Optimality of Securitized Debt with Endogenous and Flexible Information Acquisition

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This paper studies the optimality of securitized debt when information acquisition is endogenous and flexible. A seller designs an asset backed security and a buyer decides whether to buy it to provide liquidity. Rather than treating the seller as an insider endowed with information, we assume no information asymmetry before bargaining. The buyer has an expertise in acquiring information of the fundamental in the manner of rational inattention. She collects the most relevant information determined by the "shape" of the security, which may endogenously generate adverse selection. Hence, the seller deliberately designs the security in order to induce the buyer to acquire information least harmful to the seller's interest. Issuing securitized debt is uniquely optimal in raising liquidity, regardless of the stochastic interdependence of underlying assets and the allocation of bargaining powers. Fixed total risk exposure and homogeneous information cost are the key factors driving the results.
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I. Introduction

Pooling assets and issuing asset-backed securities (ABSs), in particular, issuing a securitized debt, is a popular way to raise liquidity. For example, commercial banks pool a large number of individual home mortgages or automobile loans to create a special purpose vehicle (SPV), which then issues ABSs to finance the purchase of these loans. This process can be modeled as the following story. A risk-neutral seller owns some assets generating uncertain future cash flows. She is impatient and wants to raise liquidity by issuing an asset-backed security (ABS) to a risk-neutral buyer. To raise liquidity, the seller proposes an ABS and its price, and sets it as a take-it-or-leave-it offer. Then the buyer decides whether to accept the offer or not. This simple trading game will serve as a benchmark throughout the paper, and will be greatly enriched in order to capture our key ideas featuring the optimality of securitized debt.

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The security design literature has provided insightful viewpoints in investigating this securitization process as described above. Much of this literature models sellers as "insiders" who are endowed with private information about the assets, which makes potential buyers hesitate to provide liquidity due to adverse selection. In overcoming such adverse selection, this literature considers the possibility of signaling by sellers, where buyers are passive because they cannot acquire any information about the assets. Also, various assumptions on information, assets and feasible securities to be designed are imposed in these models, which lead to various conclusions on the optimality of different forms of securities.

This paper explores another perspective to look into the problem of adverse selection, which is universal in liquidity provision. Specifically, in our benchmark model, the buyer can acquire information about the assets according to the security proposed by the seller, which endogenously generates adverse selection different from that in the security design literature. We follow (Tri Vi Dang, Gary Gorton and Bengt Holmstrom 2011) in treating the buyer as an "expert" who acquires information accordingly. In reality, buyers involved in ABS transactions are skillful and sophisticated. Their expertise in assessing investment opportunities is better modeled by endogenous information acquisition rather than exogenous information endowment. Here endogeneity means that the agents can choose from a set of information structures according to their investment opportunities. Taking this endogeneity into account, sellers design securities generating least incentive for buyers to acquire information. (Dang, Gorton and Holmstrom 2011) model such information acquisition through the costly state verification approach, in which buyers either acquire a specific signal about the future cash flows of assets or do not acquire any information. In other words, the buyer can only choose from two specific information structures. Based on this rigid information acquisition process, (Dang, Gorton and Holmstrom 2011) show that debt is the least information-sensitive and thus is an optimal contract to provide liquidity. However, there also exist infinitely many other securities, which are called "quasi-debts", as information-sensitive as the standard debt contract. Also, it is identified that some restrictive conditions are required in order to ensure the optimality of these quasi-debts when pooling is considered. As we discuss below, their non-uniqueness result stems from the rigidity of information acquisition inhabits the costly state verification approach.

This paper differs from (Dang, Gorton and Holmstrom 2011) by allowing for flexible information acquisition, which helps achieve the unique optimality of securitized debt, even if pooling of various assets is taken into account. Similar to (Dang, Gorton and Holmstrom 2011), we assume no information asymmetry at the beginning to focus on the adverse selection resulting from endogenous information acquisition. Given the security backed by the cash flows and its associated price proposed by the seller, flexibility enables the buyer to acquire information accordingly about the underlying assets. Here, specifically, flexibility means that the set of feasible information structures to be acquired by the buyer consists of all conditional distributions of signals on the underlying cash flows. It captures the ability of the buyer to allocate her attention in whatever way she wants. Hence, the buyer chooses not only the quantitative but also the qualitative nature of her information.

We model flexible information acquisition through the paradigm of rational inattention building upon (Christopher A. Sims 2003), where any information structure can be acquired at a cost proportional to reduction of entropy. This cost could result from the required time or resource to run models, do statistical tests or write reports. Flexibility enables the buyer to acquire payoff-relevant information accordingly, and the information cost requires her to optimally acquire such information in both quantitative and qualitative aspects. For example, to assess a collateralized debt with face value \$1000 and price \$800, a potential buyer would like to analyze data more carefully to see when the underlying cash flow possibly varies around \$800, while put less attention to check whether the cash flow could reach \$2000 or not. Similar to (Dang, Gorton and Holmstrom 2011), standard securitized debt is optimal for liquidity provision in our model. But our result is sharper in the sense that securitized debt is the uniquely optimal one. In (Dang, Gorton and Holmstrom 2011), only two extreme information structures are available in the setup of costly state verification while infinite forms of securities can be designed, which inevitably results in the indistinguishability of some securities. In our framework, with help of flexibility, the variety of available information structures matches the variety of potential securities to be designed, and thus the uniqueness of the standard securitized debt could be guaranteed. Quasi-debts are no longer optimal in our model. By reshaping the uneven tail above the price of a quasi-debt to a flat one, not only the buyer's information cost could be saved but also potential loss of trade from adverse selection could be mitigated. The resulted surplus could be employed by the seller to make both parties better off, and thus ultimately make a better provision of liquidity possible. Moreover, flexible information acquisition provides a unified framework to analyze securitization of multiple assets. We show that pooling and issuing securitized debt is uniquely optimal to raise liquidity, regardless of the stochastic interdependence among the underlying assets and the allocation of bargaining power.

There are two key factors determining the unique optimality of standard securitized debt. The first one is the fixed total risk exposure implicitly specified in the benchmark trading game in the sense that the total cash flows owned by the seller and buyer are invariant with respect to the success or failure of the transaction. As the total risk exposure is fixed, information acquisition is not socially valuable. Acquiring information is no more than waste of money when both parties are considered as a unity. Moreover, this trading game with fixed total risk exposure appears to be a zero-sum one, so that the information acquired by the buyer may hurt the seller through adverse selection, which further reduces the potential gain from trade. Since the buyer's incentive to acquire information is shaped by the offer proposed to her, the seller deliberately designs the ABS to optimally discourage information acquisition harmful to her own interests. Due to the limited liability, any feasible ABS is bounded above by the sum of underlying cash flows. When information cost is not too high, the flexibility allows the buyer to distinguish between any states with different payoffs. Hence the seller makes the ABS a constant whenever it is off the boundary to discourage information acquisition and thus mitigate adverse selection. This consideration gives rise to a flat tail. In states where the underlying cash flows are too low to support such constant, the ABS reaches the boundary and equals the sum of underlying cash flows. Therefore, the flat tail and the boundary component constitute a securitized debt, which is uniquely optimal for liquidity provision. We also use an example with variable total risk exposure to illustrate the importance of fixed total risk exposure in our framework. Consider the seller as an entrepreneur that raises funds from the buyer to take a project with uncertain future cash flows. They jointly expose themselves to a total risk if the buyer accepts the offer, and are not exposed to such risk if the offer is rejected. In this case, information acquisition could be socially valuable. The trading game is not a zero-sum one and the conflicting interests of the two parties could be partly reconciled. Therefore, the seller could deliberately design a contract to encourage the buyer to acquire information that helps avoid investing in states where cash flows are too low. This increases her benefit from the trade, and also leads to a more socially desirable outcome.

Another key factor is the homogeneity in information acquisition. That is, no state is more special than other states in terms of the difficulty of information acquisition. This feature stems from rational inattention and is the reason why our qualittive result does not depend on the stochastic interdependence among the underlying assets. Intuitively, if the information about some assets is much easier to acquire than the other assets, the flat part of the securitized debt cannot be preserved in the optimal ABS. We provide an example that illustrates this idea. The above two factors specify the boundary of our theory.

Finally, the origin of the uniqueness of optimal contract is not only from the flexibility itself, but from the double-sided symmetry of flexibility. In principle, general flexible choice, not necessarily restricted to flexible information acquisition, enables an economic agent to make state-contingent responses. In other words, the agent can make a best response in one state, and can make another best response in another state. Double-sided symmetry of flexibility requires that both parties engaged in a potential trade are endowed with the same level of flexibility.

How this double-sided symmetry of flexibility works can be seen by comparing our framework to (Dang, Gorton and Holmstrom 2011) and the traditional models of costly state verification like (Robert M. Townsend 1979). In all these three models, the contract designer is endowed with flexibility, in the sense that she can assign state-contingent repayment through designing any form of security. What matters to shape the different results regarding uniqueness of the optimal contract relies on the potential flexibility of the other party who decides whether to accept the offer. In our framework, ex-ante symmetric information in the form of a double-sided ignorance prevents the buyer to make a state-contingent choice if she only follows the traditional costly state verification approach to acquire information. However, the buyer in our framework is able to choose state-contingent probability of accepting the offer, namely, she can perform flexible information acquisition. In this sense the buyer enjoys the same level of flexibility as the seller. Given this double-sided symmetry of flexibility in our model, the uniqueness of an optimal contract, which is the standard securitized debt, is guaranteed. In (Dang, Gorton and Holmstrom 2011), however, the buyer can only follow the traditional costly state verification approach to acquire information, in which only two options, namely, to acquire a signal or not, are available. In other words, the buyer in (Dang, Gorton and Holmstrom 2011) cannot make state-contingent decision. Hence, the desired double-sided symmetry of flexibility fails and the uniqueness of the optimal contract fails as a consequence. Interestingly, (Townsend 1979) also employs the costly state verification approach with two options to model information acquisition, namely, to audit or not, but the unique optimality of a standard debt still emerges. Why it is this case? Different from (Dang, Gorton and Holmstrom 2011) and our framework, in (Townsend 1979) the entrepreneur has information advantage over the lender in the sense that the entrepreneur knows the realized profit of the project which the lender does not know. Thanks to the revelation principle, the lender who acquire information in the interim stage can decide whether to audit or not in any state based on the truth told by the entrepreneur who has private information. In other words, although the lender in (Townsend 1979) still only has two options to acquire information as the buyer in (Dang, Gorton and Holmstrom 2011), such two options in (Townsend 1979) are state-contingent while their counterparts in (Dang, Gorton and Holmstrom 2011) are not. Therefore, the double-sided symmetry of flexibility is still established in (Townsend 1979), and the uniqueness of the optimal contract, also a standard debt, is ensured in their model as well.

We proceed as following. Section II studies flexible information acquisition in a binary choice problem, which provides a solid foundation for analyzing players' behavior in the trading game and liquidity provision. Section III derives the uniquely optimal contract as the securitized debt in various circumstances and identifies the two key driving forces of this result. We conclude and discuss in Section IV.

Relation to Literature. We model players' information acquisition behavior through the framework of rational inattention building on (Christopher A. Sims 1998) and (Sims 2003).[1] In applied work, rational inattention is mainly studied in two cases: the linear-quadratic case (e.g., (Bartosz Mackowiak and Mirko Wiederholt 2009)), and the binary-action case. A leading example of the latter is (Michael Woodford 2009) , where firms acquire information and then decide whether to review their prices. Similar to (Ming Yang 2011), this paper also adopts the binary-action setup in a strategic framework, which is different from the single-person decision problem as employed in (Woodford 2009). Compared to (Yang 2011) where both players acquire information and move simultaneously, this paper considers a case in which players move sequentially, and only one party acquires information that results in information asymmetry. Also, this paper focuses on a specific security design problem, rather than addresses a general coordination game as (Yang 2011). Together with (Yang 2011), our work makes early attempts to incorporate rational inattention based flexible information acquisition into strategic problems and offers various new results different from this trend of rational inattention literature.

[1]To learn more about rational inattention, see (Christopher A. Sims 2005),(Christopher A. Sims 2006), (Christopher A. Sims 2010), (Yulei Luo 2008), (Bartosz Mackowiak and Mirko Wiederholt 2011), (Stijn Van Nieuwerburgh and Laura Veldkamp 2009*a*), (Stijn Van Nieuwerburgh and Laura Veldkamp 2009*b*), (Luigi Paciello 2009), (Filip Matejka 2010), (Jordi Mondria 2010), (Filip Matejka and Christopher A. Sims 2011).

This paper is also closely related to the security design literature, in much of which sellers are modeled as "insiders" exogenously endowed with private information. Sellers' information advantage over buyers result in adverse selection which further leads to inefficient trade. In order to deal with the adverse selection problem given that buyers cannot acquire information, sellers want to signal their private information out in order to partly retrieve efficient trade. In this process, appropriate security design matters. This is because signaling is costly, so that to design a security that is less information sensitive than the original asset could save the signaling cost, which in turn adds to the profit of sellers. This consideration is plausible and insightful results have been well established in literature, but there may also be other interesting possibilities worth investigating. Also, various assumptions are imposed in this literature to deliver various results. In our paper, buyers in financial markets may also actively acquire information, which could result in different interplay between the two parties and different results of security design, and we can get clearer results from a single assumption.

The key difference between our approach and much of the security design literature could be clearly seen in discussing some of their assumptions and results in details. (Gary Gorton and George G. Pennacchi 1990) shows that splitting assets into debt and equity mitigates the lemon problem between outsiders and insiders. They directly assume the existence of debt rather than considering a security design problem. In (Peter M. DeMarzo and Darrell Duffie 1999), informed sellers signal the quality of assets to competitive liquidity suppliers through retaining part of the cash flows. Equity is issued when the contractible information is not very sensitive to sellers' private information. Standard debt is optimal within the set of non-decreasing securities if the information structure allows a uniform worst case. (Bruno Biais and Thomas Mariotti 2005) studies the effects of market power on market liquidity. They derive both the optimal security and trading mechanism through the approach of mechanism design. Debt contract turns out to be optimal under distributional conditions of underlying cash flows. (Peter M. DeMarzo 2005) focuses on the consequences of pooling and tranching. Pooling has an information destruction effect that destroys the seller's ability to signal the quality of her assets separately. When tranching is possible, pooling may also have a risk diversification effect that reduces information sensitivity of the senior claim. Under specific distributional assumptions of the noise structure, (DeMarzo 2005) shows that the risk diversification effect dominates the information destruction effect as the number of underlying assets goes to infinity. In this limit case, pooling and tranching become optimal. These models also restrict their attention to non-decreasing securities[2]. (Robert D. Innes 1990) provides a standard motivation for this constraint. When the security is not monotone, a seller may cheat through borrowing from a third party, reporting a high cash flow to reduce her repayment and then repaying the side loan. The validity of this argument depends on the context. In the case of publicly traded stocks or bonds, this kind of cheat is unlikely to happen because it is difficult or even illegal for seller to manipulate the cash flows. Moreover, when the security is written on multiple underlying assets, even the

[2](Biais and Mariotti 2005) also assume dual monotonicity, i.e., both the security and the residual cash flow are non-decreasing.

concept of monotonicity is not well defined. Our framework is free of these limits.

II. Binary Choice with Endogenous and Flexible Information Acquisition

Before introducing the economic environment of security design problem, we review the logic of binary choice with flexible information acquisition, which will play a key role in the following analysis. The readers mainly interested in the security design problem can skip this section and go back to it when needed.

In our leading example, a buyer faces a take-it-or-leave-it offer. She has to acquire information and then make a binary choice. We first focus on information structures with binary signals and then show that it suffices to do so.

A. Decision Problem

Consider an agent who has to choose an action $a \in \{0, 1\}$ and will receive a payoff $u(a, \theta)$, where $\theta \in \Theta \subset \mathbb{R}$ is an unknown state distributed according to a continuous probability measure P over Θ.

The agent has access to the set of binary-signal information structures. In particular, she observes signals $x \in \{0, 1\}$ parameterized by measurable function $m : \Theta \to [0, 1]$, where $m(\theta)$ is the probability of observing signal 1 if the true state is θ (and so $1 - m(\theta)$ is the probability of observing signal 0). The conditional probability function $m(\theta)$ describes the agent's information acquisition strategy. By choosing different functional forms for $m(\theta)$, the agent can make her signal covary with fundamental in any way she would like. Intuitively, if her welfare is sensitive to fluctuation of the state within some range $A \subset \Theta$, she would pay much attention to this event by letting $m(\theta)$ be highly sensitive to $\theta \in A$. In this sense, choosing an information structure can be interpreted as hiring an analyst to write a report with emphasis on your interests. This idea will be made more clear through an example later in this section.

Quantity and Cost of Information

Following (Sims 2003), we measure the quantity of information according to information theory building on (Claude E. Shannon 1948). Information conveyed by an information structure $m(\cdot)$ is defined as the expected reduction of uncertainty through observing signals generated by $m(\cdot)$, where the uncertainty associated with a distribution is measured by Shannon's entropy.

Before observing her signal, the agent's uncertainty about θ is given by Shannon's entropy of her prior[3]

$$H(prior) = -\int_{\Theta} p(\theta) \ln p(\theta)\, d\theta\ ,$$

[3] This is essentially the unique measure of uncertainty given three axioms. See (Thomas M. Cover and Joy A. Thomas 1991) for detailed discussion.

where p is the density function of prior P[4]. After observing signal 1, the agent forms a posterior of θ

$$\frac{m(\theta)\,p(\theta)}{\int_\Theta m(\theta')\,dP(\theta')}$$

and her posterior uncertainty upon receiving signal 1 is measured by her posterior entropy

$$\begin{aligned} H(posterior|1) &= -\int_\Theta \frac{m(\theta)\,p(\theta)}{\int_\Theta m(\theta')\,dP(\theta')} \ln\left(\frac{m(\theta)\,p(\theta)}{\int_\Theta m(\theta')\,dP(\theta')}\right) d\theta \\ &= -\int_\Theta \frac{m(\theta)}{\int_\Theta m(\theta')\,dP(\theta')} \ln\left(\frac{m(\theta)\,p(\theta)}{\int_\Theta m(\theta')\,dP(\theta')}\right) dP(\theta) \ . \end{aligned}$$

Similarly, observing signal 0 leads to a posterior

$$\frac{[1-m(\theta)]\,p(\theta)}{1-\int_\Theta m(\theta')\,dP(\theta')}$$

and posterior entropy

$$H(posterior|0) = -\int_\Theta \frac{1-m(\theta)}{1-\int_\Theta m(\theta')\,dP(\theta')} \ln\left(\frac{[1-m(\theta)]\,p(\theta)}{1-\int_\Theta m(\theta')\,dP(\theta')}\right) dP(\theta) \ .$$

Then the agent's expected posterior entropy through choosing information structure $m(\cdot)$ is

$$\begin{aligned} & H(posterior) \\ = \ & \int_\Theta m(\theta')\,dP(\theta') \cdot H(posterior|1) + \left[1-\int_\Theta m(\theta')\,dP(\theta')\right] \cdot H(posterior|0) \\ = \ & -\int_\Theta m(\theta)\ln\left(\frac{m(\theta)\,p(\theta)}{\int_\Theta m(\theta')\,dP(\theta')}\right) dP(\theta) - \int_\Theta [1-m(\theta)]\ln\left(\frac{[1-m(\theta)]\,p(\theta)}{1-\int_\Theta m(\theta')\,dP(\theta')}\right) dP(\theta) \ . \end{aligned}$$

Let $I(m)$ denote the quantity of information gained through $m(\cdot)$, which equals the difference between the agent's prior entropy and expected posterior entropy, i.e.,

$$\begin{aligned} I(m) &= H(prior) - H(posterior) \\ &= \left[\int_\Theta g(m(\theta))\,dP(\theta) - g\left(\int_\Theta m(\theta)\,dP(\theta)\right)\right] , \end{aligned} \tag{1}$$

where

$$g(x) = x \cdot \ln x + (1-x) \cdot \ln(1-x) \ .$$

In information theory, $I(m)$ is called mutual information. It measures the quantity of

[4]Following the convention of information theory, we let $0 \cdot \ln 0 = 0$. This is reasonable since $\lim_{x\to 0} x \cdot \ln x = 0$.

information about θ that is conveyed by the signal.

Write

$$M \triangleq \{m \in L(\Theta, P) : \forall \theta \in \Theta, m(\theta) \in [0,1]\}$$

for the set of binary-signal information structures. Let $c : M \to \mathbb{R}_+$ be the cost (in terms of utility) of acquiring information. We assume that the cost is proportional to the quantity of information gained, i.e.,

$$c(m) = \mu \cdot I(m) \ , \tag{2}$$

where $\mu > 0$ is the marginal cost of information acquisition. It measures the difficulty in acquiring information. When $\mu = 0$, information acquisition incurs no cost and the agent can directly observe the true state. When $\mu \to \infty$, the agent cannot acquire any information at all.

It is worth noting that mutual information $I(m)$ measures function m's variability, which reflects the informativeness of actions to the fundamental. For example, when $m(\theta)$ is constant, the actions convey no information about θ and the corresponding mutual information is zero. This is because function g is strictly convex and thus $I(m)$ is zero if and only if $m(\theta)$ is constant. Hence, a nice property of our technology of information acquisition is that, there exists information acquisition if and only if $m(\theta)$ varies over θ, if and only if information cost is positive. Also note that the "shape" (functional form) of m determines not only the quantity but also the qualitative nature of information. For instance, an agent can concentrate her attention to some event through making $m(\theta)$ highly sensitive to θ within such event. In this sense, our technology of information acquisition is flexible since the agent can decide both the quantity and quality of their information through freely choosing from M. It is also worth noting that $c(\cdot)$ is convex, i.e.,

$$c(t \cdot m_1 + (1-t) \cdot m_2) \leq t \cdot c(m_1) + (1-t) \cdot c(m_2)$$

for all $m_1, m_2 \in M$ and $t \in [0,1]$. This convexity is strict when at least one of m_1 and m_2 is not a constant in θ.

Solving Binary Decision Problem with Information Acquisition

Now we are interested in the problem of an agent choosing an information structure $m \in M$ and a stochastic decision rule $f : \{0,1\} \to [0,1]$ to maximize her expected utility

(3)

$$V(m,f) = \int_\Theta \left\{ \begin{array}{c} [m(\theta) f(1) + (1-m(\theta)) f(0)] \cdot u(1,\theta) \\ + [m(\theta)(1-f(1)) + (1-m(\theta))(1-f(0))] \cdot u(0,\theta) \end{array} \right\} dP(\theta) - c(m) \ .$$

Without loss of generality, we can let $f = f^*$ where $f^*(1) = 1$ and $f^*(0) = 0$. This simplification is based on the following observation. If we let

$$m^*(\theta) = m(\theta) f(1) + (1-m(\theta)) f(0) ,$$

then $V(m^*, f^*) \geq V(m, f)$, since the first term of (3) remains the same, while, by the convexity of $c(\cdot)$, the information cost becomes smaller.[5]

Fixing $f = f^*$, we can interpret m as a joint information structure and decision rule specifying that the agent will take action 1 with probability $m(\theta)$ in state θ.

Now the agent's problem is to choose $m \in M$ to maximize

$$\begin{aligned} V^*(m) &= \int_\Theta [m(\theta) \cdot u(1,\theta) + [1 - m(\theta)] \cdot u(0,\theta)]\, dP(\theta) - c(m) \\ &= \int_\Theta m(\theta) \cdot [u(1,\theta) - u(0,\theta)]\, dP(\theta) - c(m) + \int_\Theta u(0,\theta)\, dP(\theta) \ . \end{aligned}$$

Since $\int_\Theta u(0,\theta)\, dP(\theta)$ is a constant that does not depend on m, we can redefine the agent's objective as

$$\max_{m \in M} V^*(m) = \int_\Theta \Delta u(\theta) \cdot m(\theta)\, dP(\theta) - c(m) \ ,$$

where

$$\Delta u(\theta) = u(1,\theta) - u(0,\theta)$$

is the payoff gain from taking action 1 over action 0. It shapes the agent's incentive of information acquisition.

The following lemma characterizes the optimal strategy m for the agent.[6]

PROPOSITION 1: [7]*Let* $\Pr(\Delta u(\theta) \neq 0) > 0$ *to exclude the trivial case that the agent is always indifferent between the two actions. Let* $m \in M$ *be an optimal strategy and*

$$p_1 = \int_\Theta m(\theta)\, dP(\theta)$$

be the corresponding unconditional probability of taking action 1. Then,

[5] A simple proof: the convexity of $c(\cdot)$ implies

$$\begin{aligned} c(\alpha \cdot m) &= c(\alpha \cdot m + (1-\alpha) \cdot 0) \\ &< \alpha \cdot c(m) + (1-\alpha) \cdot c(0) \\ &= \alpha \cdot c(m) \end{aligned}$$

for $\alpha \in [0, 1)$. Without loss of generality, let $\Delta f = f(1) - f(0) \geq 0$. Note that if $f(1) = 0$ or $f(1) = 1$ and $f(0) = 0$, we are done. Let $\alpha = \Delta f / f(1)$. Thus at least one of $f(1)$ and α is strictly less than 1.Then

$$\begin{aligned} & c\left(m^*\right) = c(f(1) \cdot [\alpha \cdot m + 1 - \alpha]) \\ \leq \ & f(1) \cdot c([\alpha \cdot m + 1 - \alpha]) \\ \leq \ & f(1) \cdot ([\alpha \cdot c(m) + 0]) \\ \leq \ & \Delta f \cdot c(m) < c(m) \ . \end{aligned}$$

[6] We became aware of the related work (Michael Woodford 2008) while working on this paper. Here we use Lemma 2 of (Woodford 2008) to characterize the optimal strategy. To maintain the completeness of our paper, we give a proof in our context.

[7] We do not have to require $\Theta \subset \mathbb{R}$. This proposition holds for any probability space Θ.

i) the optimal strategy is unique;
ii) there are three possibilities for the optimal strategy:
a) $p_1 = 1$ *(i.e.,* $m(\theta) = 1$ *almost surely) if and only if*

$$\int_\Theta \exp\left(-\mu^{-1}\Delta u(\theta)\right) dP(\theta) \leq 1\ ; \tag{4}$$

b) $p_1 = 0$ *(i.e.,* $m(\theta) = 0$ *almost surely) if and only if*

$$\int_\Theta \exp\left(\mu^{-1}\Delta u(\theta)\right) dP(\theta) \leq 1\ ; \tag{5}$$

c) $p_1 \in (0,1)$ *if and only if*

$$\int_\Theta \exp\left(\mu^{-1}\Delta u(\theta)\right) dP(\theta) > 1 \text{ and } \int_\Theta \exp\left(-\mu^{-1}\Delta u(\theta)\right) dP(\theta) > 1\ ; \tag{6}$$

in this case, the optimal strategy m is characterized by

$$\Delta u(\theta) = \mu \cdot \left[g'(m(\theta)) - g'(p_1)\right] \tag{7}$$

for all $\theta \in \Theta$*, where*

$$g'(x) = \ln\left(\frac{x}{1-x}\right) .$$

PROOF:
See Appendix A.

These results are intuitive. Since the information cost is convex, the agent's objective is concave, which gives rise to the uniqueness of the optimal strategy.

In case a), condition (4) holds if action 1 is very likely the ex ante best action and the cost of acquiring information is sufficiently high. Hence the agent just takes action 1 without acquiring any information. Similarly, case b) implies that if action 0 is ex ante very likely to dominate action 1 and the information cost is sufficiently high, the agent always takes action 0. In this two cases, marginal benefit of acquiring information is less than the marginal cost. Hence the decision maker chooses not to acquire any information.

In case c), as captured by the two inequalities, neither action 1 nor action 0 is ex ante dominant, thus there is information acquisition and $m(\cdot)$ is no longer a constant.

In order to get some intuition, consider an extreme case where action 1 is dominant, i.e., the payoff gain $\Delta u(\theta) > 0$ almost surely. It is obvious that the agent will always take action 1 regardless of μ, the marginal cost of information acquisition.

When neither action is dominant, i.e.,

$$\Pr(\Delta u(\theta) > 0) > 0 \text{ and } \Pr(\Delta u(\theta) < 0) > 0\ ,$$

the marginal cost of information acquisition μ plays a role.

On the one hand,

$$\lim_{\mu\to\infty}\int \exp\left(\pm\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) = 1 \ .$$

Hence Proposition 1 predicts that no information is acquired if μ is high enough.

On the other hand, since

$$\begin{aligned}
& \lim_{\mu\to 0}\frac{d}{d\mu^{-1}}\int \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) \\
= \ & \lim_{\mu\to 0}\int \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right)\Delta u\left(\theta\right) dP\left(\theta\right) \\
= \ & \lim_{\mu\to 0}\int_{\Delta u(\theta)>0} \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right)\Delta u\left(\theta\right) dP\left(\theta\right) \\
& + \Pr\left(\Delta u\left(\theta\right)=0\right) + \lim_{\mu\to 0}\int_{\Delta u(\theta)<0} \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right)\Delta u\left(\theta\right) dP\left(\theta\right) \\
= \ & +\infty + \Pr\left(\Delta u\left(\theta\right)=0\right) + 0 \\
= \ & +\infty \ ,
\end{aligned}$$

we have

$$\lim_{\mu\to 0}\int \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) > 1 \ .$$

A similar argument leads to

$$\lim_{\mu\to 0}\int \exp\left(-\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) > 1 \ .$$

Therefore, Proposition 1 reads that there must exist information acquisition if the marginal cost of information is sufficiently low. This interpretation coincides with our intuition that the agent rationally decides whether to acquire information through comparing the cost to the benefit of information acquisition.

When neither action is dominant and the marginal cost of information acquisition takes intermediate values, the agent finds it optimal to acquire some information to make her action (partially, in a random manner) contingent on θ. This is the case specified by condition (6). Since g' is strictly increasing, (7) implies that $m\left(\theta\right)$, the conditional probability of choosing action 1, is increasing with respect to payoff gain $\Delta u\left(\theta\right)$. This is intuitive. The left hand side of (7) represents the marginal benefit of increasing $m\left(\theta\right)$, while the right hand side of (7) is the marginal cost of information when increasing $m\left(\theta\right)$. Therefore, if deciding to acquire information, the agent will equate her marginal benefit with her marginal cost of doing so.

AN EXAMPLE

The following example provides some intuition behind the agent's information acquisition strategy.

Let θ distribute according to $N(t, 1)$ and

$$\Delta u(\theta) = \theta .$$

It is easy to verify that the agent always chooses action 1 (action 0) if and only if $t \geq \mu^{-1}/2$ ($t \leq -\mu^{-1}/2$). In this case, action 1 (action 0) is superior to action 0 (action 1) ex ante (i.e., $|t|$ is large) and the cost in acquiring information is relatively high (i.e., μ is large). Hence it is not worth acquiring any information at all.

Let $t = 0$, then the agent finds it optimal to acquire some information. According to (7), the optimal information acquisition strategy $m(\theta)$ satisfies

$$\theta/\mu = g'(m(\theta)) - g'\left(\int_\Theta m(\theta)\, dP(\theta)\right), \tag{8}$$

where

$$g'(m) = \ln \frac{m}{1-m} .$$

Since prior $N(0, 1)$ is symmetric about the origin and payoff gain $\Delta u(\theta)$ is an odd function, the agent is indifferent on average, i.e.,

$$\int_\Theta m(\theta)\, dP(\theta) = 1/2 .$$

Hence

$$g'\left(\int_\Theta m(\theta)\, dP(\theta)\right) = 0$$

and (8) becomes

$$\theta/\mu = \ln \frac{m(\theta)}{1-m(\theta)} .$$

Therefore,

$$m(\theta) = \frac{1}{1+\exp(-\theta/\mu)} . \tag{9}$$

First note that

$$\lim_{\mu\to 0} m(\theta) = a(\theta) \triangleq \begin{cases} 1 & \text{if } \theta \geq 0 \\ 0 & \text{if } \theta < 0 \end{cases} .$$

Step function $a(\theta)$ captures the agent's choice under complete information. In this case, the agent can observe the exact value of θ. When $\mu > 0$, the best response is characterized by (9). Since information is no longer free, the agent has to allow some mistake in her response. The conditional probability of mistake is given by

$$|m(\theta) - a(\theta)| \ ,$$

which is decreasing in $|\theta|$, the "price" of mistake. Therefore, the agent deliberately acquires information to balance the price of mistake and the cost of information.

Second, parameter μ measures the difficulty in acquiring information. Figure 1 shows how $m(\theta)$ varies with this parameter.

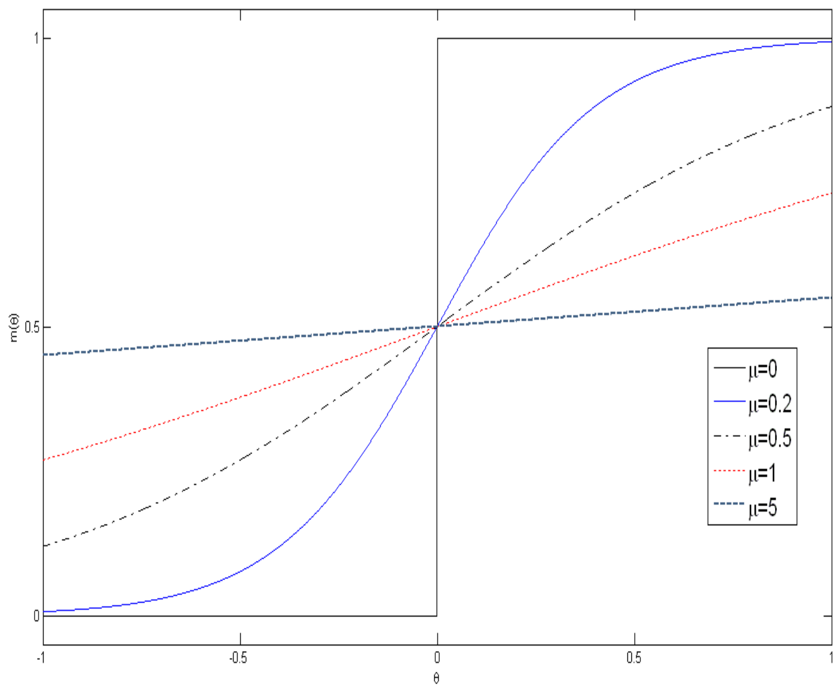

FIGURE 1. INFORMATION ACQUISITION UNDER VARIOUS INFORMATION COSTS

When $\mu = 0$, information acquisition incurs no cost and the agent's response is a step function. She never makes mistake. When μ becomes larger, she starts to compromise the accuracy of her decision to save information cost. Larger μ leads to flatter $m(\theta)$. Finally, when μ is extremely large, $m(\theta)$ is almost constant and the agent almost stops acquiring information.

Third, since the agent's action is highly sensitive to θ where slope $\left|\frac{dm(\theta)}{d\theta}\right|$ is large, $\left|\frac{dm(\theta)}{d\theta}\right|$ reflects her attentiveness around θ. Under this interpretation, Figure 1 reveals that the agent actively collects information for intermediate values of the fundamental but is rationally inattentive to values at the tails. This result coincides with our intuition. When θ is too high (low), the agent should take action 1 (action 0) anyway. Hence the information about θ on the tails are not so relevant to her payoff. When θ takes intermediate values, the agent's payoff gain from taking action 1 over action 0 depends crucially on the sign of θ. Therefore, the information about θ around zero is payoff-relevant and attracts most of her attention.

We have been focusing on binary-signal information structures. Next subsection justifies this setup.

B. Justifying the Binary-signal Information Structure

Generally, an agent can purchase any information structure $((X, \sigma), \pi)$. Here X is the set of realizations of the signal, σ is a σ-algebra on X, and $\forall \theta \in \Theta$, $\pi(\cdot|\theta)$ is a probability measure on X. $\pi(\cdot|\cdot)$ conveys information about state θ in the sense that for any event $A \subset X$, $\pi(A|\theta)$ specifies the conditional probability of A given θ. Before making a decision, the agent can acquire information about the state in the form of an information structure. An information structure specifies both the quantity and qualitative nature of the information.

The binary-signal information structure analyzed above is a special case with $X = \{0, 1\}$ and $\pi(1|\theta) = m(\theta)$ (and so $\pi(0|\theta) = 1 - m(\theta)$). For binary choice problem with flexible information acquisition, it suffices to restrict our attention to this special class of information structures. To see this, let $((X, \sigma), \pi)$ be any information structure chosen by the agent. Given $((X, \sigma), \pi)$, the agent optimally chooses her action rule as $a : X \to [0, 1]$, where $a(x)$ is the probability of taking action 1 upon receiving signal x. Let

$$\begin{aligned} X_1 &= \{x \in X : a(x) = 1\}, \\ X_0 &= \{x \in X : a(x) = 0\}, \end{aligned}$$

and

$$X_{ind} = \{x \in X : a(x) \in (0, 1)\}.$$

X_1 (X_0) is the set of signal realizations such that the agent definitely takes action 1 (0). She is indifferent when her signal belongs to X_{ind}. Then (X_1, X_0, X_{ind}) forms a partition of X. Since the only use of the signal is to make a binary decision, a signal differentiating more finely among the states just conveys redundant information and wastes the agent's attention. Hence the agent will not discern signal realizations within any of X_1, X_0 and X_{ind}. In addition, because she is indifferent between action 0 and 1 upon event X_{ind}, she would rationally pay no attention to distinguish this event from other realizations. Hence, the agent always play pure strategies upon receiving her signal. Therefore, the agent always prefers binary-signal information structures.[8]

[8](Woodford 2009) has a similar argument that the agent only needs to acquire a "yes/no" signal.

III. Security Design with Information Acquisition

A. Basic Setup

We consider a two-period game with two players. One player is a seller that owns N assets at period 0. These assets generate verifiable random cash flows $\overrightarrow{\theta} \in \Theta \subset \mathbb{R}_+^N$ in period 1[9]. The other player is a potential buyer holding consumption goods (money) at period 0. Player i's utility function is given by

$$u_i = c_{i0} + \delta_i \cdot c_{i1}, \tag{10}$$

where c_{it} denotes player i's consumption at period t and $\delta_i \in [0, 1]$ is her subjective discount factor, $i \in \{s, b\}(\{s, b\}$ stands for $\{seller, buyer\})$. We assume $\delta_b > \delta_s$ to represent that the seller has a better investment opportunity than the buyer. This assumption creates the trading demand. Both agents may benefit from transferring some goods to the seller at date 0 and compensating the buyer with repayment backed by the random cash flows $\overrightarrow{\theta}$ at date 1.

Similar to (Dang, Gorton and Holmstrom 2011), we assume no information asymmetry at period 0 to focus on the adverse selection resulting from endogenous information acquisition. Hence the two agents start with identical information about $\overrightarrow{\theta}$, which is represented by a full support common prior P over Θ. Without loss of generality, we assume that P is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}_+^N$.

A security backed by $\overrightarrow{\theta}$, the cash flows of the N assets, is a mapping $s : \Theta \to \mathbb{R}_+$ such that $\forall \overrightarrow{\theta} \in \Theta$, $s\left(\overrightarrow{\theta}\right) \in \left[0, \sum_{n=1}^{N} \theta_n\right]$. A contract $(s(\cdot), q)$ is a security $s(\cdot)$ associated with a price $q > 0$. Throughout the paper, we focus on the case where one player proposes a take-it-or-leave-it contract $(s(\cdot), q)$ to her opponent, who then acquires information and decides whether to accept it. This setup captures the idea that some agents in the markets of securitized assets are less sophisticated than others and cannot produce private information about the underlying cash flows. This separation between bargaining power and ability of information acquisition also makes our problem tractable.[10]

We first study the case where the seller designs the contract and the buyer acquires information. We then highlight two key factors driving the unique optimality of issuing securitized debt. We finally exchange the bargaining power and the ability of information acquisition to show the robustness of our main results.

[9] Here the assumption of verifiable cash flows is natural, since we generally have third parties monitor and collect the underlying loans and distribute the cash flows to the holders of asset backed securities.

[10] We would have to study a much more complicated signaling game if the issuer can produce private information before her proposal. In that case, the set of possible signals consists of all contracts, which is a functional space. To the best of our knowledge, this kind of signaling games are rarely studied before. (Peter M. DeMarzo, Ilan Kremer and Andrzej Skrzypacz 2005) does consider a security design problem where potential signals are securities. But their approach does not fit our framework of flexible information acquisition. In the literature, either the informed agent chooses finite-dimension signals (e.g., the level of debt in (Stephen A. Ross 1977), the retaining fraction of the equity in (Hayne E Leland and David H Pyle 1977), etc.), or the issuer designs the security before obtaining her information (e.g., (DeMarzo and Duffie 1999), (Biais and Mariotti 2005)).

B. Optimal Contract when the Seller Designs

Consider the particular binary choice problem where the agent is a risk neutral buyer with utility (10). Action 1 corresponds to buying the ABS $s\left(\overrightarrow{\theta}\right)$ at price q and action 0 corresponds to not buying. Write $m_{s,q}$ for the buyer's optimal strategy when facing contract (s, q). Let

$$\overline{p}_{s,q} = \int_{\Theta} m_{s,q}\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right)$$

be the buyer's unconditional probability of accepting the offer. The seller thus enjoys an expected utility

$$W(s,q) = \int_{\Theta} m_{s,q}\left(\overrightarrow{\theta}\right) \cdot \left[q - \delta_s \cdot s\left(\overrightarrow{\theta}\right)\right] dP\left(\overrightarrow{\theta}\right) . \tag{11}$$

The seller's problem is to choose a contract (s, q) satisfying $s\left(\overrightarrow{\theta}\right) \in \left[0, \sum_{n=1}^{N} \theta_n\right]$ to maximize $W(s, q)$. Let $(s^*(\cdot), q^*)$ denote the optimal contract and

$$\overline{p}_{s^*,q^*} = \int_{\Theta} m_{s^*,q^*}\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right)$$

be the corresponding probability of trade.

According to Proposition 1, there are three possible cases: a) $\overline{p}_{s^*,q^*} = 1$; b) $\overline{p}_{s^*,q^*} = 0$; and c) $\overline{p}_{s^*,q^*} \in (0, 1)$. We first argue that case b) is impossible.

PROPOSITION 2: $\overline{p}_{s^*,q^*} > 0$*, i.e., trade happens with positive probability.*

PROOF:

We prove by constructing a securitized debt that generates positive expected payoff to the seller. Let $\beta \in \left(\delta_s \delta_b^{-1}, 1\right)$ and

$$f(q) = \int_{\Theta} \min\left(\sum_{n=1}^{N} \theta_n, \beta \delta_s^{-1} q\right) dP\left(\overrightarrow{\theta}\right) .$$

Since P is a continuous distribution and $\beta^{-1}\delta_s\delta_b^{-1} < 1$, there exists $q_0 > 0$ s.t.

$$\Pr\left(\sum_{n=1}^{N} \theta_n \geq \beta \delta_s^{-1} q\right) > \beta^{-1}\delta_s\delta_b^{-1}$$

for all $q \in \left[0, q_0\right]$. Hence for any $q \in (0, q_0)$,

$$\begin{aligned} f'(q) &= \beta\delta_s^{-1} \int_{\left\{\overrightarrow{\theta} \in \Theta : \sum_{n=1}^{N} \theta_n \geq \beta\delta_s^{-1} q\right\}} 1 \cdot dP\left(\overrightarrow{\theta}\right) \\ &= \Pr\left(\sum_{n=1}^{N} \theta_n \geq \beta\delta_s^{-1} q\right) \cdot \beta\delta_s^{-1} \\ &> \beta^{-1}\delta_s\delta_b^{-1} \cdot \beta\delta_s^{-1} = \delta_b^{-1}. \end{aligned}$$

Note that

$$f(0) = 0\,,$$

which implies that

$$f(q) > \delta_b^{-1} q$$

for all $q \in (0, q_0)$.

Consider a securitized debt

$$s\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N} \theta_n, D\right)$$

with face value $D = \beta\delta_s^{-1} q$ and price $q \in (0, q_0)$. The buyer's payoff gain from accepting this offer over rejecting it is

$$\Delta u\left(\overrightarrow{\theta}\right) = \delta_b \cdot s\left(\overrightarrow{\theta}\right) - q. \tag{12}$$

By Jensen's inequality,

$$\begin{aligned} & \int_{\Theta} \exp\left(\mu^{-1} \Delta u\left(\overrightarrow{\theta}\right)\right) dP\left(\overrightarrow{\theta}\right) \\ \geq\ & \exp\left(\mu^{-1} \int_{\Theta} \Delta u\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right)\right) \\ =\ & \exp\left(\mu^{-1}\left[\delta_b \cdot \int_{\Theta} \min\left(\sum_{n=1}^{N} \theta_n, \beta\delta_s^{-1} q\right) dP\left(\overrightarrow{\theta}\right) - q\right]\right) \\ =\ & \exp\left(\mu^{-1}\left[\delta_b \cdot f(q) - q\right]\right) \\ >\ & \exp(0) = 1, \end{aligned}$$

where the last inequality comes from (12). Hence according to Proposition 1, $\overline{p}_{s,q} > 0$.

Then, the seller's expected payoff from this contract is

$$\begin{aligned} W(s,q) &= \int_\Theta m_{s,q}\left(\overrightarrow{\theta}\right)\cdot\left[q-\delta_s\cdot s\left(\overrightarrow{\theta}\right)\right]dP\left(\overrightarrow{\theta}\right) \\ &= \int_\Theta m_{s,q}\left(\overrightarrow{\theta}\right)\cdot\left[q-\delta_s\cdot\min\left(\sum_{n=1}^N\theta_n,\beta\delta_s^{-1}q\right)\right]dP\left(\overrightarrow{\theta}\right) \\ &\geq \int_\Theta m_{s,q}\left(\overrightarrow{\theta}\right)\cdot\left[q-\delta_s\cdot\beta\delta_s^{-1}q\right]dP\left(\overrightarrow{\theta}\right) \\ &= (1-\beta)\,q\cdot\overline{p}_{s,q} > 0\,. \end{aligned}$$

By definition, the seller's expected payoff through the optimal contract is $W(s^*,q^*) \geq W(s,q) > 0$. This directly implies $\overline{p}_{s^*,q^*} > 0$ since $\overline{p}_{s^*,q^*} = 0$ always generates zero expected payoff to the seller. This concludes the proof.

The key of the proof is to show that the seller can always enjoy a positive expected payoff through proposing a securitized debt. Hence her optimal contract must also generate a positive expected payoff, which can be achieved only through a successful trade. Although facing adverse selection, the seller always prefers trade. This is because she owns all bargaining power. She is able to minimize the negative effect of information acquisition through appropriately designing a contract and thus enjoy the benefit from trade.

According to Proposition 2, only case a) and c) are possible. In case a) $\overline{p}_{s^*,q^*} = 1$ and the buyer does not acquire any information. In case c), $\overline{p}_{s^*,q^*} \in (0,1)$ and the buyer does acquire some information. We first study the seller's optimal contract in case a).

Optimal Contract without Inducing Information Acquisition

A direct application of Proposition 1 suggests that any contract (s,q) that does not induce information acquisition must satisfy

$$\mathbf{E}\exp\left(-\mu^{-1}\left[\delta_b\cdot s\left(\overrightarrow{\theta}\right)-q\right]\right)\leq 1\,,$$

i.e.,

$$q \leq -\mu\ln\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s\left(\overrightarrow{\theta}\right)\right)\,. \tag{13}$$

Intuitively, the buyer just accepts the offer when the price is low enough relative to the repayment of the security. This inequality must bind for seller's optimal contract, otherwise she can benefit from increasing the price q. Hence, (13) reduces to

$$q = -\mu\ln\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s\left(\overrightarrow{\theta}\right)\right)\,. \tag{14}$$

Since the contract is always accepted, the seller's expected payoff becomes

$$\begin{aligned} & \int_{\Theta} \left[q - \delta_s \cdot s\left(\overrightarrow{\theta}\right) \right] dP\left(\overrightarrow{\theta}\right) \\ = \ & q - \delta_s \cdot \mathbf{E} s\left(\overrightarrow{\theta}\right) \\ = \ & -\mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot s\left(\overrightarrow{\theta}\right)\right) - \delta_s \cdot \mathbf{E} s\left(\overrightarrow{\theta}\right). \end{aligned}$$

Hence the seller's problem can be formalized as

$$\min_{s(\cdot)} \mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot s\left(\overrightarrow{\theta}\right)\right) + \delta_s \cdot \mathbf{E} s\left(\overrightarrow{\theta}\right)$$

subject to the feasibility condition

$$s\left(\overrightarrow{\theta}\right) \in \left[0, \sum_{n=1}^{N} \theta_n\right]. \tag{15}$$

PROPOSITION 3: *If the seller's optimal contract induces the buyer to always accept it without acquiring information, it must be a securitized debt*

$$s^*\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N} \theta_n, D^*\right)$$

with price q^**, where the face value is determined by*

$$\begin{aligned} D^* &= D\left(q^*\right) \\ &= \mu\delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q^*, \end{aligned}$$

$q^* > 0$ *is the unique fixed point of*

$$h(q) = -\mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N} \theta_n, D(q)\right)\right)$$

and the expectation is taken under common prior P.

PROOF:

See Appendix A.

First note that the face value has a lower bound, i.e.,

$$D^* > \mu\delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s].$$

Hence if the maximal cash flow

$$\sup\left\{\sum_{n=1}^{N}\theta_n : \overrightarrow{\theta} \in \Theta\right\} \leq \mu\delta_b^{-1}\cdot[\ln\delta_b - \ln\delta_s] \ ,$$

the optimal security is actually the pool of all assets. This could happen when the seller has an extremely good investment opportunity relative to the buyer (i.e., $\ln\delta_b - \ln\delta_s \gg 1$) or it is too hard for the buyer to acquire information (i.e., $\mu \gg 1$). As a direct implication, when the buyer cannot acquire any information (i.e., $\mu \to \infty$), the seller just sells the pool of all assets at price

$$\delta_b \cdot \mathbf{E}\left[\sum_{n=1}^{N}\theta_n\right]$$

and enjoys the maximal trading surplus

$$(\delta_b - \delta_s)\cdot\mathbf{E}\left[\sum_{n=1}^{N}\theta_n\right] \ .$$

Another interesting observation comes from equation (14), which implies

$$\begin{aligned} q^* &= -\mu\ln\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}\right)\right) \\ &\leq -\mu\ln\left(\exp\left(-\mu^{-1}\delta_b\cdot\mathbf{E}s^*\left(\overrightarrow{\theta}\right)\right)\right) \\ &= \delta_b\cdot\mathbf{E}s^*\left(\overrightarrow{\theta}\right) \ , \end{aligned}$$

where the inequality follows Jensen's inequality. Since the offer induces no information acquisition, both parties remain symmetrically informed and the seller should have charged the buyer $\delta_b\cdot\mathbf{E}s^*\left(\overrightarrow{\theta}\right)$. However, the seller finds it optimal to charge a lower price q^* to bribe the buyer not to acquire information.

In the rest of this section, we show that securitized debt remains uniquely optimal even if there is information acquisition.

Optimal Contract with Information Acquisition

According to Proposition 1, any contract $(s(\cdot), q)$ that induces the buyer to acquire information must satisfy

$$\mathbf{E}\exp\left(\mu^{-1}\left[\delta_b\cdot s\left(\overrightarrow{\theta}\right) - q\right]\right) > 1 \tag{16}$$

and

$$\mathbf{E}\exp\left(-\mu^{-1}\left[\delta_b\cdot s\left(\overrightarrow{\theta}\right) - q\right]\right) > 1 \ , \tag{17}$$

where the expectation is taken according to common prior P. That is, neither accepting nor rejecting the offer is dominant ex ante, and thus the buyer finds it optimal to acquire some information.

Given such a contract, Proposition 1 prescribes that the buyer's optimal strategy $m_{s,q}$ is uniquely characterized by

$$\delta_b \cdot s\left(\overrightarrow{\theta}\right) - q = \mu \cdot \left[g'\left(m_{s,q}\left(\overrightarrow{\theta}\right)\right) - g'\left(\overline{p}_{s,q}\right)\right], \tag{18}$$

where

$$\overline{p}_{s,q} = \int_\Theta m_{s,q}\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right)$$

is the buyer's unconditional probability of accepting the offer.

Taking into account of the buyer's response $m_{s,q}$, the seller chooses $(s(\cdot), q)$ to maximize her expected payoff

$$W(s,q) = \int_\Theta m_{s,q}\left(\overrightarrow{\theta}\right) \cdot \left[q - \delta_s \cdot s\left(\overrightarrow{\theta}\right)\right] dP\left(\overrightarrow{\theta}\right)$$

subject to (16), (17), (18) and the feasibility condition

$$s\left(\overrightarrow{\theta}\right) \in \left[0, \sum_{n=1}^{N} \theta_n\right]. \tag{19}$$

It is worth noting that both (16) and (17) should not bind for the optimal contract, otherwise no information will be acquired according to Proposition 1. Hence, conditional on the fact that the optimal contract does induce information acquisition, these two constraints could be ignored during optimization.

We derive the optimal contract $(s^*(\cdot), q^*)$ through calculus of variations. That is, see how the seller's expected payoff responds to the perturbation of her optimal contract.

Let $s\left(\overrightarrow{\theta}\right) = s^*\left(\overrightarrow{\theta}\right) + \alpha \cdot \varepsilon\left(\overrightarrow{\theta}\right)$ be an arbitrary perturbation of $s^*(\cdot)$. The buyer's best response $m_{s,q^*}(\cdot)$ is implicitly determined by $s(\cdot)$ through functional equation (18). Hence we need first characterize how $m_{s,q^*}(\cdot)$ varies with respect to the perturbation of $s^*(\cdot)$.

LEMMA 1: *For any perturbation $s\left(\overrightarrow{\theta}\right) = s^*\left(\overrightarrow{\theta}\right) + \alpha \cdot \varepsilon\left(\overrightarrow{\theta}\right)$, the response of the buyer's strategy $m_{s,q^*}(\cdot)$ is characterized by*

$$\begin{aligned} \left.\frac{dm_{s,q^*}\left(\overrightarrow{\theta}\right)}{d\alpha}\right|_{\alpha=0} &= \mu^{-1}\delta_b \cdot \left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} \varepsilon\left(\overrightarrow{\theta}\right) \\ &+ \frac{\left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} \mu^{-1}\delta_b \int_\Theta \left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} \varepsilon\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right)}{\left[g''\left(\overline{p}_{s^*,q^*}\right)\right]^{-1} - \int_\Theta \left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} dP\left(\overrightarrow{\theta}\right)}. \end{aligned} \tag{20}$$

PROOF:

See Appendix A.

The first term of the right hand side of (20) is the buyer's local response to $\varepsilon\left(\overrightarrow{\theta}\right)$. It is of the same sign as the perturbation $\varepsilon\left(\overrightarrow{\theta}\right)$. When the repayment increases at state $\overrightarrow{\theta}$, the buyer is more likely to accept the offer at this state. The second term measures the buyer's average response to perturbation $\varepsilon\left(\overrightarrow{\theta}\right)$ over all states. It is straightforward to verify that the denominator is positive due to Jensen's inequality. Hence, if on average the perturbation increases her repayment, the buyer would like to accept the offer more often.

Now we can calculate the variation of the seller's expected payoff $W(s,q^*)$. Taking derivative with respect to α at $\alpha=0$ for both sides of (11) leads to

(21)

$$\left.\frac{dW(s,q^*)}{d\alpha}\right|_{\alpha=0} = \int_\Theta \left.\frac{dm_{s,q^*}\left(\overrightarrow{\theta}\right)}{d\alpha}\right|_{\alpha=0} \left[q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right)\right] dP\left(\overrightarrow{\theta}\right) - \delta_s \int_\Theta m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\varepsilon\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right) .$$

Substitute (20) into (21) and manipulate we get

$$\left.\frac{dW(s,q^*)}{d\alpha}\right|_{\alpha=0} = \int_\Theta r\left(\overrightarrow{\theta}\right)\cdot\varepsilon\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right) , \tag{22}$$

where

(23)

$$r\left(\overrightarrow{\theta}\right) = -\delta_s m_{s^*,q^*}\left(\overrightarrow{\theta}\right) + \mu^{-1}\delta_b\left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1}\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right)$$

and

$$w = \frac{\int_\Theta \left[q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right)\right]\left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} dP\left(\overrightarrow{\theta}\right)}{\left[g''\left(\overline{p}_{s^*,q^*}\right)\right]^{-1} - \int_\Theta \left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} dP\left(\overrightarrow{\theta}\right)} .$$

Note that w is a constant that does not depend on $\overrightarrow{\theta}$. Its value is endogenously determined in equilibrium. Here $r\left(\overrightarrow{\theta}\right)$ is the Frechet derivative[11] of $W(s,q^*)$ at s^*, it measures the marginal contribution of any perturbation to the seller's expected payoff. The first term of (23) is the direct contribution of perturbing s^* while ignoring the variation of $m_{s^*,q^*}\left(\overrightarrow{\theta}\right)$. The second term measures the indirect contribution through the variation of $m_{s^*,q^*}\left(\overrightarrow{\theta}\right)$. This expression represents the chain rule of the calculus of variations.

[11] For the readers not familiar with this concept, just think of Frechet derivative as the gradient of $W(s,q^*)$ at "vector" s. Indeed, the gradient is a special case of Frechet derivative when $\#\Theta$ is finite.

Let

$$A_0 = \left\{ \overrightarrow{\theta} \in \Theta : \overrightarrow{\theta} \neq \overrightarrow{0}, s^*\left(\overrightarrow{\theta}\right) = 0 \right\},$$

$$A_1 = \left\{ \overrightarrow{\theta} \in \Theta : \overrightarrow{\theta} \neq \overrightarrow{0}, s^*\left(\overrightarrow{\theta}\right) \in \left(0, \sum_{n=1}^{N} \theta_n \right) \right\}$$

and

$$A_2 = \left\{ \overrightarrow{\theta} \in \Theta : \overrightarrow{\theta} \neq \overrightarrow{0}, s^*\left(\overrightarrow{\theta}\right) = \sum_{n=1}^{N} \theta_n \right\}.$$

In regions A_0 and A_2, $s^*(\cdot)$ is bounded by its lower bound and upper bound, respectively. In region A_1, $s^*(\cdot)$ is off the boundaries. Then $\{A_0, A_1, A_2\}$ is a partition of $\Theta \backslash \left\{ \overrightarrow{0} \right\}$. Since $s^*(\cdot)$ is the optimal security,

$$\left. \frac{dW(s, q^*)}{d\alpha} \right|_{\alpha=0} \leq 0$$

holds for any feasible[12] perturbation $\varepsilon\left(\overrightarrow{\theta}\right)$. Hence (22) implies

$$r\left(\overrightarrow{\theta}\right) \begin{cases} \leq 0 & \text{if } \overrightarrow{\theta} \in A_0 \\ = 0 & \text{if } \overrightarrow{\theta} \in A_1 \\ \geq 0 & \text{if } \overrightarrow{\theta} \in A_2 \end{cases}. \tag{24}$$

Since g is strictly convex, $g'' > 0$ and (24) can be rewritten as

$$\begin{aligned} & r\left(\overrightarrow{\theta}\right) \cdot g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) \\ = \quad & -\delta_s m_{s^*,q^*}\left(\overrightarrow{\theta}\right) g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ & \begin{cases} \leq 0 & \text{if } \overrightarrow{\theta} \in A_0 \\ = 0 & \text{if } \overrightarrow{\theta} \in A_1 \\ \geq 0 & \text{if } \overrightarrow{\theta} \in A_2 \end{cases}. \end{aligned} \tag{25}$$

Recall that given the optimal contract $(s^*(\cdot), q^*)$, the buyer's best response $m_{s^*,q^*}\left(\overrightarrow{\theta}\right)$ is characterized by

$$\delta_b \cdot s^*\left(\overrightarrow{\theta}\right) - q^* = \mu \cdot \left[g'\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) - g'\left(\overline{p}_{s^*,q^*}\right) \right], \tag{26}$$

where

$$\overline{p}_{s^*,q^*} = \int_{\Theta} m_{s^*,q^*}\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right)$$

[12] A perturbation ε is feasible with respect to s^* if $\exists \alpha > 0$, s.t. $\forall \overrightarrow{\theta} \in \Theta$, $s^*\left(\overrightarrow{\theta}\right) + \alpha \cdot \varepsilon\left(\overrightarrow{\theta}\right) \in \left[0, \sum_{n=1}^{N} \theta_n\right]$.

is the buyer's unconditional probability of accepting the optimal contract $(s^*(\cdot), q^*)$.

Then, (25)[13] together with (26) determines the optimal contract $(s^*(\cdot), q^*)$. Let $m = f_1(s)$ and $m = f_2(s)$ be the two continuous functions implicitly defined by

$$-\delta_s \cdot m \cdot g''(m) + \mu^{-1}\delta_b \left(q^* - \delta_s \cdot s + w\right) = 0 \tag{27}$$

and

$$\delta_b \cdot s - q^* = \mu \cdot \left[g'(m) - g'\left(\overline{p}_{s^*,q^*}\right)\right], \tag{28}$$

respectively. We have $f_1'(s) < 0$ and $f_2'(s) > 0$ since $\left[m \cdot g''(m)\right]' > 0$ and $g''(m) > 0$. By definition,

$$m_{s^*,q^*}\left(\overrightarrow{\theta}\right) = f_1\left(s^*\left(\overrightarrow{\theta}\right)\right) \text{ implies } r\left(\overrightarrow{\theta}\right) = 0\,.$$

Also note that $m_{s^*,q^*}\left(\overrightarrow{\theta}\right) = f_2\left(s^*\left(\overrightarrow{\theta}\right)\right)$ for all $\overrightarrow{\theta} \in \Theta$. Now we can characterize the optimal security through analyzing f_1 and f_2 together.

PROPOSITION 4: $\Pr(A_0) = 0$, *where* $A_0 = \left\{\overrightarrow{\theta} \in \Theta : \overrightarrow{\theta} \neq \overrightarrow{0}, s^*\left(\overrightarrow{\theta}\right) = 0\right\}$.

PROOF:

See Appendix A.

This proposition states that constraint $s\left(\overrightarrow{\theta}\right) \geq 0$ never binds. The logic underlying the proof is that on the boundary $s\left(\overrightarrow{\theta}\right) = 0$, although an increment of $s\left(\overrightarrow{\theta}\right)$ increases the seller's repayment, it increases the probability of trading even more. Hence the seller on average gains through deviating from the lower boundary. As its implication, it is not optimal to issue equity residual/call option to raise liquidity.

For those states in A_1, where the limited liability constraint

$$s\left(\overrightarrow{\theta}\right) \leq \sum_{n=1}^{N} \theta_n$$

does not bind either, both

$$m_{s^*,q^*}\left(\overrightarrow{\theta}\right) = f_1\left(s^*\left(\overrightarrow{\theta}\right)\right)$$

and

$$m_{s^*,q^*}\left(\overrightarrow{\theta}\right) = f_2\left(s^*\left(\overrightarrow{\theta}\right)\right)$$

[13] One may criticize that Equation (25) is just the first order condition of the seller's optimization problem. It only characterizes the critical points. In principle, we should characterize the largest critical point. However, our argument works for any critical point and thus our results are immune to this critique.

must hold. Since $f_1'(s) < 0$ and $f_2'(s) > 0$, $f_1(s)$ and $f_2(s)$ intersect at most once. Hence $s^*\left(\overrightarrow{\theta}\right)$ should be a constant and the buyer has no incentive to acquire information within region A_1. This result coincides with our intuition. If the limited liability constraint never binds, the seller would issue a security with constant repayment to avoid the buyer's information acquisition. However, once the underlying cash flows are too low to support such constant, $s^*\left(\overrightarrow{\theta}\right)$ reaches the limited liability boundary and equals $\sum_{n=1}^{N}\theta_n$. The next proposition shows that the optimal security must be a securitized debt.

PROPOSITION 5: *If the seller's optimal contract induces the buyer to acquire information, it must be a securitized debt* $s^*\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N}\theta_n, D^*\right)$.

PROOF:
See Appendix A.

Together with Proposition 2 and 3, this proposition enables us to conclude that pooling the assets and issuing a senior tranche is always the uniquely optimal way to raise liquidity. Pooling is directly derived from the seller's desire to maximize liquidity. It has nothing to do with the consideration of risk diversification since both agents are risk-neutral. The flat tail of the optimal security results from the seller's effort to minimize her opponent's information acquisition. In contrast to the non-uniqueness result in (Dang, Gorton and Holmstrom 2011), we can show the unique optimality of debt because of our flexible information acquisition framework. In (Dang, Gorton and Holmstrom 2011), only two extreme information structures are available in the setup of costly state verification while infinite forms of securities can be designed, which inevitably results in the indistinguishability of some securities. In our framework, with help of flexibility, the variety of available information structures matches the variety of potential securities to be designed, and thus the uniqueness of the standard securitized debt could be guaranteed. Quasi-debts are no longer optimal in our model. By reshaping the uneven tail above the price of a quasi-debt to a flat one, not only the buyer's information cost could be saved but also potential loss of trade from adverse selection could be mitigated. The resulted surplus could be employed by the seller to make both parties better off, and thus ultimately make a better provision of liquidity possible. Moreover, this flexibility also enables us to show the optimality of pooling and tranching in a broader class of environments than (Dang, Gorton and Holmstrom 2011) and without assuming a sufficiently large number of underlying assets as in (DeMarzo 2005)[14].

In addition, our qualitative result does not rely on the distributional details of underlying assets, while most models in literature are built upon specific assumptions about the cash flows. Since the stochastic interdependence among the underlying assets could be complex and violate such assumptions, our model provides a better explanation for the prevalence of securitization in financial markets.

[14](DeMarzo 2005) shows that the benefit of pooling achieves a theoretical maximum as the number of underlying assets approaches infinity.

The security design literature usually assumes Monotone Likelihood Ratio Property (MLRP) or similar conditions to guarantee a meaningful result. Our framework justifies this assumption through endogenizing the information structure. According to Proposition 5, the optimal security $s^*\left(\overrightarrow{\theta}\right)$ is non-decreasing in the sum of cash flows. Proposition 1 implies that the best information structure $m_{s^*,q^*}\left(\overrightarrow{\theta}\right)$ is increasing in the payoff gain $\delta_b \cdot s^*\left(\overrightarrow{\theta}\right) - q^*$. Hence $m_{s^*,q^*}\left(\overrightarrow{\theta}\right)$ is also non-decreasing in the sum of the cash flows. Therefore, the larger the cash flows, the higher the probability that the buyer gets a signal asking her to accept. This can be interpreted as a generalized MLRP for multi-dimensional states.

To facilitate the analysis, the security design literature usually restrict their attention to the set of "regular" securities, which are non-decreasing in the underlying cash flows (e.g., (DeMarzo and Duffie 1999), (DeMarzo 2005)). We do not have such restriction, but show that the optimal security naturally turns out to be non-decreasing.

Finally, (Dang, Gorton and Holmstrom 2011) get debt contract uniquely optimal when their fixed information cost is zero. This can be viewed as a special case of our model where marginal cost of information acquisition vanishes.

Understanding the Origin of Uniqueness

For readers familiar with the approach of costly state verification (CSV), a question naturally arises regarding the uniqueness of the optimal contract. Both (Townsend 1979) and (Dang, Gorton and Holmstrom 2011) employ CSV, why does the former but not the latter get debt uniquely optimal? In last subsection, we have attributed the non-uniqueness in (Dang, Gorton and Holmstrom 2011) to the rigidity of CSV. This argument is correct when comparing (Dang, Gorton and Holmstrom 2011) to our model, but not fully convincing when (Townsend 1979) is also considered. To fully understand the different results in (Dang, Gorton and Holmstrom 2011), (Townsend 1979) and our model, we first highlight the essence of flexibility. In principle, general flexible choice, not necessarily restricted to flexible information acquisition, enables an economic agent to make state-contingent responses. In other words, the agent can make a best response in one state, and can make another best response in another state. In all these three models, the contract designer is endowed with flexibility, in the sense that she can assign state-contingent repayment through designing any form of security. What matters to shape the different results regarding uniqueness of the optimal contract relies on the potential flexibility of the other party who decides whether to accept the offer. Through comparing these three models, we argue that the origin of the uniqueness is not only from the flexibility itself, but from the double-sided symmetry of flexibility. Here, double-sided symmetry of flexibility requires that both parties engaged in a potential trade are endowed with the same level of flexibility.

In our framework, ex-ante symmetric information in the form of a double-sided ignorance prevents the buyer to make a state-contingent choice if she only follows the traditional CSV approach to acquire information. However, the buyer in our framework is able to choose state-contingent probability (i.e., $m\left(\overrightarrow{\theta}\right)$) of accepting the offer, namely, she can perform flexible information acquisition. In this sense the buyer enjoys the same level of flexibility as the seller. Given this double-sided symmetry of flexibility in our model, the uniqueness of an optimal contract, which is the standard securitized debt, is guaranteed. In (Dang, Gorton and Holmstrom 2011), however, the buyer can only follow the traditional CSV approach to acquire information, in which only two options, namely, to acquire a signal or not, are available. Moreover, ex-ante symmetric ignorance precludes the possibility of conditioning the action on any private information. Hence the CSV makes the buyer in (Dang, Gorton and Holmstrom 2011) unable to make state-contingent decision. As a result, the desired double-sided symmetry of flexibility fails and the uniqueness of the optimal contract fails as a consequence. Interestingly, (Townsend 1979) also employs the costly state verification approach with two options to model information acquisition, namely, to audit or not, but the unique optimality of a standard debt still emerges. Why it is this case? Different from (Dang, Gorton and Holmstrom 2011) and our framework, in (Townsend 1979) the entrepreneur has information advantage over the lender in the sense that the entrepreneur knows the realized profit of the project which the lender does not know. Thanks to the revelation principle, the lender who acquire information in the interim stage can decide whether to audit or not in any state based on the truth told by the entrepreneur who has private information. In other words, although the lender in (Townsend 1979) still only has two options to acquire information as the buyer in (Dang, Gorton and Holmstrom 2011), such two options in (Townsend 1979) are state-contingent while their counterparts in (Dang, Gorton and Holmstrom 2011) are not. Therefore, the double-sided symmetry of flexibility is still established in (Townsend 1979), and the uniqueness of the optimal contract, also a standard debt, is ensured in their model as well. Figure 2 shows the relation among these three models.

This subsection explores the origin of uniqueness of the optimal contract. We address the optimality of securitized debt in next subsection.

Two Key Factors Driving the Optimality of Securitized Debt

Although our model explains the popularity of securitized debt contracts, it is important to figure out the boundary of our theory. In this subsection, we propose two key factors that drive our results. We show that issuing securitized debt is no longer optimal in absence of these factors.

The first feature of our model is its fixed total risk exposure. Before designing the contract, the seller has already owned assets $\overrightarrow{\theta}$. Hence the assets owned by the seller and the buyer as a whole is invariant with respect to the success or failure of the transaction. This fixed total risk exposure leads to a situation like "zero-sum" game, where any information acquired by the buyer makes herself better off but hurts the seller's benefit through adverse selection. That is, the buyer attempts to acquire information that helps her reject the offer once the repayment is lower than the price and accept the offer in the opposite case. However, whatever quantity and quality of information is acquired has nothing to do with their total risk exposure.

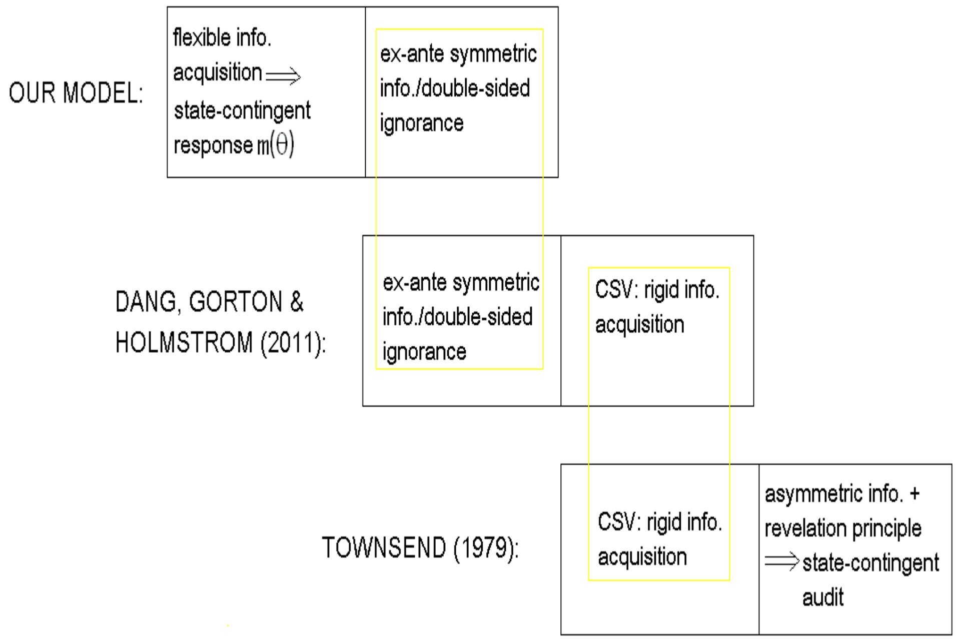

FIGURE 2. RELATION AMONG OUR MODEL, DANG, GORTON & HOLMSTROM (2011) AND TOWNSEND (1979)

The importance of this factor can be seen clearly in our derivation of the optimal security. Since the buyer's incentive to acquire information and the seller's incentive to design the security are totally shaped by their payoff gains from the success over the failure of the transaction, it makes sense to examine their payoff gains. Conditional on $\overrightarrow{\theta}$, the buyer's and seller's payoff gains are

$$\delta_b \cdot s\left(\overrightarrow{\theta}\right) - q$$

and

$$q - \delta_s \cdot s\left(\overrightarrow{\theta}\right),$$

respectively. Both these payoff gains do not explicitly depend on $\overrightarrow{\theta}$. The future cash flows $\overrightarrow{\theta}$ can affect their incentives only through the security $s\left(\overrightarrow{\theta}\right)$. This is the reason that we can define the functions $m = f_1(s)$ and $m = f_2(s)$ rather than $m = f_1\left(s, \overrightarrow{\theta}\right)$

or $m = f_2\left(s, \overrightarrow{\theta}\right)$ in (27) and (28). The simple shape of securitized debt comes from this independence of f_1 and f_2 on $\overrightarrow{\theta}$.

To make our point more clear, we consider a similar problem with variable total risk exposure. The seller is an entrepreneur who wants to raise capital q to take a project that generates cash flow θ. As before, she designs a security $s(\theta)$ and proposes a take-it-or-leave-it offer (s, q) to the bank, who is the buyer that acquires information in the present problem. The entrepreneur's project gets funded and generates future cash flow θ only if the bank accepts the offer. Hence, the total risk exposure depends on whether the transaction succeeds. In this case, the buyer's payoff gain remains the same but the seller's payoff gain becomes

$$\delta_s \cdot [\theta - s(\theta)] \ ,$$

which explicitly depends on θ. As a result, we have $m = f_1(s, \theta)$ rather than $m = f_1(s)$ and the flat part of the debt is no longer optimal. Even if $s(\theta)$ is off the boundaries, the seller would like to fluctuate $s(\theta)$ to induce the buyer to acquire some information. In general, information acquisition benefits the buyer and seller as a whole. It prevents the project to be taken when the cash flow is too low. In fact, this is a story of consulting. The seller designs a state contingent repayment to elicit information from the buyer. Their incentives are aligned rather than opposite to each other.

The second factor that drives our results is homogeneous information acquisition. That is, no state is more special than other states in terms of the difficulty of information acquisition. This property stems from rational inattention[15] and is the reason why our qualittive result does not depend on the stochastic interdependence among the underlying assets. Recall the binary decision problem in Section II, the decision maker's optimal strategy m is characterized by equation (7)

$$\Delta u(\theta) = \mu \cdot \left[g'(m(\theta)) - g'(p_1)\right] \ ,$$

where

$$p_1 = \int_\Theta m(\theta)\, dP(\theta) \ .$$

The right hand side of equation (7) is the Frechet derivative[16] of information cost. It does not explicitly depends on θ. This is the homogeneity we referred to. As an example, homogeneity fails if we replace the term

$$g'(m(\theta)) - g'(p_1)$$

with

$$g'(m(\theta)) - g'(p_1) + k(\theta)$$

for some non-constant function $k(\theta)$. In this case, we should define $m = f_2(s, \theta)$

[15]There are many information cost functions satisfying this property. For example, any strictly concave and symmetric function g in (1) corresponds to an information cost with this property.

[16]For the readers not familiar with this concept, just think of the Frechet derivative as the gradient of the cost function.

instead of $m = f_2(s)$ in (28). This dependence reflects the buyer's varying difficulties in discerning different states. Hence the optimal contract may not have a flat part as in debt.

We use a non-homogeneous information cost to illustrate our idea. Specifically, let $\theta \in [0, 1]$ and

$$c(m) = \frac{\mu}{\Pr(\theta \in [0, a])} \cdot \left[\int_{[0,a]} g(m(\theta))\, dP(\theta) - g\left(\int_{[0,a]} m(\theta)\, dP(\theta) \right) \right]$$

for some $a \in (0, 1)$. Hence the state is directly observable for $\theta \in (a, 1]$. For $\theta \in [0, a]$, the buyer can acquire its information at marginal cost $\frac{\mu}{\Pr(\theta \in [0,a])}$. Let $\delta_b = 1$ and the seller's optimal contract be (s, q). Given this contract, the buyer's optimal strategy is characterized by

$$s(\theta) - q = \mu \cdot \left[g'(m(\theta)) - g'(p_1) \right] \text{ if } \theta \in [0, a] \ ,$$

and

$$m(\theta) = \begin{cases} 1 \text{ if } \theta \in (a, 1] \text{ and } s(\theta) - q \geq 0 \\ 0 \text{ if } \theta \in (a, 1] \text{ and } s(\theta) - q < 0 \end{cases},$$

where

$$p_1 = \frac{\int_{[0,a]} m(\theta)\, dP(\theta)}{\Pr(\theta \in [0, a])} \, .$$

For $\theta \in (a, 1]$, the buyer accepts the offer if and only if $s(\theta) - q \geq 0$, thus we must have

$$s(\theta) = q$$

for $\theta \in (a, 1]$. Information remains costly in region $[0, a]$, thus a debt contract is optimal within this region according to our previous argument. However, the optimal contract on interval $[0, 1]$ is no longer a debt, as shown in Figure 3.

C. *Allocation of Bargaining Power*

One may wonder if our results are sensitive to the allocation of bargaining power. The answer is no. This subsection introduces the case where the buyer owns bargaining power and then presents the main results. Due to the similarity between the two cases, we omit most proofs here.

Suppose the buyer proposes the contract $(s(\cdot), q)$ and the seller acquires information. Write $m_{s,q}$ for the seller's optimal strategy. The uninformed buyer thus enjoys expected payoff

$$W(s, q) = \int m_{s,q}\left(\vec{\theta}\right) \cdot \left[\delta_b \cdot s\left(\vec{\theta}\right) - q \right] dP\left(\vec{\theta}\right).$$

The buyer's problem is to choose a feasible contract (s, q) satisfying $s\left(\vec{\theta}\right) \in \left[0, \sum_{n=1}^{N} \theta_n\right]$

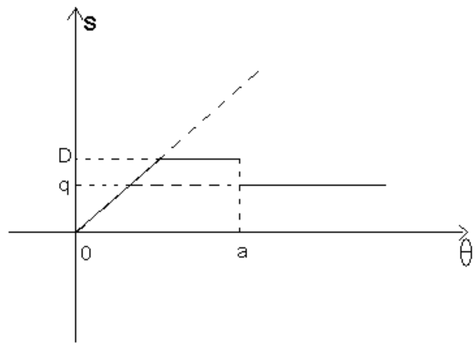

FIGURE 3. OPTIMAL CONTRACT UNDER NON-HOMOGENEOUS INFORMATION COST

to maximize $W(s,q)$. Let $(s^*(\cdot), q^*)$ denote the optimal contract for the buyer and

$$\overline{p}_{s^*,q^*} = \int_\Theta m_{s^*,q^*}\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right)$$

be the corresponding probability of trade.

PROPOSITION 6: $\overline{p}_{s^*,q^*} > 0$, *i.e., trade happens with positive probability.*

PROOF:
See Appendix A.

PROPOSITION 7: *If the buyer's optimal contract induces the seller to always accept it without acquiring information, it must be a securitized debt*

$$s^*\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N}\theta_n, D^*\right)$$

with price q^*, where

$$D^* = \mu\delta_s^{-1}\cdot[\ln\delta_b - \ln\delta_s] + \delta_s^{-1}q^*,$$

q^* is the unique fixed point of

$$h(q) = \mu\ln\mathbf{E}\exp\left(\mu^{-1}\delta_s\cdot\min\left(\sum_{n=1}^{N}\theta_n, \mu\delta_s^{-1}\cdot[\ln\delta_b - \ln\delta_s] + \delta_s^{-1}q\right)\right)$$

and the expectation is taken according to common prior P.

PROOF:

The proof is very similar to that of Proposition 3 and is omitted here.

PROPOSITION 8: *If the buyer's optimal contract induces the seller to acquire information, it must be a securitized debt* $s^*\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N} \theta_n, D^*\right)$.

PROOF:

The proof is very similar to that of Proposition 5 and is omitted here.

Proposition 3, 5, 7 and 8 show that the optimal security is always a securitized debt, no matter who owns bargaining power.[17] This result is consistent with our previous analysis. Exchanging bargaining power does not change the facts that total risk exposure is fixed and information acquisition is homogeneous.

IV. Conclusions and Discussions

This paper studies liquidity provision in presence of endogenous and flexible information acquisition. In our model, there is no information asymmetry before bargaining. Also, the buyer has an expertise in acquiring information of the fundamental in the manner of rational inattention. She collects the most payoff-relevant information according to the contract proposed to her, which may endogenously generate adverse selection. Hence, the seller deliberately designs the security in order to induce the buyer to acquire information least harmful to the seller's interest. It is shown that pooling and issuing securitized debt is the uniquely optimal way to raise liquidity, regardless of the stochastic interdependence among the underlying assets and the allocation of bargaining power. Compared to the security design literature, our results are clearer. We neither restrict our attention to non-decreasing securities nor impose various assumptions on information structures like MLRP. Instead, these properties of the optimal security are justified in equilibrium. Our results are driven by two key factors. The one is the fixed total risk exposure and the other is homogeneous information cost, without which the securitized debt may not be optimal.

The role of fixed total risk exposure sheds light on a general classification of information, namely, to classify what information is socially valuable and what information is not. In particular, flexibility enables economic agents to acquire these two types of information separately, which results in different welfare implications of information acquisition. At the level of the society, acquisition of information that is not socially valuable not only wastes social resource but also leads to endogenous adverse selection, which in turn harms social welfare. Hence, desired organizational form of the society should deter acquisition of such information. On the contrary, acquisition of socially valuable information generally increases social welfare and thus should be encouraged in principle. In our model with fixed total risk exposure, none of information is socially valuable, so that securitized debt is optimal because it best deters information acquisition. On the other hand, as the example mentioned with variable total risk exposure,

[17] However, reallocating the bargaining power does affect the face value and price of the debt, and thus affects the agents' expected payoffs.

some certain information is socially valuable as it helps prevent investing in bad states. Consequently, acquisition of such socially valuable information should be encouraged, and thus securitized debt may not be the optimal contract. This classification of information also provides a new perspective to look into the mutual existence of debt and equity, both as popular forms of financial contracts in reality. For start-ups and projects with high risk, issuing equity could be more desired because it encourages acquisition of socially valuable information, which helps to screen projects and control the total risk exposure of the entire society. In contrast, for mature corporations with robust growth, in which the provision of liquidity is of the priority, debt could be more desired as it deters unnecessary acquisition of information that is not socially valuable. This consideration is partly consistent with the well-known packing-order theory, and future work may further unify the life-cycle evolution of capital structure of corporations along the line of flexible information acquisition.

Under a similar mentality, flexibility also helps revisit the endogenous determination of capital structure in literature by specializing information acquisition. Given flexible information acquisition, agents who monitor may have different incentives in acquiring different information regarding various forms of financial contracts. Hence, different layers of financial contracts in certain capital structure enable a specialization of information acquisition. In other words, layers of capital structure correspond to specialized layers of information to be acquired. This specialization may in turn affect production of information as well as efficiency of monitoring, and further reshape the optimal capital structure. In this way, it is seen that flexibility plays an role in determining the capital structure, and more results regarding its effects on corporate finance as well as social welfare are to be expected.

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MATHEMATICAL APPENDIX

Proof of Proposition 1.

PROOF:

Suppose m is an optimal strategy. Let ε be any *feasible* perturbation function. The payoff from the perturbed strategy $m+\alpha\cdot\varepsilon$ is

$$\begin{aligned}
& V^*(m+\alpha\cdot\varepsilon) \\
= & \int_\Theta (m(\theta)+\alpha\cdot\varepsilon(\theta))\cdot\Delta u(\theta)\,dP(\theta) \\
& -\mu\cdot\left[\int_\Theta g(m(\theta)+\alpha\cdot\varepsilon(\theta))\,dP(\theta)-g\left(\int_\Theta [m(\theta)+\alpha\cdot\varepsilon(\theta)]\,dP(\theta)\right)\right],
\end{aligned}$$

where $\alpha\in\mathbb{R}$, and ε is feasible with respect to m if $\exists\alpha>0$, s.t. $\forall\theta\in\Theta$, $m(\theta)+\alpha\cdot\varepsilon(\theta)\in[0,1]$. Then the first order variation is

$$\begin{aligned}
\left.\frac{dV^*(m+\alpha\cdot\varepsilon)}{d\alpha}\right|_{\alpha=0} = & \int_\Theta \varepsilon(\theta)\cdot\Delta u(\theta)\,dP(\theta) \\
& -\mu\cdot\left[\int_\Theta \varepsilon(\theta)\cdot g'(m(\theta))\,dP(\theta)-g'\left(\int_\Theta m(\theta)\,dP(\theta)\right)\cdot\int_\Theta \varepsilon(\theta)\,dP(\theta)\right] \\
= & \int_\Theta \varepsilon(\theta)\cdot\left[\Delta u(\theta)-\mu\cdot\left(g'(m(\theta))-g'(p_1)\right)\right]dP(\theta)\ .
\end{aligned}$$

Note that

$$\Delta u(\theta)-\mu\cdot\left(g'(m(\theta))-g'(p_1)\right)$$

is the Frechet derivative of $V^*(\cdot)$ at m. Hence the tangent hyperplane at m can be expressed as

$$\left\{\widetilde{m}\in M : V^*(\widetilde{m})-V^*(m)=\int_\Theta\left[\Delta u(\theta)-\mu g'(m(\theta))+\mu g'\left(\int_\Theta m(\theta)\,dP(\theta)\right)\right](\widetilde{m}(\theta)-m(\theta))\,dP(\theta)\right\}.$$

An important observation: since $V^*(\cdot)$ is a concave functional on M, V^* is upper bounded by any hyperplane tangent at any $m \in M$, i.e., $\forall m, \widetilde{m} \in M$,

$$\begin{aligned} & V^*(\widetilde{m}) - V^*(m) \\ \leq & \int_\Theta \left[\Delta u(\theta) - \mu \cdot g'(m(\theta)) + \mu \cdot g'\left(\int_\Theta m(\theta)\, dP(\theta) \right) \right] (\widetilde{m}(\theta) - m(\theta))\, dP(\theta) \ . \end{aligned}$$

This inequality is strict when

$$m \in M^o \triangleq M \backslash \{ m \in M : m(\theta) \text{ is a constant almost surely} \}$$

and $\Pr(\widetilde{m}(\theta) \neq m(\theta)) > 0$, since $V^*(\cdot)$ is strictly concave on M^o. We will use this observation later in this proof.

The optimality of m requires $\left. \frac{dV^*(m+\alpha\cdot\varepsilon)}{d\alpha} \right|_{\alpha=0} \leq 0$ for all feasible perturbation ε. Hence we must have

$$\text{(A1)} \qquad \Delta u(\theta) - \mu \cdot \left(g'(m(\theta)) - g'(p_1) \right) \begin{cases} \geq 0 & \text{if } m(\theta) = 1 \\ = 0 & \text{if } m(\theta) \in (0,1) \\ \leq 0 & \text{if } m(\theta) = 0 \end{cases} .$$

Note that $\Pr(m(\theta) = 1) > 0$ implies $\Pr(m(\theta) = 1) = 1$. Otherwise,

$$p_1 = \int_\Theta m(\theta)\, dP(\theta) < 1$$

and for $\theta \in B = \{\theta \in \Theta : m(\theta) = 1\}$,

$$\Delta u(\theta) - \mu \cdot \left(g'(m(\theta)) - g'(p_1) \right) = -\infty$$

since $\lim_{x \to 1} g'(x) = \infty$. Then $\varepsilon(\theta) = -1_B$ is a feasible perturbation and

$$\begin{aligned} & \left. \frac{dV^*(m + \alpha \cdot \varepsilon)}{d\alpha} \right|_{\alpha=0} \\ = & \int_\Theta \left[\Delta u(\theta) - \mu \cdot \left(g'(m(\theta)) - g'(p_1) \right) \right] \cdot \varepsilon(\theta)\, dP(\theta) \\ = & \int_B (-\infty) \cdot (-1)\, dP(\theta) \\ = & +\infty , \end{aligned}$$

which contradicts the optimality of m. Hence we know that $\Pr(m(\theta) = 1) > 0$ if and only if $\Pr(m(\theta) = 1) = 1$. By the same argument, we can show that $\Pr(m(\theta) = 0) > 0$ if and only if $\Pr(m(\theta) = 0) = 1$. Therefore, the optimal strategy m must be one of the three scenarios: a) $p_1 = 1$, i.e., $m(\theta) = 1$ $a.s.$; b) $p_1 = 0$, i.e., $m(\theta) = 0$ $a.s.$; c) $p_1 \in (0,1)$ and $m(\theta) \in (0,1)$ $a.s.$.

We first search for the sufficient condition for scenario c). According to (A1), we have

$$\Delta u(\theta) - \mu \cdot \left(g'(m(\theta)) - g'(p_1)\right) = 0 \ a.s.. \tag{A2}$$

By definition,

$$g'(x) = \ln \frac{x}{1-x} ,$$

thus (A2) implies

$$m(\theta) = \frac{p_1}{p_1 + (1-p_1) \cdot \exp\left(-\mu^{-1} \Delta u(\theta)\right)} .$$

Let

$$M_1 = \left\{ m(\theta, p) = \frac{p}{p + (1-p) \cdot \exp\left(-\mu^{-1} \Delta u(\theta)\right)} : p \in [0,1] \right\} \tag{A3}$$

and

$$J(p) = \int_\Theta m(\theta, p) \, dP(\theta) ,$$

then there exists $p_1 \in [0,1]$ such that $m(\cdot, p_1) \in M_1 \subset M$ is an optimal strategy. Note that $J(p_1) = p_1$ is a necessary condition.

Since $m(\cdot, p_1) \in M_1 \subset M$, the original problem is reduced to

$$\max_{p \in [0,1]} V^*(m(\cdot, p)) = \int_\Theta \Delta u(\theta) \cdot m(\theta, p) \, dP(\theta) - c(m(\cdot, p)) .$$

The first order derivative with respect to p is

$$\begin{aligned} & \frac{dV^*(m(\cdot, p))}{dp} \\ = & \int_\Theta \Delta u(\theta) \cdot \frac{\partial m(\theta, p)}{\partial p} dP(\theta) \\ & -\mu \cdot \left[\int_\Theta g'(m(\theta, p)) \frac{\partial m(\theta, p)}{\partial p} dP(\theta) - g'\left(\int_\Theta m(\theta, p) \, dP(\theta) \right) \int_\Theta \frac{\partial m(\theta, p)}{\partial p} dP(\theta) \right] \\ = & \int_\Theta \left[\Delta u(\theta) - \mu \cdot g'(m(\theta, p)) + \mu \cdot g'(J(p)) \right] \cdot \frac{\partial m(\theta, p)}{\partial p} dP(\theta) . \end{aligned}$$

By definition,

$$\Delta u(\theta) - \mu \cdot g'(m(\theta, p)) = -\mu \cdot g'(p) ,$$

thus

$$\frac{dV^*(m(\cdot,p))}{dp} = \int_\Theta \left[-\mu \cdot g'(p) + \mu \cdot g'(J(p))\right] \cdot \frac{\partial m(\theta,p)}{\partial p} dP(\theta)$$

$$= \mu \cdot \left[g'(J(p)) - g'(p)\right] \cdot \int_\Theta \frac{\partial m(\theta,p)}{\partial p} dP(\theta) \ . \tag{A4}$$

Since

$$\begin{aligned} & \frac{\partial m(\theta,p)}{\partial p} \\ = & \left[p \cdot \exp\left(\frac{1}{2}\mu^{-1}\Delta u(\theta)\right) + (1-p) \cdot \exp\left(-\frac{1}{2}\mu^{-1}\Delta u(\theta)\right)\right]^{-2} \\ > & \ 0 \end{aligned}$$

for all $\theta \in \Theta$,

$$\frac{dV^*(m(\cdot,p))}{dp} \geq 0$$

if and only if

$$g'(J(p)) - g'(p) \geq 0 \ .$$

Since g' is strictly increasing in its argument, we have

$$\frac{dV^*(m(\cdot,p))}{dp} \geq 0$$

if and only if

$$J(p) \geq p \ .$$

In order to be a global maximum, $m(\cdot, p_1)$ must first be a local maximum within M_1. This requires

$$J(p_1) = p_1 \ . \tag{A5}$$

But (A5) is not sufficient. The sufficient condition for $m(\cdot, p_1)$ to be a local maximum within M_1 is

$$\exists \text{ neighborhood } (p_1 - \beta, p_1 + \beta),$$

$$\begin{aligned} \text{s.t. } J(p) & \geq p \text{ for all } p \in (p_1 - \beta, p_1] \\ \text{and } J(p) & \leq p \text{ for all } p \in [p_1, p_1 + \beta) \ . \end{aligned}$$

Note that

$$J(0) = 0, J(1) = 1 \ ,$$

$$\left.\frac{dJ}{dp}\right|_{p=0} = \int_\Theta \exp\left(\mu^{-1}\Delta u(\theta)\right) dP(\theta)$$

and
$$\left.\frac{dJ}{dp}\right|_{p=1} = \int_{\Theta} \exp\left(-\mu^{-1}\Delta u(\theta)\right) dP(\theta) \; .$$

Case i):
$$\int_{\Theta} \exp\left(\mu^{-1}\Delta u(\theta)\right) dP(\theta) > 1$$
and
$$\int_{\Theta} \exp\left(-\mu^{-1}\Delta u(\theta)\right) dP(\theta) > 1 \; .$$
In this case, $J(p) > p$ for p close enough to 0 and $J(p) < p$ for p close enough to 1. Since $J(p)$ is continuous, the set $\{p \in (0,1) : J(p) = p\}$ is non-empty. For any $p_1 \in \{p \in (0,1) : J(p) = p\}$, the Frechet derivative at $m(\cdot, p_1)$ is

$$\begin{aligned} & \Delta u(\theta) - \mu \cdot g'(m(\theta, p_1)) + \mu \cdot g'(J(p_1)) \\ = \; & \Delta u(\theta) - \mu \cdot g'(m(\theta, p_1)) + \mu \cdot g'(p_1) \\ = \; & 0 \end{aligned}$$

and thus $m(\cdot, p_1)$ is a critical point of functional $V^*(\cdot)$. Since $m(\cdot, p_1) \in M^o$, the observation mentioned above implies

$$\begin{aligned} & V^*(\widetilde{m}) - V^*(m(\cdot, p_1)) \\ < \; & \int_{\Theta} \left[\Delta u(\theta) - \mu \cdot g'(m(\cdot, p_1)) + \mu \cdot g'\left(\int_{\Theta} m(\cdot, p_1)\, dP(\theta)\right)\right] (\widetilde{m}(\theta) - m(\cdot, p_1))\, dP(\theta) \\ = \; & \int_{\Theta} \left[\Delta u(\theta) - \mu \cdot g'(m(\cdot, p_1)) + \mu \cdot g'(J(p_1))\right] (\widetilde{m}(\theta) - m(\cdot, p_1))\, dP(\theta) \\ = \; & 0 \end{aligned}$$

for all $\widetilde{m} \in M$ such that $\Pr(\widetilde{m}(\theta) \neq m(\theta, p_1)) > 0$. Hence, $V^*(m(\cdot, p_1))$ is strictly higher than the values achieved at any other $\widetilde{m} \in M$, i.e., $\{p \in (0,1) : J(p) = p\} = \{p_1\}$ and $m(\cdot, p_1)$ is the *unique* global maximum. This actually proves (6).

Case ii):

(A6)
$$\int_{\Theta} \exp\left(\mu^{-1}\Delta u(\theta)\right) dP(\theta) > 1$$

and

(A7)
$$\int_{\Theta} \exp\left(-\mu^{-1}\Delta u(\theta)\right) dP(\theta) \leq 1 \; .$$

(A6) implies $J(p) > p$ for p close enough to 0. Note that

$$\left.\frac{d^2 J}{dp^2}\right|_{p=1} = -2 \cdot \int_{\Theta} \left[\exp\left(-\mu^{-1}\Delta u(\theta)\right) - \exp\left(-2\mu^{-1}\Delta u(\theta)\right)\right] dP(\theta)$$
$$= -2 \cdot \left[\mathbf{E}\exp\left(-\mu^{-1}\Delta u(\theta)\right) - \mathbf{E}\exp\left(-2\mu^{-1}\Delta u(\theta)\right)\right] ,$$

where the expectation is taken according to prior P. Since

$$f(x) = x^2$$

is a strictly convex function, Jensen's inequality implies

$$\mathbf{E}\exp\left(-\mu^{-1}\Delta u(\theta)\right) \geq \mathbf{E}\exp\left(-2\mu^{-1}\Delta u(\theta)\right) .$$

The inequality is not strict only if $\Delta u(\theta)$ =constant almost surely. Since $\mathbf{E}\exp\left(-\mu^{-1}\Delta u(\theta)\right) \leq 1$, this constant must be non-negative. Moreover, since $\Pr(\Delta u(\theta) \neq 0) > 0$, this constant must be strictly positive. Hence

$$\mathbf{E}\exp\left(-\mu^{-1}\Delta u(\theta)\right) > \mathbf{E}\exp\left(-2\mu^{-1}\Delta u(\theta)\right)$$

and

$$\text{(A8)} \qquad \left.\frac{d^2 J}{dp^2}\right|_{p=1} < 0 .$$

Together with (A7), (A8) implies $J(p) > p$ for p close enough to 1. Hence there exists $\epsilon > 0$, s.t. $J(p) > p$ for all $p \in [0, \epsilon] \cup [1-\epsilon, 1]$.

We claim that $J(p) > p$ for all $p \in (0, 1)$. If this is not true, let $p_1 = \sup\{p \in (0,1) : J(p) \leq p\}$. The continuity of $J(p)$ implies $J(p_1) = p_1$. Hence $m(\cdot, p_1) \in M^o$ and it is a critical point of functional $V^*(\cdot)$. By the same argument as in Case i), $m(\cdot, p_1)$ is the unique global maximum. However, by definition, $p_1 < 1-\epsilon$ and $J(p) > p$ for all $p \in (p_1, 1)$. Then $V^*(m(\cdot, p)) > V^*(m(\cdot, p_1))$ for all $p \in (p_1, 1)$ since $\frac{dV^*(m(\cdot,p))}{dp}$ is of the same sign as $J(p) - p$. This contradicts the unique optimality of $m(\cdot, p_1)$. Therefore, $J(p) > p$ for all $p \in (0, 1)$ and the optimal strategy cannot be an interior point of M (i.e., it cannot be the case $p_1 \in (0, 1)$.) Then according to our previous discussion, only scenarios a) that $p_1 = 1$ and scenario b) that $p_1 = 0$ are possible. Since we have shown $J(p) > p$ for all $p \in (0, 1)$, we know that

$$V^*(m(\cdot, 1)) > V^*(m(\cdot, 0)) .$$

Hence, $p_1 = 1$, i.e., $m(\theta) = 1$ $a.s.$ is the *unique* optimal strategy. This actually proves (4).

case iii):

$$\int_{\Theta} \exp\left(\mu^{-1}\Delta u(\theta)\right) dP(\theta) \leq 1$$

and

$$\int_{\Theta} \exp\left(-\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) > 1 \, .$$

In this case, by the same argument as in case ii), $m(\theta) = 0 \; a.s.$ is the *unique* optimal strategy. This actually proves (5).

Now we show that it is impossible to have the case

$$\int_{\Theta} \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) \leq 1 \tag{A9}$$

and

$$\int_{\Theta} \exp\left(-\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) \leq 1 \, . \tag{A10}$$

Since

$$f(x) = x^{-1}$$

is strictly convex for $x > 0$, Jensen's inequality implies

$$\int_{\Theta} \exp\left(-\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) \geq \left[\int_{\Theta} \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right)\right]^{-1} \, .$$

The inequality is not strict only if $\Delta u(\theta) =$constant almost surely. If this is true, (A9) and (A10) implies $\Delta u(\theta) = 0$ almost surely. This is the trivial case excluded by our assumption. Hence

$$\int_{\Theta} \exp\left(-\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right) > \left[\int_{\Theta} \exp\left(\mu^{-1}\Delta u\left(\theta\right)\right) dP\left(\theta\right)\right]^{-1}$$

and (A9) and (A10) cannot be simultaneously satisfied.

Since cases i), ii) and iii) exhaust all possibilities, for each case, the corresponding conditions are not only sufficient but also necessary.

The uniqueness of the optimal strategy is proved in each case.

This concludes the proof.

Proof of Proposition 3.

PROOF:

Let $s\left(\overrightarrow{\theta}\right) = s^*\left(\overrightarrow{\theta}\right) + \alpha \cdot \varepsilon\left(\overrightarrow{\theta}\right)$ be an arbitrary perturbation of the optimal security $s^*(\cdot)$. Let

$$J(\alpha) = \mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot s\left(\overrightarrow{\theta}\right)\right) + \delta_s \cdot \mathbf{E}s\left(\overrightarrow{\theta}\right) \, .$$

Taking first order variation leads to

$$
\begin{aligned}
& \left.\frac{dJ}{d\alpha}\right|_{\alpha=0} \\
= \; & -\delta_b \left[\mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}\right)\right)\right]^{-1} \mathbf{E}\left[\exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}\right)\right) \cdot \varepsilon\left(\vec{\theta}\right)\right] + \delta_s \cdot \mathbf{E}\varepsilon\left(\vec{\theta}\right) \\
= \; & \mathbf{E}\left[\left(\delta_s - \delta_b \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}\right)\right)\right]^{-1} \exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}\right)\right)\right) \cdot \varepsilon\left(\vec{\theta}\right)\right] \\
\triangleq \; & \mathbf{E}\left[r\left(\vec{\theta}\right) \cdot \varepsilon\left(\vec{\theta}\right)\right] .
\end{aligned} \tag{A11}
$$

Let

$$A_0 = \left\{\vec{\theta} \in \Theta : \vec{\theta} \neq \vec{0}, s^*\left(\vec{\theta}\right) = 0\right\},$$

$$A_1 = \left\{\vec{\theta} \in \Theta : \vec{\theta} \neq \vec{0}, s^*\left(\vec{\theta}\right) \in \left(0, \sum_{n=1}^{N} \theta_n\right)\right\}$$

and

$$A_2 = \left\{\vec{\theta} \in \Theta : \vec{\theta} \neq \vec{0}, s^*\left(\vec{\theta}\right) = \sum_{n=1}^{N} \theta_n\right\}.$$

Then $\{A_0, A_1, A_2\}$ is a partition of $\Theta \backslash \left\{\vec{0}\right\}$. Since $s^*(\cdot)$ is the optimal security,

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} \geq 0$$

holds for any feasible perturbation $\varepsilon\left(\vec{\theta}\right)$. Hence, we have

$$r\left(\vec{\theta}\right) \begin{cases} \geq 0 & \text{if } \vec{\theta} \in A_0 \\ = 0 & \text{if } \vec{\theta} \in A_1 \\ \leq 0 & \text{if } \vec{\theta} \in A_2 \end{cases} . \tag{A12}$$

For any $\vec{\theta}' \in A_0$, (A12) implies $r\left(\vec{\theta}'\right) \geq 0$, i.e.,

$$
\begin{aligned}
\delta_s \;\; & \geq \;\; \delta_b \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}\right)\right)\right]^{-1} \exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}'\right)\right) \\
& = \;\; \delta_b \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}\right)\right)\right]^{-1} \exp\left(-\mu^{-1}\delta_b \cdot 0\right) \\
& = \;\; \delta_b \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\vec{\theta}\right)\right)\right]^{-1},
\end{aligned}
$$

i.e.,

$$\begin{aligned}\ln\delta_s &\geq \ln\delta_b - \ln\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}\right)\right)\\ &= \ln\delta_b + \mu^{-1}q^*\,,\end{aligned}$$

where the last equality comes from (14). Hence,

$$\mu^{-1}q^* \leq \ln\delta_s - \ln\delta_b < 0\,,$$

which is a contradiction. Therefore,

$$\Pr\left(A_0\right) = 0\,. \tag{A13}$$

For any $\overrightarrow{\theta}' \in A_1$, (A12) implies $r\left(\overrightarrow{\theta}'\right) = 0$, i.e.,

$$\delta_s = \delta_b\left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}\right)\right)\right]^{-1}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}'\right)\right)\,,$$

i.e.,

$$\begin{aligned}\ln\delta_s &= \ln\delta_b - \ln\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}\right)\right) - \mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}'\right)\\ &= \ln\delta_b + \mu^{-1}q^* - \mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}'\right)\,,\end{aligned}$$

where the last equality follows (14). Therefore,

$$s^*\left(\overrightarrow{\theta}'\right) = \mu\delta_b^{-1}\cdot\left[\ln\delta_b - \ln\delta_s\right] + \delta_b^{-1}q^* \tag{A14}$$

is a constant for all $\overrightarrow{\theta}' \in A_1$.

For any $\overrightarrow{\theta}' \in A_2$, (A12) implies $r\left(\overrightarrow{\theta}'\right) \leq 0$, i.e.,

$$\begin{aligned}\delta_s &\leq \delta_b\left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}\right)\right)\right]^{-1}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}'\right)\right)\\ &= \delta_b\left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}\right)\right)\right]^{-1}\exp\left(-\mu^{-1}\delta_b\cdot\sum_{n=1}^{N}\theta'_n\right)\,,\end{aligned}$$

i.e.,

$$\begin{aligned}\ln\delta_s &\leq \ln\delta_b - \ln\mathbf{E}\exp\left(-\mu^{-1}\delta_b\cdot s^*\left(\overrightarrow{\theta}\right)\right) - \mu^{-1}\delta_b\cdot\sum_{n=1}^{N}\theta'_n\\ &= \ln\delta_b + \mu^{-1}q^* - \mu^{-1}\delta_b\cdot\sum_{n=1}^{N}\theta'_n\,,\end{aligned}$$

where the last equality comes from (14). Therefore,

$$\sum_{n=1}^{N} \theta'_n \leq \mu\delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q^*. \tag{A15}$$

Let

$$D^* = \mu\delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q^*.$$

Then, (A13), (A14) and (A15) imply that

$$s^* \left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N} \theta_n, D^*\right),$$

i.e., the optimal security must be a securitized debt.

Finally, let

$$h(q) = -\mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N} \theta_n, \mu\delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q\right)\right).$$

We show that $q^* > 0$ and it is the unique fixed point of $h(q)$.

By (14), we have

$$\begin{aligned}
q^* &= -\mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot s^*\left(\overrightarrow{\theta}\right)\right) \\
&= -\mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N} \theta_n, D^*\right)\right) \\
&= -\mu \ln \mathbf{E} \exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N} \theta_n, \mu\delta_b^{-1} \cdot [\ln \delta_b - \ln \delta_s] + \delta_b^{-1} q^*\right)\right) \\
&= h\left(q^*\right).
\end{aligned}$$

Hence q^* is a fixed point of $h(q)$. First note $h(0) > 0$. Second note that

$$\begin{aligned}
& h'(q) \\
= & \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N}\theta_n,\, \mu\delta_b^{-1}\cdot[\ln\delta_b - \ln\delta_s] + \delta_b^{-1}q\right)\right)\right]^{-1} \\
& \cdot\mathbf{E}\left[\exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N}\theta_n,\, \mu\delta_b^{-1}\cdot[\ln\delta_b - \ln\delta_s] + \delta_b^{-1}q\right)\right)\cdot 1_{\left\{\sum_{n=1}^{N}\theta_n \ge \mu\delta_b^{-1}\cdot[\ln\delta_b - \ln\delta_s]+\delta_b^{-1}q\right\}}\right] \\
\le & \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N}\theta_n,\, \mu\delta_b^{-1}\cdot[\ln\delta_b - \ln\delta_s] + \delta_b^{-1}q\right)\right)\right]^{-1} \\
& \cdot\mathbf{E}\left[\exp\left(-\mu^{-1}\delta_b \cdot \min\left(\sum_{n=1}^{N}\theta_n,\, \mu\delta_b^{-1}\cdot[\ln\delta_b - \ln\delta_s] + \delta_b^{-1}q\right)\right)\cdot 1\right] \\
= & 1
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{q\to\infty} h'(q) \\
= & \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot \sum_{n=1}^{N}\theta_n\right)\right]^{-1} \cdot \mathbf{E}\left[\exp\left(-\mu^{-1}\delta_b \cdot \sum_{n=1}^{N}\theta_n\right)\cdot \lim_{q\to\infty} 1_{\left\{\sum_{n=1}^{N}\theta_n \ge \mu\delta_b^{-1}\cdot[\ln\delta_b - \ln\delta_s]+\delta_b^{-1}q\right\}}\right] \\
= & \left[\mathbf{E}\exp\left(-\mu^{-1}\delta_b \cdot \sum_{n=1}^{N}\theta_n\right)\right]^{-1} \cdot \mathbf{E}\left[\exp\left(-\mu^{-1}\delta_b \cdot \sum_{n=1}^{N}\theta_n\right)\cdot 0\right] \\
= & 0\,.
\end{aligned}$$

Hence, $h(q)$ has a unique fixed point $q^* > 0$. This concludes the proof.

Proof of Lemma 1.

PROOF:

Taking derivative with respect to α at $\alpha = 0$ for both sides of (18) leads to

$$\begin{aligned}
\mu^{-1}\delta_b \cdot \varepsilon\left(\overrightarrow{\theta}\right) &= g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) \cdot \left.\frac{dm_{s,q^*}\left(\overrightarrow{\theta}\right)}{d\alpha}\right|_{\alpha=0} \\
& \quad -g''\left(\overline{p}_b\right) \cdot \int_{\Theta} \left.\frac{dm_{s,q^*}\left(\overrightarrow{\theta}\right)}{d\alpha}\right|_{\alpha=0} dP\left(\overrightarrow{\theta}\right)\,.
\end{aligned}$$

Take integral of both sides and manipulate we get

$$\int_{\Theta} \left. \frac{dm_{s,q^*}\left(\overrightarrow{\theta}\right)}{d\alpha} \right|_{\alpha=0} dP\left(\overrightarrow{\theta}\right)$$
$$= \mu^{-1}\delta_b \left[1 - \int_{\Theta} \left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} dP\left(\overrightarrow{\theta}\right) \cdot g''\left(\overline{p}_{s^*,q^*}\right)\right]^{-1} \int_{\Theta} \left[g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right)\right]^{-1} \varepsilon\left(\overrightarrow{\theta}\right) dP\left(\overrightarrow{\theta}\right).$$

Combining the above two equations leads to (20).

Proof of Proposition 4.

PROOF:

We first prove $f_1(0) > f_2(0)$. If not, $f_1(s) < f_2(s)$ for all $s > 0$. Hence $\forall \overrightarrow{\theta} \neq \overrightarrow{0}$,

$$\begin{aligned} & r\left(\overrightarrow{\theta}\right) \cdot g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) \\ = & -\delta_s m_{s^*,q^*}\left(\overrightarrow{\theta}\right) g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ = & -\delta_s \cdot f_2\left(s^*\left(\overrightarrow{\theta}\right)\right) g''\left(f_2\left(s^*\left(\overrightarrow{\theta}\right)\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ < & -\delta_s \cdot f_1\left(s^*\left(\overrightarrow{\theta}\right)\right) g''\left(f_1\left(s^*\left(\overrightarrow{\theta}\right)\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ = & \ 0, \end{aligned}$$

where the inequality holds since $\left[m \cdot g''(m)\right]' > 0$. Then (25) implies $s^*\left(\overrightarrow{\theta}\right) = 0$ almost surely. Therefore, there is no trade, which contradicts Proposition 2.

Now we know $f_1(0) > f_2(0)$. $\forall \overrightarrow{\theta} \in A_0$,

$$\begin{aligned} & r\left(\overrightarrow{\theta}\right) \cdot g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) \\ = & -\delta_s m_{s^*,q^*}\left(\overrightarrow{\theta}\right) g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ = & -\delta_s \cdot f_2(0)\, g''(f_2(0)) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot 0 + w\right) \\ > & -\delta_s \cdot f_1(0)\, g''(f_1(0)) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot 0 + w\right) \\ = & \ 0, \end{aligned}$$

where the second equality follows the definition that $s^*\left(\overrightarrow{\theta}\right) = 0$ for $\overrightarrow{\theta} \in A_0$, the last equality comes from the definition of $f_1(s)$, and the inequality holds since $\left[m \cdot g''(m)\right]' > 0$. This result contradicts (25), which states $r\left(\overrightarrow{\theta}\right) \cdot g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) \leq 0$ for $\overrightarrow{\theta} \in A_0$. This concludes the proof.

Proof of Proposition 5.

PROOF:

Let $(\overline{s}, \overline{m})$ be the unique intersection of $f_1(s)$ and $f_2(s)$. $\forall \overrightarrow{\theta}$ such that $\sum_{n=1}^{N} \theta_n < \overline{s}$,

$$m_{s^*,q^*}\left(\overrightarrow{\theta}\right) = f_2\left(s^*\left(\overrightarrow{\theta}\right)\right) < f_2(\overline{s}) = f_1(\overline{s}) < f_1\left(s^*\left(\overrightarrow{\theta}\right)\right).$$

Then

$$\begin{aligned} & r\left(\overrightarrow{\theta}\right) \cdot g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) \\ = & -\delta_s m_{s^*,q^*}\left(\overrightarrow{\theta}\right) g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ > & -\delta_s \cdot f_1\left(s^*\left(\overrightarrow{\theta}\right)\right) \cdot g''\left(f_1\left(s^*\left(\overrightarrow{\theta}\right)\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ = & \ 0, \end{aligned}$$

where the inequality holds since $\left[m \cdot g''(m)\right]' > 0$. According to (25), $s^*\left(\overrightarrow{\theta}\right) = \sum_{n=1}^{N} \theta_n$ for all $\overrightarrow{\theta}$ such that $\sum_{n=1}^{N} \theta_n < \overline{s}$.

For any $\overrightarrow{\theta}$ such that $\sum_{n=1}^{N} \theta_n > \overline{s}$, if $s^*\left(\overrightarrow{\theta}\right) = \sum_{n=1}^{N} \theta_n$, then (25) implies

$$\begin{aligned} 0 \leq & \ r\left(\overrightarrow{\theta}\right) \cdot g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) \\ = & -\delta_s m_{s^*,q^*}\left(\overrightarrow{\theta}\right) g''\left(m_{s^*,q^*}\left(\overrightarrow{\theta}\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ = & -\delta_s \cdot f_2\left(s^*\left(\overrightarrow{\theta}\right)\right) \cdot g''\left(f_2\left(s^*\left(\overrightarrow{\theta}\right)\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ < & -\delta_s \cdot f_2(\overline{s}) \cdot g''(f_2(\overline{s})) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ < & -\delta_s \cdot f_1\left(s^*\left(\overrightarrow{\theta}\right)\right) \cdot g''\left(f_1\left(s^*\left(\overrightarrow{\theta}\right)\right)\right) + \mu^{-1}\delta_b\left(q^* - \delta_s \cdot s^*\left(\overrightarrow{\theta}\right) + w\right) \\ = & \ 0, \end{aligned}$$

which is a contradiction. Hence Proposition 4 implies $s^*\left(\overrightarrow{\theta}\right) = \overline{s}$ for all $\overrightarrow{\theta}$ such that $\sum_{n=1}^{N} \theta_n > \overline{s}$.

For any $\overrightarrow{\theta}$ such that $\sum_{n=1}^{N} \theta_n = \overline{s}$, $s^*\left(\overrightarrow{\theta}\right) = \overline{s}$ is a direct implication of Proposition 4.

Therefore, the optimal security is a securitized debt with face value $\overline{s}$, i.e., $s^*\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N} \theta_n, \overline{s}\right)$.

It is also possible that $\overline{s} = \infty$, i.e., $f_1(s)$ and $f_2(s)$ never intersects. Then the optimal security

$$s^*\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N} \theta_n, \infty\right) = \sum_{n=1}^{N} \theta_n$$

is a special securitized debt, i.e., equity. This concludes the proof.

Proof of Proposition 6.

PROOF:

Let $\beta \in \left(\delta_s \delta_b^{-1}, 1\right)$ and

$$f(q) = \delta_b \cdot \mathbf{E} \min\left(\sum_{n=1}^{N} \theta_n, \beta \delta_s^{-1} q\right),$$

where the expectation is taken according to common prior P. Since P is a continuous distribution and $\beta^{-1}\delta_s \delta_b^{-1} < 1$, there exists $q_0 > 0$ s.t.

$$\Pr\left(\sum_{n=1}^{N} \theta_n \geq \beta \delta_s^{-1} q\right) > \beta^{-1} \delta_s \delta_b^{-1}$$

for all $q \in \left[0, q_0\right]$. Hence for any $q \in (0, q_0)$,

$$\begin{aligned} f'(q) &= \beta \delta_b \delta_s^{-1} \int_{\left\{\overrightarrow{\theta} \in \Theta : \sum_{n=1}^{N} \theta_n \geq \beta \delta_s^{-1} q\right\}} 1 \cdot dP\left(\overrightarrow{\theta}\right) \\ &= \beta \delta_b \delta_s^{-1} \cdot \Pr\left(\sum_{n=1}^{N} \theta_n \geq \beta \delta_s^{-1} q\right) \\ &> \beta \delta_b \delta_s^{-1} \cdot \beta^{-1} \delta_s \delta_b^{-1} = 1. \end{aligned}$$

Note that

$$f(0) = 0,$$

which implies that

$$f(q) > q$$

for all $q \in (0, q_0)$.

Consider a securitized debt

$$s\left(\overrightarrow{\theta}\right) = \min\left(\sum_{n=1}^{N} \theta_n, D\right)$$

with face value $D = \beta \delta_s^{-1} q$ and price $q \in (0, q_0)$. Since the seller's payoff gain from accepting this offer over rejecting it is

$$\begin{aligned} & q - \delta_s s\left(\overrightarrow{\theta}\right) \\ = \; & q - \delta_s \min\left(\sum_{n=1}^{N} \theta_n, \beta \delta_s^{-1} q\right) \\ \geq \; & q - \delta_s \cdot \beta \delta_s^{-1} q \\ = \; & (1 - \beta) \cdot q > 0 \end{aligned}$$

for all $\overrightarrow{\theta} \in \Theta$, the seller will accept this offer without acquiring any information. Hence the buyer's expected payoff from proposing $(s(\cdot), q)$ is

$$\begin{aligned} W(s,q) &= \delta_b \cdot \mathbf{E} \min\left(\sum_{n=1}^{N} \theta_n, \beta\delta_s^{-1} q\right) - q \\ &= f(q) - q \\ &> 0 . \end{aligned}$$

By definition, the seller's expected payoff through the optimal contract is $W(s^*, q^*) \geq W(s,q) > 0$. This directly implies $\overline{p}_{s^*,q^*} > 0$ since $\overline{p}_{s^*,q^*} = 0$ always generates zero expected payoff to the buyer.