A THEORY OF FACTOR SHARES*

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Abstract

This paper presents a theory of how factor shares are determined. I first develop microfoundations for a unified aggregate production function that incorporates a frictional process of matching workers and firms. Firms with productivities drawn from a Pareto distribution hire capital and compete for workers. Wages are determined by Bertrand competition. In contrast with Houthakker's classic result, the aggregate production function derived here is Cobb-Douglas only in the limit as unemployment goes to zero. In general, the elasticity of substitution between capital and labor is less than one. Factor shares are asymptotically constant when unemployment disappears and also when workers' reservation wage exceeds the minimum firm productivity. In general, factor shares are variable and depend on unemployment and workers' reservation wage, as well as the firm productivity distribution. Using annual data on unemployment and eligibility for unemployment insurance, I calibrate the model and test its quantitative predictions. The theory can explain much of the behavior of factor shares in the U.S. from 1951-2003. The correlation between the data and the model's predictions during this period of over fifty years is 0.69 for the perfect foresight equilibrium and 0.73 when workers are myopic. *JEL* Codes*:* E23, E24, E25, J64.

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I. INTRODUCTION

What determines the relative income shares of workers and the owners of capital? Historically, this question was deemed so fundamental that in 1817 Ricardo called it "the principal problem in Political Economy."[1] For nearly two centuries this question has intrigued economists. One feature of the behavior of factor shares that has struck economists in particular is their apparent stability. As Solow (1958) puts it, there is a widespread belief that labor's share is "one of the great constants of nature, like the velocity of light." Of course, factor shares are not strictly constant; they exhibit significant variation over time. The problem is how to explain both the relative stability we observe and the systematic variation.

In this paper, I present a theory of how factor shares are determined. The question of what determines income shares can be broken into three distinct parts: *production*, *distribution*, and *equilibrium*. First, what is the total quantity of output produced by combining a given quantity of capital and labor inputs? Second, how is the income obtained from jointly producing a single unit of output divided between workers and firms? Third, what is the equilibrium quantity of capital and labor in the economy? To determine equilibrium factor shares, it is necessary to answer all three questions simultaneously.

I first develop microfoundations for an aggregate production function in a frictional labor market environment where workers and firms are randomly matched. The aggregate production function is *unified* in the sense that the production technology and the matching process are not separable but intertwined. This provides a simple way of modelling unemployment that is built directly into the aggregate production function itself. The key inputs determining aggregate output are (i) the capital hired or purchased by firms, and (ii) the total labor force or number of potential workers, who may end up either employed or unemployed. The unified process of matching and production plays a critical role in enabling us to see how labor market conditions, particularly unemployment, influence factor shares.

In order to produce, firms need to first purchase a machine (capital) and then search for a worker. Firms enter until expected profits are zero and their demand for capital determines the equilibrium quantity of capital. Wages are determined by Bertrand competition between firms who compete directly to hire workers. Firms' productivity levels are drawn from a Pareto distribution. When two or more firms approach an unemployed worker, the one with the highest productivity hires the worker and pays a wage equal to the second highest productivity. If exactly one firm approaches the worker, he is paid his reservation wage. If no one approaches a worker, he remains unemployed for that period. While this is not a bargaining model, the process of Bertrand competition effectively *endogenizes* a worker's

[1] Ricardo (1911), p. 1 in the 1911 edition, first published in 1817.

relative "bargaining" position.[2] Equilibrium factor shares depend on the reservation wage, the ratio of searching firms to unemployed workers, and the firm productivity distribution.

In the limit as the level of firm entry goes to infinity and unemployment disappears, it turns out that *(i)* the aggregate production function is Cobb-Douglas, and *(ii)* factors are paid their marginal products. Both of these are theoretical results, not assumptions. Jointly, they imply that factor shares are asymptotically constant. The first asymptotic result is closely connected with the classic result of Houthakker (1955) and more recent papers by Jones (2005) and Lagos (2006). The result regarding the asymptotic constancy of factor shares, which follows from both *(i)* and *(ii)*, is entirely new.

While these asymptotic results provide a neat way of embedding the standard neoclassical setting as a limiting case, what happens outside this limiting case has greater empirical relevance. This general environment, in which unemployment does exist, is the true focus of this paper, not the frictionless asymptotic results. In the presence of unemployment, factor shares are not generally constant but vary depending on labor market conditions such as the unemployment rate and workers' reservation wage. The theory is therefore flexible enough to account for both the relative stability and the systematic variation in factor shares.

Stability of Factor Shares. The constancy of factor shares in the long run is generally considered a stylized fact in macroeconomics. However, as Kaldor (1955) states, this apparent stability is puzzling and "makes the question of what determines these shares more, rather than less, intriguing." It gives rise to the question, *why* are factor shares relatively stable? There are two standard alternatives for accounting for the stability of factor shares. The most common is to assume the aggregate production function is Cobb-Douglas and factor markets are competitive. The second alternative is to use the constant-elasticity-of-substitution (CES) aggregate production function introduced by Arrow et al. (1961) and assume that technological change is purely labor-augmenting.[3]

In the theory presented here, there are two distinct ways in which constant factor shares can emerge. First, there is the asymptotic result: as the level of firm entry increases and unemployment disappears, the aggregate production function approaches a Cobb-Douglas limit and factor shares are constant. The parameter determining capital's share is inherited

[2] Since firms target workers and wages are determined through a second-price auction, there is a close relationship with the theory of competing auctions developed in Peters and Severinov (1997). The framework is also related to Postel-Vinay and Robin (2002), who use Bertrand competition to determine wages in an environment with on-the-job search. The wage determination mechanism in this paper can also be seen as a generalization of that used in directed search models of the labor market such as Julien et al. (2000) where firms are homogeneous and they are chosen at random when more than one approaches a given worker.

[3] Acemoglu (2003) provides microfoundations for the second alternative. In his paper, the economy converges in the long run to one with purely labor-augmenting technical change and constant factor shares.

from the underlying Pareto distribution of firm productivity levels. This asymptotic result is analogous, at least mathematically, to a result found in Jones (2005). In general, however, the aggregate production function derived here is *not* Cobb-Douglas. The elasticity of substitution between capital and labor is not constant at all but variable, and it is always less than one. Nonetheless, constant factor shares can arise because factors are not necessarily paid their marginal product. In this way, the possibility of constant factor shares is consistent with an aggregate production function that has an elasticity of substitution below one, a theoretical result that complements a large body of empirical work that suggests this elasticity is well below one.[4]

Factor shares are constant when workers' reservation wage equals or exceeds the minimum firm productivity. As we will see, this is because the expected division of output between workers and successful firms is endogenously linear in this case. While it has been known since Houthakker (1955) that the Pareto distribution can generate a Cobb-Douglas production function, this result is novel: in the presence of Bertrand competition for workers, the Pareto distribution leads to constant factor shares even when the aggregate production function is not Cobb-Douglas. Moreover, the parameter determining the factor income distribution is the *same* as the technology parameter in the aggregate production function, since both are inherited from the underlying distribution. There is therefore a deep and natural connection between the Pareto distribution and the tendency towards stable factor shares.

Variation in Factor Shares. The theory can explain how constant factor shares could arise, but it is also flexible enough to account for systematic variation. While constancy is a theoretical possibility, when I calibrate the model to U.S. data it predicts factor shares that are relatively stable but not constant. In the benchmark calibration, the endogenous value of workers' reservation wage is always less than the minimum firm productivity. In this case, the model predicts that labor's share is decreasing in the level of unemployment and increasing in workers' reservation wage, which depends on factors that may change over time such as future employment prospects and unemployment insurance eligibility.

Labor's share exhibits short-term fluctuations over the business cycle. In the U.S., labor's share fluctuates in a counter-cyclical manner at a quarterly frequency. However, while labor's share falls on impact in response to a positive productivity shock, it *rises* to an even higher level a year later and the positive impact persists for some time (Rios-Rull and Santaeulalia-Llopis (2010)). Shifting our attention to annual time periods, we will see that there is a slightly *negative* relationship between the unemployment rate in a given year and labor's

[4] See Antras (2004) and Chirinko (2008) for recent summaries of these estimates. An exception is Karabarbounis and Neiman (2013), who estimate the elasticity of substitution to be $\sigma = 1.25$.

share the following year. At an annual frequency, the model's predictions are consistent with this fact: when unemployment is lower, labor's share tends to rise because greater competition to hire workers increases their endogenous relative "bargaining" position.

In Section IV of this paper, I present a simple quantitative exercise where I examine the model's predictions about the relationship between unemployment rates, unemployment benefits, and factor shares. Using only data on unemployment rates and unemployment insurance eligibility, I calibrate the model and compare its predictions regarding factor shares to the U.S. data from 1951-2010.[5] The model can explain much of the behavior of factor shares in the U.S. during a period of more than fifty years from 1951-2003. During this period, the standard deviation of capital's share is 0.0068 for the perfect foresight equilibrium and 0.0075 when workers are myopic, compared with 0.0093 for the data. The correlation between the data and the model's predictions for capital's share during the period 1951-2003 is 0.69 for the perfect foresight equilibrium and 0.73 when workers are myopic. The sharp rise in capital's share from 2004-2005 appears to be due to factors outside the model's scope.

I.A. Related Literature.

While the behavior of factor shares is a topic that has been relatively neglected in recent decades, there have been signs of a resurgence of interest in the topic. Karabarbounis and Neiman (2013) document a global decline in labor's share in the corporate sector since the early 1980s. Using a CES production technology for intermediate inputs, they attribute this decline to a fall in the relative price of investment goods by estimating an elasticity of substitution between capital and labor that is greater than one. Elsby et al. (2013) discuss the recent decline in labor's share in the U.S. and identify globalization and the rise in off-shoring of labor-intensive tasks as potentially important factors.

This paper's focus on the relationship between labor market conditions and factor shares links it with papers such as Bentolila and Saint-Paul (2003), and earlier work by Blanchard (1997) which highlights the possibility of medium-run shifts in labor's share due to labor market deregulation and changes in workers' bargaining power or the degree of unionization.

In Blanchard and Giavazzi (2003), factor shares depend simultaneously on labor market deregulation, which is captured by workers' bargaining power, and product market deregulation, which is captured by the markups of imperfectly competitive firms. When more firms enter, there is greater competition in the product market and markups fall, leading to an increase in labor's share. Similarly, if there is an increase in workers' bargaining power, due to an increase in unionization for example, labor's share will increase. In the present

[5] Factor shares data is obtained from the Bureau of Labor Statistics multifactor productivity historical data release. Details are found in Section IV.

paper, when more firms enter there is greater competition between firms to hire workers, which generates both lower unemployment and a higher *endogenous* "bargaining" power of workers, thereby increasing labor's share.

Since this paper is concerned with deeper microfoundations, it is most closely related to Jones (2005) and Lagos (2006), both of whom derive Cobb-Douglas production functions from microfoundations following Houthakker (1955). While these papers take very different approaches, each of them uses an underlying distribution of productivity levels or ideas that is Pareto and aggregates across micro level production units to obtain a Cobb-Douglas aggregate production function. This paper is perhaps closest in spirit to Jones (2005), but there are at least two key points of divergence. First, Jones' paper is primarily concerned with *production*, while this paper also provides microfoundations for the *distribution* of income between capital and labor. Second, the aggregation result is itself different, as I discuss in Section II.

The unified nature of the aggregate production function distinguishes my approach not only from Jones (2005) but also Lagos (2006). While Jones abstracts from labor market frictions, they are central to both Lagos' paper and mine. In Lagos' framework, workers and vacancies are randomly matched in bilateral meetings through an exogenous matching function. The surplus generated by such matches is divided through generalized Nash bargaining in a manner that is standard in Mortensen and Pissarides (1994) style models. In this paper, firms compete directly for workers and wages are determined through a second-price auction which effectively endogenizes workers' relative "bargaining power".

We can now return to the original question: What determines the relative income shares of workers and the owners of capital? Consider two of the key components – *production* and *distribution*. Modern macroeconomics often answers this question by imposing two assumptions: *(i)* a specific aggregate production function, and *(ii)* competitive factor markets. As a consequence, the nature of the aggregate production function (i.e. its technological properties) directly determines the nature of the factor income distribution. In both Lagos (2006) and this paper, neither assumption is made. However, in Lagos' paper and other environments with an exogenous wage bargaining parameter, production and distribution are essentially divorced from each other. Here, the close relationship between factor remuneration and marginal products is not lost. In fact, the parameter governing the factor income distribution is endogenously *equal* to the production technology parameter, since both are inherited from the underlying firm productivity distribution. We will see that this result arises naturally in the unified theory of production, employment and wages presented here.

The rest of the paper is structured as follows. Section II presents the basic model. I first derive the aggregate production function and then determine the equilibrium level of firm entry. I determine the conditions under which factor shares are constant and examine how equilibrium factor shares are affected by changes in labor market conditions and the underlying productivity distribution. Section III extends the model to a simple dynamic environment and presents some comparative statics results for the steady state equilibrium. Section IV presents the quantitative analysis. All proofs are in the Appendix.

II. BASIC MODEL

Consider a static environment. There is a continuum of *ex ante* homogeneous risk-neutral potential workers of measure L and a continuum of risk-neutral firms of measure V. The ratio of firms to workers is $\theta = V/L$. The equilibrium ratio of firms to workers, θ^*, is determined by a zero profit condition.

Firms pay an entry cost, r, to obtain one unit of capital and a single productivity draw from a distribution $G(x)$. Total capital is given by firms' demand, $K = V$, and hence $\theta = k = K/L$, the capital/labor ratio. The cost r can be interpreted as the *rental rate of capital*, since it is the cost of hiring one unit of capital for a single period.

After paying the cost of capital r, firms draw a productivity level x from a distribution $G(x)$. An firm with productivity x can produce x units of output using a single unit of capital and a single worker's labor for one period. We assume that $G(x)$ is continuous and differentiable with support $(x_{\min}, \infty)$ and no mass points. We normalize $x_{\min} = 1$.

Firms can approach only a single worker. Since workers are homogeneous and firms are uncoordinated, firms make job offers to workers at random. This gives rise to a Poisson distribution with parameter θ for the number of firms approaching each worker.[6] The expected number of firms competing for a given worker is θ, so this ratio can be interpreted as *labor market tightness*, or the degree of competition for workers' labor.

Each match between a worker and a firm with productivity x produces x units of the final good with price normalized to one. From the worker's perspective, there are three possible matching outcomes: unemployment, a bilateral match, or a multilateral match. Every firm approaches a single worker, but it is *successful* only if it has the highest productivity for the particular worker it approaches.

Unemployment. If no firms arrive, the worker is unemployed and output is zero. The worker receives a payoff $z \in [0, 1]$, the *value of non-market activity.* By the Poisson distribution, this occurs with probability $e^{-\theta}$, so $u(\theta) = e^{-\theta}$ is the unemployment rate.

Bilateral match. If exactly one firm approaches worker j, it employs the worker and produces output at his own productivity level, x_j. The worker is paid his outside option, z.[7]

[6] Since firms approach each worker with equal probability, the Poisson distribution arises because we are taking the limit of a binomial distribution.

[7] This assumption can be justified by general results from auction theory found in McAfee (1993) and

The firm's net payoff is $\pi_j = x_j - z - r$.

Multilateral match. If two or more firms approach worker j, they compete for the worker's labor in Bertrand style competition. The firm with the highest productivity level, x_j^1, employs the worker and produces output at productivity x_j^1. The worker's wage equals x_j^2, the second-highest productivity, and the firm's net payoff is $\pi_j = x_j^1 - x_j^2 - r$. Firms receive a payoff of zero if they do not successfully hire a worker.

II.A. Aggregate Production Function

This process of matching and production generates an endogenous cross-sectional productivity distribution across potential workers, $H(x;\theta)$. This distribution simultaneously incorporates two dimensions: the *employment* effect of the frictional matching process, which leads to unemployment for some workers; and the *productivity* effect of the competition between firms, which leads to an allocation of labor towards higher productivity firms, thereby increasing aggregate output per worker.

The dual effect of firm entry on both employment and productivity is not new, but the channel is novel. An increase in the degree of competition for workers, θ, directly influences the entire productivity distribution across workers. This contrasts, for example, with search-theoretic models of the labor market in which an increase in the ratio of vacancies to unemployed workers leads to a simple truncation of the cross-sectional productivity distribution by increasing the cut-off productivity threshold.[8]

Consider a specific distribution of firm productivity levels, the Pareto distribution. Let $G(x) = 1 - x^{-1/\lambda}$ where $\lambda \in (0, 1)$. This distribution is used by Jones (2005), Lagos (2006), and Houthakker (1955) to derive Cobb-Douglas aggregate production functions.[9]

Suppose that n firms approach a given worker. If $n \geq 1$, the firm with the highest productivity hires the worker and the resulting productivity is the maximum of n draws from $G(x)$. If no firms approach a given worker, he is unemployed and produces zero output. Let $H(x|n) = G(x)^n$ be the distribution of the worker's productivity conditional on the number of firms arriving, n. To obtain the unconditional distribution, $H(x;\theta)$, the distribution $H(x|n)$ must be weighted by the probability that n firms approach, which is given by a Poisson

Peters and Severinov (1997). When the number of bidders is determined endogenously by a free entry condition, it is optimal for sellers to set a reserve price equal to their outside option.

[8] This will also be the case in the dynamic model of Section III, where an endogenous reservation wage greater than the minimum firm productivity leads to a truncation of the productivity distribution. However, this effect is *additional* to the primary effect described here.

[9] Gabaix (2009) provides a review of the numerous applications and results regarding the Pareto or power law distribution in economics.

distribution with parameter θ, so

$$H(x;\theta) = \sum_{n=0}^{\infty} \frac{\theta^n e^{-\theta}}{n!} G(x)^n = e^{-\theta(1-G(x))} = e^{-\theta x^{-1/\lambda}}. \tag{1}$$

The distribution $H(x;\theta) = e^{-\theta x^{-1/\lambda}}$ has continuous support $[1, \infty)$ and a mass point at zero with probability mass $u(\theta) = e^{-\theta}$, the unemployment rate. Since there is a continuum of workers, this is both the distribution of each worker's productivity and the cross-sectional distribution across *all* potential workers. As $\theta \to \infty$, this distribution converges to the Type II Extreme Value or Fréchet distribution.[10]

Aggregate Output. To obtain output per capita, $y = Y/L$, we simply take the expected value of the endogenous workers' productivity distribution, $H(x;\theta)$. Since we have $k = \theta$, output per capita can be expressed as a function of k alone. The intensive form of the production function is

$$f(k) = \gamma(1-\lambda, k)k^{\lambda}, \tag{2}$$

where $\gamma(s,x) \equiv \int_0^x t^{s-1}e^{-t}dt$, the Lower Incomplete Gamma function. The expression $\gamma(1 - \lambda, k)$ is increasing in both k and λ for $\lambda \in (0,1)$.[11] We have $f'(k) > 0$, $f''(k) < 0$, and $\lim_{k\to\infty} f'(k) = 0$ as usual, but $\lim_{k\to 0} f'(k) = (1-\lambda)^{-1}$. (See Appendix A.1.) Using the fact that $k = \theta$, we can write the aggregate production function as

$$Y = \gamma(1-\lambda, \theta)K^{\lambda}L^{1-\lambda}. \tag{3}$$

This function clearly has constant returns to scale in K and L. Importantly, however, it is *not* Cobb-Douglas since the term $\gamma(1-\lambda,\theta)$ depends on $\theta = K/L$. Only in the limit as $\theta \to \infty$ (i.e. as unemployment goes to zero) do we have $Y = \Gamma(1-\lambda)K^{\lambda}L^{1-\lambda}$ where $\Gamma(s) \equiv \lim_{x\to\infty} \gamma(s,x) = \int_0^{\infty} t^{s-1}e^{-t}dt$, the Gamma function.[12] In this limiting case, the aggregate production function is indeed Cobb-Douglas since $\Gamma(1-\lambda)$ is a constant.[13]

The fact that the production function is not generally Cobb-Douglas arises from a crucial difference between this aggregation result and that found in Jones (2005). He considers a large number of production units, each of which uses a local Leontief production technology.

[10] The Fréchet extreme value distribution is used in the model of international trade by Eaton and Kortum (2002), and it is derived from microfoundations in Kortum (1997) and Eaton and Kortum (1999). As well as Jones (2005), the Fréchet distribution is also featured in Lucas (2009) and Hsieh et al. (2011).

[11] The Lower Incomplete Gamma function is *decreasing* in s for any $s \in (0,1)$.

[12] The Gamma function is the unique function that extends the factorial function to $\mathbb{R}^+$.

[13] Strictly speaking, to say that a production function is *asymptotically Cobb-Douglas*, we require that $\lim_{k\to\infty} \frac{f(k)}{k^{\alpha}} = A$ for some constants $\alpha \in (0,1)$ and $A > 0$. Here, $\alpha = \lambda$ and $A = \Gamma(1-\lambda)$.

He then takes the convex hull of available ideas to derive a "global" production function, where ideas represent different ways of combining capital and labor to produce output.[14] Since Jones considers the convex hull across the entire economy, and takes the limit as the number of ideas becomes large, he works directly with a Type II Extreme Value or Fréchet distribution. The "global" production function is asymptotically Cobb-Douglas in the long run as the total number of ideas across the economy grows over time.

In contrast with Jones' approach, we take the highest firm productivity draw for each worker and then aggregate across all workers. Since the number of firms approaching a given worker during a single time period is relatively small, I consider the exact distribution which arises for finite θ, namely $H(x) = e^{-\theta x^{-1/\lambda}}$ with continuous support $[1, \infty)$ plus a mass point at zero which represents unemployment. While Jones' global production function becomes Cobb-Douglas over time as the number of ideas grows large, the aggregate production function derived here is not, in general, Cobb-Douglas. I focus on the general environment where unemployment exists and the elasticity of substitution is below one.

Elasticity of Substitution. Considered as a property of the aggregate production function, the elasticity of substitution between capital and labor, σ, is the percentage change in the K/L ratio for a given percentage change in the ratio of marginal products. Using the expression for σ in Arrow et al. (1961),

$$\sigma = \frac{-f'(k)(f(k) - kf'(k))}{kf(k)f''(k)}. \tag{4}$$

For the production function given by (2), the elasticity of substitution is

$$\sigma = \frac{\lambda + \varepsilon(1-\lambda, \theta)}{\lambda + \varepsilon(2-\lambda, \theta)}, \tag{5}$$

where $\varepsilon(s, x)$ denotes the elasticity of $\gamma(s, x)$ with respect to x.[15] In particular,

$$\varepsilon(1-\lambda, \theta) = \frac{\theta^{1-\lambda} e^{-\theta}}{\gamma(1-\lambda, \theta)}. \tag{6}$$

In the limit as $\theta \to 0$, we have $\sigma \to 1/2$ since $\lim_{x \to 0} \varepsilon(s, x) = s$. In the limit as $\theta \to \infty$, the elasticity of substitution $\sigma \to 1$ as $\lim_{x \to \infty} \varepsilon(s, x) = 0$. This elasticity is always less than one,

[14] See also Caselli and Coleman (2006), who use a related approach to examine technology choice and the world technology frontier.

[15] See Appendix A.0 for some properties of the elasticity $\varepsilon(s, x)$. In particular, $\varepsilon(s, x)$ is increasing in s and decreasing in x.

regardless of the value of the capital/labor ratio, $k = \theta$.[16] (See Appendix A.2 and A.3.)

Proposition 1 *If $G(x)$ is Pareto, the elasticity of substitution between capital and labor, σ, is always less than one and converges to one in the limit as $\theta \to \infty$.*

Importantly, the fact that $\sigma < 1$ is a theoretical result, not an assumption. This contrasts with the standard alternative to assuming a Cobb-Douglas aggregate production function – namely, to start with a CES production function and assume a particular value of σ, as estimated by empirical studies. While standard, this approach is problematic because in the empirical literature the value of the elasticity σ is still very much open to debate. Estimates vary widely, from as low as 0.3 to greater than one. Most empirical studies, however, indicate that the elasticity is significantly below one.[17] The theoretical result presented here therefore complements the large body of empirical work, including Antras (2004), that finds this elasticity is likely to be significantly less than one.

Unified Production and Matching. Search-theoretic models of the labor market generally feature an exogenous matching function, $M(U, V)$, which gives the number of matched worker/firm pairs – or simply "matches" – as a function of the number of unemployed workers and job vacancies. It is common to assume the matching function is constant returns to scale, so $M(U, V)/U = m(\theta)$, where $\theta = V/U$.[18] In this paper, the aggregate production function, $Y = \gamma(1-\lambda, \theta)K^{\lambda}L^{1-\lambda}$, is not just a production function multiplied by a matching function. Instead, it is a *unified* aggregate production and matching function.

A *matching* process is the process by which workers and firms are "matched" to form worker/firm pairs and the unemployment rate is determined. A *production* process is the process according to which a given match produces a certain quantity of output. In this model, matching and production are two aspects of a single process. If no firms approach a given worker, he is unemployed. If at least one firm approaches a worker, the firm with the highest productivity hires the worker and produces output at that productivity level. Both the employment status of a given worker and their expected output depend upon the number of firms competing to hire the worker.

The production process incorporates the fact that an important effect of greater competition to hire workers is an allocation of labor towards more productive firms. More firms competing for a given worker's labor implies a higher expected value of the maximum productivity, which leads to higher output for that worker and hence higher aggregate labor

[16] This result also holds when the production function and the elasticity of substitution are defined in a more conventional manner, so that $y = \tilde{f}(\kappa)$ where $\kappa \equiv K/L_e$ and L_e is the number of *employed* workers. See Appendix A.3.

[17] See footnote 3 in Acemoglu (2003) for a summary of the empirical estimates of this elasticity.

[18] See Petrongolo and Pissarides (2001) and Rogerson et al. (2005) for surveys of the relevant literature.

productivity. Seen in this light, the expression $\gamma(1-\lambda,\theta)$ can be interpreted as a kind of generalization of the matching function which includes the effect of competition between firms on aggregate output.

To understand how this unified production function works, we can isolate both the matching function and the production function by considering two limiting directions. Recall that L represents *all* workers, including the unemployed. First, we can recover the *matching* function by considering the limit as $\lambda \to 0$. This is equivalent to assuming a degenerate underlying distribution, $G(x)$. Since $\gamma(1,\theta) = 1 - e^{-\theta}$, the aggregate production function collapses to the urn-ball matching function, $m(\theta) = 1 - e^{-\theta}$.[19]

The second direction gives us the limiting *production* function. If we take the limit as $\theta \to \infty$, we obtain the Cobb-Douglas production function, $Y = \Gamma(1-\lambda)K^{\lambda}L^{1-\lambda}$. In this limiting case, there is no unemployment and hence L is the number of employed workers. The expression $\Gamma(1-\lambda)$ is the maximum level for $\gamma(1-\lambda,\theta)$ as the number of firms goes to infinity and unemployment disappears.

In general, the unified aggregate production function, $Y = \gamma(1-\lambda,\theta)K^{\lambda}L^{1-\lambda}$, incorporates both production and matching simultaneously. The marginal product of labor represents the effect on aggregate output of an extra *potential* worker, who may end up either employed or unemployed depending on the matching outcomes. The marginal product of capital represents the marginal contribution of an extra unit of hired capital, which may end up either utilized or unutilized depending on whether or not the firm is successful.

II.B. Equilibrium

The equilibrium level of firm entry, θ^*, is determined by a zero profit condition which ensures the expected payoff for firms, net of entry cost, is zero. The expected net payoff for firms, $\pi(\theta)$, is determined by both the probability of success (i.e. having the highest productivity for a given worker) and the expected payoff for a successful firm. The former is determined by the Poisson distribution, while the latter depends on the wage determination mechanism.

After a firm approaches a given worker, there are two possibilities. If the firm is alone, it employs the worker at wage z. If there are two or more firms competing, the one with the highest productivity hires the worker at a wage equal to the second highest productivity. This environment is equivalent in terms of expected payoffs to a second-price auction where

[19]The urn-ball matching function, first introduced by Butters (1977) and Hall (1979), arises endogenously in directed search models of the labor market as the economy becomes large. The urn-ball matching function arises endogenously in directed search models of the labor market. For papers related to directed search see, for example, Peters (1991), Montgomery (1991), Shimer (1996), Moen (1997), Julien et al. (2000), Burdett et al. (2001), Shi (2001), Shimer (2005a), Albrecht et al. (2006), and Galenianos and Kircher (2009).

each firm's valuation of a worker's labor equals their productivity draw from the distribution $G(x)$. There is a stochastic number of bidders determined by the Poisson distribution.[20] This environment is similar to the competing auctions framework analyzed by Peters and Severinov (1997). In Appendix A.4, I derive the following zero profit condition:

$$\pi(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1-z) - r = 0. \tag{7}$$

It is easy to see that $\pi'(\theta) < 0$, so if there exists a θ such that $\pi(\theta) > 0$, the equilibrium must be unique. Since $\pi(0) = E_G(x) - z - r$, if we have $E_G(x) > r + z$ then $\pi(0) > 0$ and there exists a unique equilibrium level of firm entry, $\theta^* > 0$, such that $\pi(\theta^*) = 0$. On the other hand, if $\pi(0) \leq 0$ no firms enter and $\theta^* = 0$. (See Appendix A.5.)

For any distribution $G(x)$, the equilibrium θ^* is decreasing in the value of non-market activity, z, and decreasing in the rental rate of capital, r. This implies that the equilibrium unemployment rate, $u(\theta^*) = e^{-\theta^*}$, is increasing in the value of non-market activity z and increasing in r. These results are intuitive. A higher value of non-market activity, z, means that firms are deterred by the lower expected profits, so θ^* is lower and unemployment is higher. A lower rental rate rate of capital, r, implies higher expected profits for firms, which leads to an increase in θ^* and a lower unemployment rate. (See Appendix A.6.)

The equilibrium level of firm entry, θ^*, determines *expected wages*, $w(\theta^*) = f(\theta^*) - r\theta^*$. This is just total wages divided by the number of potential workers, i.e. the expected payoff from market activity for all workers, including the unemployed.[21] While expected wages are pinned down by the equilibrium θ^*, the wage determination mechanism gives rise to residual wage dispersion across workers. This is because the actual wage paid to a particular worker depends on the specific productivity levels of the firms bidding for that worker. Even workers with identical productivity outcomes *ex post* can receive different wages because the profit/wages split depends on the value of the second highest productivity.

For the Pareto distribution, $G(x) = 1 - x^{-1/\lambda}$, the equilibrium ratio of firms to workers θ^* is the unique solution to

$$\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-z)e^{-\theta} = r. \tag{8}$$

(See Appendix A.7.) The equilibrium θ^* is increasing in λ and the equilibrium unemployment rate, $u(\theta^*)$, is decreasing in λ. A change in the parameter λ affects the productivity of all

[20] Auctions with a stochastic number of bidders were first studied by McAfee and McMillan (1987).

[21] The expected payoff for workers from both market *and* non-market activity is $\hat{w}(\theta^*) = f(\theta^*) - r\theta^* + ze^{-\theta}$. It is easy to verify that if workers were able to choose *ex ante* a bilateral wage, b^*, in order to maximize their expected payoff, $\hat{w}(\theta^*)$, they would choose b^* equal to their outside option, z.

matches, since it increases both the mean and the variance of the Pareto distribution.[22] Equilibrium output per capita, $f(\theta^*) = \theta^\lambda \gamma(1-\lambda, \theta)$, is increasing in λ. (See Appendix A.8.)

Factors are paid their marginal product if and only if either $z = 0$ or $\theta \to \infty$. The marginal product of capital is $MPK = f'(\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}$, so $r = MPK$ only if either $z = 0$ or $\theta \to \infty$. Expected wages are given by $w(\theta) = f(\theta) - r\theta = (1-\lambda)y - (1-z)\theta e^{-\theta}$, while the marginal product of labor is $MPL = f(\theta) - \theta f'(\theta) = (1-\lambda)y - \theta e^{-\theta}$. This is the marginal product of an extra *potential* worker, as discussed in Section II.A. If $z > 0$, workers are paid more than their marginal product. The "wage gap", $w - MPL = z\theta e^{-\theta}$, disappears only when the value of non-market activity is zero, or as $\theta \to \infty$.

In the limiting case where $\theta \to \infty$, all matches are multilateral, expected wages are $w(\theta) = (1-\lambda)y$, and the expected payoff for a successful researcher is $r\theta = \lambda y$. Both the fact that the profit/wages split is linear and the particular value of the linear split are endogenous here. This linear division of income at the aggregate level occurs simultaneously with micro-level wage dispersion across individual workers and wide variation in profits across firms. In general, however, not all matches are multilateral, so the aggregate profit/wages split is only approximately linear. We turn to the question of how to determine factor shares next.

II.C. Factor Income Shares

Imagine there are owners of capital who are paid the cost r by firms to lend them a single unit of capital for one period. Since there is a zero profit condition for firms, output is simply split between workers and owners of capital. The share of income going to capital is $s_K = rK/Y$. Using (3) and (8), capital's share is

$$s_K = \lambda + (1-z)\varepsilon(1-\lambda, \theta), \tag{9}$$

where $\varepsilon(1-\lambda,\theta)$ is the elasticity given by (6).

One way to interpret (9) is to consider the special case where $z = 0$. In this case, factors are paid their marginal product. Capital's share is equal to the elasticity of the aggregate production function, $Y = \gamma(1-\lambda,\theta)K^\lambda L^{1-\lambda}$, with respect to capital, namely $s_K = \lambda + \varepsilon(1-\lambda,\theta)$. Dissecting this further, capital is paid both the elasticity, λ, of the Cobb-Douglas production function, $Y = K^\lambda L^{1-\lambda}$, plus the elasticity, $\varepsilon(1-\lambda,\theta)$, of the generalized matching function, $\gamma(1-\lambda,\theta)$, with respect to capital. In the degenerate case where $\lambda = 0$, capital's share equals the elasticity of the matching function, $m(\theta) = 1 - e^{-\theta}$, with respect to θ. More generally, when $\lambda > 0$, capital is paid for its contribution to both matching *and* production.

[22] The expected value of the Pareto distribution $G(x) = 1 - x^{-1/\lambda}$ is $\frac{1}{1-\lambda}$ and the variance is $\frac{\lambda^2}{(1-2\lambda)(1-\lambda)^2}$ (for $\lambda < 1/2$), both of which are increasing in λ.

If we consider the direct effect of θ on factor shares *outside* of equilibrium, then capital's share is decreasing in θ. This is because the term $\varepsilon(1-\lambda,\theta)$ is decreasing in θ. (See Appendix A.9.) Equivalently, labor's share is increasing in θ. This result is intuitive since θ is a measure of the degree of *competition* for workers' labor. Greater competition for workers results in a higher labor share. As the number of firms goes to infinity, capital's share approaches its lower bound, $s_K = \lambda$. As the number of firms goes to zero, unemployment is pervasive and capital's share reaches its upper bound, $s_K = 1 - z(1-\lambda)$.

The significance of the value of non-market activity, z, decreases as $\theta \to \infty$. The intuition here is simple. As θ becomes small (i.e. unemployment is high), the value z is increasingly important, since wages equal z if exactly one firm arrives. The importance of workers' outside option, z, is negligible if θ is very high (i.e. unemployment is low) since the probability of having only one firm approach a given worker is very small. Once two or more firms compete for a particular worker, wages are bid up and workers' outside option is immaterial.

When are factor shares constant? Suppose the underlying distribution $G(x)$ is held fixed. By *constant* factor shares, I mean simply that factor shares depend only on the parameters inherited from $G(x)$. For the Pareto distribution, this means that factor shares are constant if they depend only on λ. Proposition 2 summarizes two distinct conditions under which factor shares are constant. Both are easy to see from looking at (9). In the limit as $\theta \to \infty$, we have $\gamma(1-\lambda,\theta) \to \Gamma(1-\lambda)$, so the elasticity $\varepsilon(1-\lambda,\theta)$ goes to zero and $s_K = \lambda$. If $z = 1$, the right term disappears so factor shares are again constant, $s_K = \lambda$.

Proposition 2 *Factor shares are constant, $s_K = \lambda$ and $s_L = 1-\lambda$, in two cases:*
(i) Full employment: in the limit as $\theta \to \infty$ (i.e. as unemployment goes to zero);
(ii) Workers' outside option equals minimum firm productivity: $z = x_{\min} = 1$.

In general, if factors are paid their marginal product, factor shares are constant if and only if the aggregate production function is Cobb-Douglas.[23] Equivalently, if factor shares are constant, then factors are paid their marginal product if and only if the aggregate production function is Cobb-Douglas. Both cases in Proposition 2 are consistent with this fact, but in different ways. In case (i), factors are paid their marginal product *and* we have a Cobb-Douglas aggregate production function. In case (ii), factors are *not* paid their marginal product and the aggregate production function is *not* Cobb-Douglas.

Comparative Statics. We now consider how *equilibrium* factor shares vary depending on labor market conditions and the underlying productivity distribution $G(x)$. The equilibrium

[23] The possibility of factor shares that are asymptotically constant along a balanced growth path, due to labor-augmenting technical change and a CES production function with $\sigma < 1$, is a different kind of "constancy" (see Acemoglu (2003)).

capital share is $s_K^* = \lambda + (1-z)\varepsilon(1-\lambda,\theta^*)$ where θ^* solves (8). This is clearly increasing in the rental rate of capital r, since there is only the indirect effect on s_K^* through θ. If r increases, θ^* decreases, and since $\varepsilon(1-\lambda,\theta)$ is decreasing in θ the effect is an increase in the equilibrium capital share s_K^*.

Next, how does s_K^* vary with changes in the value of non-market activity, z? If z increases, the direct effect is that s_K should decrease. However, the indirect effect is that an increase in z leads to a lower level of equilibrium firm entry, θ^*. Since $\varepsilon(1-\lambda,\theta)$ is decreasing in θ, this indirect effect should lead s_K^* to increase. The net result is unclear, but it turns out that the direct effect always dominates: the equilibrium capital share is decreasing in the value of non-market activity, z. While an increase in z increases unemployment, it also increases labor's share of output. (See Appendix A.10.)

Proposition 3 *The equilibrium labor share, s_L^*, is increasing in the value of non-market activity, z, and decreasing in the rental rate of capital r.*

We can also consider how equilibrium factor shares respond to changes in the underlying firm productivity distribution, $G(x)$. For the Pareto distribution, an increase in the shape parameter λ increases both the mean and the variance of this distribution. In turn, this shift in λ feeds into the endogenous productivity distribution across workers and thereby increases aggregate productivity, both directly and through its effect on θ.

When factor shares are "constant", (i.e. if either $\theta \to \infty$ or $z = 1$), it is clear that if λ increases, capital's share rises and labor's share falls. It is important to emphasize that this is not simply a direct "shock" to capital share, however, since it is a *result* that $s_K = \lambda$ as $\theta \to \infty$ or $z = 1$, where λ is the parameter of the underlying productivity distribution.

Outside these special cases, the effect of a change in λ is more complicated. There are two distinct channels through which changes in λ may affect equilibrium factor shares. When λ increases, there is a direct effect which implies that s_K^* should be *increasing* in λ. But there is also a more subtle effect through the term $\varepsilon(1-\lambda,\theta^*)$. There are two components to this channel. First, there is an indirect effect whereby θ^* increases with λ since more firms are attracted by the higher expected profits. This leads to greater competition for workers' labor, which has the effect of decreasing s_K^*, since $\varepsilon(1-\lambda,\theta)$ is decreasing in θ (for a given λ). Second, there is a direct effect through the function $\varepsilon(1-\lambda,\theta)$. Since $\varepsilon(1-\lambda,\theta)$ is decreasing in λ (for a given θ), an increase in λ also has the effect of decreasing s_K^*. The overall effect of an increase in λ through the term $\varepsilon(1-\lambda,\theta^*)$ is therefore one of *decreasing* s_K^*.

Proposition 4 *The equilibrium capital share, s_K^*, is increasing in λ provided the value of* *non-market activity, z, is sufficiently high:*

$$z > \frac{1}{2-\lambda}.$$

It turns out that the net effect is ambiguous: either the positive or the negative channel may dominate. The value of non-market activity z is key in determining the relative importance of these two channels. From (9), we can see that the term $(1-z)$ acts as a kind of "weight" on the component $\varepsilon(1-\lambda,\theta)$ in capital's share. A higher z gives less weight to this component and more weight to the direct influence of the parameter λ. In Appendix A.11, I derive a condition on z and λ, described in Proposition 4, which is sufficient (but not necessary) for the equilibrium capital share to be increasing in λ.

III. DYNAMIC MODEL

This section extends the model to a simple dynamic environment. One limitation of the static model is the fact that the ratio of firms to workers, $\theta = V/L$, and the capital/labor ratio, $k = K/L$, are too closely connected – in fact, they are equal to each other. This makes capital a *flow* value; all capital is newly hired by firms during a single period. The dynamic model corrects this problem by treating capital as a *stock*: the sum of existing capital in place from previously formed matches plus the capital inflow from new firms. At the same time, the minimum wage that workers are willing to accept in the dynamic setting – their *reservation wage* – is endogenous, and it reflects the value of continued search in future periods as well as the flow payoff from unemployment. This enables us to calibrate the model in Section IV and examine its predictions quantitatively.

III.A. Basic Environment

There are an infinite number of discrete time periods. In each period, there is a continuum of homogeneous risk-neutral workers of measure L. At the start of period t, a continuum of measure U_t of these workers are unemployed. In each period, there is also a continuum of risk-neutral potential firms. The measure of *entering* firms in period t is V_t and the ratio of entering firms to unemployed workers is $\phi_t = V_t/U_t$.

Unemployed workers choose whether to accept or reject job offers, and potential firms simultaneously make an entry decision. We will see that there is unique *reservation wage* $b_t \geq 0$ such that workers accept a job offer if and only if the wage offered is greater than or equal to b_t. Given the reservation wage b_t, the level of firm entry ϕ_t is pinned down by a zero profit condition. Given the level of firm entry ϕ_t, workers' reservation wage b_t is determined by an indifference condition that equates the expected payoffs from accepting and rejecting a job offer with wage b_t.

An *equilibrium* is a sequence $\{(\phi_t^*, b_t^*)\}_{t=0}^{\infty}$ that satisfies a zero profit condition for firms and an indifference condition for unemployed workers for each time t. A *steady state equilibrium* is an equilibrium where $\phi_t^* = \phi^*$ and $b_t^* = b^*$ for all t, and unemployment is constant.

To enter, potential firms must pay an upfront cost C_t, which can be interpreted as the cost of purchasing a machine or manufacturing plant with no scrap value. Each machine represents a single unit of capital. After paying the cost C_t, firms draw a productivity level x from a differentiable distribution $G(x)$ with support $(x_{\min}, \infty)$ and no mass points. We normalize $x_{\min} = 1$.

After learning their productivity x, firms can choose to search for a worker. If $x > b_t$, firms search for worker. If $x \leq b_t$, they will exit immediately since they cannot afford to pay workers their reservation wage. The ratio of *searching* firms to unemployed workers is $\theta_t = \phi_t(1 - G(b_t))$. Searching firms approach unemployed workers at random since workers are homogeneous and firms are uncoordinated. As in the static model, this gives rise to a Poisson distribution with parameter θ_t for the number of firms targeting each unemployed worker. For the sake of tractability, workers who are already employed cannot be targeted by firms.

If a firm succeeds in hiring a worker, they can produce output and earn profits until the match is destroyed. Firms discount future profits using a discount factor $\beta \in (0, 1)$. Matches are destroyed at the *end* of each period at an exogenous rate $\delta \in (0, 1]$. When a match is destroyed, the worker becomes unemployed and the firm exits. For simplicity, capital is destroyed when a match is destroyed. When the firm "exits", it re-enters the pool of potential firms who must pay a fixed cost C_t if they wish to search for workers at time t.

If no firms approach a given worker, he receives a flow payoff $z_t \in [0, 1]$ and remains unemployed. If exactly one firm approaches worker j, the firm hires him and produces output $y_j = x_j$ each period. In each period until the match is destroyed, the worker is paid his reservation wage b_t^* *at the time of hiring* t and the firm earns flow profits $\pi_j = x_j - b_t^*$. If more than one firm competes for worker j, the one with the highest productivity hires the worker and produces output $y_j = x_j^1$, the highest productivity, in each period. Unsuccessful firms exit.[24] Wages are determined by Bertrand competition, which reflects the intensity of competition to hire the worker at the time of hiring. The worker is paid a wage $w_j = x_j^2$, the second-highest productivity, and the firm earns flow profits $\pi_j = x_j^1 - x_j^2$ each period until the match is destroyed.

[24]For simplicity, I assume that a firm's capital investment is destroyed when it is unsuccessful in hiring a worker, just as it is when a match is destroyed. Failure to hire a worker can be interpreted as a failure of the new business venture. Less productive firms face a higher probability of failure.

III.B. Unemployment, Capital, and Output

The ratio of searching firms to unemployed workers, θ_t, is the key variable driving unemployment, aggregate output, and factor shares. In this section, we start by describing the evolution of unemployment, capital, and output, taking a sequence $\{(\theta_t, b_t)\}_{t=0}^{\infty}$ as given, and determine the steady state unemployment rate, capital-labor ratio, and output per capita. In Section III.C, we characterize the steady state equilibrium and present some comparative statics results.

Unemployment. Suppose that all workers are unemployed at the start of time $t = 0$, i.e. $U_0 = L$. The evolution of unemployment is given by the following:

$$U_t = U_{t-1} - (1 - e^{-\theta_{t-1}})U_{t-1} + \delta(L - U_{t-1} + (1 - e^{-\theta_{t-1}})U_{t-1}). \tag{10}$$

The number of unemployed workers at the start of period t equals the number who were unemployed at the start of period $t-1$, minus the unemployed workers who found jobs in period $t-1$, plus the number of matches that were destroyed in period $t-1$. This law of motion for unemployment simplifies to

$$U_t = (1 - \delta)\, e^{-\theta_{t-1}}\, U_{t-1} + \delta\, L. \tag{11}$$

The unemployment *rate* u_t in period t is simply the proportion of the labor force who are initially unemployed at the start of period t but who are not approached by a firm during that period, which occurs with probability $e^{-\theta_t}$. That is,

$$u_t = \frac{U_t\, e^{-\theta_t}}{L}. \tag{12}$$

This is the within-period unemployment rate, i.e. the proportion of the labor force that is not employed *within* period t. It does not include those who lose their jobs at the end of the period due to match destruction.[25]

Now consider an economy where $\theta_t = \theta$ and $b_t = b$ for all t. In the limit as $t \to \infty$, we have $U_t = U_{t-1} = U$. Substituting into (11), we obtain the following expression for *steady state unemployment,* which equates the inflows and outflows from unemployment.

$$U = \frac{\delta L}{1 - (1 - \delta)e^{-\theta}}. \tag{13}$$

[25] Since the matching process is urn-ball and match destruction takes place at the end of the period, the unemployment dynamics are the same as those which arise in some directed search models of the labor market, such as Julien et al. (2000).

The *steady state unemployment rate* is given by

$$u(\theta)=\frac{\delta e^{-\theta}}{1-(1-\delta)e^{-\theta}}. \tag{14}$$

Capital. The existing capital stock is given by the number of *active* matches, i.e. the number of matches formed in previous periods that have not yet been destroyed. This is just the number of workers who are employed at the start of period t, $L-U_t$. The quantity of "new" capital in each period equals the number of entering firms, $V_t=\phi_t U_t$, who each pay the cost C_t to purchase a machine during that period, regardless of whether or not they are successful in hiring workers. Total capital K_t is "old" plus "new" capital in period t,

$$K_t=(L-U_t)+\theta_t U_t/(1-G(b_t)). \tag{15}$$

In the steady state, each of the variables in (15) is constant and the capital/labor ratio can be obtained by substituting (13) into (15). Let $k=K/L$, where L is the total labor force. The *steady state capital/labor ratio* is

$$k(\theta,b)=\frac{\delta\theta+(1-\delta)(1-G(b))(1-e^{-\theta})}{(1-G(b))(1-(1-\delta)e^{-\theta})}. \tag{16}$$

We no longer have $\theta=k$, however k is strictly increasing in θ (taking b as given), so we still have a one-to-one mapping between k and θ in the steady state. (See Appendix A.12.)

Aggregate Output. To obtain an aggregate production function, I assume the distribution of firm productivity levels, $G(x)$, is constant over time. This assumption is useful for analytical tractability, but it rules out an examination of how changes in this distribution over time may influence factor shares. As we will see in Section IV, however, the theory can explain much of the post-war variation in factor shares in the U.S. at an annual frequency even without allowing the distribution $G(x)$ to change over time.

Again we assume $G(x)$ is Pareto, namely $G(x)=1-x^{-1/\lambda}$. Suppose that b_t is given. Let $G_{b_t}(x)$ be the (possibly) truncated distribution of *searching* firms' productivity levels, $G_{b_t}(x)=\Pr(X\leq x|x\geq b_t)$. For the Pareto distribution, the truncated distribution remains Pareto, namely $G_{b_t}(x)=1-\left(\frac{x}{x_{0t}}\right)^{-1/\lambda}$ where $x_{0t}=\max\{1,b_t\}$. The minimum of the distribution is $x_{0t}\geq 1$ since only firms with productivity greater than b_t search.

In the dynamic model, aggregate output Y_t at time t can be decomposed into output from new matches, Y_t^{new}, plus output from existing matches,

$$Y_t = Y_t^{new} + (1-\delta)\, Y_{t-1}. \tag{17}$$

The total output, Y_t^{new}, produced by new matches during period t is determined in the same manner as the static model where all matches are "new". New output Y_t^{new} is the expected value of the endogenous productivity distribution across initially unemployed workers at time t, namely $H(x;\theta_t,x_{0t}) = e^{-\theta_t\left(\frac{x}{x_{0t}}\right)^{-1/\lambda}}$ with continuous support $[x_{0t},\infty)$ plus a mass point at zero. The expected value is

$$Y_t^{new} = x_{0_t}\gamma(1-\lambda,\theta_t)\theta_t^{\lambda}U_t. \tag{18}$$

Now suppose we are in a steady state where $\theta_t = \theta$, $b_t = b$, $x_0 = \max\{1,b\}$ and $U_t = U$ for all t. In this case, Y_t^{new} is also constant. Steady state output Y is attained in the limit where the economy entered a steady state for θ infinitely many periods ago.[26] *Steady state aggregate output* is

$$Y = \frac{x_0\gamma(1-\lambda,\theta)\theta^{\lambda}U}{\delta}. \tag{20}$$

In the special case where $\delta = 1$ and $b \leq 1$, we have $Y = \gamma(1-\lambda,\theta)K^{\lambda}L^{1-\lambda}$ and the results from Section II.A hold. In particular, we obtain a Cobb-Douglas production function only in the limit where $\theta \to \infty$ and unemployment disappears.

Substituting in the steady state number of unemployed workers given by (13), we can now determine *steady state output per capita*, $y = Y/L$,

$$y(\theta,b) = \frac{x_0\gamma(1-\lambda,\theta)\theta^{\lambda}}{1-(1-\delta)e^{-\theta}}. \tag{21}$$

Output per capita is increasing in θ (taking b as given). (See Appendix A.13.) Since there is a one-to-one mapping between θ and k, we have $\theta(k)$ and can write $y = f(k)$, although we do not have a simple closed form expression for $f(k)$. This is a unified aggregate production function which incorporates both a matching process and a production technology. In the dynamic setting, output per capita is now also directly affected by the match destruction rate δ and workers' reservation wage b, since x_0 is the minimum of the searching firms' productivity distribution.

[26]To see this, imagine that the economy converged to a steady state s periods ago. Matches that were created n periods ago survive to the current period with probability $(1-\delta)^n$ so Y_t is given by the following expression. As $s \to \infty$, the right term disappears and the left term is constant.

$$Y_t = \sum_{n=0}^{s}(1-\delta)^n x_0\theta^{\lambda}\gamma(1-\lambda,\theta)\theta^{\lambda}U + \sum_{n=s+1}^{t}(1-\delta)^n x_{0_{t-n}}\theta_{t-n}^{\lambda}\gamma(1-\lambda,\theta_{t-n})\theta_{t-n}^{\lambda}U_{t-n}. \tag{19}$$

III.C. *Equilibrium*

Unemployed workers choose to either accept or reject job offers and firms simultaneously make an entry decision that determines θ_t. There will be a unique *reservation wage* b_t such that workers will accept a job offer if and only if the wage offered is greater than or equal to b_t. Given the level of firm entry ϕ_t, we will see that workers' reservation wage b_t is determined by an indifference condition that equates the expected payoffs from accepting and rejecting a job offer with wage b_t. Given a reservation wage b_t, the level of firm entry ϕ_t (and hence the key ratio θ_t) is determined by a zero profit condition.

Recall that an *equilibrium* is a sequence $\{(\phi_t^*, b_t^*)\}_{t=0}^{\infty}$ that jointly satisfies a zero profit condition for firms and an indifference condition for unemployed workers at each time t. A *steady state equilibrium* is an equilibrium where $\phi_t^* = \phi^*$ and $b_t^* = b^*$ for all t, and condition (13) for steady state unemployment holds. In this section, I focus on steady state equilibria. I prove that there exists a unique steady state equilibrium (ϕ^*, b^*) and provide some comparative static results regarding this equilibrium and and some important equilibrium outcomes such as output per capita and the unemployment rate.

Firms' Entry Decision. Suppose we are in a stationary environment where the cost of purchasing capital, C, and the flow payoff from unemployment, z, are constant over time. First consider the firms' entry decision. Taking workers' reservation wage b as given, the level of firm entry, ϕ, is pinned down by a zero profit condition. In any given period, firms will enter until the discounted present value of expected profits, net of the cost of purchasing a machine C, are zero. The zero profit condition extends the static version (7) in a straightforward manner. (See Appendix A.14.)

$$C = \frac{(1-G(b))\left(\int_{x_0}^{\infty} e^{-\theta(1-G_b(x))}(1-G_b(x))dx + e^{-\theta}(x_0 - b)\right)}{(1-\beta(1-\delta))}. \tag{22}$$

where $\theta = \phi(1-G(b))$. Firms who pay the cost C receive a productivity draw from $G(x)$. With probability $1-G(b)$, they search for a worker. Given that they are searching, their expected flow payoff is essentially the expected payoff for bidders in a second-price auction where the distribution of firm productivity levels is now the *truncated* distribution $G_b(x)$ with minimum $x_0 = \max\{1, b\}$. The expected flow payoff is discounted by the effective discount rate, $\beta(1-\delta)$, which incorporates both the discount factor β and the exogenous match survival rate. For the Pareto distribution, the zero profit condition is

$$C = \frac{x_0^{-1/\lambda}\left(x_0 \lambda \theta^{\lambda-1}\gamma(1-\lambda, \theta) + e^{-\theta}(x_0 - b)\right)}{1-\beta(1-\delta)}. \tag{23}$$

Assumption 1 implies that there is a critical value $b_c(\lambda, \beta, \delta, C) > z$ such that for any $b \leq b_c$, there exists a unique level of firm entry $\phi \geq 0$ which satisfies (23). (See Appendix A.15.) If $b > b_c$, there is no firm entry and hence $\phi = 0$. So for any given $b \geq 0$, there is a unique level of firm entry ϕ and we have a function $\phi_r : \mathbb{R}^+ \to \mathbb{R}^+$. In Appendix A.16, I show that when $b \leq b_c$, we have $\phi_r'(b) \leq 0$. Since $\phi = 0$ when $b > b_c$, the function ϕ_r is decreasing in b.

Assumption 1 *The cost of purchasing capital is not too high:* $C < \frac{1}{1-\beta(1-\delta)}\left(\frac{1}{1-\lambda} - z\right)$.

Workers' Reservation Wage. We now turn to workers' decision to accept or reject job offers, taking ϕ as given. Let V^U be the expected value of being unemployed at the start of a period, and let $V^E(x)$ be the expected value of being employed at wage x. When offered a job that pays a wage x, a worker can either accept it and receive $V^E(x)$ or reject it and remain unemployed, in which case he receives $z + \beta V^U$. Unemployed workers will accept a job offer at wage x if $V^E(x) \geq z + \beta V^U$ and reject it otherwise.

Appendix A.17 shows that for any given $\phi \geq 0$, there exists a reservation wage b such that workers will accept a job offer if and only if the wage offered is greater than or equal to b. So we have a function $b_r : \mathbb{R}^+ \to \mathbb{R}^+$ and $b_r'(\phi) \geq 0$ for all ϕ. (See Appendix A.18.) The reservation wage b satisfies the indifference condition (24),

$$V^E(b) = z + \beta V^U. \tag{24}$$

Given the existence of a reservation wage b, equations (25) jointly determine V^U and $V^E(x)$,

$$V^E(x) = x + \beta((1-\delta)V^E(x) + \delta V^U), \tag{25}$$

$$V^U = e^{-\theta}(z + \beta V^U) + (1 - e^{-\theta})V^E(w^{new}),$$

where $\theta = \phi(1 - G(b))$ and w^{new} is the expected wage,

$$w^{new} = \frac{x_0(1-\lambda)\theta^{\lambda}\gamma(1-\lambda, \theta) - \theta e^{-\theta}(x_0 - b)}{1 - e^{-\theta}}. \tag{26}$$

Equations (25) can be understood as follows. If a worker is employed at wage x in the current period, he receives a flow payoff x. If the match survives, he receives the discounted expected value of being employed at wage x in the next period. If the match is destroyed, he receives the discounted expected value of being unemployed at the start of the next period. If a worker is unemployed at the start of the current period, there are two possibilities. With probability $e^{-\theta}$, he remains unemployed and receives a flow payoff z plus the discounted value of being unemployed at the start of the next period. With probability $1 - e^{-\theta}$, the worker is employed at an expected wage of w^{new}.

If we substitute $V^E(b)$ from (25) into (24), we obtain

$$b = z(1 - \beta(1 - \delta)) + \beta(1 - \beta)(1 - \delta)V^U. \tag{27}$$

Substituting V^U from (25) into (27), the *steady state reservation wage* satisfies the following:

$$b = \frac{z(1 - \beta(1 - \delta)) + \beta(1 - \delta)(1 - e^{-\theta})w^{new}}{1 - \beta(1 - \delta)e^{-\theta}}. \tag{28}$$

where w^{new} is given by (26). Observe that when $\phi = \theta = 0$, we have $b_r(\phi) = z$, the value of non-market activity. Since $b_r'(\phi) > 0$, we have $b_r(\phi) \geq z$ for all ϕ. If $\delta = 1$, then $b = z$.

Steady State Equilibrium. A *steady state equilibrium* is a pair (ϕ^*, b^*) that simultaneously satisfies the zero profit condition (23) and workers' indifference condition (24) for each time t, as well as condition (13) for steady state unemployment. We know that for any given $\phi \geq 0$, (28) has a unique solution $b_r(\phi)$. At the same time, we know that for any given $b \leq b_c$, (23) has a unique solution $\phi_r(b)$, and if $b \geq b_c$ we have $\phi_r(b) = 0$. The function b_r is increasing in ϕ, and ϕ_r is decreasing in b. Hence there exists a unique steady state equilibrium (ϕ^*, b^*). This equilibrium satisfies both (28) and (23), and in the limit as $t \to \infty$ the economy also satisfies equation (13) for steady state unemployment.[27]

Figure I provides an example of the functions $b_r(\phi)$ and $\phi_r(b)$ and the equilibrium (ϕ^*, b^*), where $b^* < 1$ and hence $\phi^* = \theta^*$.

Proposition 5 *There is a unique steady state equilibrium* (ϕ^*, b^*) *and* $\theta^* = \phi^*(1 - G(b^*))$.

Proposition 6 contains some comparative statics results regarding the effects of the key parameters – the value of non-market activity z, the cost of purchasing capital C, and the parameter λ from the underlying productivity distribution $G(x)$ – on the equilibrium (ϕ^*, b^*). The effects of the discount factor, β, and the match destruction rate, δ, are ambiguous. (See Appendix A.19.)

We restrict our attention to the case where $b^* < 1$ and $\theta^* = \phi^*$. As we will see in Section III.D, this turns out to be the more interesting case regarding the behavior of factor shares.

Proposition 6 *If* $b^* < x_{\min} = 1$*, workers' reservation wage* b^* *is increasing in* λ *and* z*, and decreasing in* C*. The equilibrium ratio* θ^* *is decreasing in* z *and* C*, and increasing in* λ*.*

[27] Assumption 1 ensures that $b_c > z$, which rules out the possibility that $\phi^* = 0$ and $b^* = z$ is the *only* equilibrium.

Using the expressions for the steady state unemployment rate (14), output per capita (21) and the capital-labor ratio (16), we can infer some comparative statics results regarding the steady state equilibrium unemployment, $u^* = u(\theta^*)$, output per capita, $y^* = y(\theta^*)$, and the capital-labor ratio, $k^* = k(\theta^*)$, for the case where $b^* < 1$. Proposition 7 summarizes these results. (See Appendix A.20.)

Proposition 7 *If $b^* < x_{\min} = 1$, then (i) the equilibrium unemployment rate u^* is increasing in both z and C, and decreasing in λ; (ii) the equilibrium capital-labor ratio k^* is decreasing in z and C, and increasing in λ; (iii) equilibrium output per capita $y^*$is decreasing in z and C, and increasing in λ.*

III.D. Factor Income Shares

In this section, I first derive general expressions for factor shares when the economy is not necessarily in a steady state, taking a sequence $\{(\theta_t, b_t)\}_{t=0}^{\infty}$ as given. I then determine steady state factor shares and present some comparative statics results regarding the equilibrium factor shares in the steady state.

Factor shares are effectively a *weighted average* of the relative income shares determined in both new matches and previously formed matches that are still active. To start with, expected wages for all workers who are unemployed at the start of period t, including those who remain unemployed, is $w_t^{new}(1-e^{-\theta_t})$, where w_t^{new} is given by (26). Substituting in (18) for the output from new matches, Y_t^{new}, labor's share for *new* matches at time t is

$$s_{L,t}^{new} = \frac{w_t^{new}(1-e^{-\theta_t})U_t}{Y_t^{new}} = 1 - \lambda - \left(1 - \frac{b_t}{x_{0t}}\right)\varepsilon(1-\lambda, \theta_t). \tag{29}$$

Capital's share in *new* matches at time t, $s_{K,t}^{new} \equiv 1 - s_{Lt}^{new}$, is therefore

$$s_{K,t}^{new} = \lambda + \left(1 - \frac{b_t}{x_{0t}}\right)\varepsilon(1-\lambda, \theta_t). \tag{30}$$

Capital's share of aggregate output at time t is given by the following weighted average,

$$s_{K,t} = \frac{s_{K,t}^{new}\,Y_t^{new} + s_{K,t-1}(1-\delta)Y_{t-1}}{Y_t}. \tag{31}$$

Capital's share in new matches, $s_{K,t}^{new}$, is weighted by the value of new output, Y_t^{new}, and the previous period's capital share is weighted by the value of output produced by matches existing in the previous period that are still active.

Steady state capital share is given by

$$s_K = \lambda + \left(1 - \frac{b}{x_0}\right) \varepsilon(1-\lambda, \theta), \tag{32}$$

where $x_0 = \max\{1, b\}$ and $\varepsilon(1-\lambda, \theta)$ is the elasticity given by (6). Since $b = z < 1$ in the static model, equation (32) nests the earlier result for the static model (9) as a special case.

In general, steady state capital share is decreasing in θ (taking b as given) since the elasticity $\varepsilon(1-\lambda, \theta)$ is decreasing in θ. Intuitively, as the ratio of searching firms to unemployed workers rises, greater competition for workers leads to an increase in labor's share. Since greater competition for workers, reflected in a higher ratio θ, leads to lower unemployment $u(\theta)$ through (14), this gives rise to a negative relationship between unemployment and labor's share. At the same time, capital share is decreasing in b (taking θ as given) since workers are paid more in bilateral matches when the reservation wage is higher.

Proposition 8 *Factor shares are constant, $s_K = \lambda$ and $s_L = 1 - \lambda$, in two cases:*
(i) Full employment: in the limit as $\theta \to \infty$ (i.e. as unemployment goes to zero);
(ii) Reservation wage equals or exceeds the minimum firm productivity: $b \geq x_{\min} = 1$.

When are factor shares constant? Steady state factor shares are constant in two distinct cases which are analogous to Proposition 2 for the static model. First, there is an asymptotic result: factor shares are constant in the limit as $\theta \to \infty$ and unemployment disappears. Second, factor shares are constant when workers' reservation wage equals or exceeds the minimum firm productivity. The model predicts constant factor shares when workers' reservation wage b exceeds the minimum firm productivity level, $x_{\min} = 1$. In Section IV, when we calibrate the model, we will be in a position to see whether this condition is likely to actually hold. It turns out that in the benchmark calibration the endogenous reservation wage is always less than one, which allows for some variation in factor shares over time.

Decomposition of Capital Share. We treat capital share simply as the residual of labor's share, $s_K \equiv 1 - s_L$. However, capital share can be decomposed into two distinct components in the dynamic setting. The exact breakdown depends on the match destruction rate δ and the discount factor β. Each period, the share of output going to the *owners of capital*, s_C, is

$$s_C = \frac{\delta}{1 - \beta(1-\delta)} \left(\lambda + \left(1 - \frac{b}{x_0}\right) \varepsilon(1-\lambda, \theta)\right). \tag{33}$$

This is the proportion of aggregate output paid to the owners of capital at the start of each period when firms pay the cost C. Each period, the share of output, s_π, accruing to firms as a group, after wages and capital costs are paid, is

$$s_\pi = \frac{(1-\delta)(1-\beta)}{1-\beta(1-\delta)} \left(\lambda + \left(1 - \frac{b}{x_0} \right) \varepsilon(1-\lambda, \theta) \right). \tag{34}$$

If $\delta = 1$, we have $s_\pi = 0$ and there is no firm profit share, as in the static model. In general, the ratio s_π / s_C is decreasing in both the match destruction rate δ and the discount factor β, but the aggregate division of output between capital and labor does not depend directly on δ and β. The shares of output accruing to capital owners and firms are grouped together as *capital share*, $s_K = s_C + s_\pi$, where s_K is given by (32).

Comparative Statics. We now present some comparative statics results for the steady state equilibrium factor shares. For the interesting case where $b^* < 1$ and there is some variation in factor shares, the following expression represents the *steady state equilibrium capital share,*

$$s_K^* = \lambda + (1 - b^*)\, \varepsilon(1-\lambda, \theta^*), \tag{35}$$

where b^* is the equilibrium reservation wage and $\theta^* = \phi^*$ is the equilibrium ratio of searching firms to unemployed workers.

Proposition 9 *The equilibrium labor share, s_L^*, is increasing in the value of non-market activity, z, and decreasing in the cost of purchasing capital, C.*

How do equilibrium factor shares respond to a change in the cost of investing in capital? Suppose there is an increase in C. There is an indirect effect on capital share through the equilibrium ratio θ^*. A higher cost C means that entry is less attractive for firms, which decreases the equilibrium ratio θ^*. The decrease in θ^* has a positive effect on capital share since there is less competition for workers. Overall, equilibrium capital share is increasing in the cost C.

We can also compare two economies with different values of non-market activity, z. There are two opposing effects of an increase in z. When z increases, b^* increases and there is a negative effect on capital share. But there is also a positive indirect effect on capital share through θ^*, since a higher z decreases the level of firm entry, which has a positive impact on capital share. Overall, the negative effect dominates. (See Appendix A.21.)

IV. QUANTITATIVE EXERCISE

In this section, I examine *one* of the model's predictions quantitatively by asking the following question: Through the lens of the model, can variations in unemployment rates and unemployment benefits explain the behavior of factor shares in the U.S. over the period 1951-2010? To answer this question, I calibrate the model and use annual data on unemployment rates and unemployment insurance eligibility to predict the movements in factor shares during this period. The model's predictions are then compared with the data.

The theory predicts that factor shares are constant when workers' reservation wage equals or exceeds the minimum firm productivity level. In the benchmark calibration, the endogenous reservation wage is always less than the minimum firm productivity, which allows for systematic variation in factor shares. To reach the threshold value of $b_t \geq x_{\min} = 1$, a relatively high level for the value of non-market activity, z_t, is required. For more realistic values of z_t – less than around 70% of average wages – the model predicts that factor shares are not constant but they are at least relatively stable, in a sense we will now make precise.

Strategy. The basic strategy for this quantitative exercise is quite simple. I use unemployment data to directly pin down θ_t for each period, in order to match the unemployment rates exactly. The sequence of reservation wages, $\{b_t\}$, is in turn determined by the values $\{\theta_t\}$, and factor shares are determined by the entire sequence $\{(\theta_t, b_t)\}$. By matching the unemployment rates directly, I abstract from the well-known difficulties that search models have in generating sizeable fluctuations in unemployment. This enables me to examine the relationship between unemployment and factor shares predicted by the model, *taking the fluctuations in unemployment as given.*

Given a sequence of unemployment rates $\{u_t\}$ and the parameter δ, we can determine the corresponding sequence $\{\theta_t\}$ using (14) and (10), and thereby the sequence of reservation wages $\{b_t\}$. Given a sequence $\{(\theta_t, b_t)\}$, aggregate output Y_t is given by (19) and (18), and capital's share s_{Kt} is given by (32) and (30) for each period t.[28]

Workers' reservation wage b_t depends on both current labor market conditions and expectations of future values of θ_t and z_t. Let $V_t^E(x)$ be the expected value of being employed at wage x at the start of period t, and let V_t^U be the expected value of being unemployed at the start of period t. A reservation wage b_t at time t must satisfy $V_t^E(b_t) = z_t + \beta V_{t+1}^U$, where

$$V_t^E(x) = x + \beta((1-\delta)V_{t+1}^E(x) + \delta V_{t+1}^U),$$

$$V_t^U = e^{-\theta_t}(z_t + \beta V_{t+1}^U) + (1 - e^{-\theta_t})V_t^E(w_t^{new}).$$

[28] Assuming the zero profit condition for firm entry holds for all t, we could of course back out the *implied* sequence of values $\{C_t\}$ using the following expression:

$$C_t = \frac{x_{0t}^{-1/\lambda}\left(x_{0t}\lambda\theta_t^{\lambda-1}\gamma(1-\lambda,\theta_t) + e^{-\theta_t}(x_{0t} - b_t)\right)}{1-\beta(1-\delta)}, \quad x_{0t} = \max\{1, b_t\}.$$

Suppose that $z_t = z$ and $\theta_t = \theta$ from time T onwards. There exists a unique reservation wage $b_t = b \in \mathbb{R}^+$ for all $t \geq T$, and hence we can determine the steady state value of unemployment, $V_T^U = V^U(\theta, b, z)$. Given the sequence $\{\theta_t\}_{t=0}^T$, we can calculate the entire remaining sequence of reservation wages $\{b_t\}_{t=0}^{T-1}$ by working backwards from period T, provided we can also determine the sequence $\{z_t\}_{t=0}^T$.

For this simple exercise, I present two extreme cases with regard to workers' expectations. In the myopic equilibrium, workers use only current labor market conditions, as given by θ_t and z_t, to determine their reservation wage b_t. In the perfect foresight equilibrium, workers use both current and future values of θ_t and z_t to determine b_t; there is no uncertainty.

Value of Non-Market Activity. In the search literature, it is standard to think of the flow payoff from unemployment, z_t, in terms of a "replacement rate", i.e. the level of unemployment insurance benefits as a percentage of average wages, although it is understood to include the value of leisure. Generally, a single value for this replacement rate is chosen and the issue of eligibility for benefits is ignored. Over the post-war period, however, eligibility for unemployment insurance (UI) has increased dramatically in the U.S., while the after-tax replacement rate for insured workers has been relatively constant (Anderson and Meyer (1997)). For example, the UI coverage rate (or percentage of workers who are eligible for UI) was around 59% in 1948 compared with 93% in 1980.[29] As Figure II indicates, while eligibility increased steadily from 1948 until around 1980, it has been fairly stable since then.

For this quantitative exercise, I define the value of non-market activity, z_t, in a way that incorporates UI eligibility. I start by choosing a target level α for the average value of non-market activity as a percentage of average wages throughout the entire period. This enhances comparability with the search-theoretic literature, where a single value for α is generally chosen. I define the value of non-market activity in period t as $z_t = \eta p_t w_{t-1}$ where w_{t-1} is average wages in the previous period,[30] p_t is the *probability* of being eligible for UI, and η is a "normalization" parameter that enables us to match the target α.

The value of p_t is determined by both UI coverage data and long-term unemployment data. In particular, define $p_t \equiv e_t(1 - LTU_t)$, where LTU_t is the long-term unemployment rate and e_t is the UI coverage rate.[31] The inclusion of long-term unemployment is intended as a simple proxy for the fact that covered workers who become long-term unemployed may lose eligibility for UI due to exhaustion of benefits. Greater incidence of long-term unemployment

[29] Economic Report of the President (2009, 1983)

[30] Average wages for employed workers in the previous period is given by $w_{t-1} = s_{Lt-1}Y_{t-1}/(1 - u_{t-1})$.

[31] It is important to use the UI coverage rate since a worker's outside option depends on whether he or she is *eligible* for unemployment insurance. Actual receipt of UI benefits is a different matter altogether, as this depends on take-up rates which are affected by many other factors and vary across states. For details on this, see for example Blank and Card (1991).

decreases the average value of non-market activity, z_t, by decreasing p_t.

Data Sources. The factor shares data is found in the Bureau of Labor Statistics Multifactor Productivity historical release, which covers the period 1948-2010. This data is derived by the BLS from the NIPA. The BLS Multifactor Productivity release provides data for the private business sector, which excludes both general government and government enterprises.[32] The unemployment rate data is the BLS civilian annual average unemployment rate from 1948-2012. The long-term unemployment rate is the BLS measure of the annual average number of unemployed workers who are unemployed for 27 weeks or longer as a proportion of the total unemployed from 1948-2012. The UI coverage rate is the proportion of employed workers in the civilian labor force who are eligible for UI benefits. This data is obtained from the Economic Report of the President (2009, 1983) and covers the period 1948-2007.[33]

Calibration. For this quantitative exercise, I assume that all production and payments are shifted forward by one period after matching occurs so that it takes one year for current labor market conditions to be reflected in factor shares. Since I use annual data, shorter frequency movements in factor shares are not captured in this exercise.

In the initial period, before the unemployment data starts in 1948, I assume the economy's unemployment rate is the long-run average of 5.8% for the full period 1948-2012. At the end of the period 1948-2012, I assume there is a transition to a steady state equilibrium corresponding to the long-run average unemployment rate of 5.8%. In both the initial period and the end steady state, I assume the long-term unemployment rate is the long-run average of 14.7%, but the coverage rate e_t equals the 1948 level in the initial period and the 2007 level in the end steady state (due to the time trend).

During the period 1948-2012, I use the unemployment data to generate predictions for factor shares, but I start to *test* the model's predictions only from 1951 due to the weighted average nature of factor shares. By 1951, more than 90% of the matches have been formed during 1948 onwards, so the results are not sensitive to the initial unemployment rate.

[32] The BLS MFP release defines the Private Business Sector as GDP minus general government and government enterprises, minus output of household workers, nonprofit institutions, gross housing product of owner-occupied dwellings, and the rental value of nonprofit institutional real estate. Labor's share is labor compensation divided by the total of labor and capital cost. Labor compensation is defined as wages and salaries of employees plus employers' contributions for social insurance and private benefit plans, and all other fringe benefits in current dollars. An estimate of the wages, salaries, and supplemental payments of the self-employed is included. Technical details regarding the BLS MFP data can be found at http://www.bls.gov/mfp/mprtech.pdf.

[33] Through 1996, covered employment includes persons under the following programs: State, Unemployment Compensation for Federal Employees (UCFE), Railroad Retirement Board (RRB), and Unemployment Compensation for Ex-Servicemembers (UCX). From 1997, covered employment includes only the State and UCFE programs.

In the benchmark calibration, I set $\beta = 0.95$, $\delta = 0.45$ and $\alpha = 0.5$. The target α for the average value of non-market activity as a percentage of average wages is perhaps the most controversial value to choose. There has been much debate in the search literature about what this value should be. In Shimer (2005b), a value of 0.4 is chosen, while in Hagedorn and Manovskii (2008) the value of non-market activity is much higher. The approach taken here is to simply let $\alpha = 0.5$ and then perform some robustness checks where I vary this target. I also conduct robustness checks where I vary δ.[34]

After setting δ, β and α, I simultaneously choose a value for λ to match the mean capital share, a value for η to match the target α, a sequence $\{\theta_t\}_{t=1}^{T}$ to match the unemployment rates from 1948 to 2012, and values θ_0 and θ_{SS} to match the long-run average unemployment rate in the initial period and the end steady state respectively.

In the benchmark calibration, I choose λ to match the mean capital share during the period 1951-2003. I take this approach because capital's share rises sharply in 2004-2005 in the data. If I include the period 2004-2010, the mean capital share rises and the standard deviation jumps up dramatically, affecting the fit of the model for the entire period 1951-2003. As we will see, the model cannot explain the sharp rise in capital's share during 2004-2005 using a single value of λ, so I focus the analysis on the period 1951-2003.

IV.A. Results

Figure III compares the model's predictions for capital's share when workers are myopic with the U.S. data for the full period 1951-2010. Figure IV presents the same graph for the perfect foresight equilibrium. The main difference between the two alternatives is that the model's predictions for factor shares are smoothed out when workers have perfect foresight because current conditions do not have quite as strong an impact on factor shares.

One striking feature of Figures III and IV is the steep rise in capital's share in the 2000s, particularly around 2004-2005. Capital's share jumps from 0.32 in 2003 to 0.35 in 2005. This dramatic rise is not predicted by the model, despite the relatively good fit from 1951 to 2003. There are many factors outside the scope of the present model which could be relevant here. Through the lens of the model, it certainly appears that something very unusual happened to factor shares in the U.S. after 2003, something which cannot be easily reconciled with their

[34] The annual match destruction rate of $\delta = 0.45$ in the benchmark calibration represents a monthly rate of 3.75%. This includes both job-to-job transitions and transitions from employment into unemployment. (Job-to-job transitions are not explicit in the model but correspond to cases where a worker loses their job at the end of one period and instantly gains another at the start of the next period without going through any spell of unemployment.) Estimates of the job separation rate, including both quits and layoffs, are usually between 3-5% monthly or 35-60% annually. See Davis et al. (2010) for a discussion of the downward bias in the published JOLTS separations rate and an adjusted estimate of 4.65% monthly (quits plus layoffs) compared with the published JOLTS rate of 3.46% for the period 2001-2006.

behavior through the entire post-war period prior to this point.

Table I summarizes the quantitative results for the myopic and perfect foresight equilibria for 1951-2003. The correlation between the data and the model's predictions during the period of over fifty years from 1951-2003 is 0.69 for the perfect foresight equilibrium and 0.73 when workers are myopic. The standard deviation of capital's share is lower than in the data, 0.0068 for the perfect foresight equilibrium and 0.0075 when workers are myopic compared with 0.0093. At least part of this discrepancy is likely due to measurement error.

For the sake of comparison, Figure V is a scatterplot showing the weakly positive relationship between capital's share in the US from 1951-2003 and the unemployment rate one year earlier. Figure VI is a scatterplot showing the much stronger linear relationship between the model's predictions for capital share and the data on capital share from 1951-2003.

As Table I indicates, the autocorrelation of the model's predictions for capital's share is significantly higher than in the data, 0.895 for the perfect foresight equilibrium and 0.847 for the myopic model compared with 0.611. Again, this may be due partly to measurement error, but it may also be due to certain simplifying assumptions of the model. In particular, wages for existing matches cannot be re-negotiated and there is no on-the-job search, so only *new* matches respond to changes in labor market conditions. Relaxing these assumptions in future extensions of the model may somewhat reduce the persistence of factor shares.

In terms of robustness checks, I present the results of different calibrations of the model where I allow α and δ to vary. In Table II I allow α to vary while holding $\delta = 0.45$ fixed, and in Table III I allow δ to vary while holding $\alpha = 0.5$ fixed. Since the results for the perfect foresight equilibrium are not quite as strong as when workers are myopic, I proceed conservatively and focus exclusively on the perfect foresight case.

Table II shows that the main effect of varying the target replacement rate α is that the standard deviation of capital's share decreases when α increases. This is intuitive: as α increases, the value of non-market activity z_t increases and so does workers' reservation wage b_t, decreasing the variation in factor shares. Eventually, if the reservation wage is sufficiently high that it exceeds the minimum firm productivity, factor shares will be constant. Lower values of α are more consistent with the variation we see in the data but if α is too low this diminishes the importance of changes in z_t over time, which help the model explain the movements in factor shares over the long time period 1951-2003.

Table III indicates that the correlation between the data and the model's predictions for capital's share hardly changes at all for different values of the match destruction rate, δ. The main effect of increasing δ is that the standard deviation is higher due to the fact that a greater proportion of matches are newly formed each period and hence factor shares exhibit a higher degree of sensitivity to changes in labor market conditions.

V. CONCLUSION

This paper presents a novel theory designed to answer the classic question of how factor shares are determined. To answer this question, I first develop microfoundations for a unified aggregate production and matching function that incorporates unemployment. In contrast with the classic result of Houthakker (1955), and more recently Jones (2005) and Lagos (2006), the aggregate production function derived here is Cobb-Douglas only in the frictionless limiting case where unemployment disappears. In general, the elasticity of substitution between capital and labor is less than one.

The wage determination mechanism of Bertrand competition which arises naturally in this environment, together with the aggregate production and matching function, jointly determine factor shares. In theory, constant factor shares can arise in two distinct cases: in the limit as unemployment disappears, and when workers' reservation wage exceeds the minimum firm productivity. In general, factor shares are variable and they depend crucially on unemployment and workers' reservation wage, as well as on the underlying Pareto distribution of firm productivity levels.

The results of the quantitative exercise suggest that labor market conditions are indeed a key driver of the behavior of factor shares. Using only data on unemployment rates and unemployment insurance eligibility, the model can explain much of the behavior of factor shares in the U.S. over a period of more than fifty years from 1951 to 2003.

There is much further research to be done. Testing the theory's predictions for countries other than the U.S. and looking at cross-country differences in the behavior of factor shares are next on the agenda, as is attempting to explain the puzzle of why labor's share fell so sharply in the U.S. during the recent period 2004-2010. At the same time, one extension of the model which might improve its ability to match the persistence properties of the factor shares data would be to incorporate on-the-job search for workers.

The unified theory of production, employment, and wages developed in this paper is both simple and parsimonious but also rich in predictions not only regarding factor shares but many important topics in macroeconomics such as unemployment fluctuations, wage and productivity dispersion, and the nature of the aggregate production function. Exploring and testing these predictions is part of a future research agenda.

APPENDIX

A.0 Useful facts

Here I collect some useful facts that will be used repeatedly in the proofs in this Appendix. For any $s \in \mathbb{R}^+$and $x \in \mathbb{R}^+$, the Lower Incomplete Gamma function is defined in the following manner.

$$\gamma(s,x) \equiv \int_0^x t^{s-1}e^{-t}\ dt \tag{36}$$

Fact 1 *Recurrence relation:* $\gamma(s,x) = (s-1)\gamma(s-1,x) - x^{s-1}e^{-x}$

Fact 2 $\frac{\partial}{\partial x}\gamma(s,x) = x^{s-1}e^{-x}$

Fact 3 $\frac{\partial}{\partial s}\gamma(s,x) = \int_0^x t^{s-1}e^{-t}(\ln t)dt$

Fact 4 *The elasticity of* $\gamma(s,x)$ *with respect to* x *is*

$$\varepsilon(s,x) = \frac{x^s e^{-x}}{\gamma(s,x)} \tag{37}$$

Fact 5 $\varepsilon(s,x)$ *is strictly increasing in* s, $\frac{\partial}{\partial s}\varepsilon(s,x) > 0$*, for* $x > 0$.

Fact 6 $\varepsilon(s,x)$ *is decreasing in* x, $\frac{\partial}{\partial x}\varepsilon(s,x) \leq 0$

Fact 1 is obtained through integration by parts on (36). Facts 5 and 6 can be derived as follows. Differentiating (37) with respect to s using Fact 3, we obtain

$$\frac{\partial}{\partial s}\varepsilon(s,x) = x^s e^{-x}\left(\frac{\int_0^x(\ln x - \ln t)t^{s-1}e^{-t}dt}{\gamma(s,x)^2}\right)$$

If $x > 0$, since $\ln x \geq \ln t$ for all $t \leq x$ we have $\int_0^x(\ln x - \ln t)t^{s-1}e^{-t}dt > 0$, so $\frac{\partial}{\partial s}\varepsilon(s,x) > 0$. Differentiating $\varepsilon(s,x)$ with respect to x using Fact 2, we have

$$\frac{\partial}{\partial x}\varepsilon(s,x) = \frac{x^{s-1}e^{-x}}{\gamma(s,x)}\left(s - x - \frac{x^s e^{-x}}{\gamma(s,x)}\right) \tag{38}$$

So $\frac{\partial}{\partial x}\varepsilon(s,x) \leq 0$ if and only if $x \geq s - \varepsilon(s,x)$, or equivalently $x \geq s - \varepsilon(s,x)$. Applying Fact 1, this is true provided that

$$x \geq \frac{\gamma(s+1,x)}{\gamma(s,x)}$$

Rearranging and multiplying both sides by x^s, this is true if and only if $\varepsilon(s+1,x) \geq \varepsilon(s,x)$, which follows from Fact 5. So we have $\frac{\partial}{\partial x}\varepsilon(s,x) \leq 0$.

A.1 Properties of aggregate production function

Letting $\theta = k$ and taking the expected value of the distribution $H(x;k) = e^{-k(1-G(x))}$ with continuous support $[1, \infty)$ plus a mass point at zero, we have

$$f(k) = \int_1^\infty xkg(x)e^{-k(1-G(x))}dx$$

Now let $G(x) = 1 - x^{-1/\lambda}$. Changing variables, letting let $t = k(1 - G(x))$,

$$f(k) = k^\lambda \int_0^k t^{-\lambda} e^{-t} dt$$

Now, by definition, $\gamma(s, x) = \int_0^x t^{s-1}e^{-t}\, dt$, the Lower Incomplete Gamma function. Setting $x = k$ and $s = 1 - \lambda$, we have the intensive production function (2),

$$f(k) = \gamma(1 - \lambda, k)k^\lambda$$

By Fact 2, we obtain

$$f'(k) = \lambda k^{\lambda-1}\gamma(1 - \lambda, k) + e^{-k} > 0 \tag{39}$$

So $\lim_{k\to 0} f'(k) = \lim_{k\to 0} \lambda k^{\lambda-1}\gamma(1-\lambda, k)+e^{-k} = (1-\lambda)^{-1}$, $\lim_{k\to\infty} f'(k) = \lim_{k\to\infty} \lambda k^{\lambda-1}\gamma(1-\lambda, k) + e^{-k} = 0$. Finally, to derive $f''(k)$ we use Fact 1, setting $s = 2 - \lambda$ and $x = k$ to obtain

$$f''(k) = -\left(\lambda k^{\lambda-2}\gamma(2 - \lambda, k) + e^{-k}\right) < 0 \tag{40}$$

A.2 Derivation of elasticity of substitution

Inserting $f'(k)$ and $f''(k)$ from (39) and (40) into definition (4), we have

$$\begin{aligned} \sigma &= \frac{-f'(k)(f(k) - kf'(k))}{kf(k)f''(k)} \\ &= \frac{-(\lambda k^{\lambda-1}\gamma(1 - \lambda, k) + e^{-k})k^\lambda\gamma(2 - \lambda, k)}{k^{\lambda+1}\gamma(1 - \lambda, k)(-(\lambda k^{\lambda-2}\gamma(2 - \lambda, k) + e^{-k}))} \end{aligned}$$

Rearranging and simplifying, we have

$$\sigma = \frac{\lambda + k^{1-\lambda}e^{-k}/\gamma(1 - \lambda, k)}{\lambda + k^{2-\lambda}e^{-k}/\gamma(2 - \lambda, k)}$$

Using $k = \theta$, we have (5),

$$\sigma = \frac{\lambda + \varepsilon(1 - \lambda, \theta)}{\lambda + \varepsilon(2 - \lambda, \theta)}.$$

A.3 Proof of Proposition 1

To show that $\sigma < 1$, it is sufficient to show that $\varepsilon(1 - \lambda, \theta) < \varepsilon(2 - \lambda, \theta)$. This follows directly from Fact 5, which states that $\varepsilon(s, x)$ is increasing in s for $x > 0$.

Proof that $\tilde{\sigma} < 1$. The elasticity of substitution σ is defined as a property of the function $f(\theta) = \gamma(1-\lambda, \theta)\theta^{\lambda}$, where $\theta = V/L = K/L$ and L is the labor force, or total number of potential workers. We can also consider the elasticity of substitution $\tilde{\sigma}$ defined as a property of the function $f(\kappa) = \gamma(1-\lambda, \kappa)\kappa^{\lambda}$ where $\kappa \equiv K/L_e$ and $L_e \equiv (1-e^{-\theta})L$, the number of employed workers. So we have $\kappa = \theta/(1-e^{-\theta})$. Since $\kappa'(\theta) = (1-e^{-\theta} - \theta e^{-\theta})/(1-e^{-\theta})^2 > 0$, $\kappa(\theta)$ is invertible and we can write $\theta(\kappa)$. Let $g(\kappa) \equiv f(\theta(\kappa))$. The elasticity of substitution between capital and employed workers, $\tilde{\sigma}$, is given by

$$\tilde{\sigma} = \frac{-g'(\kappa)(g(\kappa) - \kappa g'(\kappa))}{\kappa g(\kappa) g''(\kappa)}$$

In the limit as $\theta \to \infty$, it is clear that $\kappa/\theta \to 1$, so we have $\tilde{\sigma} \to \sigma$ and hence $\tilde{\sigma} \to 1$ as $\theta \to \infty$. To show that $\tilde{\sigma} < 1$ in general, it suffices to show that $\tilde{\sigma} \leq \sigma$ as we have already proven that $\sigma < 1$. Now $g'(\kappa) = f'(\theta)\theta'(\kappa)$ and $g''(\kappa) = f''(\theta)(\theta'(\kappa))^2 + f'(\theta)\theta''(\kappa)$. So we have

$$\tilde{\sigma} = \frac{-f'(\theta)\theta'(\kappa)(f(\theta) - \kappa f'(\theta)\theta'(\kappa))}{\kappa f(\theta)(f''(\theta)(\theta'(\kappa))^2 + f'(\theta)\theta''(\kappa))}$$

To show that $\tilde{\sigma} \leq \sigma$, we need to prove that

$$\tilde{\sigma} = \frac{-f'(\theta)\theta'(\kappa)(f(\theta) - \kappa f'(\theta)\theta'(\kappa))}{\kappa f(\theta)(f''(\theta)(\theta'(\kappa))^2 + f'(\theta)\theta''(\kappa))} \leq \frac{-f'(\theta)(f(\theta) - \theta f'(\theta))}{\theta f(\theta) f''(\theta)} = \sigma$$

Substituting in $\kappa = \theta/(1-e^{-\theta})$ and using the fact that $f(\theta) \geq 0$ and $f'(\theta) > 0$ from (39), this is equivalent to showing that

$$\frac{\theta'(\kappa)((1-e^{-\theta})f(\theta) - \theta f'(\theta)\theta'(\kappa))}{(f''(\theta)(\theta'(\kappa))^2 + f'(\theta)\theta''(\kappa))} \geq \frac{f(\theta) - \theta f'(\theta)}{f''(\theta)} \tag{41}$$

Now $\kappa'(\theta) = (1-e^{-\theta} - \theta e^{-\theta})/(1-e^{-\theta})^2$, so we have

$$\theta'(\kappa) = \frac{(1-e^{-\theta})^2}{1-e^{-\theta} - \theta e^{-\theta}}$$

Differentiating with respect to κ and simplifying,

$$\theta''(\kappa) = \frac{e^{-\theta}(1-e^{-\theta})(2-\theta-e^{-\theta}(2+\theta))}{(1-e^{-\theta}-\theta e^{-\theta})^2}$$

Using the fact that $f''(\theta) < 0$ from (40) and rearranging (41), we need to prove that

$$\frac{Af(\theta) - \theta f'(\theta)}{1+A} \leq f(\theta) - \theta f'(\theta) \tag{42}$$

where

$$A = \frac{1-e^{-\theta}}{\theta'(\kappa)} \text{ and } B = \frac{f'(\theta)\theta''(\kappa)}{f''(\theta)(\theta'(\kappa))^2}$$

Now $1 - e^{-\theta} \leq \theta'(\kappa)$ so $A \leq 1$ and it suffices to show that $B \geq 0$. Since $f'(\theta) > 0$ and $f''(\theta) < 0$, from (39) and (40) respectively, and clearly $(\theta'(\kappa))^2 > 0$, we need only show that $\theta''(\kappa) \leq 0$, which holds if and only if $2 - \theta - e^{-\theta}(2+\theta) \leq 0$. Let $h(\theta) = 2 - \theta - e^{-\theta}(2+\theta)$. Then $h'(\theta) = -(1 - e^{-\theta} - \theta e^{-\theta}) \leq 0$ and $h(0) = 0$, so $h(\theta) \leq 0$ for all $\theta \geq 0$. Hence $A \geq 0$ and therefore (42) also holds, so we have proven that $\tilde{\sigma} \leq \sigma$ and hence $\tilde{\sigma} < 1$.

A.4 Derivation of zero profit condition

Suppose that a firm has a draw from the distribution $G(x)$ and that n firms are competing to hire a given worker j. Conditional on x and n, let $\beta(x,n)$ be the probability a firm is successful in hiring worker j, let $R(x,n)$ be the expected payoff if it is successful, and let $\pi(x,n)$ be the expected payoff for the firm net of entry cost, namely $\pi(x,n) = \beta(x,n)R(x,n) - r$. From the perspective of the firm, we have $n \geq 1$ necessarily, so there are two cases to consider: $n = 1$ and $n \geq 2$.

Case 1. $n = 1$. In this case, the wage paid is z. The expected net payoff, given a draw x from $G(x)$, is therefore $\pi(x,1) = x - z - r$. Integrating over the distribution $G(x)$, we have the expected net payoff from a bilateral match is $\pi_1 = \int_1^\infty (x - z)\ dG(x) - r$.

Case 2. $n \geq 2$. Suppose that $n \geq 2$ firms approach worker j. In this case, $R(x,n) = (x - w(x,n))$, where $w(x,n)$ is the expected value of the second-best draw given that x is the highest productivity at worker j and there are n firms. The expected net payoff for a firm in this case is

$$\pi_2(x,n) = \beta(x,n)(x - w(x,n)) - r \tag{43}$$

Here $w(x,n) = E(Y_2^n | Y_1^n = x)$, where Y_2^n is the second order statistic from n draws, and Y_1^n is the best draw from n draws. Let $H(y,n)$ be the distribution of Y_1^n, i.e. the distribution of the first-order statistic, which is just $H(y,n) = G(y)^n$. Now $E(Y_2^n | Y_1^n = x) = E(Y_1^{n-1} | Y_1^{n-1} < x)$ where Y_1^{n-1} is the first order statistic for $n-1$ draws (see p. 23 Krishna (2010)). Expected wages as a function of the highest productivity, x, and the number of firms, n, is therefore $w(x,n) = \frac{1}{H(x,n-1)} \int_1^x y\ dH(y,n-1)$. Substituting $w(x,n)$ into (43), we have

$$\pi_2(x,n) = \beta(x,n)\left(x - \frac{1}{H(x,n-1)} \int_1^x y \quad dH(y,n-1)\right) - r$$

Now $\beta(x,n)$ can be determined as follows. Given n firms at worker j, the probability that a given one of these firms, with productivity draw x, has the best idea is $G(x)^{n-1}$. This is the probability that the other $n-1$ firms at worker j all have draws less than x. (By assumption, $G(x)$ has no mass points, so the probability that two firms draw identical values is zero.) So $\beta(x,n) = G(x)^{n-1} = H(x,n-1)$. Substituting into the above, we obtain $\pi_2(x,n) = x.H(x,n-1) - \int_1^x y\ dH(y,n-1) - r$. Using integration by parts, we have $\pi_2(x,n) = \int_1^x H(y,n-1)dy - r$.

When $n \geq 2$, the expected payoff from approaching worker j is $\pi_2(n) = \int_1^\infty \pi_2(x,n)g(x)dx - r$. Again integrating by parts, we obtain

$$\pi_2(n) = [\pi_2(x,n)G(x)]_1^\infty - \int_1^\infty \frac{d}{dx}[\pi_2(x,n)]G(x)dx - r$$

Now, $\frac{d}{dx}[\pi_2(x,n)] = \frac{d}{dx}\left(\int_1^x H(y,n-1)dy - r\right) = H(x,n-1)$. Also, $[\pi_2(x,n)G(x)]_1^\infty = \lim_{x\to\infty}\pi_2(x,n)$, which is given by $\int_1^\infty H(y,n-1)dy$, since $G(x) \to 1$ as $x \to \infty$ and $G(1) = 0$. So we have

$$\pi_2(n) = \int_1^\infty H(y,n-1)dy - \int_1^\infty H(x,n-1)G(x)dx - r$$

Rearranging and substituting $H(x,n-1) = \beta(x,n)$, we have

$$\pi_2(n) = \int_1^\infty \beta(x,n)(1-G(x))dx - r \tag{44}$$

We can now determine $\pi_2(\theta)$, the expected net payoff for a firm given that $n \geq 2$.

$$\pi_2(\theta) = \frac{1}{\Pr(n_j \geq 2)}\sum_{n=2}^{\infty}\Pr(n_j = n)\pi(n) = \int_1^\infty \beta(x)(1-G(x))dx - r \tag{45}$$

where $\beta(x)$ is the probability of being successful given that $n \geq 2$. The probability $\beta(x)$ can be determined as follows.

$$\beta(x) = \frac{1}{\Pr(n_j \geq 2)}\sum_{n=2}^{\infty}\Pr(n_j = n)\beta(x,n) = \frac{1}{\Pr(n_j \geq 2)}\sum_{n=2}^{\infty}\Pr(n_j = n)G(x)^{n-1}$$

From the perspective of workers, the probability of n arrivals is given by the Poisson distribution, namely $\Pr(\hat{n}_j = n) = e^{-\theta}\theta^n/n!$ From the perspective of firms, however, this must be weighted,

$$\Pr(n_j = n) = \Pr(\hat{n}_j = n)\frac{n}{E(\hat{n}_j)} = \frac{e^{-\theta}\theta^{n-1}}{(n-1)!}$$

So $\beta(x)$ is given as follows,

$$\beta(x) = \frac{1}{1-e^{-\theta}}\sum_{n=2}^{\infty}\frac{e^{-\theta}\theta^{n-1}}{(n-1)!}G(x)^{n-1} = \frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1-e^{-\theta}}$$

Substituting $\beta(x)$ into the expression (45) for $\pi_2(\theta)$, we get

$$\pi_2(\theta) = \int_1^\infty \frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1-e^{-\theta}}(1-G(x))dx - r \tag{46}$$

We can now obtain $\pi(\theta)$, the expected net payoff for a firm, $\pi(\theta) = \Pr(n_j \geq 2)\pi_2(\theta) + \Pr(n_j = 1)\pi_1$. Substituting in π_1 and $\pi_2(\theta)$ from (46), we have

$$\pi(\theta) = (1-e^{-\theta})\int_1^\infty \left(\frac{e^{-\theta(1-G(x))} - e^{-\theta}}{1-e^{-\theta}}\right)(1-G(x))dx + e^{-\theta}\int_1^\infty (x-z)\ dG(x) - r$$

Rearranging, we get

$$\pi(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}\left(\left(\int_1^\infty x\ g(x)dx - \int_1^\infty (1-G(x))dx\right) - z\right) - r$$

Using integration by parts, we have

$$\int_1^\infty x\ g(x)dx - \int_1^\infty (1-G(x))dx = -\left[x(1-G(x))\right]_1^\infty = 1$$

provided we assume that $\lim_{x\to\infty} x(1-G(x)) = 0$. So we have (7).

$$\pi(\theta) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1-z) - r$$

A.5 Existence and uniqueness of equilibrium

Existence. Let $F(\theta) = \displaystyle\int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1-z)$. Clearly, the zero profit condition, $\pi(\theta) = 0$, holds if and only if $F(\theta) = r$, where $r > 0$. Now $F(\theta)$ is continuous in θ on $[0,\infty)$ and $F(\theta) \to 0$ as $\theta \to \infty$. If we can ensure that $F(0) > r$, the intermediate value theorem implies there must exist a $\theta > 0$ such that $F(\theta) = r$. Now, $F(0) = \displaystyle\int_1^\infty (1-G(x))dx + (1-z) = E_G(x) - 1 + 1 - z = E_G(x) - z$. So we can ensure that $F(0) > r$ provided that $E_G(x) > z + r$.

Uniqueness. To prove uniqueness of the equilibrium θ^* such that $F(\theta) = r$, it suffices to show that $F'(\theta) < 0$. Now let $f(\theta, x) = e^{-\theta(1-G(x))}(1-G(x))$. Since both $f(\theta,x)$ and $\frac{\partial}{\partial\theta}(f(\theta,x))$ are continuous in both θ and x on $[1,\infty)$, we can use Leibniz' integral rule, which implies that $F'(\theta) = \displaystyle\int_1^\infty \frac{\partial}{\partial\theta}(f(\theta,x))dx - (1-z)e^{-\theta} = -\int_1^\infty (1-G(x))^2 e^{-\theta(1-G(x))}dx - (1-z)e^{-\theta}$. Clearly, $F'(\theta) < 0$, so there exists a unique θ^* such that $F(\theta) = r$.

A.6 Proof of comparative statics for θ^* for any $G(x)$

Let $F(\theta; z, r) = \int_1^\infty e^{-\theta(1-G(x))}(1-G(x))dx + e^{-\theta}(1-z) - r = 0$. By the implicit function theorem, $\frac{\partial\theta^*}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial\theta}$. Now $\frac{\partial F}{\partial z} = -e^{-\theta}$ and $\frac{\partial F}{\partial\theta} = -\int_1^\infty e^{-\theta(1-G(x))}(1-G(x))^2dx - e^{-\theta}(1-z)$. Now since $z \le 1$, we have

$$\frac{\partial\theta^*}{\partial z} = \frac{-e^{-\theta}}{\left(\int_1^\infty e^{-\theta(1-G(x))}(1-G(x))^2dx + e^{-\theta}(1-z)\right)} < 0$$

Also, $\frac{\partial\theta^*}{\partial r} = -\frac{\partial F/\partial r}{\partial F/\partial\theta}$ where $\frac{\partial F}{\partial r} = -1$. Using $\frac{\partial F}{\partial\theta}$ from above, we have

$$\frac{\partial\theta^*}{\partial r} = \frac{-1}{\left(\int_1^\infty e^{-\theta(1-G(x))}(1-G(x))^2dx + e^{-\theta}(1-z)\right)} < 0$$

A.7 Zero profit condition – Pareto distribution

Starting with the zero profit condition and $G(x) = 1 - x^{-1/\lambda}$, let $y = G(x)$ to obtain:

$$\pi(\theta) = \lambda \int_0^1 e^{-\theta(1-y)}(1-y)^{-\lambda} dy + e^{-\theta}(1-z) - r$$

Changing variables, let $t = \theta(1-y)$, so $dy = -dt/\theta$

$$\pi(\theta) = \lambda\theta^{\lambda-1} \int_0^\theta t^{-\lambda} e^{-t} dt + e^{-\theta}(1-z) - r$$

Now by definition (36), $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\, dt$. Setting $x = \theta$ and $s = 1-\lambda$, we have (8).

$$\pi(\theta) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(1-z) - r = 0$$

A.8 Proof of comparative statics for θ^* – Pareto distribution

We have already shown in Appendix A.6 that $\frac{\partial\theta^*}{\partial z} < 0$ and $\frac{\partial\theta^*}{\partial r} < 0$ for any distribution $G(x)$. However, we include the derivations for the Pareto distribution here because the expressions for $\frac{\partial\theta^*}{\partial z}$ and $\frac{\partial\theta^*}{\partial r}$ will be used in subsequent proofs. We also prove that $\frac{\partial\theta^*}{\partial\lambda} > 0$ and output per capita is increasing in λ.

Let $F(\theta;\lambda,z,r) = \lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + (1-z)e^{-\theta} - r = 0$. By the implicit function theorem, $\frac{\partial\theta^*}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial\theta}$. Now $\frac{\partial F}{\partial z} = -e^{-\theta}$ and by differentiating and then applying Fact 1, we have

$$\frac{\partial F}{\partial\theta} = -\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) - (1-z)e^{-\theta} \tag{47}$$

Now since $z \leq 1$, we have

$$\frac{\partial\theta^*}{\partial z} = \frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + (1-z)e^{-\theta}} < 0 \tag{48}$$

We also have $\frac{\partial\theta^*}{\partial r} = -\frac{\partial F/\partial r}{\partial F/\partial\theta}$ where $\frac{\partial F}{\partial r} = -1$. Using (47), we have

$$\frac{\partial\theta^*}{\partial r} = \frac{-1}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + (1-z)e^{-\theta}} < 0 \tag{49}$$

By the implicit function theorem, $\frac{d\theta^*}{d\lambda} = -\frac{\partial F/\partial\lambda}{\partial F/\partial\theta}$, where $\frac{\partial F}{\partial\lambda}$ is given by

$$\begin{aligned}
\frac{\partial F}{\partial\lambda} &= \theta^{\lambda-1}\gamma(1-\lambda,\theta) + \lambda\left(\theta^{\lambda-1}(\ln\theta)\gamma(1-\lambda,\theta) + \theta^{\lambda-1}\frac{\partial}{\partial\lambda}\gamma(1-\lambda,\theta)\right) \\
&= \theta^{\lambda-1}\gamma(1-\lambda,\theta) + \lambda\theta^{\lambda-1}\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt
\end{aligned}$$

since $\frac{\partial}{\partial\lambda}\gamma(1-\lambda,\theta) = -\int_0^\theta t^{-\lambda}e^{-t}(\ln t)dt$, by an application of Fact 3. Using (47), we have

$$\frac{\partial\theta^*}{\partial\lambda} = \frac{\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta)+\lambda\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta-\ln t)dt\right)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+(1-z)e^{-\theta}} > 0 \tag{50}$$

since we have $\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta-\ln t)dt \geq 0$ from A.3. Finally, $f(\theta(\lambda),\lambda) = \theta^\lambda\gamma(1-\lambda,\theta)$ and let $y^* = f(\theta^*(\lambda),\lambda)$. Then $\frac{dy^*}{d\lambda} = \frac{\partial f}{\partial\theta}\frac{\partial\theta^*}{\partial\lambda} + \frac{\partial f}{\partial\lambda}$. Now $f'(\theta) > 0$ and $\frac{\partial\theta^*}{\partial\lambda} > 0$ from above, so it suffices to show that $\frac{\partial f}{\partial\lambda} > 0$. Using Fact 3, $\frac{\partial f}{\partial\lambda} = \theta^\lambda\ln\theta\gamma(1-\lambda,\theta) + \theta^\lambda\frac{\partial}{\partial\lambda}\gamma(1-\lambda,\theta)$, so we have

$$\frac{\partial f}{\partial\lambda} = \theta^\lambda\left(\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta-\ln t)dt\right) > 0. \tag{51}$$

A.9 Proof that capital share, s_K, is decreasing in θ

Since $s_K = \lambda + (1-z)\varepsilon(1-\lambda,\theta)$, it is sufficient to prove that $\varepsilon(1-\lambda,\theta)$ is decreasing in θ. This follows directly from Fact 6. To establish upper and lower bounds, recall that as $\theta\to 0$, we have $\varepsilon(s,x)\to s$, and as $\theta\to\infty$, we have $\varepsilon(1-\lambda,\theta)\to 0$. This means that as $\theta\to 0$, we have $s_K = 1-z(1-\lambda)$, and as $\theta\to\infty$, we have $s_K = \lambda$.

A.10 Proof of Proposition 3

Let $s_K^* = \lambda + (1-z)\varepsilon(1-\lambda,\theta^*(z))$. Differentiating with respect to z,

$$\begin{aligned}\frac{ds_K^*}{dz} &= \frac{\partial}{\partial z}(\lambda+(1-z)\varepsilon(1-\lambda,\theta^*(z))) + \frac{\partial\theta^*}{\partial z}\frac{\partial}{\partial\theta}\varepsilon(1-\lambda,\theta)\\ &= -\varepsilon(1-\lambda,\theta)\ +(1-z)\frac{\partial\theta^*}{\partial z}\frac{d\varepsilon(1-\lambda,\theta)}{d\theta}\end{aligned}$$

Substituting in $\frac{\partial}{\partial x}\varepsilon(s,x)$ from (38), letting $s = 1-\lambda$ and $x = \theta$, we have

$$\frac{ds_K^*}{dz} = \frac{-\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} + (1-z)\frac{\partial\theta^*}{\partial z}\frac{\theta^{-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)$$

Rearranging, this means that $\frac{ds_K^*}{dz} \leq 0$ if and only if

$$\theta \ \geq\ (1-z)\frac{\partial\theta^*}{\partial z}\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right) \tag{52}$$

$$= (1-z)\frac{\partial\theta^*}{\partial z}\left(\frac{\gamma(2-\lambda,\theta)}{\gamma(1-\lambda,\theta)}-\theta\right) \tag{53}$$

Now substituting in $\frac{\partial\theta^*}{\partial z}$ from (48), this is equivalent to

$$s_K = \lambda + (1-z)\varepsilon(1-\lambda,\theta) \geq 0$$

which is clearly always true. So we have $\frac{ds_K^*}{dz} \leq 0$.

A.11 Proof of Proposition 4

Let $s_K^* = \lambda + \frac{(1-z)\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}$ where $\theta^*(\lambda)$ solves $\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta)+(1-z)e^{-\theta}=r$. Rearranging the zero profit condition and substituting into the expression for capital share, we get

$$s_K^* = \frac{r\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}$$

So we have

$$\frac{ds_K^*}{d\lambda} = r\frac{\partial}{\partial\theta}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right)\frac{d\theta^*}{d\lambda} + r\frac{\partial}{\partial\lambda}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) \tag{54}$$

Using Fact 1, we have

$$\frac{\partial}{\partial\theta}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) = \frac{(1-\lambda)\theta^{-\lambda}}{\gamma(1-\lambda,\theta)} - \frac{\theta^{1-2\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)^2}$$

Applying Fact 3 and simplifying,

$$\frac{\partial}{\partial\lambda}\left(\frac{\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)}\right) = \frac{-\theta^{1-\lambda}}{\gamma(1-\lambda,\theta)^2}\left(\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta-\ln t)\;dt\right)$$

Letting $B=\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta-\ln t)dt$ and substituting into (54),

$$\frac{ds_K^*}{d\lambda} = \frac{r\theta^{-\lambda}}{\gamma(1-\lambda,\theta)}\left(\left((1-\lambda)-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)\frac{\partial\theta^*}{\partial\lambda} - \frac{\theta B}{\gamma(1-\lambda,\theta)}\right)$$

Applying Fact 1, we have $\frac{ds_K^*}{d\lambda}>0$ if and only if

$$\gamma(2-\lambda,\theta)\frac{d\theta^*}{d\lambda} > \theta B$$

Substituting in $\frac{d\theta^*}{d\lambda}$ from (50) and simplifying, $\frac{ds_K^*}{d\lambda}>0$ if and only if

$$\gamma(2-\lambda,\theta)\gamma(1-\lambda,\theta) > B(1-z)\theta^{2-\lambda}e^{-\theta} \tag{55}$$

Now let $F_{2,2}(a_1,a_2;b_1,b_2;z)$ be a generalized hypergeometric function defined by

$$F_{2,2}(a_1,a_2;b_1,b_2;z) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n}\frac{z^n}{n!}$$

Here $(a)_n$ is the Pochhammer symbol or ascending factorial function, defined by $(a)_n=\Gamma(a+n)/\Gamma(a)$. Calculating the integral B and then simplifying, we have the following. The last equality follows from the fact that $\lim_{x\to 0}\frac{x^{1-\lambda}}{(1-\lambda)^2}F_{2,2}(1-\lambda,1-\lambda;2-\lambda,2-\lambda;-x)=0$ and also

$\lim_{x\to 0}(\ln x)\gamma(1-\lambda, x) = 0$.

$$\begin{aligned} B &= \ln\theta \int_0^\theta t^{-\lambda}e^{-t}dt - \int_0^\theta t^{-\lambda}e^{-t}\ln t\ dt \\ &= (\ln\theta)\gamma(1-\lambda,\theta) - \left[(\ln x)\gamma(1-\lambda,x) - \frac{x^{1-\lambda}}{(1-\lambda)^2}F_{2,2}(1-\lambda,1-\lambda;2-\lambda,2-\lambda;-x)\right]_0^\theta \\ &= \frac{\theta^{1-\lambda}}{(1-\lambda)^2}F_{2,2}(1-\lambda,1-\lambda;2-\lambda,2-\lambda;-\theta) \end{aligned} \tag{56}$$

The required inequality (55) can now be stated in terms of generalized hypergeometric functions using the following identity for the incomplete gamma function, $\gamma(x,z) = z^x x^{-1} F_{1,1}(x; x+1; -z)$.

$$(1-z)e^{-\theta} < \left(\frac{1-\lambda}{2-\lambda}\right)\frac{F_{1,1}(1-\lambda;2-\lambda;-\theta)F_{1,1}(2-\lambda;3-\lambda;-\theta)}{F_{2,2}(1-\lambda,1-\lambda;2-\lambda,2-\lambda;-\theta)} \tag{57}$$

In the limit as either $\theta \to \infty$ or $z \to 1$, this inequality always holds, which is consistent with the fact that in these limiting cases we have $s_K^* = \lambda$ so $ds_K^*/d\lambda > 0$.

Now suppose that $z > 1/(2-\lambda)$, so that $1 - z < \frac{1-\lambda}{2-\lambda}$. To establish (57) and hence prove that $ds_K^*/d\lambda > 0$, it suffices to prove the following general lemma. The desired inequality follows from the special case where $a = 1-\lambda$ and $x = \theta$.

Lemma 1 *For any $a \geq 0$ and any $x \geq 0$, we have*

$$e^{-x}F_{2,2}(a,a;a+1,a+1;-x) \leq F_{1,1}(a;a+1;-x)F_{1,1}(a+1;a+2;-x) \tag{58}$$

Proof. First, we use the following result found in Miller and Paris (2012) just after Eq. (5.3), obtained by specialization of 9.1 (34) in Luke (1969).

$$F_{2,2}(a,f;b,c;-x) = \sum_{k=0}^{\infty}\frac{(a)_k(c-f)_k}{(b)_k(c)_k}\frac{x^k}{k!}F_{1,1}(a+k;b+k;-x) \tag{59}$$

Setting $f = a$ and $b = c = a+1$ in (59), and using the fact that $(1)_k = k!$, we have

$$F_{2,2}(a,a;a+1,a+1;-x) = \sum_{k=0}^{\infty}\frac{(a)_k}{(a+1)_k^2}x^kF_{1,1}(a+k;a+1+k;-x)$$

Next, we apply Kummer's first transformation, $F_{1,1}(y;z;-x) = e^{-x}F_{1,1}(z-y;z;x)$ to all $F_{1,1}$ terms. (See, for example, Andrews et al. (2000), [Eq. 4.1.11]). After replacing $F_{1,1}(1;a+2;x)$ with its definition and cancelling the term e^{-2x} from both sides, the desired inequality (58) becomes

$$\sum_{k=0}^{\infty}\frac{(a)_k\ x^k}{(a+1)_k^2}F_{1,1}(1;a+1+k;x) \leq F_{1,1}(1;a+1;x)\sum_{k=0}^{\infty}\frac{x^k}{(a+2)_k} \tag{60}$$

Since all terms in both series are positive now, we can simply compare coefficients of like powers of x. Inequality (60) holds provided that for all $k \in \mathbb{N}$, we have

$$\frac{(a)_k}{(a+1)_k^2} F_{1,1}(1; a+1+k; x) \leq F_{1,1}(1; a+1; x) \frac{1}{(a+2)_k} \tag{61}$$

It can easily be verified that the following holds:

$$\frac{(a)_k (a+2)_k}{(a+1)_k^2} = \frac{a(a+k+1)}{(a+1)(a+k)} \leq 1$$

Also, $F_{1,1}(1; a+1+k; x) \leq F_{1,1}(1; a+1; x)$ for all $k \in \mathbb{N}$ since the function $F_{1,1}(a_1; b_1; x)$ is decreasing in its second argument, $\frac{\partial F_{1,1}(a_1;b_1;x)}{\partial b_1} < 0$. (See for example Erdelyi et al. (1953) for this derivative.) So (61) holds and the lemma is proven. ■

A.12 Proof that k is increasing in θ

Here we prove that k is increasing in θ, taken b as given. Differentiating (16), we have

$$k'(\theta) = \left(\begin{array}{c} \frac{\delta + (1-\delta)(1-G(b))e^{-\theta}}{(1-G(b))(1-(1-\delta)e^{-\theta})} \\ -\frac{(1-G(b))(1-\delta)e^{-\theta}\left(\delta\theta + (1-\delta)(1-G(b))(1-e^{-\theta})\right)}{((1-G(b))(1-(1-\delta)e^{-\theta}))^2} \end{array} \right)$$

We have $k'(\theta) \geq 0$ if and only if

$$(\delta + (1-\delta)(1-G(b))e^{-\theta})(1-(1-\delta)e^{-\theta}) \geq (1-\delta)e^{-\theta} \left(\delta\theta + (1-\delta)(1-G(b))(1-e^{-\theta})\right)$$

With some algebra, this is equivalent to

$$1 - (1-\delta)e^{-\theta} - (1-\delta)\theta e^{-\theta} + (1-\delta)(1-G(b))e^{-\theta} \geq 0 \tag{62}$$

Since $(1-\delta)(1-G(b))e^{-\theta} \geq 0$, it suffices to show that

$$1 - (1-\delta)e^{-\theta} - (1-\delta)\theta e^{-\theta} \geq 0$$

Since $\delta \leq 1$, it suffices to show that $1 - e^{-\theta} - \theta e^{-\theta} \geq 0$, which is true.

A.13 Proof that output per capita is increasing in θ

Output per capita is increasing in θ (taking b as given), where $y(\theta, b) = \frac{x_0 \gamma(1-\lambda,\theta)\theta^\lambda}{1-(1-\delta)e^{-\theta}}$.Differentiating y with respect to θ, we have

$$\frac{\partial y}{\partial \theta} = \frac{x_0 \left((e^{-\theta} + \gamma(1-\lambda,\theta)\lambda\theta^{\lambda-1})(1-(1-\delta)e^{-\theta}) - \gamma(1-\lambda,\theta)\theta^\lambda(1-\delta)e^{-\theta}\right)}{\left(1-(1-\delta)e^{-\theta}\right)^2}$$

Rearranging, using the fact that $(1-\delta)e^{-\theta} < 1$, and dividing both sides by $\theta^{\lambda-1}\gamma(1-\lambda,\theta)$, we obtain $\frac{\partial y}{\partial \theta} \geq 0$ if and only if

$$\lambda + \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} \geq \frac{(1-\delta)\theta e^{-\theta}}{1-(1-\delta)e^{-\theta}} \tag{63}$$

Since $\frac{(1-\delta)\theta e^{-\theta}}{1-(1-\delta)e^{-\theta}}$ is decreasing in δ, it suffices to show that

$$\lambda + \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)} \geq \frac{\theta e^{-\theta}}{1-e^{-\theta}}$$

which is true provided that $h'(\lambda) \geq 0$ where $h(\lambda) = \lambda + \frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}$, since $h(0) = \frac{\theta e^{-\theta}}{1-e^{-\theta}}$. Differentiating $h(\lambda)$, we have

$$h'(\lambda) = \frac{d}{d\lambda}\left(\lambda + \frac{\theta e^{-\theta}}{\theta^{\lambda}\gamma(1-\lambda,\theta)}\right) = 1 - \frac{\theta e^{-\theta}}{(\theta^{\lambda}\gamma(1-\lambda,\theta))^2}\frac{d}{d\lambda}(\theta^{\lambda}\gamma(1-\lambda,\theta))$$

Substituting in $\frac{\partial}{\partial\lambda}(\theta^{\lambda}\gamma(1-\lambda,\theta))$ from (51) and simplifying, we have

$$h'(\lambda) = 1 - \frac{\theta^{1-\lambda}e^{-\theta}B}{\gamma(1-\lambda,\theta)^2}$$

where $B = \int_0^{\theta} t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt$. So $h'(\lambda) \geq 0$ if and only if

$$\theta^{1-\lambda}e^{-\theta}B \leq \gamma(1-\lambda,\theta)^2$$

Substituting expression (56) for B, we require that

$$\left(\frac{\theta^{1-\lambda}}{1-\lambda}\right)^2 e^{-\theta}F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta) \leq \gamma(1-\lambda,\theta)^2$$

Using the following identity for the incomplete gamma function (as in Appendix A.11), $\gamma(x,z) = z^x x^{-1}F_{1,1}(x; x+1; -z)$, this is equivalent to

$$e^{-\theta}F_{2,2}(1-\lambda, 1-\lambda; 2-\lambda, 2-\lambda; -\theta) \leq F_{1,1}(1-\lambda; 2-\lambda; -\theta)^2 \tag{64}$$

Now Lemma 1 implies that the left-hand side of (64) is less than or equal to $F_{1,1}(1-\lambda; 2-\lambda; -\theta)F_{1,1}(2-\lambda; 3-\lambda; -\theta)$, so it suffices to show that

$$F_{1,1}(2-\lambda; 3-\lambda; -\theta) \leq F_{1,1}(1-\lambda; 2-\lambda; -\theta)$$

Applying Kummer's first transformation, $F_{1,1}(y; z; -x) = e^{-x}F_{1,1}(z-y; z; x)$, we require that $F_{1,1}(1; 3-\lambda; \theta) \leq F_{1,1}(1; 2-\lambda; \theta)$. This is true since the function $F_{1,1}(a_1; b_1; x)$ is decreasing in its second argument (see Appendix A.11). Hence $h'(\lambda) \geq 0$, so the original inequality (63) holds and $\frac{\partial y}{\partial \theta} \geq 0$.

A.14 Derivation of zero profit condition

Let $J_1(x)$ be the discounted expected revenue net of wages for an entrepreneur with productivity x that faces no competition when hiring worker. Let $J_2(x)$ be the discounted expected revenue net of wages for a successful entrepreneur with productivity x that faces competition when hiring. If $w(x)$ is the expected value of x_2 given that x is highest, then

$$J_1(x) = \frac{x-b}{1-\beta(1-\delta)}, \quad J_2(x) = \frac{x-w(x)}{1-\beta(1-\delta)}$$

Let $\eta(\theta, x)$ be the probability of successfully hiring, given θ and x, and given that there are two or more entrepreneurs competing.

$$\eta(\theta, x) = \frac{e^{-\theta(1-G_b(x))} - e^{-\theta}}{1-e^{-\theta}}$$

The value function for entrepreneurs, $\tilde{V}$, is

$$\tilde{V} = -C + (1-G(b)) \left(\begin{array}{c} e^{-\theta} \int_{x_0}^{\infty} J_1(x) dG_b(x) \\ +(1-e^{-\theta}) \int_{x_0}^{\infty} \eta(\theta, x) J_2(x) dG_b(x) \end{array} \right)$$

Using analogous reasoning to that found in Appendix A.4, we have

$$\int_{x_0}^{\infty} \eta(\theta, x)(x - w(x)) dG_b(x) = \int_{x_0}^{\infty} \eta(\theta, x)(1 - G_b(x)) dx$$

Setting $\tilde{V} = 0$, we obtain

$$C = \frac{(1-G(b))}{1-\beta(1-\delta)} \left(\begin{array}{c} e^{-\theta} \int_{x_0}^{\infty} (x-b) dG_b(x) \\ +(1-e^{-\theta}) \int_{x_0}^{\infty} \eta(\theta, x)(1-G_b(x)) dx \end{array} \right)$$

Rearranging and using integration by parts as in Appendix A.4, the zero profit condition is

$$C = \frac{(1-G(b)) \left(\int_{x_0}^{\infty} e^{-\theta(1-G_b(x))}(1-G_b(x)) dx + e^{-\theta}(x_0 - b) \right)}{1-\beta(1-\delta)}$$

For the Pareto distribution, $G_b(x) = 1 - \left(\frac{x}{x_0}\right)^{-1/\lambda}$ where $x_0 = \max\{1, b\}$, so we have

$$C = \frac{(1-G(b)) \left(x_0 \lambda \theta^{\lambda-1} \gamma(1-\lambda, \theta) + e^{-\theta}(x_0 - b) \right)}{1-\beta(1-\delta)}$$

where $\theta = \phi(1-G(b))$.

A.15 Proof that there exists a unique ϕ for any given b

Existence. Let $F(\theta) = (1 - G(b)) \left(\int_{x_0}^{\infty} e^{-\theta(1-G_b(x))}(1 - G_b(x))dx + e^{-\theta}(x_0 - b) \right)$, where $b \in \mathbb{R}^+$ is taken as given. The zero profit condition holds if and only if $F(\theta) = C(1 - \beta(1 - \delta))$, where $C(1-\beta(1-\delta) > 0$. Now $F(\theta)$ is continuous in θ on $[0, \infty)$ and $F(\theta) \to 0$ as $\theta \to \infty$. If we can ensure that $F(0) \geq C(1-\beta(1-\delta))$, the intermediate value theorem implies there exists $\theta \geq 0$ such that $F(\theta) = C(1 - \beta(1 - \delta))$. Now, $F(0) = (1 - G(b)) \left(\int_{x_0}^{\infty}(1 - G_b(x))dx + (1 - b) \right) = (1-G(b))(E_{G_b}(x)-b)$. If $G(x)$ is Pareto, we have $F(0) \geq C(1-\beta(1-\delta))$ provided the following condition holds:

$$C \leq \frac{1 - G(b)}{1 - \beta(1 - \delta)} \left(\frac{x_0}{1 - \lambda} - b \right) \tag{65}$$

If condition (65) holds, there exists $\theta \geq 0$ and hence there exists $\phi \geq 0$ such that the zero profit condition holds, where $\phi = \theta/(1 - G(b))$. If (65) fails, then $\theta = \phi = 0$.

Critical value. There is a unique critical value $b_c(\lambda, \beta, \delta, C) > z$ such that condition (65) holds whenever $b \leq b_c$. To see this, let $f(b) = \frac{1-G(b)}{1-\beta(1-\delta)} \left(\frac{x_0}{1-\lambda} - b \right) - C$. Condition (65) holds if and only if $f(b) \geq 0$. First we prove that $f'(b) < 0$. If $b \leq 1$, we have $f(b) = \frac{1}{1-\beta(1-\delta)} \left(\frac{1}{1-\lambda} - b \right) - C$, so $f'(b) = \frac{-1}{1-\beta(1-\delta)} < 0$. If $b > 1$, we have $f(b) = \frac{\lambda b^{1-1/\lambda}}{(1-\lambda)(1-\beta(1-\delta))} - C$, so $f'(b) = \frac{-b^{-1/\lambda}}{(1-\beta(1-\delta))} < 0$. So for all $b \geq 0$, we have $f'(b) < 0$. Now if $f(0) \geq 0$, then since $f'(b) < 0$ and $f(b) \to -C$ as $b \to \infty$, there exists a unique $b_c \geq 0$ such that $f(b_c) = 0$. To ensure that $b_c > z$, we need $f(z) > 0$, which is true provided that condition (66) holds. This condition also implies that $f(0) > 0$, since $f'(b) < 0$. Hence there exists a unique critical value $b_c > z$ such that $f(b_c) = 0$, and condition (65) holds whenever $b \leq b_c$.

$$C < \frac{1}{1 - \beta(1 - \delta)} \left(\frac{1}{1 - \lambda} - z \right) \tag{66}$$

Uniqueness. To prove the uniqueness of θ where $F(\theta) = C(1 - \beta(1 - \delta))$, and hence the uniqueness of $\phi = \theta/(1 - G(b))$, it suffices to show that $F'(\theta) < 0$. Applying Leibniz' integral rule, we have $F'(\theta) = -(1 - G(b)) \left(\int_{x_0}^{\infty} (1 - G_b(x))^2 e^{-\theta(1-G_b(x))}dx + (1 - b)e^{-\theta} \right) < 0$. So for any given b, there is a unique θ and hence a unique ϕ that satisfies the zero profit condition. This gives us the function $\phi_r(b) : \mathbb{R}^+ \to \mathbb{R}^+$.

A.16 Proof that $\phi_r'(b) < 0$

Let $F_1(\theta, b) = x_0^{-1/\lambda}(x_0\lambda\theta^{\lambda-1}\gamma(1 - \lambda, \theta) + e^{-\theta}(x_0 - b)) - C(1 - \beta(1 - \delta)) = 0$ where $x_0 = \max\{1, b\}$. When $b < 1$, $F_1(\theta, b) = \lambda\theta^{\lambda-1}\gamma(1 - \lambda, \theta) + e^{-\theta}(1 - b) - C(1 - \beta(1 - \delta))$, so $\partial F_1/\partial b = -e^{-\theta}$ and $\partial F_1/\partial\theta = -(\lambda\theta^{\lambda-2}\gamma(2 - \lambda, \theta) + e^{-\theta}(1 - b))$. By the implicit function theorem, $\theta'(b) = \phi'(b) = -\frac{\partial F_1/\partial b}{\partial F_1/\partial \theta}$, which gives the following expression, which is clearly negative.

$$\phi'(b) = \theta'(b) = \frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2 - \lambda, \theta) + e^{-\theta}(1 - b)} < 0, \;\; b < 1$$

When $b \geq 1$, we have $F_1(\theta(\phi, b), b) = b^{1-1/\lambda}\lambda\theta^{\lambda-1}\gamma(1-\lambda, \theta) - C(1-\beta(1-\delta))$ where $\theta(\phi, b) = \phi(1-G(b)) = \phi b^{-1/\lambda}$ and hence $\frac{\partial \theta}{\partial b} = -\frac{1}{\lambda}\phi b^{-1/\lambda-1} = -\frac{1}{\lambda}\theta b^{-1}$. Now $\phi'(b) = -\frac{dF_1/db}{dF_1/d\phi}$, where $\partial F_1/\partial\theta = -b^{1-1/\lambda}\lambda\theta^{\lambda-2}\gamma(2-\lambda, \theta)$ is obtained by differentiating and then applying Fact 3.

$$\begin{aligned}\frac{dF_1}{db} &= \frac{\partial F_1}{\partial \theta}\frac{\partial \theta}{\partial b} + \frac{\partial F_1}{\partial b} \\ &= -b^{1-1/\lambda}\lambda\theta^{\lambda-2}\gamma(2-\lambda, \theta)\left(-\frac{1}{\lambda}\theta b^{-1}\right) - b^{-1/\lambda}(1-\lambda)\theta^{\lambda-1}\gamma(1-\lambda, \theta) \\ &= -b^{-1/\lambda}e^{-\theta}\end{aligned}$$

Also, $\frac{dF_1}{d\phi}$ is given by

$$\frac{dF_1}{d\phi} = \frac{\partial F_1}{\partial \theta}\frac{\partial \theta}{\partial \phi} = -b^{1-1/\lambda}\lambda\theta^{\lambda-2}\gamma(2-\lambda, \theta)b^{-1/\lambda}$$

So we have the following expression for $\phi'(b)$, which is again clearly negative.

$$\phi'(b) = -\frac{dF_1/db}{dF_1/d\phi} = \frac{-e^{-\theta}}{b^{1-1/\lambda}\lambda\theta^{\lambda-2}\gamma(2-\lambda, \theta)} < 0, \ \ b \geq 1$$

In general, we have

$$\phi'(b) = \frac{-e^{-\theta}}{x_0^{-1/\lambda}\left(x_0\lambda\theta^{\lambda-2}\gamma(2-\lambda, \theta) + e^{-\theta}(x_0 - b)\right)} < 0, \ \ x_0 = \max\{1, b\}$$

A.17 Reservation wage

Workers decide whether to accept or reject wage offers b, taking ϕ as given. If $V^E(b) \geq z + \beta V^U$, workers accept the wage offer b, while if $V^E(b) < z + \beta V^U$ they reject the wage offer. We show that for any given ϕ there exists a unique reservation wage b such that workers will accept a wage offer x if and only if $x \geq b$. This gives us a function $b_r(\phi) : \mathbb{R}^+ \to \mathbb{R}^+$. The reservation wage b satisfies $V^E(b) = z + \beta V^U$. Let $H(b) = V^E(b) - z - \beta V^U$. Workers accept a wage offer b if and only if $H(b) \geq 0$. Using the fact that

$$V^E(w) = \frac{w + \beta\delta V^U(w)}{1-\beta(1-\delta)}$$

from (25) and then substituting into $H(b)$ and simplifying, we have

$$H(b) = \frac{b - \beta(1-\delta)(1-\beta)V^U}{1-\beta(1-\delta)} - z$$

First, there exists at least one b such that $H(b) = 0$. To start with, observe that $H(0) < 0$. Also, we have $\lim_{b\to\infty} H(b) = +\infty$. So there exists at least one b such that $H(b) = 0$. Next, $H'(b) > 0$ clearly. Hence there exists a unique reservation wage b such that $H(b) = 0$, i.e., $V^E(b) = z + \beta V^U$.

A.18 Proof that $b_r'(\phi) \geq 0$

We start with the following expression for the reservation wage:

$$b = \frac{z(1-\beta(1-\delta)) + \beta(1-\delta)(1-e^{-\theta})w^{new}}{1-\beta(1-\delta)e^{-\theta}}.$$

Let $F_2(\theta, b) = b(1-\beta(1-\delta)e^{-\theta}) - z(1-\beta(1-\delta)) - \beta(1-\delta)w(\theta, b) = 0$ and let $w(\theta, b) = (1-e^{-\theta})w^{new} = x_0(1-\lambda)\theta^\lambda\gamma(1-\lambda,\theta) - \theta e^{-\theta}(x_0 - b)$ where $x_0 = \max\{1, b\}$. We have

$$\frac{\partial F_2}{\partial \theta} = \beta(1-\delta)\left(be^{-\theta} - \frac{\partial w}{\partial \theta}\right) \tag{67}$$

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1-\delta)e^{-\theta} - \beta(1-\delta)\frac{\partial w}{\partial b} \tag{68}$$

Case 1. $b < 1$. In this case, $w(\theta, b) = (1-\lambda)\theta^\lambda\gamma(1-\lambda,\theta) - \theta e^{-\theta}(1-b)$. Differentiating with respect to θ and b, we have

$$\frac{\partial w}{\partial \theta} = (1-\lambda)(\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}) - (1-b)(e^{-\theta}(1-\theta))$$

$$\frac{\partial w}{\partial b} = \theta e^{-\theta}$$

Substituting into (67) and (68) and simplifying, we have

$$\frac{\partial F_2}{\partial \theta} = -\beta(1-\delta)(\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta) + (1-b)\theta e^{-\theta})$$

$$\frac{\partial F_2}{\partial b} = 1 - \beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}$$

and hence using $b'(\phi) = b'(\theta) = -\frac{\partial F_2/\partial\theta}{\partial F_2/\partial b}$,

$$b'(\phi) = b'(\theta) = \frac{\beta(1-\delta)(\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta) + (1-b)\theta e^{-\theta})}{1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}}\ ,\ b < 1$$

The numerator is positive when $b < 1$ and the denominator is positive, $1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta} > 0$, since $1 - e^{-\theta} - \theta e^{-\theta} > 0$ and $\beta(1-\delta) < 1$, so $b'(\phi) > 0$ when $b < 1$.

Case 2. $b \geq 1$. In this case, we have $F_2(\theta(\phi, b), b) = b(1-\beta(1-\delta)e^{-\theta}) - z(1-\beta(1-\delta)) - \beta(1-\delta)w(\theta, b)$ where $w(\theta, b) = b(1-\lambda)\theta^\lambda\gamma(1-\lambda,\theta)$ and $\theta(\phi, b) = \phi(1-G(b)) = \phi b^{-1/\lambda}$, and hence $\frac{\partial\theta}{\partial b} = -\frac{1}{\lambda}\theta b^{-1}$. Differentiating with respect to θ and b, we have

$$\frac{\partial w}{\partial \theta} = b(1-\lambda)(\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta})$$

$$\frac{\partial w}{\partial b} = (1-\lambda)\theta^{\lambda}\gamma(1-\lambda,\theta)$$

Substituting into (67) and (68) and simplifying,

$$\frac{\partial F_2}{\partial \theta} = -\beta(1-\delta)b\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)$$

$$\frac{\partial F_2}{\partial b} = 1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)(1-\lambda)\theta^{\lambda}\gamma(1-\lambda,\theta)$$

Now $b'(\phi) = -\frac{dF_2/\partial\phi}{dF_2/\partial b}$, where

$$\frac{dF_2}{d\phi} = \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial \phi} = -b^{1-1/\lambda}\beta(1-\delta)\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)$$

$$\begin{aligned}
\frac{dF_2}{db} &= \frac{\partial F_2}{\partial \theta}\frac{\partial \theta}{\partial b} + \frac{\partial F_2}{\partial b} \\
&= -\beta(1-\delta)b\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)\left(-\frac{1}{\lambda}\theta b^{-1}\right) + 1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)(1-\lambda)\theta^{\lambda}\gamma(1-\lambda,\theta) \\
&= 1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}
\end{aligned}$$

by applying Fact 3. So we have the following expression for $b'(\phi)$, which is positive since $1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta} > 0$.

$$b'(\phi) = -\frac{dF_2/\partial\phi}{dF_2/\partial b} = \frac{b^{1-1/\lambda}\beta(1-\delta)\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)}{1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}},\ b \geq 1$$

In general, we have

$$b'(\phi) = \frac{x_0^{-1/\lambda}\beta(1-\delta)(x_0\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta) + (x_0-b)\theta e^{-\theta})}{1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta}},\quad x_0 = \max\{1,b\}$$

A.19 Proof of Proposition 6

Here we establish some comparative statics results for the equilibrium (ϕ^*, b^*) with respect to the parameters $p_i \in \mathbf{p} = (z, \lambda, C)$. We restrict our attention to the case where $b < 1$ (where $\phi = \theta$). First we define the following functions, $F_1(\theta, b; \mathbf{p})$ and $F_2(\theta, b; \mathbf{p})$.

$$F_1(\theta, b; \mathbf{p}) = x_0^{-1/\lambda}(x_0\lambda\theta^{\lambda-1}\gamma(1-\lambda,\theta) + e^{-\theta}(x_0-b)) - C(1-\beta(1-\delta))$$

$$F_2(\theta, b; \mathbf{p}) = b(1-\beta(1-\delta)e^{-\theta}) - z(1-\beta(1-\delta)) - \beta(1-\delta)w(\theta, b, \lambda)$$

where $x_0 = \max\{1, b\}$ and $w(\theta, b, \lambda) = x_0(1-\lambda)\theta^{\lambda}\gamma(1-\lambda,\theta) - \theta e^{-\theta}(x_0 - b)$. The equation $F_1(\theta, b; \mathbf{p}) = 0$ implicitly defines the best-response function $\theta_r(.)$ for $b < b_c$ and $\theta_r(b) = 0$ for any $b \geq b_c$. The equation $F_2(\theta, b; \mathbf{p}) = 0$ implicitly defines the best-response function $b_r(.)$. We know that $x^* = (\theta^*, b^*)$ is an equilibrium if and only if $F_1(\theta^*, b^*; \mathbf{p}) = 0$ and $F_2(\theta^*, b^*; \mathbf{p}) = 0$. By the implicit function theorem, for any $p_i \in \mathbf{p}$ we have

$$\frac{\partial x^*(p_i)}{\partial p_i} = -(D_x(x^*(p_i), p_i))^{-1} D_{p_i}(x^*(p_i), p_i)$$

where $D_x = \begin{bmatrix} \frac{\partial b_r}{\partial \theta} & -1 \\ -1 & \frac{\partial \theta_r}{\partial b} \end{bmatrix}$ and $D_{p_i} = \begin{bmatrix} \frac{\partial b_r}{\partial p_i} \\ \frac{\partial \theta_r}{\partial p_i} \end{bmatrix}$ for all $p_i \in \mathbf{p}$.

Before proving the comparative statics results, we first collect together some results that have already been obtained in the proofs above.

$$\begin{aligned}
\frac{\partial F_1}{\partial \theta} &= -(\lambda \theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(1-b)),\ b<1 \\
\frac{\partial F_1}{\partial b} &= -e^{-\theta},\ b<1 \\
\frac{\partial F_2}{\partial \theta} &= -\beta(1-\delta)(\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta) + (1-b)\theta e^{-\theta}),\ b<1 \\
\frac{\partial F_2}{\partial b} &= 1-\beta(1-\delta)e^{-\theta} - \beta(1-\delta)\theta e^{-\theta},\ \ b<1
\end{aligned}$$

We need $\partial b_r/\partial p_i$ and $\partial \theta_r/\partial p_i$for all $p_i \in \mathbf{p}$. Using the implicit function theorem in relation to $F_1(\theta, b; \mathbf{p})$ and $F_2(\theta, b; \mathbf{p})$, we have $\partial\theta_r/\partial p_i = -\frac{\partial F_1/\partial p_i}{\partial F_1/\partial \theta}$ and $\partial b_r/\partial p_i = -\frac{\partial F_2/\partial p_i}{\partial F_2/\partial b}$ for all $p_i \in \mathbf{p}$. So we need to first determine $\partial F_1/\partial p_i$ and $\partial F_2/\partial p_i$ for all $p_i \in \mathbf{p}$. Let $B = \int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt$.

$$\begin{aligned}
\frac{\partial F_1}{\partial \lambda} &= \theta^{\lambda-1}(\gamma(1-\lambda,\theta) + \lambda B),\ b<1 \\
\frac{\partial F_1}{\partial z} &= 0 \\
\frac{\partial F_1}{\partial C} &= -(1-\beta(1-\delta))
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_2}{\partial \lambda} &= \beta(1-\delta)\theta^{\lambda}(\gamma(1-\lambda,\theta) - (1-\lambda)B),\ b<1 \\
\frac{\partial F_2}{\partial z} &= -(1-\beta(1-\delta)) \\
\frac{\partial F_2}{\partial C} &= 0
\end{aligned}$$

Now using the fact that $\partial\theta_r/\partial p_i = -\frac{\partial F_1/\partial p_i}{\partial F_1/\partial \theta}$ and $\partial b_r/\partial p_i = -\frac{\partial F_2/\partial p_i}{\partial F_2/\partial b}$ for all i, we obtain

$$\begin{aligned}
\frac{\partial \theta_r}{\partial \lambda} &= -\frac{\partial F_1/\partial \lambda}{\partial F_1/\partial \theta} = \frac{\theta^{\lambda-1}(\gamma(1-\lambda,\theta)+\lambda B)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(1-b)},\ \ b<1 \\
\frac{\partial \theta_r}{\partial z} &= -\frac{\partial F_1/\partial z}{\partial F_1/\partial \theta} = 0 \\
\frac{\partial \theta_r}{\partial C} &= -\frac{\partial F_1/\partial C}{\partial F_1/\partial \theta} = \frac{-(1-\beta(1-\delta))}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta) + e^{-\theta}(1-b)},\ \ b<1
\end{aligned}$$

and we have

$$\begin{aligned}
\frac{\partial b_r}{\partial \lambda} &= -\frac{\partial F_2/\partial \lambda}{\partial F_2/\partial b} = \frac{-\beta(1-\delta)\theta^{\lambda}(\gamma(1-\lambda,\theta)-(1-\lambda)B)}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}, \;\; b<1 \\
\frac{\partial b_r}{\partial z} &= -\frac{\partial F_2/\partial z}{\partial F_2/\partial b} = \frac{1-\beta(1-\delta)}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}, \;\; b<1 \\
\frac{\partial b_r}{\partial C} &= -\frac{\partial F_2/\partial C}{\partial F_2/\partial b} = 0
\end{aligned}$$

Now we have $\partial\theta^*/\partial p_i = -(\det(D_x))^{-1}\left(\frac{\partial\theta_r}{\partial b}\frac{\partial b_r}{\partial p_i} + \frac{\partial\theta_r}{\partial p_i}\right)$ and $\partial b^*/\partial p_i = -(\det(D_x))^{-1}\left(\frac{\partial b_r}{\partial p_i} + \frac{\partial b_r}{\partial\theta}\frac{\partial\theta_r}{\partial p_i}\right)$. First let $D = -(\det(D_x))^{-1}$. For $b<1$, substituting in $\partial b_r/\partial\theta$ and $\partial\theta_r/\partial b$ from above, we have

$$\begin{aligned}
D &= -\frac{1}{\det(D_x)} = -\left(\frac{1}{\frac{\partial b_r}{\partial\theta}\frac{\partial\theta_r}{\partial b}-1}\right) = \frac{1}{1-\frac{\partial b_r}{\partial\theta}\frac{\partial\theta_r}{\partial b}} \\
&= \frac{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}} > 0
\end{aligned}$$

Finally, substituting in $\partial\theta_r/\partial b$, $\partial b_r/\partial\theta$, $\frac{\partial b_r}{\partial p_i}$ and $\frac{\partial\theta_r}{\partial p_i}$ into $\partial\theta^*/\partial p_i = -(\det(D_x))^{-1}\left(\frac{\partial\theta_r}{\partial b}\frac{\partial b_r}{\partial p_i} + \frac{\partial\theta_r}{\partial p_i}\right)$ and $\partial b^*/\partial p_i = -(\det(D_x))^{-1}\left(\frac{\partial b_r}{\partial p_i} + \frac{\partial b_r}{\partial\theta}\frac{\partial\theta_r}{\partial p_i}\right)$, we obtain the following.

Comparative statics for λ. For $b<1$, we have

$$\begin{aligned}
\frac{\partial\theta^*}{\partial\lambda} &= D\left(\begin{array}{c}\left(\frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right)\left(\frac{-\beta(1-\delta)\theta^{\lambda}(\gamma(1-\lambda,\theta)-(1-\lambda)B)}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}\right) \\ +\frac{\theta^{\lambda-1}(\gamma(1-\lambda,\theta)+\lambda B)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\end{array}\right), \; b<1 \\
&= \frac{\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta)+\lambda B-\frac{B\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}}\right)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta+e^{-\theta}(1-b)} \\
&= \frac{\theta^{\lambda-1}\left(\gamma(1-\lambda,\theta)+(\lambda-J)B\right)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}, \text{ where } J = \frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}}
\end{aligned}$$

where $B = \int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt$. Now we have $\frac{\partial\theta^*}{\partial\lambda} \geq 0$ if and only if

$$\gamma(1-\lambda,\theta)+(\lambda-J)B \geq 0$$

If $\lambda \geq J$, this is clearly true. Suppose instead that $J > \lambda$. Rearranging and then multiplying both sides by $1-\lambda$, the above inequality is true if and only if

$$\frac{B(1-\lambda)}{\gamma(1-\lambda,\theta)} \leq \frac{1-\lambda}{J-\lambda}$$

Now $(1-\lambda)/(J-\lambda) > 1/J$ provided $J<1$, which is true since $1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta} > 0$.

So it suffices to show that

$$\frac{B(1-\lambda)}{\gamma(1-\lambda,\theta)}\frac{\beta(1-\delta)\theta e^{-\theta}}{1-\beta(1-\delta)e^{-\theta}} \leq 1 \tag{69}$$

In Appendix A.11, we proved that $(2-\lambda)\gamma(2-\lambda,\theta)\gamma(1-\lambda,\theta) > B(1-\lambda)\theta^{2-\lambda}e^{-\theta}$, so we have

$$\frac{B(1-\lambda)}{\gamma(1-\lambda,\theta)} < \frac{(2-\lambda)\gamma(2-\lambda,\theta)}{\theta^{2-\lambda}e^{-\theta}}$$

Substituting into (69), it is sufficient to show

$$\frac{\beta(1-\delta)\theta^{\lambda-1}\gamma(2-\lambda,\theta)}{1-\beta(1-\delta)e^{-\theta}} \leq \frac{1}{2-\lambda}$$

Now let $m(\theta) = \frac{\beta(1-\delta)\theta^{\lambda-1}\gamma(2-\lambda,\theta)}{1-\beta(1-\delta)e^{-\theta}}$. It can be shown that $\hat{m} = \max\, m(\theta)$ is

$$\hat{m} = \frac{1-\zeta}{2-\lambda-\zeta}$$

where $\zeta(\beta,\delta) = \arg\max\, m(\theta)$ and ζ is the unique solution to $1-\beta(1-\delta)e^{-\zeta} = \zeta$. To ensure that $\frac{\partial\theta^*}{\partial\lambda} \geq 0$ for $b<1$, we require that

$$\frac{1-\zeta}{2-\lambda-\zeta} < \frac{1}{2-\lambda}$$

which is always true since $\lambda < 1$ and $\zeta > 0$. So we have $\frac{\partial\theta^*}{\partial\lambda} \geq 0$ for $b<1$, regardless of the values of λ, β and δ.

We also have $\frac{\partial b^*}{\partial\lambda} \geq 0$ for $b<1$.

$$\begin{aligned}\frac{\partial b^*}{\partial\lambda} &= D\left(\begin{array}{c}\frac{-\beta(1-\delta)\theta^{\lambda}(\gamma(1-\lambda,\theta)-(1-\lambda)B)}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}\\ +\frac{\beta(1-\delta)(\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)+(1-b)\theta e^{-\theta})}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}\frac{\theta^{\lambda-1}(\gamma(1-\lambda,\theta)+\lambda B)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\end{array}\right), b<1\\ &= \frac{B\beta(1-\delta)\theta^{\lambda}}{1-\beta(1-\delta)e^{-\theta}} \geq 0\end{aligned}$$

Comparative statics for z.

$$\begin{aligned}\frac{\partial\theta^*}{\partial z} &= D\left(\frac{-e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right)\left(\frac{1-\beta(1-\delta)}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}\right), b<1 \\ &= \frac{-(1-\beta(1-\delta))}{1-\beta(1-\delta)e^{-\theta}}\left(\frac{e^{-\theta}}{(\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b))}\right) < 0\end{aligned} \tag{70}$$

$$\begin{aligned}\frac{\partial b^*}{\partial z} &= D\left(\frac{1-\beta(1-\delta)}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}\right), b<1 \\ &= \frac{1-\beta(1-\delta)}{1-\beta(1-\delta)e^{-\theta}} > 0\end{aligned} \tag{71}$$

Comparative statics for C.

$$\begin{aligned}\frac{\partial \theta^*}{\partial C} &= D\left(\frac{-(1-\beta(1-\delta))}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right), b<1 \\ &= \frac{-(1-\beta(1-\delta))}{1-\beta(1-\delta)e^{-\theta}}\left(\frac{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right) < 0\end{aligned} \tag{72}$$

$$\begin{aligned}\frac{\partial b^*}{\partial C} &= D\left(\frac{\beta(1-\delta)(\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)+(1-b)\theta e^{-\theta})}{1-\beta(1-\delta)e^{-\theta}-\beta(1-\delta)\theta e^{-\theta}}\right)\left(\frac{-(1-\beta(1-\delta))}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right), b<1 \\ &= \frac{-(1-\beta(1-\delta))}{1-\beta(1-\delta)e^{-\theta}}\left(\frac{\beta(1-\delta)(\lambda\theta^{\lambda-1}\gamma(2-\lambda,\theta)+(1-b)\theta e^{-\theta})}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right) \le 0\end{aligned} \tag{73}$$

A.20 Proof of Proposition 7

Proof of (i)**.** Consider $u^* = u(\theta^*) = u(\theta^*(\lambda; z; C))$. Since $\frac{\partial\theta^*}{\partial z} \le 0$, $\frac{\partial\theta^*}{\partial C} \le 0$, and $\frac{\partial\theta^*}{\partial\lambda} \ge 0$ for $b<1$, if $u'(\theta) \le 0$ then $\frac{\partial u^*}{\partial z} = \frac{du}{d\theta}\frac{\partial\theta^*}{\partial z} \ge 0$, $\frac{\partial u^*}{\partial C} = \frac{du}{d\theta}\frac{\partial\theta^*}{\partial C} \ge 0$, and $\frac{\partial u^*}{\partial\lambda} = \frac{du}{d\theta}\frac{\partial\theta^*}{\partial\lambda} \le 0$ for $b<1$. Rearranging (14), we have $u(\theta) = \delta/(e^\theta - (1-\delta))$, which is clearly decreasing in θ so $u'(\theta) \le 0$ and the result is proven.

Proof of (ii)**.** Consider $k^* = k(\theta^*)$. We have $\frac{\partial k^*}{\partial z} = \frac{\partial k}{\partial\theta}\frac{\partial\theta^*}{\partial z} \le 0$, $\frac{\partial k^*}{\partial C} = \frac{\partial k}{\partial\theta}\frac{\partial\theta^*}{\partial C} \le 0$, and $\frac{\partial k^*}{\partial\lambda} = \frac{\partial k}{\partial\theta}\frac{\partial\theta^*}{\partial\lambda}$ for $b<1$. Start with the expression for k from (16) and rearranging, we have

$$k(\theta) = \frac{\delta\theta + (1-\delta)(1-e^{-\theta})}{1-(1-\delta)e^{-\theta}}, \; b<1$$

We know that $\frac{\partial k}{\partial\theta} \ge 0$. Since $\frac{\partial\theta^*}{\partial C} < 0$, we have $\frac{\partial k^*}{\partial C} = \frac{\partial k}{\partial\theta}\frac{\partial\theta^*}{\partial C} \le 0$. For $b<1$, we have $\frac{\partial\theta^*}{\partial\lambda} \ge 0$ and $\frac{\partial k}{\partial\theta} \ge 0$, so $\frac{\partial k^*}{\partial\lambda} = \frac{\partial k}{\partial\theta}\frac{\partial\theta^*}{\partial\lambda} \ge 0$. We also know that $\frac{\partial\theta^*}{\partial z} \le 0$, so $\frac{\partial k^*}{\partial z} = \frac{\partial k}{\partial\theta}\frac{\partial\theta^*}{\partial z} \le 0$.

Proof of (iii)**.** Consider $y^* = y(\theta^*, \lambda)$. Since $\frac{\partial\theta^*}{\partial C} \le 0$, we have $\frac{\partial y^*}{\partial C} = \frac{\partial y}{\partial\theta}\frac{\partial\theta^*}{\partial C} \le 0$, provided that $\frac{dy}{d\theta} \ge 0$, which is true. Starting with (21), we have $y(\theta,\lambda) = \frac{\gamma(1-\lambda,\theta)\theta^\lambda}{1-(1-\delta)e^{-\theta}}$ for $b<1$. So $\frac{\partial y^*}{\partial\lambda} = \frac{dy}{d\theta}\frac{\partial\theta^*}{\partial\lambda} + \frac{\partial y}{\partial\lambda}$, where $\frac{\partial\theta^*}{\partial\lambda} \ge 0$ and $y'(\theta) \ge 0$. In order to prove that $\frac{\partial y^*}{\partial\lambda} \ge 0$, it suffices to show that $\frac{\partial y}{\partial\lambda} \ge 0$. Similarly to (51) in Appendix A.8, differentiating $y(\theta,\lambda)$ with respect to λ yields:

$$\frac{\partial y}{\partial\lambda} = \frac{\theta^\lambda\left(\int_0^\theta t^{-\lambda}e^{-t}(\ln\theta - \ln t)dt\right)}{1-(1-\delta)e^{-\theta}} \ge 0$$

So we conclude that $\frac{\partial y^*}{\partial\lambda} \ge 0$ when $b<1$. Finally, $\frac{\partial y^*}{\partial z} = \frac{dy}{d\theta}\frac{\partial\theta^*}{\partial z} \le 0$ when $b<1$, since $\frac{\partial\theta^*}{\partial z} \le 0$.

A.21 Proof of Proposition 9

We show that $\frac{ds_K^*}{dC} \geq 0$ and $\frac{ds_K^*}{dz} \leq 0$ where $s_K^* = \lambda + (1-b^*)\varepsilon(1-\lambda, \theta^*)$. First, differentiating s_K^* with respect to C, we have

$$\frac{ds_K^*}{dC} = \frac{-\partial b^*}{\partial C}\varepsilon(1-\lambda,\theta) + (1-b^*)\frac{\partial \theta^*}{\partial C}\frac{\partial}{\partial \theta}\varepsilon(1-\lambda,\theta)$$

Since $\frac{db^*}{dC} \leq 0$ from (73), $\frac{\partial}{\partial\theta}\varepsilon(1-\lambda,\theta) \leq 0$ by Fact 6, and $\frac{\partial\theta^*}{\partial C} < 0$ from (72), we have $\frac{ds_K^*}{dC} \geq 0$. Next, differentiating s_K^* with respect to z, we have

$$\frac{ds_K^*}{dz} = -\frac{\partial b^*}{\partial z}\varepsilon(1-\lambda,\theta) \ + (1-b^*)\frac{\partial \theta^*}{\partial z}\frac{\partial}{\partial \theta}\varepsilon(1-\lambda,\theta)$$

Substituting in $\frac{\partial}{\partial x}\varepsilon(s,x)$ from (38), where $s = 1-\lambda$ and $x = \theta$, we have

$$\frac{ds_K^*}{dz} = -\frac{\partial b^*}{\partial z}\varepsilon(1-\lambda,\theta) + (1-b)\frac{\partial \theta^*}{\partial z}\frac{\theta^{-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)$$

Substituting in $\frac{\partial b^*}{\partial z}$ from (71) and $\frac{\partial\theta^*}{\partial z}$ from (70), we have

$$\frac{ds_K^*}{dz} = -\frac{\theta^{-\lambda}e^{-\theta}(1-\beta(1-\delta))}{\gamma(1-\lambda,\theta)(1-\beta(1-\delta)e^{-\theta})}\left(\theta + \left(\frac{e^{-\theta}(1-b)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right)\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right)\right)$$

So $\frac{ds_K^*}{dz} \leq 0$ if and only if

$$\frac{ds_K^*}{dz} = \theta + \left(\frac{e^{-\theta}(1-b)}{\lambda\theta^{\lambda-2}\gamma(2-\lambda,\theta)+e^{-\theta}(1-b)}\right)\left(1-\lambda-\theta-\frac{\theta^{1-\lambda}e^{-\theta}}{\gamma(1-\lambda,\theta)}\right) \geq 0$$

Rearranging, this holds if and only if

$$s_K = \lambda + (1-b)\varepsilon(1-\lambda,\theta) \geq 0$$

which is clearly always true, so we have $\frac{ds_K^*}{dz} \leq 0$.

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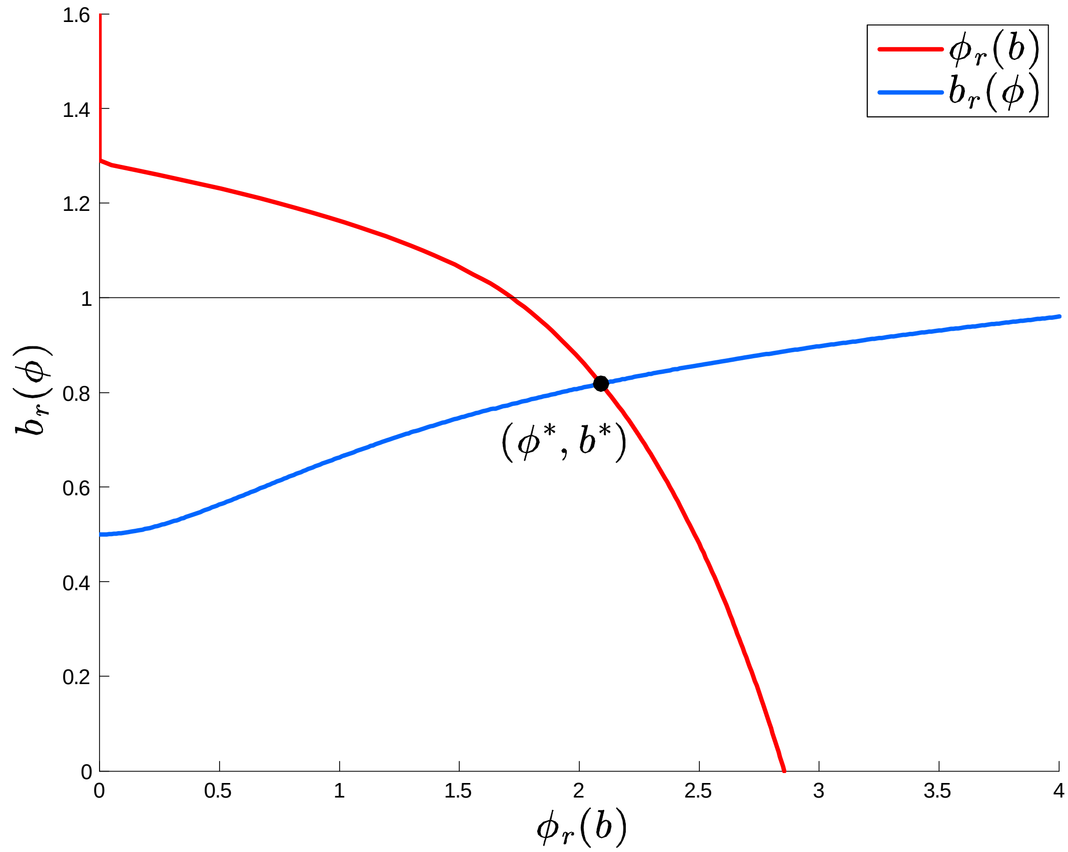

Figure I: Example of an Equilibrium (ϕ^*, b^*)

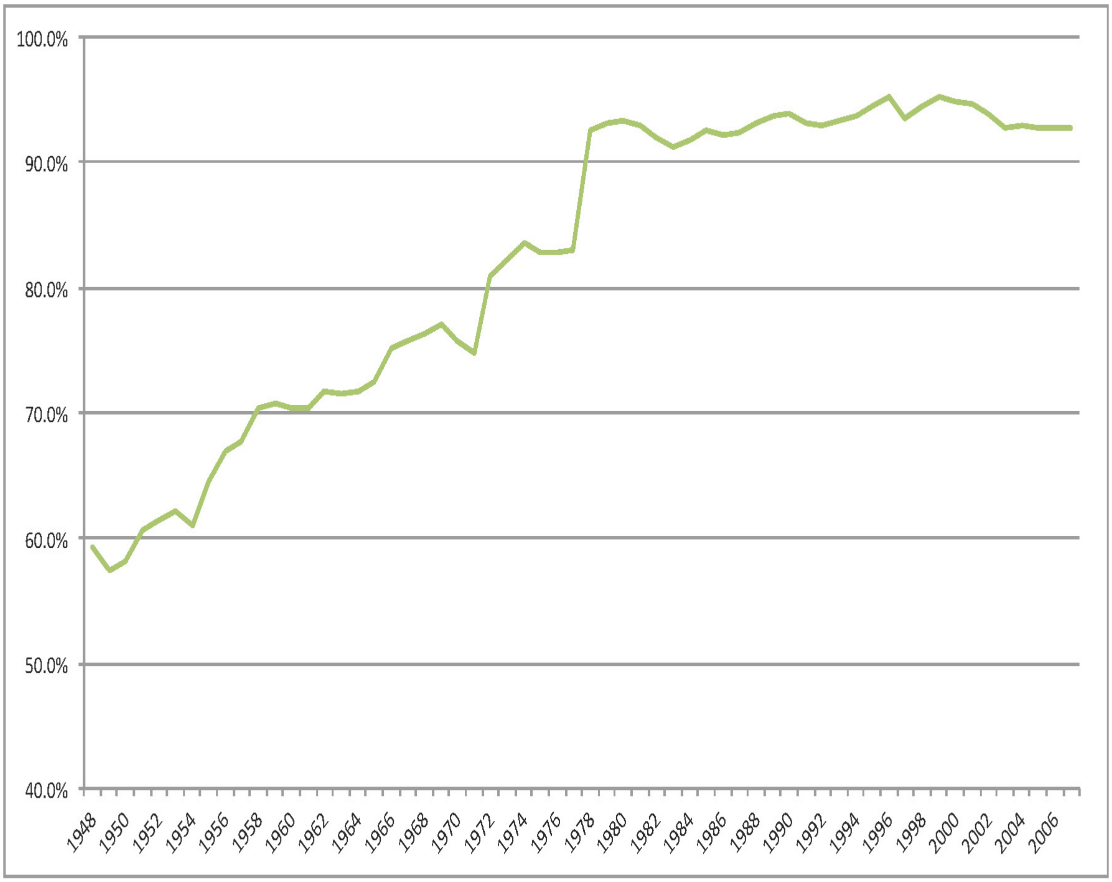

Figure II: Unemployment Insurance Coverage Rates in the US, 1948-2007

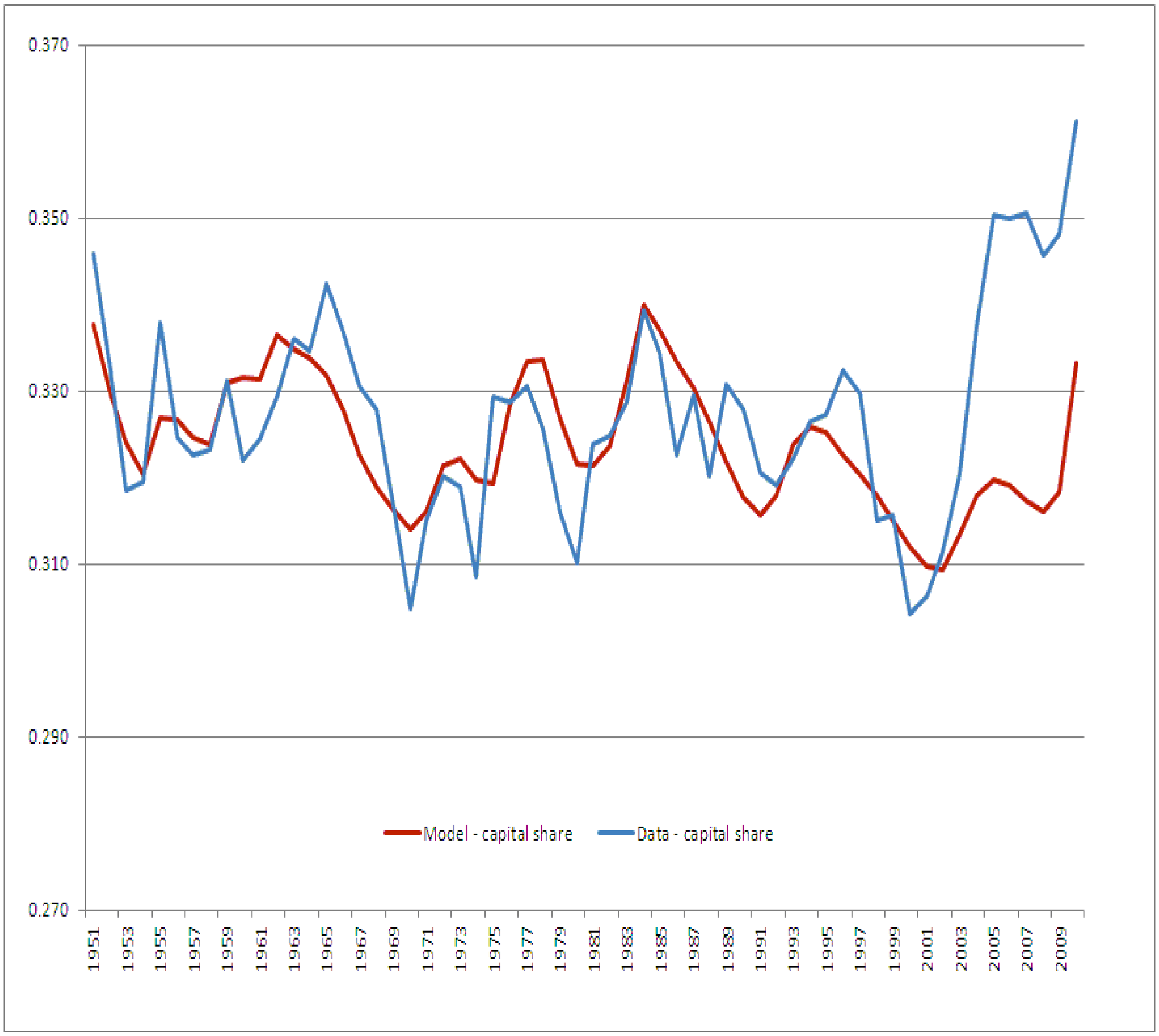

Figure III: Data vs Model Capital Share – Myopic Workers, 1951-2010

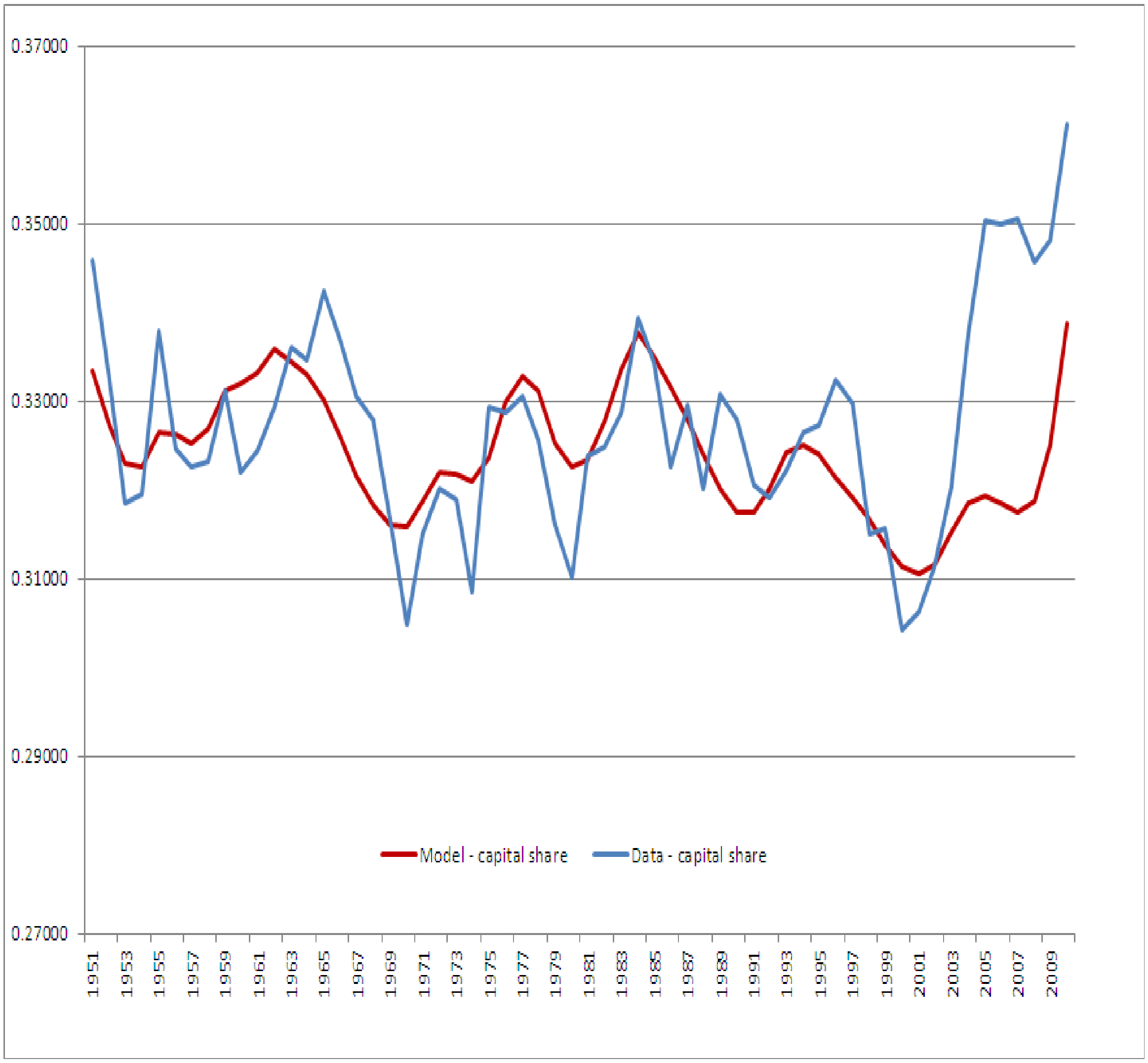

Figure IV: Data vs Model Capital Share – Perfect Foresight, 1951-2010

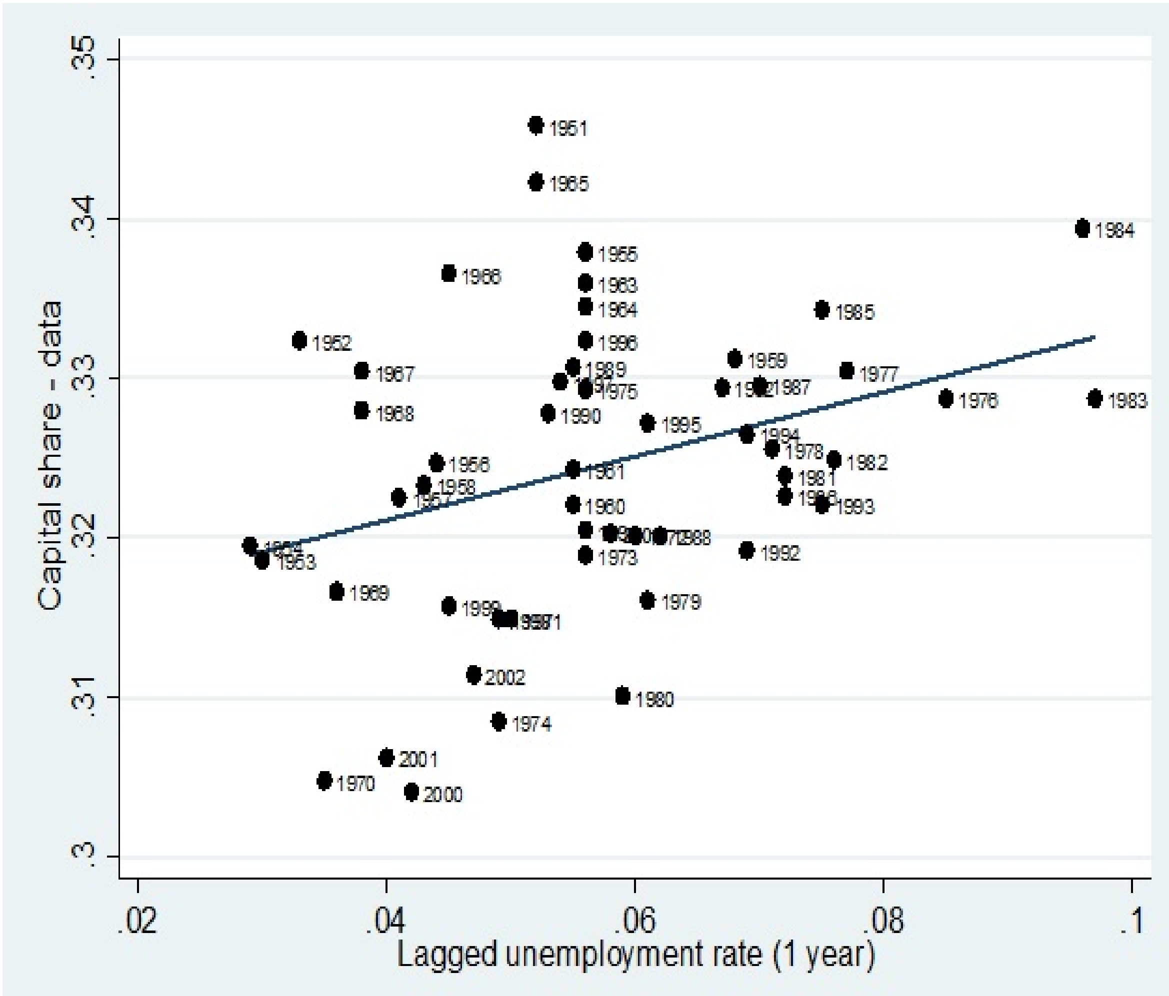

Figure V: Scatterplot of Capital Share (data) and Unemployment Rate (one year lag), 1951-2003. Line of best fit is bold.

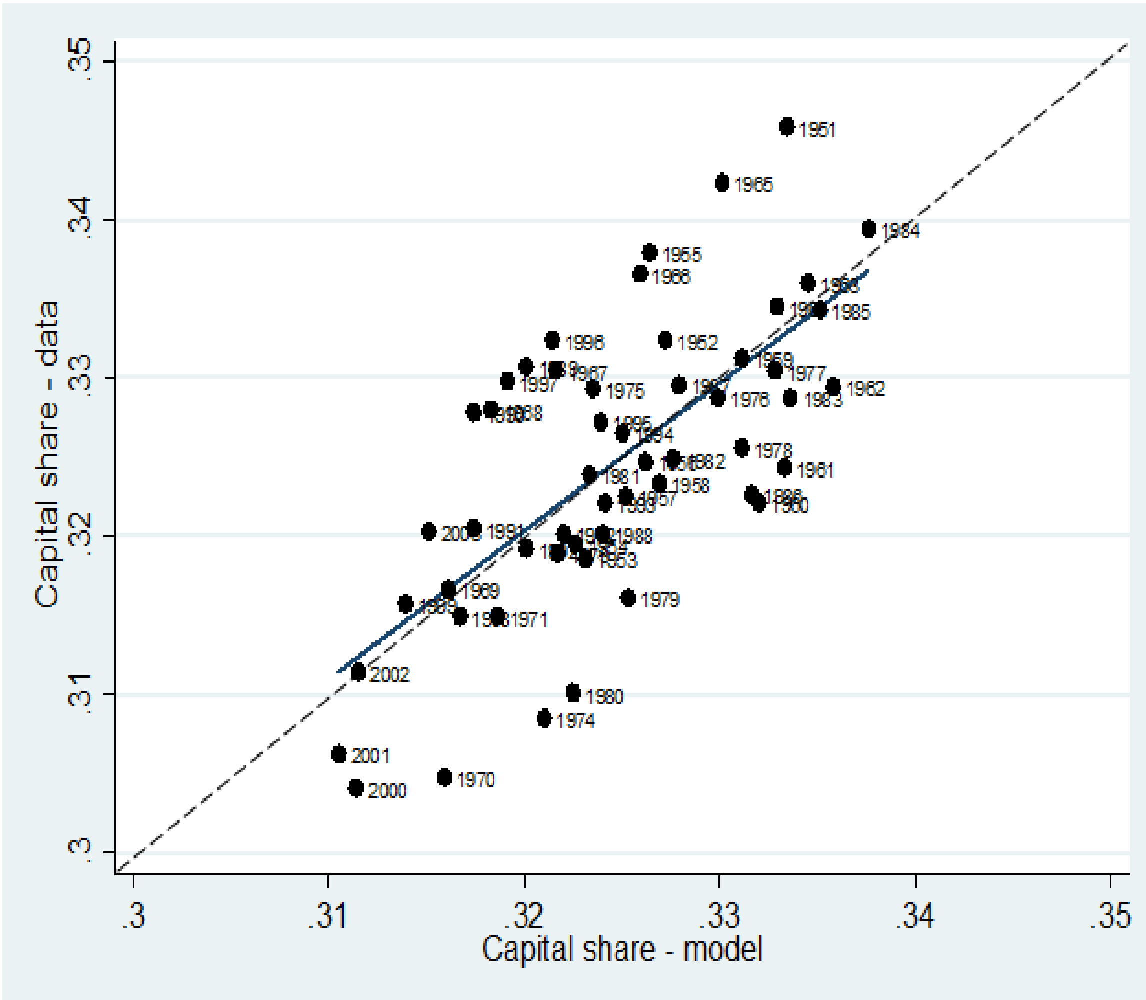

Figure VI: Scatterplot of Capital Share (data) and Capital Share (model), 1951-2003. Perfect foresight equilibrium. Bold line of best fit, 45 degree line dotted.

TABLES

	Data	Myopic model	Perfect foresight
Mean capital share	0.324	0.324	0.324
Standard deviation of capital share	0.0093	0.0075	0.0068
Correlation b/n model and data	1.000	0.731	0.690
Autocorrelation (1 period lag)	0.611	0.847	0.895

Table I: Capital Share - Data, Myopic Model, Perfect Foresight, 1951-2003

	Data	Model				
Average replacement rate, α		0.40	0.45	0.50	0.55	0.60
Mean capital share	0.324	0.324	0.324	0.324	0.324	0.324
Standard deviation capital share	0.0093	0.0084	0.0074	0.0068	0.0066	0.0061
Correlation b/n model and data	1.000	0.578	0.646	0.690	0.683	0.574
Autocorrelation (1 period lag)	0.611	0.895	0.891	0.895	0.914	0.937

Table II: Varying the Average Replacement Rate - Perfect Foresight Equilibrium, 1951-2003

	Data	Model				
Match destruction rate, δ		0.35	0.40	0.45	0.50	0.55
Mean capital share	0.324	0.324	0.324	0.324	0.324	0.324
Standard deviation capital share	0.0093	0.0061	0.0065	0.0068	0.0072	0.0078
Correlation b/n model and data	1.000	0.674	0.687	0.690	0.691	0.689
Autocorrelation (1 period lag)	0.611	0.924	0.913	0.895	0.884	0.870

Table III: Varying the Match Destruction Rate - Perfect Foresight Equilibrium, 1951-2003