trading Commission's Commitment of Traders Report*. The second, an "investor sentiment" is the bull-bear spread (proportion of traders bullish less bearish) in *Investor's Intelligence's* survey of investment newsletter writers.[12]

Line 10 include all the macroeconomic controls and the sentiment variables in a joint regression in line 11, and find an adjusted R^2 of only about 21.9 percent, far below our model. This reinforces the view that the macroeconomic variables affect the P/C nonlinearly. Finally, using all the variables along with the model and lagged P/C, leads to a very small increase in explanatory power over using just the model and lag.

[12] Both variables are measured within a week prior to the options trades. These measures are alternative measures of fear in the market and are thus compelling control variables for our measures of downside risk obtained from asset prices. While Han (2008) suggests that the significance of these measures supports a behavioral view of asset prices, we note that they could be consistent with a rational model of heterogeneous learning about the states of fundamentals such as in David (2008a).

3.3 Nonlinear Relations Between Fear Indices and Macro-Policy Variables

There is now a large and rapidly growing literature on the impact of the macroeconomy and policy variables on asset prices. We have already shown in the results above that the effect of macro-policy variables on the fear indices that we study in this paper is nonlinear. In this subsection, we build further on this theme to show clearly the nonlinear relationships between macro-policy variables and option prices. In addition, we show that the policy variables, which look generally of low importance, affect the fear indices more noticeably in periods that the policy makers stimulate the economy.

The top panels of Figure 7 show the relationship between the fear measures and expected real money growth. The x-axis in these plots has the expected money growth of investors which we have shown in Figure 4.[13] As can be seen, the ATMIV has an V-shape, with its minimum at close to one percent real money growth. The fitted curve shows a higher ATMIV when money growth is tight (around 26 percent) as opposed to when its accommodative (around 21 percent), and dips to around 17, when money growth is neutral. By comparing the estimated fear variable values in the different states in Table 2 we can see how our model generates the V-shape. Consider states states 2 and 4 for example, which have the highest ATMIV. Real money growth is strongly positive in the former state, and is strongly negative in state 4. In state 2, money growth is rapid as the central bank attempts to stimulate the economy in the regular recession, and as we will see below, the conditional relationship between money growth and ATMIV is negative. In state 4, volatility is high even as the central bank attempts to rein in high inflation even though real growth is weak and the conditional relationship is positive. Similarly, in the new economy growth rate state, the ATMIV is high as discussed, and money growth is tight, as the central bank attempts to rein in lofty expectations of real growth. On the low end, in the stable state 1 with low inflation and high real growth, real money growth is one percent and the ATMIV is the lowest.

The bottom panels of Figure 7 show the relationship between the fear measures calculated from our model using the filtered beliefs in Figure 3 and realized CU (which is a fairly smooth series). As seen, the ATMIV again has a V-shape relationship, while the P/C has an inverse V-shape. We plot the relationship for the full sample for our model since de-meaned CU never rose above 3 percent in the 1986-2008 period, when we have options data. In this period, the relationship between the data ATMIV

[13]We plot this relationship rather than the one between the ATMIV and realized money growth, since the latter is fairly noisy and provides a less precise relationship with a similar shape.

and CU is negative, while that with the P/C is positive. In the period of the options data, higher CU was taken as good news for fundamentals, which lowered uncertainty and the ATMIV and raised the P/C. For the full sample, which includes periods with very high CU, the relationships are non-monotonic as described above, since during periods of high CU, an increase in CU increased uncertainty about future fundamentals and had the opposite effects on the fear variables. The implied fear measures in the alternative states in Table 2 also show this relationship, as high ATMIV can result in recessions with high CU (state 3) or low CU (state 7). Once again, fundamentals' uncertainty is negatively related to P/C so that it has an inverse V-shape relation with CU (bottom right panel of Figure 7).

We end this discussion on the nonlinear relation between macro-finance variables and fear measures by studying their conditional relation during stimulative periods. We define stimulative periods as those where the 3-month Treasury Bill Yield is below the annualized inflation (CPI) rate. In the sub-sample of our date where we have options data (1986:2 – 2008) there are 20 quarters that we characterize as stimulative. These are periods of extreme stress in the market and it is of interest to study the response of the stock markets to money growth in these times. The left panel of Figure 8 shows that in this period ATMIV and money growth were negatively correlated. In the right panel we plot our model ATMIV and expected money growth and find a negative correlation which is somewhat stronger. The negative relation tell us that the actions of the central bank to boost money growth in such periods lowered the uncertainty and volatility in stock market, and our regime-switching model captures well this conditional relation. This role of monetary policy in reducing market uncertainty in stressful times is obviously not evident in simple linear regressions and could be the responsible for the general disillusionment with monetary aggregates in the literature (see e.g. Volcker (1977)). Our results also provide direct evidence from the options markets on the effectiveness of monetary policy during "stimulative" periods.

3.4 Does Monetary Policy Lead or Follow Market Fears?

As noted in the introduction, and shown in the left panels of Figure 2, shocks to fear measures lead to sustained impacts on future monetary policy, while monetary policy measures do not have sustained impacts on the fear measures. Is the model able to replicate such results?

The panels on the right show the analogous impulse responses when the same VAR is computed on the model fitted fear measures and policy variables. Comparing to the panels on the left, we find that our fitted model has exactly the same relationships as in the data: fear measures lead to sustained effects for six to eight quarters on policy variables, while the reverse impulses are statistically insignificant. Having a fully dynamic model that can replicate the historical impulses is useful since it can rule out certain channels for the effects. In particular, in our model the central bank and investors have exactly the same information. So, the reason why policy follows market fears is not due to differences in information of the two groups of agents.

What then explains the one-directional impulses? As seen, our results suggest that market fears have a sustained effect on industrial CU. Such a result easily follows from the real options literature, where firms delay full deployment of resources upon uncertainty shocks [see e.g. Bloom (2009)]. The learning-based Taylor rule then implies sustained impacts on interest rates (monetary policy).

Before concluding this section we add some caveats to these results. First, the impulse response functions are estimated for the options subsample, when the relationship between CU and the fear measures are monotonic as noted in Section 3.3. Over the full sample, the relationship is nonlinear and impulse responses may well have the opposite signs for the first part of our sample, when tight CU lead to monetary policy tightening. Second, we computed the impulse responses for the effects of fears on money growth and did not find significant responses in either direction. This is in line with our results in Section 3.3 where we find a strong relationship between the fear measures and money growth only in periods of stimulative monetary policy, but not in general. Finally, the impulses studied here are at a quarterly frequency, and we do not rule out that the central bank can have significant impulses at shorter horizons in higher frequency data.

4 Additional Properties of Option Prices

In this section we discuss features of observed option prices that are not directly fitted by our empirical methodology. The ability of our model to replicate these additional features provides further support for the economic mechanism that determines option prices in our model. As we will see the key variable in the model that enables to explain these additional facts is the volatility of stock variance, and we will end the section by providing its determinants.

4.1 The Volatility of ATM Volatility

In the previous section we saw that our model ATMIV was able to explain about 53 percent of the variation in the data ATMIV. In the model, the implied volatility is to a large part determined by the endogenous volatility of stock prices, which increases during periods of inflation and earnings uncertainty. In addition, looking again at the top panel of Figure 1, we see that during episodes of high volatility around the three NBER dated recessions in the options subsample, ATMIV also fluctuated by large amounts. The positive relation between volatility and the volatility of volatility is noted in Jones (2003) who further notes that it cannot arise in the Heston (1993) stochastic volatility model, which has been the workhorse of the option pricing literature. To obtain the level dependence of volatility, Jones (2003) proposes a generalization of the Constant Elasticity of Variance (CEV) model of Chan, Karolyi, Longstaff, and Sanders (1992). One drawback of the volatility processes in such models as is that they do not satisfy global growth and Lipschitz conditions, which are commonly used sufficient statistics for a number of important results. In this subsection we investigate if our model is able to shed light on the positive association between volatility and changes in volatility. Indeed as can be seen in (10), the Bayesian learning mechanism that drives volatility in our model implies that investors revise their beliefs faster during periods of high uncertainty as they have low confidence in their estimates of the current state of the fundamentals. We already established in Proposition 4 that our model stock price satisfies the regularity conditions.

The top panels of Figure 9 show the scatter plots of implied volatility and absolute changes in implied volatility for the data and model series. Both show a positive association of similar magnitude between these variables with correlations of 41 percent and 49 percent, respectively and these correlations are statistically significant. We next check if the absolute changes in implied volatility are related to the volatility of stock variance in our model. To do this we construct a time series of the model's volatility of volatility using (22) and evaluate it at each date using the filtered beliefs in Figure 3. The scatter plot of absolute changes in implied volatility (data and model) with this series are shown in the bottom panels. As seen, the model volatility of variance is highly correlated with both the data and model absolute changes in implied volatility with correlations of 34 and 61 percent respectively. Note that the model series measures the ex-ante volatility of variance at each date and is compared to the ex-post realized absolute changes in ATMIV and our model predicts a positive but not one-to-one association between these variables. This is highlighted by the fact that the correlation between these variables is only about 61 percent correlation even when both variables are generated by our model.

4.2 The Implied Volatility Premium

The volatility premium is an ex-ante measure of the stock market volatility forecast of investors' priced into options relative to a volatility forecast under the objective (P) measure, and is currently one of the most actively researched statistics in empirical option pricing. If volatility is systematically positively related to investors' pricing kernel (marginal utility of consumption), then as a priced factor it carries a negative risk premium, which leads to a higher forecast of volatility under the Q measure, or a positive volatility premium. The strong evidence that volatility is countercyclical, which we have already discussed in Section 3, suggests that the volatility premium should be positive.

The empirical finance literature now has more than one operational definition of this quantity. The first, which we call the *implied volatility premium* (IVP) is defined as the difference between at-the-money implied volatility and a forecast of future volatility to the maturity of the option under the objective (P) measure. The forecast under the P measure is constructed for specific volatility models. It must be noted that the implied volatility is a measure of the overall value of the option, which includes jumps as well as other non-Gaussian aspects of the stock return distribution, and these components also impact the IVP. A second definition, which we call the *forecast volatility risk premium* (FVRP), simply takes the difference in forecasts of future volatility under the two measures. The two different forecasts are formed with the *same* structural model and differ only by the difference in volatility drift, which is determined by the specification of the pricing kernel process.[14]

In this subsection we evaluate the ability of our model to shed light on the time variation of the IVP process. Thus our focus is different from the early papers on this topic that were focused around forecasting volatility using implied volatility and the implied volatility premium. Notable among these papers are Canina and Figlewski (1993) and Christensen and Prabhala (1998) who find that implied volatility is a useful albeit biased forecaster of future realized volatility. More recent work cited in the introduction has provided equilibrium models to understand the size of the equity premium, but not its time variation, which is our focus in this paper.

[14] In addition, Bollerslev, Tauchen, and Zhou (2009) use a measure of an ex-post volatility premium that takes the difference between implied volatility and realized volatility to predict future stock returns.

To construct a data based ex-ante IVP series we need forecasts of realized volatility, which we discuss first. We construct two forecasts using well established results in the volatility forecasting literature, which we discussed in Section 3.1. The specifications we use are similar to those in Drechsler and Yaron (2010). The first specification for our sample from 1986:Q2 to 2008:Q4 is a regression of realized volatility on its one-quarter lag, the lagged P/E ratio, and lagged returns on the S&P 500 index in periods when they are negative, which we call Projection 1. The results of this regression are:

$$\begin{array}{rcl} \text{Vol}(t+1) & = & \underset{[1.170]}{3.291} + \underset{[2.122]^*}{0.548}\,\text{Vol}(t) + \underset{[1.237]}{0.212}\,\text{P/E}(t) - \underset{[-2.004]^*}{0.470}\,\text{Ret}^-(t);\ \bar{R}^2 = 0.238 \end{array} \tag{26}$$

where $\text{Vol}(t+1)$ is the volatility realized in quarter $t+1$, which we define as the square root of the sum of squared S&P 500 index returns in the quarter, $P/E(t)$ is the S&P 500 price-to-operating earnings ratio, and $\text{Ret}^-(t)$ is the return on the S&P 500 index in periods when it is negative. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The regression $\bar{R}^2$ improves to 47 percent for the post-crash subsample starting in 1988:Q2.[15] The second forecast is similar to that constructed in Drechsler and Yaron (2010), which used the lagged implied volatility to forecast realized volatility and adjusts for the forecast bias, which we call Projection 2. The results of this regression are:

$$\begin{array}{rcl} \text{Vol}(t+1) & = & \underset{[1.659]^*}{2.334} + \underset{[0.165]}{0.019}\,\text{Vol}(t) + \underset{[4.986]}{0.709}\,\text{I. Vol}(t); \quad \bar{R}^2 = 0.393 \end{array} \tag{27}$$

The $\bar{R}^2$ of this regression improves to over 63 percent if the stock market crash is excluded. It is important to note that for each projection we use non-overlappling data by constructing one-quarter ahead volatility forecasts at the quarterly frequency so that the t-statistics are more reliable.

Using the difference between the implied volatility at the beginning of the quarter t and the expectation of quarter t realized volatility based on data available at the end of quarter $t-1$, we form the ex-ante volatility premium series. Using the two alternative forecasts of realized volatility, we have two measures of the ex-ante volatility premium, which we display in the top and middle panels of Figure

[15]Most papers on variance premium studies exclude the stock market crash from their samples. We find that the ability of our model in explaining the time variation in the variance premium is not sensitive to the exclusion of the crash.

10. The two series have a correlation of 88 percent, and their means are very similar at 2.7 and 2.9 percent respectively. We similarly construct a model based IVP series by taking the difference between the model implied volatility analyzed in Section 3.1 and the model forecast of volatility under the P-measure using simulation methods as described in equation (50) in Appendix 2. The model IVP, IVP^M, is also displayed in these panels. The mean of the IVP is 4.4 percent, which is higher than the data premium means. However, it is important to note that we did not use the data IVP in our estimation procedure to match the same sample average as in the data. It is also relevant to note that our main goal is to understand the time-variation in the variance premium, which we discuss next.

By regressing the volatility premium from projections 1 and 2, we get the following fits:

$$\begin{aligned} \mathrm{IVP}^1(t) &= \underset{[-0.910]}{-1.258} + \underset{[2.293]^*}{0.634}\,\mathrm{IVP}^M(t); \quad \bar{R}^2 = 0.168. \qquad (28) \\ \mathrm{IVP}^2(t) &= \underset{[0.015]}{0.014} + \underset{[2.673]^*}{0.539}\,\mathrm{IVP}^M(t); \quad \bar{R}^2 = 0.122. \qquad (29) \end{aligned}$$

As can be seen, the intercept terms are small and statistically insignificant, and the betas of the regression are instead strongly significant, and the model explains nearly 17 percent of the variation in the data IVP form Projection 1, which is economically significant. As seen in the plot, both data and model volatility premiums are countercyclical, are higher higher in periods of higher volatility. Once again, our model is not able to fit the fluctuation in the volatility premium in during the stock market crash or the collapse of LTCM, which were events not likely related to macroeconomic fundamentals. The model does capture the higher premiums during the recessions in the sample, and the decline in premium in good times. The fit for the IVP from Projection 2 is similar, although the explanatory power is lower and 12 percent. Looking at the middle panel, the qualitative feature of the model fit is very similar though.

We next examine if the IVP of our model is related to its FVRP. The two series are plotted in the third panel of Figure 10. The Q forecast of our model is constructed using the same methodology as the P forecast and is shown in (49) in Appendix 2. The FVRP is simply the difference in the two forecasts. Constructing the series conditional on beliefs for our full sample we find an average FVRP of 1.75 percent. As suggested at the beginning of this subsection, the IVP has two components, the FVRP, and the non-Gaussian components of the density. Our estimated time series of the two premiums suggest that the FVRP comprises about 40 percent of the total IVP. The figure also shows that the two series are highly correlated (correlation coefficient of 93.8 percent).

What explains the model's IVP? To understand the driving determinants, we investigate further some model generated time series. A natural candidate to consider is the volatility of volatility (VV).

$$\begin{aligned} \text{IVP}^M(t) &= \underset{[11.194]^*}{5.216} + \underset{[4.208]^*}{1.663} \log(\text{VV}^M)(t); \quad \bar{R}^2 = 0.401. \end{aligned} \tag{30}$$

In Figure 9 we showed that volatility fluctuations are higher in periods when the VV is higher. Equation 30 tells us that the options are priced more dearly when there is greater volatility in volatility, suggesting that volatility is itself a priced risk factor. We study this further by regressing the two components of the volatility premium on the VV:

$$\text{FVRP}^M(t) = \underset{[3.145]^*}{0.761} + \underset{[5.243]^*}{1.289} \log(\text{VV}^M)(t); \quad \bar{R}^2 = 0.530. \tag{31}$$

$$\text{IVP}^M(t) - \text{FVRP}^M(t) = \underset{[16.084]^*}{4.462} + \underset{[1.991]^*}{0.387} \log(\text{VV}^M)(t); \quad \bar{R}^2 = 0.103. \tag{32}$$

Equation (31) tells us that a large part of the fluctuation in the model IVP from the VV is captured by the fluctuation in the FVRP portion of the IVP. In fact the explanatory power of the VV for the FVRP is well over 50 percent. In contrast, the explanatory power for the the residual portion, IVP-FVRP, on the VV is only 11 percent. Therefore, only a small part of the residual premium, which essentially captures the non-Gaussian aspect of the return distribution under the risk-neutral measure is related to the VV, which directly impacts the fourth moment of returns. Finally, we note that returns in our model are non-normal also because of the jumps in the earnings process. However, the jump parameters are all constant, so that any time-variation in higher moments in the model arises from the variation in beliefs about macroeconomic states, which affects the VV and as well as other higher order moments.

4.3 What Drives the Volatility of Stock Volatility?

As seen above, VV explains the positive comovement of ATMIV and its absolute changes as well as the dynamics of the implied volatility premium. The VV is endogenously generated by the learning process in our model and is a result of the the nonlinear updating inherent in Bayes' law. In periods where investors are more uncertain about fundamentals, they put less weight on their current beliefs and more weight on incoming news so that revisions to beliefs and hence stock market volatility are higher. This implies that the VV should be directly related to measures of investors' uncertainty.

To see the relationship explicitly, we define earnings uncertainty (and analogous definitions for the other fundamentals) as

$$\mathrm{EU}(t) = \sqrt{\sum_{i=1}^{N} \pi_i(t) \left(\theta_i - \overline{\theta}(t)\right)^2}. \tag{33}$$

We plot the time series of VV and the four fundamental uncertainties in Figure 11, and Table 6 provides results on the simple OLS regressions of the VV on these variables for options subsample (1986:Q2 – 2008). While all four uncertainties are strongly countercyclical, we find that earnings uncertainty has been the single most important driver of the VV explaining more than 70 percent of its variation. Money growth uncertainty also explains a significant amount of variation in the VV, while the other two uncertainties have been of minor importance. As noted in Section 3.3, CU itself has an impact on the fear indices, but as seen here, uncertainty about CU is not a significant driver of VV. This result arises because CU is a fairly smooth process so that uncertainty about it does not drive major changes in investors uncertainty. Money growth uncertainty turns out to be an important driver as investors likely perceive changes in money as an important signal of the view of the central bank about the state of the economy.

Taken together the four fundamental uncertainties explain more than 80 percent of the variation in VV. Relatedly, Beber and Brandt (2009) show that volatility in stock and bond markets declines faster following periods of high macroeconomic uncertainty extracted from the economic derivative markets over a shorter sample from 2002-2005.

5 Conclusion

Option prices provide key forward looking information on investors' expectations, and market attention is often focused on two key fear indices, the ATMIV and the P/C. The former is measure of market turbulence, while the latter is a measure of downside risk. Standard option pricing models use exogenous stock prices and their volatilities that are unrelated to fundamentals, and are hence unable to identify specific economic factors that can explain these fears. We provide a model in which stock, bond, and option prices, are functions of investors' beliefs of the joint states of macroeconomic and policy fundamentals. The model is able to shed light on the counter (pro) cyclicality of the ATMIV (P/C), is able to explain about half their time series variation, and their compelling nonlinear relations with policy variables. In particular, the ATMIV (P/C) has a (inverse) V-shape with expected money growth and capacity utilization. The model's ability to explain the time-series properties of these fear indices is based on its inherent Bayesian learning framework in which volatility is high during periods of greater uncertainty, and bad news lead to sharper downward revisions of beliefs in good times.

Our analysis also shows that investors' uncertainty in the options market has real economic consequences, which is tempered by the efforts of the central bank to smooth fluctuations. In particular, the fear measures are able to predict future movements in interest rates. The model also explains that the relationship between the fear indices and money growth is specially strong in periods of extreme stress when the central bank follows a stimulative policy by keeping the short rate below the inflation rate.

While investors' uncertainty is explicitly extracted from the fear indices, the dynamic updating of investors' beliefs have implications for the volatility of stock market volatility and additional properties of options prices. In particular, we show our model is able to explain the positive correlation between ATMIV and absolute changes in ATMIV (a feature that is not consistent with standard option pricing models) and additionally is able to explain an economically significant amount of variation in the implied volatility premium. The model's implied volatility premium is driven to a large extent by the risk premium for volatility shocks and to a lesser extent by the fatness of tails created by the continuous shifting of moments of the return distribution from the Bayesian updating.

An important caveat is that the model, which structurally estimates the impact of investors uncertainty about the macroeconomy on the fear indices, is unable to explain some important surges in these indices at times when microstructure issues have roiled markets, such as the crash of 1987 and the collapse of LTCM in 1998. It remains a challenge to include such information into a model that already builds in the macroeconomic effects as in our model specification.

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Appendix 1

For proving Proposition 1 we will need the following lemma.

Lemma 2 *Given the process of earnings in (3) and the SPD in (8), over a small interval of time Δ we have*

$$\mathbb{E}\left[\frac{M_{t+\Delta}E_{t+\Delta}}{M_t X_t}|\nu_t = \nu_i\right] = e^{[\theta_i - \phi_i - \sigma_M \sigma_E' + \kappa(\xi_3 - \xi_1 - \xi_2)]\Delta} + o(\Delta),$$

where $\xi_3 = e^{\mu_1 + \mu_2 + 0.5\,(\sigma_1+\sigma_2)^2} - 1$.

Proof. From (3) and (8) we have

$$\begin{aligned}
\frac{E_s}{E_t} &= \exp\left(\int_t^s \theta_s - \kappa\xi_1 - 0.5\,\sigma_E\sigma_E' du + \sigma_E(W_s - W_t) + \sum_{j=L_{t+1}}^{L_s} Y_{1j}\right) \\
\frac{M_s}{M_t} &= \exp\left(\int_t^s -\phi_s - \kappa\xi_2 - 0.5\,\sigma_M\sigma_M' du - \sigma_M(W_s - W_t) + \sum_{j=L_{t+1}}^{L_s} Y_{2j}\right).
\end{aligned}$$

Multiplying the two equations we have

$$\begin{aligned}
\frac{E_s M_s}{E_t M_t} &= \exp\left(\int_t^s [\theta_s - \phi_s - \sigma_E\sigma_M' - \kappa(\xi_1+\xi_2) - 0.5\,(\sigma_E\sigma_E' + \sigma_M\sigma_M')]du + (\sigma_E - \sigma_M)(W_s - W_t)\right) \\
&\times \exp\left(\sum_{j=L_{t+1}}^{L_s} Y_{1j} + Y_{2j}\right).
\end{aligned}$$

Now for a small interval of time Δ and the fact that jumps in the drift processes and L_t are independent of each other and each occurs with probability of order $O(\Delta)$, we have

$$\begin{aligned}
\mathbb{E}[\frac{E_{t+\Delta}\, M_{t+\Delta}}{E_t M_t}|\nu_t = \nu_i] &= e^{[\theta_i - \phi_i - \sigma_E \sigma_M' - \kappa(\xi_1+\xi_2)]\Delta} \cdot \mathbb{E}\left[e^{\sum_{j=L_{t+1}}^{L_{t+\Delta}} Y_{1j}+Y_{2j}}\right] \\
&= [1 + (\theta_i - \phi_i - \sigma_E\sigma_M' - \kappa(\xi_1+\xi_2))\Delta][1-\kappa\Delta + \kappa\Delta(1+\xi_3)] + o(\Delta) \\
&= 1 + [\theta_i - \phi_i - \sigma_E\sigma_M' + \kappa(\xi_3 - (\xi_1+\xi_2))]\Delta + o(\Delta) \\
&= e^{[\theta_i - \phi_i - \sigma_M\sigma_E' + \kappa(\xi_3-\xi_1-\xi_2)]\Delta} + o(\Delta),
\end{aligned}$$

as claimed. Note in the first equality above we have used the independence property of the drift process and the jump process, while in the second we have used the definition of $e^x = 1 + x + x^2/2! \cdots$. ■

Proof of Proposition 1: The P/E ratio at time t satisfies

$$\begin{aligned}
\frac{P_t}{E_t} &= \mathbb{E}\left[\int_t^\infty \frac{M_s\, E_s}{M_t\, E_t}\, ds|\mathcal{F}_t\right] \\
&= \sum_{i=1}^N \pi_{it}\mathbb{E}\left[\int_t^\infty \frac{M_s\, E_s}{M_t\, E_t}\, ds|\nu_t = \nu_i\right] \equiv \sum_{i=1}^N \pi_{it} V_{it}.
\end{aligned}$$

Let $\hat{\theta}_i = \theta_i - \phi_i - \sigma_M\sigma_E' + \kappa(\xi_3 - \xi_1 - \xi_2)$. Using Lemma 2 to evaluate the expectations over a time interval Δ, we have

$$\begin{aligned}
V_{i,t} &= \mathbb{E}\left[\int_t^{t+\Delta} \frac{M_s\, E_s}{M_t\, E_t}\, ds|\nu_t = \nu_i\right] + \mathbb{E}\left[\frac{M_{t+\Delta}\, E_{t+\Delta}}{M_t\, E_t}\int_{t+\Delta}^\infty \frac{M_s\, E_s}{M_{t+\Delta}\, E_{t+\Delta}}\, ds|\nu_t = \nu_i\right] \\
&= \int_t^{t+\Delta} e^{\hat{\theta}_i ds} + e^{\hat{\theta}_i\Delta}\mathbb{E}\left[\int_{t+\Delta}^\infty \frac{M_s\, E_s}{M_{t+\Delta}\, E_{t+\Delta}}\, ds|\nu_t = \nu_i\right] \\
&= \frac{e^{\hat{\theta}_i\Delta} - 1}{\hat{\theta}_i} + e^{\hat{\theta}_i\Delta}\left[(1+\lambda_{ii}\Delta)V_{i,t+\Delta} + \sum_{j\neq i}\lambda_{ij}\Delta V_{j,t+\Delta}\right].
\end{aligned}$$

Since $V_{i,t}$ is time homogeneous, we have $V_{i,t} = V_{i,t+\Delta} = C_i$. Now collecting terms and taking the limit as $\Delta \to 0$, we get

$$\begin{aligned}
C_i\frac{1-e^{\hat{\theta}_i\Delta}}{\Delta} &= \frac{e^{\hat{\theta}_i\Delta - 1}}{\hat{\theta}_i\Delta} + e^{\hat{\theta}_i\Delta}\left[\lambda_{ii}C_i + \sum_{j\neq i}\lambda_{ij}C_j\right] \\
-\hat{\theta}_i C_i &= 1 + \left[\lambda_{ii}C_i + \sum_{j\neq i}\lambda_{ij}C_j\right].
\end{aligned}$$

In vector form we can write this equality as

$$\left(\text{Diag}(-\hat{\theta}) - \Lambda\right) C = \mathbf{1}_N,$$

whose solution is $C = A^{-1}\cdot \mathbf{1}_N$, as in the statement of the proposition. ■

For proving Proposition 3 we will use the algebraic result stated in the following lemma.

Lemma 3

$$\frac{\partial \overline{\theta}^{\circ}}{\partial \pi_i} = \frac{C_i\left(\theta_i - \overline{\theta}^{\circ}\right)}{\left(\sum_j \pi_j C_j\right)}.$$

Proof of Lemma 3:

$$\begin{aligned}
\frac{\partial \overline{\theta}^{\circ}}{\partial \pi_i} &= \frac{\partial\left(\frac{\sum_j \pi_j C_j \theta_j}{\sum_j \pi_j C_j}\right)}{\partial \pi_i} = \frac{C_i\theta_i\left(\sum_j \pi_j C_j\right) - C_i\left(\sum_j \pi_j C_j \theta_j\right)}{\left(\sum_j \pi_j C_j\right)^2} \\
&= \frac{C_i\theta_i}{\left(\sum_j \pi_j C_j\right)} - \frac{C_i\left(\sum_j \pi_j C_j \theta_j\right)}{\left(\sum_j \pi_j C_j\right)^2} = \frac{C_i\theta_i}{\left(\sum_j \pi_j C_j\right)} - \frac{C_i\overline{\theta}^{\circ}}{\left(\sum_j \pi_j C_j\right)} \\
&= \frac{C_i\left(\theta_i - \overline{\theta}^{\circ}\right)}{\left(\sum_j \pi_j C_j\right)},
\end{aligned}$$

which completes the proof. ■

Proof of Proposition 3: Let the second term in the variance equation be $V_2 = (\bar{\nu}^{\circ} - \bar{\nu})'(\Sigma\Sigma')^{-1}(\bar{\nu}^{\circ} - \bar{\nu})$. Then, using Lemma 3 on each element of the drift vector ν we have

$$\frac{\partial V_2}{\partial \pi_i} = 2\left[\frac{C_i(\nu_i - \overline{\nu}^{\circ})}{\sum_j \pi_j C_j} - \nu_i\right]'(\Sigma\Sigma')^{-1}(\bar{\nu}^{\circ} - \bar{\nu}).$$

Then, using the volatilities of the beliefs process in equation (11), we have $dV_2 = \mu_{V,2}dt + \sigma_{V,2}$, where

$$\begin{aligned}
\sigma_{V,2} &= \sum_i \frac{\partial V_2}{\partial \pi_i}\sigma_i \\
&= 2\sum_i \pi_i\left[\frac{C_i(\nu_i - \overline{\nu}^{\circ})}{\sum_j \pi_j\, C_j} - \nu_i\right]'(\Sigma\Sigma')^{-1}(\bar{\nu}^{\circ} - \bar{\nu})(\nu_i - \bar{\nu})'\,\Sigma'^{-1} \\
&= 2\sum_i [\pi_i^{\circ}\,(\nu_i - \overline{\nu}^{\circ}) - \pi_i\nu_i]'(\Sigma\Sigma')^{-1}(\bar{\nu}^{\circ} - \bar{\nu})(\nu_i - \bar{\nu})'\,\Sigma'^{-1}.
\end{aligned}$$

Similarly, let the third term in the variance equation be $V_3 = 2\left[(\overline{\theta}^{\circ} - \overline{\theta}) + (\overline{\beta}^{\circ} - \overline{\beta})\right]$. Then we have $dV_3 = \mu_{V,3}dt + \sigma_{V,3}$, where

$$\begin{aligned}
\sigma_{V,3} &= \sum_i \frac{\partial V_3}{\partial \pi_i}\sigma_i = 2\sum_i \frac{\partial[(\overline{\theta}^{\circ} - \overline{\theta}) + (\overline{\beta}^{\circ} - \overline{\beta})]}{\partial \pi_i}\sigma_i \\
&= 2\sum_i \pi_i\left(\left[\frac{C_i(\theta_i - \overline{\theta}^{\circ})}{\sum_j \pi_j C_j} - \theta_i\right] + \left[\frac{C_i(\beta_i - \overline{\beta}^{\circ})}{\sum_j \pi_j C_j} - \beta_i\right]\right)(\nu_i - \bar{\nu})'\Sigma'^{-1} \\
&= 2\left[(\sigma^{\circ}_{\theta_\nu} - \sigma_{\theta_\nu})' + (\sigma^{\circ}_{\beta_\nu} - \sigma_{\beta_\nu})'\right]\,\Sigma'^{-1},
\end{aligned}$$

where the second equality follows from Lemma 3, the third the definition of π_i°, and the fourth from the fact that

$$\sum_j \pi_j^{\circ}(\theta_j - \overline{\theta}^{\circ})(\beta_j - \overline{\beta}) = \sum_j \pi_j^{\circ}\theta_j\beta_j - \overline{\theta}^{\circ}\overline{\beta}^{\circ} = \sigma^{\circ}_{\theta\beta},$$

and analogous terms for the other elements of ν. Now summing $\sigma_{V,2}$ and $\sigma_{V,3}$ provides the statement of (b).

Proof of Proposition 4 Since $||\sigma^n(\pi)||$ in (17) is a continuous function of π on the N dimensional simplex, which is a compact set, it has a maximum and minimum, which we denote by $||\bar{\sigma}^n||$ and $||\underline{\sigma}^n||$. Therefore, $||S_1\sigma^n(\pi_1) - S_2\sigma^n(\pi_2)|| \leq (||\bar{\sigma}^n|| - ||\underline{\sigma}^n||) \cdot |S_1 - S_2|$ so that the Lipschitz condition is satisfied for the stock price. Similarly, $||S\,\sigma^n(\pi)||^2 \leq ||\bar{\sigma}^n||^2\, S^2 < (1 + ||\bar{\sigma}^n||^2\, S^2)$, so that the growth condition holds as well. Similarly the norm of the volatility of beliefs in (11) is bounded by $||\bar{\sigma}_i||$ and $||\underline{\sigma}_i||$ and both conditions hold for the beliefs processes, which completes the proof. ■

Proof of Proposition 5: The change of measure with respect to the Brownian motions in the context of the filtering setup has been derived in David (2008b). For brevity, we only provide the proof of the change of measure for the jump component.

Lets show the change of measure for the jump.

$$
\begin{aligned}
\kappa\mathbb{E}[\frac{M^+ - M}{M}\frac{S^+ - S}{S}|\mathcal{F}_t] &= \kappa\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{Y_1}e^{Y_2}f(Y_1,Y_2)dY_2dY_1 \\
&= \kappa\int_{-\infty}^{\infty} e^{Y_1}f(Y_1)\int_{-\infty}^{\infty} e^{Y_2}f(Y_2|Y_1)dY_2dY_1 \\
&= \kappa\int_{-\infty}^{\infty} e^{Y_1}e^{\mu_2+\frac{\sigma_2}{\sigma_1}(Y_1-\mu_1)}f(Y_1)dY_1 \\
&= \kappa e^{\mu_2-\frac{\sigma_2}{\sigma_1}\mu_1}e^{\mu_1(1+\frac{\sigma_2}{\sigma_1})+0.5(1+\frac{\sigma_2}{\sigma_1})^2\sigma_1^2} \\
&= \kappa e^{\mu_2+.5\sigma_2^2}e^{\mu_1+\sigma_1\sigma_2+0.5\sigma_1^2} \\
&= \kappa^*\mathbb{E}^*[e^{Y_1}].
\end{aligned}
$$

In the above, the second equality arises from the definition of a conditional expectation, the third because the two jump processes are perfectly correlated, and the fourth from the moment generating function of a normal distribution.

Appendix 2

1. SMM Estimation of the Regime Switching Jump-Diffusion Model

We start by providing here the details of the SMM estimation procedure, which is used to estimate the model. The procedure uses the SML methodology of Brandt and Santa-Clara (2002), which has already been extended to to learning framework in the pure diffusion setting in David (2008b). We provide here the extension to the case of observable jumps in the fundamental processes. Piazzesi (2005) has extended the procedure to a setting with jump-diffusions.

Using the definition of the inferred shocks (12) we can write the variables observed by the econometrician in (9) as perceived by the investors as $\frac{dY_t}{Y_t} = \varrho(\pi_t)dt + \Sigma_4\, dW_t + J_{4t}dL_t$. Similarly the pricing kernel in (8) under investors' filtration can be written as $dM_t/M_t = (-\bar{\phi}(\pi_t) - \kappa\xi_2)dt - \sigma_M\, d\tilde{W}_t - (e^{Y_{2t}} - 1)dL_t$, where the real rate in the economy, $\bar{\phi}(\pi_t)$, is the expected value of ϕ_t in (6) conditional on investors' filtration. Since fundamentals are stationary in growth rates, we start by defining logs of variables: $y_t = \log(Y_t)$, and $m_t = \log(M_t)$. Using these characterizations we can write

$$
dy_t = (\bar{\varrho}(\pi_t) - \frac{1}{2}\,\mathrm{diag}(\Sigma_4\Sigma_4'))dt + \Sigma_4\, d\tilde{W}_t + J_{4t}dL_t, \tag{34}
$$

$$
dm_t = (-\bar{\phi}(\pi_t) - \kappa\xi_2 - \frac{1}{2}\,\sigma_M\sigma_M')dt - \sigma_M\, d\tilde{W}_t - (e^{Y_{2t}} - 1)dL_t, \tag{35}
$$

where diag(x) is a column vector composed of the diagonal elements of a square matrix x. It is immediate that investors' beliefs π_t completely capture the state of the system (y_t, m_t) for forecasting future growth rates. The specification of the system is completed with the belief dynamics in (10).

The econometrician has data series $\{y_{t_1}, y_{t_2}, \cdots, y_{t_K}\}$. Let Ψ be the set of parameters of the model. Let

$$\mathcal{L}(\Psi) \equiv p(y_{t_1}, \cdots, y_{t_K}; \Psi) \;=\; p(\pi_{t_0}; \Psi) \prod_{k=1}^{K} p(y_{t_{k+1}} - y_{t_k}, t_{k+1} | \pi_{t_k}, t_k; \Psi),$$

where $p(y_{t_{k+1}} - y_{t_k}, t_{k+1} | \pi_{t_k}, t_k; \Psi)$ is the marginal density of fundamentals at time t_{k+1} conditional on investors' beliefs at time t_k. Since $\{\pi_{t_k}\}$ for $k = 1, \cdots, K$ is not observed by the econometrician, we maximize

$$E[\mathcal{L}(\Psi)] = \int \cdots \int \mathcal{L}(\Psi) f(\pi_{t_1}, \pi_{t_2}, \cdots, \pi_{t_K}) d\pi_{t_1}, d\pi_{t_2}, \cdots, d\pi_{t_K}, \tag{36}$$

where the expectation is over all sample paths for the fundamentals, $\tilde{y}_t$, such that $\tilde{y}_{t_k} = y_{t_k}$, $k = 1, \cdots, K$. In general, along each path, the sequence of beliefs $\{\pi_{t_k}\}$ will be different.

As a first step, we need to calculate $p(y_{t_{k+1}} - y_{t_k}, t_{k+1} | \pi_{t_k}, t_k; \Psi)$. Following Brandt and Santa-Clara (2002), we simulate paths of the state variables over smaller discrete units of time using the Euler discretization scheme (see also Kloeden and Platen 1992):

$$\begin{aligned}
\tilde{y}_{t+h} - \tilde{y}_t &= (\bar{\varrho}(\pi_t) - \frac{1}{2}(\sigma_Q \sigma_Q', \sigma_E \sigma_E')')\, h + \Sigma_2 \sqrt{h} \tilde{\epsilon}_{2t} \;+\; \mathbf{1}_{\tilde{u}_t < \kappa h} \tilde{\epsilon}_{2t}, &(37)\\
m_{t+h} - m_t &= (-\bar{\phi}(\pi_t) - \kappa \zeta_2 - \frac{1}{2} \sigma_M\, \sigma_M')\, h - \sigma_M \sqrt{h} \tilde{\epsilon}_{1t} \;+\; \mathbf{1}_{\tilde{u} < \kappa h} \tilde{\epsilon}_{2t}, &(38)\\
\pi_{t+h} - \pi_t &= \mu(\pi_t)\, h + \sigma(\pi_t) \sqrt{h} \tilde{\epsilon}_{1t}, &(39)
\end{aligned}$$

where $\tilde{\epsilon}_{1t}$ and $\tilde{\epsilon}_{2t}$ are 5- and 1- dimensional standard normal variables, respectively, $\tilde{u}_t$ is uniformly distributed, and $h = 1/M$ is the discretization interval. The Euler scheme implies that the marginal conditional density of the 4×1 fundamental growth vector y_t over h is 4-dimensional normal.

We approximate $p(\cdot|\cdot)$ with the density $p_M(\cdot|\cdot)$, which obtains when the state variables are discretized over M subintervals. Since the drift and volatility coefficients of the state variables in (10), and (34) to (35) are infinitely differentiable, and $\Sigma\Sigma'$ is positive definite, Lemma 1 in Brandt and Santa-Clara (2002) implies that $p_M(\cdot|\cdot) \to p(\cdot|\cdot)$ as $M \to \infty$. The Chapman-Kolmogorov equation implies that the density over the interval (t_k, t_{k+1}) with M subintervals satisfies $p_M(y_{t_{k+1}} - y_{t_k}, t_{k+1} | \pi_{t_k}, t_k; \Psi) \;=$

$$\int \int \phi \left(y_{t_{k+1}} - y; ; \Psi \right) \times p_M \left(y - y_{t_k}, \pi, m, t_k + (M-1)h | \pi_{t_k}, t_k \right) \, d\pi \, dy, \tag{40}$$

where $\phi(y; \psi)$, denotes the mixture-of-normals density given as:

$$\begin{aligned}
\phi(y; \psi) &= N(\varrho(\pi_t) h, \Sigma_4 \Sigma_4' h) \quad \text{with probability} \quad \kappa\, h, &(41)\\
&= N(\varrho(\pi_t) h + (0,1,0,0)' \mu_1, \Sigma_4 \Sigma_4' h + i_2 \sigma_1^2) \quad \text{with probability} \quad 1 - \kappa\, h, &(42)
\end{aligned}$$

where i_2 is the 4×4 square matrix with zero in all elements except the $(2,2)$ element, which is 1. Now $p_M(\cdot|\cdot)$ can be approximated by simulating L paths of the state variables in the interval $(t_k, t_k + (M-1)h)$ and computing the average

$$\hat{p}_M \left(y_{t_{k+1}} - y_{t_k}, t_{k+1} | \pi_{t_k}, t_k; \Psi \right) \;=\; \frac{1}{L} \sum_{l=1}^{L} \phi \left(y_{t_{k+1}} - y^{(l)}; \Psi \right). \tag{43}$$

The Strong Law of Large Numbers (SLLN) implies that $\hat{p}_M \to p_M$ as $L \to \infty$.

To compute the expectation in (36), we simulate S paths of the system (37) to (39) "through" the full time series of fundamentals. Each path is started with an initial belief, $\pi_{t_0} = \pi^*$, the stationary beliefs implied by the generator matrix Λ. In each time interval (t_k, t_{k+1}) we simulate (M-1) successive values of $\tilde{y}_t^{(s)}$ using the discrete scheme in (37), and set $\tilde{y}_{t_k}^{(s)} = y_{t_k}$. The results in the paper use $M = 90$ for quarterly data, so that shocks are approximated at roughly a daily frequency. The pricing kernel and beliefs along the entire path of the s^{th} simulation are obtained by iterating on (38) and (39). We approximate the expected likelihood as

$$\hat{\mathcal{L}}^{(S)}(\Psi) = \frac{1}{S} \sum_{s=1}^{S} \prod_{k=0}^{K-1} \hat{p}_M(y_{t_{k+1}}^{(s)} - y_{t_k}^{(s)}, t_{k+1} | \pi_{t_k}^{(s)}, t_k; \Psi), \tag{44}$$

where $\hat{p}_M(\cdot|\cdot)$ is the density approximated in (43). The SLLN implies that $\hat{\mathcal{L}}^{(S)}(\Psi) \to E[\mathcal{L}(\Psi)]$ as $S \to \infty$. We often report $\bar{\pi}_{t_k} = 1/S \sum_{s=1}^{S} \pi_{t_k}^{(s)}$, which is the econometrician's expectation of investors' belief at t_k.

To extract investors' beliefs from data on price levels and volatilities in addition to fundamentals we add overidentifying moments to the SML method above. From Proposition 1, we can compute the time series of model-implied price-earning ratios and bond yields at the discrete data points t_k, $k = 1, \cdots, K$ as

$$\widehat{\text{P/E}}_{t_k} = C \cdot \bar{\pi}_{t_k}, \qquad \widehat{i}_{t_k}(\tau) = -\frac{1}{\tau} \log\left(B(\tau) \cdot \bar{\pi}_{t_k}\right).$$

We note that the constants Cs and the functions $B(\tau)$ both depend on the parameters of the fundamental processes, Ψ. Hence, we let the pricing errors be denoted

$$e_{t_k}^P = \left(\widehat{\text{P/E}}_{t_k} - \text{P/E}_{t_k}, \widehat{i}_{t_k}(0.25) - i_{t_k}(0.25), (\widehat{i}_{t_k}(5) - \widehat{i}_{t_k}(1)) - (i_{t_k}(5) - i_{t_k}(1))\right).$$

We similarly formulate the errors from options prices as

$$e_{t_k}^O = \left(\widehat{V}_{t_k} - V_{t_k}, \widehat{(P/C)}_{t_k} - (P/C)_{t_k}\right),$$

where V is the ATMIV, and P/C is the put-call ratio as discussed. The model-implied options prices are calculated using Monte-Carlo simulations as described below.

To estimate Ψ from data on fundamentals as well as financial variables, we form the overidentified SMM objective function

$$c = \left(\frac{1}{T} \sum_{t=1}^{T} \epsilon_{\mathbf{t}}\right)' \cdot \Omega^{-1} \cdot \left(\frac{1}{T} \sum_{t=1}^{T} \epsilon_{\mathbf{t}}\right). \tag{45}$$

The moments used are the scores of the log likelihood function from fundamentals, and the pricing errors from stock, Treasury bond, and options prices. Since the number of scores in $\frac{\partial \log(\hat{\mathfrak{L}})}{\partial \Psi}(t_k)$ equals the number of parameters driving the fundamental processes in Ψ, and the number of pricing errors is five, the statistic c in (45) has a chi-squared distribution with five degrees of freedom. We correct the variance covariance matrix for autocorrelation and heteroskedasticity using the Newey-West method [see, for example, Hamilton (1994) equation 14.1.19] using a lag length of $q = 12$. A long lag length is chosen since interest rates and P/E ratios used in the error terms are highly persistent processes.

2. Options Prices

As for the likelihood function we formulate options prices as expected discounted values of their terminal payoffs under the risk-neutral measure. Expectations are approximated using Monte Carlo simulation while discretizing the dynamics of the state variables of our system along the sth sample path under the risk-neutral measure as:

$$\pi_{t+h}^{*(s)} - \pi_t^{*(s)} = \left(\mu(\pi_t^{*(s)}) - \rho(\pi_t^{*(s)})\right) h + \sigma(\pi_t^{*(s)})\sqrt{h}\tilde{\epsilon}_t^{*(s)}, \tag{46}$$

$$P_{t+h}^{n*(s)} = P_t^{n*(s)} \exp[\left(r(\pi_t^{*(s)}) - \delta(\pi_t^{*(s)})\right) h + \sigma^n(\pi_t^{*(s)})\sqrt{h}\tilde{\epsilon}_{1t}^{(s)*} + \mathbf{1}_{\tilde{u}_t < \kappa^* h}\tilde{\epsilon}_{2t}^{(s)*}], \tag{47}$$

$$B^{*(s)}_{t+h} \;=\; B^{*(s)}_{t} \exp[\,-r(\pi^{*(s)}_{t})\, h], \tag{48}$$

where $\tilde{\epsilon}^{*}_{1t}$ and $\tilde{\epsilon}^{*}_{2t}$ are 5- and 1- dimensional standard normal variables, respectively, $\tilde{u}_t$ is uniformly distributed, and $h = 1/M$ is the discretization interval. On each sample the process for the state variables is simulated starting with $\pi^{*(s)}_{t} = \pi_t$, the assumed beliefs of investors at time t. Then the value of a European call option at time t when investors have beliefs π_t that matures at $t + T$ is given by

$$C^{M\,*}(t, T, \pi_t) \;=\; \frac{1}{S} \sum_{s=1}^{S} B^{*(s)}_{t+T} \max\left[P^{n*(s)}_{t+T} - K, 0\right].$$

We report option prices for $M = 90$. To reduce the time of computations we use three variance reduction techniques: the first two, antithetic and control variate (with Black-Scholes prices), are well known. In addition, we use the expected martingale simulation technique of Duan et. al. The volatility forecast under the Q-measure is approximated from the path of forecasted beliefs under this measure as

$$\sigma^{M*}(t, T, \pi_t) \;=\; \sqrt{\frac{1}{S} \sum_{j=1}^{(T-t)\,M} \sigma^{n}(\pi^{*(s)}_{t+j\,h})\, \sigma^{n}(\pi^{*(s)}_{t+j\,h})' h}. \tag{49}$$

Similarly, using the discretized beliefs processes as in (39), volatility forecasts under the objective measure are analogously constructed as

$$\sigma^{n}(t, T, \pi_t) \;=\; \sqrt{\frac{1}{S} \sum_{j=1}^{(T-t)\,M} \sigma^{n}(\pi^{*(s)}_{t+j\,h})\, \sigma^{n}(\pi^{*(s)}_{t+j\,h})' h}. \tag{50}$$

Table 1: Eight State Regime Switching Model Estimates From SMM Procedure

Fundamental Drifts	β_1	β_2	β_3	β_4	θ_1	θ_2	θ_3	θ_4
	-0.002	0.015	0.065	0.091	-0.057	-0.052	0.061	0.077
	(0.003)	(0.008)	(0.021)	(.005)	(0.002)	(0.007)	(.007)	(0.012)
	ρ_1	ρ_2	ρ_3	ρ_4	ω_1	ω_2	ω_3	ω_4
	- 0.066	-0.025	0.01	0.086	-0.057	-0.031	0.012	0.052
	(0.010)	(0.008)	(0.002)	(0.007)	(0.001)	(0.015)	(0.007)	(.005)
Fundamental Volatilities:	$\sigma_{Q,1}$	$\sigma_{E,2}$	$\sigma_{K,1}$	$\sigma_{K,4}$	$\sigma_{H,5}$			
	0.025	0.084	0.047	0.056	0.049			
	(0.012)	(0.036)	(0.017)	(0.016)	(0.021)			
Interest Rate Rule:	α_0	α_β	α_ρ					
	0.017	0.362	0.257					
	(0.008)	(0.017)	(0.003)					
Prices of Risk:	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\sigma_{M,4}$	$\sigma_{M,5}$			
	0.045	0.421	0.300	0.3381	-0.045			
	(0.129)	(0.201)	(0.111)	(0.149)	(0.078)			
Jump Parameters:	κ	μ_1	σ_1	μ_2	σ_2			
	0.432	-0.052	0.036	0.293	-0.790			
	(0.195)	(0.027)	(0.016)	(0.133)	(0.364)			
Generator Elements:	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6		
	0.005	0.018	0.043	0.051	0.097	0.199		
	(0.002)	(0.011)	(0.007)	(0.003)	(0.023)	(0.007)		
SMM Error Value ($\chi^2(7)$):	10.470	P-Value: 0.063						

The table reports SMM estimates of the following model for CPI, Q_t, real earnings, E_t, the real pricing kernel, M_t, de-meaned capacity utilization, and real money growth:

$$
\begin{aligned}
\frac{dQ_t}{Q_t} &= \beta_t\, dt \,+\, \sigma_Q\, dW_t, \\
\frac{dE_t}{E_t} &= (\theta_t - \kappa\,\xi_1)\, dt \,+\, \sigma_E\, dW_t, + (e^{Y_{1t}} - 1)\, dL_t, \\
\frac{dM_t}{M_t} &= (-\phi_t - \kappa\,\xi_2) dt \,-\, \sigma_M\, dW_t \,+\, (e^{Y_{2t}} - 1)\, dL_t, \\
dK_t &= \rho_t\, dt \,+\, \sigma_K\, dW_t, \\
\frac{dH_t}{H_t} &= \omega_t\, dt \,+\, \sigma_H\, dW_t.
\end{aligned}
$$

W_t is a 5×1 vector of Standard Brownian Motions, L_t is the counter of a Poisson process with constant intensity κ, and Y_{it} i.i.d. $N(\mu_i, \sigma_i)$, $i = 1, 2$. The drift of the stacked state vector $\nu_t = (\beta_t, \theta_t - \kappa\,\xi_1, -\phi_t - \kappa\,\xi_2, \rho_t, \omega_t)'$, follows an eight-state unobserved regime switching model over the composite states listed in the bottom panel of Table 2 with the following generator matrix:

$$
\Lambda = \begin{pmatrix}
-\sum_j \lambda_{1j} & 0 & \lambda_6 & 0 & \lambda_2 & \lambda_1 & 0 & 0 \\
0 & -\sum_j \lambda_{2j} & \lambda_5 & \lambda_6 & 0 & \lambda_1 & \lambda_4 & 0 \\
0 & \lambda_3 & -\sum_j \lambda_{3j} & 0 & 00 & \lambda_1 & \lambda_4 & \lambda_5 \\
0 & \lambda_6 & 0 & -\sum_j \lambda_{4j} & 0 & 0 & \lambda_4 & 0 \\
\lambda_1 & 0 & \lambda_1 & 0 & -\sum_j \lambda_{5j} & \lambda_1 & 0 & 0 \\
\lambda_5 & \lambda_3 & 0 & \lambda_3 & 0 & -\sum_j \lambda_{6j} & 0 & \lambda_5 \\
\lambda_5 & 0 & 0 & 0 & 0 & \lambda_5 & -\sum_j \lambda_{7j} & 0 \\
\lambda_3 & 0 & \lambda_5 & 0 & 0 & 0 & 0 & -\sum_j \lambda_{8j}
\end{pmatrix}.
$$

The pricing kernel, M_t is observed by investors but not by the econometrician. Investors beliefs about the underlying drift states follow the filtering equation in (10). Estimates are obtained from data on the fundamentals as well five price series listed in Table 3 using the SMM methodology described in Appendix 2. Standard errors are in parentheses.

Table 2: Model Implied Transition Probabilities, Stationary Probabilities, Stock and Bond Price Valuations, Fear Indices, and Higher Moments

Implied Annual Quarterly Transition Probability (Percent) Matrix

	1	2	3	4	5	6	7	8
1	80.1	0.3	16.2	0.0	1.6	0.5	0.4	0.8
2	0.2	71.9	7.5	14.9	0.0	0.6	4.5	0.4
3	0.4	3.3	82.8	0.3	0.0	0.6	4.3	8.2
4	0.2	14.9	0.8	79.4	0.0	0.3	4.5	0.0
5	0.5	0.0	0.5	0.0	98.4	0.4	0.0	0.0
6	7.7	3.5	1.3	3.6	0.0	75.7	0.2	7.9
7	8.2	0.2	0.8	0.2	0.1	7.7	82.4	0.4
8	3.6	0.2	8.6	0.0	0.0	0.0	0.2	87.4

Implied 5-Year Transition Probability (Percent) Matrix

	1	2	3	4	5	6	7	8
1	34.7	3.6	37.8	1.3	5.2	2.0	5.1	10.3
2	3.7	30.1	16.8	27.0	0.1	3.5	13.6	5.0
3	5.2	7.3	46.1	3.5	0.2	3.3	11.4	23.0
4	3.4	27.0	8.3	42.2	0.1	3.1	14.0	1.9
5	2.0	0.33	2.6	0.22	92.3	1.5	0.3	0.9
6	16.4	8.9	14.9	9.6	1.0	25.6	3.5	20.0
7	21.4	2.6	11.4	2.3	1.2	15.5	39.3	6.2
8	9.7	2.2	26.7	0.7	0.6	0.8	3.2	56.1

Implied Stationary Probabilities, P/E Ratios, the Term Structure,
Option Prices, and Higher Order Moments

#	$\beta(\%)$	θ (%)	$\rho(\%)$	$\omega(\%)$	$\bar{\pi}(\%)$	P/E	$i_{0.25}(\%)$	S(%)	ATM(%)	P/C	Skw	Kur	$\rho_{SV}(\%)$
1	1.6	6.1	1.0	1.0	11.0	17.3	4.3	1.5	15.2	1.29	−1.9	9.4	−71.1
2	6.5	−5.2	1.0	5.3	7.6	8.9	10.9	−0.1	33.0	0.90	1.0	4.7	96.4
3	6.5	6.1	8.7	1.5	25.8	14.5	12.6	−1.9	16.3	1.21	−1.1	6.6	−78.3
4	9.1	−5.2	−2.5	−5.7	6.9	7.9	13.4	−1.1	41.6	0.76	1.4	5.1	99.1
5	1.6	7.7	1.0	−3.1	12.1	32.0	4.1	0.1	22.8	1.31	−2.3	10.8	−88.0
6	−0.2	−5.7	−6.6	5.3	4.8	12.9	0.1	2.3	21.5	0.98	−0.3	5.9	83.5
7	6.5	−5.2	−6.6	−3.0	10.5	11.4	8.7	−1.3	24.2	0.93	0.8	4.3	95.7
8	1.5	6.1	−6.6	5.3	21.2	19.7	2.3	1.3	17.5	1.34	−3.3	21.5	−92.2

The **top** and **middle panels** report the quarterly and 5-year implied transition probability matrix between the eight states implied from the generator matrix elements displayed in Table 1. Rows may not sum to one due to rounding. The **bottom panel** report the implied stationary probabilities and implied prices of the variables used in the SMM estimation procedure in the eight states. $\bar{\pi}$ is the stationary probability of each state; P/E the price-earnings ratio, and, $i_{0.25}$, the 3-month Treasury yield, and S the 5-year less 1-year Treasury yield; ATM is the at-the-money implied volatility (ATMIV); P/C is the ratio of 5% OTM put-to-call implied volatilities; Skw and Kur are the skewness and kurtosis of the 3-month risk-neutral return distribution, respectively; ρ_{SV} is the correlation between stock returns and stock variance calculated using (23). The P/E ratio and bond yields are computed as shown in Proposition 1. Implied Volatility, Put-Call Ratio and the higher order moments are for options with three months to maturity as calculated using Monte Carlo simulations as shown in Appendix 2.

Table 3: Model Fits for Expected Fundamentals, Stocks, Bonds, and Options Prices from SMM Procedure

Variable	α	β	R^2
Inflation	-0.009	1.638	0.623
	[-1.538]	[8.071]*	
Real Earnings Growth	-0.045	2.099	0.159
	[-1.501]	[3.617]*	
De-Meaned Capacity Utilization	0.002	1.353	0.751
	[0.068]	[13.540]*	
Real Money Growth	-0.005	2.15	0.365
	[-1.785]	[5.957]*	
P/E Ratio	-3.028	1.249	0.606
	[-1.915]	[19.976]*	
3-Month Yield	-0.002	0.970	0.535
	[-0.246]	[5.991]*	
5-Year Minus 1-Year Treasury Yield	0.003	0.857	0.502
	[2.456]*	[8.071]*	
ATM	0.004	0.913	0.502
	[1.522]	[5.539]*	
P/C	0.19	0.710	0.451
	[1.645]*	[8.212]*	

We display the fits of the variables used in our SMM procedure: the fundamentals, and the five pricing variables, which are used to overidentify the model. For the four fundamentals we provide the regression results for the equation $x(t) = \alpha + \beta\, E[x|\mathcal{F}_{t-1}] + \epsilon(t)$, where $x(t)$ is the realized growth and $E[x|\mathcal{F}_t]$ is investors' conditional expected growth of the fundamental under consideration. The conditional expected growth is obtained from the filtered probabilities $\pi(t)$ displayed in Figure 3, and for earnings, for example, is given by $\sum_i^N \theta_i\, \pi_i(t)$. For the price series, we present the regression results for the equation $p(t) = \alpha + \beta\, p(\pi(t)) + \epsilon(t)$, where $p(t)$ and $p(\pi(t))$ are the realized and model price conditional on investors' beliefs at t, respectively. T-statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation.

Table 4: Explaining At-The-Money Implied Volatility of 3-Month S&P 500 Options (1986:Q2 – 2008)

No.	Constant	ATMIV^M	Lag	P/E	CU	Earn	NBER	R^-_{t-1}	$\bar{R}^2$
1	1.769	0.91							0.528
	[0.454]	[3.977]*							
2	10.211		0.415						0.323
	[4.336]*		[3.346]*						
3	−1.847	0.374	0.742						0.576
	[−0.526]	[3.444]*	[2.781]*						
4	10.776			0.405					0.041
	[3.308]*			[2.989]*					
5	18.107				−0.655				0.045
	[15.277]*				[−1.719]				
6	18.888					−0.616			0.114
	[17.108]*					[−2.552]*			
7	27.409						−0.102		0.168
	[7.198]*						[−2.547]*		
8	16.593							−2.562	0.269
	[17.312]*							[−4.462]*	
9	13.781			0.343	−0.035	−0.275	−0.035	−2.128	0.361
	[2.618]*			[2.159]*	[−0.151]	[−1.486]	[−0.882]	[−7.376]*	
10	−2.002	0.990		0.089	0.760	0.098	0.000	−1.238	0.609
	[−0.518]	[3.276]*		[0.508]	[2.754]*	[0.4972]	[0.020]	[−3.065]*	
11	4.921	0.88	0.381	−0.058	0.742	0.1	0.01	−1.206	0.663
	[1.458]	[4.285]*	[2.660]*	[−0.373]	[2.755]*	[0.580]	[0.550]	[−3.169]*	

The table reports the quarterly time series regressions

$$\begin{aligned}\text{ATMIV}(t) &= \beta_0 + \beta_1\,\text{ATMIV}^M(t-1) + \beta_2\,\text{ATMIV}(t-1) + +\beta_3\,\text{P/E}(t-1) + \beta_4\,\text{C.U.}(t-1)\\ &+ \beta_5\,\text{Earn}(t-1) + \beta_6\,\text{NBER}(t-1) + \beta_7\,R^-_{t-1} \,+\, \epsilon(t).\end{aligned}$$

In different lines some of the β_i are set to zero. ATM(t) is the at-the-money Black-Scholes implied volatility on S&P 500 index options traded on the CBOE with approximately three months to maturity and trading at the beginning of the quarter. ATM^M is the at-the-money implied volatility implied by our model and calculated as shown in Appendix 2. The historical and model implied series are shown in the top panel of Figure 1. The latter are calculated conditional on investors' beliefs of fundamental drift states that are extracted and displayed in Figure 3. P/E is the price to operating income ratio of S&P 500 firms, CU is the demeaned industrial capacity utilization in the United States obtained from the Federal Reserve Board, Earn stands for the real operating earnings growth of S&P 500 firms, NBER is 100 times the quarterly expansion indicator created by the NBER, and R^- is percentage one quarter lagged returns in periods when it is negative on the S&P 500 index. Besides options prices, all other variables are measured at the end of the previous quarter. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The symbol * indicates statistical significance at the 5% level.

Table 5: Explaining the Ratio of Implied Volatilities of 5% Out-of-the-Money Puts to Calls for 3-Month S&P 500 Options (1986:Q2 – 2008)

No.	Constant	P/C^M	Lag	P/E	CU	Earn	NBER	R_{t-1}^-	COT	II	$\bar{R}^2$
1	0.385	0.717									0.454
	[3.603]*	[8.338]*									
2	0.394		0.693								0.458
	[4.886]*		[10.912]*								
3	0.237	0.42	0.406								0.538
	[2.918]*	[3.454]*	[3.570]*								
4	1.466			−0.01							0.041
	[9.536]*			[−1.344]							
5	1.297				0.015						0.048
	[37.324]*				[1.780]						
6	1.279					0.013					0.092
	[41.974]*					[2.521]*					
7	1.176						0.001				0.039
	[60.65]*						[3.170]*				
8	1.31							0.029			0.057
	[36.721]*							[4.050]*			
9	1.267								−0.901		0.008
	[27.522]*								[−0.760]		
10	53.99									−2.262	0.044
	[30.724]*									[−1.488]	
11	1.499			−0.006	−0.004	0.01	0	0.041	0.357	−0.004	0.219
	[10.53]			[−1.145]	[−0.453]	[1.315]	[0.047]	[4.846]*	[0.399]	[−2.530]*	
12	0.441	0.682		−0.001	−0.014	0.002	0.001	0.022	0.654	−0.003	0.470
	[2.128]*	[6.086]*		[−0.129]	[−2.383]*	[0.539]	[1.089]	[2.311]*	[0.897]	[−1.973]*	
13	0.264	0.412	0.392	−0.001	−0.012	0.005	0.001	0.014	0.681	−0.002	0.551
	[1.783]	[3.468]*	[3.288]*	[−0.351]	[−2.420]*	[0.126]	[1.447]	[1.952]	[1.325]	[−2.031]*	

The table reports the quarterly time series regressions

$$\begin{aligned}\text{P/C}(t) \quad &= \quad \beta_0 + \beta_1\,\text{P/C}^M(t-1) + \beta_2\,\text{P/C}(t-1) + \beta_3\,\text{P/E}(t-1) + \beta_4\,C.U.(t-1) \\ &+ \quad \beta_5\,\text{Earn}(t-1) + \beta_6\,\text{NBER}(t-1) + \beta_7\,R_{t-1}^- + \beta_8\,\text{COT}(t) \; + +\beta_8\,\text{II}(t)\epsilon(t).\end{aligned}$$

In different lines some of the β_i are set to zero. P/C(t) is the ratio of Black-Scholes implied volatilities of 5% out-of-the-money puts to calls for S&P 500 options with about three months to maturity measured at the beginning of the quarter. P/E^M is the analogous put-call ratio implied by our model and calculated as shown in Appendix 2. The historical and model implied series are shown in the top panel of Figure 1. The latter are calculated conditional on investors' beliefs of fundamental drift states that are extracted and displayed in Figure 3. P/E is the price to operating income ratio of S&P 500 500 firms; CU is the demeaned industrial capacity utilization in the United States obtained from the Federal Reserve Board; Earn stands for the real operating earnings growth of S&P firms; NBER is 100 times the quarterly expansion indicator created by the NBER; R_{t-1}^- is the one quarter lagged returns in periods when it is negative on the S&P 500 index; COT stands for the sentiment of traders measured as the net long position of large speculators on S&P 500 index futures obtained from the Commodity Futures Trading Commission's Commitment of Traders Report; II stands for investor sentiment (bullish less bearish proportion) measured in Investor's Intelligence's survey of investment newsletter writers. Besides options prices and sentiment variables, all other variables are measured at the end of the previous quarter. COT is measured on the day of the options trade, and the II on the Wednesday before the options trade. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The symbol * indicates statistical significance at the 5% level.

Table 6: Model Volatility of Stock Volatility and Fundamental Uncertainties (1986:Q2 – 2008)

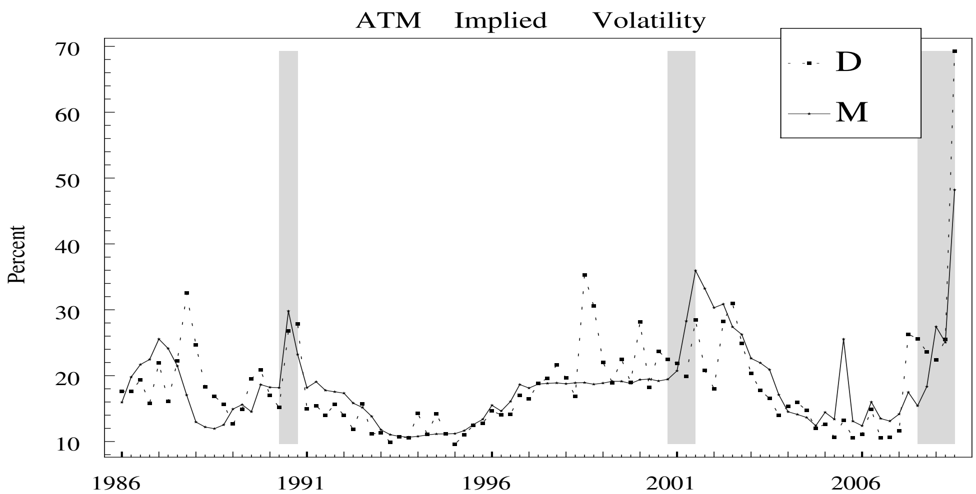
No.	Constant	Inf Unc	Earn Unc	CU Unc	MG Unc	$\bar{R}^2$
1.	2.046	0.646				0.054
	(1.535)	(1.817)*				
2.	0.933		0.346			0.725
	(1.554)		(8.347)*			
3.	1.767			0.677		0.037
	(1.023)			(1.729)*		
4.	-0.226				1.952	0.234
	(-0.143)				(2.553)	
5.	0.989	0.0313	0.353	-1.116	1.751	0.891
	(1.032)	(0.182)	(14.295)*	(-3.444)*	(5.910)	

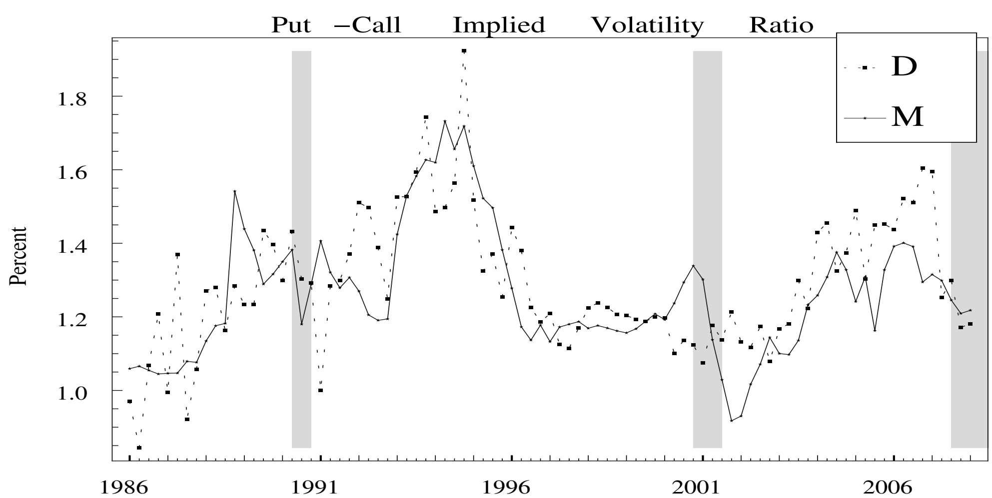
The table reports the quarterly time series regressions

$$\mathrm{VV}(t) \;=\; \beta_0 + \beta_1\,\mathrm{IU}(t) + \beta_2\,\mathrm{EU}(t) + \beta_3\,\mathrm{CU}(t) + \beta_4\,\mathrm{MG}(t) + \epsilon(t). \tag{51}$$

In different lines some of the β_i are set to zero. Inf Unc stands for inflation uncertainty, Earn Unc for earnings uncertainty, CU Unc for Capacity Utilization uncertainty, and MG Unc for money growth uncertainty. Uncertainty for each fundamental variable is measured using equation (33) and the models volatility of stock volatility is computed using (22). Time series of all variables are evaluated at the filtered belief series in Figure 3. T-Statistics are in parenthesis and are adjusted for heteroskedasticity and autocorrelation using the Newey and West (1987) method. The symbol * indicates statistical significance at the 5% level.

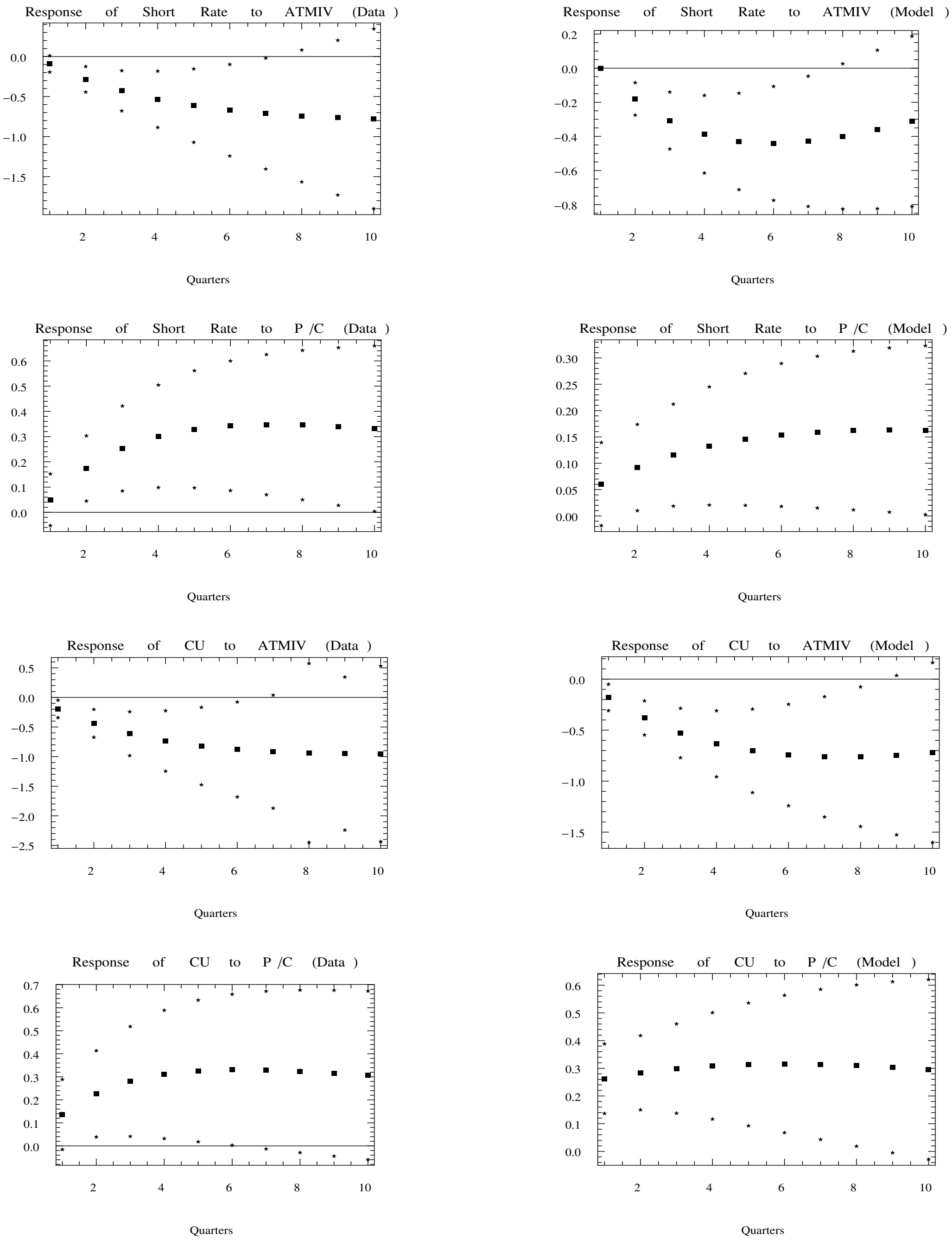
Figure 1: Data and Model Fitted At-the-Money Implied Volatility (ATMIV) and 5% Out-of-the-Money Put-Call Implied Volatility Ratio (P/C) from (1986:Q2-2008)





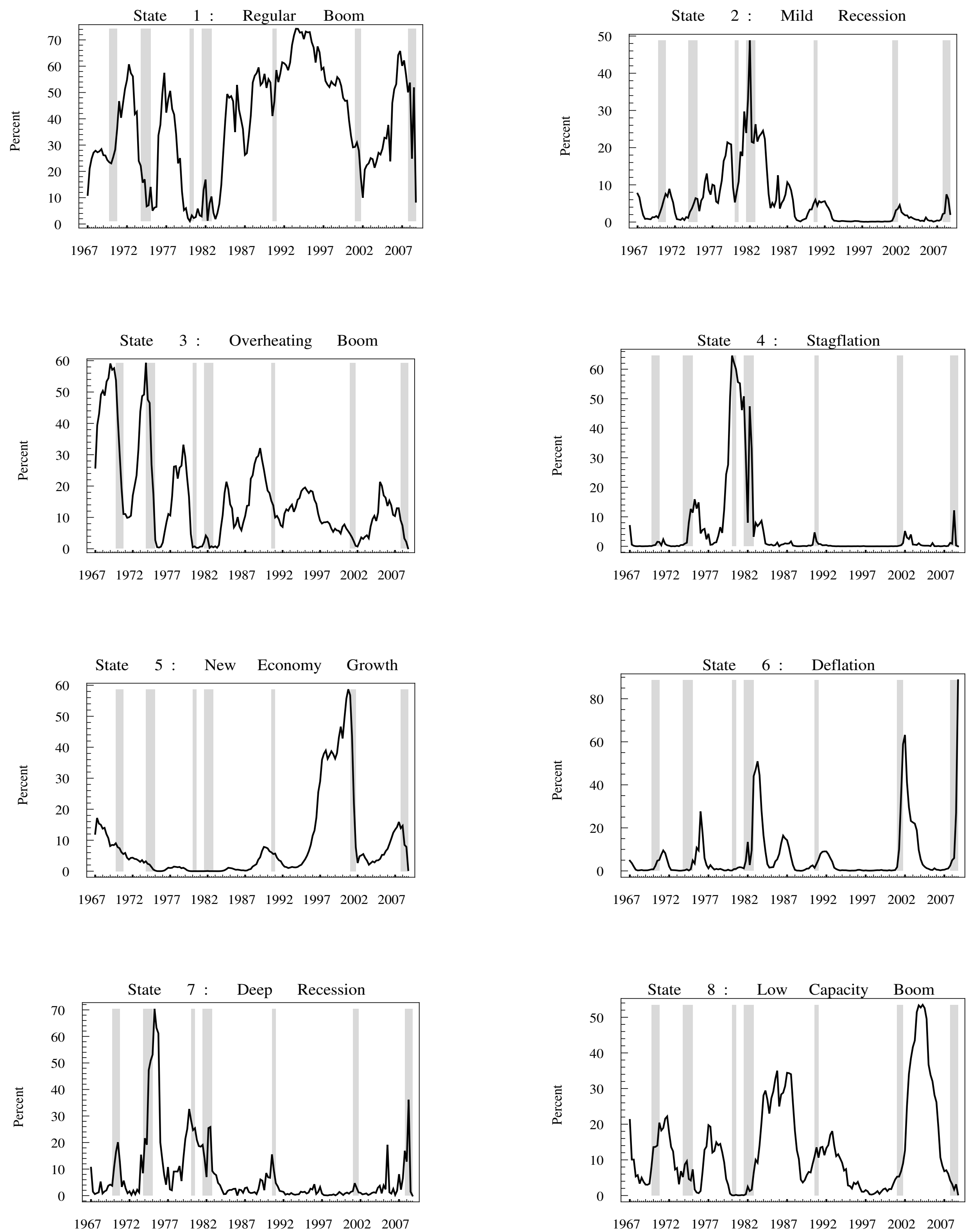
The at-the-money implied volatility (ATMIV) and ratio of 5% OTM put-to-call implied volatilities (P/C) at about three months to maturity are constructed at a quarterly frequency from S&P 500 index options prices as discussed in Section 2.2. The legend "D" denotes the historical data series, while "M" denotes those from our model. The model series are calculated using Monte Carlo simulations as shown in Appendix 2. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 2: Impulse Responses of Short Rates and Industrial Capacity Utilization to Shocks to Fear Measures (1986:Q2-2008)



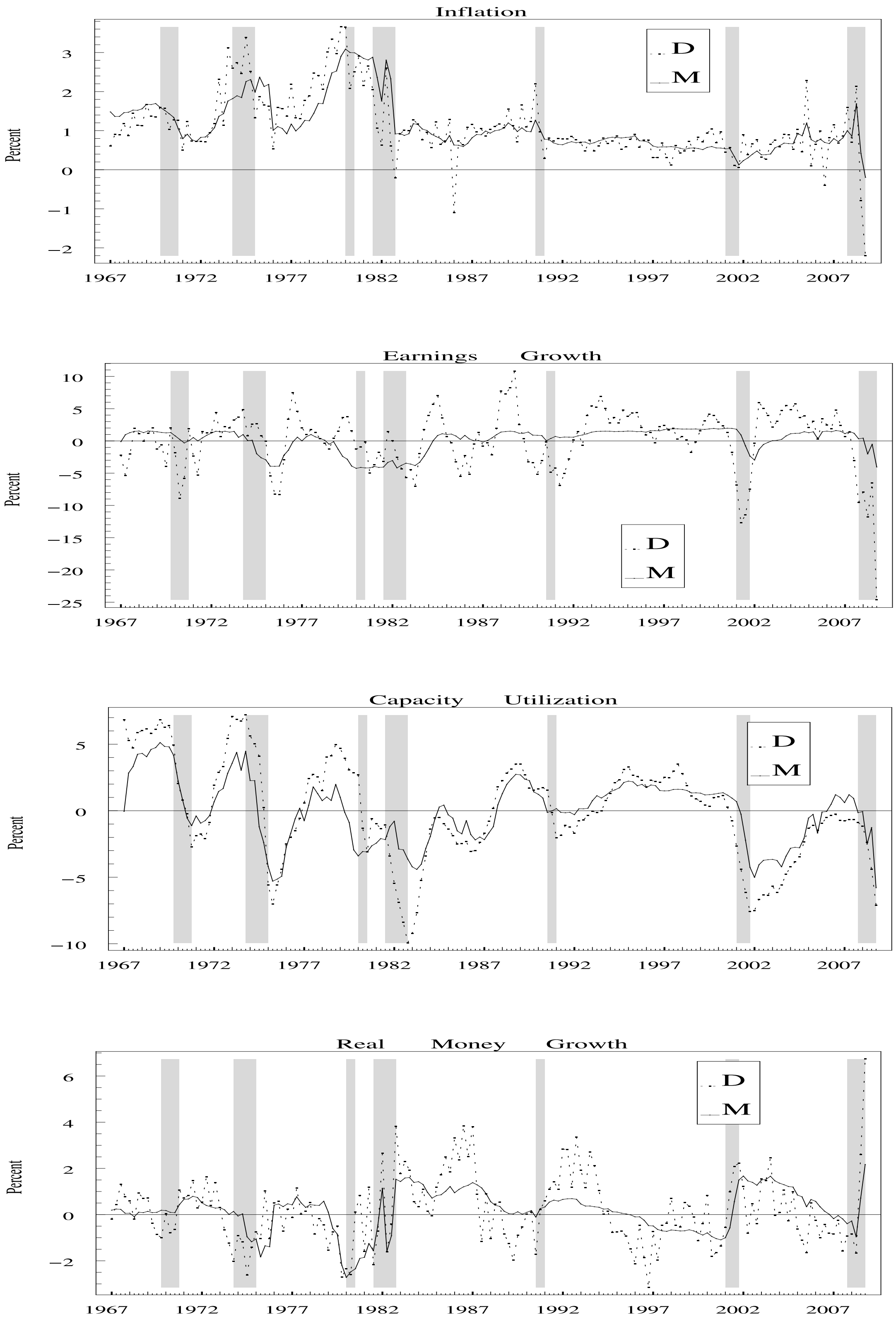
We report the generalized impulse response function (IRF) of Pesaran and Shin (1998) for the 1st order VAR system with the two variables in each panel. The IRF using this definition is independent of the order of variables in the VAR. Two standard error bands using bootstrap with 5000 repetitions are also displayed.

Figure 3: Conditional Probabilities of Eight States (1967-2008)



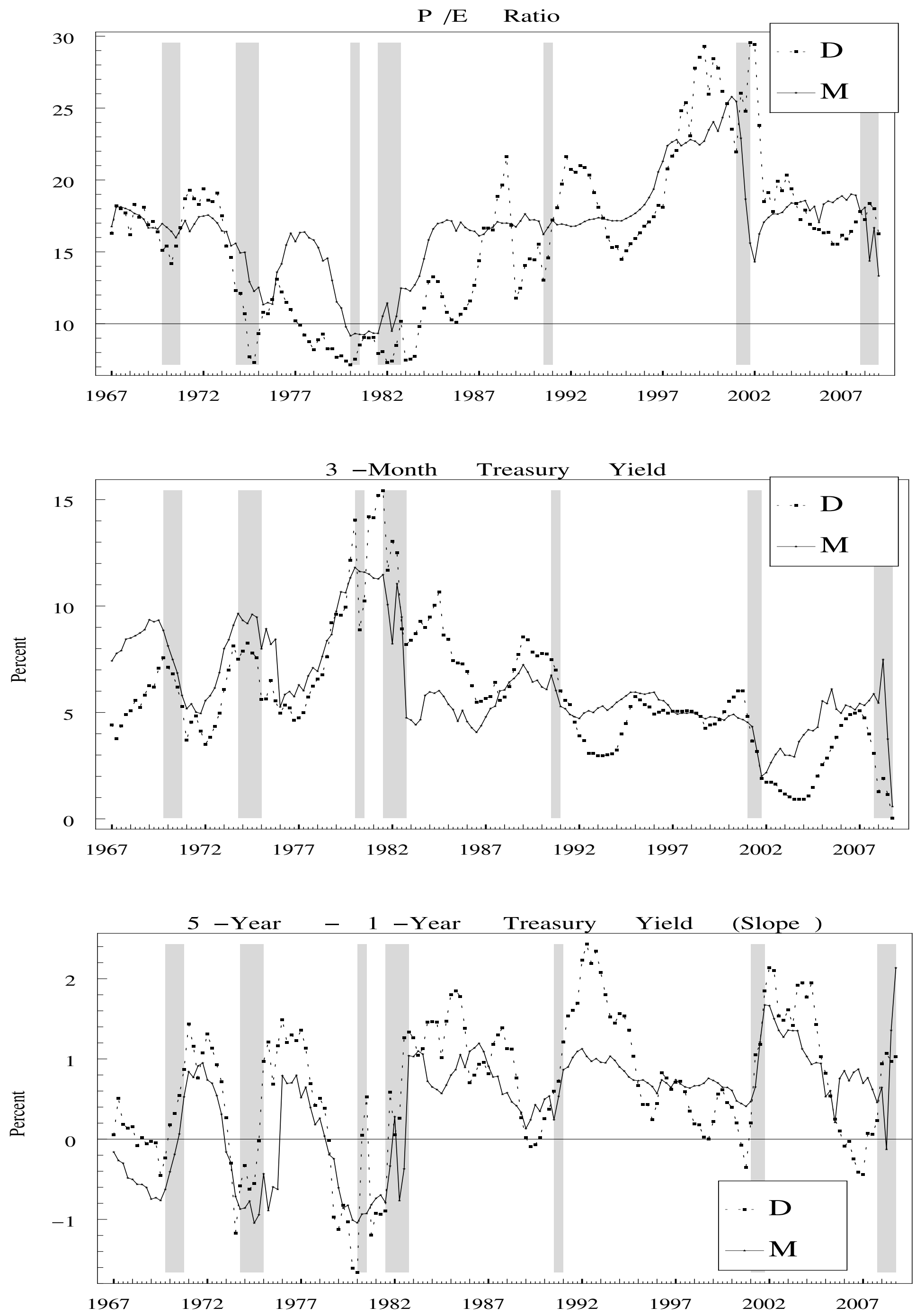
The state definitions are in Table 2. β, θ, ρ, and ω are the drifts of inflation, earnings growth, de-meaned capacity utilization, and real money growth, respectively. The filtered beliefs are obtained from the SMM procedure in Appendix 2. The calibrated values of the parameters are shown in Table 1. Shaded areas represent NBER-dated recessions.

Figure 4: Fundamentals: Empirical and Model Fitted (1960-2008)



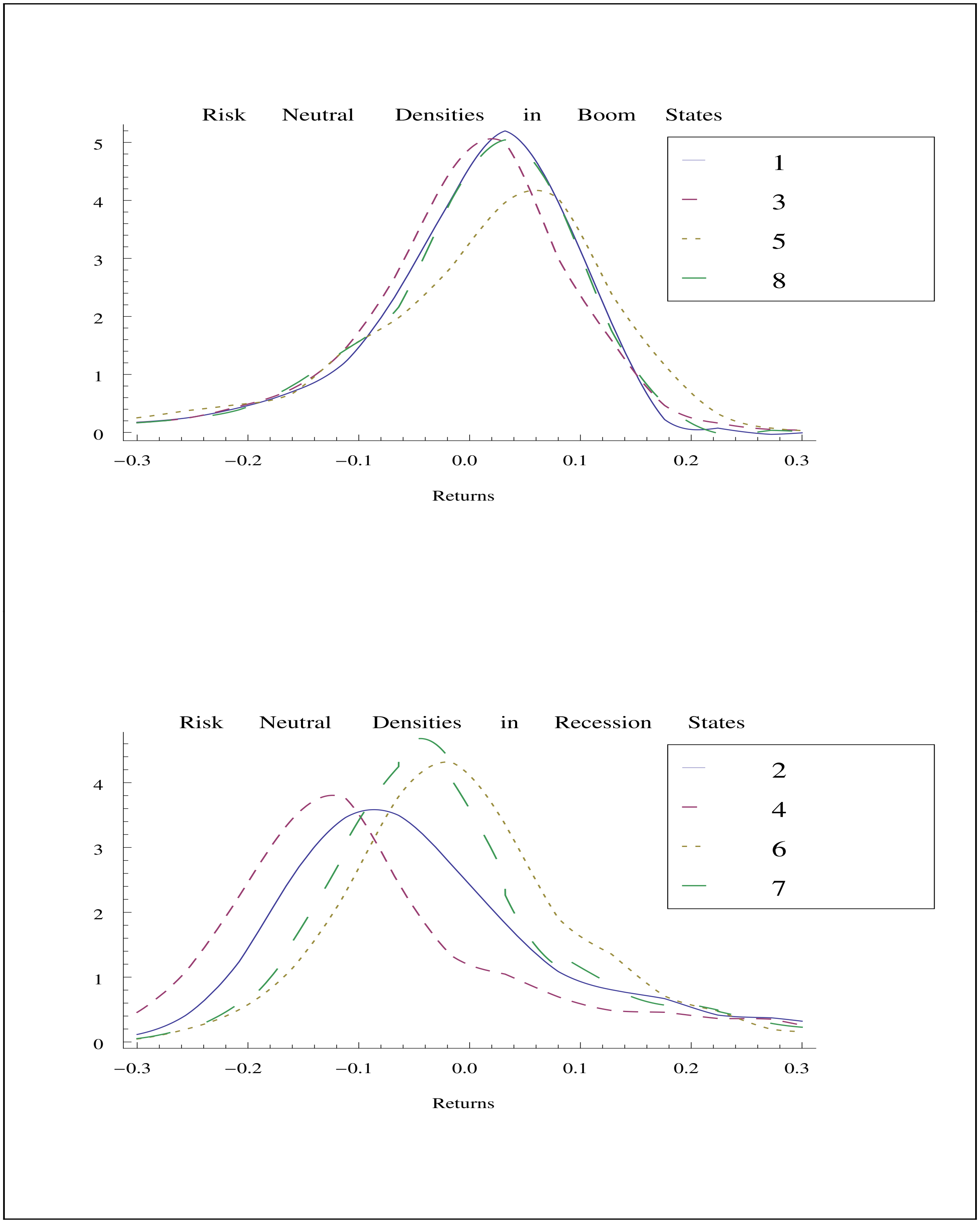
Historical values of financial and fundamental variables series (D) are in solid lines and their fitted values (M) from the SMM estimation procedure in Appendix B are in dashed lines. The calibrated values of the parameters are shown in Table 1. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 5: Stock and Bond Prices: Empirical and Model Fitted (1960-2008)



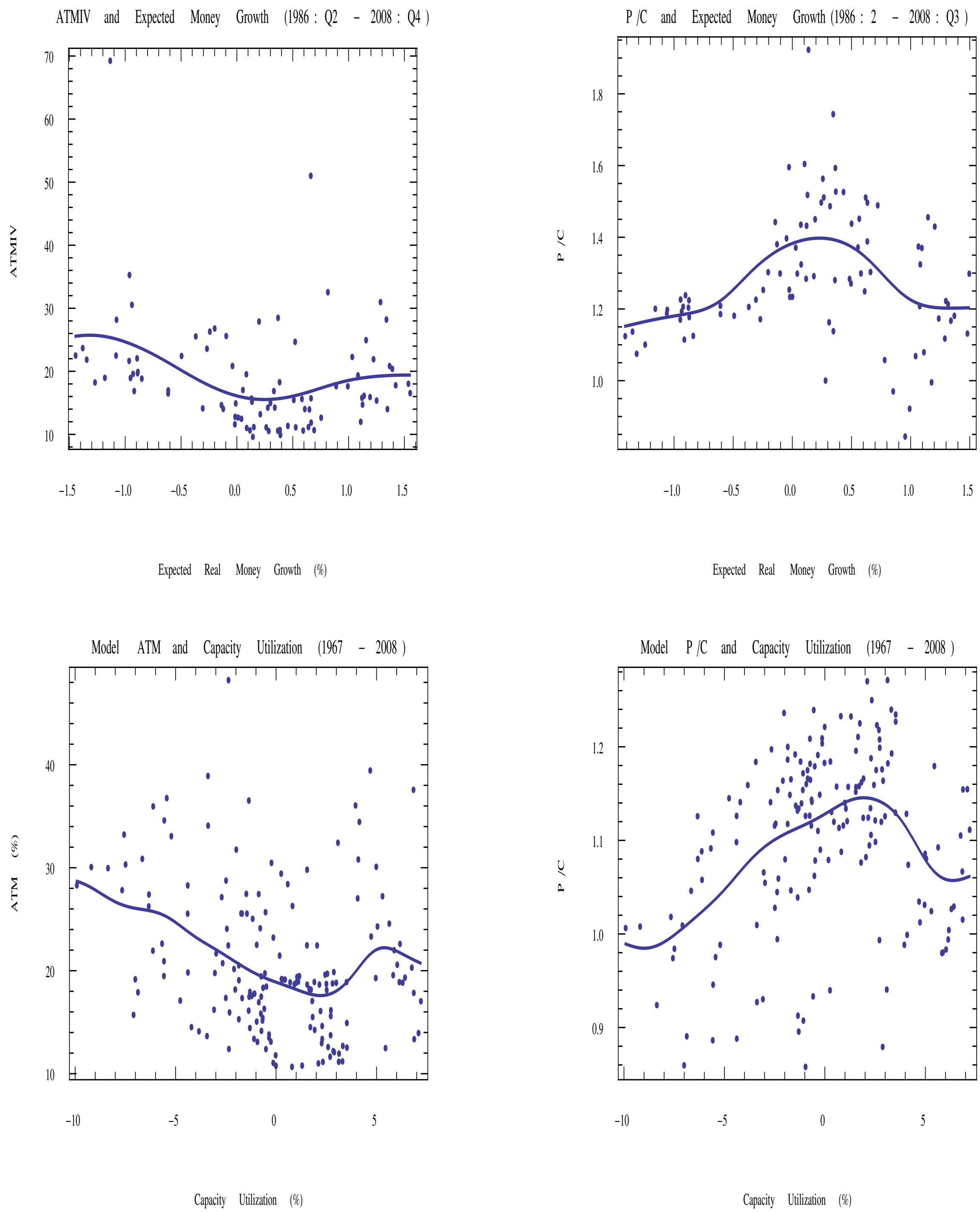
Historical values of financial and fundamental variables series (D) are in solid lines and their fitted values (M) from the SMM estimation procedure in the Appendix are in dashed lines. The estimated values of the parameters are shown in Table 1 and the implied asset price valuations are in Table 2 The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. Shaded areas represent NBER-dated recessions.

Figure 6: Strike-Adjusted 3-Month Densities of Stock Returns Under the Risk-Neutral Measures in the Eight States



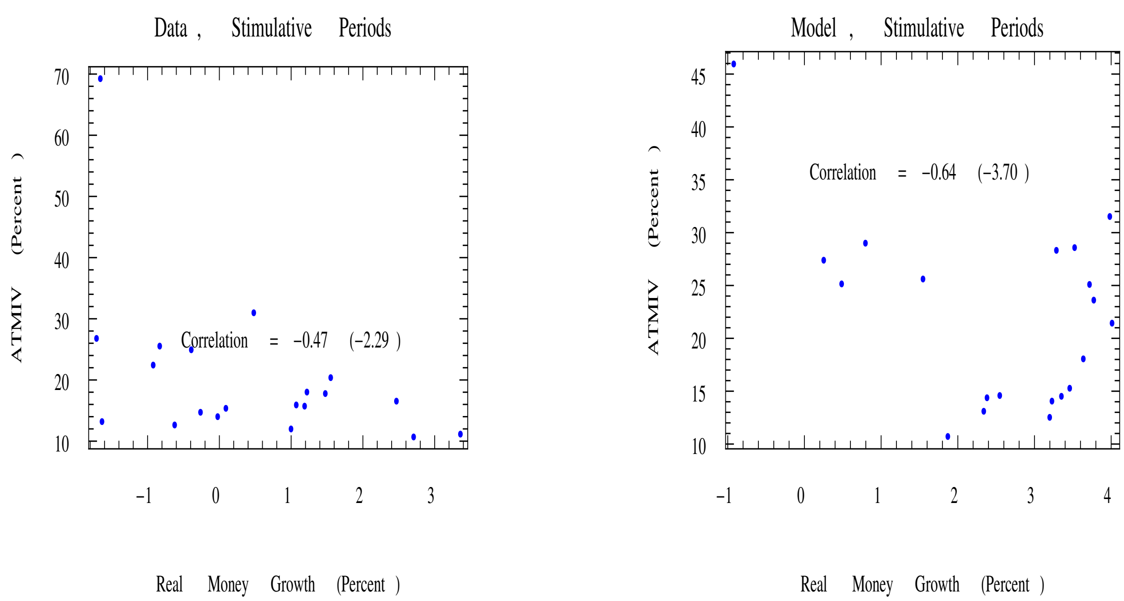
Stock return risk-neutral densities conditional on investors having 80 percent probability of each of the eight states (and equal probabilities for the other states) are calculated using our estimated model parameters in Table 1 using Monte Carlo simulations as shown in Appendix 2.

Figure 7: Relationship Between Fear Indices and Policy Variables From Kernel Regressions



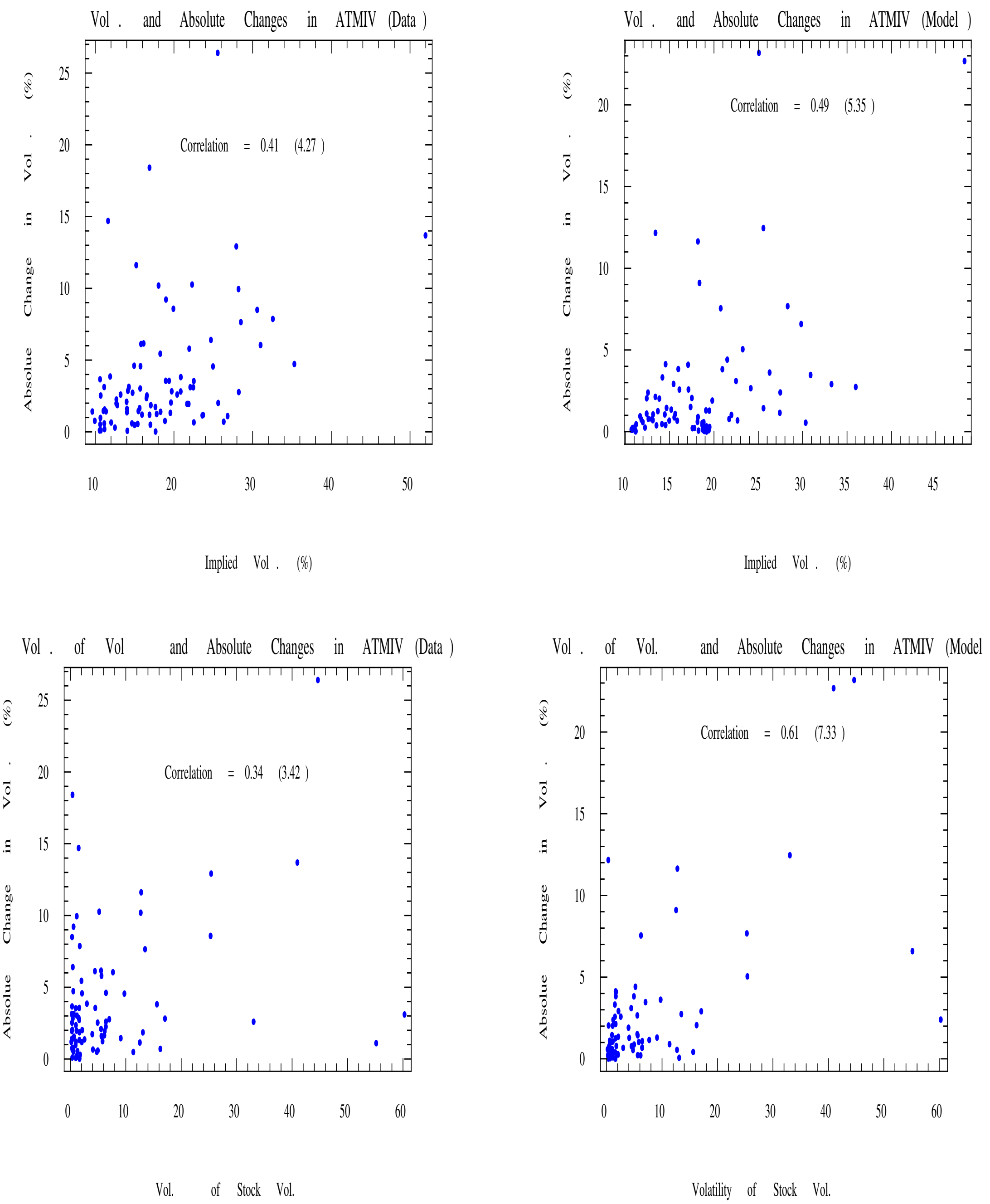
Data and model expected fundamentals are shown in Figure 4. The at-the-money implied volatility (ATMIV) and the ratio of 5% OTM put-to-call implied volatilities (P/C) at about three months to maturity is constructed at a quarterly frequency from S&P 500 index options prices as discussed in Section 2.2. The model option price series are calculated using Monte Carlo simulations as shown in Appendix 2. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3. The fitted lines from the nonparametric regressions are estimated with a Gaussian kernel [see e.g. Campbell, Lo, and MacKinlay (1997)].

Figure 8: Relationship Between ATMIV and Money Growth During Stimulative Periods



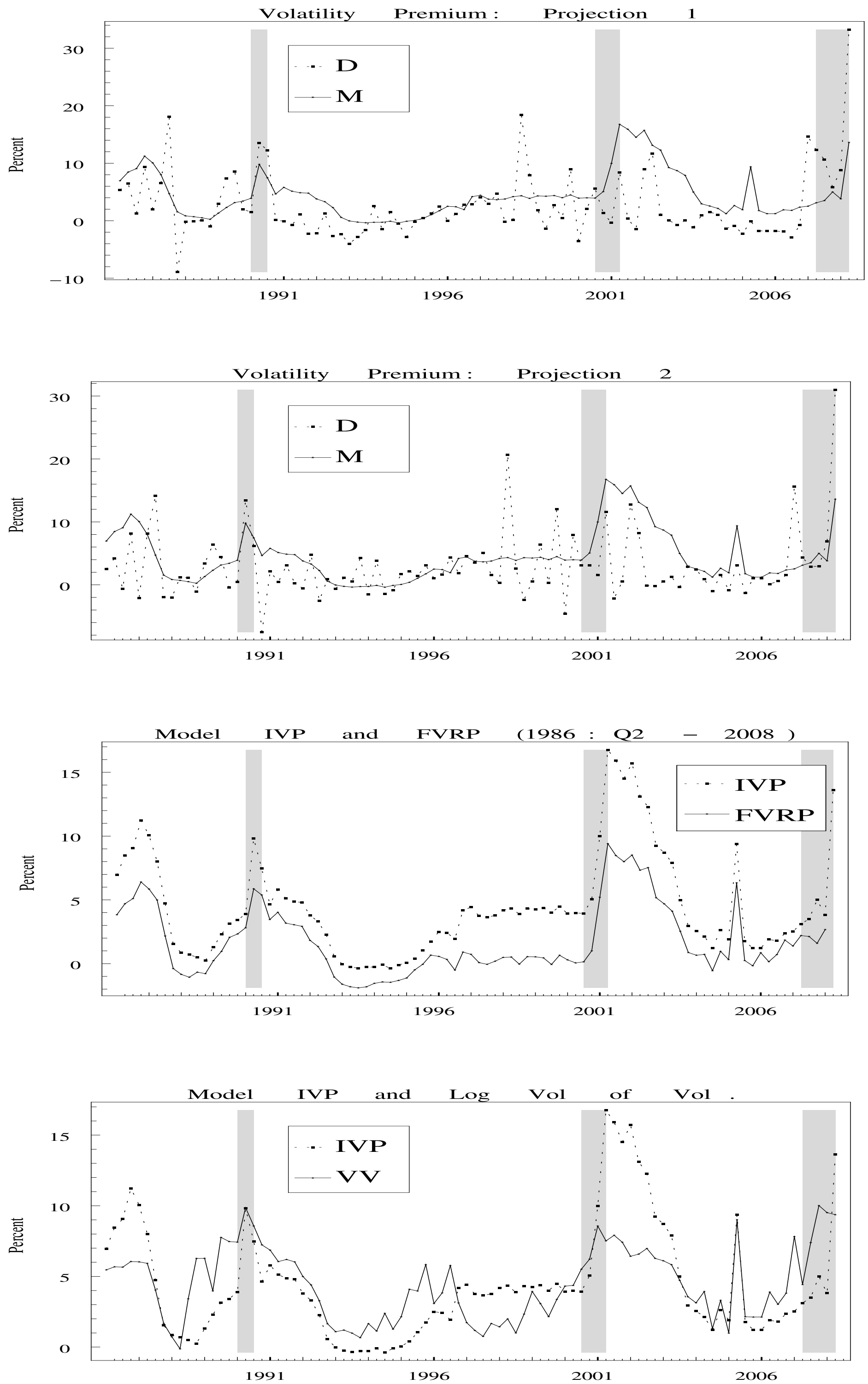
We define "Stimulative" periods as those where the 3-month Treasury Bill Yield is below the annualized inflation (CPI) rate. In the sub-sample of our date where we have options data (1986:2 – 2008) there are 20 quarters that we characterize as stimulative. The panels report the sample correlation and its t.statistic in parenthesis. The at-the-money implied volatility (ATMIV) at about three months to maturity is constructed at a quarterly frequency from S&P 500 index options prices as discussed in Section 2.2. The model series are calculated using Monte Carlo simulations as shown in Appendix 2. The filtered beliefs series of investors used to generate the fitted values are shown in Figure 3.

Figure 9: Relationship Between ATM Implied Volatility and Absolute Changes in ATM Implied Volatility, (1986:Q2-2008)



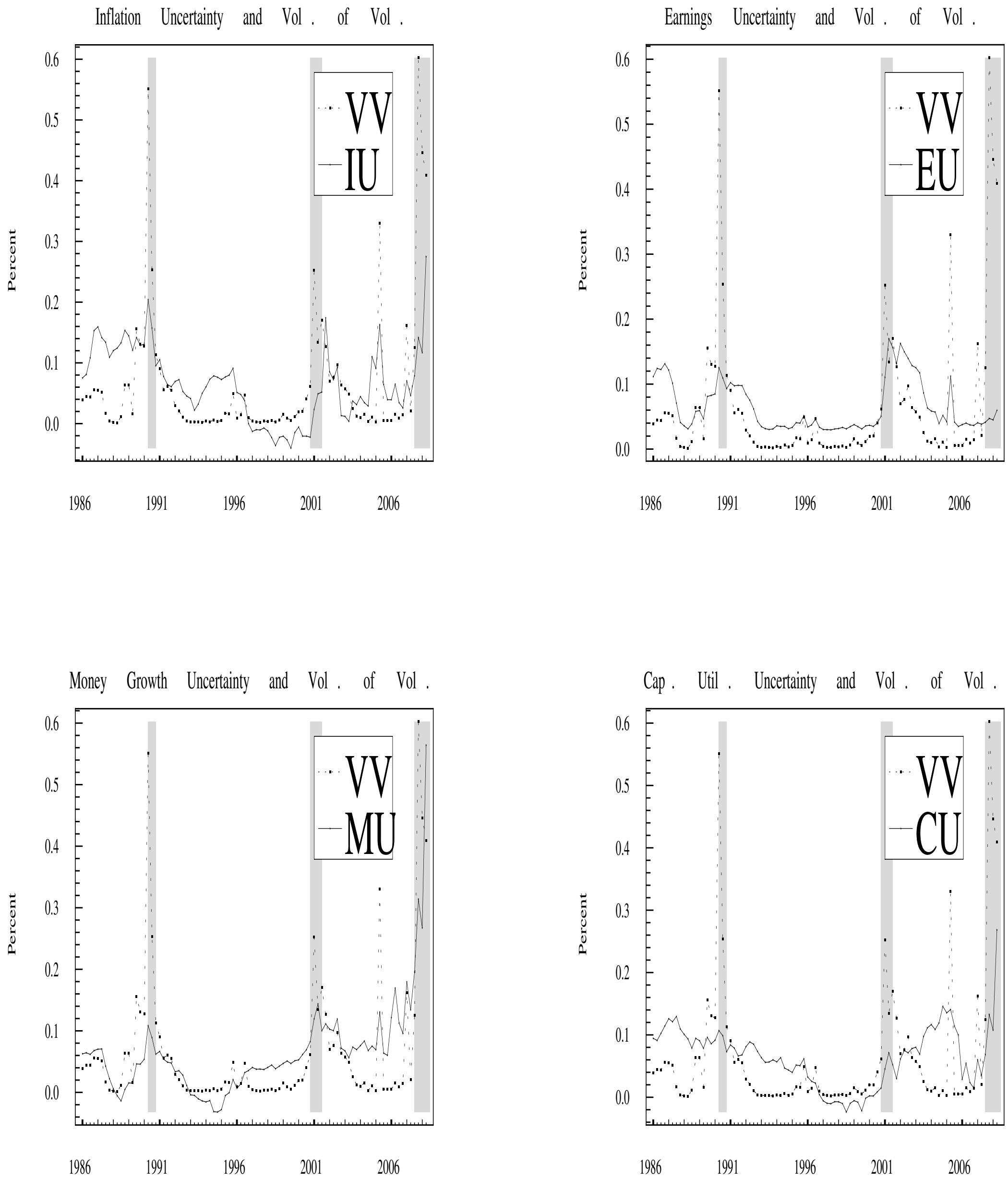
Data and model ATM implied volatility are shown in the top panel of Figure 1. The model volatility of volatility is computed using (22) and the filtered belief series in Figure 3.

Figure 10: Volatility Premium (1986:Q2-2008)



The **first** and **second** panels show the data implied volatility premiums (IVP), which are the difference between the ATMIV in Figure 1 and the P-measure forecast of realized volatility from from Projection 1 (equation (26)) and Projection 2 (equation (27)), respectively. The panels also show the analogous implied volatility premium from our model which is the difference between the model implied volatility in Figure 1 and the model forecast of volatility under the P-measure using simulation methods as described in equation (50) in Appendix 2. The **third** panel shows the IVP and the Forward Volatility Risk Premium (FVRP) from our model. The FVRP is the difference in volatility forecasts under the Q- and P-measures respectively. The **fourth** panel shows the model's IVP and volatility of volatility series as computed using (22). All model variables are computed using the filtered belief series in Figure 3.

Figure 11: Volatility of Stock Market Volatility and Fundamental Uncertainties (1986:Q2-2008)



The model volatility of volatility is computed using (22) and fundamental RMSE uncertainties are computed using (33). All variables are computed using the filtered belief series in Figure 3.