Housing assignment with restrictions: theory and and evidence from Stanford campus*

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1 Introduction

Within narrow geographic areas, housing markets assign movers with different characteristics to indivisible houses that differ by quality. This paper studies housing assignment when a subset of *eligible* buyers have exclusive access to a subset of houses that form a *restricted area*. In our leading example, buyers affiliated with Stanford University have exclusive access to houses on campus. Similar issues arise whenever a subset of buyers receives much lower utility from a subset of houses, for example, families with children may not consider neighborhoods with very bad access to schools.

This paper has two parts. We first present evidence on house prices on and right around Stanford campus over the last decade. Using both a simple comparables approach and nearest neighbor regressions, we show that houses on campus trade at a substantial discount to similar properties off campus. The discount is smaller for higher quality houses. We then interpret the evidence using an assignment model with a continuum of houses in which buyer types differ not only by eligibility but also by the marginal utility of house quality. Houses in the restricted area trade at a discount if the relationship between house quality and buyer type distributions is sufficiently different for the restricted area.[1]

Without the access restriction, our model has an efficient equilibrium in which higher types buy higher quality houses. House prices reflect the relative dispersion of house quality and buyer types. The cost of an additional unit of quality depends on the marginal buyer type; it rises at a faster rate if more distinct buyers must be assigned to similar houses. When there are more eligible buyers than houses in the restricted area, the efficient equilibrium survives even with the restriction as long as the dispersion of quality in the restricted area relative to the dispersion of type among eligible buyers is everywhere sufficiently similar to the relative dispersion in the economy at large.

Once pairs of distributions are sufficiently different, arbitrage by eligible agents across areas becomes impossible and houses in the restricted area trade at a discount. We consider economies in which house quality in the restricted area is relatively low. Eligible buyers who do not buy in the restricted area then buy higher quality houses outside. Moreover, eligible buyers of the best restricted houses buy lower quality houses than non-eligible buyers with the same preferences. Price discounts thus compensate those high type buyers for compromising on quality inside the restricted area, rather than, say, help them buy a better house that they would not choose at outside market prices. If the latter effect is at work, we would expect lower price discounts at the low end of the quality spectrum. The evidence on this is weak.

[1] Assignment models are surveyed by Sattinger (1993). We consider two-sided assignement with a continuum of houses and multiple dimensions of mover heterogeneity, as in Landvoigt, Piazzesi and Schneider (2013). In such a setting, a change in the characteristics of a subset of movers (a change in credit condition there, reducing eligibility here) has potentially uneven effects on prices across house qualities.

2 House prices on and around Stanford campus

We obtain house prices at the property level from deeds data for the years 2002-2012. We match deeds to assessor data that contain house characteristics such as lot size, building size, the number of bathroom and bedrooms. Since we have coordinates of for each house, we can also use the American Community Survey to measure neighborhoods characteristics at the blockgroup level. We restrict attention to a narrow area around Stanford campus. Figure 1 shows the Stanford campus (zipcode 94305) together with the areas of Palo Alto and Menlo Park close to campus (census tracts 5109, 5113, 5114, 5115, 5116, 5130, 6125, 6126, 6127, 6128 and 6129). The campus is the grey shaded area. The campus transactions are mostly in the south-east corner of the grey shaded area.

The right panels of Figure 1 show a histogram of house prices on and off campus. The main point here is that the support of the campus price distribution is narrower than that of the surrounding area. On the one hand, the left tail of the campus distribution does not include a few cheap houses that are available off campus. On the other hand, the upper tail of the

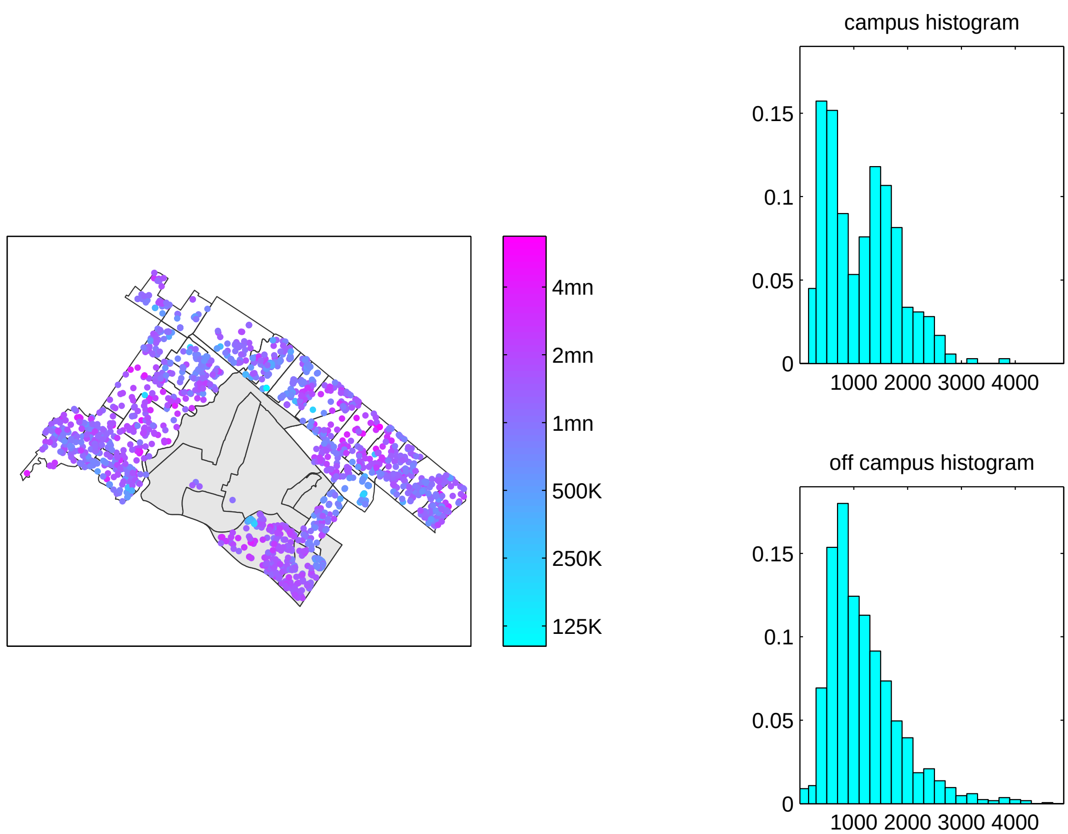

Figure 1: Left: Map of transactions, 2002-2012. The color-coding uses cold colors for cheap and warm colors for expensive houses. The map shows the Stanford campus together with areas of Palo Alto and Menlo Park in close proximity to campus. Right: Campus and off-campus histograms of house prices in these transactions.

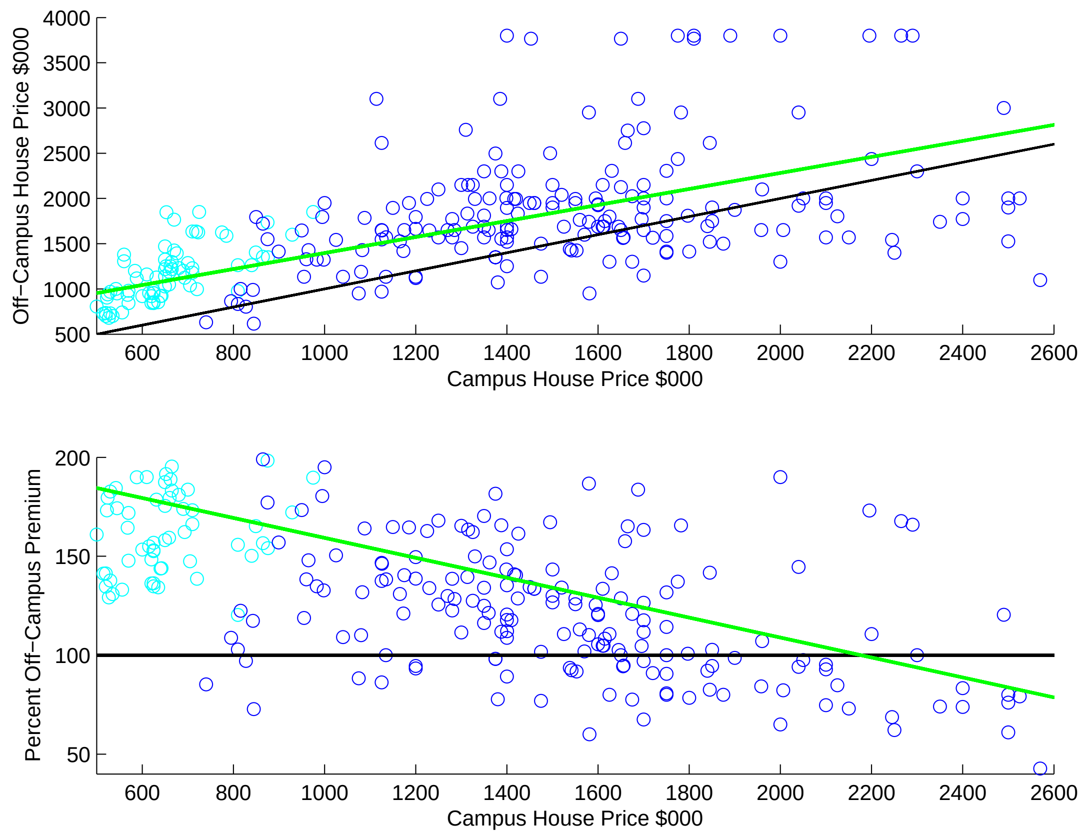

Figure 2: **Top panel**: Campus house transactions measured on the horizontal axis together with the median value of their off-campus comparables on the vertical axis. The light blue color is condos. The black 45 degree line indicates equal pricing on and off campus. The green OLS regression line uses all dotted observations. The data are from years 2002-2012. **Bottom panel**: The percent premium in off-campus transactions, 2002-2012. Light blue dots are condots.The black horizontal line indicates equal pricing on and off campus. The green line is an OLS regression through the dots.

campus distribution does not include mansions as large as those in nearby neighborhoods.[2]

Do similar houses trade at different prices on campus? An answer to this question requires estimating the hypothetical price of an on-campus house if it were located off-campus. Figure 2 takes a first crack at this by comparing prices of campus homes to those of their off-campus comparables. Each dot in the figure represents a campus transaction during the years 2002-2012. Condos are light blue whereas single family homes are dark blue. The horizontal axis measures the transaction price for the campus house. The vertical axis measures the median price of comparable off campus houses. We select comparables from transactions that occurred within

[2]For example, the off-campus area on the map in Figure 1 features private residences such as the home of Facebook Co-Founder and CEO Mark Zuckerberg, who bought a $7 million property in 2011 and added the four surrounding properties for $30 million in 2013. Other private residences are the $7 million home of Google Co-Founder and CEO Larry Page or the $11.2 property that Yahoo! CEO Marissa Mayer bought recently. Such houses are simply not available on Stanford campus.

180 days based on similarity by building area, lot size as well as the number of bathrooms and bedrooms.

The majority of dots are located above the 45 degree line that would indicate equal pricing on and off campus. Off-campus comparables are thus typically more expensive than the house on campus. This premium is particularly large for condos at the low end of the price distribution. This visual impression is confirmed by an OLS regression though the cloud of dots in Figure 2: the green regression line has an intercept of \$512K and a slope coefficient of 0.89 (which is highly significant, but insignificantly different from one.)

The bottom panel of Figure 2 shows the premium in off-campus transactions as a percentage of the campus house prices. The black horizontal line at 100% indicates equal pricing on and off campus. The green OLS regression line suggests that campus houses are mostly cheaper percentage-wise than their off-campus comparables. The discount is larger for low-end houses on campus. To buy a condo or small house off campus, a faculty member would have to pay almost double as much as on campus. For medium houses in the 1-1.5 million dollar range, a faculty member has to pay roughly 150% more to buy off campus. The campus discount disappears at the high end of the house quality spectrum. Houses worth more than 2 million Dollars on campus are not cheaper than those off campus.

Table 1 reports results from an alternative approach to estimating the hypothetical off-campus value of campus properties. Rather than use median comparable prices, we use predicted values from a nearest neighbor regression as in Caplin et al (2008). The regression is run year by year and regressors include the above characteristics as well as geographic and neighborhood information; in particular, we include latitude and longitude of the property, the shares of units in the census block group that are rented and that are in multi-unit buildings and the share of households in the highest (topcoded) income bracket of the census blockgroup. The latter variables help predict prices in neighborhoods with diverse individual properties.

The quantitative findings based on this alternative approach confirm the visual impressions from Figure 2. The percentage premium for houses outside campus is highest at the low end of the house quality spectrum. A faculty member who wants a house outside of campus comparable to a house in the bottom quartile of the quality distribution on campus pays 160% of what he would pay on campus. This premium declines and reaches zero for high-end houses (in the top quartile of the campus quality distribution, where houses cost more than \$1.6 million in year 2010 Dollars.) We estimate the absolute dollar premium for a house outside campus to be roughly constant: \$400K across the board.

Table 1: How much more expensive are houses outside campus?

Price range of campus houses		Difference between off and on campus			
		in percent	s.e.	in Dollars	s.e.
in lowest quartile	$\leq$ \$550K	162.1%	17.1	\$446	138
second quartile	\$550K – \$1,2 mio	127.8%	4.5	\$380	79
third quartile	\$1,2 mio – \$1,6 mio	116.6%	5.3	\$400	119
top quartile	$\geq$ \$1,6 mio	107.1%	6.5	\$323	175
# observations	357				

Note: Nearest neighbor matching with small sample standard errors. The imputation of nearest neighbor is done by year and uses information on lot size, building size, # bathrooms and geographical information (longitude, latitude). The imputation controls for the share of housing units, renters and households in the highest income bracket of the census block.

3 An assignment model with restricted access

The model describes the interaction of movers and sellers in a single time period.

Setup

A continuum of houses of measure one has been put up for sale. Houses differ by quality, measured by a one dimensional index h. A share ρ of houses are located in a *restricted area* that only a subset of buyers have access to. The distribution of quality inside and outside the restricted area is described by cdfs $G_r(h)$ and $G(h)$, respectively. The cdf G is smooth and strictly increasing on $[0, \bar{h}]$ with $G(0) = 0$. The cdf G_r is smooth and strictly increasing on $[\underline{h}_r, \bar{h}_r] \subset [0, \bar{h}]$. We thus allow the quality range inside the restricted area to be narrower than outside. Throughout we denote densities by lower case letters.

There is a continuum of buyers of measure one. Everyone buys at most one house. A share $\eta \geq \rho$ of *eligible buyers* can buy anywhere. The remaining buyers must buy outside the restricted area. Utility from housing does not depend on location: anyone who buys a house of quality h at price p receives surplus $\theta h - p$.[3] Buyers differ by their marginal utility of house quality θ.[4] The distribution of types θ for eligible and other buyers is described by cdfs $F_e(\theta)$ and $F(\theta)$, respectively. The cdf $F(\theta)$ is smooth and strictly increasing on $[0, \bar{\theta}]$, with $F(0) = 0$. The cdf $F_e(\theta)$ is strictly increasing on an interval $[\underline{\theta}_e, \bar{\theta}]$, where $\bar{\theta} > \underline{\theta}_e \geq 0$.

An equilibrium consists of buyers' house choices h as well as prices for restricted and unrestricted houses $p(h)$ and $p_r(h)$ so that all buyers optimize given prices and markets clear. We consider Pareto-efficient equilibria such that house quality is strictly increasing in type θ. We

[3] This assumption serves to zero in on the role of distributions on prices. Allowing eligible agents to obtain higher utility from restricted houses introduces an additional force that works to increase house prices in the restricted area. For the application we consider, this force must be weak enough and is omitted.

[4] Equivalently, we can think of $1/\theta$ as the marginal cost of funds to a buyer of type θ.

further require that all buyers obtain nonnegative surplus from buying a house, so $p(0) = 0$. We also have $p_r(h) \leq p(h)$ in equilibrium since eligible agents do not lose from buying outside the restricted area.

Equilibrium without restrictions

As a benchmark, consider equilibrium without the restriction. The overall distributions of types and houses in the economy, regardless of eligibility, are given by

$$\begin{aligned} F_u(\theta) &= \eta F_e(\theta) + (1-\eta) F(\theta), \\ G_u(h) &= \rho G_\rho(h) + (1-\rho) G(h). \end{aligned}$$

Without restrictions, buyers are assigned to houses according to the strictly increasing QQ plot of F against G, that is, $\theta_u(h) = F_u^{-1}(G_u(h))$.

Suppose the price function is convex. The optimal choice for a buyer of type θ is characterized by the first order condition

$$p'(h) = \theta. \tag{1}$$

The marginal buyer at quality h prefers a slightly higher (lower) quality house if the price schedule increases by less (more) than θ at quality h. Prices follow from integrating equation (1) given the initial condition $p(0) = 0$. The resulting price function is in fact convex because $\theta_u(h)$ is strictly increasing.

The assignment is steep when the distribution of types is more dispersed than the distribution of house qualities. Indeed, the slope $\theta_u'(h)$ is given by the density ratio $\theta_u'(h) = g(h) / f(\theta_u(h))$. When it is high, there are relatively more similar houses close to h than there are buyers of similar type close to $\theta_u(h)$. Similar houses must thus be assigned to buyer types with rather different marginal utilities. Prices must increase at a faster rate $p''(h) = \theta_u'(h)$ near h to induce those different buyers not to prefer h itself.

The equilibrium surplus $\theta h - p$ earned by buyers increases with their distance from type 0 who is just indifferent between buying and not buying. As quality increases, buyers receive additional surplus $\theta_u'(h) h$. This additional compensation for buying a house of quality h is higher if there are less buyers relative to houses.

Market clearing with a restricted area

If quality is increasing in type, the assignment must be the same for all buyers of type θ who buy outside the restricted area, regardless of whether they are eligible or not. We thus define house quality assignments $\theta_r(h)$ and $\theta(h)$ inside and outside the restricted area, respectively. The function θ_r is strictly increasing on $[\underline{h}_r, \bar{h}_r]$ and the function θ is strictly increasing on $[0, \bar{h}]$.

Let $\tilde{f}_e(\theta)$ denote the (endogenous) density of eligible agents who buy in the restricted area. Markets must clear at every quality level both inside and outside the restricted area:

$$\begin{aligned} \rho g_r(h) &= \rho \tilde{f}_e(\theta_r(h)) \theta_r'(h), \\ (1-\rho) g(h) &= (f_u(\theta(h)) - \rho \tilde{f}_e(\theta(h)) \theta'(h). \end{aligned} \tag{2}$$

Houses for sale in the restricted area at quality h must be bought by eligible agents who are assigned those houses in the restricted area. Moreover, houses for sale outside the restricted area must be bought by buyers who are not assigned houses in the restricted area.

In addition, the number of eligible agents who locate outside the restricted area must be nonnegative, that is, for all $\theta \in [\underline{\theta}_e, \bar{\theta}]$

$$\rho \tilde{f}_e(\theta) \leq \eta f^e(\theta). \tag{3}$$

If $p_r(h) < p(h)$ then this condition holds with quality at $\theta = \theta_r(h)$. All eligible buyers buy in the restricted area when quality is strictly cheaper there. In contrast, if prices are the same across areas at some quality, then eligible buyers are indifferent between areas.

Equilibrium with equal prices

We first ask whether the restriction is binding, that is, whether it makes the unrestricted equilibrium infeasible. Suppose that prices are the same across areas for all quality levels. The equilibrium assignment θ_u implies a unique density $\tilde{f}_e$ that clears the market in equation (2). The question is whether there are always enough eligible agents to buy the restricted houses at every quality level.

Condition (3) now restricts the slope of the assignment:

$$\theta_u'(h) = \frac{\rho g_r(h) + (1-\rho)\, g(h)}{\eta f_e(\theta_u(h)) + (1-\eta)\, f(\theta_u(h))} \geq \frac{\rho}{\eta} \frac{g_r(h)}{f_e(\theta_u(h))}.$$

Since $\rho \leq \eta$, the condition is always satisfied if the distributions of houses and buyers are identical. If $\rho = \eta$, it says that the density ratios $g_r(h)/f_e(\theta)$ and $g(h)/f(\theta)$ must be equal across areas. This is the knife edge condition that implies equal prices if the two areas were completely segmented markets.

With $\rho < \eta$, the predictions of the model differ from one with segmented markets: an equal price equilibrium may also exist when the density ratios are different. Indeed, arbitrage by eligible agents can work to equate prices. For example, suppose the house quality densities are the same. Consider a quality range around h with many more eligible than ineligible agents. With segmented markets, prices rise less with h in the restricted area the relative demand for more expensive houses is lower there. In the present model, some eligible agents can move out of the restricted area and thus equate the relative demands.

Equilibrium with discount for the restricted area

We now investigate why houses in the restricted area can be strictly cheaper for all quality levels. In this case, if a quality level is available in the restricted area, no eligible buyer will buy it outside. The $\eta - \rho$ eligible buyers who nevertheless buy outside the restricted area thus choose qualities that are not available inside. We focus on equilibria such that all eligible buyers who move outside the restricted area buy higher quality houses than those available inside.[5]

The assignment of houses inside the restricted area is given by

$$\theta_r(h) = F_e^{-1}\left(\frac{\rho}{\eta} G_r(h)\right). \tag{4}$$

[5]This assumption works for our Stanford application as long as eligible employees who live off campus are either renters or buy higher quality housing. We do not model tenure choice but consider only the assignment conditional on movers' decision to own. More generally, allowing in addition for some eligible buyers who buy lower quality houses would not affect our interpretation of valuation at the high end, but could lead to different predictions about the low end.

In particular, there is a highest type $\theta^* = \theta_r\left(\bar{h}_r\right) = F_e^{-1}(\rho/\eta)$ who is indifferent between buying the highest restricted house $\bar{h}_r$ at price $p_r\left(\bar{h}_r\right)$ and buying a higher quality $h^* > \bar{h}_r$ outside the restricted area.[6]

For all types beyond θ^*, the restriction does not bite and the assignment is given by $\theta_u(h)$. Below the house quality $h^* = \theta_u^{-1}(\theta^*)$, houses outside the restricted area are assigned to ineligible buyers according to

$$\theta(h) = F^{-1}\left(\frac{1-\rho}{1-\eta}G(h)\right). \tag{5}$$

Since $h^* > \bar{h}_r$ an equilibrium with equal prices cannot exist. Indeed, since assignments are monotonic we must have $\theta_r\left(\bar{h}_r\right) > \theta_u\left(\bar{h}_r\right)$ which is incompatible with (3). With the distributions assumed here, the unrestricted assignment asks relatively low types to move into the restricted area. However, not enough of those types are eligible to support an equilibrium with equal prices.

First order conditions of the type (1) hold both inside and outside the restricted area. At quality levels available in the restricted area, prices are found by integration using the indifference of type θ^* between $\bar{h}_r$ and h^*:

$$\begin{aligned} p(h) &= \int_0^h \theta(\tilde{h})d\tilde{h}, \\ p_r(h) &= p(h) - \int_h^{h^*} \left(\theta_r(\tilde{h}) - \theta(\tilde{h})\right) d\tilde{h}. \end{aligned}$$

A price discount exists at h in the restricted area as long as the average assignment between h and h^* is higher there. Intuitively, low prices must induce relatively high types to buy the relatively low quality houses inside the restricted area. To establish that the resulting prices support an equilibrium, we also need to show that eligible types optimally choose their area. Sufficient conditions for the existence of an equilibrium are provided in the appendix.

Figure 3 provides a concrete numerical illustration of the above type of equilibrium, with distributions loosely motivated by the Stanford evidence. The top left panel displays two type densities, for f a lognormal (in blue) and for f_e a lognormal truncated below at $\underline{\theta}_e$ that also has a smaller right tail. The top middle panel displays a uniform house density g_r with narrow support for inside the (shaded) restricted area together with a beta density g for outside. The top right panel shows equilibrium assignments in the restricted area (in red) and outside (in blue). Types higher than the critical θ^* shown on the top left follow the efficient assignment beyond h^* shown on the top right.

Eligible buyers at the top end of the restricted area buy lower quality houses than ineligible buyers with the same preferences; for the same threshold marginal utility θ^*, for example,

[6] We can view introduction of the restriction as a comparative static on preferences: noneligible agents no longer like restricted houses. This helps relate our exercise to the graph theoretic analysis of equilibria with a finite number of agents and indivisible goods in Caplin and Leahy (2010). Those authors represent equilibrium allocations as graphs connecting goods with vertices indicating indifference by some agent and show that comparative statics can be characterized in terms of five possible market transitions. The introduction of the restriction here resembles a "prune and graft" transition: the branch of restricted houses is sliced off from the tree representing the unrestricted assignment and grafted on at h^*.

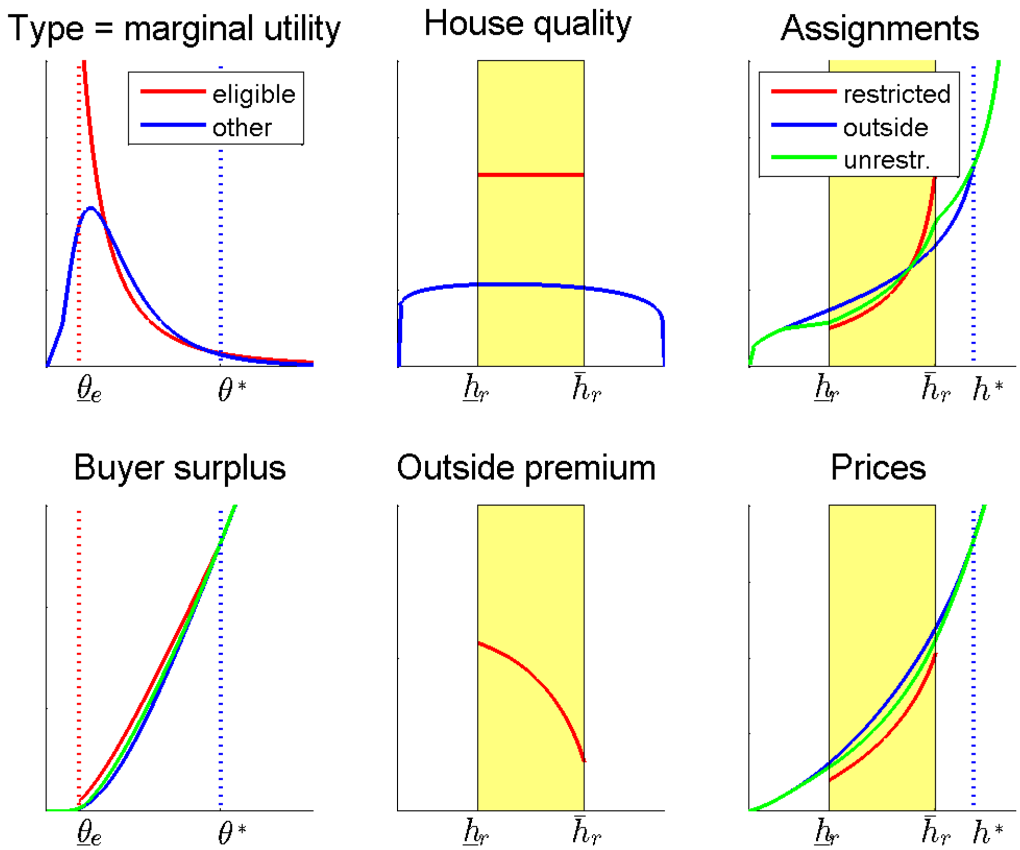

Figure 3: Illustrative example. *Top left*: *type densities;* red for eligible buyers f_e, blue for other buyers f. *Top middle*: *house densities*; red for restricted houses g_r, blue for other houses g. *Top right*: *assignments*; green for $\theta_u(h)$ (unrestricted), red for $\theta_r(h)$ (with restriction, inside restricted area), blue for $\theta(h)$ (with restriction, outside). *Bottom left: buyer surplus;* green: unrestricted, red: with restriction, inside restricted area, blue: with restriction, outside. *Bottom middle: outside premium* $(p(h) - p_r(h))/p_r(h)$, *Bottom right:prices*, same legend as bottom left.

eligible buyers buy $\bar{h}_r$ while ineligible buyers buy $h^* > \bar{h}_r$. The effect of the restriction in this range is not to enable buyers to buy higher quality houses than their ineligible counterparts, but instead to offer lower quality houses at a discount. Price discounts can be read off the plot of price functions at the bottom right.

As we move down to lower house qualities, the price discount increases in size if house quality inside the restricted area remains low relative to the distribution of eligible types. As long as $\theta_r(h) > \theta(h)$, ever deeper discounts are needed for eligible types to prefer lower quality houses inside the restricted area to higher quality houses outside. In the example, θ_r actually falls below θ for low qualities. In this range, the absolute size of the price discount declines; at the same time, eligible types buy higher quality houses than their ineligible counterparts. Of course, the percentage premium earned by outside houses may still increase as quality declines, as shown in the bottom middle panel.

The bottom left panel shows the surplus earned by different buyers in equilibrium. The restriction favors eligible buyers who must be at least as well off as their ineligible counterparts. At qualities available inside the restricted area, equilibrium surplus is

$$\theta_r(h)\,h - p_r(h) = \int_0^h (\theta_r(h) - \theta_r(\tilde{h}))d\tilde{h} + \int_0^{h^*} (\theta_r(\tilde{h}) - \theta(\tilde{h}))d\tilde{h}.$$

The first term takes the form that typically obtains in a simple assignment of eligible agents to restricted houses. Surplus is higher the further away a buyer is from the lowest type. Here the interaction with ineligible buyers implies that eligible buyers receive an additional rent represented by the second term.

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Appendix

A Sufficient conditions for existence

Here we provide sufficient conditions for the existence of an equilibrium with unequal prices of the type presented in the text.

Proposition. Let $h^* = \theta^{-1}\left(\theta_r\left(\bar{h}_r\right)\right)$ and suppose that for all $h \in [\underline{h}_r, \bar{h}_r]$,

$$\int_h^{h^*} \left(\theta_r(\tilde{h}) - \theta(\tilde{h})\right) d\tilde{h} > 0 \tag{A-1}$$

There is an equilibrium such that the assignment θ_r is given by (4) and the assignment θ is given by (5) for $h \leq h^*$ and $\theta(h) = \theta_u(h)$ for $h > h^*$. Moreover, eligible buyers buy outside the restricted area if and only if $\theta > \theta^*$.

Proof. Markets clear for all qualities below h^* by construction of θ and θ_r. Markets clearing above h^* follows because $h^* > \bar{h}_r$ (so there are no restricted houses above h^*) and $\theta = \theta_u$.

For noneligible agents, the first order condition is necessary and sufficient for optimality given differentiable and convex price functions.

For eligible agents, the first order condition $p_r'(h) = \theta_r(h)$ says that h is the best house in the restricted area. We must also establish that they choose the area optimally. Consider first an eligible buyer $\theta_r(h)$ who buys quality h in the restricted area. He must prefer this quality to any house $\tilde{h}$ outside the restricted area:

$$\theta_r(h)\, h - p_r(h) \geq \theta_r(h)\, \tilde{h} - p(\tilde{h}).$$

Consider first $\tilde{h} \leq h^*$. Using the first order condition and indifference of type θ^*, we have

$$\theta_r(h)\,(h - \tilde{h}) + \int_h^{h^*} \theta_r(\hat{h}) d\hat{h} \geq \int_{\tilde{h}}^{h^*} \theta(\hat{h}) d\hat{h}. \tag{A-2}$$

Condition (A-1) holds for all $h \leq h^*$. Indeed, it holds at $h = \bar{h}_r$ by assumption and we have $\theta_r\left(\bar{h}_r\right) = \theta(h^*)$. It follows that (A-2) is true for any $\tilde{h} \leq h^*$.

For $\tilde{h} > h^*$, we have

$$\begin{aligned}
\theta_r(h)\, h - p_r(h) &\geq \theta_r(h)\, \bar{h}_r - p_r(h) \\
&= \theta_r(h)\, \bar{h}_r + \theta^*\left(h^* - \bar{h}_r\right) - p(h^*) \\
&\geq \theta_r(h)\, \bar{h}_r + \theta^*(\tilde{h} - \bar{h}_r) - p(\tilde{h}) \\
&> \theta_r(h)\, \tilde{h} - p(\tilde{h})
\end{aligned}$$

where the first line follows because $\theta_r(h)$'s first order condition holds, the equality follows from the indifference condition for θ^*, and the second inequality follows because ineligible buyers of type θ^* is choose optimally.

It remains to show that eligible buyers $\theta(h)$ who choose h outside the restricted area do not prefer $\tilde{h}$ inside the restricted area. By a similar argument to above, we have

$$\begin{aligned}
\theta(h)\,h - p(h) &\geq \theta(h)\,h^* - p(h^*) \\
&= \theta(h)\,h^* + \theta^*\left(\bar{h}_r - h^*\right) - p_r\left(\bar{h}_r\right) \\
&\geq \theta(h)\,h^* + \theta^*(\tilde{h} - h^*) - p(\tilde{h}) \\
&> \theta(h)\,\tilde{h} - p(\tilde{h}).
\end{aligned}$$

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