Slow Moving Debt Crises*

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October 2013

Abstract

What circumstances or policies leave sovereign borrowers at the mercy of self-fulfilling increases in interest rates? To answer this question, we study the dynamics of debt and interest rates in a model where default is driven by insolvency. Fiscal deficits and surpluses are subject to shocks but influenced by a fiscal policy rule. Whenever possible the government issues debt to meet its current obligations and defaults otherwise. We show that low and high interest rate equilibria may coexist. Higher interest rates, prompted by fears of default, lead to faster debt accumulation, validating default fears. We call such an equilibrium a slow moving crisis, to distinguish it from rollover crises in which investor runs precipitate immediate default. We investigate how the existence of multiple equilibria is affected by the fiscal policy rule, the maturity of debt, and the level of debt.

1 Introduction

Yields on sovereign bonds for Italy, Spain and Portugal shot up dramatically in late 2010 with nervous investors suddenly casting the debt sustainability of these countries into doubt. An important concern for policy makers was the possibility that higher interest rates were self-fulfilling. High interest rates, the argument goes, contribute to the rise in debt over time, eventually driving countries into insolvency, thus justifying higher interest rates in the first place.

News coverage reflected the fact that uncertainty about future interest rates and debt dynamics were at the center of investors' concern. For example, a *Financial Times*' report

*We thank comments and suggestions from Manuel Amador, Fernando Broner, Hal Cole, Emmanuel Farhi, Pablo Kurlat, Hugo Hopenhayn and Guido Sandleris. Nicolás Caramp and Greg Howard provided valuable research assistance.

on the Italian bond market cites a pessimistic observer expecting "Italian bonds to perform worse than Spanish debt this summer, as investors focus on the sustainability of Italy's debt burden," given Italy's high initial debt-to-GDP ratio. A more optimistic investor in the same report argues that Italy "can cope with elevated borrowing costs for some time particularly when shorter-dated bond yields remain anchored" and that "it's critical to bring these yields down, but there is time for Italy to establish that its policies are working."[1] A number of reports referred to an "Italian Debt Spiral" webpage by Thomson Reuters in which users could compute the primary surplus needed by Italy to stabilize its debt-to-GDP ratio under different scenarios.[2]

Yields subsided in the late summer of 2012 after the European Central Bank's president, Mario Draghi, unveiled plans to purchase sovereign bonds to help sustain their market price. A view based on self-fulfilling crises can help justify such lender-of-last-resort interventions to rule out bad equilibria. Indeed, this notion was articulated by Draghi during the news conference announcing the Outright Monetary Transactions (OMT) bond-purchasing program (September 6th, 2012),

> "The assessment of the Governing Council is that we are in a situation now where you have large parts of the Euro Area in what we call a bad equilibrium, namely an equilibrium where you have self-fulfilling expectations. You may have self-fulfilling expectations that generate, that feed upon themselves, and generate adverse, very adverse scenarios. So there is a case for intervening to, in a sense, break these expectations [...]"

If this view is correct, a credible announcement to do "whatever it takes" is all it takes to rule out bad equilibria, no bond purchases need to be carried out. To date, this is exactly how it seems to have played out. There have been no purchases by the ECB and no countries have applied to the OMT program.

In this paper we investigate the possibility of self-fulfilling crises in a dynamic sovereign debt model in which investors use a simple forecasting model to form expectations about future debt sustainability. Calvo (1988) first formalized the feedback between interest rates and the debt burden, showing that it opens the door to multiple equilibria.[3] Our contribution is to cast this feedback mechanism in a dynamic setting, focusing on the conditions for multiple equilibria.

[1] "Investors wary of Italy's borrowing test", *Financial Times*, July 16, 2012.

[2] The widget can be found at http://graphics.thomsonreuters.com/11/07/BV_ITDBT0711_VF.html

[3] For recent extensions of this framework applied to the European crisis see Corsetti and Dedola (2011) and Corsetti and Dedola (2013).

In our model, default is driven by insolvency and occurs only when the government is unable to finance debt payments. The government faces random shocks to its budgetary needs and attempts to finance itself by selling bonds to a large group of risk-neutral investors. We assume that government's policy is described by a fiscal rule, as in the literature on debt sustainability (Bohn, 2005; Ghosh et al., 2011) and in work studying the interaction of fiscal and monetary policy (Leeper, 1991). The fiscal rule specifies deficits and surpluses as functions of exogenous shocks and endogenous state variables, such as the debt level. Default occurs when the government's need for funds exceeds its borrowing capacity. The borrowing capacity, in turn, is limited endogenously by the probability of future default.

We begin by studying a case where only short-term debt is allowed. We show that the equilibrium bond price function (mapping the state of the economy into bond prices) and borrowing capacity is uniquely determined in this case. However, this does not imply that the equilibrium is unique. Multiplicity still arises in this case from what we call a *Laffer curve effect:* revenue from a bond auction is non-monotone in the amount of bonds issued. Since the borrower needs to target a given level of revenue, there are multiple bond prices consistent with an equilibrium.

We then turn to a model with a flexible debt maturity. Interestingly, with long-term bonds, the price function and borrowing capacity are no longer uniquely determined, instead, lower bond prices in the future feed back into current bond prices. This highlights an intertemporal coordination problem among investors, since they must now worry about future market conditions. By implication, even if all current investors were gathered in a room, in an effort to coordinate their actions, this will not prevent the bad equilibrium.

Along a bad equilibrium the government faces higher interest rates, leading to increased debt accumulation. This raises the probability of insolvency and default, ultimately justifying investors' demands for a higher interest rates. We call such self-fulfilling high interest rate equilibria "slow moving crises" to capture the fact that it develops over time through the accumulation of debt. The label helps distinguishes this type of crisis from rollover debt crises, which have been extensively studied in the literature, starting with Giavazzi and Pagano (1989), Alesina et al. (1992), Cole and Kehoe (2000) and more recently in Conesa and Kehoe (2012) and Aguiar et al. (2013). A rollover crisis is essentially a "run" on the borrowing government by current investors, who pull out of the market entirely, leading to a failed bond auction and triggering immediate default.[4] We

[4]Chamon (2007) argues that the coordination problem leading to a "run" could be prevented in practice by the manner in which bonds are underwritten and offered for purchase to investors by investment banks.

see rollover crises and slow moving crises as complementary ingredients to interpret turbulence in sovereign debt markets.[5] Indeed, rollover crises are also possible in our model, but for most of the paper we leave them aside to focus on slow moving crises.

How can slow moving crises be avoided? We identify a safe region of initial conditions and parameters for which the equilibrium is unique. The equilibrium is unique whenever debt is low enough. This result is intuitive since low debt mutes the feedback from interest rates to the cost of debt service. The equilibrium is also unique for fiscal rules that actively respond by reducing deficits when debt rises. This responsiveness directly counters the feedback effect from rising debt. Finally, longer debt maturity also helps guarantee a unique equilibrium. Shorter maturities require greater refinancing which potentiates the effects of high interest rates on debt accumulation.

A noteworthy feature of slow moving crises is that the existence of both a good and bad equilibrium may be transitory, in the sense that, if one goes down the path of a bad equilibrium for a sufficiently long time, debt may reach a level at which there exists a unique continuation equilibrium with high interest rates; the bad equilibrium may set in. Although the debt crises is initially triggered by self-fulfilling pessimistic expectations, the government becomes trapped into a bad outcome, due to the poor fundamentals it develops. The government may blame the vagaries of the market for some time, but eventually the market's mistreatment does real and irreparable damage. By implication, policy attempts to rule out bad equilibria must be put in place swiftly to avoid going down a bad equilibrium's detrimental path for too long.

Relative to the most recent literature on sovereign debt crises, which builds on the formalizations in Eaton and Gersovitz (1981) and Cole and Kehoe (2000), our approach differs in two ways.

The first difference is that for most of the analysis we model the government as following a fiscal rule, rather than model it as an optimizing agent. This modeling choice is not essential for the emergence of slow moving crises (as we show in Section 5) but we think it is useful for several reasons. First, it allows us to focus on the coordination problem between investors, since, as we show, multiple equilibria may arise even when the government is not strategic. Second, fiscal policy rules allow us to consider situations with partial commitment. For example, a government may promise efforts to increase surpluses when debt rises, but to a limited degree due to political economy constraints. The end-product of these considerations may be embedded in the fiscal policy rule. Un-

[5] Failed tesobonos auctions during Mexico's 1994 crises provided a motivation for the rollover crises literature. In the recent case of Italy, on the other hand, market participants seem clearly worried about adverse debt dynamics and bond yield, suggesting the forces at work in a slow moving crisis.

derstanding the positive implications of different policy rules constitutes an important first step towards a normative analysis. Third, although making fiscal policy endogenous is desirable, it may be difficult to capture in stylized optimization problems a number of constraints and biases coming from the political process. Finally, at a more practical level, fiscal policy rules seem descriptive of the debt-sustainability forecasting models actually used by market participants; we also hope to provide a bridge to the academic literature estimating fiscal rules.

The more fundamental difference with most of the current literature is in our timing assumptions in the debt market. The typical way sovereign-debt models are set up assumes borrowers can commit, within a period, to the amount of bonds issued, in keeping with the convention introduced by Eaton and Gersovitz (1981). As a side effect, this rules out slow moving crises, because it allows the borrower to always select the path of lower debt accumulation. Instead, we follow Calvo (1988) and assume that the government's resource needs from the financial market are determined first and that it then adjusts bond issuances to meet these financing needs, given market prices. This makes the size of the issuance endogenous to bond prices. As we show, this timing assumption is crucial to capturing slow moving crises, both in models with fiscal rules and in optimizing models.

At first glance, it may appear intuitively reasonable to adopt the standard timing assumption, letting borrowers commit to the amount of bonds issued. Certainly in the very short run, say, during any given market transaction or offer, the issuer is able to commit to a given bond size offering. However, this is not the relevant time frame. To see why, consider a borrower showing up to market with some given amount of bonds to sell. If the price turns out to be lower than expected, the borrower may quickly return to offer additional bonds for sale in order to make up the difference in funding. The important point here is that the overall size of the bond issuance remains endogenous to the bond price.

In Section 5 we provide a microfoundation for our timing assumption, by studying two explicit game-theoretic models of an optimizing government issuing bonds in repeated rounds. The first game is cast in continuous time and assumes a potentially unlimited number of bond auctions within each "period". The government loses all ability to commit to its bond issuance, since it can always reverse or increase issuances in the next round. The subgame-perfect equilibrium of this game provides an explicit microfoundation for the timing assumption used in the rest of the paper and shows how our approach can be extended to models with an optimizing government. The second game, set in discrete time, features only a finite number of periods and rounds, but introduces preferences that are not additively separable. Lower funds acquired in the market today increase the desire or need for funds tomorrow. This seems directly relevant for such things as infrastructure spending, but it may also capture elements of the financing of payroll, where a temporary shortfall in payments may be possible but must be repaid eventually. For this game, we show that there may be multiple subgame-perfect equilibria with different bond prices, similar to the equilibria we isolate in this paper. The second game shows that our form of multiplicity can arise even in environments where the government has some ability to commit to bond issuances, as long as one captures intertemporal linkages.

2 Solvency, Default and Debt Dynamics

In this section we introduce the basic sovereign debt model that we build on in later sections. We start by assuming that all borrowing is short term, that the primary surplus is completely exogenous and that there is zero recovery after default. All these assumptions are relaxed later.

2.1 Borrowers and Investors

Time is discrete with periods $t = 1, 2, \ldots, T$. A finite horizon is not crucial, but makes arguments simpler and ensures that multiplicity is not driven by an infinite horizon.

Government. The government generates a sequence of primary fiscal surpluses $\{s_t\}$, representing total taxes collected minus total outlays on government purchases and transfers (s_t is negative in the case of a deficit). We take the stochastic process $\{s_t\}$ as exogenously given and assume it is bounded above by $\bar{s} < \infty$. Let $s^t = (s_1, s_2, \ldots, s_t)$ denote a history up to period t. In period t, s_t is drawn from a continuous c.d.f. $F\left(s_t | s^{t-1}\right)$.[6]

The government attempts to finance $\{s_t\}$ by selling non-contingent debt to a continuum of investors in competitive credit markets. Absent default, the government budget constraint in period $t < T$ is

$$q_t(s^t) \cdot b_{t+1}(s^t) = b_t(s^{t-1}) - s_t, \tag{1}$$

where b_t represents debt due in period t and q_t is the price of a bond issued at t that is

[6] In applications it will be convenient to make the Markov assumption and write $F(s_t \mid s_{t-1})$, but at this point nothing is gained by this restriction.

due at $t+1$. In the last period, $b_{T+1}(s^T) = 0$ and avoiding default requires

$$s_T \geq b_T(s^{T-1}).$$

We write this last period constraint as an inequality, instead of an equality, to allow larger surpluses than those needed to service the debt. Of course, the resulting slack would be redirected towards lower taxes or increased spending and transfers, but we abstain from describing such a process.[7]

We assume that the government honors its debts whenever possible, so that default occurs only if the surplus and potential borrowing are insufficient to refinance outstanding debt. For now, we assume that if a default does occur bond holders lose everything; this assumption will be relaxed later. Let $\chi(s^t) = 1$ denote full repayment and $\chi(s^t) = 0$ denote default.

Our focus is on the debt dynamics during normal times or during crises leading up to a default. Consequently, we characterize the outcome up to the first default episode and abstain from describing the post-default outcomes. Specifically, for any realization of surpluses $\{s_t\}_{t=0}^T$ we only specify the outcome for debt and prices $b_{t+1}(s^t)$ and $q_t(s^t)$ in period t if $\chi(s^\tau) = 1$ for all $\tau \leq t$.[8] Similarly, one can interpret s_t as the surplus in periods t prior to default; default may alter future surpluses, but we need not model this fact to solve for the evolution of debt before default.[9]

Investors and Bond Prices. Each period there is a group of wealthy risk-neutral investors that compete in the credit market and ensure that the equilibrium price of a short term debt equals

$$q_t(s^t) = \beta \mathbb{E}[\chi_{t+1}(s^{t+1}) \mid s^t].$$

2.2 Equilibrium in Debt Markets

An equilibrium specifies $\{b_{t+1}(s^t), q_t(s^t), \chi_t(s^t)\}$ such that for all histories s^t with no current or prior default—i.e., with $\chi_\tau(s^\tau) = 0$ for all s^τ prior or equal to s^t—the government budget constraint (1) holds and the bond price satisfies $q_t(s^t) = \beta \mathbb{E}[\chi_{t+1}(s^{t+1}) \mid s^t]$.

[7]It is best not to interpret the finite horizon literally. One can imagine, instead, that the last "period" T represents an infinite continuation of periods. As long as all uncertainty is realized by T one can collapse the remaining periods from T onwards into the last period.

[8]This is possible because we abstract from modeling government welfare. In models where default is the result of an optimizing government, future variables enter its decision.

[9]Perhaps default alters future primary surpluses—for example, if creditors punish debtors or if defaulting debtors adjust taxes and spending to the new financial circumstances.

In addition, default occurs only when inevitable, a notion formalized by the following backward-induction argument.

In the last period the government repays if and only if $s_T \geq b_T$. The price of debt equals

$$q_{T-1} = \beta\left(1 - F\left(b_T|s^{T-1}\right)\right) \equiv Q_{T-1}(b_T, s^{T-1}).$$

Define the *maximal debt capacity* by[10]

$$m_{T-1}(s^{T-1}) \equiv \max_{b'} Q_{T-1}(b', s^{T-1})b',$$

where b' represents next period's debt, b_T in this case.

The government seeks to finance $b_{T-1} - s_{T-1}$ in period $T-1$ by accessing the bond market. This is possible if and only if

$$b_{T-1} - s_{T-1} \leq m_{T-1}(s^{T-1}).$$

We assume that whenever this condition is met the government does indeed manage to finance its needs and avoid default; otherwise, when $b_{T-1} - s_{T-1} > m_{T-1}(s^{T-1})$, the government defaults on its debt.

Turning to period $T-2$, investors anticipate that the government will default in the next period whenever $s_{T-1} < b_{T-1} - m_{T-1}(s^{T-1})$. Thus, the bond price equals

$$q_{T-2} = \beta \Pr\left(s_{T-1} \geq b_{T-1} - m_{T-1}(s^{T-1})|s^{T-2}\right) \equiv Q_{T-2}(b_{T-1}, s^{T-2}).$$

The maximal debt capacity in period $T-2$ is then

$$m_{T-2}(s^{T-2}) \equiv \max_{b'} Q_{T-2}(b', s^{T-2})b'.$$

Again, default is avoided if and only if $b_{T-2} - s_{T-2} \leq m_{T-2}(s^{T-2})$. The probability of this event determines bond prices in period $T-3$.

Continuing in this way we can solve for the debt limits and price functions in all earlier periods by the recursion

$$m_t(s^t) = \max_{b'} \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) \mid s^t\right) \cdot b'$$

[10]The maximum is well defined because the function involved is continuous and we can restrict the maximization to $0 \leq b \leq \bar{s}$, since $b < 0$ yields negative values and $b > \bar{s}$ yields zero.

and the associated price functions

$$Q_t(b', s^t) \equiv \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) \mid s^t\right).$$

Returning to the conditions for an equilibrium sequence $\{b_{t+1}(s^t), q_t(s^t), \chi_t(s^t)\}$, we require that for all histories s^t where $b_t(s^{t-1}) - s_t \leq m_t(s^t)$ that $\chi_t(s^t) = 1$ and $b_{t+1}(s^t)$ solve

$$Q_t(b_{t+1}(s^t), s^t) \cdot b_{t+1}(s^t) = b_t(s^{t-1}) - s_t. \tag{2}$$

Interestingly, both the maximal debt capacity function $\{m\}$ and the price functions $\{Q\}$ are uniquely determined. As we show next, this does not imply that the equilibrium path for debt is unique.

3 Self-Fulfilling Debt Crisis

In this section we study the model laid out in the previous section. We first show that there are multiple equilibria, with different self-fulfilling interest rates and debt dynamics. We then extend the model by including a recovery value and by allowing surpluses to react to debt levels.

3.1 Multiple Equilibria in the Basic Model

Define the correspondence

$$B_t(b, s^t) = \{b' \mid Q_t(b', s^t)b' = b - s_t\}.$$

Note that $B_t(b, s^t)$ is nonempty for all $b \leq m_t\left(s^t\right) + s_t$ and empty for $b > m_t\left(s^t\right) + s_t$. When $b < m_t\left(s^t\right) + s_t$ the set $B_t(b, s^t)$ contains at least two values, since $Q_t(b', s^t)b'$ attains a strictly positive maximum $m_t\left(s^t\right)$ for some $b' \in [0, T\bar{s}]$ and $Q_t(b', s^t)b' \to 0$ as both $b' \to 0$ and $b' \to \infty$.

Define the policy function with the lowest debt

$$\underline{B}_t(b, s^t) = \min B_t(b, s^t),$$

and let $\{\underline{b}_{t+1}(s^t)\}$ to be the path generated by

$$\underline{b}_t(s^{t-1}) = \underline{B}_t(\underline{b}_{t+1}(s^t), s^t).$$

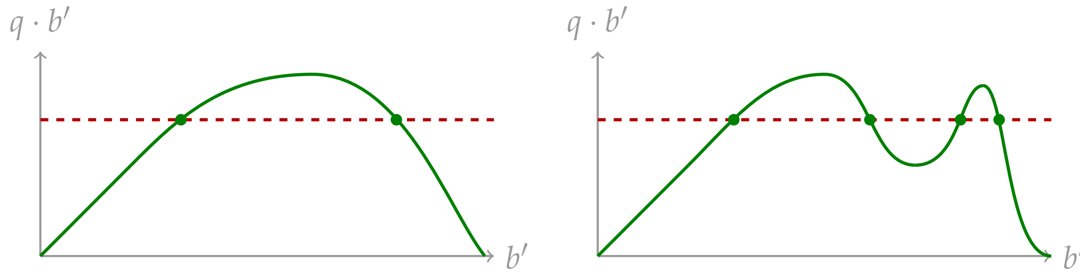


Figure 1: The revenue function without recovery value. Left panel shows a case with 2 equilibria; right panel shows a case with 4 equilibria.

Eaton and Gersovitz (1981) and much of the subsequent literature on sovereign debt proceeds by selecting this low debt equilibrium outcome. Here we are concerned with the possibility of other outcomes with higher debt.

Proposition 1 (Multiplicity)**.** *Any sequence for debt $\{b_{t+1}\}$ satisfying*

$$b_{t+1}(s^t) \in B_t(b(s^{t-1}), s^t)$$

until $B_t(b(s^{t-1}), s^t)$ is empty is part of an equilibrium. If $b_1 - s_1 < m_1(s_1)$ or $T \geq 3$ then there are at least two equilibrium paths. In any equilibrium

$$b_{t+1}(s^t) \geq \underline{b}_{t+1}(s^t) \qquad \text{for all } s^t.$$

Figure 1 plots two possibilities for the revenue from issuing bonds $Q_t(b', s^t)b'$ as a function of b'. This function achieves a maximum at an interior debt level because higher debt increases the probability of default, which destroys bond holder value. We refer to this curve as the Laffer curve for debt issuance. The left panel shows a case with a unique local maximum; the right panel shows a case with several local maxima.

The government needs to finance $b_t - s_t$. In the figure, this level is represented by the dashed horizontal line. In the case depicted in the left panel, there are two values of b_{t+1} that raise the needed revenue. In the case depicted in the right panel, there are four solutions. Notice that the bond price q_t corresponds to the slope of a line going through the origin and the equilibrium point (not drawn in the figure), so larger values of b_{t+1} correspond to lower prices and thus higher interest rates.

It seems reasonable to rule out equilibria for which the Laffer curve is locally decreasing. These equilibria are "unstable" in the Walrasian sense that any small increase in the price of bonds would reduce the supply of bonds issued by the government and increase the demand by investors (to infinity, since investors are risk neutral). These equilibria are also unlikely to be stable with respect to most formalizations of learning dynamics. Moreover, Frankel, Morris and Pauzner (2003) show that global games would not select such equilibria. Finally, these equilibria lead to counterintuitive comparative statics. For example, near an unstable equilibrium, an increase in the current debt level b_t lowers the equilibrium interest rate, i.e. it increases q_t.

Adopting the stability criterion just discussed, the panel on the left features a unique equilibrium, while the panel on the right features two equilibria. In the second case, the high-debt equilibrium is sustained by a higher interest rate that is self fulfilling: a lower bond price forces the government to sell more bonds to meet its financial obligations; this higher debt leads to a higher probability of default in the future, lowering the price of the bond and justifying the pessimistic outlook. This two-way feedback between high interest rates and debt sustains multiple equilibria.

We can then study how the model parameters and initial conditions affect multiplicity. In particular, the presence of multiple equilibria depends on the initial debt level b_t. Looking at the right panel of Figure 1, a reduction in the initial debt level b_t shifts the dashed red line downwards, eliminating the bad (stable) equilibrium.

In the model with only short term bonds, multiple stable equilibria can only be obtained with primitives that produce a non-single-peaked Laffer curve. As we shall see, in the model with long term debt this is no longer the case and multiple stable equilibria are possible also with a single-peaked Laffer curve.

Prior to default an equilibrium makes a selection from the correspondence B_t in each period. The entire set of equilibria is generated by considering all the permutations of these selections for $t = 1, 2, \ldots, T-1$. Note that the current period's correspondence B_t, maximum debt capacity $m_t(s^t)$, and the Laffer curve $Q_t(b', s^t)b'$ are all independent of the equilibrium that is selected in past or future periods. This implies that expectations of a bad equilibrium arising in the future has no consequence on the ability of the government to raise funds today. As we shall see, this property rests on the assumption of short term debt and no longer holds in Section 4 when we introduce longer term debt. However, even in the setting with short-term debt, past interest rates have an effect on current interest rates through inherited debt. Thus, if the the bad equilibrium interest rate was selected yesterday this raises the interest rate today, even if the good equilibrium is being played today.

3.2 Discussion and Relation to the Literature

The presence of a two-way feedback between higher interest rates and lower probability of repayment is reminiscent of Calvo (1988), who analyzed this feedback in a two-period model of an optimizing government who can choose partial default and faces convex costs of taxation. In particular, Calvo adopted a "price taking" assumption similar to the one made here, in which the government chooses the amount of financing it needs (in our case this is just given mechanically by the fiscal rule) and can end up financing it at different equilibrium prices, depending on market's expectations. However, most of the sovereign debt literature, such as Eaton and Gersovitz (1981), Cole and Kehoe (2000) and Arellano (2008), set up the problem assuming that the government chooses the amount of bonds issued b_{t+1} each period, rather than the amount of financing it needs. This had the—perhaps unintended—consequence of allowing the government to select the low-debt equilibrium in Figure 1, ruling out the high-debt equilibrium. In Section 5, we formulate a game between the government and the investors that offers a microfoundation to our approach and we emphasize that the crucial assumption is an assumption about commitment on bond issuances in the near future.

The discussion of stability above raises a question. Consider the left panel of Figure 1. Suppose we start immediately to the right of the bad, unstable equilibrium. At that point, there is an excess supply of bonds as the government needs additional funds to satisfy its financing needs. So we can think the price will keep adjusting downwards, pushing us further to the right until we reach a level of bond issuance at which there is a zero probability of repayment. This suggests the presence of a third, stable equilibrium, with a zero bond price and default occurring in period t. To correctly formulate this possibility we need to relax the assumption made so far that default only happens when there is no b' that allows the government to finance $b_t - s_t$. Allowing for this possibility means introducing the possibility of rollover crises, that is an event in which bad expectations by investors lead to immediate default *today* and where default today implies zero expected repayment at all future dates. This is precisely what happens in Cole and Kehoe (2000), where a bad equilibrium corresponds to a situation in which at zero bond prices the government prefers to default today. In the context of our model, in which s_t is not a choice, such a rollover crisis can only be avoided when $b_t < s_t$, i.e. when the government can fully repay its debt using the current surplus. In the next subsection we introduce a recovery value for debt in the event of default. In that case, the conditions for a run equiliria to exists are more restrictive. For the remainder of the paper, we leave aside the possibility of rollover crises, to concentrate on slow moving crises and simplify the exposition.

3.3 Recovery Value

We now generalize the model by adding a recovery value for debt. We assume that if the government defaults debtors seize a fraction $\phi \in [0, 1)$ of the available surplus, so that

$$Q_{T-1}(b_T, s^{T-1}) = \beta \left(1 - F\left(b_T | s^{T-1}\right)\right) + \frac{\beta}{b_T}\phi \int_0^{b_T} s_T \, dF\left(s_T | s^{T-1}\right)$$

Defining the revenue function

$$G(b_T, s^{T-1}) \equiv Q_{T-1}(b_T, s^{T-1}) b_T = \beta \left(1 - F\left(b_T | s^{T-1}\right)\right) b_T + \beta\phi \int_0^{b_T} s_T \, dF\left(s_T | s^{T-1}\right)$$

note that

$$\frac{\partial}{\partial b_T} G(b_T, s^{T-1}) = \beta \left(1 - F\left(b_T | s^{T-1}\right)\right) - \beta(1-\phi) f\left(b_T | s^{T-1}\right) b_T$$

may be positive or negative. However,

$$\lim_{b_T \to \infty} G(b_T, s^{T-1}) = \beta \mathbb{E}[s_T | s^{T-1}] > 0,$$

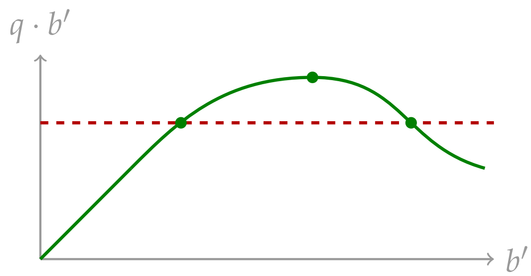
implying that there is a region of low current debt with a unique equilibrium. The same is true in earlier periods.

Proposition 2. *Suppose the recovery value from default is positive, $\phi > 0$. Given any history s^t, then* for low enough current debt $b_t(s^{t-1})$ there exists a unique value for $b_{t+1}(s^t)$ satisfying

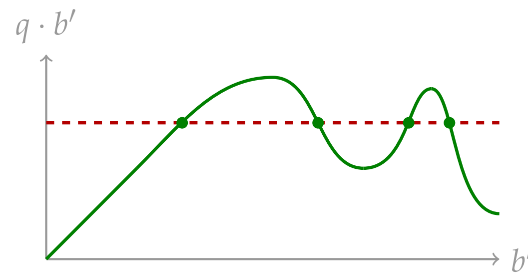
$$Q_t(b_{t+1}(s^t), s^t) \cdot b_{t+1}(s^t) = b_t(s^{t-1}) - s_t.$$

Multiple solutions may exist for high enough levels of current debt $b_t(s^{t-1})$.

Figure 2 illustrates the situation. In both panels, for high debt there may still be multiple equilibria, but for sufficiently small debt only the good side of the Laffer curve is available. Once again, two panels are displayed. In the first, the Laffer curve is single peaked, and in the the second panel, the Laffer curve has multiple peaks. The important point is that, in both cases, for low enough debt levels of $b_t(s^{t-1}) - s_t$, there exists a unique equilibrium—even without invoking a refinement based on stability.



(a) A case with 2 equilibria.



(b) A case with 4 equilibria.

Figure 2: The revenue function with recovery value displaying two equilibrium points.

3.4 Fiscal Rules

When debt is high, governments tend to make efforts to increase surpluses in order to stabilize debt. To capture this we make surpluses partially endogenous, by assuming a dependence with the current debt level.

The distribution of fiscal surplus now depends on the current level of debt, in addition to the past history,

$$s_t \sim F(s_t \mid s^{t-1}, b_t).$$

Fiscal policy rules of this kind are commonly adopted in the literature studying solvency (e.g. Bohn, 2005; Ghosh et al., 2011) as well as the literature studying the interaction of monetary and fiscal policy (e.g. Leeper, 1991).

The recursion defining an equilibrium is similar

$$m_t(s^t) = \max_{b'} \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) | s^t, b'\right) b',$$

$$Q_t(b', s^t) \equiv \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) | s^t, b'\right).$$

Fiscal rules may have an important impact on debt limits $m_t(s^t)$ as well as on the existence of multiple equilibria. Rather than explore this idea in the present context, we will do so in the model with long-term bonds in Section 4.

4 Long Term Debt and Slow Moving Crises

We now generalize the model to allow for bonds of longer maturity. This is important for a number of reasons. First of all, short term debt is not a realistic assumption for most advanced economies (e.g. Arellano and Ramanarayanan, 2012). For example, the average maturity from 2000-2009 for Greece, Spain, Portugal and Italy was 5-7 years. Second, a common concern with short term debt is that it makes the government more susceptible to debt crises. Cole and Kehoe (1996) discuss this idea, in the context of roll-over runs. Since the source of multiplicity is different in our model, it is of interest to understand whether long term debt reduces the potential for multiplicity. Third, as we showed, in our model with short term debt the current equilibria are unaffected by the selection of future equilibria. Thus, the expectation of a bad equilibrium being selected in the future does nothing to current borrowing limits or interest rates. We shall see that this conclusion is special to the short term debt assumption. Finally, long-term debt creates the possibility of multiple stable equilibria for a different reason than what was discussed in the case of short term debt.

4.1 Adding Long Term Bonds to the Basic Model

We assume that the government issues bonds with geometrically decreasing coupons: a bond issued at t promises to pay a sequence of coupons $\kappa, (1-\delta)\kappa, (1-\delta)^2\kappa, \ldots$ where $\delta \in (0,1)$ and $\kappa > 0$ are fixed parameters. Of course, these payments are made only in the absence of default. This well-known formulation of long-term bonds is useful because it avoids having to carry the entire distribution of past vintages of long-term bonds (see Hatchondo and Martinez, 2009). A bond of this kind issued at time $t-j$ is equivalent to $(1-\delta)^j$ bonds issued at time t. As a result, there is a unique state variable for the entire distribution of past bonds; likewise, we need only keep track of one (normalized) price.

The entire issuance of past bonds can be summarized by the level of current bond equivalents which we denote by b_t with budget constraint

$$q_t(s^t) \cdot \left(b_{t+1}(s^t) - (1-\delta)\, b_t(s^{t-1})\right) = \kappa b_t(s^{t-1}) - s_t.$$

One can interpret this as follows. Current bond equivalents pay a coupon κ but depreciate at rate δ. As a result, if $b_{t+1}(s^t) = (1-\delta)\, b_t(s^{t-1})$ this corresponds to the situation where no new bond issuances are taking place.

We assume that the current surplus is affected by last period's surplus and the level of current debt

$$s_t \sim F(s_t \mid s_{t-1}, b_t);$$

to simplify, this drops the potential dependence on the past history s^{t-2}.

We allow for some positive recovery in the event of default. Namely, we assume that if default occurs the total value of debt is negotiated down to a recovery value $v\,(s_t)$. The pricing condition now takes the form

$$q_t = \beta E_t \left[\kappa + (1-\delta)\, q_{t+1} | \text{No default at } t+1\right] \Pr\left[\text{No default at } t+1\right] + \frac{1}{b_{t+1}} E_t \left[v\left(s_{t+1}\right) | \text{Default at } t+1\right] \Pr\left[\text{Default at } t+1\right].$$

The second term is divided by b_{t+1}, reflecting the fact that each bond holder gets a proportion of the recovery value $v(s_{t+1})$.

Since bonds are eternal, we cannot assume a finite horizon. Instead, we assume that the horizon is infinite, but that all uncertainty is resolved after a finite horizon T: in all periods $t \geq T$ the surplus is constant at the value s_T. This effectively allows us to start our analysis at time T and solve for an equilibrium backwards, as before.

At date T if default is avoided the price of long-term bonds equals

$$q^* \equiv \frac{\beta\kappa}{1-\beta(1-\delta)} = 1,$$

where we have adopted the normalization $\kappa = 1/\beta - 1 + \delta$ to ensure that $q^* = 1$.

From period T onwards, the country is able to repay the coupons due and keep debt constant whenever

$$s_T \geq \kappa b_T - \delta b_T = r b_T,$$

where we have defined $r \equiv \kappa - \delta = \frac{1}{\beta} - 1$.[11] In period $T-1$, the price of long term bonds is then

$$Q_{T-1}\left(b_T, s_{T-1}\right) = 1 - F(r b_T | s_{T-1}, b_T) + \frac{\beta}{b_T} \int_{-\infty}^{r b_T} v\left(s_T\right) dF\left(s_T | s_{T-1}, b_T\right).$$

Using this function we define the maximal revenue from debt issuance at $T-1$ by

$$m_{T-1}\left(b_{T-1}, s_{T-1}\right) \equiv \max_{b_T} \left\{Q_{T-1}\left(b_T, s_{T-1}\right)\left(b_T - (1-\delta) b_{T-1}\right)\right\}.$$

We assume that no default occurs at $T-1$ whenever the government needs to issue less than the maximal possible so that

$$b_{T-1} \leq m_{T-1}\left(b_{T-1}, s_{T-1}\right) + s_{T-1}.$$

Let R_{T-1} denote the subset of pairs (b_{T-1}, s_{T-1}) where this inequality holds, so that the

[11] Once again, the budget constraint is written as an inequality in the last period. Of course, if the inequality holds with slack we can interpret the true surplus as adjusting to reach equality.

government is able to meet its financial obligations; we assume that default occurs otherwise.

Unlike the case with short-term date, before proceeding to earlier periods we need to select an equilibrium at $T-1$, picking a value for b_T that satisfies

$$Q_{T-1}\left(b_T, s_{T-1}\right)\left(b_T-(1-\delta)\, b_{T-1}\right)=\kappa b_{T-1}-s_{T-1}, \tag{3}$$

for each b_{T-1} and s_{T-1}. Let $B_T\left(b_{T-1}, s_{T-1}\right)$ denote any such selection. The domain of the function B_T is precisely R_{T-1}, all situations where default is avoidable.

We now describe the recursion for earlier periods $t \leq T-2$. Given Q_{t+1}, m_{t+1}, R_{t+1}, B_{t+2}, we can compute the price

$$\begin{aligned} Q_t\left(b_{t+1}, s_t\right)=\beta \int_{R_{t+1}} & \left(\kappa+(1-\delta) Q_{t+1}\left(B_{t+2}\left(b_{t+1}, s_{t+1}\right), s_{t+1}\right)\right) d F\left(s_{t+1} \mid s_t, b_{t+1}\right) \\ & +\frac{\beta}{b_{t+1}} \int_{R_{t+1}^c} v\left(s_{t+1}\right) d F\left(s_{t+1} \mid s_t, b_{t+1}\right), \end{aligned}$$

the debt limit

$$m_t\left(b_t, s_t\right) \equiv \max_{b_{t+1}} Q_t\left(b_{t+1}, s_t\right) \cdot\left(b_{t+1}-(1-\delta) b_t\right)$$

the set $R_t=\left\{\left(b_t, s_t\right) \mid b_t \leq m_t(b_t, s_t)+s_t\right\}$ of repayment and a new selection $B_{t+1}(b_t, s_t)$ function defined over the domain R_t solving

$$Q_t\left(B_{t+1}(b_t, s_t), s_t\right) \cdot\left(B_{t+1}(b_t, s_t)-(1-\delta)\, b_t\right)=\kappa b_t-s_t .$$

Proceeding in the same way we can compute $\{Q_t, m_t, R_t, B_{t+1}\}$.

The dynamics for debt can now be computed by iterating

$$b_{t+1}(s^t)=B_{t+1}(b_t(s^{t-1}), s_t)$$

until

$$\left(b_t, s_t\right) \notin R_t$$

at which point default occurs.

The introduction of long-term bonds produces important differences with the model of Section 2. With long-term bonds it is no longer possible to define the maximal revenue m without having a rule for selecting equilibria in the future. A simple approach is to assume that whenever multiple solutions to (3) are possible, we select the one with the lowest level of b_t. But other selections are possible, leaving to different paths for the maximal debt revenue m. This means that by selecting equilibria in different ways, one obtains a range of maximal debt revenues. This also means that a country's debt capacity at time t is influenced by investors' expectations about the potential for multiple equilibria in the future.

Laffer Curves. When long-term bonds are present, we can distinguish two different types of coordination failure among investors. The first is the case in which the country could reduce the amount of bonds issued and still be able to cover its financing needs $\kappa b_t - s_t$, if all the investors who are purchasing bonds at date t bid a higher price for these bonds. This is the case in which the expression

$$Q_t\left(b_{t+1}, s_t\right) \cdot \left(b_{t+1} - (1-\delta)\, b_t\right) \tag{4}$$

is a decreasing function of b_{t+1} at $b_{t+1}(s^t)$. The second is the case in which all the investors who are purchasing bonds at date t and all the investors who purchased bonds in the past would get a higher expected repayment if they coordinated on reducing the face value of the debt b_{t+1}. This is the case in which the expression

$$Q_t\left(b_{t+1}, s_t\right) \cdot b_{t+1} \tag{5}$$

is a decreasing function of b_{t+1} at $b_{t+1}(s^t)$. We call the expression (4) the "issuance Laffer curve" and expression (5) the "stock Laffer curve". Notice that a country can very well be on the decreasing side of the stock Laffer curve and yet still be on the increasing side of the issuance Laffer curve.

4.2 An Application Motivated by Italy

We now study a continuous-time version of the model with long-term bonds, under a deterministic, linear fiscal rule. The adaptation to continuous time is convenient numerically, but is of no substantive consequence.

The first objective of this section is to illustrate the dynamics of a slow moving crisis with long-term bonds where multiplicity appears during the build-up phase of the crisis; there is a good equilibrium with a high price for the bond and a bad one with a low price. At some point in time the continuation equilibrium becomes unique: the bad equilibrium path features a high probability of default because of the high debt accumulated, but there is no other equilibrium. Likewise, along the good equilibrium path debt is low and eventually the only equilibrium features a low probability of default. The second objective is to show how the fiscal rule, the initial debt level, and debt maturity affect the presence of multiplicity.

Time is continuous. Investors are risk neutral and have discount factor r. Bonds issued at time t pay a coupon $\kappa e^{-\delta(\tau - t)}$ in each $\tau > t$, which is the continuous time equivalent of the long-term bonds introduced in the previous sections. Similarly, we assume $\kappa = r + \delta$, so the bond price under no default is equal to 1.

Between times 0 and T, the country surplus evolves deterministically following the differential equation

$$\dot{s} = -\lambda \left(s - \alpha \left(b - b^* \right) \right). \tag{6}$$

The country has some target debt level b^*, when current debt exceeds the target the country adjusts its fiscal surplus towards the value $\alpha \left(b - b^* \right)$. The speed of adjustment to the target surplus is determined by the parameter λ. A larger coefficient α implies a more aggressive response to high debt. After time T, the country's long-run surplus is constant at $s(t) = rS$, where S is the long-run present value of surplus which is drawn randomly at time T from a continuous distribution with c.d.f. $F(S)$.

At time T, if the stock of accumulated debt $b(T)$ is smaller than S there is no default and the bond price is 1. If $S < b(T)$, the bond holders share equally the recovery value ϕS, with $\phi < 1$. Therefore, the bond price immediately before the resolution of uncertainty at time T is given by

$$q(T) = 1 - F(b(T)) + \phi \int_0^{b(T)} \frac{S}{b(T)} dF(S). \tag{7}$$

We focus on cases in which default never occurs before time T. Therefore, the bond price satisfies the differential equation

$$(r + \delta) q = \kappa + \dot{q}, \tag{8}$$

and the government's budget constraint is

$$q \left(\dot{b} + \delta b \right) = \kappa b - s. \tag{9}$$

To characterize the equilibria, we proceed as follows. The initial values for the debt stock and for the surplus, $b(0)$ and $s(0)$, are given. Choosing an initial value $q(0)$ we can then solve forward the system of ODEs in s, q, b given by (6), (8) and (9) and find the terminal values $b(T)$ and $q(T)$. If these values satisfy (7) we have an equilibrium. It is convenient to represent this construction graphically in terms of two loci for the terminal value of debt $b(T)$ and the terminal value of debt $q(T) b(T)$. In Figure 3 we plot two curves. The curve with an interior maximum is a Laffer curve similar to the one analyzed in Section 3, showing the relation between $b(T)$ and $q(T)\,b(T)$ implied by the bond pricing equation (7), namely

$$q(T)\,b(T) = (1 - F(b(T)))b(T) + \phi \int_0^{b(T)} S\, dF(S). \tag{10}$$

The downward sloping curve plots the values of $b(T)$ and $q(T)\,b(T)$ that come from solving the ODEs (6), (8) and (9) for different values of the initial price $q(0)$. The curves are plotted for a numerical example with the following parameters:

$$T = 10, \quad \delta = 1/7, \quad r = 0.02, \quad \phi = 0.7, \quad \log S \sim N\left(0.3, 0.1^2\right).$$

Taking the time period as a year, we consider a country in which uncertainty will be resolved in 10 years and the average debt maturity is 7 years. The risk-free interest rate is 2% and the recovery rate in case of default is 70%. The distribution of the present value of surplus, after uncertainty is resolved has mean 1.357 and standard deviation 0.136. The initial conditions are

$$s(0) = -0.1, \quad b(0) = 1,$$

and the fiscal policy parameters are

$$\lambda = 1, \quad \alpha = 0.02, \quad b^* = 0.$$

Figure 3 shows the presence of three equilibria. Note that both the first and third equilibrium are "stable", under various notions of stability discussed earlier. Thus, the model with long-term debt features multiple stable equilibria even when the Laffer curve is singled peaked. Figure 4 shows the dynamics of the primary surplus, debt and bond prices for the two stable equilibria, which we term "good" (solid lines) and the "bad" (dashed lines).

The model captures various features of recent episodes of sovereign market turbulence. Sovereign bond spreads experience a sudden and unexpected jump, in moving from the good to the bad equilibrium. The debt-to-GDP ratio increases slowly but steadily. Auctions of new debt issues do not show particular signs of illiquidity, yet, interest rates climb along with the level of debt. Large differences in debt dynamics appears gradually, as bond prices diverge and a larger fraction of debt is issued at crisis prices.

A characteristic feature of a slow moving crisis is that multiplicity plays out in the early phase of a crisis. This is unlike the case of liquidity crises, where multiplicity in the

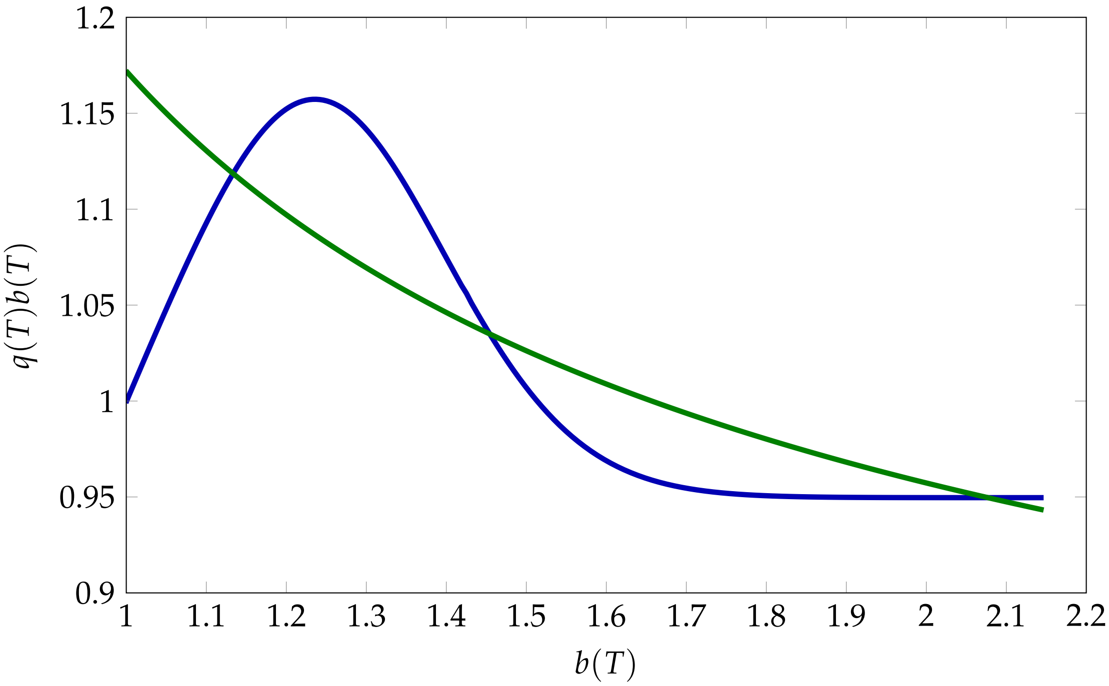


Figure 3: Three equilibria in example economy.

rollover crisis occurs in the terminal phase that ultimately triggers default. In our model, instead, along either equilibrium path, multiplicity eventually disappears.

Figure 5 illustrates this point. It overlays Figure 3, with four new dashed lines. Each dashed line corresponds to a different time horizon and initial debt condition. In particular, we plot them for $t = 1.2$, and $t = 2.9$ and use as initial conditions the values of $s(t)$ and $b(t)$ reached under the good and the bad equilibrium paths shown in Figure 4 (which coincide at $t = 0$). Notice that at $t = 1.2$ multiplicity is still present, so it is possible, for example, for the economy to follow the bad path between $t = 0$ and $t = 1.2$ and then to switch to a good path.[12] However, at $t = 2.9$ a switch is no longer possible. There are two reasons multiplicity disappears as we approach T. First, the remaining time horizon shrinks, leaving less time to accumulate or decumulate debt. Second, debt may have reached a high enough level to ensure the bad equilibrium, or viceversa.

Fiscal Rules. How does the fiscal policy rule affect the equilibrium or the existence of multiple equilibria? In Figure 6, we look at the effects of increasing α. To better illustrate the power of a more responsive fiscal policy, we adjust the debt target b^* so that for each of the three values of α the country reaches a good equilibrium with the same $q(T)$ and

[12]Clearly, the switch needs to be unexpected for prices to be in equilibrium between $t = 0$ and $t = 1.2$.

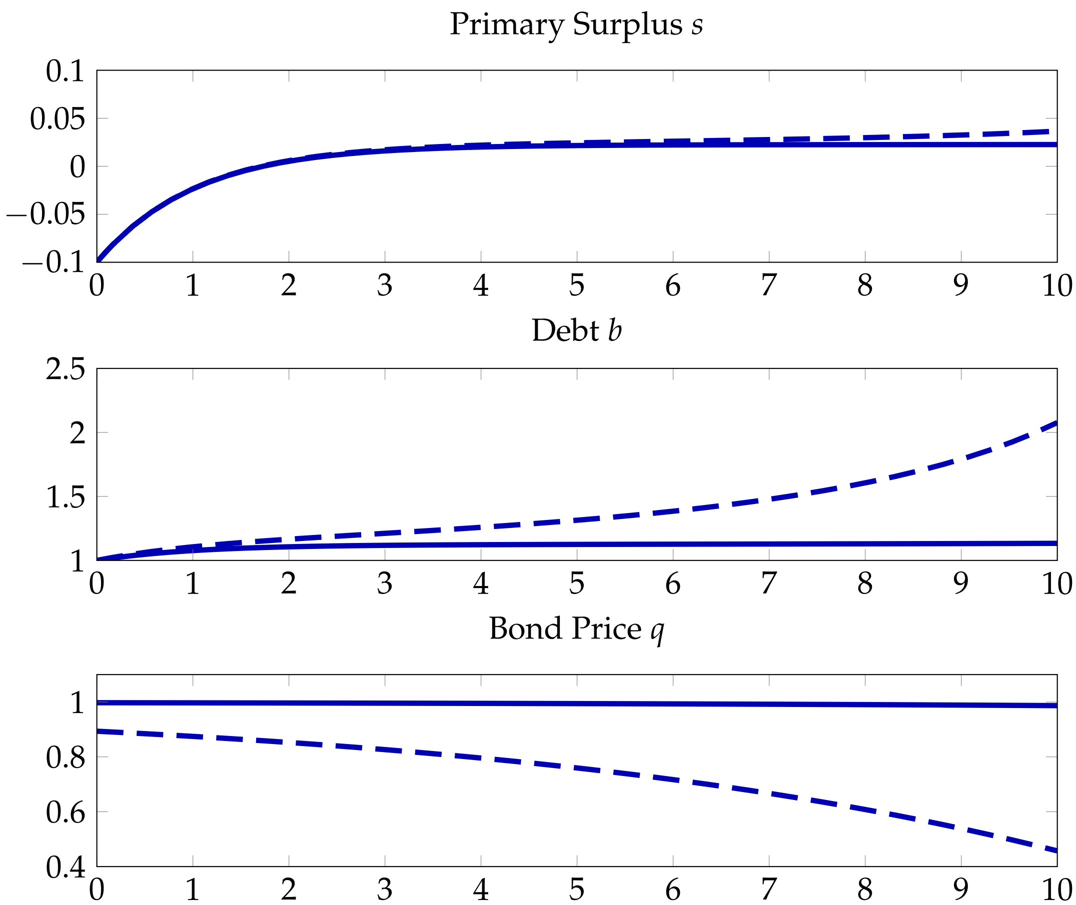


Figure 4: Dynamics of surplus, debt and bond price in good (solid line) and bad (dashed line) equilibrium.

$b(T)$. A sufficiently high value of α rules out the bad equilibrium, because as the investors contemplate the effect of lower bond prices they realize that the government would react more aggressively to a faster increase in b and thus eventually reach a lower level of $b(T)$.

A different way to look at policy rules is to ask how aggressive the rules need to be to make a given initial debt level immune to bad equilibria. In particular, in Figure 7 we look at the parameter space (α, b_0) and divide it into four regions, making no adjustments to b^*. In the red region there is a single equilibrium, in the bottom portion debt is low and on the good side of the Laffer curve, while in the upper portion (above pink region) the unique equilibrium lies on the bad side of the Laffer curve. There are three equilibria in the pink region, just as in our calibrated example. In the yellow region no equilibrium with debt exists, implying immediate default at $t = 0$.

Consider for example, the case $\alpha = 0.01$ in the graph, in which four cases are possible. For low levels of b_0, we get a unique equilibrium on the increasing portion of the Laffer

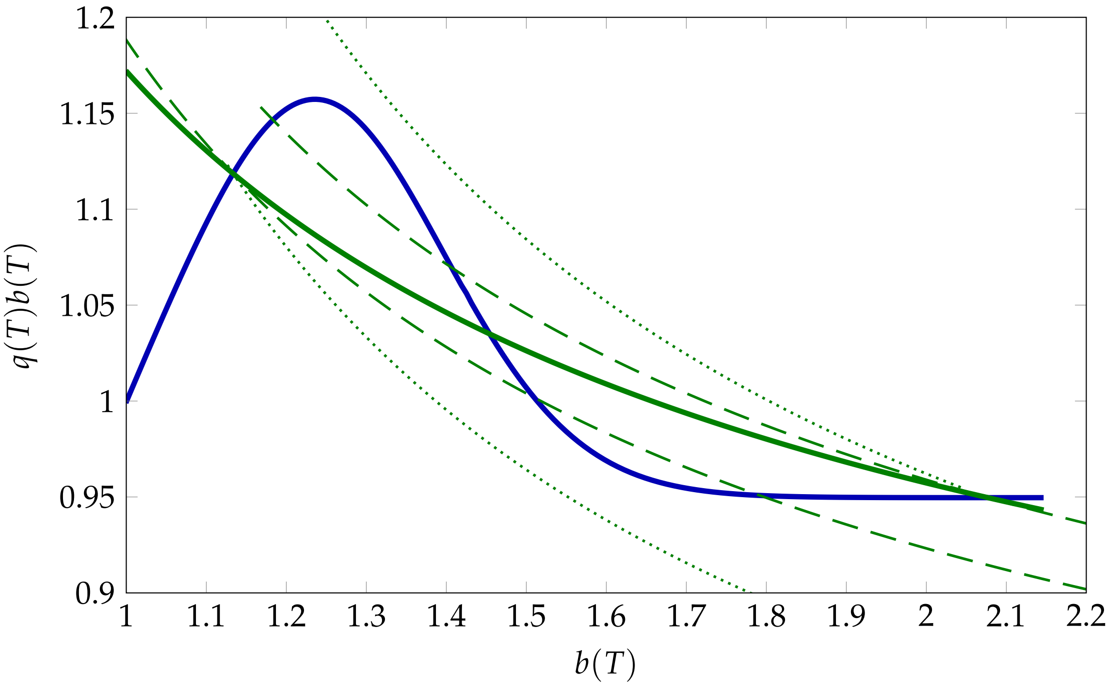


Figure 5: Solid green line shows $t = 0$, dashed green line $t = 1.2$, dotted green line $t = 2.9$.

curve (lower portion of the red region). For higher levels of b_0, we have three equilibria, as depicted in Figure 3 (pink region). For even higher levels of b_0, we have a unique equilibrium again, but this time on the decreasing portion of the Laffer curve. Finally, for very high values of b_0, there is no equilibrium without default.

Debt Maturity. Consider next the impact of debt maturity, captured by δ. Figure 8 shows the effects of varying δ around our benchmark value, while adjusting b^* to keep the same low-debt equilibrium. A longer maturity, with a low enough value for δ, leads to a unique equilibrium. Intuitively, shorter maturities require greater refinancing, increasing the exposure to self-fulfilling high interest rates. The debt burden of longer maturities, in contrast, is less sensitive to the interest rate.

Figure 9 is similar to Figure 7, but over the parameter space (δ, b_0) instead of (α, b_0). Again, we divide the figure into four regions. There are three equilibria in the pink region, just as in our calibrated example. In the red region there is a single equilibrium. In the bottom portion of the red region the equilibrium lies on the good side of the Laffer curve, while in the upper portion (above the pink region) it is on the bad side of the Laffer curve. In the yellow region no equilibrium exists, implying immediate default at $t = 0$.

In the figure, for given δ, the good equilibrium is unique for low enough levels of debt

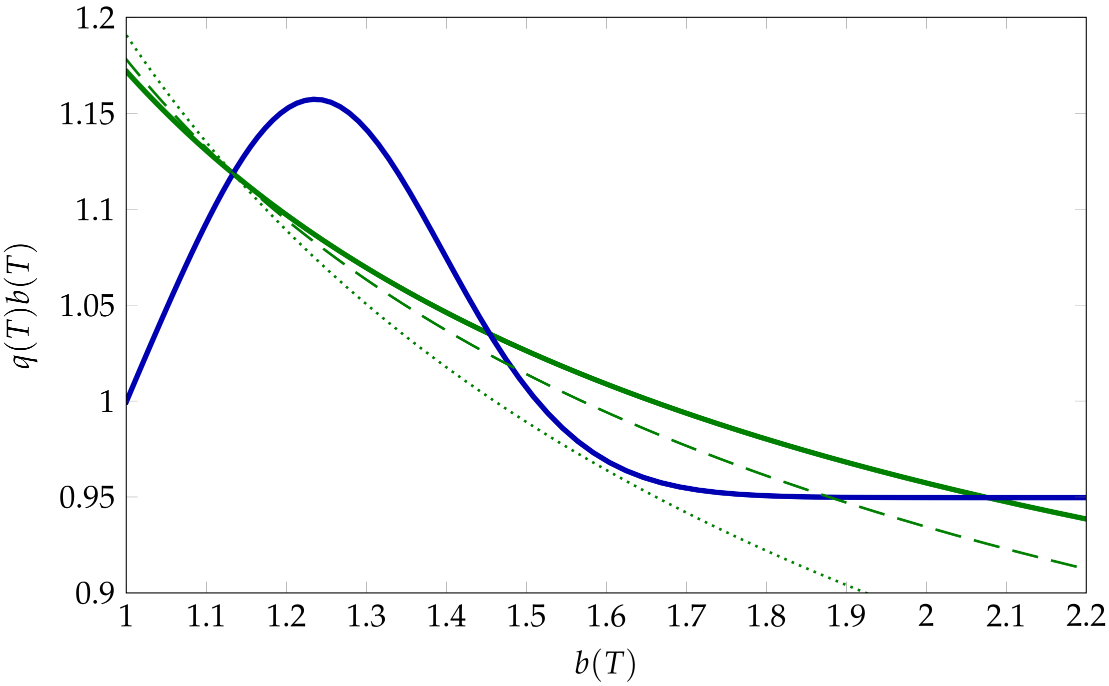


Figure 6: Solid green line $\alpha = 0.02$, dashed green line $\alpha = 0.03$, dotted green line $\alpha = 0.05$.

b_0. For a given initial debt b_0, a longer maturity for debt, a lower value for δ, also leads to a unique good equilibrium (lower red region). Shorter maturities, higher values for δ, may place the economy in the intermediate "danger zone" (pink region) with 3 equilibrium values for the interest rate. Still higher values for δ may lead to a unique bad equilibrium (upper red region) or to non-existence prompting immediate default (upper right, yellow region).

We conclude that according to Figure 7, shorter maturities place the borrower in danger: in some cases vulnerable to a possible bad equilibrium, in others certain of a bad equilibrium and in still others in an immediate situation of default.

Slow Moving Crises and Liquidity Crises. It is interesting to compare the slow moving crisis in our model to liquidity induced crises featured in Cole and Kehoe (2000) and related work, such as Cole and Kehoe (1996) and Conesa and Kehoe (2012). In these models, when debt is high enough borrowers become vulnerable to a run by investors, who may decide not to rollover debt, prompting default. If this run comes unexpectedly, there would be no rise in interest rates, just a sudden crisis, a zero price for bonds and default as in Giavazzi and Pagano (1989) and Alesina et al. (1992). Cole and Kehoe (2000) extended these models by studying sunspot equilibria with a constant arrival probability

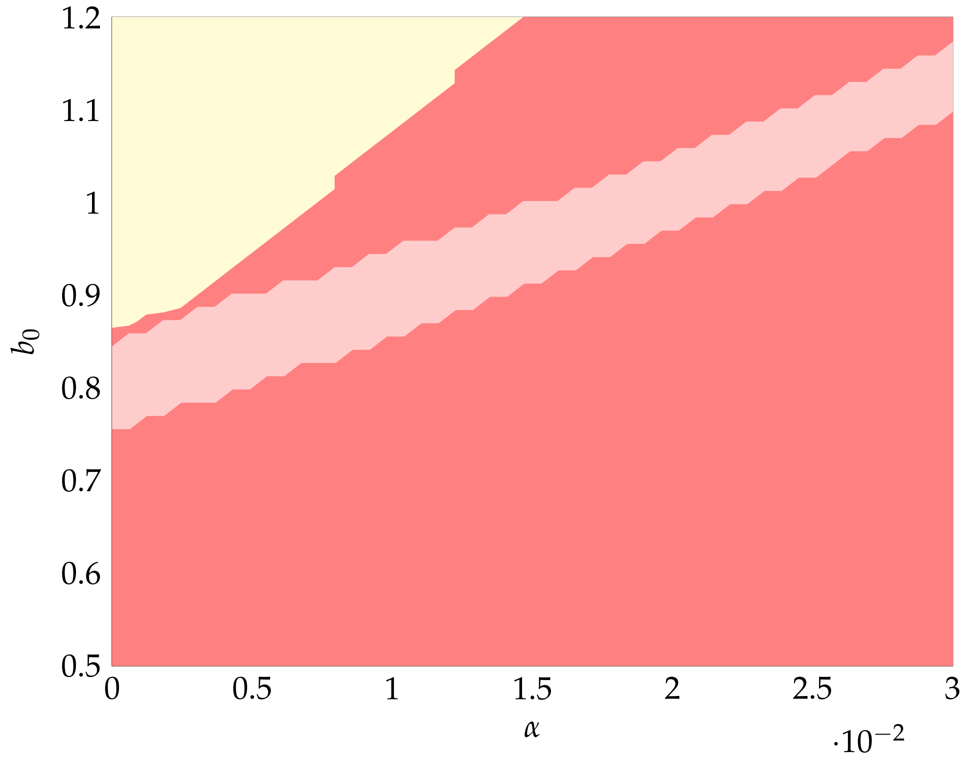


Figure 7: Regions with unique equilibrium (red), three equilibria (pink) or immediate default (yellow).

for the liquidity crisis. When this probability is not zero, the interest rate rises and the government makes an effort to reduce debt to a safe level that excludes investor runs and lowers the interest rate. Thus, high interest rates in liquidity crisis models may be present even with a decreasing path for debt.[13] In contrast, in our model debt rises along the bad, high interest equilibrium path. Indeed, the rising path for debt and higher interest rates are intimately related, the one implying the other.

Another interesting distinction is that the multiplicity from liquidity crises is broader and more pervasive than the multiplicity due to slow moving crises. In the example above, we found three equilibrium interest rates. However, only two of these can be considered part of a stable equilibrium. In contrast, liquidity crises open the door to a continuum of sunspot equilibria, indexed by the constant arrival probability of the run.

[13] Conesa and Kehoe (2012) extend liquidity crisis models to include uncertainty in income and find that debt may be increasing in some cases. Nevertheless, high interest rates are driven by the sunspot probability of a run, not by the accumulation of debt.

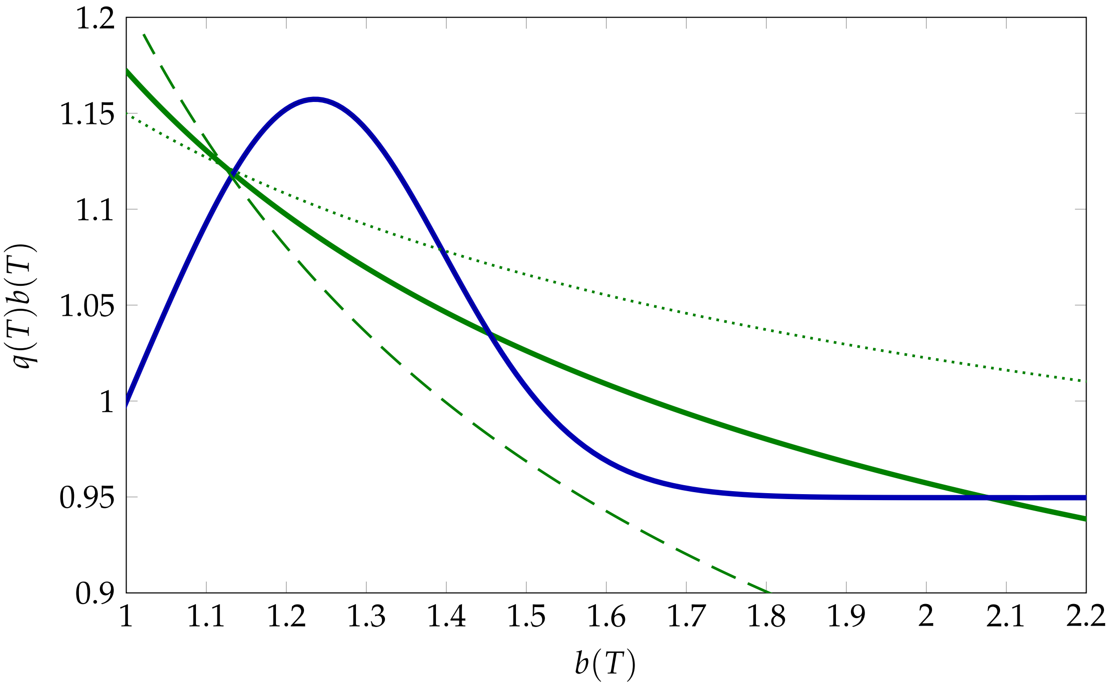


Figure 8: Solid green line $\delta = 1/7$, dashed green line $\delta = 1/10$, dotted green line $\delta = 1/5$.

5 Commitment and Multiplicity

In the previous sections, we have assumed that whenever the government budget constraint can be satisfied at multiple bond prices, all these prices constitute potential equilibria. That is, we have assumed that the government cannot commit to the amount of bonds issued in a given period. In this section, we consider a model in which the government *can* commit to bond issuance in the very short run and yet multiple equilibria arise. The idea is to split a period of the models in the previous sections into shorter subperiods and to assume that the government can only commit to bond issuances in a subperiod. For a concrete example, a period in the model of the previous sections could be interpreted as a month, in which the government borrowing needs are determined by fiscal policy decisions that adjust slowly, while the subperiods may be different days in which auctions of Treasury bonds can take place. The government can commit to sell a fixed amount of bonds in each auction, but cannot commit to run future auctions if it hasn't reached its objective in terms of resources raised.

Given the purposes of this section, we will work with fully specified games in which the government's behavior is derived from maximization.

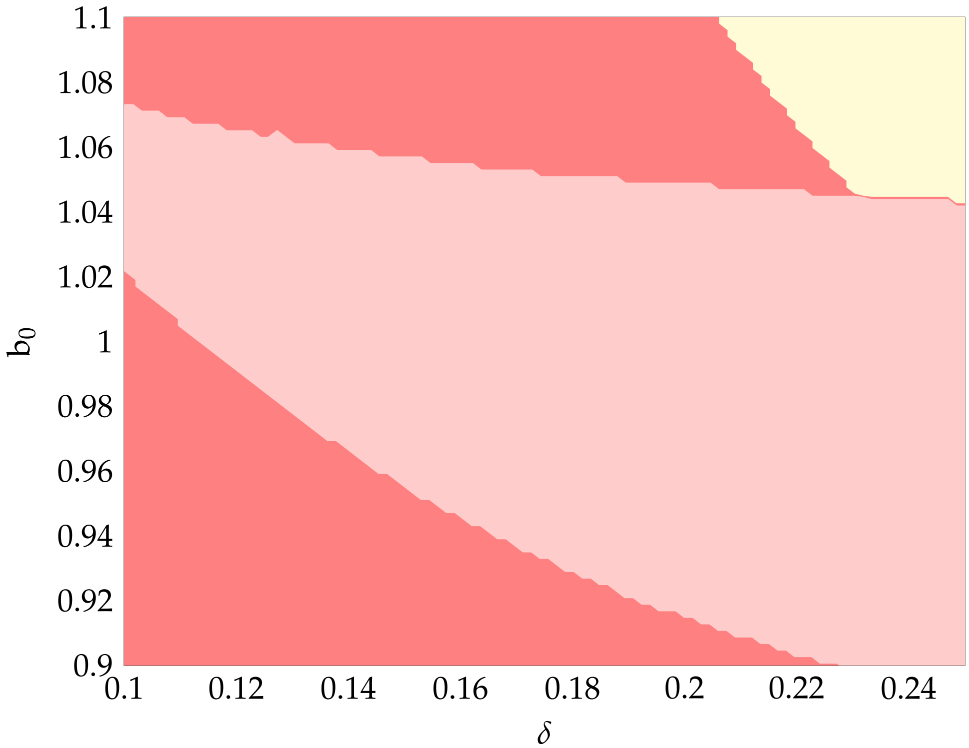


Figure 9: Regions with unique equilibrium (red), three equilibria (pink) or immediate default (yellow).

5.1 A Game with No Commitment

Consider a two-period model in which the government's objective function is

$$u(s) + \beta V(b'),$$

where s is current primary surplus and b' is the stock of bonds issued in the first period, to be repaid in the second period. Both u and V are decreasing, differentiable and concave functions. We could interpret u as the payoff resulting from a full specification of the benefits of public expenditure and the costs of taxation and V as as the expectation of a value function in an optimizing model with an infinite horizon.

The government also has a stock of bonds b inherited from the past that it needs to repay at the beginning of the first period. Thus, in the first period it must finance $b - s$ from outside investors.

There is a continuum of atomistic investors that are assumed to be risk neutral with discount factor β. Because of risk neutrality, and because we assume all bond holders are treated equally without seniority clauses, only the expected payment by the government need be specified to price bonds. If the total debt owed to investors equals b', then their total expected repayment is given by the function

$$G\left(b'\right).$$

Naturally, $G(0) = 0$. Each bond obtains an expected repayment equal to $G\left(b'\right)/b'$ and it is useful to think of $G\left(b'\right)/b'$ as a decreasing function, although we do not need this assumption for our results.[14] The function G encapsulates all the relevant considerations regarding repayment, including the probability of default as well as the recovery value in the case of default and how these vary with the level of indebtedness. Note that this framework could capture strategic default or moral hazard by the government, all of these are embedded in V and G.

The first period is divided into infinite rounds $i = 1, 2, \ldots$ and the government can run an auction in each round. We can think of auctions taking place in real time at $t = 1/2, 2/3, 3/4, \ldots$ At $t = 1$, the government collects the revenue from all these auctions, uses them to pay $b - s$ and the payoff $u(s)$ is realized. Finally, at $t = 2$ the payoff $V\left(b'\right)$ is realized. Letting d_i denote bond issuances in round i, total bonds issued in period 1 are then

$$b' = \sum_{i=0}^{\infty} d_i.$$

At each round i the investors bid price q_i for the issuance d_i.

The crucial assumption we make is that in each auction the government cannot commit to the size of debt issuances in future auctions. We capture this by studying this setup as a game and employ an equilibrium concept close to sub-game perfection.[15] Formally, strategies are described by functions $d_i = D_i(d^{i-1}, q^{i-1})$ and $q_i = Q_i(d^i, q^{i-1})$, where superscripts denote sequences up to round i. An equilibrium requires that

i. in round i, after any history (d^{i-1}, q^{i-1}), the government strategy D_j for the remaining rounds $j = i, i+1, \ldots$ is optimal, given that future prices satisfy $q_j = Q(d^j, q^{j-1})$ at $j = i, i+1, \ldots$

ii. the price in round i after history (d^i, q^{i-1}) satisfies $d_i = Q(d^i, q^{i-1}) = G(\sum_{i=0}^{\infty} d_i)$ where $\{d_i\}$ is computed using the government strategy D_j for $j = i, i+1, \ldots$ and future bond prices Q_j for $j = i+1, \ldots$

Observe that in along an equilibrium path the bond price is constant across rounds $q_i = q$.

[14] This is guaranteed, for example, if $G(b')$ is concave.

[15] The only difference is due to the fact that investors are assumed to be atomistic.

Denote the outcome of the game by (s^*, b'^*, q^*). Our main result in this section is a tight characterization of the possible outcomes in all equilibria of this game.

Proposition 3. *A triplet (s^*, b'^*, q^*) is the outcome of an equilibrium if and only if*

$$(s^*, b'^*) \in \arg\max_{s,b'} u(s) + \beta V(b') \qquad s.t. \qquad s + b = q^* b'$$

and

$$q^* = \beta \frac{G(b'^*)}{b'^*}.$$

Proof. We start with the sufficiency part, by describing a particular equilibrium which implements an outcome satisfying these two requirements. The equilibrium strategy of the investors is to set the price $Q(d^i, q^{i-1}) = q^*$ for any history (d^i, q^{i-1}) with $q^{i-1} = \{q^*, ..., q^*\}$. The strategy of the government is to issue $b'^* - \sum_{j=0}^{i} d_j$ after any history with $q^{i-1} = \{q^*, ..., q^*\}$. The resulting equilibrium outcome is that the government issues b'^* in the first auction and no further auction takes place. Since at each round the price is independent of the amount of bonds issued, the government cannot gain by changing its bonds issuances. Investors on the other hand expect that if the government deviates and offers anything other than a total issuance of b'^* in the current round, then it will not end the game and return for a further auction and issue exactly b'^*. This justifies their bid being independent of the amount of bonds issued in the current round.

Turning to the necessary part, suppose we have an equilibrium with outcome (s, b') and define $q = (b - s)/b'$. We want to prove that $q = MRS \equiv V'(b)/u'(s)$. Suppose, towards a contradiction, that we have a proposed equilibrium where instead $q \neq MRS$. For concreteness suppose $q > MRS$. The other case is symmetric.

In equilibrium the borrower is supposed to exit with (b, c) at some round. Suppose that instead, upon reaching this round, the government considers a deviation, does not exit and instead issues a small extra $\varepsilon > 0$ amount of debt in the next round, for a current total of

$$\tilde{b} = b + \varepsilon.$$

The market must then respond with a price $\tilde{q}$ for this round. The current price yields a current revised consumption

$$\tilde{c} = c + \tilde{q}\varepsilon.$$

In the equilibrium of the ensuing sub-game, the price in all future rounds must be constant and given by $\tilde{q} = \tilde{q}^* = G(\tilde{b}^*)$ where $\tilde{b}^*$ is the end outcome for debt following this sub-game. The associated end outcome for consumption is then $\tilde{c}^* = c + \tilde{q}(\tilde{b}^* - b)$.

The following inequalities hold

$$u(c) + V(b) \geq u(\tilde{c}^*) + V(\tilde{b}^*) \geq u(\tilde{c}) + V(\tilde{b}). \tag{11}$$

The first inequalities follows because (b, c) is an equilibrium outcome. The second because otherwise in the sub-game the borrower would prefer to stop after the initial deviation.

We can now prove that the end outcome of the sub-game cannot have more total debt than the initial deviation, that is, $\tilde{b}^* \leq \tilde{b}$. Suppose, by contradiction, that $\tilde{b}^* > \tilde{b}$. Let

$$\lambda = \frac{\tilde{b} - b}{\tilde{b}^* - b} \in (0, 1),$$

and notice that

$$(\tilde{b}, \tilde{c}) = (1 - \lambda)(b, c) + \lambda(\tilde{b}^*, \tilde{c}^*).$$

Strict concavity and the first inequality in (11) then imply $u(\tilde{c}^*) + V(\tilde{b}^*) < u(\tilde{c}) + V(\tilde{b})$, which contradicts the second inequality in (11).

Since the function $G(b)/b$ is non-increasing in b, $\tilde{b}^* \leq \tilde{b}$ implies

$$\tilde{q} = \frac{G(\tilde{b}^*)}{\tilde{b}^*} \geq \frac{G(\tilde{b})}{\tilde{b}}.$$

By choosing an initial deviation with $\varepsilon > 0$ small enough, the borrower can ensure that the lower bound on $\tilde{q}$ is arbitrarily close to q, since $G(\tilde{b})/\tilde{b} \to q$ as $\tilde{b} \to b$. But then, since $q > MRS$, this implies that along this deviation the borrower can sell bonds at a price $\tilde{q} > MRS$, which implies $u(\tilde{c}) + V(\tilde{b}) > u(c) + V(b)$. Therefore, if the proposed equilibrium satisfies $q > MRS$, there is a profitable deviation by the borrower, a contradiction. $\square$

The assumption that time is perfectly divisible, so that a further auction round is always available, delivers equilibria with outcomes that are equivalent to that of a price-taking government. The price-taking government in the first condition of Proposition 3 is the polar opposite of a government that can fully commit to b' and solve

$$\max_{s,b'} u(s) + \beta V(b') \qquad \text{s.t.} \qquad s + b = Q(b')b'$$

with $Q(b') = \beta G(b')/b'$. This is the assumption typically adopted in the literature. Instead, we assume that the government can commit to bond issuances in each round, but find that the outcome is *as if* it lacked any such commitment.

It is of interest to look at intermediate cases in which some degree of commitment is available. In the remainder of this section, we explore one such case using a simple example.

5.2 A Game with Partial Commitment

We now turn to a game with partial commitment. For simplicity, we focus on a simple three-period model. Our results show that the possibility to raise funds in future rounds of issuance can jeopardize the borrower's attempt to stay away from the wrong side of the Laffer curve.

5.2.1 The Game

There are three periods, $t = 0, 1, 2$. Debt is long-term and is a promise to pay 1 at date 2. In period 0, the government chooses how many bonds b_1 to sell. Next, an auction takes place and risk neutral investors bid q_0 for the bonds, the government receives $q_0 b_1$ from the investors and uses it to finance spending[16]

$$g_0 = q_0 b_1.$$

In period 1, the government chooses b_2, the investors bid q_1, the government then raises $q_1 (b_2 - b_1)$ and uses it to finance spending

$$g_1 = q_1 (b_2 - b_1).$$

Finally, in period 2 the surplus s is randomly drawn from an exponential distribution with CDF $F(s) = 1 - e^{-\lambda s}$ on $[0, \infty)$. The government repays if $s \geq b_2$, defaults otherwise and there is no recovery.

The government objective is to maximize

$$\alpha \min \{g_0, \overline{g}\} + \theta \min \{g_0 + g_1, \overline{g}\} + \int_{b_2}^{\infty} (s - b_2) \, dF(s),$$

that is, the government needs to finance a target level of spending $\overline{g}$ and has a preference for early spending. The parameter $\theta > 1$ captures the loss from not meeting the target $\overline{g}$, α captures the gain from early spending, g_0 and g_1 are restricted to be non-negative.[17]

[16] In following the timing of the game, one could find a bit confusing the fact that the government first chooses the issuance b_1 and then the investors choose the price q_0. But we stick to the subscripts to keep the notation consistent throughout the paper.

[17] Both assumptions seem reasonable. For example, investment spending on infrastructure requires some

Investors are risk neutral and do not discount future payoffs.

5.2.2 Strategies and Equilibrium

The government's strategy is given by a b_1 and a function $B_2(b_1, q_0)$ that gives b_2 for each past history (b_1, q_0). The investors' strategy is given by two functions $Q_0(b_1)$ and $Q_1(b_1, q_0, b_2)$.

We analyze subgame perfect equilibria moving backward in time, starting from period 1. In period 1, investors are willing to pay

$$Q_1(b_1, q_0, b_2) = 1 - F(b_2),$$

given the stock b_2 of government bonds. In period 1, given the stock of bonds b_1 and the price q_0, the government solves

$$\max_{g_1, b_2} \theta \min\{g_0 + g_1, \bar{g}\} + \int_{b_2}^{\infty} (s - b_2)\, dF(s) \tag{12}$$

subject to

$$g_1 = (1 - F(b_2))(b_2 - b_1)$$

and $g_0 = q_0 b_1$. The solution to this problem gives us the best response $B_2(b_1, q_0)$. Going back to period 0, investors' optimality requires

$$q_0 = 1 - F(B_2(b_1, q_0)). \tag{13}$$

We will construct equilibria in which a solution to (13) always exists. However, depending on the value of b_1, there may be multiple values of q_0 that solve (13). Let $\mathcal{Q}_0(b_1)$ be a map that selects a solution of (13) for each b_1 and let $\mathcal{B}_2(b_1) = B_2(b_1, \mathcal{Q}_0(b_1))$ denote the associated value of b_2.[18] To check that the choice of b_1 at date 0 is optimal, we need to check that it maximizes

$$\alpha \min\{[1 - F(\mathcal{B}_2(b_1))]\, b_1, \bar{g}\} + \theta \min\{[1 - F(\mathcal{B}_2(b_1))]\, \mathcal{B}_2(b_1), \bar{g}\} + \int_{\mathcal{B}_2(b_1)}^{\infty} (s - \mathcal{B}_2(b_1))\, dF(s).$$

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total outlay over an extended time horizon, but with a preference for early completion. As another example consider the payment of government wages. Suppose payment can be delayed, if needed, but at a cost, because workers are impatient and demand compensation.

[18] We could easily extend the analysis to allow a stochastic selection of equilibria.

5.2.3 Multiple equilibria

We now proceed to show that multiple equilibria are possible under some parametric assumptions. We begin by characterizing the government optimal behavior $B_2(b_1, q_0)$ at $t = 1$, for given values of b_1 and q_0.

Lemma 1. *Given q_0 and b_1, the optimal choice of b_2 must satisfy either*

$$q_0 b_1 + (1 - F(b_2))(b_2 - b_1) < \overline{g}$$

and

$$\theta(1 - \lambda(b_2 - b_1)) = 1$$

or

$$q_0 b_1 + (1 - F(b_2))(b_2 - b_1) = \overline{g}$$

and

$$\theta(1 - \lambda(b_2 - b_1)) \geq 1.$$

Proof. It is easy to show that in equilibrium we always have $q_0 b_1 \leq \overline{g}$. Therefore, the marginal benefit of increasing b_2 is

$$\begin{aligned} \theta(1 - F(b_2) - f(b_2)(b_2 - b_1)) - (1 - F(b_2)) &= \\ (1 - F(b_2))[\theta(1 - \lambda(b_2 - b_1)) - 1] \end{aligned}$$

if $g_0 + (1 - F(b_2))(b_2 - b_1) < \overline{g}$ and 0 otherwise. The statement follows immediately. □

The Laffer curve for total debt issued in this game is given by

$$(1 - F(b))b = e^{-\lambda b} b.$$

We assume

$$\overline{g} < \max_b e^{-\lambda b} b = (\lambda e)^{-1} \tag{14}$$

so in equilibrium the government can reach the target $\overline{g}$. Under assumption (14) there are two solutions to

$$e^{-\lambda b} b = \overline{g},$$

which we label $\underline{b}$ and $\overline{b}$. The two solutions satisfy $\underline{b} < 1/\lambda < \overline{b}$. Assume also that

$$\theta(1 - \lambda \underline{b}) > 1, \tag{15}$$

which implies that the government has a sufficiently strong incentive to spend in periods 0 and 1. Define the cutoff

$$\hat{b}_1 = \overline{b} - \frac{1}{\lambda}\left(1 - \frac{1}{\theta}\right) \in \left(0, \overline{b}\right), \tag{16}$$

where the inequalities follows from $\overline{b} > 1/\lambda$ and $\theta > 1$ (from (15)).

We can now characterize the continuation equilibria that arise after the choice of b_1 by the government at date 0, that is, we look for candidates for the equilibrium selections $\mathcal{Q}_0(b_1)$ and $\mathcal{B}_2(b_1)$. We first consider the case in which b_1 is below the cutoff $\hat{b}_1$.

Lemma 2. *If $b_1 < \hat{b}_1$ there is a unique continuation equilibrium, with $b_2 = \underline{b}$.*

Proof. The equilibrium exists because $(1 - F(b_2))\, b_2 = \overline{g}$ at $b_2 = \underline{b}$ and assumption (15) implies $\theta(1 - \lambda(b_2 - b_1)) > 1$ for any $b_1 \geq 0$. To prove uniqueness notice that we cannot have $b_2 \in (\underline{b}, \overline{b})$ in equilibrium, otherwise $e^{-\lambda b_2} b_2 > \overline{g}$, we cannot have $b_2 \geq \overline{b}$, otherwise $\theta(1 - \lambda(b_2 - b_1)) < 1$, and we cannot have $b_2 < \underline{b}$, otherwise $e^{-\lambda b_2} b_2 < \overline{g}$ and $\theta(1 - \lambda(b_2 - b_1)) > 1$ (always using Lemma 1). □

Lemma 3. *If $b_1 \geq \hat{b}_1$ there are two continuation equilibria, one with $b_2 = \underline{b}$ and one with $b_2 = \overline{b}$.*

Proof. The good equilibrium exists as in the previous claim. The second equilibrium exists because $b_1 \geq \hat{b}_1$ is equivalent to

$$\theta\left(1 - \lambda\left(\overline{b} - b_1\right)\right) \geq 1.$$

□

The previous two lemmas imply that the following is a possible selection for continuation equilibria

$$\mathcal{B}_2(b_1) = \begin{cases} \underline{b} & \text{if } b_1 \leq \hat{b}_1 \\ \overline{b} & \text{if } b_1 > \hat{b}_1 \end{cases}. \tag{17}$$

Now we can go back to period 0 and study the government's optimization problem when the continuation equilibria are selected as in (17). The government chooses b_1 to maximize

$$\alpha e^{-\lambda \mathcal{B}_2(b_1)} b_1 + \theta \min\left\{e^{-\lambda \mathcal{B}_2(b_1)} \mathcal{B}_2(b_1), \overline{g}\right\} + \frac{1}{\lambda} e^{-\lambda \mathcal{B}_2(b_1)}.$$

The government faces a trade-off here. If it chooses $b_1 \leq \hat{b}_1$ it ensures that in the continuation game investors will expect low issuance of bonds in period 1 and so only $\underline{b}$ bonds will be eventually issued, keeping the government on the good side of the Laffer curve.

However, to keep b_1 low the government foregoes the benefits from early spending α. In particular, choosing $0 \leq b_1 \leq \hat{b}_1$ we have

$$\alpha e^{-\lambda \underline{b}} b_1 + \theta \overline{g} + \frac{1}{\lambda} e^{-\lambda \underline{b}}.$$

While choosing $\hat{b}_1 < b_1 \leq \overline{b}$ we have

$$\alpha e^{-\lambda \overline{b}} b_1 + \theta \overline{g} + \frac{1}{\lambda} e^{-\lambda \overline{b}}.$$

Clearly, the only possible optimal choices are $b_1 = \hat{b}_1$ and $b_1 = \overline{b}$. It is optimal to choose $b_1 = \overline{b}$ if

$$\alpha e^{-\lambda \overline{b}} \overline{b} + \frac{1}{\lambda} e^{-\lambda \overline{b}} > \alpha e^{-\lambda \underline{b}} \hat{b}_1 + \frac{1}{\lambda} e^{-\lambda \underline{b}}.$$

Using (16) to substitute for $\hat{b}_1$ in this inequality we obtain the following proposition. Define the cutoff

$$\hat{\alpha} \equiv \frac{1}{\lambda} \frac{e^{-\lambda \underline{b}} - e^{-\lambda \overline{b}}}{\overline{g} - e^{-\lambda \underline{b}} \left(\overline{b} - \frac{1}{\lambda} \left(1 - \frac{1}{\theta} \right) \right)}$$

if the expression at the denominator is positive and let $\hat{\alpha} = \infty$ otherwise.[19]

Proposition 4. *If $\alpha > \hat{\alpha}$ there is an equilibrium in which the stock of bonds is constant at $b_1 = b_2 = \overline{b}$, on the wrong side of the Laffer curve.*

The game also admits a good equilibrium in which $\mathcal{B}_2(b_1) = \underline{b}$ for all b_1. Notice that also in this good equilibrium all bonds are issued at date 0, and we have $b_1 = b_2 = \underline{b}$. Therefore, bond issuance in period 1 only matters for off-the-equilibrium-path dynamics. However, off-the-equilibrium-path dynamics are crucial to determine the amount of bonds the government issues in the first period.

The government can commit not to issue more bonds than b_2 in period 2, given that it is the final date before the resolution of uncertainty. So the government will never reach a b_2 such that a reduction in b_2 can increase current revenues, in other words, it will always be on the increasing side of the issuance Laffer curve:

$$1 - \lambda (b_2 - b_1) \geq 0. \tag{18}$$

However, this condition is not enough to rule out an equilibrium with total debt on the wrong side of the Laffer curve, because the slope of the stock Laffer curve is $1 - \lambda b_2$,

[19] It is easy to find combinations of model parameters that ensure $\hat{\alpha} < \infty$.

which can be negative in spite of (18) if $b_2 - b_1$ is small. Moreover, the government at date 0 cannot try to move away from the bad equilibrium by reducing b_1 below $\bar{b}$, because, if it does, the market expects the government to issue $\bar{b} - b_1 > 0$ at date 1, and therefore the pricing function $\mathcal{Q}_0 (b_1)$ is flat for b_1 near $\bar{b}$. The only option is to reduce b_1 all the way to $\hat{b}_1$, which is enough to eliminate the bad equilibrium. But this is too costly in terms of delayed spending.

6 Concluding Remarks

Based on our analysis it seems difficult to dismiss the concern that a country may find itself in a self-fulfilling "bad equilibrium" with high interest rates. In our model, bad equilibria are not driven by the fear of a sudden rollover crisis, as commonly modeled in the literature following Giavazzi and Pagano (1989), Alesina et al. (1992) and Cole and Kehoe (1996) and others. Thus, the problems these "bad equilibria" present are not resolved by attempts to rule out such investor runs. Instead, high interest rates can be self fulfilling because they imply a slow but perverse debt dynamic. Our results highlight the importance of fiscal policy rules and debt maturity in determining whether the economy is safe from the threat of crises.

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