Public's Inflation Expectations and Monetary Policy

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Abstract

The paper develops a DSGE model in which the monetary policy rate transfers information about the central bank's view on macroeconomic developments to incompletely informed price setters. This model is used to study to what extent such a signal channel of monetary transmission affects the Federal Reserve's ability to stabilize the economy in the face of short-run disturbances. To this end, the model is estimated with likelihood methods on a U.S. data set including the *Survey of Professional Forecasters* as a measure of price setters' expectations. The paper carries out an econometric evaluation of the model, showing that the signal channel is empirically relevant. While the signal channel enhances the Federal Reserve's ability to stabilize the economy in the face of demand shocks, monetary policy is found to be quite ineffective at stabilizing the economy in the aftermath of technology shocks.

Keywords: Incomplete information, heterogeneous expectations, higher-order beliefs, Bayesian econometrics, persistent real effects of nominal shocks.

JEL classification: E5, C11, D8.

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1 Introduction

An important feature of economic systems is that information is dispersed across market participants and policy makers. Policy makers often take actions that transfer information to market participants. A prominent example is a central bank setting the policy rate that can be publicly and cheaply observed. Observing the policy rate conveys information about the central bank's view on macroeconomic developments. Such an information transfer may strongly influence the transmission of monetary impulses and the central bank's ability to stabilize the economy. Consider the case in which a central bank expects that a disturbance will shrink economic activity in the next few quarters. Cutting the policy rate, on the one hand, has the effect of stimulating the economy. On the other hand, lowering the policy rate might push the economy into a recession if this action convinced market participants that a contractionary shock will hit the economy. The paper develops a dynamic stochastic general equilibrium (DSGE) model to study the empirical relevance of this signal channel of monetary transmission and its implications for the macroeconomic effects of monetary policy.

The model developed in the paper is characterized by the following two ingredients: (1) firms have limited and dispersed information about aggregate developments and (2) the nominal interest rate set by the central bank can be publicly observed. The model features technology, monetary, and demand shocks. Technology shocks are idiosyncratic but they have a persistent aggregate component that is not observed by price-setting firms. Price setters observe their idiosyncratic productivity, which conveys information about the aggregate component of technology, and an exogenous private signal about the current demand shock. Furthermore, price setters face costs of price adjustment, implying that price setters need to forecast the evolution of their nominal marginal costs. When firms expect the price level to go up, they, ceteris paribus, find it optimal to raise their price. Such a coordination motive in price setting as well as the availability of private information makes it optimal for price setters to forecast the forecast of other firms (Townsend 1983a and 1983b). Furthermore, price setters observe the interest rate, which is set by the central bank according to a Taylor rule. Hence, this signal provides public information about the central bank's view on current inflation and output to price setters.

It is important to note that the model has two channels of monetary transmission. The first channel is the traditional New Keynesian channel based on the fact that the central bank can affect the real interest rate due to price stickiness. The second channel is based on the perfect observability of the policy rate, which conveys information about the central bank's view about inflation and output to firms. We call this second channel the *signal channel* of monetary transmission. How firms interpret the change in the policy rate is of first-order importance for the response of firms' inflation expectations and hence for the propagation of shocks. A rise in the policy rate can be interpreted by firms in two alternative ways. First, a monetary tightening might imply that the central bank is responding to a monetary shock leading the central bank to deviate from the Taylor rule. Second, an interest rate rise may also be interpreted as the response of the central bank to an inflationary non-policy shock, which, in the model, is either an adverse technology shock or a positive demand shock. If the first interpretation prevails, the central bank can conduct a successful stabilization policy in the face of short-run fluctuations as tightening monetary policy curbs firms' inflation expectations and hence inflation. If the second interpretation prevails, a rise in the policy rate induces firms to expect higher inflation. In this case, monetary policy cannot successfully stabilize inflation. In fact, any attempt by the central bank to counter the inflationary effects of non-policy shocks by raising the policy rate ends up bringing about even higher inflation. The paper shows that a strong systematic response to inflation is critical to allow the central bank to stabilize prices. The reason is that an aggressive policy toward inflation stabilization mitigates the expected inflationary consequences of non-policy shocks that a rise in the policy rate may lead firms to expect.

The model is estimated through likelihood methods on a U.S. data set that includes the *Survey of Professional Forecasters* (SPF) as a measure of price setters' inflation expectations. The data range includes the 1970s, which were characterized by one of the most notorious episodes of a substantial rise in inflation and inflation expectations in recent U.S. economic history. When the model is taken to the data, we find that an interest rate rise signals that either a positive demand shock or a contractionary monetary shock may have hit the economy. Firms, however, do not sensibly change their expectations about aggregate technology shocks when they observe a monetary tightening. It follows that the Federal Reserve has limited ability to counter the inflationary consequences of technology shocks. The reason is that when the Federal Reserve raises the policy rate to counter a negative technology shock, firms believe that the central bank is reacting to either a positive demand shock or a contractionary monetary shock. These two shocks have conflicting effects upon inflation expectations, and it turns out that these effects cancel each other out. In contrast, the signal channel turns out to improve the effectiveness of stabilization policies in the face of demand shocks. The reason is that when the central bank raises the interest rate, firms start believing that a contractionary monetary shock might have occurred. Expecting a contractionary monetary shock tends to lower inflation expectations, curbing the inflationary consequences of the positive demand shock. The change in the policy rate does not lead firms to update their expectations about the occurrence of a demand shock very much because they also learn about it from their exogenous private signal.

The paper also documents that the model with the signal channel fits the data better than a canonical New Keynesian DSGE model (e.g., Rotemberg and Woodford (1997) and Rabanal and Rubio-Ramirez, 2005) in which firms have perfect information about the history of shocks and hence the signal channel is inactive. Furthermore, we find that the model with the signal channel does a better job of explaining the dynamics of the observed inflation expectations (e.g., the SPF).

The paper also makes a methodological contribution, providing an algorithm to solve a DSGE model in which agents form higher-order expectations. The algorithm exploits the fact that conditional on the law of motion for higher-order expectations about the exogenous state variables, the model can be solved as a standard linear rational expectations model (e.g., the algorithm proposed by Blanchard and Kahn, 1980, Uhlig, 1998, and Sims 2002).[1] The solution routine proposed in the paper turns out to be sufficiently fast and reliable to allow us to estimate a medium-scale DSGE model with higher-order expectations. Estimating such models is a very computationally demanding task, requiring the econometrician to solve the model thousands of times at radically different points in the parameter space.

The idea that the monetary authority sends a public signal in an economy in which agents have dispersed information has been pioneered by Morris and Shin (2003a). Morris and Shin (2003b), Angeletos and Pavan (2004), Hellwig (2005), and Angeletos and Pavan (2007) focus on the welfare effects of disclosing public information in models with dispersed information and complementarities. Hellwig (2002) derives impulse responses in a price-setting economy with monopolistic competition and incomplete information. Angeletos, Hellwig, and Pavan (2006) study the signaling effects of policy decisions in a coordination game. While this theoretical literature has produced several fascinating results, the models that have been introduced are too stylized to be taken to the data.

Coibion and Gorodnichenko (2011) find that the Federal Reserve tends to raise the policy rate more gradually if private sector's inflation expectations are lower than the Federal Reserve's forecasts of inflation. This evidence can be reconciled with a model in which monetary policy has signaling effects, such as the model studied in this paper, and the central bank acts strategically to stabilize the economy. Walsh (2010) shows that (perceived or actual) signaling effects of monetary policy alter the central bank's decisions, resulting in a bias (i.e., an opacity bias) that distorts the central bank's optimal response to shocks.

Lorenzoni (2009 and 2010) studies a model in which aggregate fluctuations are driven

[1]Unlike the method proposed by Nimark (2008), the approach proposed in this paper does not require solving a system of non-linear equations, which can become quite complicated as the degree of complexity of the DSGE model of interest increases.

by the private sector's uncertainty about the economy's fundamentals. The main difference from my paper is the imperfect observability of the policy rate. The key feature of the model studied in this paper is the fact that the monetary policy rate is an endogenous public signal, conveying non-redundant information to price setters. In Lorenzoni's papers, while monetary policy affects agents' incentives to respond to private and public signals, the signaling effect of monetary policy is not investigated.

The model studied in this paper is similar to the model studied by Nimark (2008). The nice feature of this model is that the supply side of this economy can be analytically worked out and turns out to be characterized by an equation that resembles the standard New Keynesian Phillips curve. The model studied in this paper shares this feature. Nonetheless, in Nimark (2008) the signal channel does not arise because assumptions on the Taylor-rule specification imply that the policy rate conveys only redundant information to price setters. The paper is also related to the macroeconomic literature of incomplete common knowledge studied, for instance, in Woodford (2002) and Mackowiak and Wiederholt (2009 and 2010). Melosi (2010) conducts an econometric analysis of a simple DSGE model with dispersed information. Bianchi (2010) studies how agents' beliefs react to shifts in monetary policy regime and the associated implications for the transmission mechanism of monetary policy. Bianchi and Melosi (2010) develop a DSGE model in which agents have to learn the persistence of the realized monetary regimes. They use this model to study how public expectations and uncertainty react to monetary policy decisions and communication.

The paper is also related to the literature that uses incomplete information models for studying the persistence in economic fluctuations (Townsend, 1983a, 1983b; Adam, 2009; Angeletos and La'O, 2009; and Rondina, 2008) and the propagation of monetary disturbances to real variables and prices (Phelps, 1970; Lucas, 1972; Woodford, 2002; Adam, 2007; and Gorodnichenko, 2008).[2]

The paper is related to Del Negro and Eusepi (2010), who perform an econometric evaluation of the extent to which the inflation expectations generated by DSGE models are in line with the observed inflation expectations. The main differences with this paper are as follows. First, in our setting, price setters have heterogeneous and dispersed higher-order expectations as they observe private signals (e.g., firm-specific productivity). Second, this paper fits the model to a data set that includes the 1970s, whereas Del Negro and Eusepi (2010) use a data set starting from the early 1980s.

The paper is organized as follows. Section 2 describes the incomplete-information model, in which monetary impulses propagate through the signal channel. In that section, we also

[2] See Mankiw and Reis (2002a, 2002b, 2006, 2007), and Reis (2006a, 2006b, 2009) for models with information frictions that do not feature imperfect common knowledge but can generate sizeable persistence.

describe a model in which firms have complete information. The latter model will be used as a benchmark to evaluate the empirical performance of the model with incomplete information. The empirical analysis of the paper is dealt with in Section 3. Section 4 concludes.

2 Models

Section 2.1 introduces the model that features the signal channel (i.e., the policy rate is a public signal conveying non-redundant information to price setters). In section 2.2, I present the time protocol of the model. Section 2.3 presents the problem of households. Section 2.4 presents the price-setting problem of firms, which have incomplete information. In Section 2.5, the central bank's and government's behavior is modeled. Section 2.6 deals with the log-linearization of the model equations. Section 2.7 briefly discusses how to solve DSGE models in which agents have higher-order uncertainty. Section 2.8 presents the perfect information model, which will turn out to be useful in evaluating the empirical significance of the signal channel. Finally, Section 2.9 sheds light on how the signal channel works.

2.1 The Incomplete-Information Model (IIM)

The economy is populated by a continuum $(0, 1)$ of monopolistically competitive firms, a continuum $(0, 1)$ of households, a central bank (or monetary authority), and a government (or fiscal authority). A Calvo lottery establishes which firms are allowed to re-optimize their prices in any given period t (Calvo, 1983). Those firms that are not allowed to re-optimize index their price to the steady-state inflation. The outcome of the Calvo lotteries is assumed to be common knowledge among agents. Households consume the goods produced by firms, demand money holdings and government bonds, pay taxes to or receive transfers from the fiscal authority, and supply labor to the firms in a perfectly competitive labor market. Firms sell differentiated goods to households. The fiscal authority has to finance maturing government bonds. The fiscal authority can collect lump-sum taxes and issue new government bonds. The central bank supplies money so as to set the interest rate at which the government's bonds pay out their return.

Aggregate and idiosyncratic shocks hit the model economy. The aggregate shocks are a technology shock, a monetary policy shock, and a demand shock. All of these shocks are orthogonal to each other at all leads and lags. Idiosyncratic shocks include a firm-specific technology shock that determines the level of technology for firm j at time t, and the outcome of the Calvo lottery for price optimization. The idiosyncratic technology shocks are correlated with the level of aggregate technology A_t.

2.2 The Time Protocol

Any period t is divided into three stages. All actions that are taken in any given stage are simultaneous. At stage 0 $(t, 0)$, shocks are realized and the central bank observes the realization of the aggregate shocks and sets the interest rate. At stage 1 $(t, 1)$, firms observe (i) their idiosyncratic productivity $A_{j,t}$, (ii) their signal about preference shocks $g_{j,t}$, and (iii) the interest rate set by the central bank and then set their price. At stage 2 $(t, 2)$, households learn the realization of all the shocks in the economy and decide their consumption, C_t, money holdings, M_t, demand for (one-period) nominal government bonds, B_t and labor supply, N_t. At this stage, firms hire labor and produce so as to deliver the demanded quantity $Y_{j,t}$ at the price they have set at stage 1. Then, markets clear.

2.3 Households

Households have perfect information and hence we can use the representative household to solve their problem:

$$\max_{C_{t+s},M_{t+s},B_{t+s},N_{t+s}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^{t+s} g_{t+s} \left[\ln C_{t+s} + \frac{\chi_m}{1-\gamma_m} \left(\frac{M_{t+s}}{P_{t+s}} \right)^{1-\gamma_m} - \chi_n N_{t+s} \right]$$

where β is the deterministic discount factor, g_t denotes a preference shifter that scales up or down the overall period utility, and P_t is the price level of the composite good consumed by households. The parameters γ_m, χ_m, and χ_n are parameters of the utility function. The preference shifter follows an AR process: $\ln g_t = \rho_g \ln g_{t-1} + \sigma_g \varepsilon_{g,t}$ with Gaussian shocks $\varepsilon_{g,t} \backsim \mathcal{N}(0, 1)$. These shocks play the role of demand shocks in the economy.

The flow budget constraint of the representative household in period t reads

$$P_t C_t + B_t + M_t = W_t N_t + R_{t-1} B_{t-1} + M_{t-1} + \Pi_t + T_t$$

where W_t is the (competitive) nominal wage rate, R_t stands for the nominal (gross) interest rate, Π_t are the (equally shared) dividends paid out by the firms, and T_t stands for government transfers. Composite consumption in period t is given by the Dixit-Stiglitz aggregator

$$C_t = \left(\int_0^1 C_{j,t}^{\frac{\nu-1}{\nu}} di \right)^{\frac{\nu}{\nu-1}},$$

where $C_{j,t}$ is consumption of a differentiated good j in period t.

At the stage 3 of every period t, the representative household chooses a consumption vector, labor supply, money holdings, and bond holdings subject to the sequence of the flow budget constraints and a no-Ponzi-scheme condition. The representative household takes as given the nominal interest rate, the nominal wage rate, nominal aggregate profits, nominal lump-sum taxes, and the prices of all consumption goods. It can be shown that the demand for the good produced by firm j is:

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\nu} Y_t \tag{1}$$

where the price level of the composite good is defined as

$$P_t = \left(\int (P_{j,t})^{1-\nu}\, di\right)^{\frac{1}{1-\nu}} \tag{2}$$

2.4 Firms' Price-Setting Problem

Firms are endowed with a linear technology:

$$Y_{j,t} = A_{j,t} N_{j,t} \tag{3}$$

where $N_{j,t}$ is the amount of labor employed by firm j at time t and $A_{j,t}$ is the technology shock that can be decomposed into a persistent aggregate component, A_t, and a white-noise firm-specific component, $\varepsilon^a_{j,t}$. More specifically, we have:

$$A_{j,t} = A_t e^{\widetilde{\sigma}_a \varepsilon^a_{j,t}} \tag{4}$$

with $\varepsilon^a_{j,t} \overset{iid}{\backsim} \mathcal{N}(0,1)$, and $A_t = \gamma^t a_t$ where $\gamma > 1$ is the linear trend of the aggregate technology and a_t is the de-trended level of aggregate technology that follows an AR process $\ln a_t = \rho_a \ln a_{t-1} + \sigma_a \varepsilon_{a,t}$ with Gaussian shocks $\varepsilon_{a,t} \overset{iid}{\backsim} \mathcal{N}(0,1)$.

Following Calvo (1983), we assume that a fraction θ of firms are not allowed to re-optimize their price in any given period. Those firms that are not allowed to re-optimize are assumed to index their prices to the steady-state inflation rate.

We assume that the firms that are allowed to re-optimize take their price-setting decisions based on incomplete knowledge about the history of shocks that have hit the economy. More specifically, it is assumed that firms' information set at stage 1 of time t (i.e., when prices are set) includes the history of firm-specific productivity, the history of the private signal on the preference shifter, the history of the nominal interest rate set by the central bank, and the history of the price set by the firm. To put it in symbols, the information set $\mathcal{I}_{j,t}$ of firm

j at stage 1 of time t is given by

$$\mathcal{I}_{j,t} \equiv \{A_{j,\tau}, g_{j,\tau}, R_{\tau}, P_{j,\tau} : \tau \leq t\} \tag{5}$$

where $g_{j,t}$ denotes the exogenous private signal concerning the preference shifter g_t. This signal is defined as follows:

$$g_{j,t} = g_t e^{\widetilde{\sigma}_g \varepsilon^g_{j,t}} \tag{6}$$

where $\varepsilon^g_{j,t} \overset{iid}{\backsim} \mathcal{N}(0,1)$. This signal is meant to capture the fact that firms arguably conduct some market analysis to gather information about demand before setting their price.[3] It is important to emphasize that if the noise variances are equal to zero (i.e., $\widetilde{\sigma}_a = 0$ and $\widetilde{\sigma}_g = 0$), firms are perfectly informed about all the model variables (also the ones that have yet to be realized at stage 1) when setting their price . Finally, firms are assumed to know the model transition equations and their structural parameters.

Let us denote the steady-state (gross) inflation rate as π_*, the nominal marginal costs for firm j as $MC_{j,t} = W_t/A_{j,t}$, the time t value of one unit of the composite consumption good in period $t+s$ to the representative household as $\Xi_{t|t+s}$, and the expectation operator conditional on firm j's information set $\mathcal{I}_{j,t}$ as $\mathbb{E}_{j,t}$. At stage 1, an arbitrary firm j that is allowed to re-optimize its price solves

$$\max_{P^*_{j,t}} \mathbb{E}_{j,t} \left[\sum_{s=0}^{\infty} (\beta\theta)^s \, \Xi_{t|t+s} \left(\pi^s_* P^*_{j,t} - MC_{j,t+s} \right) Y_{j,t+s} \right]$$

subject to the firm's specific demand in equation (1). When solving their price-setting problem, firms take as given the nominal wage rate W_t and the price for the composite good P_t that will clear the market at stage 2. Furthermore, firms commit themselves to satisfying any demanded quantity that will arise at stage 2 at the price they have set at the stage 1.

2.5 The Monetary and Fiscal Authority

There is a monetary authority and a fiscal authority. The monetary authority sets the nominal interest rate according to the reaction function

$$R_t = (r_* \pi_*) \left(\frac{\pi_t}{\pi_*} \right)^{\phi_\pi} \left(\frac{Y_t}{Y^*_t} \right)^{\phi_y} \eta_{r,t} \tag{7}$$

[3] Note that observing the history of their price $P_{j,t}$ does not convey any information about the state of the economy to firms because their price is either adjusted at the steady-state inflation rate or simply a function of the history of the signals they have observed.

where r_* is the steady-state real interest rate, π_t is the (gross) inflation rate, and Y_t^* is the potential output, that is, the output level that is realized if prices were perfectly flexible (i.e., $\theta = 0$). $\eta_{r,t}$ is a stochastic process that affects the nominal interest rate in period t. This process is assumed to follow an AR process: $\ln \eta_{r,t} = \rho_r \ln \eta_{r,t-1} + \sigma_r \varepsilon_{r,t}$, with Gaussian shocks $\varepsilon_{r,t} \overset{iid}{\backsim} \mathcal{N}(0,1)$. We refer to the innovation ε_{Rt} as a monetary policy shock. We choose to model the log of $\eta_{r,t}$ as a first-order autoregressive process. The alternative would have been to include a lagged nominal interest rate on the right-hand side of equation (7) and specify the $\ln \eta_{r,t}$ as a white noise process. The reason for our modeling choice is that we want to treat symmetrically the three exogenous state variables of the model (i.e., the state of technology a_t, the state of monetary policy $\eta_{r,t}$, and the preference shifter g_t), and therefore, we model each exogenous stochastic process as a first-order autoregression.

The central bank observes the contemporaneous realizations of aggregate shocks (i.e., $\varepsilon_{a,t}$, $\varepsilon_{r,t}$, and $\varepsilon_{g,t}$) in every period and sets the interest rate R_t. The central bank cannot simply tell firms the history of shocks since there is an incentive for the central bank to lie to firms to generate surprise inflation with the aim of pushing output growth above the trend γ.[4] Unexpected inflation raises output because some prices are sticky. This rise in output has benefits because producers have monopoly power and the unexpected inflation reduces the monopoly distortion. Since there is no commitment device that would back up the central bank's words, then any central bank's statements about real output, inflation, and shocks are not deemed as credible by price setters.

The flow budget constraint of the fiscal authority in period t reads

$$R_{t-1}B_{t-1} - B_t + M_{t-1} - M_t = T_t$$

The fiscal authority has to finance maturing government bonds. The monetary authority supplies money so as to set the nominal interest rate R_t, at which the government's bonds pay out their return, according to the equation (7). The fiscal authority can collect lump-sum taxes or issue new government bonds. We assume that the fiscal authority follows a Ricardian fiscal policy. Since there is no capital accumulation, the resource constraint implies $Y_t = C_t$.

[4] The fact that the central bank sets the interest rate before firms set their prices cannot be considered as a viable commitment device to communicate current inflation and output to firms. The reason is that the Taylor rule makes the interest rate depend on output and inflation only up to the shock $\eta_{r,t}$, which is not observed by firms.

2.6 Detrending and Log-linearization

First, I solve firms' and households' problems that are described in Sections 2.3 and 2.4 and obtain the consumption Euler equation and a price-setting equation. Second, I detrend the non-stationary variables before log-linearizing the model equations around their value at the non-stochastic steady-state equilibrium. Let us define the de-trended real output as $y_t \equiv Y_t/\gamma^t$. We denote the log-deviation of (stationary) variables from their steady-state value with $\widehat{\ }$. From the linearized price-setting equation, one can obtain an expression that resembles the New Keynesian Phillips curve, which is reported below (detailed derivations are in Appendix A).

$$\hat{\pi}_t = (1-\theta)(1-\beta\theta)\sum_{k=1}^{\infty}(1-\theta)^k \widehat{mc}_{t|t}^{(k)} + \beta\theta\sum_{k=1}^{\infty}(1-\theta)^k \widehat{\pi}_{t+1|t}^{(k)} \tag{8}$$

where $\widehat{\pi}_{t+1|t}^{(k)}$ denotes the average k-th order expectations about next period's inflation rate, $\widehat{\pi}_{t+1}$, that is, $\widehat{\pi}_{t+1|t}^{(k)} \equiv \underbrace{\int \mathbb{E}_{j,t} \ldots \int \mathbb{E}_{j,t}}_{k} \widehat{\pi}_{t+1}$, any integer $k>1$. $\widehat{mc}_{t|t}^{(k)}$ denotes the average k-th order expectations about the real aggregate marginal costs, $\widehat{mc}_t \equiv \int \widehat{mc}_{j,t}dj$, which evolve according to the equation:

$$\widehat{mc}_{t|t}^{(k)} = \hat{y}_{t|t}^{(k)} - \widehat{a}_{t|t}^{(k-1)} \tag{9}$$

any integer $k>1$. The log-linearized Euler equation is standard and reads

$$\widehat{g}_t - \widehat{y}_t = \mathbb{E}_t\widehat{g}_{t+1} - \mathbb{E}_t\widehat{y}_{t+1} - \mathbb{E}_t\hat{\pi}_{t+1} + \hat{R}_t \tag{10}$$

where $\mathbb{E}_t(\cdot)$ denotes the expectation operator conditional on the complete information set, which includes the history of the three aggregate shocks. The Taylor rule is also standard and boils down to

$$\hat{R}_t = \phi_\pi\hat{\pi}_t + \phi_y(\hat{y}_t - \widehat{y}_t^*) + \widehat{\eta}_{r,t} \tag{11}$$

The preference shifter evolves according to

$$\widehat{g}_t = \rho_g\widehat{g}_{t-1} + \sigma_g\varepsilon_{g,t} \tag{12}$$

The process for technology becomes

$$\widehat{a}_t = \rho_a\widehat{a}_{t-1} + \sigma_a\varepsilon_{a,t} \tag{13}$$

The process leading the state of monetary policy becomes

$$\widehat{\eta}_{r,t} = \rho_r \widehat{\eta}_{r,t-1} + \sigma_r \varepsilon_{r,t} \tag{14}$$

We de-trend and then log-linearize the signal equation concerning the aggregate level of technology (4) and obtain

$$\widehat{a}_{j,t} = \widehat{a}_t + \widetilde{\sigma}_a \varepsilon^a_{j,t} \tag{15}$$

We log-linearize the signal equation concerning the preference shifter (6):

$$\widehat{g}_{j,t} = \widehat{g}_t + \widetilde{\sigma}_g \varepsilon^g_{j,t} \tag{16}$$

The signal about monetary policy is given by the log-linearized Taylor rule (11). Firms use these equations to solve their signal extraction problem, which will be examined in the next section. The parameter set of the log-linearized incomplete-information model is given by the vector

$$\Theta_{IIM} = \left[\theta, \phi_\pi, \phi_y, \beta, \rho_g, \rho_a, \rho_r, \widetilde{\sigma}_g, \sigma_g, \sigma_a, \widetilde{\sigma}_a, \sigma_r, \gamma\right]'$$

2.7 Firms' Signal Extraction Problem and Model Solution

Firms need to form beliefs about the current realization and the future dynamics of their nominal marginal costs given the observable signals in their information set $\mathcal{I}_{j,t}$. Characterizing how firms form such beliefs requires solving a signal extraction problem. It is important to notice that firms need to form expectations on the price level to estimate nominal costs. Since the price level is the aggregate outcome of other firms' price setting, firms also need to form expectations about what other firms expect about nominal marginal costs and on what other firms expect that other firms expect and so on (i.e., the so-termed higher-order expectations).

To solve the model we assume common knowledge of rationality[5] and focus on equilibria where the higher-order expectations about the exogenous state variables, $\varphi^{(0:k)}_{t|t} \equiv \left[a^{(s)}_{t|t}, \eta^{(s)}_{r,t|t}, g^{(s)}_{t|t} : 0 \leq s \leq k\right]'$ follow a VAR(1) process:

$$\varphi^{(0:k)}_{t|t} = \mathbf{M}\varphi^{(0:k)}_{t-1|t-1} + \mathbf{N}\boldsymbol{\varepsilon}_t \tag{17}$$

where $\boldsymbol{\varepsilon}_t \equiv \left[\begin{array}{ccc} \varepsilon_{a,t} & \varepsilon_{r,t} & \varepsilon_{g,t} \end{array}\right]'$ and $\mathbf{M}$ and $\mathbf{N}$ are matrices that are not yet known. We assume that $\mathbf{M}$ is invertible. Note that we have truncated the order of the average expectations at

[5] See Nimark (2008), Assumption 1 p. 373 for a formal formulation of the assumption of common knowledge of rationality in this context.

$k < \infty$.[6] We found that the results of the paper would not change sensibly by keeping track of an additional order of average expectations. Furthermore, we guess the matrix $\mathbf{v}_0$ which determines the dynamics of the endogenous variables $\mathbf{s}_t \equiv \left[\widehat{y}_t, \widehat{\pi}_t, \widehat{R}_t\right]$:

$$\mathbf{s}_t = \mathbf{v}_0 \varphi_{t|t}^{(0:k)} \tag{18}$$

Given the guessed matrices $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{v}_0$, the structural equations (8)-(11) imply that (see Appendix B.2)

$$\Gamma_0 \mathbf{s}_t = C + \Gamma_1 \mathbb{E}_t \mathbf{s}_{t+1} + \Gamma_2 \varphi_{t|t}^{(0:k)} \tag{19}$$

For given parameters Θ, take the following steps:

Step 0 Set $i = 1$ and guess the matrices $\mathbf{M}^{(i)}$ $\mathbf{N}^{(i)}, \mathbf{v}_0^{(i)}$.

Step 1 Solve equations (17) and (19) with a standard linear rational expectations model solver (Blanchard and Kahn, 1980, Uhlig, 1998, and Sims 2002). The solver will deliver the matrices Λ_0, Λ_1, and Λ_2:

$$\begin{bmatrix} \varphi_{t|t}^{(0:k)} \\ \mathbf{s}_t \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \Lambda_0 \end{bmatrix} + \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \Lambda_1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \varphi_{t-1|t-1}^{(0:k)} \\ \mathbf{s}_{t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{N} \\ \Lambda_2 \end{bmatrix} \boldsymbol{\varepsilon}_t \tag{20}$$

Step 2 Use the matrices Λ_1 and Λ_2 to obtain the new matrix $\mathbf{v}_0^{(i+1)} = \Lambda_2 \mathbf{M}^{-1}$.[7]

Step 3 Given the law of motion (17) for $\varphi_{t|t}^{(0:k)}$, equation (15) for the signal concerning the aggregate technology, equation (16) for the signal concerning the preference shifter, and equation (20) for the monetary policy signal $\widehat{R}_t \in \mathbf{s}_t$, solve firms' signal extraction problem through the Kalman filter as shown in Appendix C. This delivers a new set of matrices $\mathbf{M}^{(i+1)}$ $\mathbf{N}^{(i+1)}$

Step 4 If $\left\|\mathbf{M}^{(i)} - \mathbf{M}^{(i+1)}\right\| < \varepsilon_m$, $\left\|\mathbf{N}^{(i)} - \mathbf{N}^{(i+1)}\right\| < \varepsilon_n$, and $\left\|\mathbf{v}^{(i)} - \mathbf{v}^{(i+1)}\right\| < \varepsilon_v$ with $\varepsilon_m > 0$, $\varepsilon_n > 0$, $\varepsilon_v > 0$ and small, STOP or else set i=i+1 and go to STEP 1.

2.8 The Model with Perfect Information (PIM)

If the noise variance of the private exogenous signals $\widetilde{\sigma}_a$ and $\widetilde{\sigma}_g$ is equal to zero, higher-order uncertainty would fade away (i.e., $\varphi_{t|t}^{(k)} = \varphi_t$, any integer k) and the model would boil down to

[6] When we estimate the model, we set $k = 10$. Nimark (2011) shows that for k bounded but sufficiently high, the approximation of the equilibrium dynamics is very accurate. I find that estimating the model with $\kappa = 11$ would lead to almost identical posterior distributions for model parameters.

[7] Note that there exists an additional over-identifying restriction: $\mathbf{v}_0 \mathbf{N} = \Lambda_3$. This condition can be shown to hold true if $\mathbf{v}_0 = \Lambda_2 \mathbf{M}^{-1}$.

a canonical three-equation New Keynesian model with Calvo sticky prices (e.g., Rotemberg and Woodford (1997), and Rabanal and Rubio-Ramirez, 2005). More specifically, the Phillips curve equation (8) would become $\hat{\pi}_t = \kappa_{pc}\widehat{mc}_t + \beta \mathbb{E}_t \widehat{\pi}_{t+1}$, where $\kappa_{pc} \equiv (1-\theta)(1-\theta\beta)/\theta$ and the real marginal costs $\widehat{mc}_t = \hat{y}_t - \widehat{a}_t$. The IS equation and the Taylor rule would be the same as in the incomplete information model. See equations (10) and (11). In this perfect information, model the monetary shock propagates by affecting the intertemporal allocation of consumption. The real effects of money solely emerge as a result of price-stickiness as opposed to the sluggish adjustments of firms' expectations. We call this canonical New Keynesian model the perfect information model.

The parameter set of the log-linearized perfect information model is given by the vector

$$\Theta_{PIM} = \left[\theta, \phi_\pi, \phi_y, \beta, \rho_g, \rho_a, \rho_r, \sigma_g, \sigma_a, \sigma_r, \gamma\right]'$$

2.9 The Signal Channel of Monetary Transmission

A salient feature of the imperfect information model is that the central bank can transfer information about output and inflation to price setters by setting its policy rate. We call this new transmission channel signal channel. Price setters use the policy rate as a signal that helps them to figure out the history of non-policy shocks (namely, the technology and preference shocks) and, at the same time, to infer potential exogenous deviations from the rule (i.e., $\eta_{r,t}$).

To shed light on how the signal channel works, we conduct a numerical experiment. For simplicity, we shut down the preference shock, that is, $\sigma_g = 0$, and set the value of the other parameters as indicated in Table 1.[8] Figure 1 shows the propagation of a monetary shock to inflation (top left plot) and to the one-quarter-ahead inflation expectations (top right plot). The bottom graphs report the response of the average expectations about the state of technology (bottom left plot) and about the state of monetary policy (bottom right plot) from order 0[9] up to the fourth order. The plots at the bottom show that average expectations about aggregate technology promptly fall into negative territory after a contractionary monetary shock. This means that the rise in the policy rate is mostly interpreted by firms as the central bank responding to a negative technology shock. Average expectations about a monetary shock respond only a little to a monetary shock. As a result, firms expect inflation to go up in the next quarter. See the top right plot of Figure 1.

The response of inflation is negative upon a contractionary monetary policy shock (see top

[8] These values are the ones we will use to center the prior distribution. See Section 3.2.

[9] Conventionally, the zero-order average expectation about a variable is the realization of the variable itself.

left plot). The colored bars in the top left plot report the effects of the change of the higher-order beliefs about technology on inflation (in blue) and those about the state of monetary policy (in red). To compute these effects, denote $\partial\pi_{t+h}/\partial a_{t|t}^{(0:k)}$ as the vector of responses of inflation to the average higher-order beliefs about the aggregate technology and $\partial a_{t|t}^{(0:k)}/\partial\varepsilon_{r,t}$ as the vector of responses of the average higher-order beliefs about the aggregate technology to a monetary shock. The former vector can be easily derived by equation (18), while the latter vector can be obtained by equation (17). Note that under perfect information, the vector $\partial a_{t|t}^{(0:k)}/\partial\varepsilon_{r,t}$ would be equal to a null vector. Thus, we can write:

$$\begin{aligned}\frac{\partial\pi_{t+h}}{\partial\varepsilon_{r,t}} &= \underbrace{\frac{\partial\pi_{t+h}}{\partial a_{t+h|t+h}^{(0:k)}}\frac{\partial a_{t+h|t+h}^{(0:k)}}{\partial\varepsilon_{r,t}}}_{\equiv \text{ effects of the HOEs about technology}} \\ &+ \underbrace{\frac{\partial\pi_{t+h}}{\partial\eta_{r,t|t}^{(0:k)}}\frac{\partial\eta_{r,t|t}^{(0:k)}}{\partial\varepsilon_{r,t}}}_{\equiv \text{ effects of the HOEs about monetary policy}}\end{aligned}$$

where h is the number of periods after the monetary shock. Note that if firms were perfectly informed about the nature of shocks that hit the economy in every period, the former effect would be equal to zero and the signal channel would be inactive. In this numerical example, the signal channel seems to play an important role by reducing the response in inflation in the aftermath of a contractionary monetary policy shock. The reason is that firms interpret the rise of the policy rate as the central bank's response to a negative technology shock. Such a concern dampens the disinflationary effects associated with a monetary tightening.

Figure 2 reports the case in which the signal-to-noise ratio $\sigma_a/\widetilde{\sigma}_a$ is equal to ten, implying that idiosyncratic productivity conveys very precise information about aggregate technology shocks. As a result, firms will use the policy signal to mainly learn about what they do not know: the monetary policy shocks. This situation boosts the effectiveness of a monetary tightening as reported by the top left plot in Figure 2. The contribution of the higher-order expectations about the process of monetary shocks is dominant, pushing the response of inflation down. This is because firms learn from their idiosyncratic productivity that it is unlikely that a technology shock has occurred. Consequently, firms will be more prone to interpret the rise in the policy rate as resulting from a negative monetary shock rather than a response of the central bank to an adverse technology shock. If firms have very precise information about the non-policy shocks, the signal channel is very weak.

Figures 3 and 4 deal with the cases in which firms have less and less precise information about the dynamics of aggregate technology: the signal-to-noise ratio $\sigma_a/\widetilde{\sigma}_a$ is set to be equal to 0.5 and 0.2, respectively. As firms become less and less informed about the state of aggregate technology, the signal channel has a stronger and stronger effect because firms use the policy signal to extract information about technology shocks. In both cases depicted in Figures 3 and 4, inflation responds positively to a monetary tightening. Note also that inflation expectations respond more and more positively to a contractionary monetary shock as the firms' lack of information is exacerbated. Two effects lead the signal channel to be so strong as to produce a positive response of inflation in the aftermath of a monetary contraction. First is the fact that firms are poorly informed about technology shocks. Second is the informative content of the policy rate. As far as the first effect, when the signal-to-noise ratio $\sigma_a/\widetilde{\sigma}_a$ is small, firms have only one reliable signal to learn about the state of technology: the policy signal. The second effect has to do with the variance decomposition of the Taylor rule. If the variability of the current interest rate (conditional on the past interest rate R_{t-1}) is mostly explained by the technology shock, then firms will mostly rely on the central bank's actions to learn about the state of technology. In the numerical examples we are studying about 91% of the variability of the policy rate stems from the aggregate technology shock.[10] Hence, firms can extract a lot of information about technology shocks from observing the interest rate. When firms have poor information about the technology shock and the policy rate is very informative about this shock, the signal channel is very powerful, leading inflation to respond positively to a contractionary monetary shock.

The more information about monetary shocks that firms are able to collect from observing the policy rate, the weaker the signal channel. Figure 5 shows what happens when firms are poorly informed about the process of aggregate technology a_t (i.e., $\sigma_a/\widetilde{\sigma}_a = 0.2$) and the state of monetary policy $\eta_{r,t}$ is $\sigma_r = 0.5$ (i.e., five times bigger than that in Table 1). The informative content of the policy rate about the monetary shock is now 65%, as opposed to 9% when $\sigma_r = 0.1$. The fact that the policy rate is much more informative about monetary shocks makes a monetary tightening more effective in reducing the inflation rate. The inflation rate goes down after a monetary contraction. The signal channel will have even weaker effects if firms are precisely informed about the process of aggregate technology (i.e., $\sigma_a/\widetilde{\sigma}_a$ is large). This result is shown in Figure 6.

The inflation risk associated with the technology shocks also changes the importance of the signal channel. One parameter that clearly influences the response of inflation after a negative technology shock is the policy parameter ϕ_π. The smaller ϕ_π is, the more accommodative the central bank is, and the more inflation rises after a negative technology shock. When the inflation coefficient of the Taylor rule is small, firms will be very concerned when

[10] Such a larger information flow about the process of aggregate technology from the policy signal is explained by the quite small size of the monetary shock, $\sigma_r = 0.10$.

they expect a technology shock because they know that the central bank will not effectively fight the inflationary consequences of such a shock. Thus, the signal channel dampens the fall in inflation or might even cause inflation to rise in the aftermath of a monetary tightening. Figure 7 shows that a more accommodative monetary policy (i.e., $\phi_\pi = 1$) substantially strengthens the distortions exerted by the signal channel upon the transmission of monetary impulses. This figure shows that inflation responds positively to a monetary tightening when $\phi_\pi = 1$.[11]

To sum up, the functioning of the signal channel is influenced by three factors: (*i*) The quality of private information about non-policy shocks (i.e., technology and demand shocks), (*ii*) the informative content of the policy rate R_t, and (*iii*) the inflation risk associated with a non-policy shock.

3 Empirical Analysis

This section contains the quantitative analysis of the model. I combine a prior distribution for the parameter set of the three models with their likelihood function and conduct Bayesian inference and evaluation.

In Section 3.1, I present the data set and the state-space model for the econometrician. In Section 3.2, I discuss the prior distribution for the model parameters. Section 3.3 presents the posterior distribution. In Section 3.4, we conduct an econometric evaluation of the imperfect information model and the signal channel for monetary transmission. Section 3.5 studies the impulse response functions of the observables (i.e., GDP growth rate, inflation, Federal Funds rate, and inflation expectations) to an unanticipated monetary shock. Section 3.6 deals with how the signal channel affects the propagation of non-policy shocks, such as the technology shock and the demand shock.

3.1 Econometrician's State-Space model

The data set includes five observable variables: U.S. GDP growth rate, U.S. inflation rate (from the GDP deflator), the federal funds rate, one-quarter-ahead inflation expectations, and four-quarters-ahead inflation expectations. The last two observables are obtained from

[11] Changing the inflation coefficient also affects the informative content of the policy rate. The variance decomposition of the Taylor rule reveals that a fall in the inflation coefficient ϕ_π from two to one raises the information content of the policy rate about technology by about 4%. The reason is that weak responses to inflation tend to raise the variability of inflation in the aftermath of technology shocks. Nonetheless, the impact of more accommodative policy upon the informative content of the policy rate seems to be small - at least in this numerical example. The response of inflation thus seems to be mainly driven by the higher risk of inflation associated with technology shocks, as described in the main text.

the *Survey of Professional Forecasters* (SPF). A detailed description of the data set is provided in Table 2. The data set ranges from 1970:3 to 2008:4. The measurement equations are:[12]

$$\left[\ln\left(\frac{GDP_t}{POP_t^{\geq 16}}\right)-\ln\left(\frac{GDP_{t-1}}{POP_{t-1}^{\geq 16}}\right)\right]\cdot 100=100\ln\gamma+\hat{y}_t-\hat{y}_{t-1}$$

$$100\ln\frac{PGDP_t}{PGDP_{t-1}}=100\ln\pi_*+\hat{\pi}_t$$

$$100\cdot FEDRATE_t=\hat{R}_t+100\ln R_*$$

$$\ln\left(\frac{PGDP3_t}{PGDP2_t}\right)100=\mathbf{1}_1^T\left[\mathbf{v}_0\mathbf{T}^{(1)}\mathbf{M}\varphi_{t|t}^{(0:k)}\right]+100\ln\pi_*+\sigma_{m_1}\varepsilon_t^{m_1}$$

$$\ln\left(\frac{PGDP6_t}{PGDP2_t}\right)25=\mathbf{1}_1^T\left(\mathbf{v}_0\mathbf{T}^{(1)}\mathbf{M}^4\varphi_{t|t}^{(0:k)}\right)+100\ln\pi_*+\sigma_{m_2}\varepsilon_t^{m_2}$$

where $PGDP2_t$, $PGDP3_t$, $PGDP6_t$ are the *Survey of Professional Forecasters*' forecasts of the current, one-quarter-ahead, and four-quarters-ahead GDP price index and the truncation matrix

$$\mathbf{T}^{(s)}\equiv\begin{bmatrix}\mathbf{0}_{3(k-s+1)\times 3s} & \mathbf{I}_{3(k-s+1)}\\ \mathbf{0}_{3s\times 3s} & \mathbf{0}_{3s\times(k+1-s)3}\end{bmatrix}$$

The vectors $\varphi_t^{(0:k)}$, $\mathbf{s}_t$ and the matrix $\mathbf{v}_0$ have been defined in Section 2.7. The vector $\mathbf{1}_1^T$ is a (3×1) column vector whose elements are all equal to zero but the first one.

We relate these statistics with the first moment of the distribution of firms' expectations implied by the model. To avoid stochastic singularity, we introduce two i.i.d. Gaussian measurement errors $\varepsilon_t^{m_1}$ and $\varepsilon_t^{m_2}$. Furthermore, these errors are meant to capture the difference between the observed expectations, which are the mean of the interviewed professional forecasters' inflation expectations, and their model concepts, $\hat{\pi}_{t+1|t}^{(1)}$ and $\hat{\pi}_{t+4|t}^{(1)}$.

3.2 Priors

The prior medians and the 95% credible intervals are reported in Table 3. In the steady state the discount factor β depends on the linear trend of real output γ and the steady-state real interest rate R_*/π_*. Hence, I fix the value for this parameter so that the steady-state nominal interest rate R_* matches the sample mean of the $FEDRATE_t$ in the sample.

Note that the prior medians for the variance of the idiosyncratic productivity $a_{j,t}$ and that of the private signal concerning the preference shifter are set so that the model can match the cross-sectional variance of the expectations about current inflation and output as reported in the *Survey of Professional Forecasters*. The prior for the standard deviation of

[12] Note that the standard deviations of shocks are rescaled by a factor of 100.

technology shock, σ_a, is centered at 0.70, which is consistent with the real business cycle literature (e.g., Kydland and Prescott, 1982).

The prior distribution puts a probability mass to values for the Calvo parameter θ, implying that firms adjust their prices about every three quarters. This belief is derived from micro studies on price setting (Bils and Klenow, 2004, Klenow and Kryvtsov, 2008, Nakamura and Steinsson, 2008, and Klenow and Malin, 2010).

The priors for the autoregressive parameters ρ_a, ρ_r, and ρ_g reflect the belief that the corresponding exogenous processes may exhibit sizeable persistence as is usually observed in the macroeconomic data. Nonetheless, these priors are broad enough to accommodate a wide range of persistence degrees for these exogenous processes.

Priors for the parameters of the Taylor equation (i.e., response to inflation, ϕ_π, response to economic activity, ϕ_y, autoregressive parameter, ρ_r, and the standard deviation of the i.i.d. monetary shock, σ_r) are chosen as follows. The priors for ϕ_π and ϕ_y are centered at 2.00 and 0.25, respectively, and imply a fairly strong response to inflation and the output gap.

The volatility of the monetary policy shock, σ_r, and the demand shock, σ_g is informally taken according to the rule proposed by Del Negro and Schorfheide (2008) that the overall variance of endogenous variables is roughly close to that observed in the pre-sample, ranging from 1960:1 to 1970:2. The prior median for the measurement errors (i.e., σ_{m1}, σ_{m2}) is set so as to match the variance of inflation expectations reported in the *Livingston Survey*.

3.3 Posteriors

A closed-form expression for the posterior distribution is not available (Fernandez-Villaverde and Rubio-Ramirez, 2004), but we can approximate the moments of the posterior distributions via the Metropolis-Hastings algorithm. We obtain $100,000$ posterior draws. The posterior moments for the parameters of the imperfect information model (IIM) and the perfect information model (PIM) are reported in Table 4. The posterior median for the Calvo parameter θ implies quite flexible price contracts, which is in line with what is found in micro studies (Bils and Klenow, 2004, Klenow and Kryvtsov, 2008, Nakamura and Steinsson, 2008, and Klenow and Malin, 2010). The posterior median for the autoregressive parameters ρ_a and ρ_g is larger than what is conjectured in the prior. In particular, the autoregressive parameter of technology is close to unity, suggesting that the process of technology is almost a unit root.

The posterior median for the variance of firm-specific technology shock $\widetilde{\sigma}_a$ implies that the signal-to-noise ratio $\sigma_a/\widetilde{\sigma}_a$ is very close to unity. The posterior median for the noise variance $\widetilde{\sigma}_g$ implies a signal-to-noise ratio that is larger than unity. As discussed in Section 2.9, these noise variances are important as they strongly affect how firms interpret changes in the interest rate and hence the signal channel of monetary transmission.

The posterior median for the inflation coefficient of the Taylor rule, ϕ_π, is substantially smaller than its prior median. In Section 2.9 we show that this policy parameter affects the inflationary effects of non-policy shocks and hence plays an important role in shaping the signal channel of monetary transmission. In particular, when firms have imprecise private information about non-policy shocks (i.e., signal-to-noise ratios $\sigma_a/\widetilde{\sigma}_a$ and $\sigma_g/\widetilde{\sigma}_g$ are small), accommodative monetary policy may even lead to a rise in inflation after a monetary tightening.

The posterior median for the variance of the monetary shock σ_r is bigger than that conjectured in the prior by a factor of six. As observed in Section 2.9, this parameter affects how the signal channel works by influencing the informative content of the policy rate.

3.4 Model Evaluation

In Section 3.4.1, we investigate the imperfect information model's ability to fit the data relative to the perfect information model. Since the signal channel naturally emerges in a model with dispersed information in which the policy rate can be perfectly observed, evidence that the imperfect information model is at odds with the data would question the importance of the signal channel. In Section 3.4.2, we assess how well the imperfect information model fares at fitting the observed inflation expectations relative to the perfect information model. Since the signal channel imposes tight restrictions on how the policy rate influences average inflation expectations (see Figures 1-7), this exercise is very informative about whether the channel is empirically relevant. Furthermore, Del Negro and Eusepi (2010) show that including the SPF in the data set tends to sharpen the prediction of model selection tests.

3.4.1 Marginal Data Density Comparison

Bayesian tests rely on computing the marginal data density (MDD). The marginal data density is needed for updating prior probabilities over a given model space. Denote the data set, presented in Section 3.1, as Y. The MDD associated with the IIM is defined as $P(Y|\mathcal{M}_{IIM}) = \int \mathcal{L}(Y|\Theta_{IIM})\, p(\Theta_{IIM})\, d\Theta_{IIM}$, where $\mathcal{L}(Y|\Theta_{IIM})$ denotes the likelihood function derived from the model and $p(\Theta_{IIM})$ is the prior density, whose moments are described in Section 3.2.

A Bayesian test of the null hypothesis that the imperfect information model is at odds with the data can be performed by comparing the MDDs associated with this model ($\mathcal{M}_{IIM}$)

and the perfect information model ($\mathcal{M}_{PIM}$). Under a $0-1$ loss function the test rejects the null that the imperfect information model is at odds with the data, if the imperfect information model has a larger posterior probability than the alternative model, namely, the perfect information model (Schorfheide, 2000). The posterior probability of the model $\mathcal{M}_s$, where $s \in \{IIM, PIM\}$, is given by:

$$\pi_{T,\mathcal{M}_s} = \frac{\pi_{0,\mathcal{M}_s} \cdot P\left(Y|\mathcal{M}_s\right)}{\sum_{s\in\{IIM,PIM\}} \pi_{0,\mathcal{M}_s} \cdot P\left(Y|\mathcal{M}_s\right)} \tag{21}$$

where $\pi_{0,\mathcal{M}_s}$ stands for the prior probability of the model $\mathcal{M}_s$. $P\left(Y|\mathcal{M}_s\right)$ is the MDD associated with the model $\mathcal{M}_s$. We use Geweke's harmonic mean estimator (Geweke, 1999) to estimate the marginal data density. Table 5 shows that the imperfect information model attains a larger posterior probability and hence the null can be rejected. The null hypothesis can be rejected unless the prior probability in favor of the incomplete information model (i.e., $\pi_{0,\mathcal{M}_{IIM}}$) is as small as $8.5E-7$. Such a low prior probability suggests that only if one has extremely strong a-priori information against the imperfect information model, one can conclude that the null cannot be rejected.

3.4.2 Predicting the Observed Inflation Expectations

The top plot in Figure 8 reports the four-quarters-ahead inflation expectations predicted by the imperfect information model estimated without including the observed inflation expectations. This predicted path is compared with the observed inflation expectations obtained from the SPF. Analogously, the bottom plot reports the predictive path from the perfect information model and compares it with the SPF. Figure 9 does the same for the one-quarter-ahead inflation expectations. These plots shed light on how well the imperfect information model and the perfect information model fit the observed inflation expectation. Since the signal channel relies on affecting inflation expectations, it is very important to assess whether the imperfect information model delivers empirically consistent predictions for the inflation expectations. While it is hard to establish a winner by visual inspection of the figures, it seems that the imperfect information model produces much smoother inflation expectations than the perfect information model. Data on inflation expectations are quite smooth, favoring the dynamics implied by the imperfect information model.

A synthetic measure of models' ability to fit the observed inflation expectations is the root mean square errors associated with the models' predictions. This statistic is reported in Table 6 for the incomplete information model (IIM) and the perfect information model (PIM). The table considers both the full sample and the first part of the sample, which has been characterized by the largest volatility of the observed inflation expectations. In both samples and for both measures of inflation expectations, the incomplete information model does better than the perfect information model at fitting the observed inflation expectation.

3.5 Propagation of Monetary Shocks

Figure 10 shows the impulse response functions (and their 95% posterior credible sets in gray) of GDP growth rate, inflation rate, interest rate, one-quarter-ahead inflation expectations, and four-quarters-ahead inflation expectations to a 25-basis-point rise in the interest rate. The responses are reported as deviations from the balanced-growth path in units of percentage points of annualized rates. Two features of these impulse response functions have to be emphasized. First, four-quarters-ahead inflation expectations respond positively to a monetary policy shock. Second, inflation and especially inflation expectations seem to react very sluggishly to shocks even though the estimated average duration of the price contract is very short. Both these facts have to do with the presence of the signal channel.

To understand how the signal channel affects the response of the five observables to a monetary shock, we report the response of the higher-order expectations about the three exogenous state variables (i.e., the aggregate technology a_t, the state of monetary policy $\eta_{r,t}$, and the preference shifter g_t) in Figure 11. Average first-order expectations about aggregate technology go down only moderately (by around 27% of the posterior median of σ_a). Average first-order expectations about the preference shifter rise by 50% of the posterior median of σ_g. The latter is quite a substantial deviation from the zero level, which can be explained by the fact that firms have relatively less precise information about the preference shock. The posterior medians for the signal-to-noise ratios $\sigma_a/\widetilde{\sigma}_a$ and $\sigma_g/\widetilde{\sigma}_g$ are 0.96 and 0.65, respectively. Furthermore, the response of average expectations about the preference shifter is not surprising given the variance decomposition of the Taylor rule reported in Table 7.

Figure 11 reports the decomposition of the response of inflation to a monetary shock into the effect of average higher-order expectations about monetary policy, the aggregate technology, and the preference shifter (see the top left plot). Using the notation introduced in Section 2.9, this decomposition reads

$$
\begin{aligned}
\frac{\partial \pi_{t+h}}{\partial \varepsilon_{r,t}} \quad = \quad & \underbrace{\frac{\partial \pi_{t+h}}{\partial a^{(0:k)}_{t+h|t+h}} \frac{\partial a^{(0:k)}_{t+h|t+h}}{\partial \varepsilon_{r,t}}}_{\equiv \text{ effects of the HOEs about technology}} \\
+ \quad & \underbrace{\frac{\partial \pi_{t+h}}{\partial \eta^{(0:k)}_{r,t+h|t+h}} \frac{\partial \eta^{(0:k)}_{r,t+h|t+h}}{\partial \varepsilon_{r,t}}}_{\equiv \text{ effects of the HOEs about monetary policy}} \\
+ \quad & \underbrace{\frac{\partial \pi_{t+h}}{\partial g^{(0:k)}_{t+h|t+h}} \frac{\partial g^{(0:k)}_{t+h|t+h}}{\partial \varepsilon_{r,t}}}_{\equiv \text{ effects of the HOEs about the preference shifter}} \qquad (22)
\end{aligned}
$$

for $h = 0, 1, ..., 20$. Note that under perfect information (i.e.,$\widetilde{\sigma}_a = \widetilde{\sigma}_g = 0$), then

$$
\frac{\partial a^{(0:k)}_{t+h|t+h}}{\partial \varepsilon_{r,t}} = \frac{\partial g^{(0:k)}_{t+h|t+h}}{\partial \varepsilon_{r,t}} = 0, \text{ all } k \text{ and } h.
$$

and the signal channel would be inactive. This means that the effect of the higher-order expectations (HOEs) about the state of technology and that about the preference shifter on inflation is equal to zero in the aftermath of a monetary shock. In contrast, under incomplete information, a large component associated with the HOEs about the preference shifter can be interpreted as a situation in which price setters mistakenly believe that the interest rate has changed as a result of a preference shock, which has inflationary effects.

Figure 11 sheds light on the distortion introduced by the signal channel. The rise in the interest rate causes the four-quarters-ahead inflation expectations to rise because firms interpret this monetary policy action as the response of the central bank to a positive demand shock. Such an interpretation of the rise in the interest rate tends to produce inflationary pressures, which are captured by the red bars lying in positive territory. Such inflationary pressures have very important consequences for the effectiveness of monetary policy aimed at stabilizing output and inflation in the face of short-run disturbances, such as technology and demand shocks. We will focus on this interesting issue in the next section.

A quite striking feature of the top left plot of Figure 11 is that expecting an adverse technology shock implies disinflationary consequences. These consequences are quite small but the sign seems to be wrong. Nevertheless, this result is correct and actually very illustrative of how the imperfect information model works. When price setters observe a rise in the policy rate, they believe with a certain (small) probability that the central bank is responding to a negative technology shock. This is captured by a negative sign for the gradient vector $\frac{\partial a_{t|t}^{(0:k)}}{\partial \varepsilon_{r,t}}$. Yet, if the inflation coefficient ϕ_π is sufficiently large, firms are confident of the central bank's ability to control the inflationary consequences of the expected negative technology shock. Thus, the effect of the HOEs about technology is overall $\frac{\partial \pi_t}{\partial a_{t|t}^{(0:k)}} \frac{\partial a_{t|t}^{(0:k)}}{\partial \varepsilon_{r,t}} < 0$. As a result, no inflation comes from the response of the higher-order beliefs about technology to a monetary shock. Yet, this result does not always hold. In fact, it can be shown that a more accommodative monetary policy would revert the sign of the element in the vector $\partial \pi_t / \partial a_{t|t}^{(0:k)}$. When the central bank adjusts the policy rate less than one for one with the inflation rate (while the remaining parameters are set to equal the posterior medians in Table 4), expecting a negative technology shock can be shown to contribute to raising inflation rate in the model.[13]

To sum up, the data suggest that the signal channel has the effect of mitigating the fall in the inflation rate and pushing inflation expectations up in the aftermath of a monetary tightening. The reason is that firms tend to attach a non-negligible probability that the central bank has adjusted the policy rate to react to a demand shock.

3.6 Propagation of Non-Policy Shocks

Figure 12 shows the response of GDP growth rate, inflation rate, interest rate, one-quarter-ahead inflation expectations, and four-quarter-ahead inflation expectations to a one-standard-deviation technology shock. Figure 13 plots the decomposition of the response of inflation to the technology shock (top left plot) and the response of average higher-order expectations about the three exogenous state variables (i.e., a_t, $\eta_{r,t}$, and g_t) to the technology shock. The stars in the top left graph denote the response of inflation when the signal channel is inactive, that is, when firms cannot observe the policy rate set by the central bank. The stars in the top right plot denote the response of the average first-order expectations about the aggregate technology that would arise if the signal channel were inactive.

Let us focus, first, on the response of the average higher-order expectations (i.e., the top right plot and the bottom plots). The signal channel has two effects: First, it gives more information to firms about the size of the aggregate technology shock. Second, observing the policy rate also conveys information about the state of monetary policy and the preference shifter. As far as the first effect is concerned, in the top right plot of Figure 13 observe

[13] In the IIM the determinacy region is wider than that of the perfect-information model. Inflation coefficients ϕ_π that are smaller than one may well ensure determinacy. This result has to do with the fact that the imperfect-common-knowledge Phillips curve (8) includes higher-order beliefs about real marginal costs and inflation. Heuristically, this type of expectation is quite more sluggish in adjusting to shocks than the perfect-information rational expectations operator appearing in the standard New Keynesian Phillips curve.

that the response of the average first-order expectations is closer to the actual level of the aggregate technology (i.e., order 0) than the stars, which denote the response of average first-order expectations when the signal channel is inactive. This suggests that firms attain somewhat better information about the technology by observing the policy signal. From the perspective of a central bank that wants to stabilize inflation, the fact that average expectations about aggregate technology respond more promptly to a technology shock is clearly bad news. As far as the second effect is concerned, the signal channel confuses firms about shocks that have not occurred. More specifically, observing the rise of the interest rate in response to an unobserved aggregate technology shock leads firms to believe that a contractionary monetary policy shock or a positive preference shock might have occurred. It is worthwhile emphasizing that if the policy signal were not observed by firms, average expectations about the state of monetary policy and about the preference shifter would not move. In contrast, when the signal channel is active, the bottom plots of Figure 13 show that average expectations about the state of monetary policy and the preference shifter respond to technology shocks. If firms become persuaded that a contractionary monetary policy shock has occurred, then the confusion generated by the signal channel would be a good thing from the perspective of a central bank that wants to limit the response of inflation to technology shocks. Firms' inflation expectations would go down and, hence, the technology shock would have a smaller impact on inflation. However, if the monetary tightening led firms to believe that a positive demand shock has hit the economy, the opposite de-stabilizing effect would prevail. Firms' inflation expectations would take off and inflation would go up.

The top left plot in Figure 13 shows that the response of average expectations about the state of monetary policy and that of the preference shifter contribute to the adjustment of inflation by similar amounts. Thus, the two effects of the signal channel on inflation basically cancel each other out. Comparing the solid line with the stars in the top left plot reveals that the response of inflation to a technology shock is very similar regardless of the signal channel being active or inactive. This implies that the signal channel has basically very limited effects on the ability to stabilize inflation in the aftermath of a technology shock. Upon the occurrence of the shock, the signal channel contributes to reducing the response of inflation only by a factor of 1.51.

The propagation of a preference shock is described in Figure 14. The response of inflation and inflation expectations is hump-shaped. Figure 15 shows that the signal channel gives rise to such a pattern. As we discussed for the technology shock, there are three effects. First, the signal channel provides better information about the preference shock that has hit the economy (see bottom right plot of Figure 15). Second, the signal channel confuses firms, inducing them to believe that a contractionary monetary shock has prompted the central bank to raise the policy rate. Third, the signal channel also leads firms to believe that a negative technology shock might be the reason behind the observed rise in the interest rate. As discussed above for the technology shock, the first and the third effects amplify the response of inflation to a preference shock and hence increase the de-stabilizing extent of such a shock. In contrast, the second effect pushes inflation expectations down and hence limit the adjustment of inflation after a positive preference shock. The top left plot of Figure 15 tells us which effect prevails: the second effect. While the third effect (captured by the blue bars) has a very limited impact on inflation, the second effect (captured by the green bars) seems to substantially contribute to pushing inflation down. Comparing the solid line with the stars in the top left plot reveals that the response of inflation to a technology shock is much less pronounced when the signal channel is active. This implies that the signal channel enhances the central bank's ability to stabilize inflation in the aftermath of a preference shock. Upon the occurrence of the shock, the signal channel contributes to reducing the response of inflation by a factor of 5.22.

4 Concluding Remarks

The paper studies a DSGE model in which incompletely informed price setters use the interest rate set by the central bank to infer the nature of shocks that have hit the economy. Since there are a coordination motive in price setting and price setters observe private signals, the model features dispersed information and higher-order uncertainty. In this model, monetary impulses propagate through two channels: the traditional New Keynesian channel based on price stickiness and the signal channel. The latter arises because changing the policy rate conveys non-redundant information about inflation and the output gap to price setters.

First, the paper fits the model to a data set that includes the *Survey of Professional Forecasters* (SPF) as a measure of price setters' inflation expectations. Second, the paper performs a formal econometric evaluation of this new transmission channel and finds that it fits the data better than a canonical New Keynesian model with perfect information. In particular, the model with the signal channel seems to predict well the inflation expectations measured by the SPF. This provides an important empirical validation for the signal channel as it imposes tight restrictions on inflation expectations in the imperfect information model.

After having established the empirical importance of the new channel, the paper turns to studying how the signal channel works and affects the effectiveness of monetary policy aimed at stabilizing inflation. We find that firms interpret an interest rate rise as the central bank's response to either a positive demand shock or a contractionary monetary shock. Firms, however, do not sensibly change their expectations about aggregate technology shocks when they observe a monetary tightening. The paper shows that this finding implies that the Federal Reserve has limited ability to counter the inflationary consequences of technology shocks. In contrast, the signal channel turns out to improve the effectiveness of monetary policy stabilization in the face of demand shocks.

In the model, the central bank communicates with price setters only by setting the policy rate. In other words, the central bank is not allowed to vocally communicate the state of the economy to price setters. On theoretical grounds, this feature of the model is justified because the central bank has an incentive to lie and to create surprise inflation in order to raise output and reduce the monopolistic distortion. Consequently, any announcement made by the central bank will not be regarded as truthful by price setters unless a credible commitment device is in place. On practical grounds, however, it is well known that market participants react (and sometimes over-react) to the central bank's announcements. Empirically assessing how the central bank's communication affects the transmission mechanism of monetary impulses is beyond the scope of this paper and is left for future research.

Finally, the paper relies on a number of assumptions that have been made to improve the model's tractability. Model tractability is essential for conducting reliable econometric inference. For instance, the paper does not study how households' beliefs adjust to new information coming from the central bank. Estimating a DSGE model where both households and firms have incomplete information is a fascinating topic for future research.

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Tables and Figures

Table 1: Baseline Calibration for the Numerical Exercise

Name	Value	Name	Value
θ	0.65	ρ_a	0.85
ϕ_π	2.00	σ_a	0.70
ϕ_y	0.50	σ_r	0.10
ρ_r	0.50		

Table 2: Observables

Variables	**Description**	**Source**
GDP_t	Gross Domestic Product - Quarterly	BEA (GDPC96)
$POP_t^{\geq 16}$	Civilian noninstitutional population - 16 years and over	BLS (LNS10000000)
$PGDP_t$	Consumer Price Index - Averages of Monthly Figures	BLS (CPIAUCSL)
$FEDRATE_t$	Effective Federal Funds Rate - Averages of Daily Figures	Board of Governors (FEDFUNDS)
$PGDP2_t$	Mean of Expections of current GDP price index	SPF in mean.xls (PGDP2)
$PGDP3_t$	Mean of Expections of one-quarter-ahead GDP price index	SPF in mean.xls (PGDP3)
$PGDP6_t$	Mean of Expectations of one-year-ahead GDP price index	SPF in mean.xls (PGDP6)

Table 3: Prior Distributions

Name	Support	Density	Median	95% Credible Set
θ	$[0, 1]$	Beta	0.65	$(0.28, 0.99)$
ϕ_π	$\mathbb{R}+$	Gamma	2.0	$(1.61, 2.40)$
ϕ_y	$\mathbb{R}+$	Gamma	0.25	$(0.00, 0.65)$
ρ_r	$[0, 1]$	Beta	0.50	$(0.15, 0.90)$
ρ_a	$[0, 1]$	Beta	0.85	$(0.30, 0.99)$
ρ_g	$[0, 1]$	Beta	0.50	$(0.15, 0.90)$
σ_a	$\mathbb{R}+$	InvGamma	0.70	$(0.35, 1.70)$
$\widetilde{\sigma}_a$	$\mathbb{R}+$	InvGamma	1.40	$(0.95, 2.20)$
σ_g	$\mathbb{R}+$	InvGamma	1.00	$(0.50, 2.40)$
$\widetilde{\sigma}_g$	$\mathbb{R}+$	InvGamma	1.00	$(0.67, 1.55)$
σ_r	$\mathbb{R}+$	InvGamma	0.10	$(0.05, 0.85)$
σ_{m_1}	$\mathbb{R}+$	InvGamma	0.45	$(0.22, 1.10)$
σ_{m_2}	$\mathbb{R}+$	InvGamma	0.45	$(0.22, 1.10)$
$\ln \gamma$	$\mathbb{R}$	Normal	0.00	$(-0.20, 0.20)$
$\ln \pi_*$	$\mathbb{R}$	Normal	0.00	$(-0.20, 0.20)$

Table 4: Posterior Distributions

Name	IIM			PIM		
		95% Interval			95% Interval	
	Median	Lower	Upper	Median	Lower	Upper
θ	0.46	0.39	0.53	0.61	0.57	0.65
ϕ_π	1.07	1.03	1.12	1.33	1.22	1.45
ϕ_y	0.25	0.17	0.33	0.24	0.16	0.35
ρ_r	0.71	0.66	0.75	0.49	0.42	0.55
ρ_a	0.99	0.98	0.99	0.98	0.97	0.99
ρ_g	0.77	0.74	0.80	0.84	0.81	0.87
σ_a	1.10	0.94	1.26	1.03	0.92	1.16
$\widetilde{\sigma}_a$	1.14	0.90	1.40	NA	NA	NA
σ_g	1.21	1.05	1.31	0.81	0.67	0.95
$\widetilde{\sigma}_g$	1.57	0.94	2.52	NA	NA	NA
σ_r	0.61	0.50	0.70	0.57	0.50	0.65
σ_{m_1}	0.16	0.15	0.19	0.19	0.17	0.22
σ_{m_2}	0.16	0.14	0.18	0.18	0.16	0.21
$100\ln\gamma$	0.32	0.28	0.35	0.31	0.26	0.34
$100\ln\pi_*$	0.80	0.62	0.99	0.81	0.59	1.01

Table 5: Marginal-Data-Density Comparisons

	$\mathcal{M}_{IIM}$	$\mathcal{M}_{PIM}$
MDD	-252.3	-266.3

Table 6: Forecasting Performance of the Smoothed Estimates

	RMSEs			
	1Q-ahead SPF		4Q-ahead SPF	
	IIM	PIM	IIM	PIM
1970:3-1986:4	1.18	1.49	1.25	1.75
Full Sample	0.90	1.18	0.97	1.34

Note: The table provides the root mean squared errors (RMSEs) for the smoothed estimates of the one-quarter-ahead inflation expectations

Table 7: Informative Content of the Public Signal at the Posterior Medians of the IIM

	$\varepsilon_{a,t}$	$\eta_{r,t}$	$\varepsilon_{g,t}$
Informative content of R_t	63.00%	8.44%	28.56%

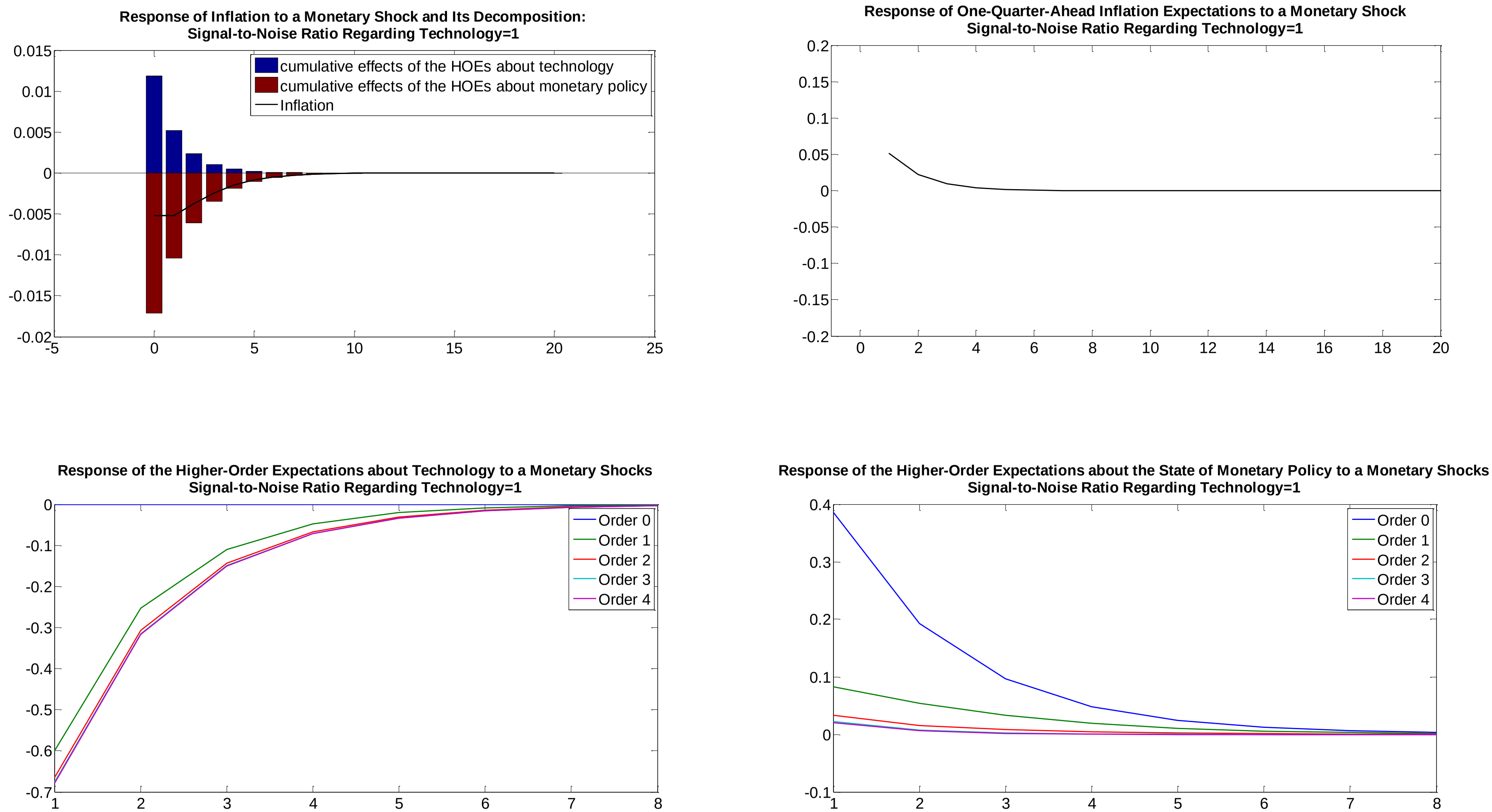


Figure 1: The top left plot reports the response of inflation to a monetary shock. The top right plot reports the response of one-quarter-ahead inflation expectations to a monetary shock. The bottom left panel reports the response of the average expectations about technology from order 0 to 4 to a monetary shock. The bottom right panel reports the response of the average expectations about the state of monetary policy from order 0 to 4 to a monetary shock. The monetary shock is normalized so that it produces a 25-basis-point rise in the interest rate.

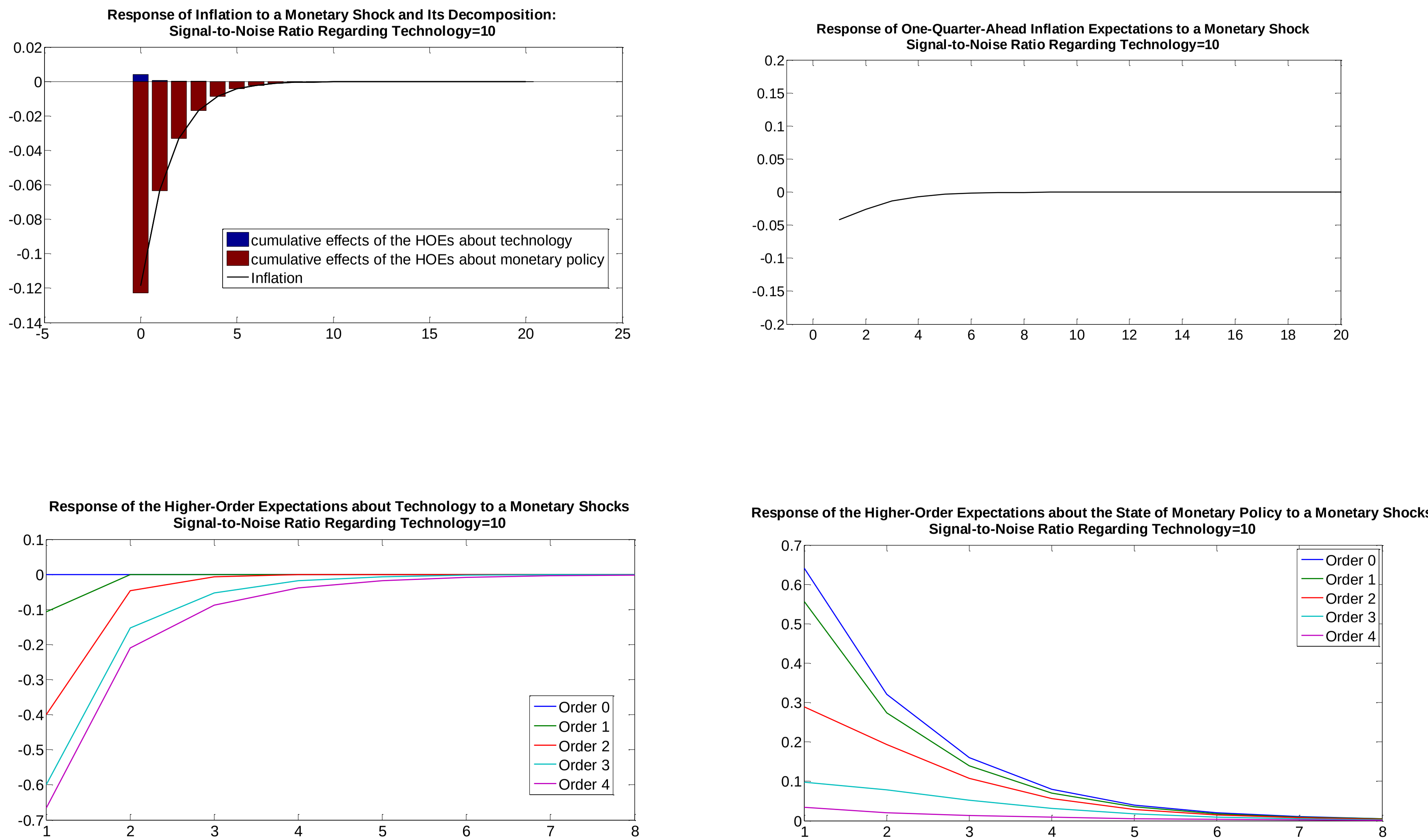


Figure 2: The top left plot reports the response of inflation to a monetary shock. The top right plot reports the response of one-quarter-ahead inflation expectations to a monetary shock. The bottom left panel reports the response of the average expectations about technology from order 0 to 4 to a monetary shock. The bottom right panel reports the response of the average expectations about the state of monetary policy from order 0 to 4 to a monetary shock. The monetary shock is normalized so that it produces a 25-basis-point rise in the interest rate.

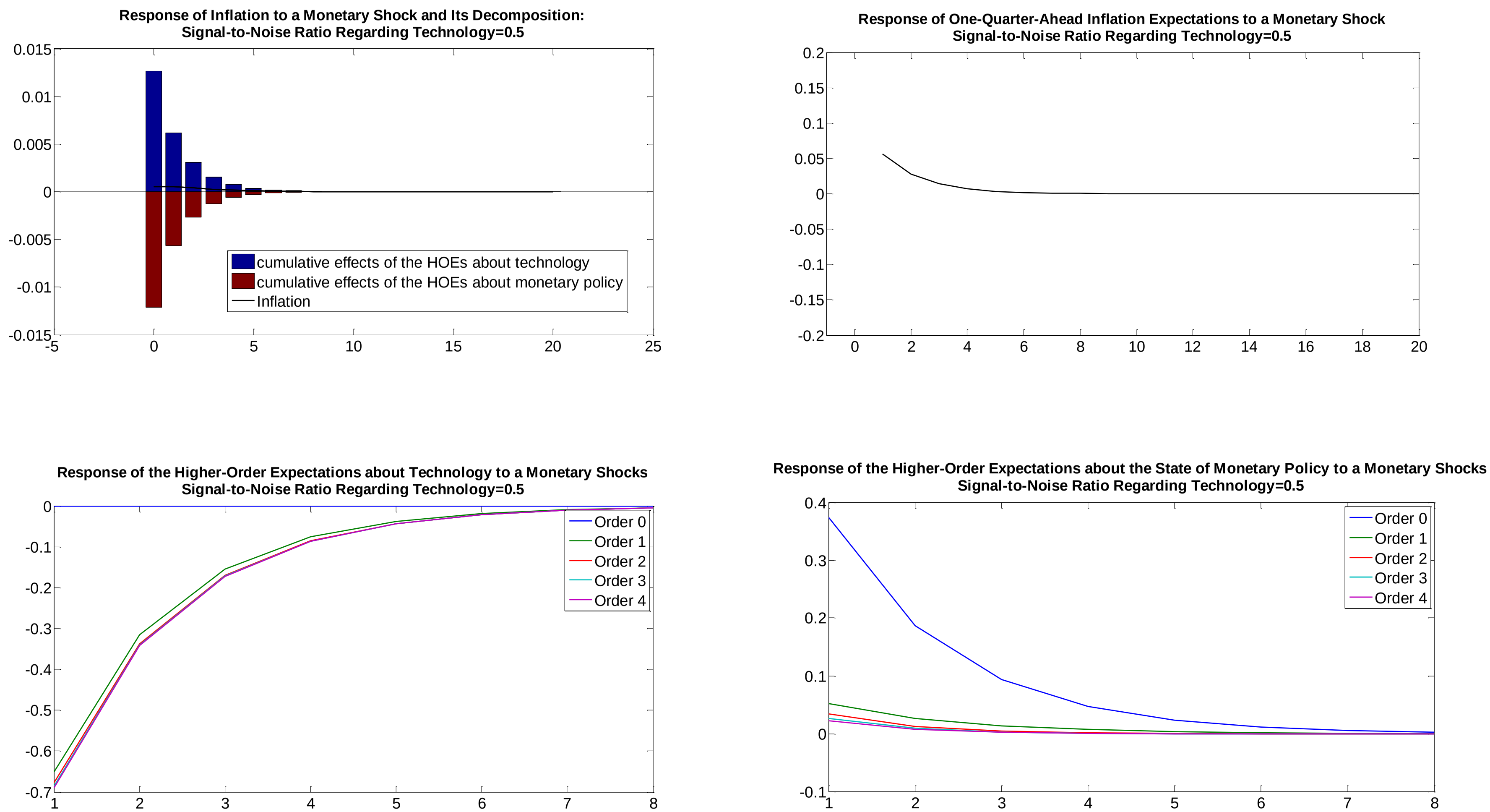


Figure 3: The top left plot reports the response of inflation to a monetary shock. The top right plot reports the response of one-quarter-ahead inflation expectations to a monetary shock. The bottom left panel reports the response of the average expectations about technology from order 0 to 4 to a monetary shock. The bottom right panel reports the response of the average expectations about the state of monetary policy from order 0 to 4 to a monetary shock. The monetary shock is normalized so that it produces a 25-basis-point rise in the interest rate.

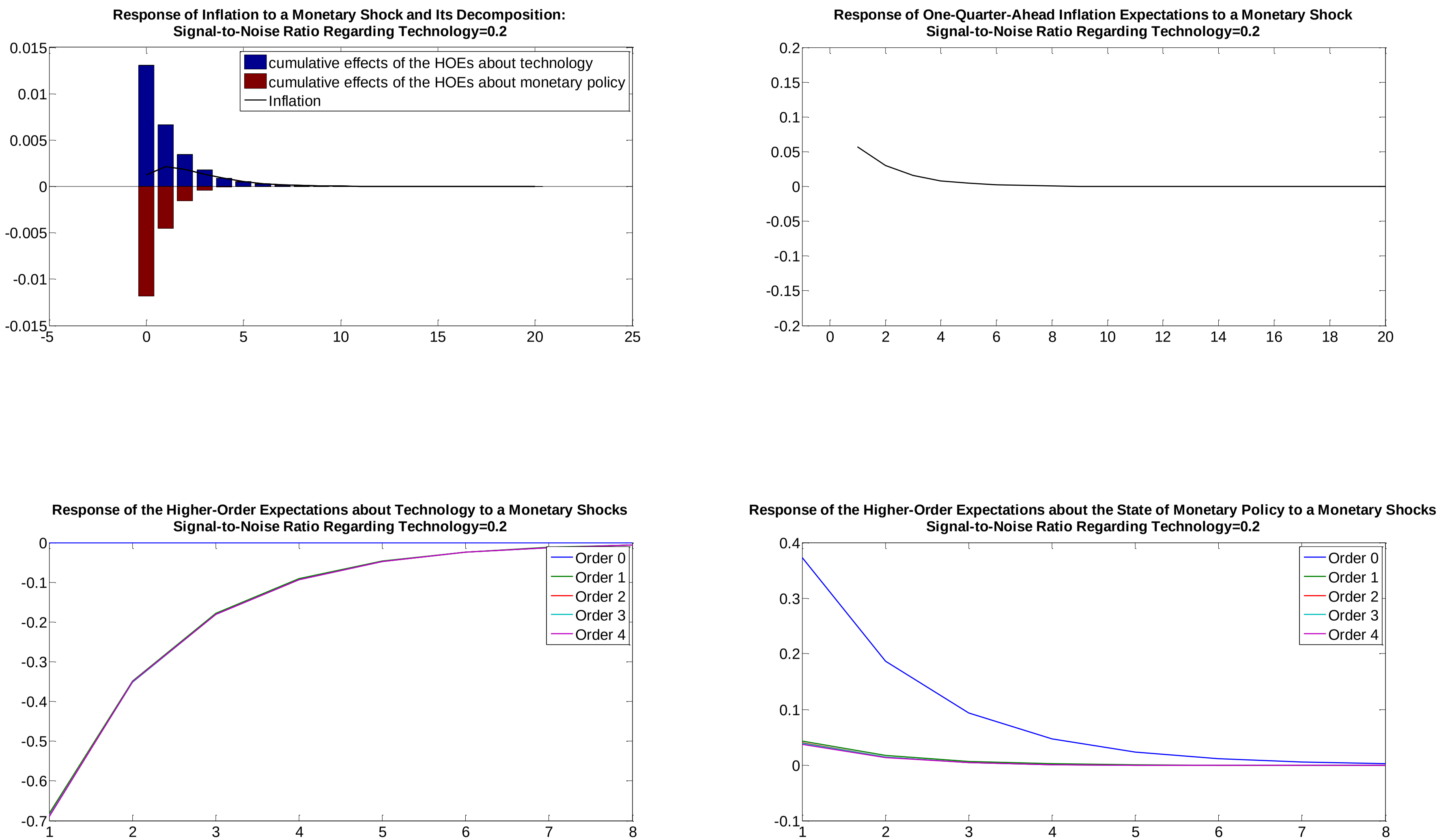


Figure 4: The top left plot reports the response of inflation to a monetary shock. The top right plot reports the response of one-quarter-ahead inflation expectations to a monetary shock. The bottom left panel reports the response of the average expectations about technology from order 0 to 4 to a monetary shock. The bottom right panel reports the response of the average expectations about the state of monetary policy from order 0 to 4 to a monetary shock. The monetary shock is normalized so that it produces a 25-basis-point rise in the interest rate.

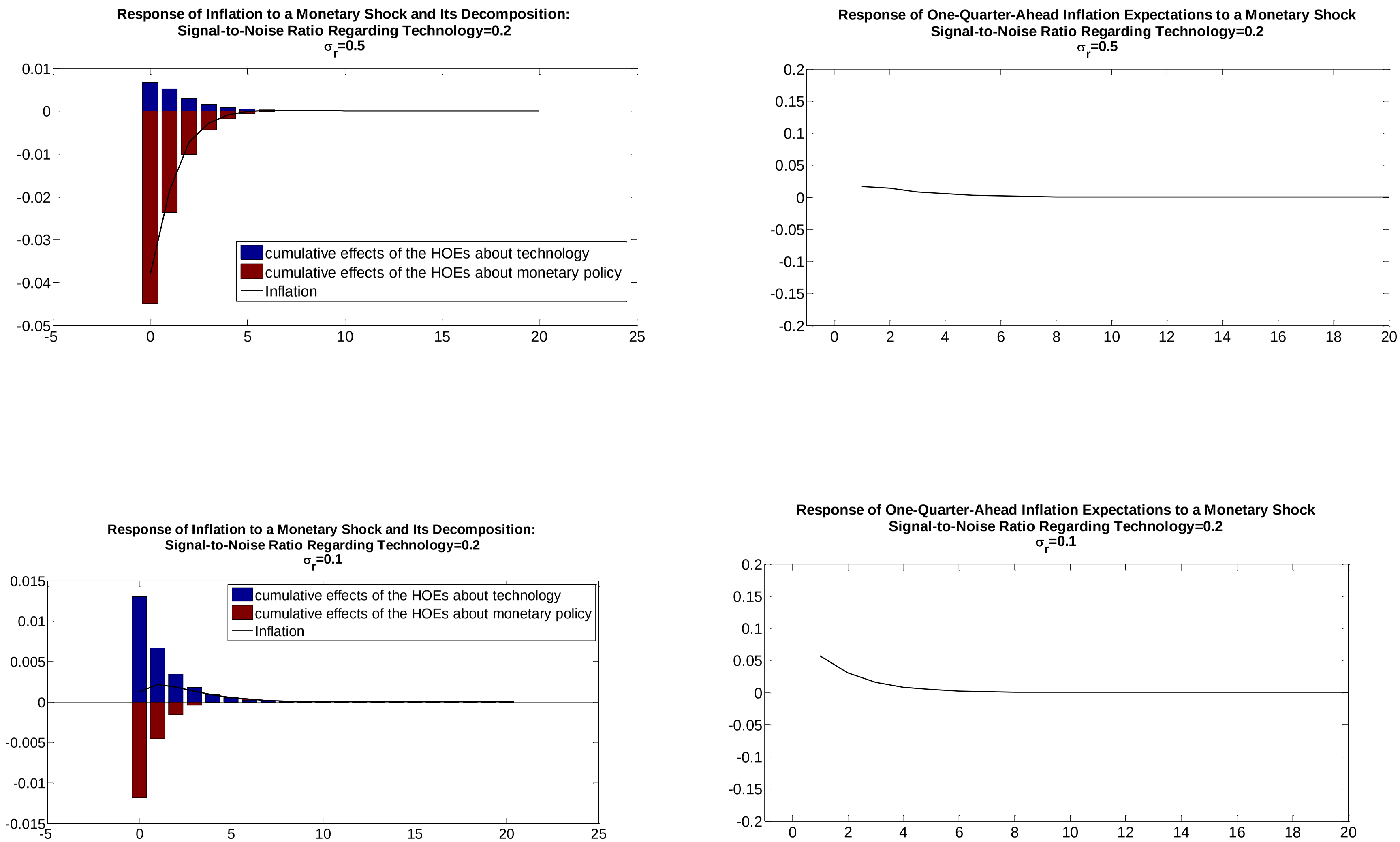


Figure 5: The plots of the left column report the response of inflation and its decomposition to a monetary policy shock for the specified parameterization. The plots of the right column report the impulse response of the one-quarter-ahead inflation expectations for the specified parameterization. The monetary shock is normalized so that it produces a 25-basis-point rise in the interest rate.

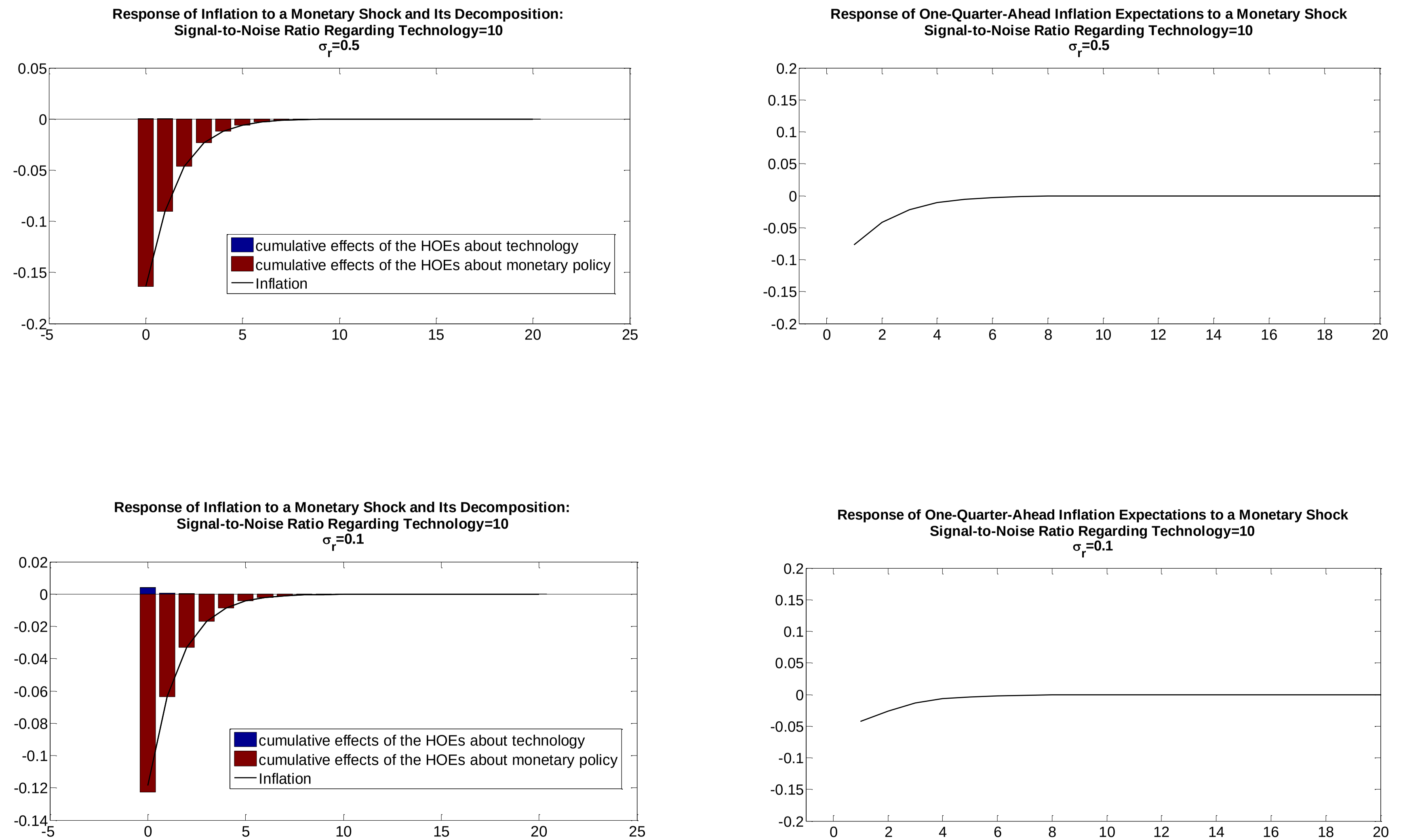


Figure 6: The plots of the left column report the response of inflation and its decomposition to a monetary policy shock for the specified parameterization. The plots of the right column report the impulse response of the one-quarter-ahead inflation expectations for the specified parameterization. The monetary shock is normalized so that it produces a 25-basis-point rise in the interest rate.

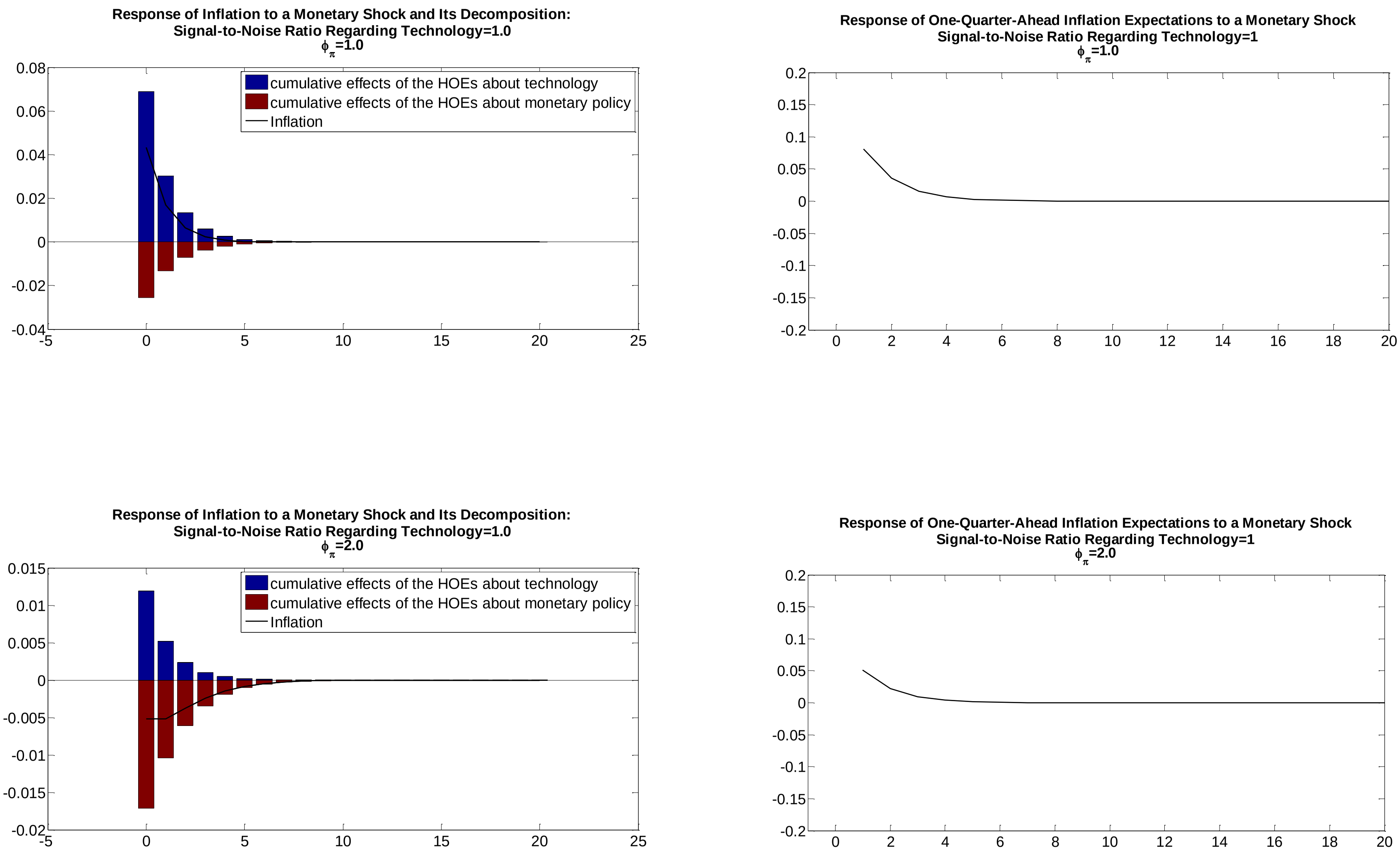
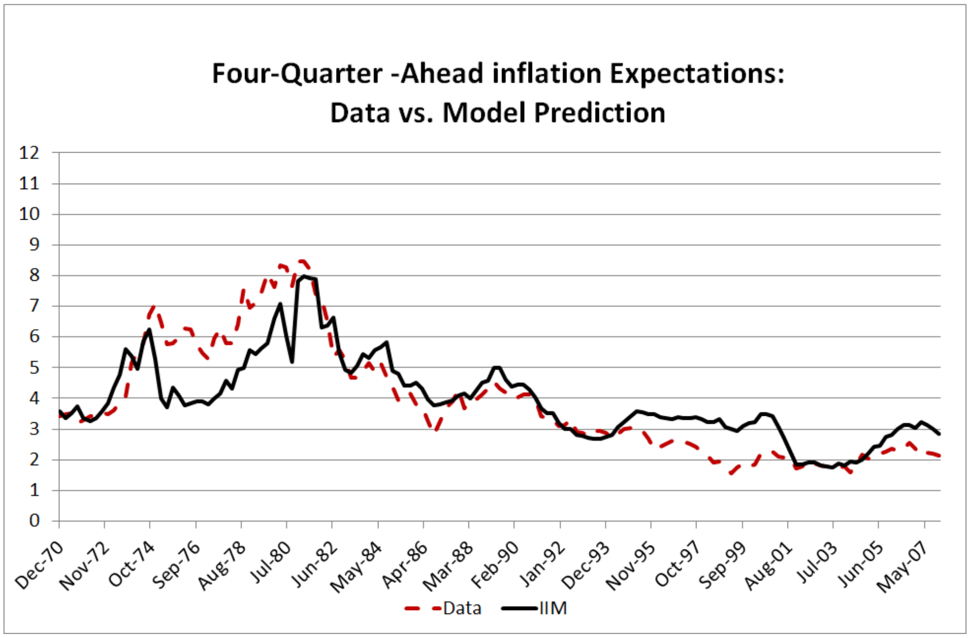


Figure 7: The plots of the left column report the response of inflation and its decomposition to a monetary policy shock for the specified parameterization. The plots of the right column report the impulse response of the one-quarter-ahead inflation expectations for the specified parameterization. The monetary shock is normalized so that it produces a 25-basis-point rise in the interest rate.



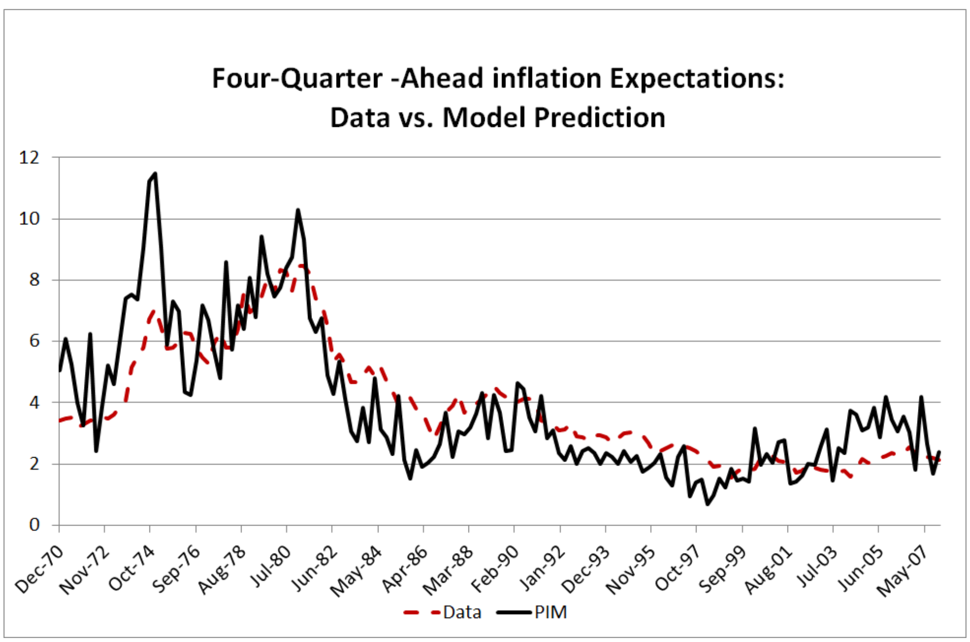
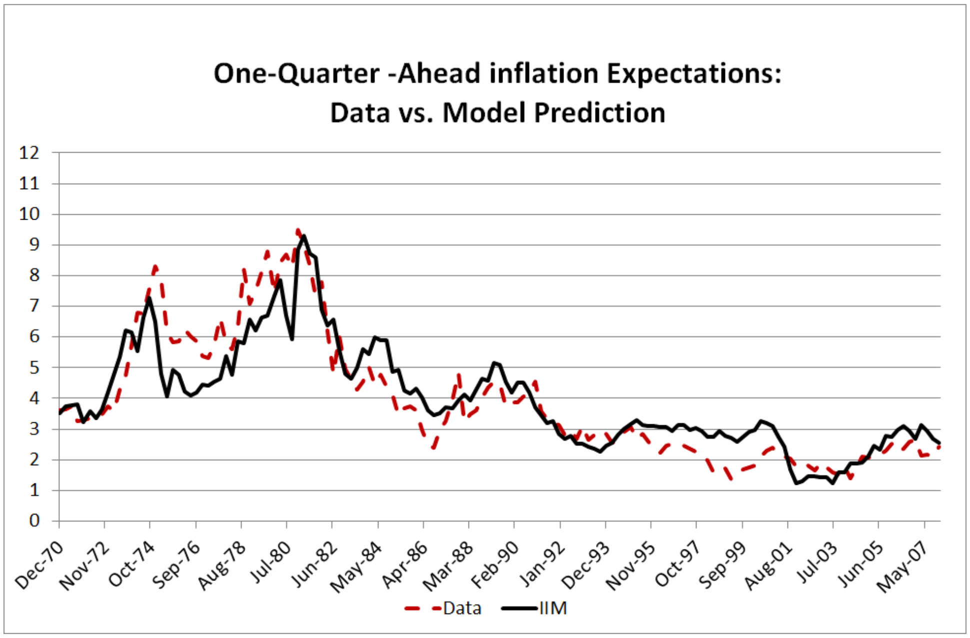


Figure 8: The figure shows the four-quarters-ahead median forecast for the GDP deflator for the SPFs (red dashed line), together with four-quarters-ahead expected inflation generated by the IIM estimated to a data set that does not include the SPF.



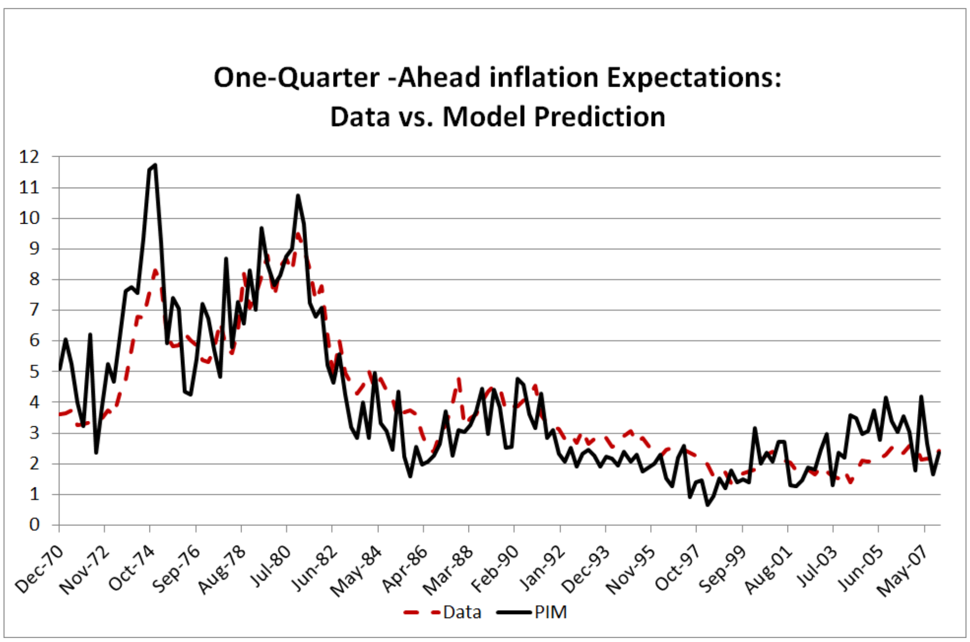


Figure 9: The figure shows the one-quarter-ahead median forecast for the GDP deflator for the SPFs (red dashed line), together with one-quarters-ahead expected inflation generated by the IIM estimated to a data set that does not include the SPF.

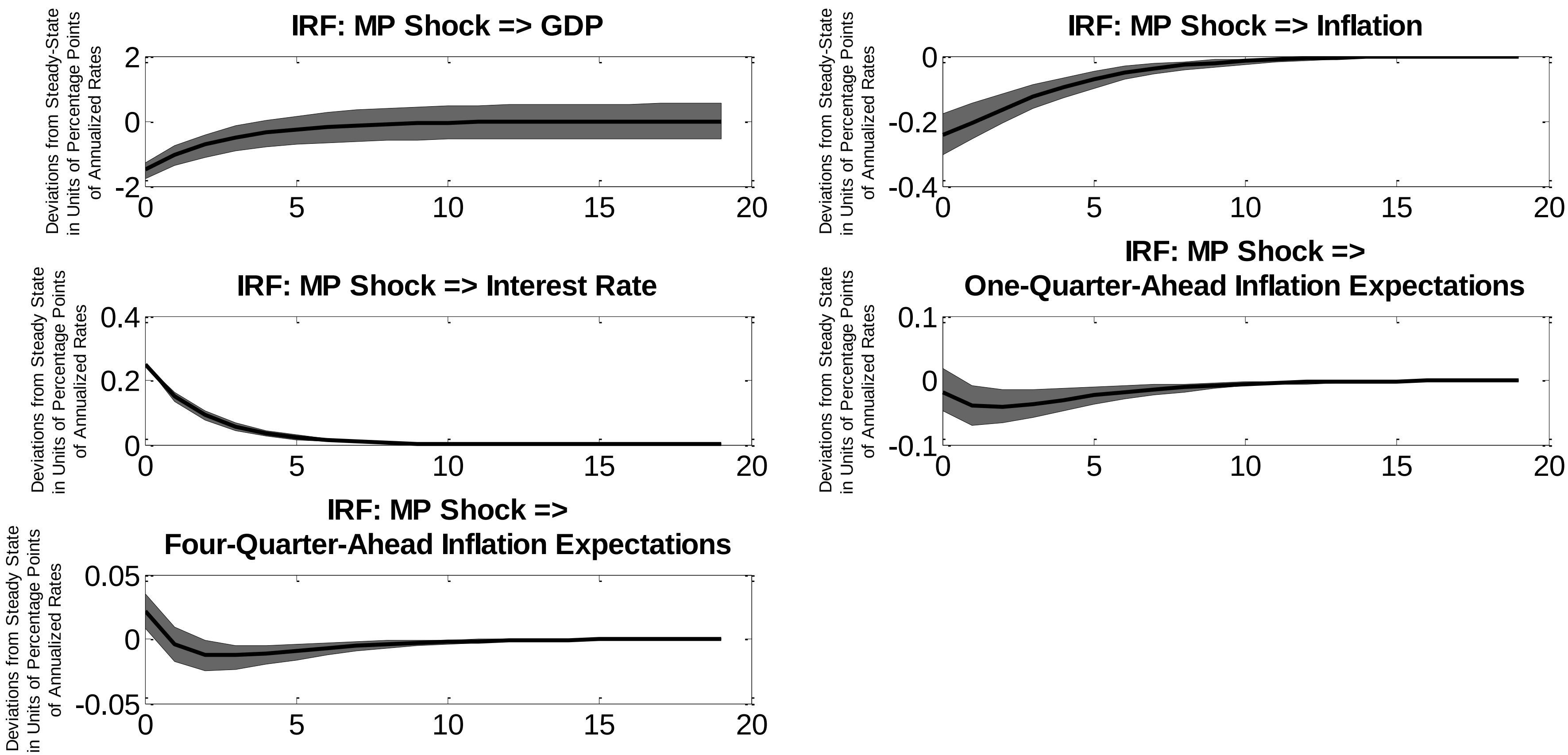


Figure 10: Impulse response functions of the observables to a one-standard-deviation monetary policy shock and their 95 percent posterior credible sets.

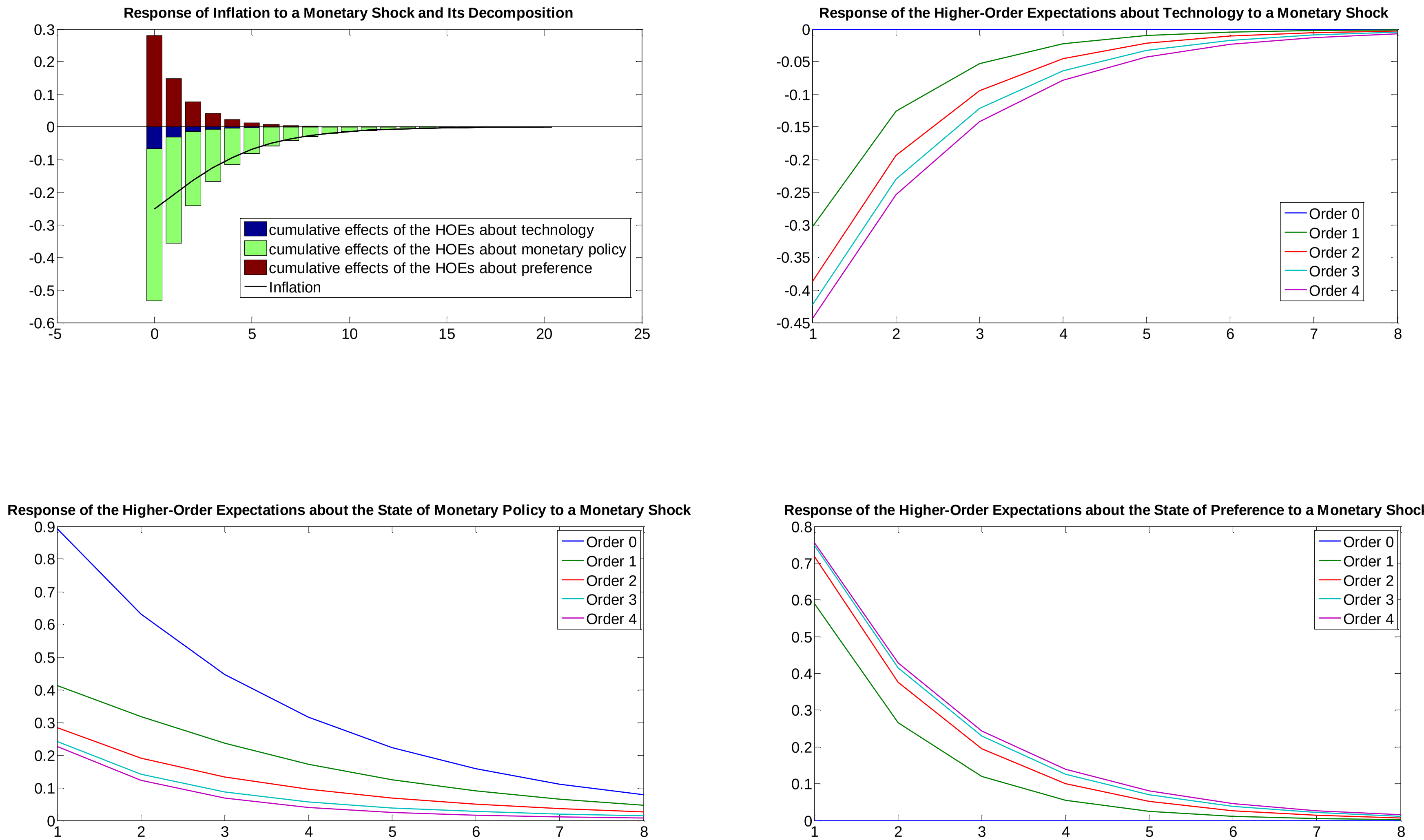


Figure 11: Impulse Response Functions of Inflation to a Monetary Shock and its Decompositions. The top left plot reports the response of inflation and its decomposition into the effects of the HOEs about the three exogenous state variables on inflation. The top right plot reports the response of the higher-order expectations about the aggregate technology to the monetary shock. The bottom left plot reports the response of the higher-order expectations about the state of monetary policy to the monetary shock. The bottom right plot reports the response of the higher-order expectations about the preference shifter to the monetary shock. The size of the initial shock is normalized so that it produces a 25-basis-point rise in the interest rate. All the figures are obtained by evaluating the model at the posterior median.

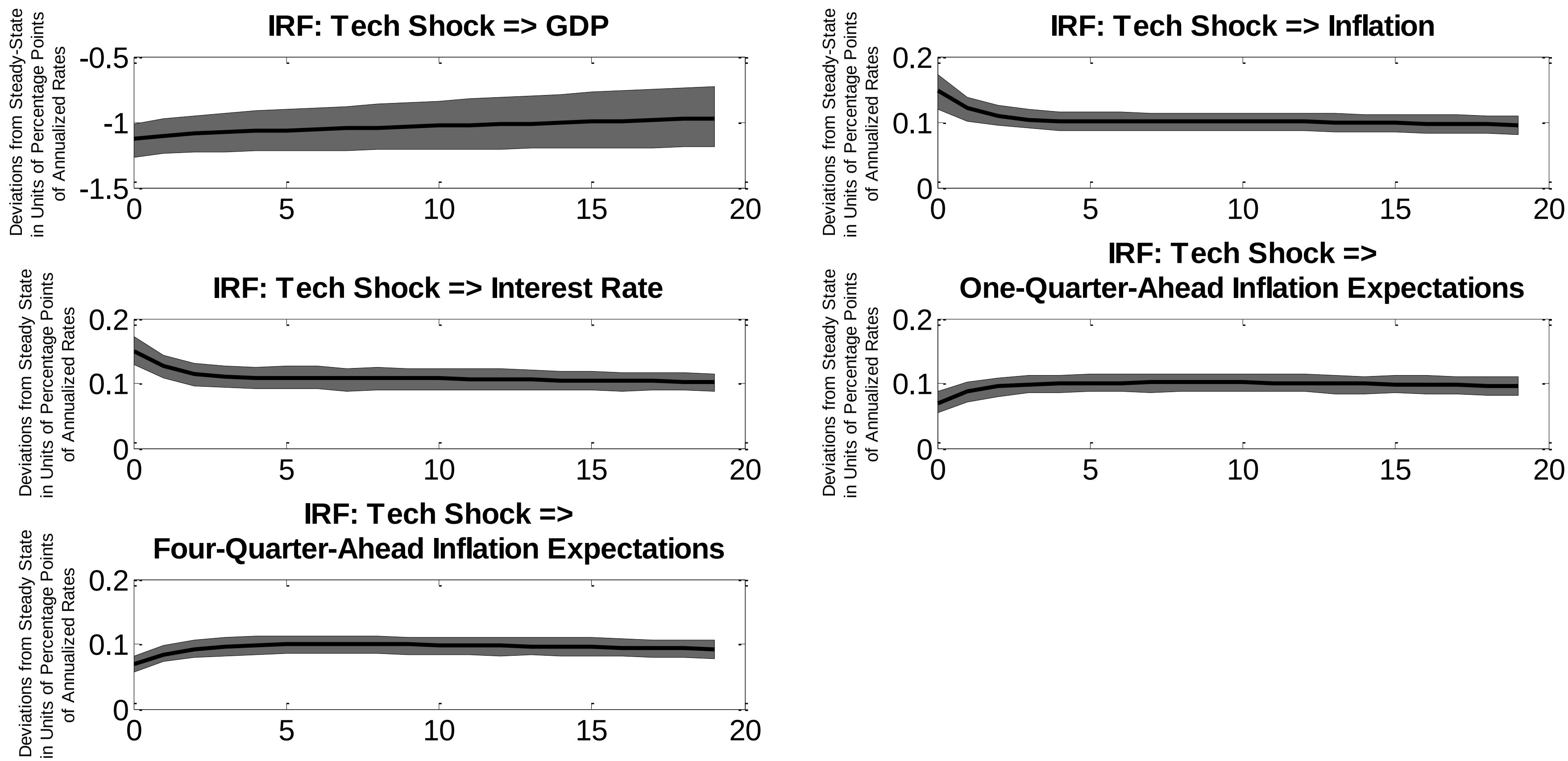


Figure 12: Impulse response functions of the observables to a one-standard-deviation aggregate technology shock and their 95 percent posterior credible sets.

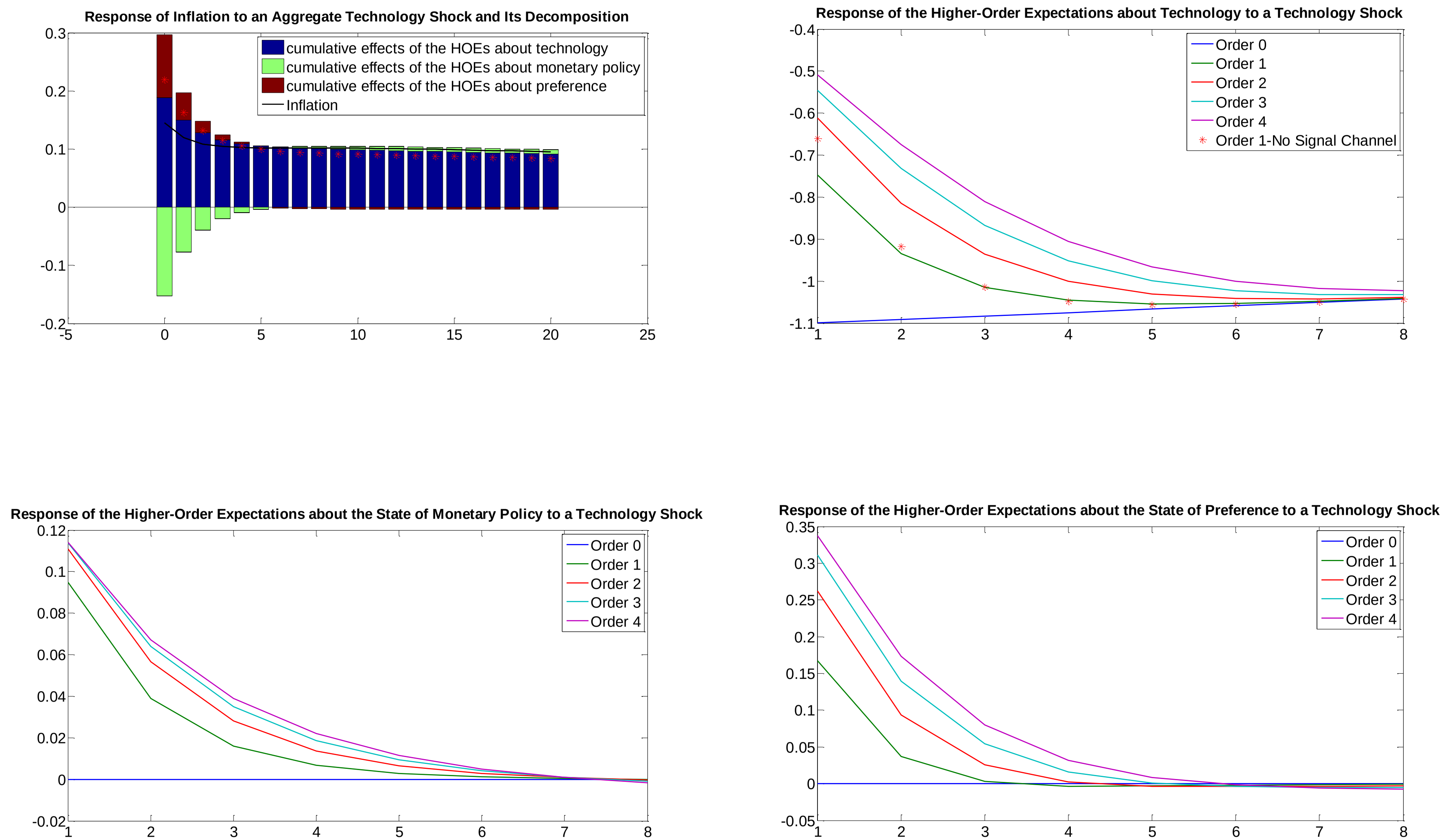


Figure 13: Impulse Response Functions of Inflation to aTwo-Standard-dDeviation Aggregate Technology Shock and its Decompositions. The top left plot reports the response of inflation and its decomposition into the effects of the HOEs about the three exogenous state variables on inflation. The top right plot reports the response of the higher-order expectations about the aggregate technology to the technology shock. The bottom left plot reports the response of the higher-order expectations about the state of monetary policy to the technology shock. The bottom right plot reports the response of the higher-order expectations about the preference shifter to the technology shock. All the figures are obtained by evaluating the model at the posterior median.

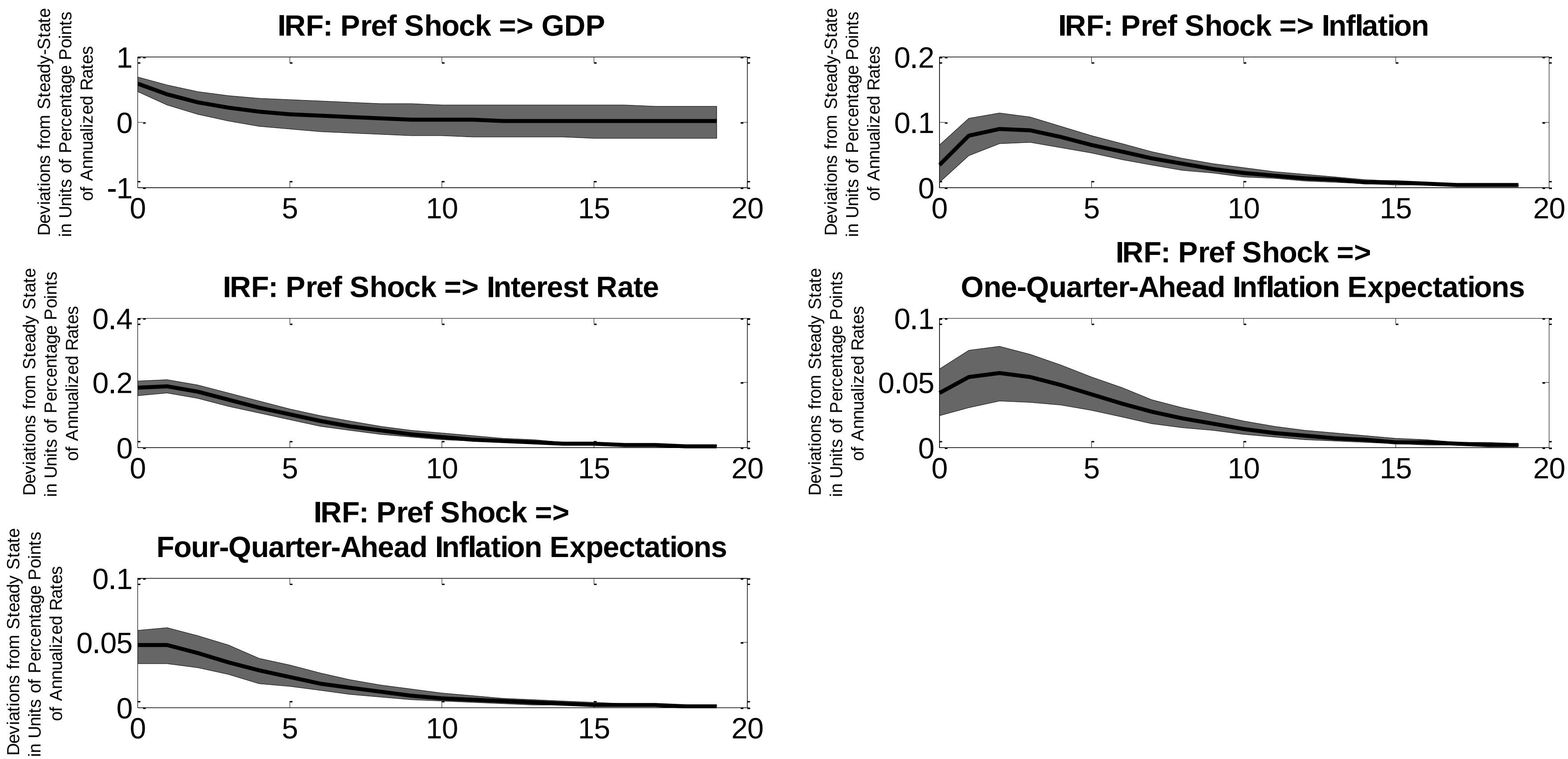


Figure 14: Impulse response functions of the observables to a one-standard-deviation preference shock and their 95 percent posterior credible sets.

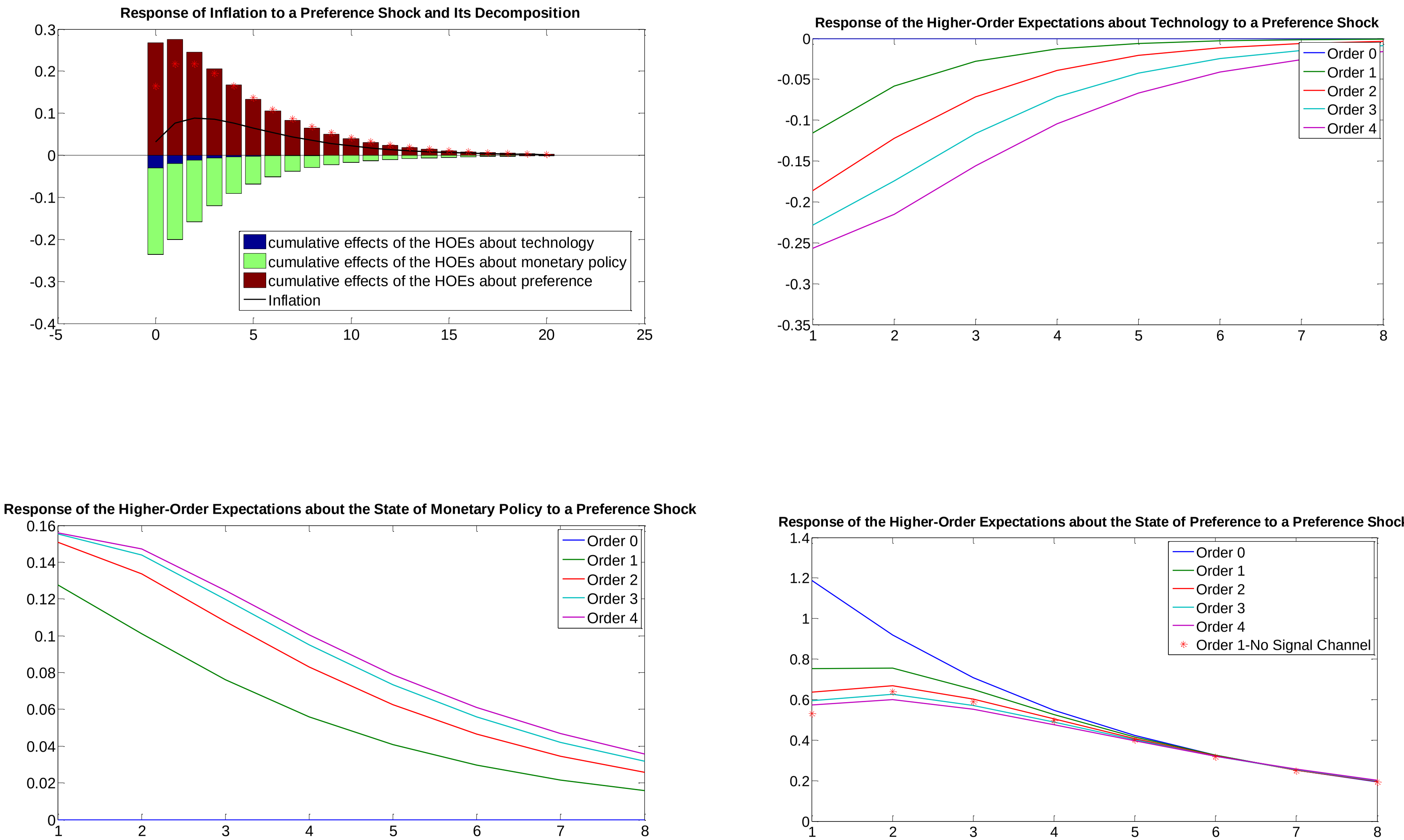


Figure 15: Impulse Response Functions of Inflation to a Two-Standard-Deviation Preference Shock and its Decompositions. The top left plot reports the response of inflation and its decomposition into the effects of the HOEs about the three exogenous state variables on inflation. The top right plot reports the response of the higher-order expectations about the aggregate technology to the preference shock. The bottom left plot reports the response of the higher-order expectations about the state of monetary policy to the preference shock. The bottom right plot reports the response of the higher-order expectations about the preference shifter to the preference shock. All the figures are obtained by evaluating the model at the posterior median.

Appendix

In Section A, I provide derivation of the imperfect-common-knowledge Phillips curve (8). In Section B, I show how to characterize the laws of motion for the three endogenous state variables (i.e., inflation $\hat{\pi}_t$, real output $\hat{y}_t$ and the interest rate $\hat{R}_t$). In Section C, I characterize the transition equations for the average higher-order expectations about the exogenous state variables.

A The Imperfect Common Knowledge Phillips Curve

The log-linear approximation of the labor supply can be shown to be given by $\widehat{c}_t = \widehat{w}_t$. Recalling that the resource constraint implies that $\hat{y}_t = \hat{c}_t$, then the labor supply can be written as follows:

$$\hat{y}_t = \hat{w}_t \tag{23}$$

Log-linearizing the equation for the real marginal costs yields

$$\widehat{mc}_{j,t} = \hat{w}_t - \widehat{a}_t - \varepsilon^a_{j,t}$$

Recall that $(\ln A_{j,t} - \ln \gamma \cdot t) \in \mathcal{I}_{j,t}$ and write:

$$\mathbb{E}_{j,t}\widehat{mc}_{j,t} = \mathbb{E}_{j,t}\hat{w}_{j,t} \underbrace{-\widehat{a}_t - \varepsilon^a_{j,t}}_{\ln A_{j,t} - \ln \gamma \cdot t}$$

where $\mathbb{E}_{j,t}$ are expectations conditioned on firm j's information set at time t, $\mathcal{I}_{j,t}$, defined in (5). Using the equation (23) for replacing $\hat{w}_t$ yields:

$$\mathbb{E}_{j,t}\widehat{mc}_{j,t} = \mathbb{E}_{j,t}\hat{y}_t - \widehat{a}_t - \varepsilon^a_{j,t}$$

By integrating across firms, we obtain the average expectations on marginal costs:

$$\widehat{mc}^{(1)}_{t|t} = \hat{y}^{(1)}_{t|t} - \widehat{a}_t$$

The linearized price index can be written as:

$$0 = -\theta\hat{\pi}_t + (1-\theta)\int \hat{p}^*_{j,t} dj$$

By rearranging:

$$\int \hat{p}^*_{j,t} dj = \frac{\theta}{1-\theta}\hat{\pi}_t$$

Recall that we defined $\hat{p}^*_{j,t} = \ln P^*_{j,t} - \ln P_t$ and $\hat{\pi}_t = \ln P_t - \ln P_{t-1} - \ln \pi_*$,

$$\int \ln P^*_{j,t} dj - \ln P_t = \frac{\theta}{1-\theta}\left(\ln P_t - \ln P_{t-1} - \ln \pi_*\right)$$

and then

$$\int \ln P^*_{j,t} dj = \frac{1}{1-\theta}\ln P_t - \frac{\theta}{1-\theta}\left(\ln P_{t-1} + \ln \pi_*\right)$$

By rearranging:

$$\ln P_t = \theta\left(\ln P_{t-1} + \ln \pi_*\right) + (1-\theta)\int \left(\ln P^*_{j,t}\right) dj \tag{24}$$

The price-setting equation is:

$$\mathbb{E}\left[\sum_{s=0}^{\infty} (\beta\theta)^s \frac{\Xi_{j,t+s}}{P_{t+s}} \left[(1-\nu)\pi_*^s + \nu \frac{MC_{j,t+s}}{P^*_{j,t}}\right] Y_{j,t+s} | \mathcal{I}_{j,t}\right] = 0$$

Define

$$\begin{aligned}
y_t &= \frac{Y_t}{\gamma^t},\ c_t = \frac{C_t}{\gamma^t};\ p^*_{j,t} = \frac{P^*_{j,t}}{P_t},\ y_{j,t} = \frac{Y_{j,t}}{\gamma^t} \\
w_t &= \frac{W_t}{\gamma^t P_t},\ a_t = \frac{A_t}{\gamma^t}, R_t = \frac{R_t}{R_*},\ mc_{j,t} = \frac{MC_{j,t}}{P_t} \\
\xi_{j,t} &= \gamma^t \Xi_{j,t}
\end{aligned}$$

Hence, write:

$$\mathbb{E}\left\{\xi_{j,t}\left[1-\nu+\nu\frac{mc_{j,t}}{p^*_{j,t}}\right] y_{j,t} + \sum_{s=1}^{\infty}(\beta\theta)^s \xi_{j,t+s}\left[(1-\nu)\pi_*^s + \nu\frac{mc_{j,t+s}}{p^*_{j,t}}\left(\Pi_{\tau=1}^s \pi_{t+\tau}\right)\right] y_{j,t+s}|\mathcal{I}_{j,t}\right\} = 0 \tag{25}$$

First realize that the square brackets are equal to zero at the steady state and hence we do not care about the terms outside them. We can write

$$\mathbb{E}\left[\left[1-\nu+\nu mc_{j,*}e^{\widehat{mc}_{j,t}-\widehat{p}^*_{j,t}}\right] + \sum_{s=1}^{\infty}(\beta\theta)^s\left[(1-\nu)\pi_*^s + \nu mc_{j,*}e^{\widehat{mc}_{j,t+s}-\widehat{p}^*_{j,t}+\sum_{\tau=1}^s \hat{\pi}_{t+\tau}}\right]|\mathcal{I}_{j,t}\right] = 0$$

Taking the derivatives yield:

$$\mathbb{E}\left[\widehat{mc}_{j,t} - \widehat{p}^*_{j,t} + \sum_{s=1}^{\infty}(\beta\theta)^s\left[\left(\widehat{mc}_{j,t+s} - \widehat{p}^*_{j,t} + \sum_{\tau=1}^{s}\hat{\pi}_{t+\tau}\right)\right]|\mathcal{I}_{j,t}\right] = 0$$

We can take the term $\widehat{p}_{j,t}$ out of the sum operator in the second term and gather the common term to obtain:

$$\mathbb{E}\left[\widehat{mc}_{j,t} - \frac{1}{1-\beta\theta}\widehat{p}^*_{j,t} + \sum_{s=1}^{\infty}(\beta\theta)^s\left(\widehat{mc}_{j,t+s} + \sum_{\tau=1}^{s}\hat{\pi}_{t+\tau}\right)|\mathcal{I}_{j,t}\right] = 0$$

Recall that $\widehat{p}^*_{j,t} \equiv \ln P^*_{j,t} - \ln P_t$ and cannot be taken out of the expectation operator. We write:

$$\ln P^*_{j,t} = (1-\beta\theta)\,\mathbb{E}\left[\widehat{mc}_{j,t} + \frac{1}{1-\beta\theta}\ln P_t + \sum_{s=1}^{\infty}(\beta\theta)^s\left(\widehat{mc}_{j,t+s} + \sum_{\tau=1}^{s}\hat{\pi}_{t+\tau}\right)|\mathcal{I}_{j,t}\right] \tag{26}$$

Rolling this equation one step ahead yields:

$$\ln P^*_{j,t+1} = (1-\beta\theta)\,\mathbb{E}\left[\widehat{mc}_{j,t+1} + \frac{1}{1-\beta\theta}\ln P_{t+1} + \sum_{s=1}^{\infty}(\beta\theta)^s\left(\widehat{mc}_{j,t+s+1} + \sum_{\tau=1}^{s}\hat{\pi}_{t+\tau+1}\right)|\mathcal{I}_{j,t+1}\right]$$

Take firm j's conditional expectation at time t on both sides and apply the law of iterated expectations:

$$\mathbb{E}\left(\ln P^*_{j,t+1}|\mathcal{I}_{j,t}\right) = (1-\beta\theta)\,\mathbb{E}\left[\widehat{mc}_{j,t+1} + \frac{1}{1-\beta\theta}\ln P_{t+1} + \sum_{s=1}^{\infty}(\beta\theta)^s\left(\widehat{mc}_{j,t+s+1} + \sum_{\tau=1}^{s}\hat{\pi}_{t+\tau+1}\right)|\mathcal{I}_{j,t}\right]$$

We can take $\widehat{mc}_{j,t+1}$ inside the sum operator and write:

$$\mathbb{E}\left(\ln P^*_{j,t+1}|\mathcal{I}_{j,t}\right) = (1-\beta\theta)\,\mathbb{E}\left[\frac{1}{1-\beta\theta}\ln P_{t+1} + \frac{1}{\beta\theta}\sum_{s=1}^{\infty}(\beta\theta)^s\,\widehat{mc}_{j,t+s} + \sum_{s=1}^{\infty}(\beta\theta)^s\sum_{\tau=1}^{s}\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right]$$

Therefore,

$$\sum_{s=1}^{\infty}(\beta\theta)^s\,\mathbb{E}\left[\widehat{mc}_{j,t+s}|\mathcal{I}_{j,t}\right] = \frac{\beta\theta}{1-\beta\theta}\left[\mathbb{E}\left(\ln P^*_{j,t+1}|\mathcal{I}_{j,t}\right) - \mathbb{E}\left(\ln P_{t+1}|\mathcal{I}_{j,t}\right)\right] - \beta\theta\sum_{s=1}^{\infty}(\beta\theta)^s\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right] \tag{27}$$

The equation (26) can be rewritten as:

$$\begin{aligned}\ln P^*_{j,t} &= (1-\beta\theta)\left\{\mathbb{E}\left[\widehat{mc}_{j,t}|\mathcal{I}_{j,t}\right] + \frac{1}{1-\beta\theta}\mathbb{E}\left[\ln P_t|\mathcal{I}_{j,t}\right] + \sum_{s=1}^{\infty}(\beta\theta)^s\,\mathbb{E}\left[\widehat{mc}_{j,t+s}|\mathcal{I}_{j,t}\right]\right\}\\ &\quad + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^s\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau}|\mathcal{I}_{j,t}\right]\end{aligned}$$

By substituting the result in equation (27) we obtain:

$$\begin{aligned}\ln P^*_{j,t} &= (1-\beta\theta)\left[\mathbb{E}\left[\widehat{mc}_{j,t}|\mathcal{I}_{j,t}\right] + \frac{1}{1-\beta\theta}\mathbb{E}\left[\ln P_t|\mathcal{I}_{j,t}\right]\right]\\ &\quad + \beta\theta\left[\mathbb{E}\left(\ln P^*_{j,t+1}|\mathcal{I}_{j,t}\right) - \mathbb{E}\left(\ln P_{t+1}|\mathcal{I}_{j,t}\right)\right] - (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right]\\ &\quad + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^s\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau}|\mathcal{I}_{j,t}\right]\end{aligned}$$

Consider the last term:

$$\begin{aligned}(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^s\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau}|\mathcal{I}_{j,t}\right] &= (1-\beta\theta)\,\beta\theta\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right] + (1-\beta\theta)\sum_{s=2}^{\infty}(\beta\theta)^s\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau}|\mathcal{I}_{j,t}\right]\\ &= (1-\beta\theta)\,\beta\theta\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right] +\\ &\quad + (1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\left(\sum_{\tau=1}^{s}\left[\left(\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right]\right)\right] + \mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right]\right)\end{aligned}$$

Therefore we can write that

$$\begin{aligned}
(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau}|\mathcal{I}_{j,t}\right] &= (1-\beta\theta)\,\beta\theta\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right] \\
&\quad +(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right] \\
&\quad +(1-\beta\theta)\left(\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\right)\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right]
\end{aligned}$$

Note that

$$\left(\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\right)=\frac{(\beta\theta)^{2}}{1-\beta\theta}$$

Hence,

$$\begin{aligned}
(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau}|\mathcal{I}_{j,t}\right] &= (1-\beta\theta)\,\beta\theta\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right] \\
&\quad +(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right] \\
&\quad +(\beta\theta)^{2}\,\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right]
\end{aligned}$$

and by simplifying:

$$\begin{aligned}
(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau}|\mathcal{I}_{j,t}\right] &= \beta\theta\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right] \\
&\quad +(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right]
\end{aligned}$$

We substitute this result into the original equation to get:

$$\begin{aligned}
\ln P_{j,t}^{*} &= (1-\beta\theta)\left[\mathbb{E}\left[\widehat{mc}_{j,t}|\mathcal{I}_{j,t}\right]+\frac{1}{1-\beta\theta}\mathbb{E}\left[\ln P_{t}|\mathcal{I}_{j,t}\right]\right] \\
&\quad +\beta\theta\left[\mathbb{E}\left(\ln P_{j,t+1}^{*}|\mathcal{I}_{j,t}\right)-\mathbb{E}\left(\ln P_{t+1}|\mathcal{I}_{j,t}\right)\right]-(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right] \\
&\quad +\beta\theta\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right]+(1-\beta\theta)\sum_{s=1}^{\infty}(\beta\theta)^{s+1}\sum_{\tau=1}^{s}\mathbb{E}\left[\hat{\pi}_{t+\tau+1}|\mathcal{I}_{j,t}\right] \qquad (28)
\end{aligned}$$

After simplifying we get:

$$\begin{aligned}
\ln P_{j,t}^{*} &= (1-\beta\theta)\left[\mathbb{E}\left[\widehat{mc}_{j,t}|\mathcal{I}_{j,t}\right]+\frac{1}{1-\beta\theta}\mathbb{E}\left[\ln P_{t}|\mathcal{I}_{j,t}\right]\right] \\
&\quad +\beta\theta\left[\mathbb{E}\left(\ln P_{j,t+1}^{*}|\mathcal{I}_{j,t}\right)-\mathbb{E}\left(\ln P_{t+1}|\mathcal{I}_{j,t}\right)\right]+\beta\theta\mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right] \qquad (29)
\end{aligned}$$

We can rearrange:

$$\begin{aligned}\ln P^*_{j,t} &= (1-\beta\theta)\,\mathbb{E}\left[\widehat{mc}_{j,t}|\mathcal{I}_{j,t}\right] + \mathbb{E}\left[\ln P_t|\mathcal{I}_{j,t}\right] \\ &\quad +\beta\theta\left[\mathbb{E}\left(\ln P^*_{j,t+1}|\mathcal{I}_{j,t}\right) + \mathbb{E}\left[\hat{\pi}_{t+1}|\mathcal{I}_{j,t}\right] - \mathbb{E}\left(\ln P_{t+1}|\mathcal{I}_{j,t}\right)\right] \end{aligned} \tag{30}$$

Note that by definition $\hat{\pi}_{t+1} \equiv \ln P_{t+1} - \ln P_t - \ln \pi_*$. Hence we can show that

$$\begin{aligned}\ln P^*_{j,t} &= (1-\beta\theta)\cdot\mathbb{E}\left[\widehat{mc}_{j,t}|\mathcal{I}_{j,t}\right] + (1-\beta\theta)\,\mathbb{E}\left[\ln P_t|\mathcal{I}_{j,t}\right] \\ &\quad +\beta\theta\cdot\mathbb{E}\left(\ln P^*_{j,t+1}|\mathcal{I}_{j,t}\right) - \beta\theta\ln\pi_* \end{aligned} \tag{31}$$

We denote the firm j's average k-th order expectation about an arbitrary variable $\hat{x}_t$ as

$$\mathbb{E}^{(k)}\left(\hat{x}_t|\mathcal{I}_{j,t}\right) \equiv \int \mathbb{E}\left(\int \mathbb{E}\left(\ldots\left(\int \mathbb{E}\left(\hat{x}_t|\mathcal{I}_{j,t}\right)dj\right)\ldots|\mathcal{I}_{j,t}\right)dj|\mathcal{I}_{j,t}\right)dj$$

where expectations and integration across firms are taken k times.

Let us denote the **average reset price** as $\ln P^*_t = \int \ln P^*_{j,t}dj$. We can integrate equation (31) across firms to obtain an equation for the average reset price:

$$\begin{aligned}\ln P^*_t &= (1-\beta\theta)\cdot\widehat{mc}^{(1)}_{t|t} + (1-\beta\theta)\ln P^{(1)}_{t|t} \\ &\quad +\beta\theta\ln P^{*(1)}_{t+1|t} - \beta\theta\ln\pi_* \end{aligned} \tag{32}$$

where we use the claim of the proposition above. Keep in mind that the price index equation can be manipulated to get equation (24)

$$\ln P_t = \theta\left(\ln P_{t-1} + \ln\pi_*\right) + (1-\theta)\ln P^*_t \tag{33}$$

Let us plug the equation (32) into the equation (33):

$$\begin{aligned}\ln P_t &= \theta\ln P_{t-1} + \left(\theta - (1-\theta)\,\beta\theta\right)\ln\pi_* \\ &\quad + (1-\theta)\left[(1-\beta\theta)\cdot\widehat{mc}^{(1)}_{t|t} + (1-\beta\theta)\ln P^{(1)}_{t|t} + \beta\theta\ln P^{*(1)}_{t+1|t}\right] \end{aligned} \tag{34}$$

Use the fact that $\ln P_t = \widehat{\pi}_t + \ln P_{t-1} + \ln \pi_*$ and from the price index (24):[14]

$$\ln P^*_{t+1} = \frac{\hat{\pi}_{t+1}}{1-\theta} + \ln P_t + \ln \pi_*$$

Furthermore, the following fact is easy to establish:

$$\ln P_{t+1} = \widehat{\pi}_{t+1} + \ln P_t + \ln \pi_*$$

Applying these three results to equation (34) yields:

$$\begin{aligned} \widehat{\pi}_t + \ln P_{t-1} + \ln \pi_* &= \theta \ln P_{t-1} + (\theta - (1-\theta)\,\beta\theta) \ln \pi_* \\ &\quad + (1-\theta) \left[(1-\beta\theta) \cdot \widehat{mc}^{(1)}_{t|t} + (1-\beta\theta) \ln P^{(1)}_{t|t} + \beta\theta \left(\frac{\hat{\pi}^{(1)}_{t+1|t}}{1-\theta} + \ln P^{(1)}_{t|t} + \ln \pi_* \right) \right] \end{aligned} \tag{35}$$

and after simplifying:

$$\widehat{\pi}_t = (1-\theta)\,(1-\beta\theta) \cdot \widehat{mc}^{(1)}_{t|t} + (1-\theta)\,\widehat{\pi}^{(1)}_{t|t} + \beta\theta \left(\hat{\pi}^{(1)}_{t+1|t} \right) \tag{36}$$

By repeatedly taking firm j's expectation and average the resulting equation across firms:

$$\widehat{\pi}^{(k)}_{t|t} = (1-\theta)\,(1-\beta\theta) \cdot \widehat{mc}^{(k)}_{t|t} + (1-\theta)\,\widehat{\pi}^{(k+1)}_{t|t} + \beta\theta \left(\hat{\pi}^{(k+1)}_{t+1|t} \right)$$

Repeatedly substituting these equations for $k \geq 1$ back to equation (36) yields: the imperfect-common-knowledge Phillips curve:

$$\hat{\pi}_t = (1-\theta)\,(1-\beta\theta) \sum_{k=1}^{\infty} (1-\theta)^k\, \widehat{mc}^{(k)}_{t|t} + \beta\theta \sum_{k=1}^{\infty} (1-\theta)^k\, \widehat{\pi}^{(k)}_{t+1|t}$$

B The Laws of Motion for the Endogenous State Variables

In this section I, first, introduce some useful results and, second, characterize the law of motion for the endogenous state variables $(\widehat{\pi}_t, \widehat{y}_t, \widehat{R}_t)$, which are inflation $\hat{\pi}_t$, real output $\hat{y}_t$, and the (nominal) interest rate $\hat{R}_t$. It will be shown that this law of motion depends on model parameters and the

[14] This last result comes from observing that

$$\ln P_t = \theta\,(\ln P_{t-1} + \ln \pi_*) + (1-\theta) \ln P^*_t$$

By using the fact that $\ln P_t = \widehat{\pi}_t + \ln P_{t-1} + \ln \pi_*$:

$$\widehat{\pi}_t + \ln P_{t-1} + \ln \pi_* = \theta\,(\ln P_{t-1} + \ln \pi_*) + (1-\theta) \ln P^*_t$$

Rolling one period forward:

$$\widehat{\pi}_{t+1} = (\theta - 1)\,(\ln P_t + \ln \pi_*) + (1-\theta) \ln P^*_{t+1}$$

and finally by rearranging we get the result in the text.

coefficient matrices, $\mathbf{M}$ and $\mathbf{N}$, of the transition equation for the average higher-order expectations about the exogenous variables.

B.1 Preliminaries

Recall that the *assumption of common knowledge in rationality* ensures that agents use the actual law of motion of higher-order expectations to forecast the dynamics of the higher-order expectations. The following claims turn out to be useful:

Proposition 1 *If one neglects the effect of average beliefs of order larger than k, then the following is approximately true:*

$$\varphi_{t|t}^{(s:k+s)} = \mathbf{T}^{(s)} \varphi_{t|t}^{(0:k)}$$

where

$$\mathbf{T}^{(s)} \equiv \begin{bmatrix} \mathbf{0}_{3(k-s+1)\times 3s} & \mathbf{I}_{3(k-s+1)} \\ \mathbf{0}_{3s\times 3s} & \mathbf{0}_{3s\times(k+1-s)3} \end{bmatrix}$$

Proof. It is straightforward but help to fix some notation. Since we neglect the average beliefs of order larger than k

$$\varphi_{t|t}^{(s:k+s)} \equiv \begin{bmatrix} \varphi_{t|t}^{(s:k)} \\ \varphi_{t|t}^{(s:k+s)} \end{bmatrix}_{3(k+1)\times 1} = \begin{bmatrix} \varphi_{t|t}^{(s:k)} \\ \mathbf{0}_{3s\times 1} \end{bmatrix}_{3(k+1)\times 1}$$

Note that

$$\varphi_{t|t}^{(s:k+s)} = \begin{bmatrix} \mathbf{0}_{3(k-s+1)\times 3s} & \mathbf{I}_{3(k-s+1)} \\ \mathbf{0}_{3s\times 3s} & \mathbf{0}_{3s\times(k+1-s)3} \end{bmatrix} \underbrace{\begin{bmatrix} \varphi_{t|t}^{(0:s-1)} \\ \varphi_{t|t}^{(s:k)} \end{bmatrix}}_{\varphi_{t|t}^{(0:k)}}$$

■

Proposition 2 $\mathbf{s}_{t|t}^{(s)} = \mathbf{v}_0 \mathbf{T}^{(s)} \varphi_{t|t}^{(0:k+s)}$, *for any $0 \leq s \leq k$.*

Proof. We conjectured that $\mathbf{s}_t = \mathbf{v}_0 \varphi_{t|t}^{(0:k)}$. Then common knowledge in rationality implies:

$$\mathbf{s}_{t|t}^{(s)} = \mathbf{v}_0 \varphi_{t|t}^{(s:k+s)}$$

■

Since we truncate beliefs after the k-th order we have that

$$\mathbf{s}_{t|t}^{(s)} = \mathbf{v}_0 \mathbf{T}^{(s)} \varphi_{t|t}^{(0:k)}, \text{for any } 0 \leq s \leq k$$

Proposition 3 *The following holds true for any $h \in \{0 \cup \mathbb{N}\}$*

$$\mathbf{s}_{t+h|t}^{(1)} = \mathbf{v}_0 \mathbf{M}^h \mathbf{T}^{(1)} \varphi_{t|t}^{(0:k)}$$

Proof. Consider

$$\mathbf{s}_t = \mathbf{v}_0 \varphi_{t|t}^{(0:k)}$$

Then it is easy to see that by taking agents' expectations and then averaging across them we obtain by the *assumption of common knowledge in rationality*:

$$\mathbf{s}^{(1)}_{t|t} = \mathbf{v}_0 \varphi^{(1:k+1)}_{t|t}$$

and by neglecting the contribution of beliefs of order higher than k we can write: $\mathbf{T}^{(1)}\varphi^{(0:k)}_{t|t} = \varphi^{(1:k+1)}_{t|t}$. This leads to write:

$$\mathbf{s}^{(1)}_{t|t} = \mathbf{v}_0 \mathbf{T}^{(1)} \varphi^{(0:k)}_{t|t} \tag{37}$$

Furthermore, consider $\mathbf{s}_{t+1}$:

$$\mathbf{s}_{t+1} = \mathbf{v}_0 \varphi^{(0:k)}_{t+1|t+1}$$

By taking agents' expectations and then averaging across them we obtain:

$$\mathbf{s}^{(1)}_{t+1|t} = \mathbf{v}_0 \varphi^{(1:k+1)}_{t+1|t}$$

First note that by the assumption of common knowledge in rationality we can write: $\varphi^{(1:k+1)}_{t+h|t} = \mathbf{M}^h \varphi^{(1:k+1)}_{t|t}$. Second, recall that we neglect the contribution of beliefs of order higher than k. These two facts lead us to

$$\mathbf{s}^{(1)}_{t+1|t} = \mathbf{v}_0 \mathbf{M} \mathbf{T}^{(1)} \varphi^{(0:k)}_{t|t}$$

Consider now $\mathbf{s}_{t+2}$. By taking agents' expectations and then averaging across them we obtain:

$$\mathbf{s}^{(1)}_{t+2|t} = \mathbf{v}_0 \varphi^{(1:k+1)}_{t+2|t}$$

and substituting $\mathbf{s}^{(1)}_{t+1|t}$ that we have characterized above yields:

$$\mathbf{s}^{(1)}_{t+2|t} = \mathbf{v}_0 \mathbf{M}^2 \mathbf{T}^{(1)} \varphi^{(0:k)}_{t|t}$$

Keeping on deriving $\mathbf{s}^{(1)}_{t+h|t}$ for any other $h \in \{0 \cup \mathbb{N}\}$ as shown above leads at the formula in the claim. ■

B.2 The Laws of Motion of the Endogenous State Variables

The laws of motion of the three endogenous state variables, which are inflation $\hat{\pi}_t$, real output $\hat{y}_t$, and the (nominal) interest rate $\hat{R}_t$, are given by the IS equation (10), the Phillips curve (8), and the Taylor Rule (11). We want to write this system of linear equations as:

$$\Gamma_0 \mathbf{s}_t = C + \Gamma_1 \mathbb{E}_t \mathbf{s}_{t+1} + \Gamma_2 \varphi^{(0:k)}_{t|t} \tag{38}$$

where $\mathbf{s}_t \equiv \left[\hat{\pi}_t, \hat{y}_t, \hat{R}_t\right]'$. It is obvious how to write equations (10) and (11) in the form (38). However, how to write Phillips curve (8) in the form (38) is not obvious and requires a bit of work. However, note that given the results derived in the previous section and the definition (9), this equation can be rewritten as:

$$
\begin{aligned}
\mathbf{a}_0\varphi_{t|t}^{(0:k)} &= (1-\theta)(1-\beta\theta)\sum_{s=0}^{k-1}(1-\theta)^s\,\mathbf{1}_2^T\left[\mathbf{v}_0\mathbf{T}^{(s+1)}\varphi_{t|t}^{(0:k)}\right]+\\
&-(1-\theta)(1-\beta\theta)\sum_{s=0}^{k-1}(1-\theta)^s\left[\boldsymbol{\gamma}_a^{(s)\prime}\varphi_{t|t}^{(0:k)}\right]\\
&+\beta\theta\sum_{s=0}^{k-1}(1-\theta)^s\,\mathbf{1}_1^T\left[\mathbf{v}_0\mathbf{M}\mathbf{T}^{(s+1)}\varphi_{t|t}^{(0:k)}\right]
\end{aligned}
$$

where $\boldsymbol{\gamma}_a^{(s)}=\left[\mathbf{0}_{1\times 3s},(1,1,0,0,0,0),\mathbf{0}_{1\times 3(k-s)}\right]'$. The following restrictions upon vectors of coefficients $\mathbf{a}_0$ and $\mathbf{a}_1$ can be derived from the Phillips curve above:

$$
\widehat{\pi}_t=\left[(1-\theta)(1-\beta\theta)\left[\boldsymbol{\nu}\mathbf{m}_1-\left(\sum_{s=0}^{k-1}(1-\theta)^s\boldsymbol{\gamma}_a^{(s)\prime}\right)\right]+\beta\theta\boldsymbol{\nu}\mathbf{m}_2\right]\varphi_{t|t}^{(0:k)} \tag{39}
$$

where I define:

$$
\begin{aligned}
\mathbf{m}_1 &\equiv \begin{bmatrix}\left[\mathbf{1}_2^T\mathbf{v}_0\mathbf{T}^{(1)}\right]\\ (1-\theta)\left[\mathbf{1}_2^T\mathbf{v}_0\mathbf{T}^{(2)}\right]\\ (1-\theta)^2\left[\mathbf{1}_2^T\mathbf{v}_0\mathbf{T}^{(3)}\right]\\ \vdots\\ (1-\theta)^{k-1}\left[\mathbf{1}_2^T\mathbf{v}_0\mathbf{T}^{(k)}\right]\end{bmatrix},\ \mathbf{m}_2\equiv\begin{bmatrix}\left[\mathbf{1}_1^T\mathbf{v}_0\mathbf{M}\mathbf{T}^{(1)}\right]\\ (1-\theta)\left[\mathbf{1}_1^T\mathbf{v}_0\mathbf{M}\mathbf{T}^{(2)}\right]\\ (1-\theta)^2\left[\mathbf{1}_1^T\mathbf{v}_0\mathbf{M}\mathbf{T}^{(3)}\right]\\ \vdots\\ (1-\theta)^{k-1}\left[\mathbf{1}_1^T\mathbf{v}_0\mathbf{M}\mathbf{T}^{(k)}\right]\end{bmatrix},\\
\boldsymbol{\nu} &\equiv \mathbf{1}_{1\times k}
\end{aligned}
$$

Equation (39) is a linear function of the vector of average higher-order expectations $\varphi_{t|t}^{(0:k)}$.

C Transition Equation of High–Order Expectations

In this section, we show how to derive the law of motion of the average higher-order expectations of the exogenous variables (i.e., $\widehat{a}_t$, $\varepsilon_{r,t}$, g_t,) for given parameter values and the matrix of coefficients $\mathbf{v}_0$. We focus on equilibria where the HOEs evolve according to:

$$
\varphi_{t|t}^{(0:k)}=\mathbf{M}\varphi_{t-1|t-1}^{(0:k)}+\mathbf{N}\boldsymbol{\varepsilon}_t \tag{40}
$$

where $\boldsymbol{\varepsilon}_t\equiv\left[\begin{array}{ccc}\varepsilon_{a,t} & \eta_{r,t} & \varepsilon_{g,t}\end{array}\right]'$. Denote $\mathbf{X}_t\equiv\varphi_{t|t}^{(0:k)}$, for notational convenience. Firms' reduced-form state-space model can be concisely cast as follows:

$$
\mathbf{X}_t=\mathbf{M}\mathbf{X}_{t-1}+\mathbf{N}\boldsymbol{\varepsilon}_t \tag{41}
$$

$$
\mathbf{Z}_t=\mathbf{D}_1\mathbf{X}_t+\mathbf{Q}e_{j,t} \tag{42}
$$

where

$$
\mathbf{D}_1=\left[\begin{array}{ccc}\mathbf{d}_1' & \mathbf{d}_2' & \left(\mathbf{1}_3^T\mathbf{v}_0\right)'\end{array}\right]'
$$

with $\mathbf{d}_1' = \left[1, \mathbf{0}_{1\times 3(k+1)-1}\right]$, $\mathbf{d}_2' = [\mathbf{0}_{1\times 2}, 1, \mathbf{0}_{1\times 3k}]$, and $e_{j,t} = \left[\varepsilon_{j,t}^a, \varepsilon_{j,t}^g\right]'$ and

$$\mathbf{Q} = \begin{bmatrix} \widetilde{\sigma}_a & 0 \\ 0 & \widetilde{\sigma}_g \\ 0 & 0 \end{bmatrix}$$

.

Solving firms' signal extraction problem requires applying the Kalman filter. The Kalman equation and the conditional variance and covaraince matrix can be easily derived and read:

$$\begin{aligned} \mathbf{X}_{t|t}(j) &= \mathbf{X}_{t|t-1}(j) + \mathbf{P}_{t|t-1}\mathbf{D}_1'\mathbf{F}_{t|t-1}^{-1}\left[\mathbf{Z}_t - \mathbf{Z}_{t|t-1}(j)\right] && (43) \\ \mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}\mathbf{D}_1'\mathbf{F}_{t|t-1}^{-1}\mathbf{D}_1\mathbf{P}_{t|t-1}' && (44) \end{aligned}$$

where

$$\mathbf{P}_{t|t-1} = \mathbf{W}\mathbf{P}_{\mathbf{t-1|t-1}}\mathbf{W}' + \mathbf{U}\mathbf{U}' \tag{45}$$

Therefore, combining equation (44) with equation (45) yields:

$$\mathbf{P}_{t+1|t} = \mathbf{W}\left[\mathbf{P}_{t|t-1} - \mathbf{P}_{t|t-1}\mathbf{D}_1'\mathbf{F}_{t|t-1}^{-1}\mathbf{D}_1\mathbf{P}_{t|t-1}'\right]\mathbf{W}' + \mathbf{U}\mathbf{U}' \tag{46}$$

Denote the Kalman-gain matrix as $\mathbf{K}_t \equiv \mathbf{P}_{t|t-1}\mathbf{D}_1'\mathbf{F}_{t|t-1}^{-1}$. Recall equation (42) and write the law of motion of the firm j's first-order beliefs about $\mathbf{X}_t$ as

$$\mathbf{X}_{t|t}(j) = \mathbf{X}_{t|t-1}(j) + \mathbf{K}_t\left[\mathbf{D}_1\mathbf{X}_t + \mathbf{Q}e_{j,t} - \mathbf{D}_1\mathbf{X}_{t|t-1}(j)\right]$$

where we have combined equations (43) and (42). By recalling that $\mathbf{X}_{t|t-1}(j) = \mathbf{W}\mathbf{X}_{t-1|t-1}(j)$, we have:

$$\mathbf{X}_{t|t}(j) = \mathbf{W}\mathbf{X}_{t-1|t-1}(j) + \mathbf{K}_t\left[\mathbf{D}_1\mathbf{X}_t + \mathbf{Q}e_{j,t} - \left(\mathbf{D}_1\mathbf{W}\mathbf{X}_{t-1|t-1}(j)\right)\right]$$

By rearranging one obtains:

$$\mathbf{X}_{t|t}(j) = (\mathbf{W} - \mathbf{K}\mathbf{D}_1\mathbf{W})\,\mathbf{X}_{t-1|t-1}(j) + \mathbf{K}\left[\mathbf{D}_1\mathbf{W}\cdot\mathbf{X}_{t-1} + \mathbf{D}_1\mathbf{U}\cdot\boldsymbol{\varepsilon}_t + \mathbf{Q}e_{j,t}\right] \tag{47}$$

The vector $\mathbf{X}_{t|t}(j)$ contains firm j's first-order expectations about model's state variables. Integrating across firms yields the law of motion of the average expectation about $\mathbf{X}_{t|t}^{(1)}$:

$$\mathbf{X}_{t|t}^{(1)} = (\mathbf{W} - \mathbf{K}\mathbf{D}_1\mathbf{W})\,\mathbf{X}_{t-1|t-1}^{(1)} + \mathbf{K}\left[\mathbf{D}_1\mathbf{W}\cdot\mathbf{X}_{t-1} + \mathbf{D}_1\mathbf{U}\cdot\boldsymbol{\varepsilon}_t\right]$$

Note that $\varphi_{t|t}^{(0:\infty)} = \left[\varphi_t, \varphi_{t|t}^{(1:\infty)}\right]'$ and that:

$$\varphi_t = \underbrace{\begin{bmatrix} \rho_a & 0 & 0 & \mathbf{0} \\ 0 & \rho_r & 0 & \mathbf{0} \\ 0 & 0 & \rho_g & \mathbf{0} \end{bmatrix}}_{\mathbf{R}_1}\varphi_{t-1|t-1}^{(0:k)} + \underbrace{\begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_g \end{bmatrix}}_{\mathbf{R}_2}\cdot\boldsymbol{\varepsilon}_t$$

So by using the assumption of common knowledge in rationality, we can fully characterize the matrices $\mathbf{M}$ and $\mathbf{N}$:

$$\mathbf{M} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3k} \\ \mathbf{0}_{3k\times 3} & \left(\mathbf{W} - \mathbf{K}\mathbf{D}_1\mathbf{W}\right)|_{(1:3k,1:3k)} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{K}\left(\mathbf{D}_1\mathbf{W}\right)|_{(1:3k,1:3(k+1))} \end{bmatrix} \quad (48)$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{R}_2 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{K}\mathbf{D}_1\mathbf{U}|_{(1:3k,1:3)} \end{bmatrix} \quad (49)$$

where $\cdot|_{(n_1:n_2,m_1:m_2)}$ denotes the submatrix obtained by taking the elements lying between the n_1-th row and the n_2-th row and between the m_1-th column and the m_2-th column. Note that $\mathbf{K}$ in the above equation denotes the steady-state Kalman gain matrix, which is obtained by iterating the equations (44)-(46) and the equation for the Kalman-gain matrix below:

$$\mathbf{K} = \mathbf{P}_{t|t-1}\mathbf{D}_1'\mathbf{F}_{t|t-1}^{-1}$$

until convergence.