Robust Contracts in Continuous Time*

Jianjun Miao† Alejandro Rivera‡

April 16, 2013

Abstract

We study two types of robust contracting problem under hidden action in continuous time. In type I problem, the principal is ambiguous about the project cash flows, while he is ambiguous about the agent's beliefs in type II problem. The principal designs a robust contract that maximizes his utility under the worst-case scenario subject to the agent's incentive and participation constraints. We implement the optimal contract by cash reserves, debt and equity. In addition to receiving ordinary dividends when cash reserves reach a threshold, outside equity holders also receive special dividends or inject cash in the cash reserves to hedge against model uncertainty. Ambiguity aversion lowers outside securities value and raises the credit yield spread. It generates equity premium for type I problem, but not for type II problem. The equity premium and the credit yield spread are state dependent and high for distressed firms with low cash reserves.

JEL Classification: D86, G12, G32, J33

Keywords: robustness, ambiguity, moral hazard, principal-agent problem, capital structure, equity premium, asset pricing

---

*We thank Zhiguo He, Juan Ortner, Yuliy Sannikov, and Noah Williams for helpful comments. We have also benefited from comments by the seminar participants at Boston University and the University of Wisconsin at Madison. First version: December 2012.

†Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215. Tel.: 617-353-6675. Email: miaoj@bu.edu. Homepage: http://people.bu.edu/miaoj.

‡Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215.

1 Introduction

Uncertainty and information play an important role in principal-agent problems. Consistent with the rational expectations hypothesis, the traditional approach to these problems typically assumes that both the principal and the agent share the same belief about the uncertainty underlying an outcome, say output. The agent can take unobservable actions to influence the output distribution. This distribution is common and known to both the principal and the agent. This approach has generated important economic implications and found increasingly widespread applications in practice, e.g., managerial compensation, insurance contracts, and lending contracts, etc.

However, there are several good reasons for us to think about departures from the traditional approach. First, the Ellsberg (1961) paradox and related experimental evidence demonstrate that there is a distinction between risk and uncertainty (or ambiguity). Risk refers to the situation where there is a known probability distribution over the state of the world, while ambiguity refers to the situation where the information is too vague to be adequately summarized by a single probability distribution (Knight (1921)). As a result, a decision maker may have multiple priors in mind (Gilboa an Schmeidler (1989)). Second, as Anderson, Hansen and Sargent (2003) and Hansen and Sargent (2001, 2008) point out, economic agents view economic models as an approximation to the true model. They believe that economic data come from an unknown member of a set of unspecified models near the approximating model. Concern about model misspecification induces a decision maker to want robust decision rules that work over that set of nearby models.[1]

The goal of this paper is to study how to design robust contracts with hidden action in a dynamic environment. We adopt a continuous-time framework to address this question. More specifically, our model is based on DeMarzo and Sannikov (2006). The continuous-time framework is analytically convenient for several reasons. First, it allows us to represent belief distortions by perturbations of the drift of the Brownian motion using the powerful Girsanov Theorem.[2] Second, it allows us to adapt and extend the martingale approach to the dynamic contracting problems recently developed by DeMarzo and Sannikov (2006), Sannikov (2008), and Williams (2009, 2011). Third, it allows us to express solutions in terms of ordinary differential equations (ODEs) which can be numerically solved tractably. Finally, it allows us to conduct capital structure implementation so that we can analyze the impact of robustness on asset pricing transparently.

[1]There is a growing literature on the applications of robustness and ambiguity to finance and macroeconomics, e.g., Epstein and Wang (1994), Hansen (2007), Hansen and Sargent (2010), Ilut and Schneider (2011), and Ju and Miao (2012), among others.

[2]See Karatzas and Shreve (1991).

When formulating robust contracting problems in continuous time, we face two important issues. The first issue is that we have to consider who faces model ambiguity in our two-party contracting problems, unlike in the representative agent models. We study two possibilities. First, the agent knows the output distribution chosen by himself. But the principal faces model uncertainty in the sense that he believes that there may be multiple distributions surrounding the output distribution. In this case, we call the contracting problem *type I robust contracting problem*. Second, the principal trusts the output distribution chosen by the agent. But the principal is ambiguous about what beliefs the agent has. This case corresponds to *type II robust contracting problem*.

The second issue is how to model decision making under ambiguity. There are several approaches in decision theory. A popular approach is to adopt the maxmin expected utility model of Gilboa and Schmeidler (1989). Chen and Epstein (2002) formulate this approach in a continuous-time framework. We find that this approach is hard to work with in our continuous-time contracting problems because two types of inequality constraints (the constraint on the set of priors and the incentive constraint) are involved in optimization. We thus adopt the approach proposed by Anderson, Hansen, and Sargent (2003) and Hansen et al. (2006).[3] This approach is especially useful for our analysis since model discrepancies are measured by entropy, which is widely used in statistics and econometrics for model detection.

For both types of robust contracting problems, we assume that the principal copes with model uncertainty by designing a robust contract that maximizes his utility in the worst-case scenario subject to the agent's incentive and participation constraints. For type I robust contracting problem, the principal's utility function is modeled as the multiplier preferences proposed by Anderson, Hansen and Sargent (2003) and Hansen and Sargent (2001, 2008) and axiomatized by Maccheroni, Marinacci and Rustichini (2006a,b) and Strzalecki (2011). For type II robust contracting problem, we adopt Woodford's (2010) approach.[4] In this approach, the principal evaluates his utility using his trusted approximating model. But there is a penalty term in the utility function arising from the principal's concerns about the agent's belief distortions. For both types of contracting problems, the principal solves maxmin problems, which are related to the zero-sum differential game literature (e.g., Fleming and Souganidis (1989))

We find the following new results. First, unlike the DeMarzo and Sannikov (2006) model, our model of type I robust contracting problem implies that the optimal sensitivity of the agent's continuation value to the cash flow uncertainty is not always at the lower bound

[3]See Hansen and Sargent (2008) for a textbook treatment of this approach in discrete time.

[4]Woodford (2010) originally studies a discrete-time linear-quadratic problem. Hansen and Sargent (2012) study a continuous-time limit of his problem.

to ensure incentive compatibility. By contrast, for type II robust contracting problem, the optimal sensitivity is always at the lower bound. The intuition is the following. In type I robust contracting problem, the principal is ambiguous about the probability distribution of the project cash flows. He wants to remove this ambiguity and transfer uncertainty to the agent. But he does not want the agent to bear too much uncertainty since this may generate excessive volatility and a high chance of liquidation. When the agent's continuation value is low, the principal is more concerned about liquidation and hence the optimal sensitivity is at the lower bound so that the incentive constraint just binds. But when the agent's continuation value is high, the principal is more concerned about model uncertainty and hence the optimal contract allows the agent to bear more uncertainty. In this case, the optimal sensitivity of the agent's continuation value to the cash flow is state dependent and exceeds its lower bound. The situation is different for type II robust contracting problem in which the principal trusts the distribution of cash flows but he has ambiguity about the agent's beliefs. The robust contract penalizes the agent's belief distortions and makes the agent bears the minimal uncertainty about the cash flow.

Second, we show that the robust contracts can be implemented by cash reserves, debt, and equity as in Biais et al. (2007).[5] Unlike their implementation, the equity payoffs consist of regular dividends (paid only when the cash reserves reach a threshold level) and special dividends (or new equity injection if negative). The special dividends or new equity injection are used as a hedge against model uncertainty. They ensure that the cash reserves track the agent's continuation value so that the payout time and the liquidation time coincide with those in the optimal contract. For type I robust contracting problem, special dividends or cash injection occur only when the cash reserves are sufficiently high. In this case, when the project performs well, outside equity holders inject cash to raise the cash reserves. But when the project performs bad, outside equity holders receive special dividends, which lower the cash reserves. By contrast, for type II robust contracting problem, special dividends are distributed when the cash reserves are low and new equity injection occur when the cash reserves are high.

Third, incorporating model uncertainty has important asset pricing implications. For type I robust contracting problem, the principal's worst-case belief generates a market price of model uncertainty, which contributes to the uncertainty premium and hence the equity premium. The uncertainty premium lowers the stock price and debt value and hence makes some profitable projects unfunded. It also raises the credit yield spread. Importantly, the

[5]Our implementation and interpretation are also similar to those in DeMarzo et al. (2012). We can also implement the robust contracts by credit lines, debt and equity as in DeMarzo and Sannikov (2006). We have not pursued this route in this paper.

equity premium and the credit yield spread are state dependent and high for distressed firms with low cash reserves. This also implies that the equity premium and the credit yield spread are high in recessions since cash reserves are low in bad times. By contrast, there is no uncertainty premium for type II robust contracting problem just as in DeMarzo and Sannikov (2006) and Biais et al. (2007). The reason is that outside investors (or the principal) are risk neutral and have no distortion of beliefs about cash flows in type II problem.

To generate time-varying equity premium or credit yield spread, the existing literature typically introduces one of the following assumptions: time-varying risk aversion as in the Habit formation model of Campbell and Cochrane (1999), time-varying economic uncertainty as in the long-run risk model of Bansal and Yaron (2004), or regime-switching consumption and learning under ambiguity as in Ju and Miao (2012). By contrast, in our contracting model, investors are risk neutral with endogenously distorted beliefs, dividends are endogenous, and the driving state process is identically and independently distributed.

Finally, unlike DeMarzo and Sannikov (2006) and Biais et al. (2007), we show that the stock price in both types of robust contracting problems is convex for low levels of cash reserves and concave for high levels of cash reserves. Intuitively, after a sequence of low cash-flow realizations, cash reserves are close to the liquidation boundary. The robust contract has already taken into account the worst-case scenario. The equity price does not have to respond strongly to a decrease in the cash reserves when they are low in the sense that the marginal change in the stock price decreases when the cash reserves fall. But for high levels of cash reserves, the principal pessimistically believes that the firm does not perform that well. Thus, the stock price reacts strongly to a decrease in the cash reserves when they are large in the sense that the marginal change in the stock price increases when the cash reserves fall. This result implies that asset substitution problem is more likely to occur for financially distressed firms or newly established firms with low cash reserves.

Our paper is related to a fast growing literature on dynamic contracting problems in continuous time.[6] Our paper is most closely related to the seminal contributions by DeMarzo and Sannikov (2006) and Biais et al. (2007). Our main contribution is to introduce robustness into their models and study capital structure implementation and asset pricing implications. Our paper is also related to the microeconomic literature that introduces robustness into static mechanism design problems (see Bergemann and Schlag (2011) and Bergemann and Morris

[6]See Holmstrom and Milgrom (1987), Schattler and Sung (1993), Ou-Yang (2003), DeMarzo and Sannikov (2006), Biais (2007, 2010), Sannikov (2008), He (2009, 2011), Williams (2009, 2011), Zhang (2009), Piskorski and Tchistyi (2010), Prat and Jovanovic (2010), DeMarzo et al. (2012), Cvitanic and Zhang (2012), Zhu (2012), He, Wei and Yu (2012), Ju and Wan (2012), and Szydlowski (2012). This literature is closely related to that on discrete-time dynamic contracts, e.g., Green (1987), Spear and Srivastava (1987), Thomas and Worrall (1990), Atkeson and Lucas (1992), Albuquerque and Hopenhayn (2004), Quadrini (2004), Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007a,b).

(2012) and references cited therein). This literature typically focuses on static models with hidden information instead of hidden action. Szydlowski (2012) introduces ambiguity into a dynamic contracting problem in continuous time. He assumes that the principal is ambiguous about the agent's effort cost. His modeling of ambiguity is quite different from ours and can be best understood as a behavioral approach. His utility model cannot be subsumed under the decision-theoretic setting of Gilboa and Schmeidler (1989) and its continuous time version by Chen and Epstein (2002).

Our modeling of two types of robust contracting problems is inspired by Hansen and Sargent (2012) who classify four types of ambiguity in robust monetary policy problems in which a Ramsey planner faces private agents. They argue that "a coherent multi-agent setting with ambiguity must impute possibly distinct sets of models to different agents, and also specify each agent's understanding of the sets of models of other agents." This point is particularly relevant for contracting problems because such problems must involve at least two parties. Their types II and III ambiguity corresponds to our types I and II, respectively. Both types generate endogenous belief heterogeneity and deliver interesting contract dynamics and asset pricing implications. We thus focus on these two types in the paper.

The remainder of the paper proceeds as follows. Section 2 lays out the conceptual framework in both a static and a dynamic discrete-time settings. Section 3 presents the continuous-time model. Sections 4 and 5 analyze types I and II robust contracting problems, respectively. Section 6 studies capital structure implementation. Section 7 concludes. Technical details are relegated to appendices.

2 A Conceptual Framework

In this section, we lay out a conceptual framework for studying robust contracts. We start with a static setup and then present a dynamic discrete-time setup. The discrete-time framework is helpful for understanding the continuous-time model studied later.

2.1 Static Setup

Consider a textbook model of the principal-agent problem under moral hazard (e.g., Mas-Colell, Whinston and Green (1995)). A principal owns a technology of production and hires an agent to manage this technology. The technology can generate stochastic output. Output depends on the agent's effort. Suppose that output x is drawn from a distribution with pdf $f(\cdot|a)$ where the distribution depends on the effort level a. The principal does not observe the agent's effort. He receives output and pays the agent compensation $c(x)$ contingent on the output level x. His realized utility is given by $v(x - c(x))$ for some function v. The agent's realized utility is given by $u(c(x), a)$ for some function u. A contract (c, a) specifies a compensation scheme c and an effort choice a.

An optimal contract solves the following:

Problem 2.1 *(standard static contract)*

$$\max_{(c,a)} \int v(x - c(x)) f(x|a) \, dx$$

subject to

$$\int u(c(x), a) f(x|a) \, dx \geq \int u(c(x), \hat{a}) f(x|\hat{a}) \, dx, \ \forall \hat{a}, \tag{1}$$

$$\int u(c(x), a) f(x|a) \, dx \geq u_0. \tag{2}$$

Equation (1) is an incentive constraint and equation (2) is an individual rationality constraint or a participation constraint, where u_0 is an outside utility level.

In this standard model, an important assumption is that both the principal and the agent believe that $f(\cdot|a)$ is the true output distribution and use this common distribution to evaluate their expected utility. Suppose that the true output distribution is unknown. Both the principal and the agent view the distribution f as an approximating model. They may fear that this model is misspecified. Surrounding this model there is a set of distributions including the true distribution. The principal and the agent want to design a robust contract that is less fragile to misspecification.

We start with the Gilboa-Schmeidler approach and call the resulting problem the type 0 robust contracting problem.

Problem 2.2 *(type 0 static robust contract)*

$$\max_{c,a} \min_{g \in \mathcal{G}(a)} \int v(x - c(x)) g(x|a) \, dx, \tag{3}$$

subject to (1) and (2).

In this problem, the principal believes that there is a set of distributions $\mathcal{G}(a)$ containing the reference distribution f conditional on a. He does not trust f and his utility function is given by the maxmin utility model of Gilboa and Schmeidler (1989). By contrast, the agent trusts f and his utility function is given by the expected utility model. Thus, the incentive and participation constraints are given by (1) and (2).

We modify type 0 robust contracting problem in two ways in the spirit of Hansen and Sargent (2008). First, we impose some structure on the set $\mathcal{G}(a)$. Specifically, following Hansen and Sargent (2008), we use relative entropy to measure discrepancies between distributions and let the set of distributions $\mathcal{G}(a)$ be

$$\mathcal{G}(a) = \left\{ g : \int g(x|a) \ln\left(\frac{g(x|a)}{f(x|a)}\right) dx \leq \eta, \ \int g(x|a)\, dx = 1 \right\}, \ \eta > 0. \tag{4}$$

Second, instead of using the maxmin expected utility model, we use the multiplier model as the principal's utility function. This model removes the constraint set $\mathcal{G}(a)$ in (3) and imposes an entropy penalty on utility.

Problem 2.3 *(type I static robust contract)*

$$\max_{c,a} \min_{g} \int v(x - c(x))\, g(x|a)\, dx + \theta \int g(x|a) \ln\left(\frac{g(x|a)}{f(x|a)}\right) dx,$$

subject to (1) and (2).

The penalty parameter $\theta > 0$ captures the degree of concerns for robustness. When θ approaches infinity, $g = f$ and type I robust contracting problem reduces to the standard problem 2.1. A small value of θ implies a large degree of concerns for robustness. Alternatively, one may view $1/\theta$ as a parameter for ambiguity aversion following Maccheroni, Marinacci, and Rustichini (2006a). A small value of θ means a large degree of ambiguity aversion. Type I robust contracting problem is closely related to type 0 problem and the parameter θ is related to the Lagrange multiplier associated with the constraint set $\mathcal{G}(a)$.

In the spirit of Woodford (2010), we also study a type II robust contract problem. In this problem, the principal trusts the output distribution f and uses this distribution to evaluate his expected utility. But he has ambiguity about agent's beliefs and thinks the agent may use a distorted model g. The degree of distortion is measured by an entropy criterion as in (4).

Problem 2.4 *(type II static robust contract)*

$$\max_{c,a} \min_{g} \int v(x - c(x))\, f(x|a)\, dx + \theta \int g(x|a) \ln\left(\frac{g(x|a)}{f(x|a)}\right) dx, \tag{5}$$

subject to

$$\int u(c(x), a)\, g(x|a)\, dx \geq \int u(c(x), \hat{a})\, g(x|\hat{a})\, dx, \ \forall \hat{a}, \tag{6}$$

$$\int u(c(x), a)\, g(x|a)\, dx \geq u_0. \tag{7}$$

The second term in (5) describes the cost of distortion of the agent's beliefs. The parameter $\theta > 0$ may be viewed as the Lagrange multiplier associated with the constraint set (4) and can be interpreted as the degree of concerns for robustness. A small value of θ means a great degree of concerns for robustness. Alternatively, we may interpret $1/\theta$ as an ambiguity aversion parameter.

Note that in the incentive and participation constraints (6) and (7), the distorted distribution g is used to evaluate the agent's utility. For each possible distribution g, the contract (c, a) must be incentive compatible and individually rational to the agent. The agent is not averse to ambiguity and is an expected utility maximizer with distorted beliefs. The principal is averse to the agent's belief ambiguity and selects an optimal contract that works for the worst-case distribution.

2.2 Dynamic Discrete-Time Setup

We now move on to a dynamic discrete-time setup closely related to Spear and Srivastava (1987). Denote time by $t = 0, 1, 2....$ At time t, the agent chooses effort a_t. Then output x_t is drawn from the fixed distribution with pdf $f(x|a_t)$. For any fixed action, output is independently and identically distributed over time. The principal pays the agent compensation c_t at time t. His period utility at time t is $v(x_t - c_t)$ and the agent's period utility at time t is $u(c_t, a_t)$. To incorporate moral hazard, we assume that the principal only observes the past and current output and he does not observe the agent's effort. Let $x^t = \{x_0, x_1, ..., x_t\}$ denote a history of realized output. Contracted compensation at time t depends on x^t only and is given by $c_t = c(x^t)$ for some function c. The agent's effort choice a_t depends on x^{t-1} and is given by $a_t = a(x^{t-1})$ for some function a. Let $a(x^{-1}) = a_0$.

Given $(c, a) = \{(c(x^t), a(x^{t-1})) : \text{all } x^t\}$, we can construct a sequence of probability distributions $\{\pi(x^t; a)\}$ recursively for each history x^t as follows:

$$d\pi(x^0; a) = f(x_0|a_0)\, dx_0,$$

$$d\pi(x^{t+1}; a) = f(x_{t+1}|a(x^t))\, d\pi(x^t; a)\, dx_{t+1},\ t \geq 0.$$

Assume that the subjective discount factors for the principal and the agent are given by β and α respectively. We can now state the following:

Problem 2.5 *(standard contract in discrete time)*

$$\max_{(c,a)} \sum_{t=0}^{\infty} \beta^t \int v(x_t - c(x^t))\, d\pi(x^t; a),$$

subject to

$$\begin{aligned}
& \sum_{t=0}^{\infty} \alpha^t \int u\left(c\left(x^t\right), a\left(x^{t-1}\right)\right) d\pi\left(x^t; a\right) \qquad (8)\\
\geq & \sum_{t=0}^{\infty} \alpha^t \int u\left(c\left(x^t\right), \hat{a}\left(x^{t-1}\right)\right) d\pi\left(x^t; \hat{a}\right), \ \forall \hat{a},
\end{aligned}$$

$$\sum_{t=0}^{\infty} \alpha^t \int u\left(c\left(x^t\right), a\left(x^{t-1}\right)\right) d\pi\left(x^t; a\right) \geq u_0. \qquad (9)$$

Inequalities (8) and (9) are incentive and participation constraints, respectively. To incorporate model ambiguity, we have to extend the previous static setup to a dynamic environment. The first step is to construct multiple distributions in a dynamic environment.

2.2.1 Martingale Representation of Distortions

Given an effort choice a, an approximating model f and an alternative distorted distribution g, we define the likelihood ratio as

$$m_{t+1}\left(x^{t+1}; a\right) = \frac{g\left(x_{t+1} | a\left(x^t\right)\right)}{f\left(x_{t+1} | a\left(x^t\right)\right)}.$$

It is the Radon-Nikodym derivative of g with respect to f and satisfies

$$\int m_{t+1}\left(x^{t+1}; a\right) f\left(x_{t+1} | a\left(x^t\right)\right) dx_{t+1} = 1. \qquad (10)$$

We then define a joint likelihood ratio recursively as

$$M_{t+1}\left(x^{t+1}; a\right) = m_{t+1}\left(x^{t+1}; a\right) M_t\left(x^t; a\right), \ M_0 = 1. \qquad (11)$$

$\{M_t\}$ is a martingale under the distribution induced by f and is the Radon-Nikodym derivative for joint distributions on information up to date t. A process of densities $\{m_t\}$ induces a distorted distribution that is absolutely continuously with respect to f. The choice of a distorted distribution is equivalent to the choice of densities.

Following Hansen and Sargent (2008), we define the infinite-horizon relative entropy as

$$\begin{aligned}
& (1-\beta) \sum_{t=0}^{\infty} \beta^t \int M_t\left(x^t; a\right) \ln M_t\left(x^t; a\right) d\pi\left(x^t; a\right) \qquad (12)\\
= & \ \beta \sum_{t=0}^{\infty} \beta^t \int M_t\left(x^t; a\right) m_{t+1}\left(x^{t+1}; a\right) \ln m_{t+1}\left(x^{t+1}; a\right) d\pi\left(x^{t+1}; a\right),
\end{aligned}$$

where the equality follows from the definition of $\{M_t\}$. Note that we have used the principal's subjective discount factor β to discount future entropy.

Woodford (2010) introduces a different measure of the infinite-horizon entropy. In our context, his measure corresponds to

$$\sum_{t=0}^{\infty} \beta^t \int m_{t+1}\left(x^{t+1}; a\right) \ln m_{t+1}\left(x^{t+1}; a\right) d\pi\left(x^{t+1}; a\right). \tag{13}$$

As Hansen and Sargent (2012) point out, this is not a measure of the usual relative entropy. But it is an expected value of the discounted local entropy for one-step-ahead measures.

2.2.2 Robust Contracts

We are now ready to formulate the three types of robust contracting problems in a dynamic setup.

Problem 2.6 *(type 0 robust contract in discrete time)*

$$\max_{(c,a)} \min_{\{m_{t+1}\} \in \mathcal{M}(a)} \sum_{t=0}^{\infty} \beta^t \int M_t\left(x^t; a\right) v\left(x_t - c\left(x^t\right)\right) d\pi\left(x^t; a\right),$$

subject to (8), (9), (10), and (11).

The set $\mathcal{M}(a)$ corresponds to a set of distributions that contains the one induced by f conditional on a. It describes the principal's model ambiguity. The principal's utility function in the above problem belongs to the class of recursive multiple priors utility introduced by Epstein and Wang (1994) and axiomatized by Epstein and Schneider (2003).

Problem 2.7 *(type I robust contract in discrete time)*

$$\begin{aligned}\max_{(c,a)} \min_{\{m_{t+1}\} \in \mathcal{M}(a)} & \sum_{t=0}^{\infty} \beta^t \int M_t\left(x^t; a\right) v\left(x_t - c\left(x^t\right)\right) d\pi\left(x^t; a\right) \\ & + \theta\beta \sum_{t=0}^{\infty} \beta^t \int M_t\left(x^t; a\right) m_{t+1}\left(x^{t+1}; a\right) \ln m_{t+1}\left(x^{t+1}; a\right) d\pi\left(x^{t+1}; a\right),\end{aligned}$$

subject to (8), (9), (10), and (11).

In type I dynamic robust contracting problem, we use the infinite-horizon entropy in (12) as a measure of the cost of belief distortions. The principal's utility function in this problem is the dynamic multiplier model introduced in Hansen and Sargent (2001, 2008) and Maccheroni, Marinacci and Rustichini (2006b).

Finally, in type II robust contracting problem, the principal trusts the output distribution chosen by the agent. But he has ambiguity about the agent's beliefs. The agent's belief distortions impose costs to the principal, which are measured by the entropy in (13).

Problem 2.8 *(type II dynamic robust contract in discrete time)*

$$\begin{aligned}\max_{(c,a)} \min_{\{m_{t+1}\}\in\mathcal{M}(a)} & \sum_{t=0}^{\infty}\beta^t \int v\left(x_t - c\left(x^t\right)\right) d\pi\left(x^t;a\right) \\ & + \theta\sum_{t=0}^{\infty}\beta^t \int m_{t+1}\left(x^{t+1};a\right)\ln m_{t+1}\left(x^{t+1};a\right) d\pi\left(x^{t+1};a\right),\end{aligned}$$

subject to (10), (11), and

$$\begin{aligned}& \sum_{t=0}^{\infty}\alpha^t \int M_t\left(x^t;a\right) u\left(c\left(x^t\right), a\left(x^{t-1}\right)\right) d\pi\left(x^t;a\right) \\ \geq & \sum_{t=0}^{\infty}\alpha^t \int M_t\left(x^t;\hat{a}\right) u\left(c\left(x^t\right), \hat{a}\left(x^{t-1}\right)\right) d\pi\left(x^{t-1};\hat{a}\right), \ \forall\hat{a}\end{aligned}$$

$$\sum_{t=0}^{\infty}\alpha^t \int M_t\left(x^t;a\right) u\left(c\left(x^t\right), a\left(x^{t-1}\right)\right) d\pi\left(x^t;a\right) \geq u_0.$$

The preceding two inequalities represent the incentive and participation constraints. The expected values are computed using the agent's distorted beliefs corresponding to $\{M_t\}$ or $\{m_t\}$.

3 A Continuous-Time Model

In this section, we present a continuous-time model, which is our main focus. We start with a benchmark model with common beliefs, which follows from DeMarzo and Sannikov (2006). We then introduce belief distortions.

3.1 Benchmark

Time is continuous in the interval $[0,\infty)$. Fix a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \ \bar{P}\right)$, on which a one-dimensional standard Brownian motion $\left(\bar{B}_t\right)_{t\geq 0}$ is defined. Define a state process as

$$X_t = x + \sigma\bar{B}_t,$$

where $x > 0$ and $\sigma > 0$. Here $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by $\bar{B}$ or equivalently by X.

Contracting Problem. An agent (or entrepreneur) owns a technology (or project) that can generate a cumulative cash-flow process represented by (X_t) [7] The project needs initial capital $K > 0$ to be started. The agent has no initial wealth and needs financing from outside investors (the principal). Once the project is started, the agent affects the technology performance by taking an action or effort level $a_t \in [0,1]$, which changes the distribution of cash flows. Specifically, let

$$\begin{aligned} B_t^a &= B_t - \frac{\mu}{\sigma}\int_0^t a_s ds, \\ M_t^a &= \exp\left(\int_0^t a_s dB_s - \frac{1}{2}\int_0^t a_s^2 ds\right), \quad \frac{dP^a}{d\bar{P}}|_{\mathcal{F}_t} = M_t^a, \end{aligned}$$

where $\mu > 0$. Then by the Girsanov Theorem, B^a is a standard Brownian motion under measure P^a and we have

$$dX_t = \mu a_t dt + \sigma dB_t^a, \tag{14}$$

Note that the triple (X, B^a, P^a) is a weak solution to the above stochastic differential equation (SDE).

The agent can derive private benefits $\lambda\mu(1-a_t)\,dt$ from the action a_t, where $\lambda \in (0,1]$. Due to linearity, this modeling is also equivalent to the binary effort setup where the agent can either shirk, $a_t = 0$, or work, $a_t = 1$. Hence, we adopt this simple assumption throughout the paper. Alternatively, we can interpret $1 - a_t$ as the fraction of cash flow that the agent diverts for his private benefit, with λ equal to the agent's net consumption per dollar diverted. In either case, λ represents the severity of the agency problem. The choice of the agent's action is unobservable to the principal, creating the moral hazard issue. The principal only observes past and current cash flows and his information set is represented by the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by (X_t).

Both the principal and the agent are risk neutral and discount the future cash flows according to r and γ respectively. Assume that $r < \gamma$ so that the agent is more impatient than the principal. The technology can be liquidated. If it is liquidated, the principal obtains L and the agent gets outside value zero. Assume that $L < \mu/r$ so that liquidation is inefficient.

The principal offers to contribute capital K in exchange for a contract (C, τ, a) that specifies a termination (stopping) time τ, a cash compensation $C = \{C_t : 0 \leq t \leq \tau\}$ to the agent, and a suggested effort choice $a = \{a_t \in \{0,1\} : 0 \leq t \leq \tau\}$. Assume that C and a are adapted to $(\mathcal{F}_t)$ and that C is a right continuous with left limits, and increasing process

[7] All processes in the paper are assumed to be progressively measurable with respect to $\{\mathcal{F}_t\}$. Equalities and inequalities in random variables or stochastic processes are understood to hold almost surely.

satisfying[8]

$$E^{P^a}\left[\left(\int_0^t e^{-\gamma s} dC_s\right)^2\right] < \infty,\ t \geq 0,\ C_0 \geq 0.$$

The monotonicity requirement reflects the fact that the agent has limited liability.

Fix a contract (C, τ, a) and assume that the agent follows the recommended choice of effort. His continuation value W_t at date t is defined as

$$W_t = E_t^{P^a}\left[\int_t^\tau e^{-\gamma(s-t)}(dC_s + \lambda\mu(1-a_s)ds)\right], \tag{15}$$

where $E_t^{P^a}$ denotes the conditional expectation operator with respect to the measure P^a given the information set $\mathcal{F}_t$. His total expected utility at date 0 is equal to W_0.

We now formulate the continuous-time contracting problem.

Problem 3.1 *(DeMarzo and Sannikov (2006))*

$$\max_{(C,\tau,a)} \quad E^{P^a}\left[\int_0^\tau e^{-rs}(dX_s - dC_s) + e^{-r\tau}L\right], \tag{16}$$

subject to:

$$E^{P^a}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] \geq E^{P^a}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-\hat{a}_s)ds)\right],\ \forall \hat{a}_s \in \{0,1\}, \tag{17}$$

$$E^{P^a}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] = W_0, \tag{18}$$

where $W_0 \geq 0$ is given.

The principal's utility is given by (16). Consistent with the rational expectations hypothesis, both the principal and the agent use the measure P^a to evaluate expected utility. Inequality (17) is the incentive constraint and equation (18) is the promising-keeping or participation constraint.[9] Assume that the agent cannot save and both the principal and the agent have full commitment.

Solution. We briefly outline the solution in DeMarzo and Sannikov (2006), which can also be obtained as a special case in our analysis in Sections 4 and 5. We first transform the above problem into a stochastic control problem with a single state variable: the continuation value

[8] The sequare integrability is imposed to ensure W_t defined in (15) has a martingale representation (see Cvitanic and Zhang (2013), Chapter 7).

[9] It is technically more convenient to write the participation constraint as equality instead of inequality "≥ 0" in (18).

W_t. By the Martingale Representation Theorem (Cvitanic and Zhang (2013), Lemma 10.4.6), the continuation value W_t satisfies the SDE:

$$dW_t = \gamma W_t dt - dC_t - \lambda\mu(1 - a_t)dt + \phi_t dB_t^a,\ 0 \leq t \leq \tau, \tag{19}$$

where ϕ_t is the $\mathcal{F}_t$-measurable sensitivity of the agent's continuation value to the project performance.

Suppose that the principal finds it optimal to never induce shirking.[10] If the agent deviates and chooses low effort $a_t = 0$ for an instant dt, output decreases by μdt. Since effort is not observable to the principal, this is equivalent to dB_t^a being reduced by $\mu/\sigma dt$. Thus, the agent incurs a loss of $\phi_t \mu/\sigma dt$ and gets private benefit $\lambda\mu dt$ by (19). Therefore, exerting high effort is optimal for the agent if and only if

$$\phi_t \mu/\sigma \geq \lambda\mu \ \text{ or } \phi_t \geq \sigma\lambda. \tag{20}$$

This is the incentive compatibility condition.

Let $F(W_0)$ denote the principal's value function in Problem 3.1 when the agent's continuation value is equal to W_0. DeMarzo and Sannikov (2006) show that $F'(W) \geq -1$ for all W. Define $\bar{W}$ as the lowest value such that $F'\left(\bar{W}\right) = -1$. The value function F satisfies the following differential equation on the interval $[0, \bar{W}]$:

$$rF(W) = \max_{\phi \geq \sigma\lambda} \mu + F'(W)\gamma W + \frac{F''(W)}{2}\phi^2, \tag{21}$$

with boundary conditions:

$$F(0) = L, \quad F'(\bar{W}) = -1, \quad F''(\bar{W}) = 0.$$

DeMarzo and Sannikov (2006) show that $F(W)$ is strictly concave so that it is optimal to set $\phi = \sigma\lambda$. The intuition is that it is not optimal to make the agent bear more risk than the minimal amount required for him to exert effort. Inducing a higher volatility to the agent's continuation value will increase the probability of triggering inefficient liquidation of the project. For $W_t \in [0, \bar{W}]$, the principal makes no payments to the agent, and only pays him when W_t hits the boundary $\bar{W}$. The payment dC_t is such that the process W_t reflects on that boundary. When $W > \bar{W}$, $F(W) = F\left(\bar{W}\right) - \left(W - \bar{W}\right)$. The principal pays $W - \bar{W}$ immediately to the agent and the contract continues with the agent's new initial value $\bar{W}$.

[10]Sufficient conditions for this to hold are given in Propositions 1 and 8 of DeMarzo and Sannikov (2006). Zhu (2012) relaxes this condition and solves for the optimal contract with shirking.

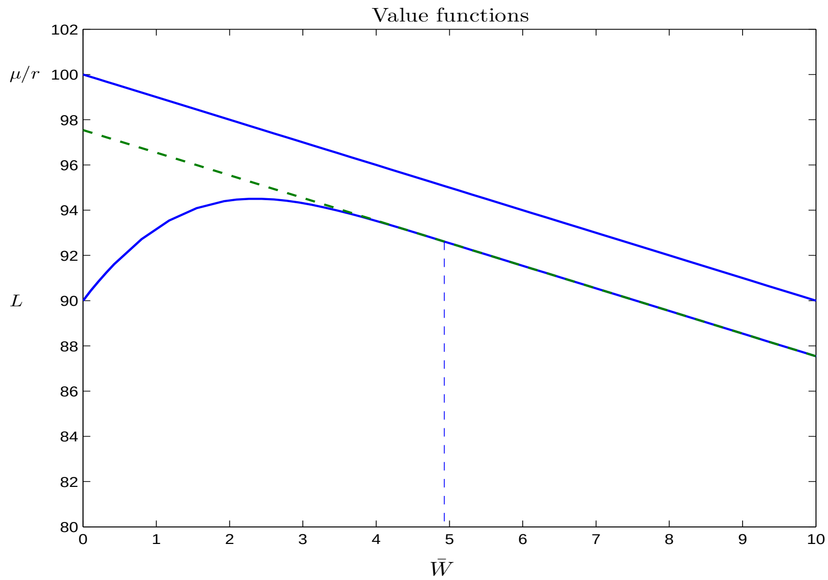

Figure 1: **Value function for the benchmark model**. The top straight line is the first-best value function $F(W) = \mu/r - W$. The curve is the value function with agency. Parameter values are $\mu = 10$, $r = 0.10$, $\gamma = 0.15$, $\lambda = 0.20$, $\sigma = 5$, and $L = 90$.

Once W_t hits 0 for the first time, the contract is terminated. The point $W_t = 0$ is an absorbing boundary.

Figure 1 depicts a typical value function $F(W)$ as a function of the state variable W. It also plots the first-best solution $F(W) + W = \mu/r$. In the first-best case, the principal delivers the agent a lump-sum value W immediately. The agent always exerts high effort and the project is never liquidated.

Turn to the initial startup stage. The project can be funded if and only if $\max_{w\geq 0} F(w) \geq K$. If the agent has all bargaining power due to competition of principals, he extracts the maximal W_0 such that $F(W_0) = K$. If the principal has all bargaining power due to competition of agents, he delivers the agent W^* such that $F'(W^*) = 0$.

Capital Structure Implementation. Now, we study the implementation of the optimal contract. There are several ways of implementation. Instead of following DeMarzo and Sannikov (2006), we follow the approach of Biais et al. (2007) that uses cash reserves, debt and equity. Let the cash reserves (M_t) satisfy

$$dM_t = rM_t dt + dX_t - dC_t - d\Psi_t,$$

where $M_0 = W_0/\lambda$ and

$$d\Psi_t = \underbrace{[\mu - (\gamma - r) M_t]\, dt}_{\text{coupon}} + \underbrace{\frac{1-\lambda}{\lambda} dC_t}_{\text{dividends}}.$$

The cash reserves are held on a bank account, and earn interests at the rate r. Project cash flows dX_t are added to the cash reserves. The agent holds a fraction λ of nontradable equity and receives dividends dC_t. Outside equity holders hold a fraction $1-\lambda$ of tradable equity and receive dividends $\frac{1-\lambda}{\lambda} dC_t$. Outside debt holders receive coupon payments $[\mu - (\gamma - r) M_t]\, dt$.[11] The agent uses proceeds from security issuance to finance investment costs K and hoard cash $M_0 = W_0/\lambda$. He distributes total dividends dC_t/λ when cash reserves meet an upper bound $\bar{M} = \bar{W}/\lambda$. The firm is liquidated if its cash reserves are exhausted (i.e., $M_t = W_t/\lambda = 0$). By (19), $M_t = W_t/\lambda$ for $a_t = 1$ and $\phi_t = \sigma\lambda$. In addition, $W_t = E_t^{P^1}\left[\int_t^\tau e^{-r(s-t)} dC_s\right]$, where P^1 is the measure corresponding to high effort $a = 1$. Thus, the above implementation is incentive compatible and implements the optimal contract.

The equity price is given by

$$S_t = E_t^{P^1}\left[\int_t^\tau e^{-r(s-t)} \frac{1}{\lambda} dC_s\right], \tag{22}$$

where τ is the liquidation time. Note that the expected return on equity is equal to r which is less than the agent's discount rate γ. The agent has an incentive to sell the stock. Following Biais (2007), we assume that the agent's held equity cannot be traded.

The bond price D_t is given by

$$D_t = E_t^{P^1}\left[\int_t^\tau e^{-r(s-t)} [\mu - (\gamma - r) M_s]\, ds + e^{-r(\tau - t)} L\right]. \tag{23}$$

Bond holders obtain the liquidation value L when the cash reserves hit the liquidation boundary.

As a measure of the default risk at time $t \in [0, \tau)$, we take the credit yield spread Δ_t on a console bond that pays one unit of account at each date until the time of default. For any $t \in [0, \tau)$, Δ_t is defined by the following formula:

$$\int_t^\infty e^{-(r+\Delta_t)(s-t)} ds = E_t^{P^1}\left[\int_t^\tau e^{-r(s-t)} ds\right]. \tag{24}$$

[11] Note that M_t is bounded. The coupon payment is positive when γ is close to r.

Solving yields

$$\Delta_t = \frac{rT_t}{1 - T_t},$$

where $T_t = E_t\left[e^{-r(\tau - t)}\right]$ for all $t \in [0, \tau]$ represents Arrow-Debreu price at time t of one unit claim paid at the time of default.

3.2 Belief Distortions

We now consider the possibility of belief distortions due to concerns about model misspecifications or model ambiguity. Both the principal and the agent view the probability measure P^a as an approximating model. Either one of them may not trust this model and consider alternative models as possible. We shall use the continuous-time analogue of the martingale representation in Section 2.2.1 to express belief distortions.

Suppose that all distorted beliefs are described by mutually absolutely continuous measures with respect to P^a over any finite time intervals. Denote the set of such measures by P^a. We can then use the Girsanov Theorem to construct these measures. Define a *density generator* to be a real-valued process (h_t) satisfying $\int_0^t h_s^2 ds < \infty$ for all $t > 0$ such that the process (z_t) defined by

$$z_t = \exp\left(\int_0^t h_s dB_s^a - \frac{1}{2}\int_0^t h_s^2 ds\right) \tag{25}$$

is a $(P^a, \mathcal{F}_t)$-martingale.[12] Denote the set of density generators by $\mathcal{H}^a$. By the Girsanov Theorem, there is a measure Q^h corresponding to h defined on $(\Omega, \mathcal{F})$ such that z_t is the Radon-Nikodym derivative of Q^h with respect to P^a when restricted to $\mathcal{F}_t$,

$$\frac{dQ^h}{dP^a}|_{\mathcal{F}_t} = z_t, \tag{26}$$

and the process $\left(B_t^h\right)$ defined by

$$B_t^h = B_t^a - \int_0^t h_s ds,$$

is a standard Brownian motion under the measure Q^h. Under measure Q^h, cash flows follow dynamics:

$$dX_t = \mu a_t dt + \sigma\left(dB_t^h + h_t dt\right). \tag{27}$$

For the continuous-time version of the type 0 robust contracting problem, we follow Chen

[12] See Hansen et al (2006) for construction.

and Epstein (2002) and define a set of priors as

$$\left\{Q^h \in \mathcal{P}^a : |h_t| \leq \kappa\right\}. \tag{28}$$

For the continuous-time version of type I problem, we follow Anderson, Hansen and Sargent (2002) and Hansen and Sargent (2012) and define the continuous-time discounted relative entropy as

$$rE^{P^a}\left[\int_0^\infty e^{-rs} z_t \ln z_t dt\right] = \frac{1}{2}E^{P^a}\left[\int_0^\infty e^{-rs} z_t h_t^2 dt\right],$$

where the equality follows from (25) and integration by parts. Note that we use the principal's subjective discount rate to compute discounted entropy. Finally, we follow Hansen and Sargent (2012) to describe a continuous-time analogue of Woodford's (2010) discrepancy measure in Section 2.2.1. Define this measure in discrete time $t = 0$, dt, $2dt$, $3dt$....as

$$\sum_{j=0}^{\infty} e^{-r(j+1)dt} E^{P^a}\left[\frac{z_{(j+1)dt}}{z_{jdt}} \ln\left(\frac{z_{(j+1)dt}}{z_{jdt}}\right)\right].$$

As $dt \to 0$, the above expression approaches[13]

$$\frac{1}{2}E^{P^a}\left[\int_0^\infty e^{-rs} h_t^2 dt\right],$$

where we have applied Ito's Lemma to show that

$$\lim_{dt\downarrow 0} \frac{1}{dt} E^{P^a}\left[\frac{z_{t+dt}}{z_t} \ln\left(\frac{z_{t+dt}}{z_t}\right)\right] = \frac{1}{2}h_t^2.$$

In the next section, we turn to the study of how the principal and the agent design robust contracts. We shall focus on formulating and solving types I and II robust contracting problems in continuous time. We find that type 0 robust contracting problem in continuous time is technically challenging because the incentive constraint (17) and the constraint on belief distortions (28) may occasionally bind, which complicate the analysis and potentially introduce kinks in the value function.

4 Type I Robust Contracts

We formulate type I robust contracting problem under moral hazard as follows:

[13] Hansen and Sargent (2012) point out that the limit is relative entropy with a reversal of probability models.

Problem 4.1 *(type I robust contract in continuous time)*

$$\sup_{(C,\tau,a)} \inf_{h} \quad E^{Q^h}\left[\int_0^\tau e^{-rt}(dX_t - dC_t) + e^{-r\tau}L\right] + \frac{\theta}{2}E^{P^a}\left[\int_0^\tau e^{-rs}z_t h_t^2 dt\right], \tag{29}$$

subject to (17), (18), and (25).

The parameter $\theta > 0$ describes the degree of concern for robustness. As in Section 2, we may also interpret $1/\theta$ as an ambiguity aversion parameter. A small value of θ implies a large degree of ambiguity aversion or a large degree of concern for robustness. When θ approaches infinity, this problem reduces to the standard contracting problem 3.1 analyzed by DeMarzo and Sannikov (2006). Mathematically, Problem 4.1 is a combined singular control and stopping problem. As Hansen et al. (2006) point out, it is also related to the zero-sum stochastic differential game problem (e.g., Fleming and Souganidis (1989)). We shall proceed heuristically to derive the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation for optimality and then provide a formal verification theorem.[14] We finally analyze several numerical examples to illustrate economic intuition. It is technically challenging and quite involved to provide a rigorous proof of the HJBI equation. Such a proof is beyond the scope of this paper.

4.1 First-Best Robust Contract

Before analyzing the optimal contract with agency, we start with the first-best case in which the principal observes the agent's effort choice and hence the incentive constraint (17) in problem 4.1 is not valid.

The derivations of the HJBI equation consist of several steps. First, we ignore the incentive constraint (17) or (20) and keep the participation constraint as in the benchmark model. We shall rewrite the dynamics of (W_t) in terms of the distorted belief Q^h. Using Girsanov's Theorem, we rewrite (19) as

$$dW_t = \gamma W_t dt - dC_t - \lambda\mu\left(1 - a_t\right)dt + h_t\phi_t dt + \phi_t dB_t^h, \tag{30}$$

where B_t^h is a standard Brownian motion under the measure Q^h.

[14]See Fleming and Soner (1993) for stochastic optimal control theory. One can verify that the Bellman-Isaacs condition may not be satisfied in our models (see Hansen et al. (2006) for a discussion of this condition). Thus, in general one cannot exchange the order of max and min operators without affecting the solution. This condition is also violated in Szydlowski (2012).

Second, we write the objective function in (16) under the measure Q^h as

$$
\begin{aligned}
& E^{Q^h}\left[\int_0^\tau e^{-rt}(dX_t - dC_t) + e^{-r\tau}L\right] + \frac{\theta}{2}E^{P^a}\left[\int_0^\infty e^{-rs}z_t h_t^2 dt\right] \\
= \; & E^{Q^h}\left[\int_0^\tau e^{-rt}(\mu a_t dt + \sigma h_t dt - dC_t) + e^{-r\tau}L\right] + \frac{\theta}{2}E^{Q^h}\left[\int_0^\infty e^{-rs}h_t^2 dt\right],
\end{aligned}
$$

where have used the fact that $dX_t = (\mu a_t + \sigma h_t)\, dt + \sigma dB_t^h$.

Third, define $F(W_0)$ as the value function for Problem 4.1 without the incentive constraint (17) when we vary the promised value W_0 to the agent. We use the dynamic programming principle to write an approximate Bellman equation:

$$
rF(W_t)\, dt = \sup_{dC_t, \phi_t, a_t \in \{0,1\}} \inf_{h_t} \; \mu a_t dt + \sigma h_t dt - dC_t + \frac{\theta}{2}h_t^2 dt + E_t^{Q^h}\left[dF(W_t)\right], \tag{31}
$$

subject to (30). This equation has an intuitive economic interpretation. The left-hand side represents the mean return required by the principal. The right-hand side represents the total return expected by the principal. It consists of the cash flow plus the expected capital gain or loss $E_t^{Q^h}\left[dF(W_t)\right]$. The optimality requires the expected return equals the required mean return. Note that all expected values are computed using the measure Q^h.

Now we use Ito's Lemma and (30) to derive

$$
E_t^{Q^h}\left[dF(W_t)\right] = F'(W_t)(\gamma W_t dt - dC_t - \lambda\mu(1-a_t)\, dt + h_t\phi_t dt) + \frac{F''(W_t)}{2}\phi_t^2 dt.
$$

Plugging this equation into (31) yields:

$$
\begin{aligned}
rF(W_t)\, dt \; = \; & \sup_{a_t \in \{0,1\}, dC_t, \phi_t} \inf_{h_t} \; \mu a_t dt + \sigma h_t dt - dC_t + \frac{\theta}{2}h_t^2 dt \\
& + F'(W_t)(\gamma W_t dt - dC_t - \lambda\mu(1-a_t)\, dt + h_t\phi_t dt) + \frac{F''(W_t)}{2}\phi_t^2 dt.
\end{aligned}
$$

Suppose that $dC_t = c_t dt$, where $c_t \geq 0$. Removing the time subscripts and cancelling dt, we obtain the HJBI equation:

$$
\begin{aligned}
rF(W) \; = \; & \sup_{a \in \{0,1\}, c \geq 0, \phi} \inf_{h} \; \mu a + \sigma h - \left(1 + F'(W)\right)c \\
& + F'(W)(\gamma W + h\phi - \lambda\mu(1-a)) + \frac{F''(W)}{2}\phi^2 + \frac{\theta h^2}{2}.
\end{aligned} \tag{32}
$$

Clearly, for this problem to have a finite solution, we must have $F'(W) \geq -1$. We then get $c > 0$ if and only if $F'(W) = -1$. This equation defines a boundary point $\bar{W}$. This illustrates the feature of the singular control problem: the principal makes payments to the agent if only if W_t reaches the point $\bar{W}$. The payments make the process (W_t) reflects at this point.

The objective function in (32) is convex in h. Solving for the worst-case density generator yields:

$$h^* = -\frac{\phi F'(W) + \sigma}{\theta}. \tag{33}$$

Substituting it back into (32) yields:[15]

$$rF(W) = \sup_{a\in\{0,1\},\phi} \mu a + F'(W)(\gamma W - \lambda\mu(1-a)) + \frac{\phi^2}{2}F''(W) - \frac{[\phi F'(W) + \sigma]^2}{2\theta}. \tag{34}$$

Assuming that[16]

$$\theta F''(W) - F'(W)^2 < 0, \tag{35}$$

so that the expression on the right-hand side of equation (34) is concave in ϕ, we can derive the optimal sensitivity:

$$\phi^*(W) = \frac{F'(W)\sigma}{\theta F''(W) - F'(W)^2}. \tag{36}$$

Note that the concavity of F is sufficient for (35) to hold. Since $\lambda \in (0,1]$ and $F'(W) \geq -1$, it follows that $\lambda F'(W) + 1 \geq 0$ and hence implementing high effort $a_t = 1$ is optimal.

The following result characterizes the first-best robust contract for type I problem.

Proposition 1 *Consider the first-best type I robust contracting problem. Suppose that*

$$L < \frac{\mu}{r} - \frac{\sigma^2}{2r\theta}, \tag{37}$$

and that there is a unique twice continuously differentiable solution F to the ODE on $[0, \bar{W}]$:

$$rF(W) = \mu + F'(W)\gamma W - \frac{[F'(W)\sigma]^2}{2\theta\left[\theta F''(W) - F'(W)^2\right]} - \frac{\sigma^2}{2\theta},$$

with the boundary conditions,

$$F(0) = \frac{\mu}{r} - \frac{\sigma^2}{2r\theta}, \tag{38}$$

$$F'\left(\bar{W}\right) = -1,\ F''\left(\bar{W}\right) = 0,$$

[15] Notice that this equation is not equivalent to that from a risk-sensitive control problem in Anderson, Hansen, and Sargent (2003) due to the presence of σ in the last quadratic term. The reason is that in our model the instantaneous utility is dX_t which contains Brownian uncertainty and is affected by distorted beliefs, but in the standard robust control model the instantaneous utility is locally riskless (i.e., of order dt), which is unaffected by distorted beliefs.

[16] If F is concave, one can check that the Bellman-Isaacs condition is satisfied and hence the max and min operators in (32) can be exchanged. A similar remark applies to the problems analyzed later.

such that condition (35) holds. Then:

(i) When $W \in \left[0, \bar{W}\right]$, *the principal's value function is given by* $F(W)$, *the first-best sensitivity* $\phi^*(W)$ *is given by (36), the worst-case density generator is given by*

$$h^*(W) = -\frac{\phi^*(W) F'(W) + \sigma}{\theta}. \tag{39}$$

and the agent always exerts high effort $a^*(W) = 1$ *at all times. The contract initially delivers* $W \in \left[0, \bar{W}\right] \geq 0$ *to the agent whose continuation value* (W_t) *follows the dynamics*

$$dW_t = \gamma W_t dt - dC_t^* + \phi^*(W_t) dB_t^1, \ W_0 = W, \tag{40}$$

for $t \geq 0$, *where the optimal payments are given by*

$$C_t^* = \int_0^t \mathbf{1}_{\left\{W_s = \bar{W}\right\}} dC_s^*, \tag{41}$$

and the project is never liquidated.

(ii) When $W > \bar{W}$, *the principal's value function is* $F(W) = F\left(\bar{W}\right) - \left(W - \bar{W}\right)$. *The principal pays* $W - \bar{W}$ *immediately to the agent and the contract continues with the agent's new initial value* $\bar{W}$.

This proposition shows that we cannot achieve the first-best solution in the benchmark model. The intuition is as follows. The principal is ambiguity averse and would like to transfer uncertainty to the agent when designing a contract. Ideally, the risk-neutral agent should insure the principal by making the principal's payoff flows constant. This means that the agent should absorb all risk from the project cash flows. However, this contract is not feasible due to limited liability. The project cash flows can be negative and the agent can incur losses. Without limited liability we can achieve the first-best solution in the benchmark model.[17] With limited liability, uncertainty sharing is limited. The marginal cost $|F'(W)|$ to the principal from delivering an additional unit of value to the agent must be greater than or equal to 1. The principal makes payments to the agent when and only when the marginal cost is equal to 1 at some point $\bar{W}$. The tradeoff is the following: On the one hand, the principal wants to make payments to the agent earlier because the agent is more impatient. On the other hand, the principal wants to delay payments, allowing the agent's continuation value W_t to get larger. This benefits the principal because if W_t is closer to zero, the principal has to bear more the project cash flows uncertainty. On the boundary $W_t = 0$, the principal bears full uncertainty and his value is given by (38). The term $\sigma^2 / (2r\theta)$ represents the discount due

[17]The principal pays the agent W_0 immediately and then $dX_t - \mu dt$ thereafter.

to model uncertainty. It increases with volatility σ and ambiguity aversion parameter $1/\theta$.

Proposition 1 formalizes the above intuition. It shows that the worst-case density generator and the sensitivity of the agent's continuation value to the cash flow are state dependent. The agent bears large cash flow uncertainty, but he does not absorb all uncertainty due to limited liability. Because the principal also bears uncertainty, his value function F is nonlinear and the last two nonlinear terms in the ODE reflect the value discount due to model ambiguity.

We emphasize that in two-party contracting problems, model ambiguity generates endogenous belief heterogeneity. Specifically, in type I robust contracting problem, the agent trusts the approximating model P^a and his value follows the dynamics (40) under P^a. However, the principal has doubt about the approximating model P^a and the agent's continuation value under the principal's worst-case model Q^{h^*} follows the dynamics:

$$dW_t = \gamma W_t dt - dC_t^* + \phi^* \left(W_t\right) h^* \left(W_t\right) dt + \phi^* \left(W_t\right) dB_t^{h^*}. \tag{42}$$

This point has important pricing implications when we implement the optimal contract with agency in later analysis.

Note that the concavity of the value function $F\left(W\right)$ is not needed in Proposition 1. For a wide range of parameter values in our numerical examples, we find that F is concave. However, unlike in the benchmark model, we are unable to prove it formally.

4.2 Robust Contract with Agency

Turn to the case with moral hazard in which the principal does not observe the agent's effort choice and hence the incentive constraint (17) must be imposed in Problem 4.1. Without risk of confusion, we still use $F\left(W_0\right)$ to denote the value function for Problem 4.1 when we vary the promised value W_0 to the agent. Suppose that implementing high effort is optimal. In this case, the incentive constraint is equivalent to (20). Using a similar argument to that in the previous subsection, we can proceed heuristically to derive the HJBI equation for optimality. Imposing constraint (20) and setting $a_t = 1$ in the associated equations in the previous subsection, we can show that the HJBI equation is given by[18]

$$\begin{aligned} rF(W) &= \sup_{c\geq 0,\phi\geq\sigma\lambda} \inf_h \ \mu+\sigma h-\left(1+F'(W)\right)c \\ &\quad +F'(W)(\gamma W+h\phi)+\frac{F''(W)}{2}\phi^2+\frac{\theta h^2}{2}. \end{aligned}$$

Thus, the worst-case density generator is still given by (33) and there is a boundary point $\bar{W}$ satisfying $F'\left(\bar{W}\right)=-1$ such that $c>0$ if $F'\left(\bar{W}\right)=-1$ and $c=0$ if $F'(W)>-1$. We can then rewrite the HJBI equation as

$$rF(W)=\sup_{\phi\geq\sigma\lambda} \ \mu+F'(W)\gamma W+\frac{\phi^2}{2}F''(W)-\frac{\left[\phi F'(W)+\sigma\right]^2}{2\theta}. \tag{43}$$

Under condition (35), the optimal sensitivity is given by

$$\phi^*(W)=\max\left\{\frac{F'(W)\sigma}{\theta F''(W)-F'(W)^2},\sigma\lambda\right\}. \tag{44}$$

Note that (43) is identical to (21) in the benchmark model, when the last nonlinear term in (43) is removed (e.g., $\theta\to\infty$). This term reflects the cost of model uncertainty.

The following result characterizes the second-best robust contract for type I problem.

Proposition 2 *Consider the second-best type I robust contracting problem. Suppose that implementing high effort is optimal and that condition (37) holds. Suppose that there exists a unique twice continuously differentiable solution F to the ODE (43) on $\left[0,\bar{W}\right]$ with boundary conditions*

$$F'\left(\bar{W}\right)=-1,\ F''\left(\bar{W}\right)=0,\ F(0)=L,$$

such that condition (35) is satisfied. Then:

(i) When $W\in\left[0,\bar{W}\right]$, $F(W)$ is the value function for Problem 4.1, the optimal sensitivity $\phi^*(W)$ is given by (44), and the worst-case density generator is given by

$$h^*(W)=-\frac{\phi^*(W)F'(W)+\sigma}{\theta}. \tag{45}$$

[18] For type 0 robust contracting problem with Chen and Epstein (2002) recurisve multiple-priors utility, the HJBI equation is given by

$$\begin{aligned} rF(W) &= \max_{c\geq 0,\phi\geq\sigma\lambda} \min_{|h|\leq\kappa} \ \mu+\sigma h-(1+F'(W))c \\ &\quad +F'(W)(\gamma W+h\phi)+\frac{F''(W)}{2}\phi^2. \end{aligned}$$

This problem is hard to analyze due to the two constrained optimization problems.

The contract delivers the value $W \in \left[0, \bar{W}\right]$ to the agent whose continuation value (W_t) follows the dynamics:

$$dW_t = \gamma W_t dt - dC_t^* + \phi^* (W_t) \, dB_t^1, \; W_0 = W, \tag{46}$$

for $t \in [0, \tau]$, where the optimal payments are given by

$$C_t^* = \int_0^t \mathbf{1}_{\left\{W_s = \bar{W}\right\}} dC_s^*.$$

The contract terminates at time $\tau = \inf \{t \geq 0 : W_t = 0\}$.

(ii) When $W > \bar{W}$, the principal's value function is $F(W) = F\left(\bar{W}\right) - \left(W - \bar{W}\right)$. The principal pays $W - \bar{W}$ immediately to the agent and the contract continues with the agent's new initial value $\bar{W}$.

Unlike the first-best case, the incentive constraint requires that the sensitivity ϕ_t be at least as large as a lower bound $\sigma\lambda$ as in the benchmark model. In the benchmark model, the choice of ϕ_t reflects the following tradeoff: a large ϕ_t is needed to provide incentives to the agent. But a large ϕ_t also raises the volatility of the agent's continuation value and hence raises the chance of liquidation. The optimal sensitivity just achieves the lower bound $\sigma\lambda$. However, unlike the benchmark model, this lower bound does not always bind in the presence of model ambiguity. The reason is that there is an uncertainty and incentive tradeoff. The robust contract should transfer uncertainty from the ambiguity averse principal to the risk neutral agent as much as possible. Thus, the agent should expose more to the uncertainty so that the optimal sensitivity may exceed the lower bound.

Under what situation does this happen? For a low value of W, the principal is more concerned about inefficient liquidation. Thus, the optimal contract will set ϕ_t at the lower bound. When W is large and close to the payout boundary $\bar{W}$, the principal is more concerned about model uncertainty and hence he would like the agent to be exposed more to the cash flow uncertainty by providing him more incentives so that

$$\phi^* (W) = \frac{F'(W) \, \sigma}{\theta F''(W) - F'(W)^2} > \sigma\lambda. \tag{47}$$

From the above analysis, the agent is more likely to be overincentivized when his continuation value is high.

Figure 2 plots the value functions for type I robust contracting problem with and without agency. The payout boundary is given by $\bar{W}^{FB}$ for the first-best case. It is lower than that for the contract with agency, implying that moral hazard generates inefficient delay in payout. Both value functions are concave and become linear after the payout boundaries with a slope

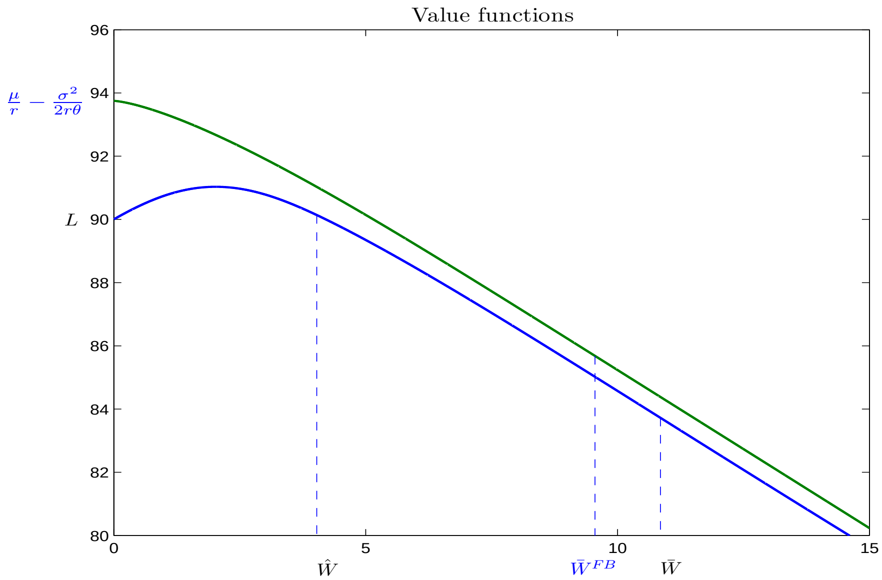

Figure 2: **Value functions for type I robust contracting problem**. The upper curve is the first-best value function and the payout boundary is $\bar{W}^{FB}$. The lower curve is the value function with agency and the payout boundary is $\bar{W}$. The optimal sensitivity changes value at $\hat{W}$. Parameter values are $\mu = 10$, $r = 0.10$, $\gamma = 0.15$, $\lambda = 0.20$, $\sigma = 5$, $L = 90$, and $\theta = 20$.

-1. Figure 3 plots the worst-case density generator h^* and the optimal sensitivity ϕ^* for the contract with agency. Consistent with the previous intuition, the figure shows that there is a cutoff value $\hat{W}$, such that the sensitivity $\phi^*(W)$ reaches the lower bound $\sigma\lambda$ for all $W \in [0, \hat{W}]$ and it is given by (47) for all $W \in [\hat{W}, \bar{W}]$.[19] Figure 3 also shows that $h^*(W)$ increases with W and $h^*(W) < 0$ for all $W \in \left[0, \bar{W}\right]$. Intuitively, the principal's aversion to model uncertainty leads to his pessimistic behavior. The local mean of the Brownian motion is shifted downward under the principal's worst-case belief.

Figure 4 illustrates that the value function may not be globally concave. In particular, it is convex when the agent's continuation value is close to the liquidation boundary. To see the intuition, we rewrite (43) as

$$\frac{\phi^*(W)^2}{2}F''(W) = \left[rF(W) - \ \mu - F'(W)\gamma W\right] + \frac{\left[\phi^*(W)F'(W) + \sigma\right]^2}{2\theta}.$$

When $\theta \to \infty$, the second expression on the right-hand side of the above equation vanishes and the model reduces to the benchmark one analyzed in Section 3.1 so that the first square

[19] We are unable to prove this result formally. But it is quite robust for a wide range of parameter values in the numerical solutions.

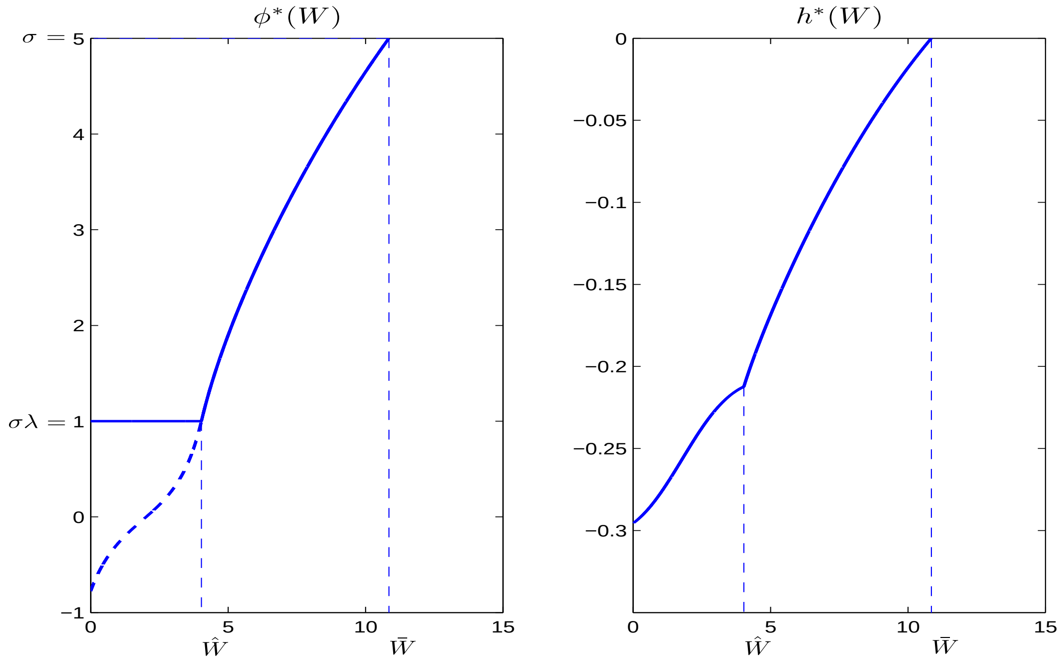

Figure 3: **Optimal sensitivity and the worst-case density generator for type I robust contracting problem.** Parameter values are $\mu = 10$, $r = 0.10$, $\gamma = 0.15$, $\lambda = 0.20$, $\sigma = 5$, $L = 90$, and $\theta = 20$.

bracket expression is negative and $F''(W) < 0$. However, when the principal is sufficiently ambiguity averse (i.e., $1/\theta$ is sufficiently large), the second expression may dominate so that $F''(W) > 0$. This case can happen when W is sufficiently small for the first square bracket expression to be small. In this case, public randomization in the sense of stochastic termination of the project is optimal to the principal, as illustrated by the dashed line from the origin in Figure 4.[20] By contrast, when W is sufficiently large, $F'(W)$ is close to -1 and $\phi^*(W)$ is close to σ so that the second square bracket expression is close to zero. Thus, $F''(W) < 0$ when W is sufficiently large.

The following proposition gives a necessary and sufficient condition for implementing high effort.

Proposition 3 *Implementing high effort is optimal at all times for Problem 4.1 if and only if*

$$rF(W) \geq \max_{\phi \leq \sigma\lambda} F'(W)(\gamma W - \lambda\mu) + \frac{\phi^2}{2} F''(W) - \frac{[\phi F'(W) + \sigma]^2}{2\theta}, \tag{48}$$

[20]Stochastic liquidation is common in the discrete-time models, e.g., Clementi and Hopenhayn (2006), Biais et al. (2007), and DeMarzo and Fishman (2007a,b). Since such an analysis is standard, we omit it here.

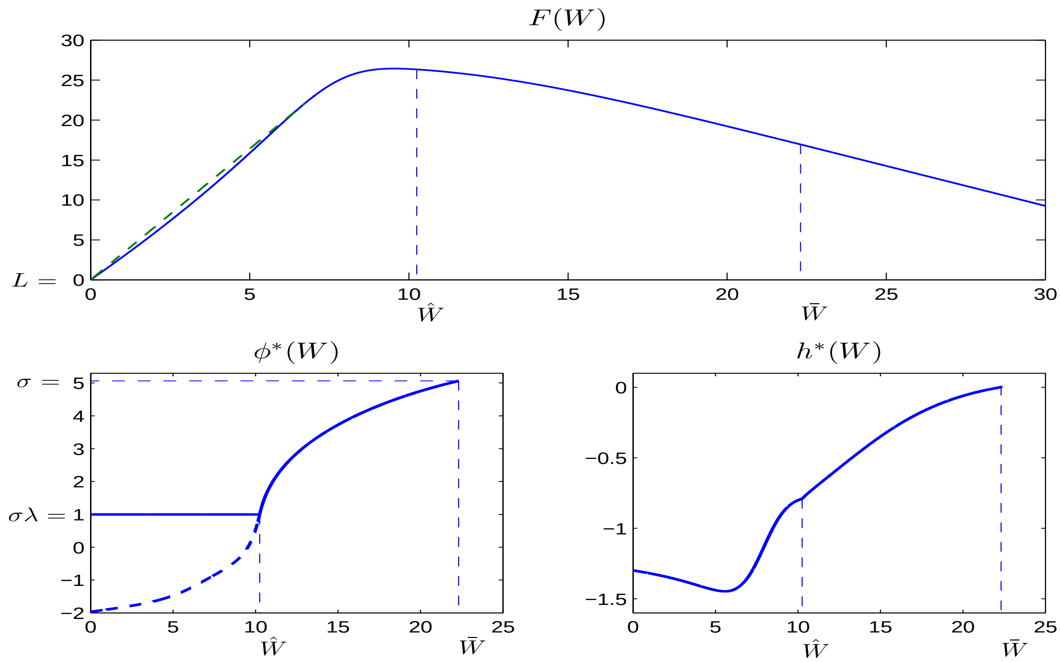

Figure 4: **Value function, optimal sensitivity, and the worst-case density generator for type I robust contracting problem.** Parameter values are $\mu = 5$, $r = 0.10$, $\gamma = 0.15$, $\lambda = 0.20$, $\sigma = 5$, $L = 0$, and $\theta = 6$.

for $W \in \left[0, \bar{W}\right]$, *where* F *is given in Proposition 2 and satisfies condition (35).*

When $\theta \to \infty$, the condition in (48) reduces to the one in Proposition 8 in DeMarzo and Sannikov (2006) because F is concave and $\phi = 0$ in the limit. In all our numerical examples, condition (48) is satisfied.

The following proposition shows that the value function F decreases if the degree of concern for robustness or the degree of ambiguity aversion increases, i.e., $1/\theta$ increases. The intuition is that model uncertainty is costly to the principal and hence reduces his value. The last term in (43) gives this cost, which is the local entropy $\theta h^* (W)^2 /2$.

Proposition 4 *The value function* $F(W)$ *on* $\left[0, \bar{W}\right]$ *in Problem 4.1 increases with the parameter* θ.

5 Type II Robust Contracts

We formulate type II robust contracting problem under moral hazard as follows:

Problem 5.1 *(type II robust contract in continuous time)*

$$\sup_{(C,\tau,a)} \inf_{h} \quad E^{P^a}\left[\int_0^\tau e^{-rt}(dX_t - dC_t) + e^{-r\tau}L\right] + \frac{\theta}{2}E^{P^a}\left[\int_0^\tau e^{-rs}h_t^2 dt\right], \tag{49}$$

subject to

$$E^{Q^h}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] \geq E^{Q^h}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-\hat{a}_s)ds)\right], \ \forall \hat{a}_s \in \{0,1\}, \tag{50}$$

$$E^{Q^h}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] = W_0, \tag{51}$$

where $W_0 \geq 0$ *is given and* Q^h *is the measured defined in (26).*

The interpretation of the parameter θ is identical to that in the discrete-time counterpart. Unlike the standard contracting problem or the type I robust contracting problem, here the agent's continuation value is computed using the distorted belief Q^h. Define this value as

$$W_t = E_t^{Q^h}\left[\int_t^\tau e^{-\gamma(s-t)}(dC_s + \lambda\mu(1-a_s)ds)\right]. \tag{52}$$

Following DeMarzo and Sannikov (2006), we can use the Martingale Representation Theorem to show that (W_t) follows the dynamics:

$$dW_t = \gamma W_t dt - dC_t - \lambda\mu(1-a_t)dt + \phi_t dB_t^h, \tag{53}$$

where $\left(B_t^h\right)$ is the standard Brownian motion under the measure Q^h. Let $F(W)$ be the value function for Problem 5.1 when we vary $W_0 = W$.

As in the previous section, we first analyze the first-best case by removing the incentive constraint in Problem 5.1.

5.1 First-Best Robust Contract

We proceed heuristically to derive the HJBI equation for optimality in several steps. First, we rewrite the dynamics of (W_t) in terms of the principal's probability model P^a. Using Girsanov's Theorem, we rewrite (53) as

$$dW_t = \gamma W_t dt - dC_t - \lambda\mu(1-a_t)dt - h_t\phi_t dt + \phi_t dB_t^a, \tag{54}$$

where $dB_t^a = dB_t^h + h_t dt$ is a standard Brownian motion under P^a.

Second, without risk of confusion, we define $F(W_0)$ as the value function for Problem 5.1

without the incentive constraint (50) when we vary the promised value W_0 to the agent. we use the dynamic programming principle to write an approximate Bellman equation:

$$rF\left(W_t\right)dt = \sup_{dC_t,\phi_t,a_t\in\{0,1\}} \inf_{h_t} \ \mu a_t dt - dC_t + \frac{\theta}{2}h_t^2 dt + E_t^{P^a}\left[dF\left(W_t\right)\right], \tag{55}$$

subject to (54). Unlike type I contracting problem, here we use the principal's trusted probability model P^a to derive the Bellman equation.

Now we use Ito's Lemma and (54) to derive

$$E_t^{P^a}\left[dF\left(W_t\right)\right] = F'(W_t)(\gamma W_t dt - dC_t - \lambda\mu\left(1-a_t\right)dt - h_t\phi_t dt) + \frac{F''(W_t)}{2}\phi_t^2 dt.$$

Plugging this equation into (55) yields:

$$\begin{aligned} rF\left(W_t\right)dt &= \sup_{dC_t,\phi_t,a_t\in\{0,1\}} \inf_{h_t} \ \mu a_t dt - dC_t + \frac{\theta}{2}h_t^2 dt \\ &\quad + F'(W_t)(\gamma W_t dt - dC_t - \lambda\mu\left(1-a_t\right)dt - h_t\phi_t dt) + \frac{F''(W_t)}{2}\phi_t^2 dt. \end{aligned}$$

Suppose that $dC_t = c_t dt$, where $c_t \geq 0$. Removing the time subscripts and cancelling dt, we obtain the equation:

$$\begin{aligned} rF(W) &= \sup_{c\geq 0,\phi,a\in\{0,1\}} \inf_{h} \ \ \mu a - \left(1+F'\left(W\right)\right)c \\ &\quad + F'(W)(\gamma W - h\phi - \lambda\mu\left(1-a\right)) + \frac{F''(W)}{2}\phi^2 + \frac{\theta h^2}{2}. \end{aligned} \tag{56}$$

Clearly, for this problem to have a finite solution, we must have $F'\left(W\right) \geq -1$. We then get $c > 0$ if and only if $F'\left(W\right) = -1$. This equation defines a boundary point $\bar{W}$. The principal makes payments to the agent if only if W_t reaches the point $\bar{W}$. The payments make the process (W_t) reflect at this point.

The objective function in (56) is convex in h. Solving for the worst-case density generator yields:

$$h^* = \frac{F'\left(W\right)\phi}{\theta}. \tag{57}$$

Substituting it back into (56) yields:

$$rF(W) = \sup_{\phi,a\in\{0,1\}} \mu a + F'(W)\left(\gamma W - \lambda\mu\left(1-a\right)\right) + \frac{\phi^2}{2}F''(W) - \frac{\phi^2 F'\left(W\right)^2}{2\theta}. \tag{58}$$

Assuming condition (35) holds, we obtain the optimal sensitivity $\phi^*\left(W\right) = 0$. In this case, the agent is not exposed to any risk. Thus, the agent's belief distortion does not matter so that $h^* = 0$. If $\lambda \in [0, 1]$, implementing high effort is optimal since $F'(W) \geq -1$.

The following result characterizes the solution for the first-best type II robust contracting problem.

Proposition 5 *Consider the first-best type II robust contracting problem. Suppose that $L < \mu/r$. For any $W_0 = W \geq 0$, the principal initially gives a lump-sum payment W to the agent and does not pay him in the future. The agent always exerts high effort $a_t = 1$. The principal's value function is given by $F(W) = \mu/r - W$.*

Unlike type I first-best robust contract, type II first-best robust contract is the same as the first-best contract in the benchmark model without ambiguity. The intuition is that the principal can design a contract such that the agent does not get exposed to uncertainty so that the principal does not need to worry about ambiguity about the agent's expectations.

5.2 Robust Contract with Agency

Turn to the case with moral hazard. Using a proof similar to that for Lemma 3 in DeMarzo and Sannikov (2006), we can establish the following result:

Lemma 1 *Consider type II robust contracting problem with agency. For any belief $Q^h \in \mathcal{P}^a$, implementing high effort is optimal to the agent if and only if condition (20) holds.*

The key to the proof is to apply the Martingale Representation Theorem under the measure Q^h. We omit the details here.

Without risk of confusion, we use $F(W_0)$ to denote the value function for Problem 5.1 when we vary the promised value W_0 to the agent. We proceed heuristically to derive the HJBI equation for optimality. Suppose that implementing high effort is optimal. We then impose condition (20) and set $a_t = 1$ when performing derivations as in the previous subsection. We then obtain the HJBI equation:

$$rF(W) = \sup_{c \geq 0, \phi \geq \sigma\lambda} \inf_{h} \quad \mu - \left(1 + F'(W)\right) c + F'(W)(\gamma W - h\phi) + \frac{F''(W)}{2}\phi^2 + \frac{\theta h^2}{2}. \quad (59)$$

Thus, we must have $F'(W) \geq -1$, $c = 0$ if $F'(W) > -1$ and $c \geq 0$ if $F'(W) = 1$. This defines a payout boundary $\bar{W}$ satisfying $F'\left(\bar{W}\right) = 1$. The worst-case density generator is given by (57). Substituting this solution into the above equation yields:

$$rF(W) = \sup_{\phi \geq \sigma\lambda} \quad \mu + F'(W)\gamma W + \frac{F''(W)}{2}\phi^2 - \frac{F'(W)^2}{2\theta}\phi^2. \quad (60)$$

We then obtain the following result:

Proposition 6 *Consider type II robust contracting problem. Suppose that implementing high effort is optimal and that condition (37) holds. Suppose that there exists a unique twice continuously differentiable solution F to the ODE on $[0, \bar{W}]$:*

$$rF(W) = \quad \mu + F'(W)\gamma W + \frac{F''(W)}{2}(\sigma\lambda)^2 - \frac{F'(W)^2}{2\theta}(\sigma\lambda)^2, \tag{61}$$

with boundary conditions

$$F'\left(\bar{W}\right) = -1,\ F''\left(\bar{W}\right) = 0,\ F(0) = L,$$

such that condition (35) is satisfied. Then:

(i) When $W \in \left[0, \bar{W}\right]$, $F(W)$ is the value function for Problem 5.1, the optimal sensitivity $\phi^*(W) = \sigma\lambda$, and the worst-case density generator is given by

$$h^*(W) = \frac{F'(W)\sigma\lambda}{\theta}.$$

The contract delivers the value $W \geq 0$ to the agent whose continuation value (W_t) follows the dynamics:

$$dW_t = \gamma W_t dt - dC_t^* - h^*(W_t)\sigma\lambda dt + \lambda\sigma dB_t^1,\ W_0 = W, \tag{62}$$

for $t \in [0, \tau]$, where the optimal payments are given by

$$C_t^* = \int_0^t \mathbf{1}_{\left\{W_s = \bar{W}\right\}} dC_s^*.$$

The contract terminates at time $\tau = \inf\{t \geq 0 : W_t = 0\}$.

(ii) When $W > \bar{W}$, the principal's value function is $F(W) = F\left(\bar{W}\right) - \left(W - \bar{W}\right)$. The principal pays $W - \bar{W}$ immediately to the agent and the contract continues with the agent's new initial value $\bar{W}$.

Unlike in type I robust contracting problem, here the agent faces model ambiguity. The agent does not trust the principal's approximating model for the project cash flows and uses distorted beliefs to evaluate expected payoffs. The optimal contract tries to remove the agent's model ambiguity and thus specifies the minimal sensitivity (i.e., $\phi^*(W) = \sigma\lambda$) to provide incentives to the agent. The principal is concerned about robustness of the agent's beliefs and hence chooses the contract that is optimal given the worst-case belief of the agent.

Unlike in type I robust contracting problem, under the approximating model P, the agent's continuation value (W_t) follows the dynamics in (62). But the agent has doubt about this model and his continuation value under the principal's worst-case belief Q^{h^*} follows the dynamics:

$$dW_t = \gamma W_t dt - dC_t^* + \sigma\lambda dB_t^{h^*},\ W_0 = W. \tag{63}$$

The following proposition gives a necessary and sufficient condition for the optimality of implementing high effort. We choose parameter values such that this condition is always satisfied in all our numerical examples.

Proposition 7 *Implementing high effort is optimal in Problem 5.1 if and only if*

$$rF(W) \geq F'(W)(\gamma W - \lambda\mu) \tag{64}$$

for all $W \in \left[0, \bar{W}\right]$, *where* F *is given in Proposition 6 and satisfies condition (35).*

As in type I robust contracting problem, model ambiguity is costly to the principal and lowers his value. The last term in (61) reflects this cost, which is equal to the local entropy $\theta h^*(W)^2/2$.

Proposition 8 *The value function* $F(W)$ *on* $\left[0, \bar{W}\right]$ *in Problem 5.1 increases with the parameter* θ.

Figure 5 plots the value function $F(W)$ and the worst case density generator h^*. This figure shows that F may not be globally concave, unlike in the benchmark model. In particular, it may be convex for low values of W. The intuition is the following. For low values of W, inefficient liquidation is more likely. The principal is more concerned about model uncertainty faced by the agent. He would like to share uncertainty with the agent in the optimal contract. Thus, he is willing to absorb uncertainty and hence his value function is convex. We also find that convexity is more likely to happen when the principal is more averse to uncertainty faced by the agent (i.e., θ is smaller). In this case, the principal is more willing to share this uncertainty.

The right two panels show the worst-case density generators. Unlike in type I robust contracting problems, here the worst-case density generators are positive for small values of W and negative for large values of W. The intuition is that the agent's worst-case belief is chosen by the principal to minimize his utility. When W is small, a positive $h^*(W)$ is chosen because it lowers the agent's continuation value (see (62)), making the project more likely to be liquidated. When W is large, a negative $h^*(W)$ is chosen because it raises the

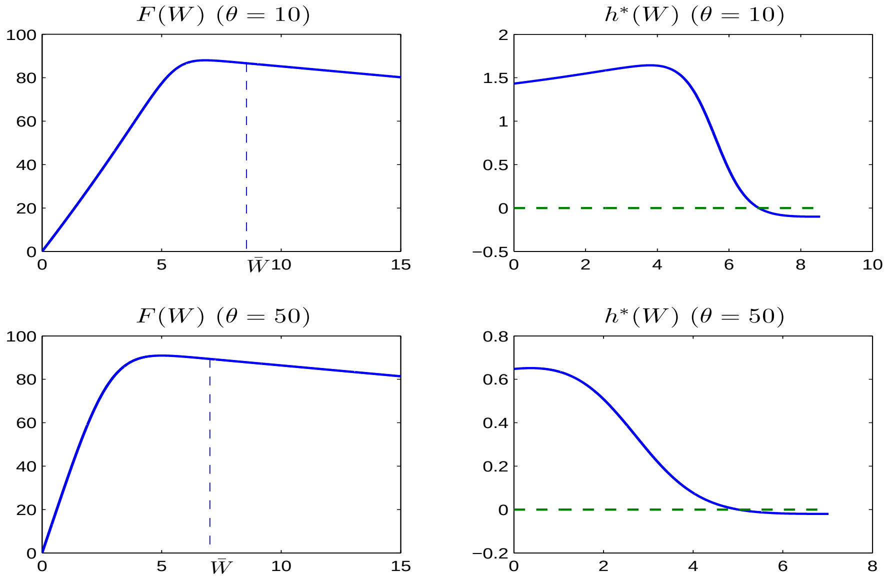

Figure 5: **Value functions and worst-case density generators for type II robust contracting problem**. The parameter values for the top two panels are $\mu = 10$, $r = 0.10$, $\gamma = 0.15$, $\lambda = 0.20$, $\sigma = 5$, $L = 0$, and $\theta = 10$. The parameter values for the lower two panels are the same as the preceding values, except for $\theta = 50$.

agent's continuation value to the payout boundary, which also lowers' principal value. There is no distortion $h^*(W) = 0$ when $F(W)$ reaches the maximum. Intuitively, any pessimistic distortion of the agent's beliefs due to ambiguity aversion lowers the principal's value.

6 Asset Pricing Implications

We use the same approach as in the benchmark model to implement the optimal contracts for Problems 4.1 and 5.1. We shall show that ambiguity aversion generates some new insights in asset pricing.

6.1 Implementing Type I Robust Contract

As in the benchmark model, we still use the cash reserves, debt and equity to implement the optimal contract characterized in Proposition 2. The cash reserves follow dynamics:

$$dM_t = rM_tdt + dX_t - \underbrace{dC_t^*}_{\text{inside dividends}} - d\Psi_t,\ M_0 = W_0/\lambda,$$

for $0 \leq t \leq \tau = \inf \left\{ t \geq 0 : M_t = \bar{W}/\lambda \right\}$, where

$$d\Psi_t = \underbrace{\left[\mu - (\gamma - r) M_t\right] dt}_{\text{coupon}} + \underbrace{\frac{1-\lambda}{\lambda} dC_t^*}_{\text{outside dividends}} + \underbrace{\left[\sigma - \frac{\phi^*(W_t)}{\lambda}\right] dB_t^1}_{\text{special dividends}},$$

and $\bar{W}$, C^* and ϕ^* are given in Proposition 2. Unlike the benchmark model, there is a new term in the cash reserve dynamics:

$$\left[\sigma - \phi^*(W_t)/\lambda\right] dB_t^1 = \left[\sigma - \phi^*(W_t)/\lambda\right] (dX_t - \mu dt)/\sigma.$$

The interpretation of the other terms are the same as in the benchmark model. We interpret the new term as special dividends paid only to the outside equity holders. Note that this term can be negative and we interpret this case as new equity injection as in Leland (1994) style models.[21] The expected value of special dividends is equal to zero under the measure P^1. One reason that we assign the new term to the outside equity holders is to keep limited liability of the agent and the bond holders.

By Proposition 2, $\phi^*(W_t) \geq \sigma\lambda$. In addition, when the agent's continuation value W_t is small, $\phi^*(W_t) = \sigma\lambda$. But when W_t is large, $\phi^*(W_t) > \sigma\lambda$. Thus, special dividends occur only when the continuation value is sufficiently large. In this case, when the project performs well (i.e., $dB_t^1 > 0$), outside equity holders inject cash in the firm in order to raise the cash reserves.[22] But when the project performs bad (i.e., $dB_t^1 < 0$), outside equity holders receive positive special dividends, which lower the cash reserves. This payout policy is used to as a hedge against model uncertainty so that the cash reserves track the agent's continuation value, i.e., $M_t = W_t/\lambda$. This ensures that the liquidation time and the payout time coincide with those in the optimal contract.

We can rewrite the cash reserves dynamics as

$$dM_t = \gamma M_t dt + \frac{\phi^*(\lambda M_t)}{\lambda} dB_t^1 - \frac{1}{\lambda} dC_t^*,$$

and use (46) to show that $M_t = W_t/\lambda$. We can also show that $W_t = E_t^P \left[\int_t^\tau e^{-\gamma(s-t)} dC_s^*\right]$. Thus, as in the benchmark model, the above capital structure is incentive compatible and implements the optimal contract.

Unlike in the benchmark model, we price assets using the principal's pricing kernel which

[21] See DeMarzo et al (2012) for a similar similar implementation and interpretation.

[22] In the Leland (1994) model, equity holders inject new equity for the purpose of avoiding costly bankruptcy.

is based on his worst-case belief Q^{h^*}. Specifically, outside equity value per share is given by

$$S_t = E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} \frac{1}{\lambda} dC_s^* + \frac{1}{1-\lambda} \int_t^\tau e^{-r(s-t)} \left(\sigma - \frac{\phi^* (\lambda M_s)}{\lambda} \right) dB_s^1 \right],$$

where $\tau = \inf \{ t \geq 0 : M_t = 0 \}$ is the liquidation time. By a similar analysis in Anderson, Hansen and Sargent (2003), we can show that the principal's fear of model misspecification generates a market price of model uncertainty. This market price of model uncertainty is given by the absolute value $|h^* (W)|$ of the worst-case density generator in (45), which contributes to the equity premium.

Proposition 9 *The local expected equity premium under the measure P is given by*

$$\frac{-1}{1-\lambda} \left[\sigma - \frac{\phi^* (\lambda M)}{\lambda} \right] \frac{h^* (\lambda M)}{S(M)} - \frac{\phi^* (\lambda M)}{\lambda} \frac{S' (M)}{S(M)} h^* (\lambda M),$$

where h^* and ϕ^* are given in Proposition 2 and $S_t = S(M_t)$ for a function S given in Appendix B1.

The equity premium contains two components. The first component is due to the exposure of special dividends to the Brownian motion uncertainty. This component is negative because the factor loading $[\sigma - \phi^* (\lambda M_s) / \lambda] / (1 - \lambda) < 0$ and special dividends are intertemporal hedges. The second component is due to the exposure of the stock price to the Brownian motion uncertainty. This component is positive whenever $S' (M) > 0$. Since the first component is zero for sufficiently small values of M, the equity premium is positive for these values. But we are unable to derive the sign of the sum of the two components analytically for high values of M.

Debt value satisfies

$$D_t = E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} [\mu - (\gamma - r) M_t] ds + e^{-r(\tau - t)} L \right].$$

The credit yield spread Δ_t is defined as follows:

$$\int_t^\infty e^{-(r+\Delta_t)(s-t)} ds = E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} ds \right].$$

Solving yields:

$$\Delta_t = \frac{r T_t}{1 - T_t},$$

where $T_t = E_t^{Q^{h^*}} \left[e^{-r(\tau - t)} \right]$ for all $t \in [0, \tau]$ represents the Arrow-Debreu price at time t of one unit claim paid at the time of default.

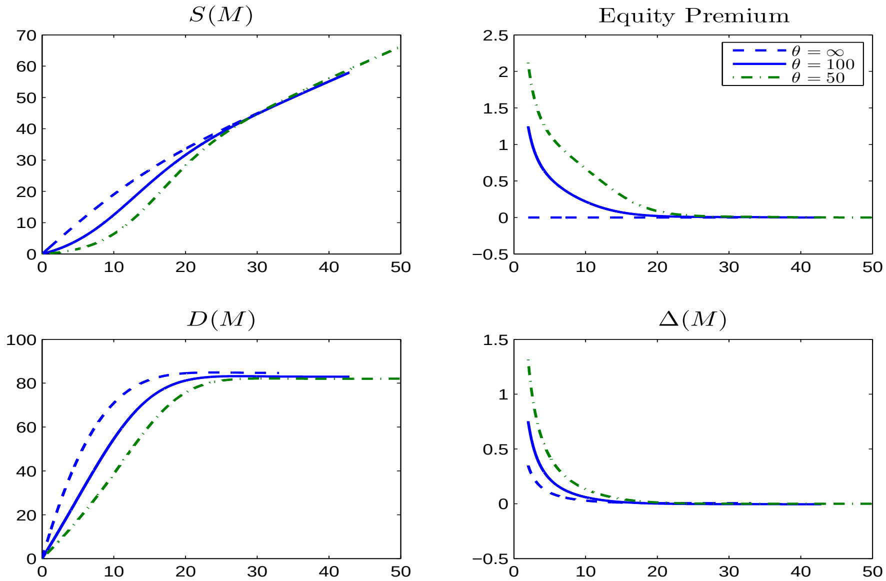

Figure 6: **Stock prices, equity premiums, debt value, and credit yield spreads for type I robust contracting problem**. The parameter values are $\mu = 10$, $r = 0.10$, $\gamma = 0.15$, $\lambda = 0.20$, $\sigma = 5$, and $L = 0$.

In Appendix B1, we show that the stock price, equity premium, debt value, and credit yield spreads are functions of the state variable, the level of cash reserves M. Figure 6 plots these functions for three values of θ. The benchmark model corresponds to $\theta = \infty$. The figure shows that the stock price is an increasing function of M, while the equity premium and the credit yield spread are decreasing functions of M. The principal's aversion to model uncertainty generates a positive equity premium, which approaches infinity as M goes to zero and decreases to zero as M rises to the payout boundary. This implies that the equity premium is high for financially distressed or recently established firms with low cash reserves. This also implies that the equity premium is high in recessions since cash reserves are low in bad times. Intuitively, when M is low, the incentive constraint binds and the ambiguity averse principal bears more uncertainty and hence demanding a higher equity premium. But when M is large, the agent can share the principal's uncertainty since the optimal sensitivity ϕ_t^* is state dependent. This leads the principal to bear less uncertainty, thereby reducing the equity premium.

Figure 6 also shows that debt value decreases with the ambiguity aversion parameter $1/\theta$, while the equity premium and the credit yield spread increase with $1/\theta$. Interestingly, unlike in the benchmark model, here the equity price is not a concave function of the cash reserves. Under model uncertainty, the stock price is convex for low levels of cash reserves and concave for high levels of cash reserves. Intuitively, after a sequence of low cash-flow realizations, cash reserves are low. The robust contract has already taken into account the worst-case scenario. The equity price does not have to respond strongly to a decrease in the cash reserves when they are low in the sense that the marginal change in the stock price decreases when the cash reserves fall. But for high levels of cash reserves, the principal pessimistically believes that the firm does not perform that well. Thus, the stock price reacts strongly to a decrease in the cash reserves when they are large in the sense that the marginal change in the stock price increases when the cash reserves fall. This result implies that asset substitution problem is more likely to occur for financially distressed firms or newly established firms with low cash reserves.

The following proposition is similar to Proposition 6 in Biais et al. (2007).

Proposition 10 *At any time $t \geq 0$, the following holds:*

$$D_t + (1-\lambda) S_t = F(W_t) + M_t - \frac{\theta}{2} E^{P^1}\left[\int_0^\tau e^{-rs} z_t h^*(W_t)^2 \, dt\right]. \tag{65}$$

In addition, $D_t + (1-\lambda) S_t$ increases with θ for any fixed cash reserves $M_t = W_t/\lambda$.

The left-hand side of (65) is the market value of outside securities, i.e., the present value of the cash flows these securities will distribute. The right-hand side of (65) represents the assets generating these cash flows. The last term is the entropy cost, which is subtracted to obtain the operating cash flows allocated to the principal (outside investors),

$$E_t^{Q^{h^*}}\left[\int_t^\tau e^{-r(s-t)}(dX_s - dC_s^*) + e^{-r(\tau-t)}L\right].$$

Proposition 10 also shows that the market value of outside securities decreases with the ambiguity aversion parameter $1/\theta$. The intuition is that the principal's aversion to model uncertainty generates an ambiguity premium, which lowers the market value of outside securities. An immediate implication of this result is that aversion to model uncertainty can make some profitable projects unfunded.

6.2 Implementing Type II Robust Contract

As before, we still use cash reserves, debt and equity to implement the optimal contract. By a similar argument to that in the previous subsection, the cash reserves follow the dynamics:

$$dM_t = rM_t dt + dX_t - \underbrace{dC_t^*}_{\text{inside dividends}} - d\Psi_t, \; M_0 = W_0/\lambda,$$

for $0 \leq t \leq \tau \equiv \inf\left\{t \geq 0 : M_t = \bar{W}/\lambda\right\}$, where

$$d\Psi_t = \underbrace{\left[\mu - (\gamma - r) M_t\right] dt}_{\text{coupon}} + \underbrace{\frac{1-\lambda}{\lambda} dC_t^*}_{\text{outside dividends}} + \underbrace{\sigma h^* (\lambda M_t)\, dt}_{\text{special dividends}},$$

and $\bar{W}$, C^* and h^* are given in Proposition 6. By (62), we can show that $M_t = W_t/\lambda$ and $W_t = E_t^{Q^{h^*}}\left[\int_t^\tau e^{-\gamma(s-t)} dC_s^*\right]$. Thus, the above capital structure is incentive compatible and implements the optimal contract.

Unlike in type I robust contract, the firm pays locally riskless special dividends $\sigma h^* (W_t)\, dt$ to the outside equity holders only. When this term is negative, it is interpreted as new equity injection. As Figure 5 shows, when W is small, $h^* (W) > 0$, the outside equity holders obtain special dividends to lower the cash reserves. But when W is large, $h^* (W) < 0$, the outside equity holders inject cash in the firm to raise the cash reserves. In this way, the cash reserves track the agent's continuation value under the principal's belief P^1.

Since the principal (outside investors) trusts the model, he uses the belief P^1 to price assets. We can then compute debt value and the credit yield spread as in (23) and (24), respectively, with $C = C^*$. However, outside equity value per share is given by

$$S_t = E_t^{P^1}\left[\int_t^\tau e^{-r(s-t)} \frac{1}{\lambda} dC_s^* + \int_t^\tau e^{-r(s-t)} \frac{\sigma h^* (W_s)}{1-\lambda} ds\right].$$

Unlike in type I robust contracting problem, there is no equity premium here. The reason is that outside investors (the principal) are risk neutral and they have no belief distortion. Even though there is no equity premium, investors' concerns about the robustness of the agent's beliefs still have pricing implications. Specifically, the agent's belief distortions represented by the worst-case density generator h^* affect the dynamics of the state variable, i.e., the cash reserves:

$$dM_t = (\gamma M_t - \sigma h^* (W))\, dt + \sigma dB_t^1 - \frac{1}{\lambda} dC_t^*. \tag{66}$$

Because this state variable determines asset prices, the agent's belief distortions influence asset prices.

Figure 7 plots the stock price, debt value and the credit yield spread as functions of the cash reserves for three values of θ. The benchmark model corresponds to $\theta = \infty$. This figure shows similar patterns to those in Figure 6 for type I contracting problem. However, these patterns are generated by the distorted cash reserve dynamics perturbed by the worst-case density generator. Even though there is no equity premium, there is countercyclical credit yield spread. The reason is that for low continuation values, the firm pays out special dividends to hedge against model uncertainty and hence the cash reserves are low. Thus, the firm is

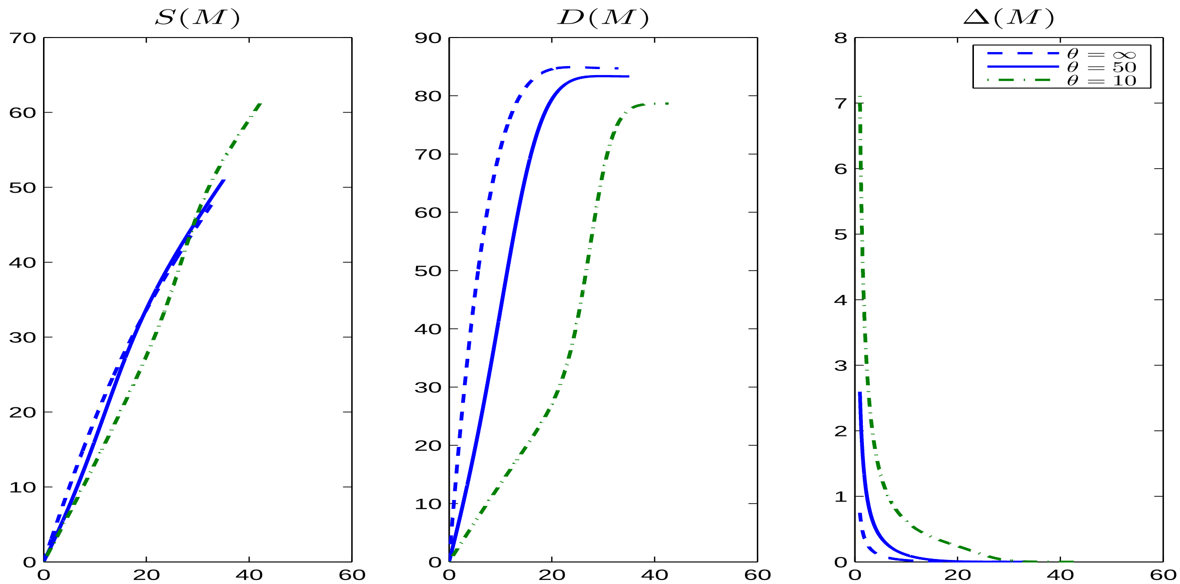

Figure 7: **Stock prices, debt values, and credit yield spreads for type II robust contracting models.** The parameter values for the top two panels are $\mu = 10$, $r = 0.10$, $\gamma = 0.15$, $\lambda = 0.20$, $\sigma = 5$, and $L = 0$.

more likely to be liquidated and the credit yield spread is high.

The following result is similar to Proposition 10. The intuition and implications are also similar.

Proposition 11 *At any time $t \geq 0$, the following holds:*

$$D_t + (1 - \lambda) S_t = F(W_t) + M_t - \frac{\theta}{2} E^{P^1} \left[\int_0^\tau e^{-rs} h^* (W_t)^2 \, dt \right].$$

In addition, $D_t + (1 - \lambda) S_t$ increases with θ for any fixed cash reserves $M_t = W_t / \lambda$.

7 Concluding Remarks

Contracting problems involve at least two parties. Introducing ambiguity and robustness into such problems must consider which party faces ambiguity and what it is ambiguous about. In this paper, we have focused on two types of ambiguity. In type I problem, the principal does not trust the distribution of the project cash flow chosen by the agent. But the agent trusts it. The principal is averse to model ambiguity. In type II problem, the principal trusts the cash-flow distribution chosen by the agent, but he has ambiguity about what beliefs the agent might have. The agent does not face ambiguity and is an expected utility maximizer. We find type I contracting problem particularly interesting because it generates countercyclical firm-level equity premium and has interesting asset pricing implications. In particular, the equity premium and the credit yield spread are high for firms with severe incentive problems and low cash reserves.

In future research, it would be interesting to consider other types of ambiguity. For example, the agent may face ambiguity about the project cash flows or both the principal and the agent may face ambiguity. Our paper focuses on contracting problems under moral hazard with binary actions. It would be interesting to generalize our analysis to a more general principal-agent problem such as that in Sannikov (2008). Finally, it would be interesting to extend our approach to dynamic contracts with hidden information and study robust mechanism design problems in continuous time.

Appendices

A Proofs

Proof of Proposition 1: Define $\mathcal{H}^a$ as the set of density generators associated with the effort level a. Let $Q^h \in \mathcal{P}^a$ be the measure induced by $h \in \mathcal{H}^a$. Define $\Gamma(w)$ as the set of progressively measurable processes (ϕ, C, a) such that (i) ϕ satisfies

$$E^{Q^h}\left[\int_0^t \left(e^{-\gamma s}\phi_s\right)^2 dt\right] < \infty \text{ for all } t > 0,$$

(ii) C is increasing, right continuous with left limits and satisfies

$$E^{Q^h}\left[\left(\int_0^t e^{-rs} dC_s\right)^2\right] < \infty, \text{ for all } t > 0,$$

(iii) $a_t \in \{0, 1\}$, and (iv) W_t satisfies (30), with boundary conditions $W_0 = w$ and $W_t = 0$ for $t \geq \tau \equiv \inf\{t \geq 0 : W_t = 0\}$. For any $(\phi, C, a) \in \Gamma(w)$ and $h \in \mathcal{H}^a$, define the principal's objective function as:

$$\begin{aligned}
& J(C, a, h) \\
= \; & E^{Q^h}\left[\int_0^\tau e^{-rt}(dX_t - dC_t) + e^{-r\tau}L\right] + \frac{\theta}{2}E^{P^a}\left[\int_0^\tau e^{-rs} z_t h_t^2 dt\right] \\
= \; & E^{Q^h}\left[\int_0^\tau e^{-rt}(\mu a_t dt + h_t\sigma dt - dC_t) + e^{-r\tau}L\right] + \frac{\theta}{2}E^{Q^h}\left[\int_0^\tau e^{-rs} h_t^2 dt\right], \qquad \text{(A.1)}
\end{aligned}$$

where we have used the fact that $dX_t = \mu a_t dt + h_t\sigma dt + \sigma dB_t^h$ where B_t^h is a standard Brownian motion under the measure Q^h. We can then write the first-best type I robust contracting problem as follows:

$$F(w) = \sup_{(\phi,C,a)\in\Gamma(w)} \inf_{h\in\mathcal{H}^a} J(C, a, h), \; w \geq 0. \qquad \text{(A.2)}$$

Define an operator as

$$\mathcal{D}^{(\phi,a,h)}F(W) = \mu a + \sigma h + F'(W)(\gamma W + h\phi - \lambda\mu(1-a)) + \frac{F''(W)}{2}\phi^2 + \frac{\theta h^2}{2}. \qquad \text{(A.3)}$$

We can describe the optimality conditions stated in the proposition as variational inequalities:

$$0 = \min\left\{ rF(W) - \sup_{a\in\{0,1\},\phi\in\mathbb{R}} \inf_{h\in\mathbb{R}} \mathcal{D}^{(\phi,a,h)}F(W),\ F'(W)+1 \right\}, \tag{A.4}$$

for all $W > 0$ and the boundary conditions are given in the proposition. One can check that under condition (35), the policies (ϕ^*, a^*, h^*) stated in the proposition satisfy

$$rF(W) = \sup_{a\in\{0,1\},\phi\in\mathbb{R}} \inf_{h\in\mathbb{R}} \mathcal{D}^{(\phi,a,h)}F(W) = \mathcal{D}^{(\phi^*,a^*,h^*)}F(W).$$

We now show that F is the value function in 4 steps. **Step 1.** Define the following process:

$$G_t^{(\phi,C,a,h)} = \int_0^t e^{-rs}(dX_s - dC_s) + \theta\int_0^t e^{-rs}\frac{h_s^2}{2}ds + e^{-rt}F(W_t), \tag{A.5}$$

where (W_t) satisfies (30).

Step 2. Fix a process h^* such that $h_t^* = h^*(W_t)$. Consider any candidate choice $(\phi, C, a) \in \Gamma(w)$. By Ito's Lemma under Q^{h^*},

$$\begin{aligned}
e^{rt}dG_t^{(\phi,C,a,h^*)} &= \mu a_t dt + \sigma h_t^* dt + \sigma dB_t^{h^*} - dC_t^c + \frac{\theta h_t^{*2}}{2}dt \\
&\quad + F'(W_t)\left[\gamma W_t dt - dC_t^c - \lambda\mu(1-a_t)dt + h_t^*\phi_t dt + \phi_t dB_t^{h^*}\right] \\
&\quad + \frac{1}{2}F''(W_t)\phi_t^2 dt - rF(W_t)dt + \Delta F(W_t) - \Delta C_t \\
&= \left[\mathcal{D}^{(\phi_t,a_t,h_t^*)}F(W_t) - rF(W_t)\right]dt - \left(1+F'(W_t)\right)dC_t^c \\
&\quad + \left(\sigma + \phi_t F'(W_t)\right)dB_t^{h^*} + \Delta F(W_t) - \Delta C_t,
\end{aligned}$$

where C^c is the continuous part of C, $\Delta C_t = C_t - C_{t-}$ is the jump, and $\Delta F(W_t) = F(W_t) - F(W_{t-})$. By the variational inequalities (A.4) and $dC_t^c \geq 0$,

$$\mathcal{D}^{(\phi_t,a_t,h_t^*)}F(W_t) - rF(W_t) \leq 0,\quad \left(1+F'(W_t)\right)dC_t \geq 0.$$

Since $F'(W_t)$ is bounded on $\left[0,\bar{W}\right]$,

$$E^{Q^{h^*}}\left[\int_0^t e^{-rs}\left(1+\phi_s F'(W_s)\right)\sigma dB_s^{h^*}\right] = 0.$$

Since $F'(W) \geq -1$,

$$\begin{aligned}
\Delta F(W_t) - \Delta C_t &= F(W_t) - F(W_t + \Delta C_t) - \Delta C_t \\
&= -\int_{W_t}^{W_t+\Delta C_t} \left[F'(c) + 1\right] dc \leq 0.
\end{aligned}$$

It follows that $G_t^{(\phi,C,a,h^*)}$ is a $\left(Q^{h^*}, \mathcal{F}_t\right)$-supermartingale. This implies that $G_0^{(\phi,C,a,h^*)} \geq E^{Q^{h^*}}\left[G_{t\wedge\tau}^{(\phi,C,a,h^*)}\right]$ for any finite time $t \geq 0$. Taking limit as $t \to \infty$, we have

$$G_0^{(\phi,C,a,h^*)} \geq E^{Q^{h^*}}\left[G_\tau^{(\phi,C,a,h^*)}\right] \geq \inf_{h\in\mathcal{H}^a} E^{Q^h}\left[G_\tau^{(\phi,C,a,h)}\right].$$

Taking supremum for $(\phi, C, a) \in \Gamma(w)$ and using (A.5), we obtain

$$F(w) = F(W_0) = G_0^{(\phi,C,a,h^*)} \geq \sup_{(\phi,C,a)\in\Gamma(w)} \inf_{h\in\mathcal{H}^a} E^{Q^h}\left[G_\tau^{(\phi,C,a,h)}\right].$$

Step 3. Fix (ϕ^*, C^*, a^*) and consider any process $(h_t) \in \mathcal{H}^{a^*}$. Use Ito's Lemma to derive

$$\begin{aligned}
e^{rt} dG_t^{(\phi^*,C^*,a^*,h)} &= \mu a_t^* dt + \sigma h_t dt + \sigma dB_t^h - dC_t^{c*} + \frac{\theta h_t^2}{2} dt \\
&\quad + F'(W_t)\left[\gamma W_t dt - dC_t^{c*} - \lambda\mu(1 - a_t^*) dt + h_t \phi_t^* dt + \phi_t^* dB_t^h\right] \\
&\quad + \frac{1}{2} F''(W_t) \phi_t^{*2} dt - rF(W_t) dt + \Delta F(W_t) - \Delta C_t \\
&= \left[\mathcal{D}^{(\phi_t^*,a_t^*,h_t)} F(W_t) - rF(W_t)\right] dt - \left(1 + F'(W_t)\right) dC_t^{c*} \\
&\quad + \left(1 + \phi_t F'(W_t)\right) \sigma dB_t^h + \Delta F(W_t) - \Delta C_t \\
&\geq \left[\mathcal{D}^{(\phi_t^*,a_t^*,h_t^*)} F(W_t) - rF(W_t)\right] dt - \left(1 + F'(W_t)\right) dC_t^{c*} \\
&\quad + \left(\sigma + \phi_t F'(W_t)\right) dB_t^h + \Delta F(W_t) - \Delta C_t.
\end{aligned}$$

Note that $\mathcal{D}^{(\phi_t^*,a_t^*,h_t^*)} F(W_t) - rF(W_t) = 0$. In addition, by (41),

$$\int_0^t e^{-rt}\left(1 + F'(W_s)\right) dC_s^{c*} = \int_0^t e^{-rt}\left(1 + F'(W_s)\right) \mathbf{1}_{\{W_s = \bar{W}\}} dC_s^{c*} = 0.$$

Thus, $G_t^{(\phi^*,C^*,a^*,h)}$ is a $\left(Q^h, \mathcal{F}_t\right)$-submartingale. This implies that $G_0^{(\phi^*,C^*,a^*,h)} \leq E^{Q^h}\left[G_{t\wedge\tau}^{(\phi^*,C^*,a^*,h)}\right]$ for any finite time t. Taking limit as $t \to \infty$ yields

$$F(w) = G_0^{(\phi^*,C^*,a^*,h)} \leq E^{Q^h}\left[G_\tau^{(\phi^*,C^*,a^*,h)}\right].$$

Taking infimum for $h \in \mathcal{H}^a$ yields:

$$F(w) \leq \inf_{h \in \mathcal{H}^a} E^{Q^h} \left[G_\tau^{(\phi^*, C^*, a^*, h)} \right] \leq \sup_{(\phi, C, a) \in \Gamma(w)} \inf_{h \in \mathcal{H}^a} E^{Q^h} \left[G_\tau^{(\phi, C, a, h)} \right]$$

Step 4. By Steps 2 and 3, we know that

$$F(w) = \sup_{(\phi, C, a) \in \Gamma(w)} \inf_{h \in \mathcal{H}^a} E^{Q^h} \left[G_\tau^{(\phi, C, a, h)} \right].$$

Since $F(W_\tau) = F(0) = L$, it follows from (A.5) and (A.1) that $E^{Q^h} \left[G_\tau^{(\phi, C, a, h)} \right] = J(C, a, h)$. We then obtain (A.2).

Evaluating at the solution (ϕ^*, C^*, a^*, h^*) in Proposition 1, we can easily check that $G_t^{(\phi^*, C^*, a^*, h^*)}$ is a $\left(Q^{h^*}, \mathcal{F}_t\right)$-martingale. Condition (35) ensures that ϕ^* achieves the maximum in (43). Thus, we obtain

$$F(w) = \max_{(\phi, C, a) \in \Gamma(w)} \min_{h \in \mathcal{H}^a} E^{Q^h} \left[G_\tau^{(\phi, C, a, h)} \right],$$

and (ϕ^*, C^*, a^*, h^*) is optimal for the first-best type I robust contracting problem. Note that if condition (37) holds, the project is never liquidated. To deliver $W_0 = 0$ to the agent, he always exerts high effort and never gets paid. Q.E.D.

Proof of Propositions 2 and 3: Define J as in (A.1). We modify condition (iii) in the definition of the feasible set $\Gamma(w)$ to incorporate the incentive constraint as follows: if $a_t = 0$, then $\phi_t \leq \sigma\lambda$ and if $a_t = 1$, then $\phi_t \geq \sigma\lambda$. The optimality condition described in Propositions 2 and 3 can be summarized by the following variational inequalities:

$$0 = \min \left\{ rF(W) - \sup_{(a,\phi) \in \Lambda} \inf_{h \in \mathbb{R}} \mathcal{D}^{(\phi, a, h)} F(W), \ F'(W) + 1 \right\}, \tag{A.6}$$

for all $W > 0$, where

$$\Lambda = \{(0, \varphi) : \varphi \leq \sigma\lambda\} \cup \{(1, \varphi) : \varphi \geq \sigma\lambda\}.$$

The boundary conditions are given in Proposition 2. It is easy to verify that under conditions (35) and (48), $a^*(W) = 1$, $\phi^*(W)$, and $h^*(W)$ described in Proposition 2 achieves the above maxmin. In particular, condition (48) ensures that, for $W \in \left[0, \bar{W}\right]$,

$$rF(W) \geq \sup_{(0,\phi) \in \Lambda} \inf_{h \in \mathbb{R}} \mathcal{D}^{(\phi, 0, h)} F(W).$$

We want to show that F is the value function for Problem 4.1.

We follow similar steps to those in the proof of Proposition 1. We only need to modify Step 2. Fix a process h^* such that $h_t^* = h^*(W_t)$. Consider any candidate choice $(\phi, C, a) \in \Gamma(w)$. By Ito's Lemma under Q^{h^*},

$$\begin{aligned}
e^{rt} dG_t^{(\phi,C,a,h^*)} &= \mu a_t dt + \sigma h_t^* dt + \sigma dB_t^{h^*} - dC_t^c + \frac{\theta h_t^{*2}}{2} dt \\
&\quad + F'(W_t)\left[\gamma W_t dt - dC_t - \lambda\mu(1-a_t)dt + h_t^*\phi_t dt + \phi_t dB_t^{h^*}\right] \\
&\quad + \frac{1}{2} F''(W_t)\phi_t^2 dt - rF(W_t)dt + \Delta F(W_t) - \Delta C_t \\
&= \left[\mathcal{D}^{(\phi_t,a_t,h_t^*)} F(W_t) - rF(W_t)\right] dt - \left(1 + F'(W_t)\right) dC_t^c \\
&\quad + \left(\sigma + \phi_t F'(W_t)\right) dB_t^{h^*} + \Delta F(W_t) - \Delta C_t.
\end{aligned}$$

By by the variational inequalities (A.6),

$$rF(W_t) \geq \mathcal{D}^{(\phi_t,a_t,h_t^*)} F(W_t),$$

In addition, $(1 + F'(W_t))\, dC_t^c \geq 0$. Thus, $G_t^{(\phi,C,a,h^*)}$ is a $\left(Q^{h^*}, \mathcal{F}_t\right)$-supermartingale. The rest of the proof is the same as that for Proposition 1. Q.E.D.

Proof of Proposition 4: We adapt Lemma 6 in DeMarzo and Sannikov (2006). We use the Envelope Theorem to differentiate ODE (43) with respect to θ to obtain:

$$\begin{aligned}
r\frac{\partial F(W)}{\partial \theta} &= \frac{\partial F'(W)}{\partial \theta}\gamma W + \frac{\phi^*(W)^2}{2}\frac{\partial F''(W)}{\partial \theta} + \frac{\left[\phi^*(W) F'(W) + \sigma\right]^2}{2\theta^2} \\
&\quad - \frac{\left[\phi^*(W) F'(W) + \sigma\right]\phi^*(W)}{\theta}\frac{\partial F'(W)}{\partial \theta}.
\end{aligned}$$

Under measure Q^{h^*}, it follows from (45) and (46) that (W_t) satisfies

$$dW_t = \gamma W_t dt - dC_t^* - \frac{\left[\phi^*(W) F'(W) + \sigma\right]\phi^*(W)}{\theta} dt + \phi^*(W_t)\, dB_t^{h^*},$$

where $\left(B_t^{h^*}\right)$ is a standard Brownian motion under Q^{h^*}. Using the Feynman-Kac formula, we obtain that the solution to the above ODE for $\partial F(W)/\partial\theta$ is

$$\frac{\partial F(W)}{\partial \theta} = E^{Q^{h^*}}\left[\int_t^\tau e^{-r(s-t)}\frac{\left[\phi^*(W) F'(W) + \sigma\right]^2}{2\theta^2} ds | W_t = W\right] \geq 0,$$

as desired. Q.E.D.

Proof of Proposition 5: Define $\Gamma(w)$ as the set of progressively measurable processes (ϕ, C, a) such that (i) ϕ satisfies

$$E^{P^a}\left[\int_0^t \left(e^{-\gamma}\phi_s\right)^2 dt\right] < \infty \text{ for all } t,$$

(ii) C is increasing, continuous and satisfies

$$E^{P^a}\left[\left(\int_0^t e^{-rs} dC_s\right)^2\right] < \infty, \text{ for all } t,$$

(iii) $a_t \in \{0,1\}$, and (iv) W_t satisfies (54), with boundary conditions $W_0 = w$ and $W_t = 0$ for $t \geq \tau \equiv \inf\{t \geq 0 : W_t = 0\}$. Define the principal's objective function as:

$$J(C, a, h) = E^{P^a}\left[\int_0^\tau e^{-rt}(dX_t - dC_t) + e^{-r\tau}L\right] + \frac{\theta}{2}E^{P^a}\left[\int_0^\tau e^{-rs}h_t^2 dt\right]. \tag{A.7}$$

We can then write down the first-best type I robust contracting problem as follows:

$$F(w) = \sup_{(\phi,C,a)\in\Gamma(w)} \inf_{h\in\mathcal{H}^a} J(C, a, h), \ w \geq 0. \tag{A.8}$$

Define an operator as

$$\mathcal{D}^{(\phi,a,h)}F(W) \equiv \mu a + F'(W)(\gamma W + h\phi - \lambda\mu(1-a)) + \frac{F''(W)}{2}\phi^2 + \frac{\theta h^2}{2}. \tag{A.9}$$

We can describe the optimality conditions derived in Section 5.1 as variational inequalities:

$$0 = \min\left\{rF(W) - \sup_{a\in\{0,1\},\phi\in\mathbb{R}} \inf_{h\in\mathbb{R}} \mathcal{D}^{(\phi,a,h)}F(W), \ F'(W) + 1\right\},$$

for all $W > 0$.

As in the proof of Proposition 1, the solution to the above variational inequalities gives the value function for the first best type II robust contracting problem. As described in Section 5.1, we need to solve ODE (58). Assuming condition (35) holds, we then obtain the optimal $\phi^* = 0$. Since $F'(W) \geq -1$, optimal $a^* = 1$. ODE (58) becomes

$$rF(W) = \mu + F'(W)\gamma W,$$

for $W \in [0, \bar{W}]$, with boundary conditions $F'(\bar{W}) = -1$, and $F''(\bar{W}) = 0$. The general solution to this ODE is

$$F(W) = AW^{\frac{r}{\gamma}} + \frac{\mu}{r},$$

for some constant A. For the boundary condition to hold, $\bar{W} = 0$. Since $F(0) = \mu/r > L$, the project is never liquidated. Since $F'(W) = -1$ for $W > \bar{W}$, we obtain $F(W) = \mu/r - W$. Q.E.D.

Proof of Propositions 6 and 7: Define J as in (A.7). We modify condition (iii) in the definition of the feasible set $\Gamma(w)$ in the proof of Proposition 5 to incorporate the incentive constraint as follows: if $a_t = 0$, then $\phi_t \leq \sigma\lambda$ and if $a_t = 1$, then $\phi_t \geq \sigma\lambda$. The optimality condition described in Propositions 6 and 7 can be summarized by the following variational inequalities:

$$0 = \min\left\{ rF(W) - \sup_{(a,\phi)\in\Lambda} \inf_{h\in\mathbb{R}} \mathcal{D}^{(\phi,a,h)}F(W),\ F'(W) + 1 \right\}. \tag{A.10}$$

for all $W > 0$, where $\mathcal{D}^{(\phi,a,h)}F(W)$ is defined in (A.9) and

$$\Lambda = \{(0,\varphi) : \varphi \leq \sigma\lambda\} \cup \{(1,\varphi) : \varphi \geq \sigma\lambda\}.$$

The boundary conditions are given in the proposition. When $a = a^* = 1$, given condition (35) and ODE (61), we can check that

$$\sup_{(1,\phi)\in\Lambda} \inf_{h\in\mathbb{R}} \mathcal{D}^{(\phi,1,h)}F(W) = \mathcal{D}^{(\phi^*,a^*,h^*)}F(W) = rF(W),\ W \in \left(0, \bar{W}\right),$$

where ϕ^* and h^* are given in Proposition 6. By condition (64), we can easily check that

$$\begin{aligned}
\mathcal{D}^{(\phi^*,a^*,h^*)}F(W) &= rF(W) \geq F'(W)(\gamma W - \lambda\mu) \\
&= \sup_{(0,\phi)\in\Lambda} \inf_{h\in\mathbb{R}} \mathcal{D}^{(\phi,0,h)}F(W) = \mathcal{D}^{(0,0,0)}F(W),
\end{aligned}$$

where we have used condition (35) to derive that the above extremization problem. The above two equations imply that

$$rF(W) = \sup_{(a,\phi)\in\Lambda} \inf_{h\in\mathbb{R}} \mathcal{D}^{(\phi,a,h)}F(W) = \mathcal{D}^{(\phi^*,a^*,h^*)}F(W). \tag{A.11}$$

Now, as in the proof for Propositions 1-3, we proceed in 4 steps. **Step 1.** Define the following process:

$$G_t^{(\phi,C,a,h)} = \int_0^t e^{-rs}\left(dX_s - dC_s\right) + \theta\int_0^t e^{-rs}\frac{h_s^2}{2}ds + e^{-rt}F\left(W_t\right), \tag{A.12}$$

where (W_t) satisfies (54).

Step 2. Fix a process h^* such that $h_t^* = h^*(W_t)$. Consider any candidate choice $(\phi, C, a) \in \Gamma(w)$. By Ito's Lemma under measure P,

$$\begin{aligned}
e^{rt}dG_t^{(\phi,C,a,h^*)} &= \mu a_t dt + \sigma dB_t^a - dC_t^c + \frac{\theta h_t^{*2}}{2}dt \\
&\quad + F'(W_t)\left[\gamma W_t dt - dC_t^c - \lambda\mu(1-a_t)dt - h_t\phi_t dt + \phi_t dB_t\right] \\
&\quad + \frac{1}{2}F''(W_t)\phi_t^2 dt - rF(W_t)dt + \Delta F(W_t) - \Delta C_t \\
&= \left[\mathcal{D}^{(\phi_t,a_t,h_t^*)}F(W_t) - rF(W_t)\right]dt - \left(1+F'(W_t)\right)dC_t^c \\
&\quad + \left(\sigma + \phi_t F'(W_t)\right)dB_t^a + \Delta F(W_t) - \Delta C_t.
\end{aligned}$$

By (A.10), $rF(W_t) \geq \mathcal{D}^{(\phi_t,a_t,h_t^*)}F(W_t)$. In addition, $(1+F'(W_t))\,dC_t^c \geq 0$. Thus, $G_t^{(\phi,C,a,h^*)}$ is a $(P^a, \mathcal{F}_t)$-supermartingale. This implies that

$$F(w) = G_0^{(\phi,C,a,h^*)} \geq E^{P^a}\left[G_\tau^{(\phi,C,a,h^*)}\right] \geq \inf_{h\in\mathcal{H}^a} E^{P^a}\left[G_\tau^{(\phi,C,a,h)}\right].$$

Taking supremum for $(\phi, c, a) \in \Gamma(w)$ yields:

$$F(w) \geq \sup_{(\phi,C,a)\in\Gamma(w)}\ \inf_{h\in\mathcal{H}^a}\ E^{P^a}\left[G_\tau^{(\phi,C,a,h)}\right].$$

Step 3. Fix (ϕ^*, C^*, a^*) and consider any process $(h_t) \in \mathcal{H}^a$. Use Ito's Lemma to derive

$$\begin{aligned}
e^{rt}dG_t^{(\phi^*,C^*,a^*,h)} &= \mu a_t^* dt + \sigma dB_t^{a^*} - dC_t^{c*} + \frac{\theta h_t^2}{2}dt \\
&\quad + F'(W_t)\left[\gamma W_t dt - dC_t^{c*} - \lambda\mu(1-a_t^*)dt - h_t\phi_t^* dt + \phi_t^* dB_t^{a^*}\right] \\
&\quad + \frac{1}{2}F''(W_t)\phi_t^{*2}dt - rF(W_t)dt + \Delta F(W_t) - \Delta C_t^* \\
&= \left[\mathcal{D}^{(\phi_t^*,a_t^*,h_t)}F(W_t) - rF(W_t)\right]dt - \left(1+F'(W_t)\right)dC_t^{c*} \\
&\quad + \left(1+\phi_t F'(W_t)\right)\sigma dB_t^{a^*} + \Delta F(W_t) - \Delta C_t^*, \\
&\geq \left[\mathcal{D}^{(\phi_t^*,a_t^*,h_t^*)}F(W_t) - rF(W_t)\right]dt - \left(1+F'(W_t)\right)dC_t^* \\
&\quad + \left(\sigma + \phi_t F'(W_t)\right)dB_t^{a^*} + \Delta F(W_t) - \Delta C_t^*.
\end{aligned}$$

Thus, $G_t^{(\phi^*,C^*,a^*,h)}$ is a $\left(P^{a^*},\mathcal{F}_t\right)$-submartingale. This implies that

$$F\left(w\right)=G_0^{(\phi^*,C^*,a^*,h)}\leq E^{P^{a^*}}\left[G_\tau^{(\phi^*,C^*,a^*,h)}\right].$$

Taking infimum for h yields:

$$F\left(w\right)\leq\inf_{h\in\mathcal{H}^{a^*}}E^{P^{a^*}}\left[G_\tau^{(\phi^*,C^*,a^*,h)}\right]\leq\sup_{(\phi,C,a)\in\Gamma(w)}\inf_{h\in\mathcal{H}^a}E^{P^a}\left[G_\tau^{(\phi,C,a,h)}\right].$$

Step 4. By Steps 2 and 3, we know that

$$F\left(w\right)=\sup_{(\phi,C,a)\in\Gamma(w)}\inf_{h\in\mathcal{H}^a}E^{P^a}\left[G_\tau^{(\phi,C,a,h)}\right].$$

Since $F\left(W_\tau\right)=F\left(0\right)=L$, it follows from (A.12) and (A.7) that $E^{P^a}\left[G_\tau^{(\phi,C,a,h)}\right]=J\left(C,a,h\right).$ We then obtain (A.8).

Evaluating at the solution (ϕ^*,C^*,a^*,h^*) in Proposition 5, we can easily check that $G_t^{(\phi^*,C^*,a^*,h^*)}$ is a $(P,\mathcal{F}_t)$-martingale. Condition (35) ensures that ϕ^* achieves the maximum in (43). Thus, we obtain

$$F\left(w\right)=\sup_{(\phi,C,a)\in\Gamma(w)}\inf_{h\in\mathcal{H}^a}E^{P^a}\left[G_\tau^{(\phi,C,a,h)}\right],$$

and (ϕ^*,C^*,a^*,h^*) is optimal for type II robust contracting problem with agency. Q.E.D.

Proof of Proposition 8: We adapt Lemma 6 in DeMarzo and Sannikov (2006). We differentiate ODE (61) with respect to θ to obtain:

$$r\frac{\partial F(W)}{\partial\theta}=\gamma W\frac{\partial F'(W)}{\partial\theta}+\frac{(\sigma\lambda)^2}{2}\left[\frac{\partial F''(W)}{\partial\theta}-2\frac{F'(W)}{\theta}\frac{\partial F'(W)}{\partial\theta}+\frac{F'(W)^2}{\theta^2}\right].$$

Using the Feynman-Kac formula, we obtain that the solution to the above ODE for $\partial F(W)/\partial\theta$ is

$$\frac{\partial F(W)}{\partial\theta}=E^P\left[\int_t^\tau e^{-r(s-t)}\frac{\left(\sigma\lambda\right)^2F'(W_s)^2}{2\theta^2}ds|W_t=W\right]\geq 0,$$

where (W_t) follows (62) on the interval $[0,\bar{W}]$. Q.E.D.

Proof of Proposition 9: The equity premium is defined as

$$\frac{1}{S_t}\left(\underbrace{\frac{dC_t}{\lambda}+\frac{1}{1-\lambda}\left[\sigma-\frac{\phi^*\left(W_t\right)}{\lambda}\right]dB_t^1}_{\text{dividends}}+\underbrace{dS_t}_{\text{capital gains}}-rS_t dt\right). \tag{A.13}$$

By Ito's Lemma,

$$\begin{aligned} dS_t &= dS\left(M_t\right)=S'\left(M_t\right)\gamma M_t dt+S'\left(M_t\right)\frac{\phi^*\left(W_t\right)}{\lambda}dB_t^1 \\ &\quad -\frac{S'\left(M_t\right)}{\lambda}dC_t+\frac{\left[\phi^*\left(\lambda M_t\right)\right]^2}{2\lambda^2}S''\left(M_t\right)dt. \end{aligned} \tag{A.14}$$

Plugging (A.14) and (B.1) into (A.13) and noting the fact that C_t increases only when $S'\left(M_t\right)=1$, we can compute the local expected equity premium under measure P given in the proposition. Q.E.D.

Proof of Proposition 10: It follows from (46) and Girsanov's Theorem that

$$dW_t=\gamma W_t dt-dC_t^*+\phi^*\left(W_t\right)h^*\left(W_t\right)dt+\phi^*\left(W_t\right)dB_t^{h^*}.$$

By Ito's Lemma,

$$\begin{aligned} e^{-rT\wedge\tau}W_{T\wedge\tau} &= e^{-rt}W_t+\int_t^{T\wedge\tau}e^{-rs}\left(\gamma-r\right)W_s ds+\int_t^{T\wedge\tau}e^{-rs}\phi^*\left(W_s\right)dB_s^{h^*} \\ &\quad -\int_t^{T\wedge\tau}e^{-rs}dC_s^*+\int_t^{T\wedge\tau}e^{-rs}\phi^*\left(W_s\right)h^*\left(W_s\right)ds, \end{aligned}$$

for any $T>t$, where $\tau=\inf\left\{t\geq 0:W_t=0\right\}$. Taking expectations with respect to Q^{h^*} and letting $T\to\infty$, we use $M_t=W_t/\lambda$ and $W_\tau=0$ to derive

$$M_t=E_t^{P^1}\left[\int_t^{\tau}e^{-r(s-t)}\left[\frac{1}{\lambda}dC_s^*-\left(\gamma-r\right)M_s ds-\frac{\phi^*\left(W_s\right)h^*\left(W_s\right)}{\lambda}ds\right]\right].$$

It follows that

$$
\begin{aligned}
& D_t + (1-\lambda) S_t \\
= \; & E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} \left(\mu - (\gamma - r) M_s\right) ds + e^{-r(\tau - t)} L \right] \\
& + E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} \frac{1-\lambda}{\lambda} dC_s^* \right] + E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} \left(\sigma - \frac{\phi^*(W_s)}{\lambda} \right) dB_s^1 \right] \\
= \; & E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} \left(dX_t - dC_s^*\right) + e^{-r(\tau - t)} L \right] \\
& + E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} \left(\frac{1}{\lambda} dC_s^* - (\gamma - r) M_s ds - \frac{\phi^*(W_s) h^*(W_s)}{\lambda} ds \right) \right],
\end{aligned}
$$

as desired.

To show that $D_t + (1-\lambda) S_t$ increases with θ, we only need to show that

$$
G(W_t) \equiv E_t^{Q^{h^*}} \left[\int_t^\tau e^{-r(s-t)} \left(dX_t - dC_s^*\right) + e^{-r(\tau-t)} L \right]
$$

increases with θ for any fixed W_t. By Section 4.2, $G(W)$ satisfies ODE (43) after subtracting the local entropy term $\theta h^*(W)^2/2:$

$$
rG(W) = \mu + G'(W)\gamma W + \frac{\phi^*(W)^2}{2} G''(W) - \frac{\left[\phi^*(W) G'(W) + \sigma\right]^2}{\theta}. \tag{A.15}
$$

We then adapt Lemma 6 in DeMarzo and Sannikov (2006) or Proposition 4 to show that $G(W)$ increases with θ. Q.E.D.

Proof of Proposition 11: It follows from (62) and Ito's Lemma that

$$
\begin{aligned}
e^{-rT\wedge\tau} W_{T\wedge\tau} \; = \; & e^{-rt} W_t + \int_t^{T\wedge\tau} e^{-rs} (\gamma - r) W_s ds + \int_t^{T\wedge\tau} e^{-rs} \sigma\lambda dB_s - \int_t^{T\wedge\tau} e^{-rs} dC_s^* \\
& - \int_t^{T\wedge\tau} e^{-rs} h^*(W_s) \sigma\lambda ds,
\end{aligned}
$$

for any $T > t$, where $\tau = \inf\{t \geq 0 : W_t = 0\}$. Taking expectations with respect to P and letting $T \to \infty$, we use $M_t = W_t/\lambda$ to obtain

$$
M_t = E_t^{P^1} \left[\int_t^\tau e^{-r(s-t)} \left[\frac{1}{\lambda} dC_s^* - (\gamma - r) M_s ds + \sigma h^*(W_s) ds \right] \right].
$$

We can then derive that

$$
\begin{aligned}
& D_t + (1-\lambda) S_t \\
= \; & E_t^{P^1}\left[\int_t^{\tau} e^{-r(s-t)} \left(\mu - (\gamma - r) M_s\right) ds + e^{-r(\tau-t)} L\right] \\
& + E_t^{P^1}\left[\int_t^{\tau} e^{-r(s-t)} \frac{1-\lambda}{\lambda} dC_s^*\right] + E_t^{P}\left[\int_t^{\tau} e^{-r(s-t)} \sigma h^*\left(W_s\right) ds\right] \\
= \; & E_t^{P^1}\left[\int_t^{\tau} e^{-r(s-t)} \left(dX_t - dC_s^*\right) + e^{-r(\tau-t)} L\right] \\
& + E_t^{P}\left[\int_t^{\tau} e^{-r(s-t)} \left(\frac{1}{\lambda} dC_s^* - (\gamma - r) M_s ds + \sigma h^*\left(W_s\right) ds\right)\right],
\end{aligned}
$$

as desired.

To show that $D_t + (1-\lambda) S_t$ increases with θ, we only need to show that

$$
G\left(W_t\right) \equiv E_t^{P^1}\left[\int_t^{\tau} e^{-r(s-t)} \left(dX_t - dC_s^*\right) + e^{-r(\tau-t)} L\right]
$$

increases with θ for any fixed W_t. By Section 5.2, $G(W)$ satisfies ODE (61) after subtracting the local entropy term $\theta h^*(W)^2/2$:

$$
rG(W) = \quad \mu + G'(W)\gamma W + \frac{G''(W)}{2} (\sigma\lambda)^2 - \frac{G'(W)^2}{\theta} (\sigma\lambda)^2, \tag{A.16}
$$

We then adapt Lemma 6 in DeMarzo and Sannikov (2006) or Proposition 8 to show that $G(W)$ increases with θ. Q.E.D.

B Asset Pricing Formulas

In this appendix, we follow DeMarzo and Sannikov (2006) and Biais et al. (2007) to represent asset prices as ODEs. We use the cash reserves M as a state variable and write debt value, equity price and credit yield spreads as functions of M.

B. 1 Type I Robust Contract

Under the worst-case belief Q^{h^*}, we use Girsanov's Theorem to write the cash reserve dynamics as

$$
dM_t = \gamma M_t dt + \frac{\phi^*\left(\lambda M_t\right) h^*\left(\lambda M_t\right)}{\lambda} dt + \frac{\phi^*\left(\lambda M_t\right)}{\lambda} dB_t^{h^*} - \frac{1}{\lambda} dC_t^*.
$$

Thus, the equity price $S_t = S(M_t)$ satisfies the ODE:

$$rS(M) = \frac{1}{1-\lambda}\left[\sigma - \frac{\phi^*(\lambda M_s)}{\lambda}\right] h^*(\lambda M) + \left(\gamma M + \frac{\phi^*(\lambda M)\, h^*(\lambda M)}{\lambda}\right) S'(M) + \frac{[\phi^*(\lambda M)]^2}{2\lambda^2} S''(M), \tag{B.1}$$

with the boundary conditions:

$$S(0) = 0, \quad S'\left(\frac{\bar{W}}{\lambda}\right) = 1.$$

The bond price $D_t = D(M_t)$ satisfies the ODE:

$$rD(M) = \mu - (\gamma - r) M + \left(\gamma M + \frac{\phi^*(\lambda M)\, h^*(\lambda M)}{\lambda}\right) D'(M) + \frac{[\phi^*(\lambda M)]^2}{2\lambda^2} D''(M),$$

with boundary conditions:

$$D(0) = L, \ D'\left(\frac{\bar{W}}{\lambda}\right) = 0.$$

The Arrow-Debreu price of one unit claim paid at the time of default, $T_t = T(M_t)$, satisfies the ODE:

$$rT(M) = \left(\gamma M + \frac{\phi^*(\lambda M)\, h^*(\lambda M)}{\lambda}\right) T'(M) + \frac{[\phi^*(\lambda M)]^2}{2\lambda^2} T''(M),$$

subject to the boundary conditions:

$$T(0) = 1, \ T'\left(\frac{\bar{W}}{\lambda}\right) = 0.$$

B. 2 Type II Robust Contract

For type II robust contracting problem, the cash reserves follow the dynamics:

$$dM_t = (\gamma M_t - \sigma h^*(W))\, dt + \sigma dB_t^1 - \frac{1}{\lambda} dC_t.$$

Thus, the equity price $S_t = S(M_t)$ satisfies the ODE:

$$rS(M) = \frac{\sigma h^*(\lambda M)}{1-\lambda} + (\gamma M - \sigma h^*(\lambda M))\, S'(M) + \frac{\sigma^2}{2} S''(M),$$

with the boundary conditions:

$$S(0) = 0, \quad S'\left(\frac{\bar{W}}{\lambda}\right) = 1.$$

The bond price $D_t = D(M_t)$ satisfies the ODE:

$$rD(M) = \mu - (\gamma - r) M + (\gamma M - \sigma h^* (\lambda M)) D'(M) + \frac{\sigma^2}{2} D''(M),$$

with boundary conditions:

$$D(0) = L, \; D'\left(\frac{\bar{W}}{\lambda}\right) = 0.$$

The Arrow-Debreu price of one unit claim paid at the time of default, $T_t = T(M_t)$, satisfies the ODE:

$$rT(M) = (\gamma M - \sigma h^* (\lambda M)) T'(M) + \frac{\sigma^2}{2} T''(M),$$

subject to the boundary conditions:

$$T(0) = 1, \; T'\left(\frac{\bar{W}}{\lambda}\right) = 0.$$

References

Albuquerque, Rui and Hugo Hopenhayn, 2004, Optimal Lending Contracts and Firm Dynamics, *Review of Economic Studies* 71, 285-315.

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent, 2003, A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection, *Journal of the European Economic Association*, 1, 68-123.

Atkeson, Andrew, and Robert E. Lucas, Jr., 1992, On Efficient Distribution with Private Information, *Review of Economic Studies* 59, 427-453.

Bansal, Ravi and Amir Yaron, 2004, Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles, Journal of Finance 59, 1481-1509.

Bergemann, Dirk and Stephen Morris, 2012, *Robust Mechanism Design: The Role of Private Information and Higher Order Beliefs*, World Scientific, Singapore.

Bergemann, Dirk and Karl Schlag, 2011, Robust Monopoly Pricing, *Journal of Economic Theory* 146, 2527-2543.

Biais, Bruno, Thomas Mariotti, Guillaume Plantin, and Jean-Charles Rochet, 2007, Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications, *Review of Economic Studies* 74, 345-390.

Biais, Bruno, Thomas Mariotti, Jean-Charles Rochet, and Stephane Villeneuve, 2010, Large Risks, Limited Liability, and Dynamic Moral Hazard, *Econometrica* 78, 73–118.

Campbell, John Y., and John H. Cochrane, 1999, By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior, Journal of Political Economy 107, 205-251.

Chen, Zengjing and Larry G. Epstein, 2002, Ambiguity, Risk, and Asset Returns in Continuous Time, *Econometrica* 70, 1403–1443.

Clementi, Gian Luca, and Hugo A. Hopenhayn, 2006, A Theory of Financing Constraints and Firm Dynamics, *Quarterly Journal of Economics* 121, 229-265.

Cvitanic, Jaksa and Jianfeng Zhang, 2013, *Contract Theory in Continuous-Time Models*, Springer-Verlag, Berlin.

DeMarzo, Peter and Michael Fishman, 2007a, Agency and Optimal Investment Dynamics, *Review of Financial Studies* 20, 151-188.

DeMarzo, Peter and Michael Fishman, 2007b, Optimal Long-Term Financial Contracting, *Review of Financial Studies* 20, 2079-2128.

DeMarzo, Peter M., Michael Fishman, Zhiguo He, and Neng Wang, 2012, Dynamic Agency and the q Theory of Investment, *Journal of Finance* 67, 2295-2340.

DeMarzo, Peter M. and Yuliy Sannikov, 2006, Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model, *Journal of Finance* 61, 2681–2724.

Ellsberg, Daniel, 1961, Risk, Ambiguity and the Savage Axiom, *Quarterly Journal of Economics* 75, 643-669.

Epstein, Larry G. and Martin Schneider, 2003, Recursive multiple-priors, *Journal of Economic Theory* 113, 1-31.

Epstein, Larry G. and Tan Wang, 1994, Intertemporal Asset Pricing under Knightian Uncertainty, *Econometrica* 62, 283-322.

Fleming, Wendell H. and H. Mete Soner, 1993, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York.

Fleming, W. H. and Panagiotis E. Souganidis, 1989, On the Existence of Value Functions of Two-player, Zero-sum Stochastic Differential Games, *Indiana University Mathematics Journal* 38, 293–314.

Gilboa, Itzhak and David Schmeidler, 1989, Maxmin Expected Utility with Non-unique Prior, *Journal of Mathematical Economics* 18, 141–153.

Green, Edward, 1987, Lending and the Smoothing of Uninsurable Income, in *Contractual Arrangements for Intertemporal Trade*, ed. by E.C. Prescott and N. Wallace. Minneapolis: University of Minnesota Press, 3-25.

Hansen, Lars Peter, 2007, Beliefs, Doubts and Learning: The Valuation of Macroeconomic Risk, *American Economic Review* 97, 1-30.

Hansen, Lars Peter and Thomas J. Sargent, 2001, Robust Control and Model Uncertainty, *American Economic Review* 91, 60-66.

Hansen, Lars Peter and Thomas J. Sargent, 2008, *Robustness*, Princeton University Press, Princeton and Oxford.

Hansen, Lars Peter and Thomas J. Sargent, 2010, Fragile Beliefs and the Price of Uncertainty, *Quantitative Economics* 1, 129–162.

Hansen, Lars Peter and Thomas J. Sargent, 2012, Three Types of Ambiguity, *Journal of Monetary Economics* 59, 422-445.

Hansen, Lars Peter, Thomas J. Sargent, Gauhar A. Turmuhambetova, and Noah Williams, 2006, Robust Control and Model Misspecification, *Journal of Economic Theory* 128, 45-90.

He, Zhiguo, 2009, Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion, *Review of Financial Studies* 22, 859-892.

He, Zhiguo, 2011, A Model of Dynamic Compensation and Capital Structure, *Journal of Financial Economics* 100, 351-366.

He, Zhiguo, Bin Wei, and Jianfeng Yu, 2012, Optimal Long-term Contracting with Learning, available at SSRN 1991518.

Holmstrom, Bengt and Paul Milgrom, 1987, Aggregation and Linearity in the Provision of Intertemporal Incentives, *Econometrica* 55, 303–328.

Ilut, Cosmin and Martin Schneider, 2011, Ambiguous Business Cycles, working paper, Stanford University.

Ju, Nengjiu and Jianjun Miao, 2012, Ambiguity, Learning, and Asset Returns, *Econometrica* 80, 559-591.

Ju, Nengjiu, and Xuhu Wan, 2011, Optimal Compensation and Pay-Performance Sensitivity in a Continuous-Time Principal-Agent Model, *Management Science*, forthcoming.

Karatzas, Ioannis and Steven E. Shreve, 1991, *Brownian Motion and Stochastic Calculus*, 2nd Edition, Springer-Verlag, New York.

Knight, Frank, 1921, *Risk, Uncertainty, and Profit*, Boston: Houghton Mifflin.

Leland, Hayne, 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance* 49, 1213–1252.

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini, 2006a, Ambiguity Aversion, Robustness and the Variational Representation of Preferences, *Econometrica* 74, 1447-1498.

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini, 2006b, Dynamic Variational Preferences, *Journal of Economic Theory* 128, 4-44.

Mas-Colell, Andreu, Michael D.Whinston, and Jerry R.Green, 1995, *Microeconomic Theory*, Oxford University Press.

Ou-Yang, Hui, 2003, Optimal Contracts in a Continuous-Time Delegated Portfolio Management Problem, *Review of Financial Studies* 16, 173–208.

Piskorski, Tomasz, and Alexei Tchistyi, 2010, Optimal Mortgage Design, *Review of Financial Studies* 23, 3098-3140.

Prat, Julien and Boyan Jovanovic, 2010, Dynamic Incentive Contracts under Parameter Uncertainty, NBER working paper 16649.

Quadrini, Vincenzo, 2004, Investment and Liquidation in Renegotiation-Proof Contracts with Moral Hazard, *Journal of Monetary Economics* 51, 713-751.

Sannikov, Yuliy, 2008, A Continuous-Time Version of the Principal-Agent Problem, *Review of Economic Studies* 75, 957–984.

Schattler, Heinz and Jaeyoung Sung, 1993, The First-Order Approach to the Continuous-Time Principal-Agent Problem with Exponential Utility, *Journal of Economic Theory* 61, 331–371.

Spear, Stephen E. and Sanjay Srivastava, 1987, On Repeated Moral Hazard with Discounting, *Review of Economic Studies* 54, 599–617.

Strzalecki, Tomasz, 2011, Axiomatic Foundations of Multiplier Preferences, *Econometrica* 79, 47-73.

Szydlowski, Martin, 2012, Ambiguity in Dynamic Contracts, working paper, Northwestern University.

Thomas, Jonathan and Tim Worrall, 1990, Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem, *Journal of Economic Theory* 51, 367-390.

Woodford, Michael, 2010, Robustly Optimal Monetary Policy with Near-Rational Expectations, *American Economic Review* 100, 274-303.

Williams, Noah, 2009, On Dynamic Principal-Agent Problems in Continuous Time, working paper, University of Wisconsin at Madison.

Williams, Noah, 2011, Persistent Private Information, *Econometrica* 79, 1233-1274.

Zhang, Yuzhe, 2009, Dynamic Contracting with Persistent Shocks, *Journal of Economic Theory* 144, 635-675.

Zhu, John, 2012, Optimal Contracts with Shirking, *Review of Economic Studies*, forthcoming.