Optimal Unemployment Insurance and Cyclical Fluctuations*

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Abstract

We study the design of optimal unemployment insurance in an environment with moral hazard and cyclical fluctuations. The optimal unemployment insurance contract balances the insurance motive to provide consumption for the unemployed with the provision of incentives to search for a job. This balance is affected by aggregate conditions, as recessions are characterized by reductions in job finding rates. We show how benefits should vary with aggregate conditions in an optimal contract. In a special case of the model, the optimal contract can be solved in closed form. We show how this contract can be implemented in a rather simple way: by allowing unemployed workers to borrow and save in a bond (whose return depends on the state of the economy), providing flow payments which are constant over an unemployment spell but vary with the aggregate state, and giving additional lump sum payments (or charges) upon finding a job or when the aggregate state switches. We then consider a calibrated version of the model and study the quantitative impact of changing from the current unemployment system to the optimal one. In a recession, the optimal system reduces unemployment rates by roughly 2.5 percentage points and shortens the duration of unemployment by about 50%.

1 Introduction

In the United States, unemployment insurance is implemented at the state level, with a typical program providing 26 weeks of benefits, at a level determined as a fraction of previ-

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ous earnings (the replacement rate) up to a cap. However in times of recession, the federal government has typically provided extended unemployment benefits which kick in once state benefits are exhausted. During the most recent recession, these extensions happened in several rounds. In June 2008 Congress instituted the Emergency Unemployment Compensation program. This provided an additional 34 weeks of benefits to all states (in two tiers), a third tier providing a further 13 weeks for states with a 3-month average unemployment rate of 6% or greater, and a fourth tier with an additional 6 weeks for states with a 3-month average unemployment rate of 8.5% or greater.[1] Then in 2009, a federal-state Extended Benefits program was passed (and extended in July 2010), which provided an additional 20 weeks by essentially adding 13 weeks to tier three and 7 weeks to tier four. Thus in high unemployment states, benefits were provided for a maximum of 99 weeks. It is intuitive that unemployment insurance should change with changes in the labor market, as economic slowdowns naturally lead to longer unemployment spells. However the observed pattern of extensions and the threshold unemployment rates of the different tiers are rather arbitrary, and the whole program has been subject to uncertainties in its implementation. In this paper, we study the design of optimal unemployment insurance when the economy experiences cyclical fluctuations in job finding and job loss rates.

In particular, we build on previous work by Shavell and Weiss [1979] and Hopenhayn and Nicolini [1997], who studied optimal unemployment insurance with moral hazard.[2] As in their papers, a risk averse unemployed worker puts forth effort to search for a new job, with effort increasing the likelihood of finding a job, but being costly in utility terms. A risk-neutral unemployment agency provides unemployment benefits to help the worker smooth his consumption over the unemployment spell, and seeks to minimize the cost of providing a given level of expected utility to the worker. However the agency cannot observe the level of search effort, and so must structure the benefits in order to provide appropriate search incentives.

Our model adds business cycle fluctuations by having the job finding rate switch according to an exogenous Markov process. When the economy enters a recession state, job finding rates fall for all levels of search effort, while finding rates rise once the recession ends. We also allow for changes in job loss rates over the business cycle, although the recent literature such as Shimer [2012] has suggested that these may be less important. We analyze how the cyclical fluctuations affect the optimal level and duration of benefits provided over an unemployment spell, and how those benefits change when the aggregate state of the economy changes. We also compare the optimal unemployment program with a version of the current

[1]The information on the unemployment programs comes from the Department of Labor website, `http://ows.doleta.gov/unemploy`.

[2]He [2012] develops a related continuous time model of management compensation.

system to evaluate potential gains from reform.

Similar issues have been addressed recently in the literature. However ours is the most complete analysis of the optimal contracting problem in unemployment insurance design, which complements some of the recent work which is more applied and empirical. Hopenhayan and Nicolini [2001] considered the impact of cyclical changes like ours in a two-period model. A similar discrete time model was analyzed by Sanchez [2008], who showed that benefits decrease faster in booms than recessions. However he did not characterize the differences in benefit levels across states. Kroft and Notowidigdo [2011] analyze how unemployment benefits (but not durations) should vary in a related job search model. Landais et al. [2012] study a general equilibrium matching model with search effort and focus on characterizing the optimal benefit level over the cycle. They also consider an extension where benefits expire. Mitman and Rabinovich [2013] consider the cyclical behavior of unemployment benefits in a Diamond-Mortensen-Pissarides model where business cycles are driven by productivity shocks.

After laying out the model, we turn to a special case which yields an explicit solution. Here we assume that workers have exponential utility and the unemployment insurance agency has an exponential cost function. This case allows us to contrast in a simple way how the information frictions affect the insurance system, as we contrast the optimal system under moral hazard with one where job search effort is observable. As in the previous literature, we show that consumption declines over an unemployment spell, and the consumption that the agent would obtain upon finding a job also decreases the longer he is unemployed. Our results also illustrate how the contract varies over the business cycle, with the resulting variations in search effort and consumption. Under moral hazard, the optimal contract trades off insurance with incentives, and this relationship varies with the business cycle. That is, in a recession the required insurance is greater, as for a given amount of search effort the a worker will be unemployed for a longer period of time. However the incentive problem is also magnified, as a worker will get fewer job contacts and so generate less information. The optimal contract resolves this tension by inducing more search effort when it is more productive in a boom, but with a greater reward (in promised utility terms) when the worker finds a job in a recession.

Continuing with the special case, we show how to implement the contract via relatively simple instruments. That is, rather than a direct implementation where we equate consumption and unemployment benefits, we show that the contract can be implemented by allowing the worker to save and borrow, providing benefits which are constant in each aggregate state, providing a re-employment bonus, and giving an additional bonus or charge when the aggregate state changes. This implementation is similar to Werning [2002] and Shimer and Werning [2008] who find that constant unemployment benefits are optimal in related models with exponential utility. Werning [2002] studies a model like ours, absent business cycle fluctuations but allows the agent hidden savings.[3] Throughout we suppose that the agent's wealth is observable, and one instrument in our implementation is the effective rate of return on wealth. Allowing hidden savings would shut down this channel, as the agency could not effectively tax (or subsidize) the agent's wealth. Williams [2013] provides an example along these lines in a moral hazard model. We abstract from hidden savings in order to focus directly on search incentives, and thus suppose that workers provide the unemployment agency with access to their financial records in exchange for receiving unemployment benefits.

Shimer and Werning [2008] study a related sequential search model with unobservable offers (but no search decisions) and exponential utility. Our implementation shares the features that they highlight, namely the distinction between consumption and liquidity which the insurance benefits provide. However in our environment benefits and the interest rate on savings are constant over an unemployment spell within a given aggregate state, but vary across states. In addition, our information friction differs. Since our contract must provide incentives for search, it includes a re-employment bonus upon finding a job.[4] Finally, when an agent is unemployed but the aggregate state switches, we find that the agency provides a payment (or charge) reflecting the impact of the switch on his future income. Thus our implementation mixes elements of the current system of constant benefits, the unemployment insurance savings accounts of Feldstein and Altman [2007], and the re-employment bonuses which have been tried as experiments in various locations. The effects of these programs in US states were studied by Robins and Spiegelman [2001] and OLeary et al. [2005] among others and in Canada by Card and Hyslop [2005]. Cyclical fluctuations are new to our study, and necessitate the additional payment when the state of the economy switches, which captures changes in incentives and costs of insurance over the cycle.

We then turn to a quantitative version of the model to illustrate the impact of reforming the unemployment insurance system. We calibrate the model using a stylized version of the current system, which provides constant replacement rate of 47% for six months in a boom and nine months in a recession. We simulate a large population of workers, and match the cyclical behavior of job finding rates documented by Shimer [2012]. We also target an elasticity of unemployment duration with respect to an increase in benefits between the estimates of Meyer [1990] which has been commonly used in the literature and the more recent findings of Kroft and Notowidigdo [2011] and Chetty [2008]. We show that the optimal unemployment insurance contract leads to a significant increase in job finding rates and

[3]Mitchell and Zhang [2010] study a related model with exponential utility and hidden savings, but unlike the monetary cost of effort here and in Werning [2002], they consider a separable utility cost. They find a rather different optimal contract, with increasing benefits during unemployment spells.

[4]Pavoni [2009] finds that with duration dependence in unemployment spells it may be optimal to subsidize the wages of the long-term unemployed.

corresponding reduction the unemployment rate and the average duration of unemployment. In a recession, the optimal system reduces unemployment rates by roughly 2.5 percentage points and shortens the duration of unemployment by roughly 50%.

To gain insight on the impact of the reform of unemployment insurance in the most recent recession, we simulate a recession seventy weeks long and compar the performance of the benchmark current system with 39 weeks of benefits, the current system with benefits extended to 99 weeks, and our optimal system. Under the benchmark system, unemployment rates increase by 1.4 percentage points as the economy goes from a boom to the long recession. However, consistent with the empirical evidence of Rothstein [2011], we find the extension of benefits has a very small impact, increasing unemployment rates by only an additional 0.1 percentage point. However that does not imply that unemployment insurance has no impact, as we find that under our optimal system the unemployment rate increases by only 0.4 percentage points in the recession, and throughout the recession remains 2.7 percentage points lower than the current system.

2 The Model

In this section, we layout the model. The model is essentially a continuous time version of Hopenhayn and Nicolini [1997] with multiple unemployment spells and business cycle fluctuations.

2.1 The Setup

We assume that an infinitely-lived worker transits over time between employment and unemployment. When he is in an unemployment spell, he may exert effort to find a job, with effort being costly but increasing the arrival rate of a job. An unemployed worker does not have any income except perhaps a minimal amount of consumption, $\alpha \geq 0$. This can be interpreted as the consumption the agent derives from sources other than unemployment insurance. For example Hopenhayn and Nicolini [1997] consider variants where the agent has assets which yield a constant flow of income for consumption.[5] For simplicity, we assume that all jobs are identical and pay a constant wage ω. Employed workers lose their jobs according to an exogenous separation rate.[6] When employed, the worker does not take effort. In addition, we assume that the state of the economy s_t switches between a *good* or

[5]Technically, with α being positive, we prevent the utility level from being $-\infty$ if consumption is 0 when the utility function is unbounded below.

[6]An extension of the model, following Wang and Williamson [1996] would incorporate costly effort to retain the job.

boom state ($s_t = G$) and a *bad* or recession state ($s_t = B$). In the good state, job finding rates are higher and separation rates are lower.

Let a_t be the search effort chosen from $[0, \bar{a}]$ by the worker at t.[7] If the economy is currently in state s and an unemployed worker puts forth search effort a_t, the Poisson arrival intensity of a job offer is

$$q_s(a_t) \equiv q_{s0} + q_{s1} a_t \text{ with } q_{s0},\, q_{s1} > 0.$$

To interpret this, suppose that the state does not change and the agent puts forth a constant effort a_0. Then the waiting time until a job offer arrives is an exponentially distributed random variable with rate $q_s(a_0)$. Equivalently, over a short time interval of length $\Delta > 0$ in which a_0 is taken, the probability that the worker gets a job is approximately $q_s(a_0)\Delta$. We assume that finding rates are higher in booms for all levels of effort:

$$q_B(\hat{a}) < q_G(\hat{a}) \text{ for all } \hat{a} \in [0, \bar{a}].$$

Note that since $q_G(\cdot)$ and $q_B(\cdot)$ are affine, this is equivalent to $q_{B0} < q_{G0}$ and $q_{B0} + q_{B1} < q_{G0} + q_{G1}$.

For simplicity, we assume that an employed worker loses his job with an exogenous separation rate p_s, where $p_B > p_G$. The aggregate state of the economy switches between booms and recessions according to a two-state Markov switching process with switching rate λ_s.

2.2 Preferences and Incentive Compatible Contracts

If the worker consumes c and takes effort a, his instantaneous utility is $u(c, a) = \bar{u}(c + \alpha, a)$ where u is strictly increasing and concave in c and strictly decreasing and convex in a with $u(c, 0) = u(c)$. Note that we incorporate the lower bound on consumption $\alpha \geq 0$ here and thus require $c \geq 0$. We also assume that workers die stochastically, which is governed by an idiosyncratic Poisson process with constant arrival rate $\kappa > 0$. When the shock hits, the worker dies, receives zero utility forever and the contract is terminated. The subjective discount rate of the worker is $\hat{\rho} > 0$, but incorporating the death probability the effective discount rate is $\rho = \hat{\rho} + \kappa$.

We assume that an insurance agency ("the principal") provides unemployment insurance to help the worker ("the agent") smooth his consumption. The worker's employment status and the aggregate state are publicly observable, but the search effort is not observable to the insurance agency. So moral hazard arises and the agency needs to offer an unemployment

[7] We set an upper bound of effort, $\bar{a}$, to make the maximization problem well defined. In all the examples, $\bar{a}$ is sufficiently large that the optimal effort is always interior.

insurance contract which induces the worker to take appropriate search effort. A contract is a pair of processes $(c, a) \equiv (\{c_t\}_{t\in[0,\infty)}, \{a_t\}_{t\in[0,\infty)})$. Specifically, c is the consumption process with c_t being the amount of instantaneous consumption *promised* by the contract at t. We assume that $c_t \in [0, \bar{c}]$ for all $t \in [0, \infty)$. We impose the upper bound $\bar{c}$ as a resource constraint on the insurance agency. The process a is the effort process with a_t the instantaneous effort level *suggested* by the contract, with $a_t = 0$ if the worker is employed. The contract is history dependent in the sense that c_t and a_t depend on the entire history of the worker's employment status and the aggregate state up to time t. Technically, if we denote the filtration generated by the history as $\{\mathcal{F}_t\}_{t\in[0,\infty)}$, then (c, a) is $\mathcal{F}_t$-adapted.

We can now describe the worker's utility maximization problem. Under a contract (c, a), the true effort level is not verifiable, so the worker chooses effort to maximize his expected utility:

$$\max_{\hat{a}\in A} E^{\hat{a}} \left[\rho \int_0^\infty e^{-\rho t} u(c_t, \hat{a}_t) dt \right] \tag{1}$$

Here, A is the set of all $\mathcal{F}_t$-adapted processes with values in $[0, \bar{a}]$. $E^{\hat{a}}[\cdot]$ is the expectation operator induced by the effort process $\hat{a} \in A$. A contract is *incentive compatible* if and only if a is the optimal choice in problem (1).

If the worker is unemployed, to promise consumption c_t the agency needs to deliver c_t to the worker. However if he is employed, the agency needs only to deliver $c_t - \omega$. The discount rate of the agency is also $\hat{\rho}$ and so the effective rate is ρ. Although most studies of social insurance consider a risk-neutral agency, we allow for the agency to have a cost function v which is increasing and convex. Thus the objective of the agency is to design an insurance contract as follows.

$$\max_{(c,a)\in C} E^a \left[-\rho \int_0^\infty e^{-\rho t} v(c_t - \mathbf{1}(\{\text{the worker is employed}\})\omega) dt \right]$$

such that

$$(c, a) \text{ is incentive compatible}$$

and

$$E^a \left[\rho \int_0^\infty e^{-\rho t} u(c_t, a_t) dt \right] \geq \bar{W}.$$

Here, C is the set of all feasible contracts, $\mathbf{1}(\cdot)$ is the indicator function of an event and $\bar{W}$ is the reservation utility of the worker.[8] The first constraint is the incentive constraint and the second is the participation constraint. We assume that enforcement and commitment are complete, so once the contract is signed neither party can leave it until the worker dies.

[8]A contract (c, a) is *feasible* if c and a are $\mathcal{F}_t$-adapted and $c_t \in [0, \bar{c}]$ and $a_t \in [0, \bar{a}]$ for all $t \in [0, \infty)$.

3 The Optimal Contract

We now derive the law of motion for the agent's promised utility, the key endogenous state variable in our setting. Then we derive the Hamilton-Jacobi-Bellman equations determining the optimal contract.

3.1 The Worker's Incentives and Promised Utility

We first show how the worker chooses his optimal search effort under a contract and how the agency provides incentives to induce appropriate effort. Under a contract, consumption can in general depend on the worker's entire employment history. Thus traditional dynamic programming methods cannot be applied directly because of the lack of a recursive structure. Therefore, as in Sannikov [2008] and Schättler and Sung [1993], we use a martingale approach to formulate the key incentive compatibility condition. This allows us to show that the agent's promised utility captures the relevant history, which in turn allows us to use recursive methods.[9] The martingale approach for controlled Markov jump process was developed by Boel et al. [1975] and Davis [1976].

We have already defined the aggregate state $s_t \in \{G, B\}$, which we now assign the numerical values $G = 0$ and $B = 1$. Similarly, we define $j_t \in \{E, U\}$ as the worker's job status, with E being employed (numerical value of one) and U unemployed (numerical value of zero). All the information observable to the insurance agency is characterized by the two processes j and s, which are two-state Markov jump processes with *unpredictable* jumps between 0 and 1.[10] The associated compensated jump martingales, $\{m_t^J\}_{t\in[0,\infty)}$ and $\{m_t^S\}_{t\in[0,\infty)}$, are defined as:

$$m_t^j = \int_0^t \left(-(1-j_{\hat{t}})\left[(1-s_{\hat{t}})\, q_G(a_{\hat{t}}) + s_{\hat{t}} q_B(a_{\hat{t}})\right] + j_{\hat{t}}\left[(1-s_{\hat{t}})\, p_G + s_{\hat{t}} p_B\right]\right) d\hat{t} + j_t$$

and

$$m_t^S = \int_0^t \left[(1-s_{\hat{t}})\, \lambda_G + s_{\hat{t}} p_G\right] d\hat{t} + j_t.$$

Equivalently

$$dm_t^J = \left(-(1-j_t)\left[(1-s_t)\, q_G(a_t) + s_t q_B(a_t)\right] + j_t\left[(1-s_t)\, p_G + s_t p_B\right]\right) dt + \Delta j_t$$

and

$$dm_t^S = \left[-\left(1-s_t\right)\lambda_G + j_t \lambda_B\right] dt + \Delta s_t,$$

[9] The same results hold using a stochastic maximum principle as in Williams [2011].

[10] See Elliott [1982] for details on jump processes. For technical reasons, we assume that their trajectories are right continuous.

with Δj_t and Δs_t being the indicators of job finding or loss events and switches of the aggregate state. For example, if at date t the worker is unemployed ($j_t = 0$) and the economy is in a boom ($s_t = 0$), then:

$$dm_t^J = -q_G(a_t)dt + \Delta j_t.$$

Thus the martingale m_t^J has a negative drift but a positive jump when the worker finds a job, so that its expectation is zero.

Given a feasible contract (c, a) and arbitrary effort process $\hat{a}$, we define the promised utility of the worker as

$$W_t \equiv E^{\hat{a}}\left[\rho \int_t^\infty e^{-\rho t} u(c_t, \hat{a}_t) dt | \mathcal{F}_t\right] \text{ for } t \in [0, \infty]$$

which is the expected utility of the worker under the contract from date t on. We then have the following result which is derived from a martingale representation theorem.

Proposition 1 *Under a contract $(c, a) \in C$, suppose that the effort process $\hat{a}$ is chosen. Then, there exist two $\mathcal{F}_t$-predictable processes $\{g_t^J\}_{t\in[0,\infty)}$ and $\{g_t^S\}_{t\in[0,\infty)}$ such that*

$$E^{\hat{a}}\left[\int_0^\infty \left(e^{-\rho t} g_t^J\right)^2 dt\right] < \infty \text{ and } E^{\hat{a}}\left[\int_0^\infty \left(e^{-\rho t} g_t^S\right)^2 dt\right] < \infty \tag{2}$$

and

$$dW_t = \rho(W_t - u(c_t, \hat{a}_t))dt + \rho g_t^J dm_t^J + \rho g_t^S dm_t^S \text{ for } t \in [0, \infty]. \tag{3}$$

Proof. *See Appendix A.1.* ■

Here, g_t^J is the sensitivity of promised utility to changes in employment status while g_t^S is the sensitivity to changes in the aggregate state. In fact, the worker's promised utility encodes all the relevant information in the observable history. With this state variable, we can make the model recursive and so can use dynamic programming methods.

Since search effort is costly but affects the worker's employment status, $\{g_t^J\}_{t\in[0,\infty)}$ governs the incentive to take search effort. This is demonstrated by the following Proposition.

Proposition 2 *Given a contract $(c, a) \in C$, suppose that $\{g_t^J\}_{t\in[0,\infty)}$ is the process given in Proposition 1. Then the contract is incentive compatible if and only if at any date t when the worker is unemployed:*

$$a_t \in \arg\max_{\tilde{a}\in[0,\bar{a}]} g_t^J q_{s_t}(\tilde{a}) + u(c_t, \tilde{a}) \tag{4}$$

Proof. *See Appendix A.2* ■

Thus the proposition shows that the local or instantaneous incentive constraint (4) is sufficient to characterize full incentive compatibility. To see how the incentive constraint captures the worker's tradeoff between the utility cost of effort against the benefit of an increase in promised utility, suppose that the optimal effort choice from (4) is interior. Then we have:

$$g_t^J q_{s_t 1} = -u_a(c_t, a_t)$$

By (3), the worker's promised utility increases by ρg_t^J if he finds a job at t. Increases in search effort make job finding more likely, and the expected marginal benefit is $\rho g_t^J q_{s1}$, which is equated to the marginal cost $-\rho u_a(c_t, a_t)$. Thus the contract provides search incentives by increasing promised utility by g_t^J when an unemployed worker finds a job.

3.2 The Value Functions and the Optimal Contract

In this section, we derive the conditions the value function of the insurance agency must satisfy, which is key to solving for the optimal contract. Define $V(W, j, s)$ as the maximum expected payoff the agency can attain under an incentive compatible contract which promises utility W to the worker when his employment status is j and the current aggregate state is s. Since j and s are binary variables, we effectively have four separate value functions $V(\cdot, j, s)$ for $j = e, u$ and $s = G, B$.

3.2.1 The Boundary Points of the Value Functions

As a first step in the construction of the value functions, we compute their boundary points. The left boundary points are generated by the harshest contracts, which promise the lowest expected utility to the worker, and the right boundary points are generated by the most generous contracts, which promise the highest expected utility to the worker.

The left boundary points are denoted by the pairs $(W_l^{js}, V(W_l^{js}, j, s))$ with W_l^{js} being the lowest expected utility that the insurance agency can promise to the worker in the four different cases. Since the worker's utility is strictly increasing and consumption is bounded by α, the harshest punishment to the worker provides zero (additional) consumption when he is unemployed and takes away all his wage income when he is employed. Under this contract the worker has no incentive to search because his consumption is constant at α, regardless of his employment status. His expected utility is thus constant at:

$$\underline{u} \equiv \rho \int_0^\infty e^{-\rho t} u(0, 0) dt = \bar{u}(\alpha)$$

Under this contract, the insurance agency's income is 0 when the worker is unemployed and ω if he is employed. We show how to compute the agency's expected payoff in the four cases in Proposition 4 in the appendix.

Symmetrically, the right boundary points are denoted $(W_r^{js}, V(W_r^{js}, j, s))$, with W_r^{js} the highest expected utility that the insurance agency can promise to the worker. The most generous reward to the worker is providing the maximum consumption $\bar{c}$ constantly. Therefore, the agency's value is always $-v(\bar{c})$. In an unemployment spell, the worker's consumption is $\alpha + \bar{c}$ and in an employment spell it is $\alpha + \bar{c} + \omega$. Therefore, the worker has incentive to take search effort when unemployed. The details of the calculation of the worker's expected utility are given by Proposition 4 in the appendix.

3.2.2 The Hamilton-Jacobi-Bellman Equations

With the boundaries established, we now characterize the value functions in the interior of their domains. We derive the Hamilton-Jacob-Bellman (HJB) equations they satisfy, which allows us to construct the optimal contract by solving a system of differential equations.

First, it is convenient to change the control variables. Given a contract (c, a), let $\{g_t^J\}_{t\in[0,\infty)}$ and $\{g_t^S\}_{t\in[0,\infty)}$ be the sensitivity processes defined in Proposition 1. By (3), if an unemployed worker finds a job at time t then $\Delta j_t = 1$ and his promised utility jumps up by ρg_t^J. Similarly, when an employed worker loses his job, $\Delta j_t = -1$ and his promised utility falls by ρg_t^J. So if we define W_t^J as the worker's promised utility immediately after the change of his job status, then:

$$g_t^J \Delta j_t = \frac{W_t^J - W_t}{\rho}. \tag{5}$$

Then we can re-write the incentive constraint (4) as:

$$\max_{\hat{a}} \frac{W_t^J - W_t}{\rho} q_s(\hat{a}) + u(c_t, \hat{a}). \tag{6}$$

Note that this incentive constraint effectively ties down one of the potential degrees of freedom of the contract, as a and W^J as they are linked by the incentive constraint. We define the function $a^s(W, W^J)$ as the unique solution of (6).

In addition, all workers experience a change in promised utility when the aggregate state switches. By (3), promised utility jumps by ρg_t^S if the economy transits from boom into recession ($\Delta s_t = 1$) and falls $-\rho g_t^S$ if the economy goes from a recession to a boom. So if W_t^S is the promised utility right after the switch in the aggregate state then

$$g_t^S \Delta s_t = \frac{W_t^S - W_t}{\rho}.$$

Thus we now view the agency's instruments in designing a contract as: c_t, consumption above the minimal level; W_t^J, promised utility after a change in the worker's employment status; and W_t^S, promised utility after a change in the aggregate state.

Since the proofs of the existence, differentiability and concavity of the value functions are highly technical, and follow standard methods, we assume these properties without proof. The following proposition characterizes the HJB equations that the value functions satisfy.

Proposition 3 *Suppose that the value functions $V(\cdot, j, s)$ exist, are differentiable, and concave. Then their left and right boundaries satisfy the conditions in Proposition 4(in Appendix A.3). Furthermore, they satisfy the following set of HJB equations*[11]*:*

$$
\begin{aligned}
\rho V(W,u,s) \quad = \quad & \max_{\substack{\hat{c}\in[0,\bar{c}] \\ W^J\in[W_l^{es},W_r^{es}] \\ W^S\in[W_l^{us'},W_r^{us'}]}} \quad -\rho v(\hat{c}) && (7)\\
& +\rho V_W(W,u,s)\left[W - u(\hat{c}, a^s(W,W^J)) - q_s(a^s(W,W^J))\frac{W^J-W}{\rho} - \lambda_s\frac{W^S-W}{\rho}\right] \\
& +q_s(a^s(W,W^J))\big(V(W^J,e,s) - V(W,u,s)\big) && (8)\\
& +\lambda_s\big(V(W^S,u,s') - V(W,u,s)\big)
\end{aligned}
$$

for any $W \in [W_l^{us}, W_r^{us}]$ with $s = G, B$;

$$
\begin{aligned}
\rho V(W,e,s) \quad = \quad & \max_{\substack{\hat{c}\in[0,\bar{c}] \\ W^J\in[W_l^{us},W_r^{us}] \\ W^S\in[W_l^{es'},W_r^{es'}]}} \quad -\rho v(\hat{c}-\omega) + \rho V_W(W,e,s)\left(W - u(\hat{c}) - p_s\frac{W^J-W}{\rho} - \lambda_s\frac{W^S-W}{\rho}\right) \\
& +p_s\big(V(W^J,u,s) - V(W,e,s)\big) + \lambda_s\big(V(W^S,e,s') - V(W,e,s)\big) && (9)
\end{aligned}
$$

for any $W \in [W_l^{es}, W_r^{es}]$ and $s = G, B$.
Proof. *See Appendix A.4.* ■

The HJB equations capture the expected change in the agency's value. The first terms summarize the instantaneous cost of providing consumption and the expected marginal cost of the change in a the worker's promised utility. The final two terms capture the expected changes in costs when the worker's employment status or aggregate state switch.

We now show how the worker's promised utility reacts to the switches of the aggregate state, which are out of his control. The jumps in promised utility follow the "slope matching" rule documented in Piskorski and Tchistyi [2011] and Li [2012]. That is, the contract equates the marginal value (to the agency) of promised utility before and after a change in the aggregate state.

Corollary 1 *Suppose that the value functions exist, are differentiable and concave. Given the current state s, the worker's employment status j and promised utility W_t, then W_t^S the*

[11]See Proposition 4 for notations.

promised utility after the aggregate state switches to s' satisfies:

$$W_t^S = \begin{cases} W_l^{js'} \text{ if } V_W(W_t, j, s) \geq V_W(W_l^{js'}, j, s') \\ W_r^{js'} \text{ if } V_W(W_t, j, s) \leq V_W(W_r^{js'}, j, s') \end{cases}$$

or else W_t^S solves

$$V_W(W_t, j, s) = V_W(W_t^S, j, s').$$

Proof. *In each HJB equation listed in Proposition 3, W_t independently solves the following:*

$$\max_{W^S} -\rho V_W(W, j, s)\lambda_s \frac{W^S - W}{\rho} + \lambda_s V(W^S, j, s'),$$

whose solution gives the result. ■

4 A Solvable Special Case

While the results above hold for general utility functions, we typically require numerical methods to solve for the optimal contracts. In this section we consider a special case which allows for explicit solutions. This allows us to gain insight on the structure of the optimal contract and how consumption and effort should respond to cyclical fluctuations.

For simplicity, we focus on a single unemployment spell with permanent jobs, so we set $p_s = 0$ for $s = G, B$.[12] We also assume the job finding rate is linear, $q_s(a) = q_s a$. That is, we set $q_{0s} = 0$ and we drop the subscript "1" on q_{1s}. We also assume that both workers and the unemployment agency have exponential preferences. Thus workers' utility function is:

$$u(c, a) = -\exp\left(-\theta_A \left(c - h(a)\right)\right)$$

where h in increasing and convex with $h(0) = 0$. The agency's cost function is given by:

$$v(c) = \exp\left(\theta_P c\right).$$

Of these special assumptions, the most important are the exponential utility and cost functions, which often give explicit solutions (see Holmstrom and Milgrom [1987] and Williams [2011], for example). Models of optimal social insurance typically consider risk neutral cost functions for the government, as we will also do below in our quantitative study. However the unemployment agency may well care about the variability of unemployment insurance payments in order to making budgeting decisions. Below we will see how variation in θ_P, the agency's coefficient of absolute risk aversion, affects the structure of the contract.

[12]The same form of the solution holds with $p_s \neq 0$.

4.1 The Value of an Employed Worker

With permanent employment, the agency's value of an employed worker is easy to establish. First, note that the HJB equation (9) simplifies to:

$$\rho V(W, e, s) = \max_{c} -\rho v(c-\omega) + \rho V_W(W, e, s)[W - u(c)]$$

The optimality equation for c is thus:

$$-v'(c-\omega) = V_W(W, e, s)u_c(c).$$

Since the worker values consumption smoothing and there is no more risk or need to provide incentives once a worker is employed, it is easy to see that the optimal contract provides constant consumption for an employed worker. Thus agency's value function is independent of the aggregate state, with promised utility simply determining how much consumption the agency must provide. In particular, we show in Appendix B.1 that the value function is given by:

$$V(W, e, s) = V(W, e) = -\exp(-\theta_P \omega)(-W)^{-\frac{\theta_P}{\theta_A}}$$

and the consumption of an employed agent is:

$$c(W, e) = -\frac{1}{\theta_A}\log(-W).$$

Thus we just invert the utility function in order to determine how much consumption is needed to deliver the promised utility to the agent.

4.2 Full Information

Although we did not consider it above, in order to analyze the effect of moral hazard in unemployment insurance contracts, it is useful to first analyze the case when the agent's effort is observable and contractible. With full information, we can simply dispense with the incentive constraint in our analysis above. Therefore the HJB equation (8) for the value of an unemployed worker is:

$$\begin{aligned}\rho V(W, u, s) \quad = \quad & \max_{c, a, W^J, W^S} -\rho v(c) + \rho V_W(W, u, s)\left[W - u(c, a) - q_s a\frac{W^J - W}{\rho} - \lambda_s \frac{W^S - W}{\rho}\right] \\ & + q_s a[V(W^J, e) - V(W, u, s)] + \lambda_s[V(W^S, u, s') - V(W, u, s)]\end{aligned}$$

The optimality conditions for c and a are:

$$\begin{aligned}-v'(c) \quad &= \quad V_W(W, u, s)u_c(c, a) \\ \rho V_W(W, u, s)\left[u_a(c, a) + q_s \frac{W^J - W}{\rho}\right] \quad &= \quad q_s[V(W^J, e) - V(W, u, s)].\end{aligned}$$

The optimality conditions for W^J and W^S imply the "slope matching" conditions:

$$\begin{aligned} V_W(W,u,s) &= V_W(W^J,e) \\ V_W(W,u,s) &= V_W(W^S,u,s'). \end{aligned}$$

In Appendix B.2 we show that the value function for an unemployed worker takes the same form as that for an employed worker, but with a different leading constant that depends on the aggregate state. That is:

$$V(W,u,s) = -V_u(s)(-W)^{-\frac{\theta_P}{\theta_A}}.$$

We also show that the optimal choice of effort is independent of W, that is $a = \bar{a}(s)$. In the appendix, we use the optimality condition for effort and the HJB equation to derive the four equations which determine the four unknowns $(\bar{a}(G), V_u(G), \bar{a}(B), V_u(B))$. Our results also imply that the optimal consumption and the adjustments in promised utility can be written:

$$\begin{aligned} c(W,u,s) &= \frac{\log(V_u(s)) + \theta_A h(\bar{a}(s))}{\theta_P + \theta_A} - \frac{1}{\theta_A}\log(-W) \\ W^J(W,s) &= \left(\frac{V_u(s)}{\exp(-\theta_P \omega)}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} W \\ W^S(W,s) &= \left(\frac{V_u(s)}{V_u(s')}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} W. \end{aligned}$$

Thus the consumption function again inverts the utility function, but now also compensates the worker for putting forth costly effort. In addition, the agency's marginal value of providing consumption varies with the aggregate state s through $V_u(s)$, so that changes the consumption delivery. Upon finding a job or having a switch in the aggregate state, there is a multiplicative adjustment in promised utility, capturing changes in the relative costs of providing utility to the worker.

4.3 Moral Hazard

Under moral hazard, a is no longer a free choice variable for the unemployment agency, but instead must satisfy the incentive constraint (6). The agent's optimality condition captures the incentive constraint here:

$$-u_a(c,a) = q_s \frac{W^J - W}{\rho}.$$

Above we used this relation to determine $a^s(W, W^J)$, but here it is easiest to solve for W^J:

$$W^J = W - \frac{\rho}{q_s} u_a(c,a)$$

Imposing the incentive constraint, the agency's value function for an unemployed worker satisfies the HJB equation (8), which now can be written:

$$\begin{aligned}\rho V(W,u,s) &= \max_{c,a,W^S} -\rho v(c) + \rho V_W(W,u,s)\left[W - u(c,a) + a u_a(c,a) - \lambda_s \frac{W^S - W}{\rho}\right] \\ &\quad + q_s a\left[V\left(W - \frac{\rho}{q_s} u_a(c,a), e\right) - V(W,u,s)\right] + \lambda_s[V(W^S,u,s') - V(W,u,s)]\end{aligned}$$

The optimality condition for c is then:

$$-v'(c) = u_c(c,a)V_W(W,u,s) - a u_{ac}(c,a)\left[V_W(W,u,s) - V_W\left(W - \frac{\rho}{q_s}u_a(c,a), e\right)\right],$$

while the optimality condition for a is:

$$\rho a u_{aa}(c,a)\left[V_W(W,u,s) - V_W\left(W - \frac{\rho}{q_s}u_a(c,a), e\right)\right] = q_s\left[V(W,u,s) - V\left(W - \frac{\rho}{q_s}u_a(c,a), e\right)\right].$$

In Appendix B.3 we show that the value function takes the same form as in the full information case, but with different leading constants. That is:

$$V(W,u,s) = -V^*(s)(-W)^{-\frac{\theta_P}{\theta_A}}.$$

We also show that the optimal choice of effort is again independent of W: $a = a^*(s)$ and the consumption policy function is of the same form as in the full information case:

$$c(W,u,s) = c^*(s) + h(a^*(s)) - \frac{1}{\theta_A}\log(-W). \tag{10}$$

The appendix shows how to determine the six constants $(V^*(s), a^*(s), c^*(s))$ for $s = G, B$, using the optimality conditions for c and a and the HJB equation. We also show that the adjustments in continuation utility are again multiplicative and can be written:

$$\begin{aligned}W^J(W,u,s) &= w_J(s)W \\ W^S(W,u,s) &= w_S(s)W.\end{aligned}$$

The consumption function captures the same factors as under full information: delivering utility to the agent, compensating him for putting forth effort, and reflecting variations in the agency's costs. However now the constant term $c^*(s)$ also reflects the incentive effects and the corresponding compensation for the additional employment risk the worker must bear. The utility adjustment factor w_S for the change in the aggregate state is as before, reflecting changes in the agency's marginal cost of providing utility. However now the adjustment term w_J captures the incentive effects of increases in utility upon finding a job.

Therefore under the optimal contract, promised utility when unemployed evolves as:

$$\begin{aligned} dW_t &= \rho\left[W_t - u(c_t,a_t) - q_{s_t}a_t\frac{W_t^J - W_t}{\rho} - \lambda_{s_t}\frac{W_t^S - W_t}{\rho}\right]dt \\ &\quad +(W_t^J - W_t)\Delta s_t^J + (W_t^S - W_t)\Delta s_t^S \\ &= W_t\left[\rho + \rho u(c^*(s_t)) - q_{s_t}a^*(s_t)(w_J(s_t)-1) - \lambda_{s_t}(w_S(s_t)-1)\right]dt \\ &\quad +W_t(w_J(s_t)-1)\Delta s_t^J + W_t(w_S(s_t)-1)\Delta s_t^S \\ &= W_t\left[\mu_W(s_t)dt + (w_J(s_t)-1)\Delta s_t^J + (w_S(s_t)-1)\Delta s_t^S\right], \end{aligned}$$

where the last line defines μ_W. Under the contract an unemployed worker's promised utility grows at a constant rate in each aggregate state, and experiences proportional jumps when the state switches or when the worker finds a job. For later use we define a new variable $X = -\log(-W)$, which converts promised utility to physical units. Using a generalized version of Ito's lemma it evolves as[13]:

$$dX_t = -\mu_W(s_t)dt - \log(w_J(s_t))\Delta s_t^J - \log(w_S(s_t))\Delta s_t^S.$$

In our implementation below X_t will be tied to the worker's wealth.

4.4 Illustration

Figure 1 illustrates the dynamics of the contract over an unemployment spell during a recession. In this and the following figures we use some parameters from our calibration below: $q_G = 0.0038$, $q_B = 0.0035$, $\lambda_G = 0.0058$, $\lambda_B = 0.0078$, $\rho = 0.001$ and $\omega = 495$. We also choose these baseline preference parameters: $\theta_A = 0.00015$, $\theta_P = 0.00005$, $h(a) = v\frac{a^{1+\phi}}{1+\phi}$ with $v = 0.01$ and $\phi = 1.7$. The low levels of absolute risk aversion imply plausible levels of relative risk aversion given the relatively high levels of consumption and wages in the model. Figure 1 illustrates the differences between the contract under full information and with moral hazard, plotting the levels of consumption and effort over the unemployment spell, the level of consumption the agent would obtain were he to find a job, and the change in promised utility upon finding a job.

We see that under full information, all of these variables are constant over the spell, as the contract effectively insures the worker. By contrast, under moral hazard consumption declines over time in order to provide search incentives, as was emphasized by Shavell and Weiss [1979] and Hopenhayn and Nicolini [1997]. We also see that the more rapidly the unemployed worker can find a job the higher will be his consumption in the moral hazard case. In addition, we see that (at least for these parameter values) effort is slightly higher under moral hazard than under full information. The reason is that the information friction

[13] See Theorem 1.14 and Example 1.15 in Øksendal and Sulem [2005]

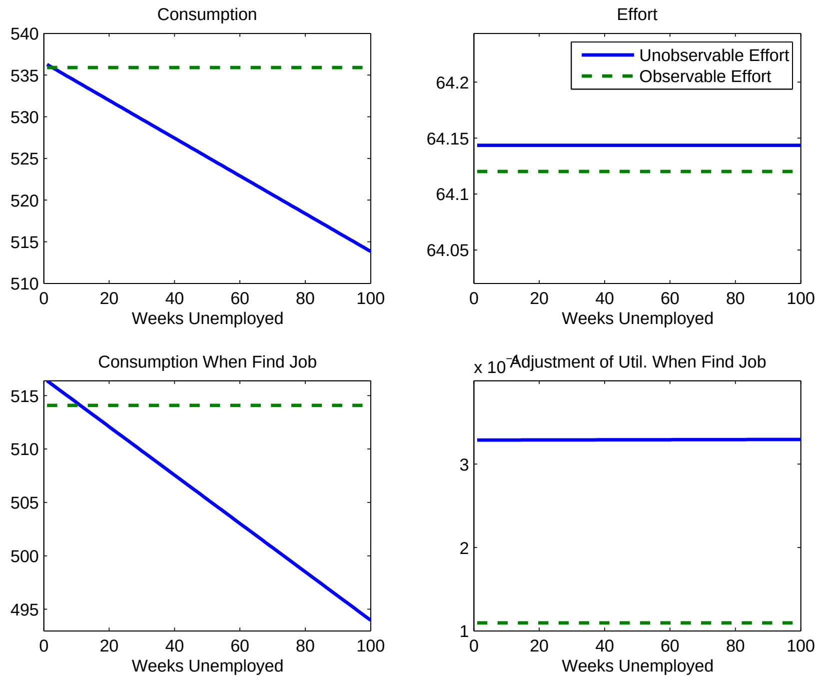

Figure 1: Behavior of selected variables in the optimal contracts with full information and moral hazard during a recession.

reduces the efficiency of the contract during a unemployment spell and increases the cost of staying unemployed. By inducing the agent to search harder, unemployment spells would be shortened and hence the costs of benefits would be reduced.[14] In order to induce this higher level of effort, the agency promises the agent higher consumption upon finding a job, at a level which is larger (at least at the outset) than under full information, and increases his promised utility to a greater extent.

Figure 2 plots the same variables, now focusing on how the optimal contract under moral hazard differs in booms and recessions. Since search is more productive in a boom, the optimal contract has agents search harder in a booms, and so provides greater consumption as compensation for increased effort. Consumption upon finding a job is nearly identical in booms and recessions, but decreases at a slightly more rapid rate in recessions. In addition the agent gets a larger increase in promised utility upon finding a job in a recession than a

[14]This is different from the case of CRRA preferences in Section 6 below, where the effort level is lower with moral hazard. In that case the contract becomes inefficient if the agent's promised utility becomes too high or too lower because of the boundaries of the agent's compensation and the income effect. Inducing high search effort makes the promised utility move to inefficient regions and increases the cost of incentive provision. However, in the current CARA case consumption is unbounded and there is no income effect, so incentive provision is less costly.

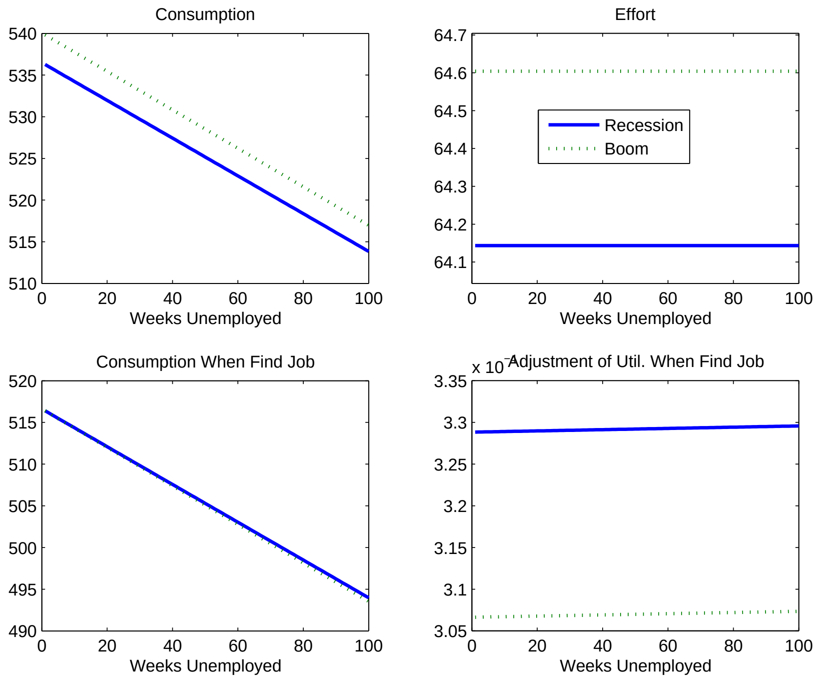

Figure 2: Behavior of selected variables in the optimal contracts with moral hazard in booms and recessions.

boom. Both of these effects capture the worsening of the information friction in a recession. That is, in a recession with the reduction in the effect of search on job finding, it is harder for the agency to tell whether agents remain unemployed due to bad luck or shirking. Thus the contract must provide additional incentives relative to the full information case, making consumption decline at a slightly faster rate but at the same time increasing the promised utility reward for successful search.

With our explicit solutions for the optimal contract, we can also analyze how the contract varies with changes in any of the parameters of the model. Figure 3 summarizes some of these comparative statics, plotting the optimal level of effort $a^*(s)$ and the constant $c^*(s)$ from the consumption function. Recall that the constant (independent of the promised utility) in the consumption function is $c^*(s) + h(a^*(s))$, which includes the compensation for effort as well. Thus $c^*(s)$ accounts for purely the changes in consumption which are not compensating search effort. In each panel of the figure we fix all of the parameters at the baseline levels except for the one shown, and plot the optimal policies in booms and recessions. We see that as the agent's risk aversion parameter θ_A increases, consumption increases and effort falls in both states. As the principal's risk aversion parameter θ_P increases, consumption falls and effort decreases. In both of these cases, it becomes costlier to provide incentives so less effort is induced, and the benefit to the party whose risk aversion has gone up is increased. When

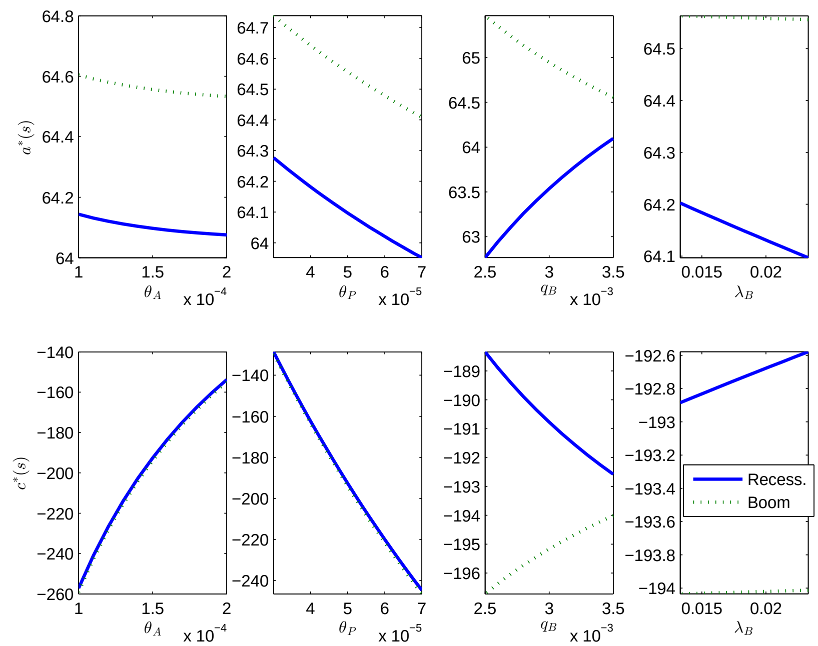

Figure 3: Comparative statics of effort $a^*(s)$ and the consumption constant $c^*(s)$ for variations in the parameters.

the productivity of search in recession q_B increases, the recession becomes less severe and the information friction is also lessened. In this case the effort and consumption levels in booms and recessions get closer together, with effort increasing in a recession and falling in a boom and consumption falling in a recession and increasing in a boom. When recessions are of shorter duration as λ_B increases, there is little effect on consumption or effort in a boom. Effort decreases and consumption increases in the recession, as the shorter durations make recessionary periods less influential in the overall costs of the optimal system.

5 Implementation of the Optimal Contract

We now show how to implement the optimal contract via some simple instruments. Previous studies of optimal unemployment insurance contracts have generally focused on a direct implementation, where the unemployment agency tracks the worker's promised utility and makes payments conditional upon it. Promised utility and consumption decline over an unemployment spell, providing a rationale for declining unemployment insurance benefits. However in this section we show that the optimal contract can be implemented by allowing the worker to save and borrow, providing unemployment benefits which are constant in each aggregate state, giving a re-employment bonus, and incorporating an additional bonus or charge when the aggregate state changes. Thus our implementation mixes elements of the current system of constant benefits with the unemployment insurance savings accounts of Feldstein and Altman [2007], and the re-employment bonuses which were tried in some US states and whose effects were studied by Robins and Spiegelman [2001]. Cyclical fluctuations are new to our study, and necessitate the additional payment or charge when the state of the economy switches to capture changes in incentives and costs of insurance.

As discussed above, our implementation is related to Werning [2002] and Shimer and Werning [2008], who find that constant benefits are optimal with exponential utility in related search models when agents can save and borrow. Although our setting differs, our results are similar and provide the same distinction between benefits and consumption, and thus insurance and liquidity, as in their papers. The re-employment bonus and the payments when the aggregate state switches are new to our paper.

5.1 A Worker's Consumption-Savings-Effort Problem

We consider an implementation of the optimal contract via a consumption-savings-effort problem for a worker. The worker has wealth x_t and has access to a bond with a state-dependent instantaneous rate of return $r(s_t)$ when unemployed and r^e when employed. In addition, the worker gets a flow payment each instant, which is time-independent but depends on the worker's employment status and the aggregate state. Let b^e be the constant flow payment when employed and $b^u(s_t)$ be the payment when unemployed. Finally, when the worker is unemployed he gets a lump sum payment of $B(s_t)$ when he finds a job and $A(s_t, x_t)$ when the aggregate economy switches state. Note that although all of these terms are referred to as "payments" they may be negative, in which case they are charges or taxes the agent must pay. When the agent is employed his wealth evolves as:

$$dx_t = [r^e x_t - c_t + b^e]dt. \tag{11}$$

When the worker is unemployed his wealth follows:

$$dx_t = [r(s_t)x_t - c_t + b^u(s_t)]dt + B(s_t)\Delta s_t^J + A(s_t, x_t)\Delta s_t^S. \tag{12}$$

The goal of our implementation is to choose the parameters $(r^e, b^e, r(s_t), b^u(s_t), B(s_t), A(s_t, x_t))$ so that the agent's optimal choices agree with those given above under the contract.

5.2 An Employed Worker

We now consider the problem of a worker who chooses his consumption and effort optimally, given the wealth evolution above. We begin with an employed worker, later turning to an unemployed worker. We denote the worker's value function $J(x, e)$ and note that it satisfies the HJB equation:

$$\rho J(x,e) = \max_c \rho u(c) + J_x(x,e)[r^e x - c + b^e],$$

and the first order condition is:

$$\rho u'(c) = J_x(x,e).$$

In Appendix C.1, we show that the value function is $J(x,e) = -J_e \exp(-r^e \theta_A x)$, where:

$$J_e = \frac{\rho}{r^e} \exp\left(\frac{r^e - \rho - r^e \theta_A b^e}{r^e}\right).$$

We also show that the consumption function is:

$$c(x,e) = \frac{\rho - r^e}{\theta_A r^e} + b^e + r^e x.$$

To implement the optimal contract, we need to ensure that consumption and wealth are constant (as they are under the optimal contract), and that the levels of consumption and expected utility agree with those in the contract. For the promised utility levels to agree we need:

$$W = J(x,e) = -J_e \exp(-r^e \theta_A x)$$

and therefore note that we have:

$$c(W(x),e) = -\frac{1}{\theta_A} \log(-J(x,e)) = -\frac{1}{\theta_A} \log(J_e) + r^e x.$$

Thus for $c(W(x),e) = c(x,e)$ we need $r^e = \rho$, so the interest rate must equal the rate of time preference. In this case:

$$c(x,e) = b^e + \rho x$$

and wealth x is indeed constant. In addition, we can simplify the agent's value function to:

$$J(x,e) = -\exp(-\theta_A(b^e + \rho x)).$$

There is an indeterminacy in the implementation at this stage, as to deliver a given level of promised utility W we can trade off the constant payment b^e with the initial wealth x. We can interpret b^e as the after-tax wage, so one implementation would set $x = 0$ and choose the labor income tax (or subsidy) in order to deliver the appropriate level of utility. But a different implementation would set $b^e = 0$, completely taxing away labor income, with the agent financing his consumption out of an appropriately chosen stock of wealth x. We resolve this indeterminacy below when we consider the unemployed.

5.3 An Unemployed Worker

We now suppose that once the worker finds a job, he receives a constant payment b^e which is tied to his wealth and the aggregate state at the date he becomes employed. That is, if the worker finds a job at some date T and his wealth upon beginning the job is x_T (after he the receives the bonus $B(s_T)$ as in (12)), then in employment he gets the constant payment $b^e = (r(s_T) - \rho)x_T$. Thus:

$$J(x_T, e) = -\exp(-\theta_A(b^e(x_T) + \rho x_T)) = -\exp(-\theta_A r(s_T)x_T).$$

Note that once employed, the worker's wealth is constant at $x_t = x_T$ for $t > T$, so the above expression determines $J(x, e)$ for all x. We also specify that the payment upon a switch of the aggregate state $A(s, x)$ takes the form:

$$A(s, x) = \left(\frac{r(s)}{r(s')} - 1\right) x + \frac{r(s)}{r(s')}\hat{A}(s)$$

for some constants $\hat{A}(s)$. Thus upon the switch of the aggregate state, the worker is given a payment $\hat{A}$ independent of wealth, as well as an additional term proportional to wealth which accounts for the gain or loss to the worker from the change in interest rates.

We now consider the problem of an unemployed worker, who must choose consumption and effort over his unemployment spell, with wealth evolution given by (12) and post-employment value function determined above. The HJB equation for the unemployed worker in aggregate state s is then:

$$\begin{aligned}\rho J(x, u, s) &= \max_{c,a} \rho u(c, a) + J_x(x, u, s)[r(s)x - c + b^u(s)] \\ &\quad + q_s a[J(x + B(s), e) - J(x, u, s)] + \lambda_s[J(x + A(s, x), u, s') - J(x, u, s)].\end{aligned}$$

The first order condition for c is as above:

$$\rho u_c(c, a) = J_x(x, u, s),$$

while the first order condition for a is:

$$-\rho u_a(c, a) = q_s[J(x + B(s), e) - J(x, u, s)].$$

We now set the parameters of the policy $(b^u(s), \hat{A}(s), B(s), r(s))$ to implement the optimal contract. To do so, we guess that the value function takes the form $J(x, u, s) = -\exp(-r(s)\theta_A x)$. In Appendix C.2, we show that this implies that effort is $a = a(s)$ independent of x. We also show that the optimal consumption function can be written:

$$c(x, u, s) = -\frac{1}{\theta_A}\log\frac{r(s)}{\rho} + h(a(s)) + r(s)x.$$

Thus to implement the optimal contract, we must clearly have the same effort and consumption choices, and we must have the utility levels match as well, so:

$$W = -\exp(-r(s)\theta_A x),$$

and thus from (10) we have:

$$c(W(x), u, s) = c^*(s) + h(a^*(s)) + r(s)x.$$

So to have the consumption policies agree $c(W(x), u, s) = c(x, u, s)$ we must have:

$$c^*(s) = -\frac{1}{\theta_A} \log \frac{r(s)}{\rho}.$$

This determines the required interest rates $r(s)$ for the implementation:

$$r(s) = -\rho u(c^*(s)) \tag{13}$$

Finally, the evolution of x_t must agree with the evolution of promised utility above. Note that we have:

$$x_t = -\frac{\log(-W_t)}{r(s_t)\theta_A} = \frac{X_t}{r(s_t)\theta_A}$$

and therefore, using a generalized Ito's lemma[15]:

$$\begin{aligned}
dx_t &= -\frac{\mu_W(s_t)}{r(s_t)\theta_A}dt - \frac{\log(w_J(s_t))}{r(s_t)\theta_A}\Delta s_t^J - \frac{1}{\theta_A}\left(\frac{\log(w_S(s_t))}{r(s_t)} - \left(\frac{1}{r(s_t')} - \frac{1}{r(s_t)}\right)(X_t - \log(w_S(s_t)))\right)\Delta s_t^S \\
&= -\frac{\mu_W(s_t)}{r(s_t)\theta_A}dt - \frac{\log(w_J(s_t))}{r(s_t)\theta_A}\Delta s_t^J - \left(\frac{\log(w_S(s_t))}{r(s_t')\theta_A} + \frac{\log(-W_t)}{r(s_t')\theta_A} + x_t\right)\Delta s_t^S. \\
&= -\frac{\mu_W(s_t)}{r(s_t)\theta_A}dt - \frac{\log(w_J(s_t))}{r(s_t)\theta_A}\Delta s_t^J - \left(\frac{\log(w_S(s_t))}{r(s_t')\theta_A} + \left(1 - \frac{r(s_t)}{r(s_t')}\right)x_t\right)\Delta s_t^S.
\end{aligned}$$

At the same time, under the optimal policies derived here we have:

$$dx_t = [\frac{1}{\theta_A}\log\frac{r(s_t)}{\rho} - h(a^*(s_t)) + b^u(s_t)]dt + B(s_t)\Delta s_t^J + A(s_t, x_t)\Delta s_t^s.$$

Thus, recalling the form of $A(s, x)$ above, we must have:

$$b^u(s) = -\frac{\mu_W(s)}{r(s)\theta_A} - \frac{1}{\theta_A}\log\frac{r(s)}{\rho} + h(a^*(s)) \tag{14}$$

$$B(s) = -\frac{\log(w_J(s_t))}{r(s_t)\theta_A} \tag{15}$$

$$\hat{A}(s) = -\frac{\log(w_S(s_t))}{r(s_t)\theta_A}. \tag{16}$$

In Appendix C.2 we show that under this policy the HJB equation for the unemployed worker's problem is satisfied, which verifies the guess of the functional form of the value function. Therefore this policy implements the optimal contract.

[15] See Theorem 1.16 in Øksendal and Sulem [2005].

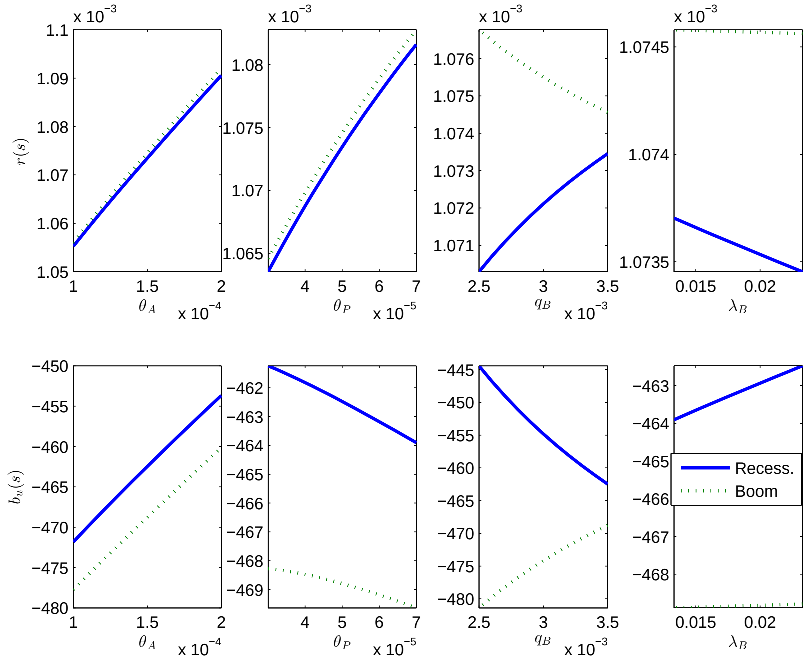

Figure 4: Comparative statics of the effective interest rate $r(s)$ and the benefit $b_u(s)$ for variations in the parameters.

5.4 An Illustration

In figures 4 and 5 we illustrate the implementation, showing the changes in some of the key implementation parameters when there are variations in the model parameters. We use the same baseline parameters and variations as in our illustration of the contract above. In figure 4 we plot the effective interest rate $r(s)$ and the constant benefit payment $b_u(s)$ for unemployed workers. First, note that $r(s) > \rho$ in all of the settings, so the contract provides an interest rate subsidy. In this example the wealth levels consistent with a given level of utility are relatively large, and with high interest rates the effective consumption out of wealth is thus fairly substantial, so the benefit levels b_u are actually negative. Clearly these features depend on the particular parameter specification shown. In figure 5 we plot the lump sum payments $B(s)$ when the worker finds a job and $\hat{A}(s)$ when the aggregate state switches. The re-employment bonuses are quite substantial, and are larger in recessions than booms in order to provide incentives. Interestingly, the payment $\hat{A}(s)$ is positive when the economy switches from a boom to a recession and negative when the economy enters a recession. This payment is at least an order of magnitude smaller than the re-employment bonus, and seems to reflect the fact that consumption falls is a recession.

Qualitatively, the comparative statics are similar to those shown above in the optimal contract. When the risk aversion parameters increase, interest rates increase and re-employment

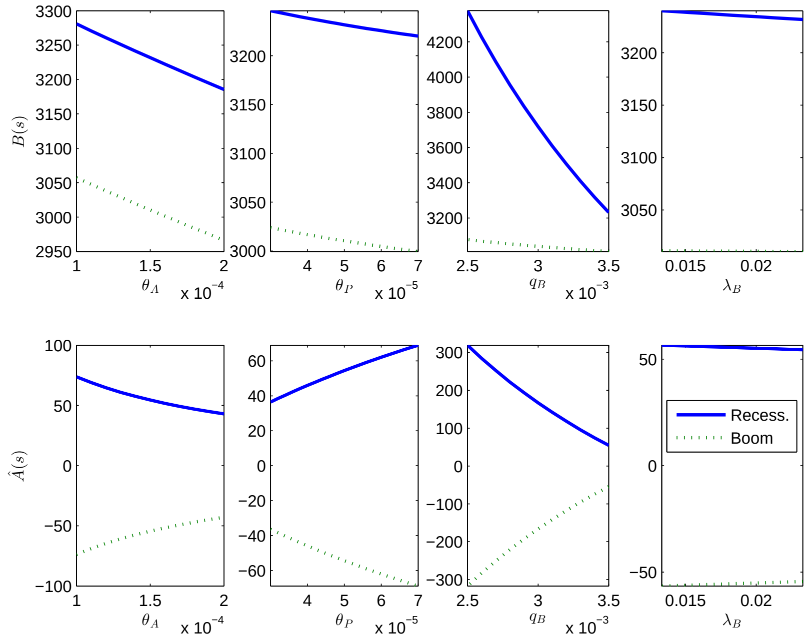

Figure 5: Comparative statics of the payment for finding a job $B(s)$ and the payment when the state switches $\hat{A}(s)$ for variations in the parameters.

bonuses decrease, and benefits increase when it is the agent's risk aversion and decrease when it is the principal's. When the productivity of search in a recession increases, again all of the parameters get closer together across business cycle states, due to the direct effect of the smaller difference across states and the lessening of the information friction.

6 A Quantitative Example

While the previous sections provided useful insight the optimal contract and its implementation, the assumptions we made there were rather special. In this section we study the quantitative implications of the optimal contract under more standard assumptions in a calibrated version of the model. We now assume workers' preferences are additively separable and of the form:

$$u(c,a) = u(c) - h(a) = \frac{(c+\alpha)^{1-\gamma}}{1-\gamma} - \frac{a^{1+\phi}}{1+\phi}$$

with $\gamma, \phi > 0$. Here $\gamma > 0$ is the coefficient of relative risk aversion, $\alpha > 0$ is the minimal consumption of the worker, and ϕ governs the elasticity of job search. We also now assume that the unemployment agency is risk neutral, so $v(c) = c$. In addition to having job loss ($p_s > 0$), we reintroduce the (small) constant terms q_{0s} in the job finding rate in order to ease computation.

6.1 The Benchmark Contract

To calibrate the model and measure the effects of switching to the optimal unemployment insurance system, we consider a stylized version of the current system, which we call the benchmark contract. Under the benchmark contract, an unemployed worker receives the constant benefit c^B for a fixed length of time. To capture the regular extensions of benefits during recessions, we assume that the worker receives benefits for at most T_G weeks in a boom and $T_B > T_G$ weeks in a recession. Once the benefits have expired, the agency provides no further consumption. In Appendix D we show how to calculate the worker's utility and search effort under the benchmark contract.

6.2 Data and Calibration

We take a time period to be one week. First, we fix a few parameters following the literature. Following Hopenhayn and Nicolini [1997], we set the coefficient of risk aversion to be $\gamma = 0.5$ and the weekly discount rate $\rho = 0.001$, which corresponds to annual discount rate of 5%. Since utility is bounded below by zero, we set $\alpha = 0$, which is interpretable as the worker having no outside assets or other sources of income. We set the constants in the job finding rate to very small numbers $q_{s0} = 10^{-5}$, which prevents some singularity problems but has no impact on our results. We set the maximum compensation that the UI agency can give to a worker (in addition to his wage if any), at $\bar{c} = \omega$, the wage. Thus the worker can consume at most ω when unemployed and 2ω when employed. This is simply used to determine the upper bounds on the worker's utility and the agency's costs. The data set for our estimation below ends in 2007, and the median annual wage then was \$25,737 so we set $\omega = \$495$, the corresponding weekly value.[16]

For the benchmark contract, we set the benefit length to be $T_G = 26$ weeks in booms, which is the average duration across US states, and $T_B = 39$ weeks in recessions, corresponding to the length of the regular federal extended unemployment benefits program. We consider an example below where we extend benefits to $T_B = 99$ weeks, the maximum length with the emergency benefits extension during the most recent recession. We set the unemployment benefit to $c^b = 0.47\omega$, consistent with the 47% average replacement ratio in the US in fiscal 2009.[17]

We estimate the Markov process for the aggregate state, and the corresponding job finding and job loss rates, using the data from Shimer [2012], which consists of quarterly averages of monthly job finding and separation rates from 1948-2006. As Shimer emphasized, most of the cyclicality in the data comes from the job finding rates. To focus on this cyclical

[16]This data comes from the SSA at http://www.ssa.gov/oact/cola/central.html.

[17]See http://workforcesecurity.doleta.gov/unemploy/ui_replacement_rates.asp

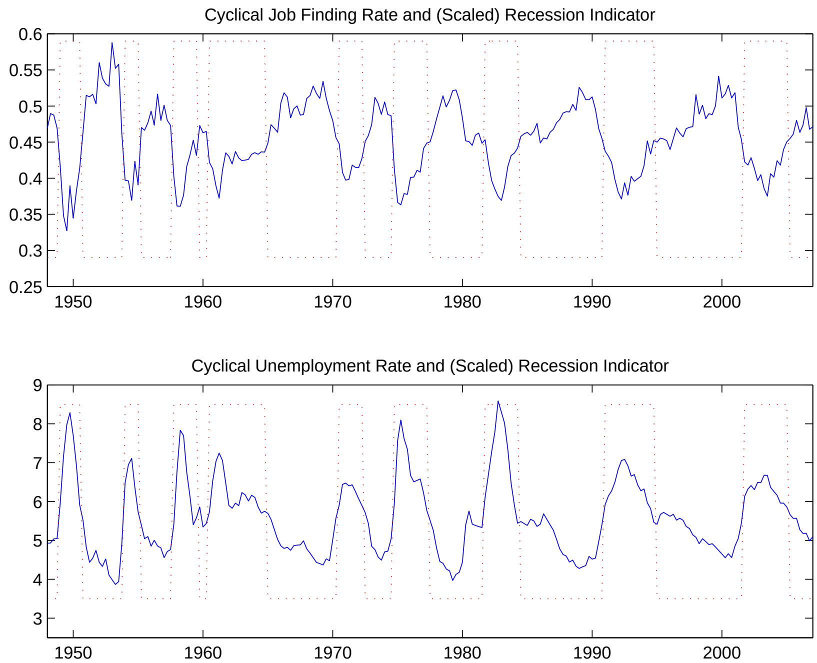

Figure 6: Cyclical job finding and unemployment rates, along with the estimated recession indicator.

component, we use a Hodrick-Prescott filter to remove a low frequency trend from the job finding rate data. Then we estimate a two-state Markov chain on this data, following the approach of Hamilton [1989]. That is, letting the H-P filtered job finding rate be f_t we estimate:

$$f_t = m_{s_t} + \epsilon_t,$$

where s_t is the aggregate state and ϵ_t is an error. We obtain the estimates $m_G = 0.487$ and $m_B = 0.411$ for the mean job finding rates in booms and recessions, and diagonals of the transition matrix are 0.933 and 0.911. Thus, for example, the probability of remaining in a boom in the next quarter conditional on being in a boom in the current quarter is 0.933. These transition probabilities imply switching rates of the aggregate state of $\lambda_G = 0.0173$ and $\lambda_B = 0.0233$ on a weekly basis.

The cyclical job finding rates and the estimated recession indicator, which is one when the smoothed (full-sample) probability of a recession is greater than 0.5, are shown in the top panel of Figure 6. Using this recession indicator, we then read off the mean separation rates in booms and recessions from the H-P filtered separation rate data. These are 0.0337 and 0.0353 in booms and recessions, respectively, which imply weekly job loss rates of $p_G = 0.0085$ and $p_B = 0.0089$. Although we do not use it in the calibration, we also extracted the cyclical unemployment rate, which is shown in the bottom panel of Figure 6. There we see that our

	Benchmark		Optimal	
	Boom	Recess	Boom	Recess
Unemployment Rate (%)	5.33	6.57	3.60	4.00
Unemployment Duration (weeks)	6.21	7.33	4.44	4.67
Finding Rate (month)	0.49	0.41	0.64	0.61
Separation Rate (month)	0.033	0.035	0.033	0.035
Net Cost/Worker (% of ω)	2.50	3.09	1.95	2.21

Table 1: Summary statistics from our simulation of the benchmark contract and the optimal contract

recession indicator corresponds quite well to periods of high unemployment, as the average unemployment rate is 4.93 in booms and 6.56 in recessions.

We calibrate the rest of the parameters by simulating a population of 2 million workers and computing the average job finding rates as well as directly calculating the elasticities of unemployment duration with respect to an increase in benefits. For this simulation, we assume a mortality rate of 0.1% per year, which is significantly less than the 0.8% average in US data, reflecting that we focus on working-age adults. In practice, this means that we terminate and re-start the contracts for 0.1% of our population at an annual rate. For the unemployment elasticity, as discussed by Landais et al. [2012] and Chetty [2008], the typical range of estimates is 0.5-1, and we target an elasticity in the middle of this range at 0.7. This is midway between the value of 0.9 of Meyer [1990] which has been commonly used in the literature and the more recent estimates around 0.55 by Kroft and Notowidigdo [2011] and Chetty [2008]. We find this elasticity in our simulations by computing the average unemployment duration under our benchmark contract as well as a contract with benefits increased by 1%. The job average job finding rates are largely driven by the parameters governing the impact of job search on finding rates, and we find that $q_{G1} = 0.0038$ and $q_{B1} = 0.0035$ match the data quite well. The elasticity is largely driven by the effort cost function parameter, and we find that $\phi = 0.16$ gives us an elasticity of 0.72.

6.3 Quantitative Implications

In Table 1 we summarize the results of our simulations. We report the average over booms and recessions of unemployment rate, unemployment duration, job finding and separation rates, and the net cost per worker of the unemployment system. In the simulations, we initialize the optimal contract at the utility level provided by the benchmark contract. That is, we keep workers indifferent between remaining in the current system and switching to the optimal system. Overall, the unemployment durations we find are short relative to the data, where under the current system average durations are roughly 10 weeks in booms and 15 in recessions. In our model there is very little heterogeneity, so average durations are essentially the inverse of the finding rates. We choose to match the relatively high average finding rate which is observed in the data, so as a consequence we find durations that are too short relative to the data. Incorporating heterogeneity in jobs and workers could help to generate more realistic unemployment durations.

The table shows that switching from the current system to the optimal one would yield substantial reductions in unemployment, mainly driven by large increases in the job finding rate. For example, in a recession, the optimal system reduces unemployment rates by roughly 2.5 percentage points and shortens the duration of unemployment by roughly 50% relative to the benchmark system. Moreover, the cyclical fluctuations in unemployment rates and durations are substantially dampened under our optimal system. Under the benchmark system, the unemployment rate increases by 1.2 percentage points and the average duration increases by more than one week. However under the optimal system, the unemployment rate only increases by 0.4 percentage points and the duration only increases by one quarter of a week on average. In addition to providing better economic performance, the optimal system costs less. On average the net cost of the current system, measured as the average net payments to the unemployed workers, is 2.5-3% of wages, which is roughly equal to paying the replacement benefit of 47% of wages to 5.3-6% of the population. By contrast, the optimal system costs only around 2% of wages, even though (as we show below) the effective replacement rate is significantly higher under the optimal system. In addition to paying out benefits to a smaller base of unemployed workers, the net cost of the optimal system is reduced because it taxes the income of employed workers.

In figures 7-9 we illustrate how the optimal contract differs from the benchmark contract, and how it varies over the business cycle. The characteristics of the optimal contract here are similar to the solvable case. In Figure 7 we plot an unemployed worker's consumption over an unemployment spell under the optimal contract, the benchmark contract, and the optimal contract with observable effort. With observable effort consumption is constant over an unemployment spell, as there is no need to provide incentives, and the replacement ratio is very close to one. In order to provide incentives, the optimal contract has declining consumption over the unemployment spell, as we discussed above. Quantitatively, the rate of decline is very low and the replacement ratio is very high in the calibrated model. By contrast, the replacement rate is fixed at 47% in the benchmark contract, and falls to zero after 39 weeks in a recession. Thus the optimal contract provides substantially more consumption smoothing than the current system.

In Figure 8 we plot the job finding rate in a recession over an unemployment spell under

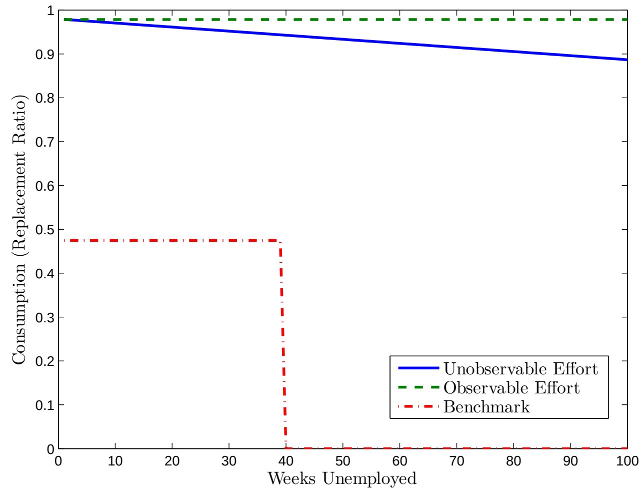

Figure 7: Consumption over an unemployment spell in a recession under the optimal contract, the benchmark contract, and the optimal contract with observable effort.

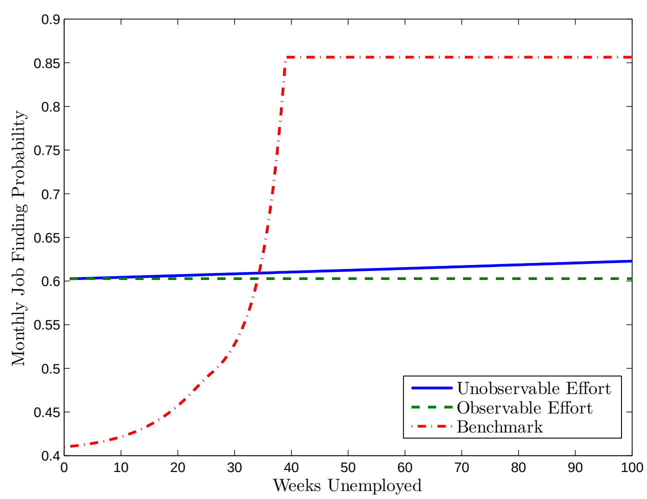

Figure 8: The job finding rate over an unemployment spell in a recession under the optimal contract, the benchmark contract, and the optimal contract with observable effort.

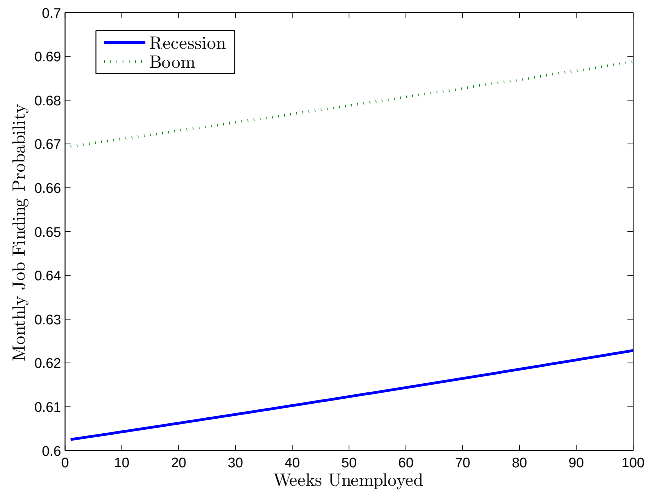

Figure 9: The job finding rate over an unemployment spell in a boom and a recession under the optimal contract.

the optimal contract, the benchmark contract, and the optimal contract with observable effort. We report the finding rate instead of the effort level directly, as it is easier to interpret. With observable effort, the finding rate is constant at 0.6 over the spell, and with unobservable effort it increases over time. The increasing rate reflects the impact of incentives in the optimal contract, but again moral hazard has a relatively small quantitative effect relative to full information. Under the benchmark contract, the finding rate starts very low, around 0.4, and remains low until the near the expiration of benefits when increases rapidly. However since typical unemployment spells are short, 7.3 weeks on average, relatively few workers stay unemployed long enough to experience the large increase in finding rates.

Figure 9 shows the job finding rate over an unemployment spell in a boom and a recession under the optimal contract. Here we see that the job finding rate in a boom is essentially a parallel shift up of the job finding rate in a recession. This reflects two factors: first, for any given effort level by assumption the finding rate is higher in a boom, and second, the effort level in a boom is higher by a roughly constant amount at each point in time. Search effort is more productive in a boom, and thus it is efficient for the agent to search harder in a boom. In both aggregate states, the incentive effects lead to an increase in effort over time at a similar rate.

Finally, we consider a simulation which helps to gauge the potential impact the reform of unemployment insurance could have had during the recession of 2007-2009. For this exercise, we start the economy in a boom, then suppose that the economy enters a recession seventy weeks long. We then compare the performance of the benchmark current system with 39

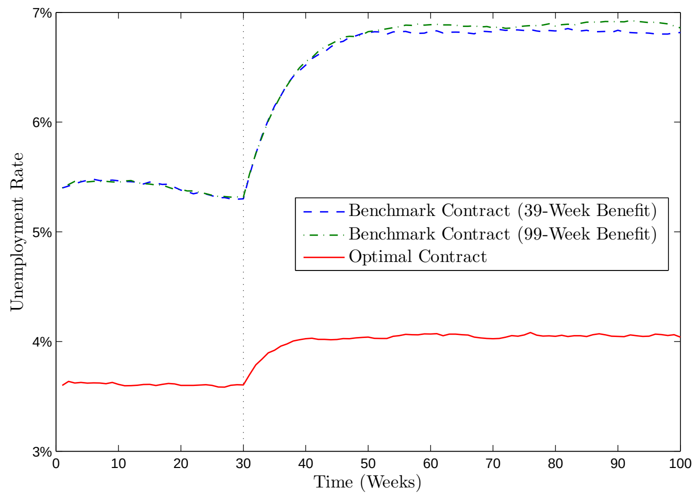

Figure 10: Unemployment rates in a long recession under the optimal contract, the benchmark contract, and a contract with benchmark contract with benefits extended to 99 weeks.

weeks of benefits, a version of the current system with benefits extended to 99 weeks, and our optimal system. Our model is calibrated to match typical business cycles from 1948-2006, with an average increase of about 1.5 percentage points in a recession, and so cannot account for the 5 percentage point increase in unemployment which occurred in 2008-2009. Nonetheless, our results, shown in Figure 10 (where the recession begins at week 30), are indicative of the effects that reform may have. Under the benchmark system, unemployment rates increase by 1.4 percentage points, from 5.3% to 6.7% as the economy goes from the boom to the long recession. The benchmark system with extended benefits has only a very small impact on unemployment, as the unemployment rate in the recession increases by 0.1 percentage point to 6.8%. This is similar to the empirical evidence of Rothstein [2011], who estimated that the benefits extensions increased unemployment rates by 0.2-0.6 percentage points. As the overall increase in the unemployment rate in 2008 was roughly three times that in the model, the proportion of the increase attributable to benefits extensions in the model is similar to Rothstein [2011]. However the relative insensitivity of unemployment rates to benefit durations clearly does not imply that that unemployment insurance overall has a minor impact on the unemployment rate. We find that under our optimal system the unemployment rate increases by only 0.4 percentage points in the recession, from 3.6% to 4.0%, and remains 2.7 percentage points lower than the current system in the long recession. Thus our findings, although clearly only illustrative, suggest that a reformed unemployment insurance system would have led to a lower and more stable unemployment rate.

7 Conclusion

In this paper we have characterized the optimal unemployment insurance system in an economy with cyclical fluctuations. An optimal unemployment insurance contract trades off the provision of insurance through consumption smoothing with providing incentives to search, and this tradeoff is affected by aggregate conditions. After laying out a relatively general model, we showed that with exponential utility and exponential costs, the contract could be solved explicitly. This allowed us to characterize how the contract varies over the business cycle, as well as to study the dependence of the contract on the various parameters of the model, such as the depth and duration of recessions. We also showed that the optimal contract could be implemented in a rather simple way: by allowing unemployed workers to borrow and save in a bond (whose return depends on the state of the economy), providing flow payments which are constant over an unemployment spell but vary with the aggregate state, and giving additional lump sum payments (or charges) upon finding a job or when the aggregate state switches. Finally, we showed in a calibrated version of the model that reform of the unemployment system could yield substantial gains in economic performance. Even though the unemployment rate may be relatively insensitive to the length of benefits in the current system, an unemployment insurance system which provides more smoothing of consumption and at same time provides better search incentives can lead to substantial reductions in unemployment.

The simplicity of our model allowed us to isolate the effects of insurance and incentives on job search as a determinant of unemployment. However it also meant that we were unable to address several important issues bearing on the impact of unemployment insurance on job search and employment. For example, following Hopenhayn and Nicolini [1997] we assumed that all jobs were identical and paid the same constant wage. While this yielded tractability, it meant that there was no scope for mismatch between unemployed workers and job vacancies in our model, or the related issue of workers potentially accepting jobs that require less education or training than they possess. In addition, our model assumed that all unemployed workers remained in the labor force and continued to search, even when their benefits expired. Both the issue of mismatch and changes in labor force participation have been important in the recent recession, and our model could be extended to allow us to address these issues.

A Proofs

A.1 Proof of Proposition 1

Let

$$\begin{aligned} \varphi_t(c,a) &= E_t^a[\rho \int_0^\infty e^{\rho \hat{t}} u(c_{\hat{t}}, a_{\hat{t}}) d\hat{t} | \mathcal{F}_t] \\ &= \rho \int_0^t e^{-\rho \hat{t}} u(c_{\hat{t}}, a_{\hat{t}}) d\hat{t} + e^{-\rho t} W_t. \end{aligned} \quad (17)$$

In words, $\varphi_t(c,a)$ is the conditional expected total utility of the worker based on the information unfolded up to time t. Therefore, process $\{\varphi_t(c,a)\}_{t\in[0,\infty)}$ is a $\mathcal{F}_t$-adapted martingale. According to the *martingale representation theorem*,[18] there exist two $\mathcal{F}_t$-predictable and square integrable (equation (2)) processes $\{g_t^J\}_{t\in[0,\infty)}$ and $\{g_t^S\}_{t\in[0,\infty)}$ such that

$$d\varphi_t(c,a) = \rho e^{-\rho t} g_t^J dm_t^J + \rho e^{-\rho t} g_t^S dm_t^S. \quad (18)$$

So (18) and (17) imply (3).

A.2 Proof of Proposition 2

Suppose not and that the process a does not satisfy (4) for some $\tilde{t} > 0$ with strictly positive probability. Let $\{W_t\}_{t\in[0,\infty)}$ be the promised utility process generated by a under the contract (c,a). We define

$$\hat{\varphi}_t(a') = \rho \int_0^t e^{-\rho s} u(c_s, a'_s) ds + e^{-\rho t} W_t \text{ for } t \in [0,\infty)$$

for some alternative feasible effort process a'. Obviously, $\hat{\varphi}_0(a') = W_0$. According to (3), we have

$$\begin{aligned} d\hat{\varphi}_t(a') &= \rho e^{-\rho t} u(c_t, a'_t) dt - \rho e^{-\rho t} W_t dt + e^{-\rho t} dW_t \\ &= \rho e^{-\rho t} \left\{ [u(c_t, a'_t) - u(c_t, a_t)] \, dt + g_t^J dm_t^J + g_t^S dm_t^S \right\}. \end{aligned}$$

Without loss of generality, we assume that the worker is unemployed at t. Let $\left\{m_t'^J\right\}_{t\in[0,\infty)}$ be the compensated jump martingale associated with j under the effort process a'. Then

$$dm_t^J = \left\{ [(1-s_t)q_G(a'_t) + s_t q_B(a'_t)] - [(1-s_t)q_G(a_t) + s_t q_B(a_t)] \, dt + dm_t'^J \right\}.$$

Therefore

$$\begin{aligned} d\hat{\varphi}_t(a') = \; & \rho e^{-\rho t} \{ [u(c_t, a'_t) - u(c_t, a_t) \\ & + g_t^J \left((1-s_t) q_G(a'_t) + s_t q_B(a'_t) \right) - g_t^J \left((1-s_t) q_G(a_t) + s_t q_B(a_t) \right)] \, dt \\ & + g_t^J dm_t'^J + g_t^S dm_t^S \} . \end{aligned}$$

Note that a', $\{m_t'^J\}_{t\in[0,\infty)}$ and $\{m_t^S\}_{t\in[0,\infty)}$ are two martingales. The drift of $\{\hat{\varphi}(a')\}$ has the same sign as

$$[g_t^J((1-s_t)q_G(a'_t) + s_t q_B(a'_t)) + u(c_t, a'_t)] - [g_t^J((1-s_t)q_G(a_t) + s_t^S q_B(a_t)) + u(c_t, a_t)].$$

[18] Note that, $u(c_t, a_t)$ is bounded in compact intervals, $\{\varphi_t(c,a)\}_{t\in[0,\infty)}$ is a uniformly integrable martingale, so the martingale representation theorem is valid here. See Elliott [1982] for technical details.

So we choose a' such that (4) is satisfied. Then $\{\hat{\varphi}_t(a')\}_{t\in[0,\infty)}$ is a sub-martingale under the measure generated by a'. Therefore

$$E^{a'}[\hat{\varphi}_{\tilde{t}}(a')] > \hat{\varphi}_0(a') = W_0$$

which implies that a is dominated by the effort plan that adopting a' from 0 to $\tilde{t}$ and then switching to a. So a is not optimal.

To prove the other direction, suppose that a satisfies the incentive compatibility conditions in Proposition 2. Then, by definition, $\{\hat{\varphi}_t(a')\}_{t\in[0,\infty)}$ is a super-martingale under a' for any alternative effort process a'. Since c_t, a_t, and a'_t are bounded in compact intervals according to the feasibility conditions, $\hat{\varphi}_\infty(a')$ is bounded. Then

$$W_0 = \hat{\varphi}_0(a') \geq E^{a'}[\hat{\varphi}_\infty(a')]$$

and a dominates a'.

A.3 Proposition 4

Proposition 4 *The lower bounds of the expected utility of the worker are $W_l^{js} = \underline{u}$ for $j = e, u$ and $s = G, B$ and the corresponding values of the insurance agency, $V(W_l^{js}, j, s)$, satisfies:*

$$\rho V(W_l^{us}, u, s) = \lambda_s \left(V(W_l^{us'}, u, s') - V(W_l^{us}, u, s) \right) + q_{s0} \left(V(W_l^{es}, e, s) - V(W_l^{u}, u, s) \right) \tag{19}$$

$$\begin{aligned} \rho V(W_l^{es}, e, s) &= \rho v(\omega) + \lambda_s \left(V(W_l^{es'}, e, s') - V(W_l^{es}, e, s) \right) \\ &\quad + p_s \left(V(W_l^{us}, u, s) - V(W_l^{es}, e, s) \right) \end{aligned} \tag{20}$$

for $s = G, B$ and $s' \neq s$. The upper bounds of the expected utility of the worker W_r^{js} satisfy the following:

$$\rho W_r^{us} = \max_{\hat{a}\in[0,\hat{a}]} \rho u(\bar{c}, a) + \lambda_s (W_r^{us'} - W_r^{us}) + q_s(\hat{a}) \left(W_r^{es} - W_r^{us} \right) \tag{21}$$

$$\rho W_r^{es} = \rho u(\bar{c} + \omega) + \lambda_s \left(W_r^{es'} - W_r^{es} \right) + p_G \left(W_r^{us} - W_r^{es} \right) \tag{22}$$

and the corresponding expected values of the insurance agency are $V(W_r^{js}, j, s) = -v(\bar{c})$.

For the proof, we start with the left boundaries and assume that the worker is unemployed. Let $\tau = \tau^S \wedge \tau^J$ with τ^S being the time of the next economic state change and τ^J the time the next job offer. Then, for any $\Delta > 0$, note that the current instantaneous payoff is 0 and we have

$$\begin{aligned} V(W_l^{us}, u, s) &= e^{-\rho(\Delta\wedge\tau)} \Pr(\tau > \Delta) V(W_l^{us}, u, s) + e^{-\rho(\Delta\wedge\tau)} \Big[\Pr\left(\Delta < \tau, \tau = \tau^S\right) V(W_l^{us'}, u, s') \\ &\quad + \Pr\left(\Delta < \tau, \tau = \tau^J\right) V(W_l^{es}, e, s) \Big] \end{aligned}$$

or

$$\begin{aligned} V(W_t^{us}, u, s) &= e^{-\rho(\Delta\wedge\tau)} e^{-(\lambda_s + q_{s0})\Delta} V(W_l^{us}, u, s) \\ &\quad + e^{-\rho(\Delta\wedge\tau)} \left(1 - e^{-(\lambda_s + q_{s0})\Delta}\right) \left[\frac{\lambda_s}{\lambda_s + q_{s0}} V(W_l^{us'}, u, s') + \frac{q_{s0}}{\lambda_s + q_{s0}} V(W_l^{es}, e, s) \right]. \end{aligned}$$

So

$$\begin{aligned} 0 &= e^{-\rho(\Delta\wedge\tau)} e^{-(\lambda_s + q_{s0})\Delta} \left(V(W_l^{us'}, u, s') - V(W_l^{us}, u, s) \right) \\ &\quad + e^{-\rho(\Delta\wedge\tau)} \left(1 - e^{-(\lambda_s + q_{s0})\Delta}\right) \left[\frac{\lambda_s}{\lambda_s + q_{s0}} V(W_l^{us'}, u, s') + \frac{q_{s0}}{\lambda_s + q_{s0}} V(W_l^{es}, e, s) \right]. \end{aligned}$$

Let $\Delta \to 0$, then we have (19). (20) can be proved by a similar procedure.

Now, we prove the result on the right boundary point, focusing first on an unemployed worker who solves the following problem:

$$\begin{aligned}\max_{a\in A} \quad & E_0[\rho \int_0^{\Delta\wedge\tau} e^{-\rho t} u(\bar{c}, a) dt] \\ & + e^{-\rho(\Delta\wedge\tau)} \left[\Pr(\tau > \Delta) W_r^{us} + \Pr\left(\tau > \Delta, \tau = \tau^S | a_t\right) W_r^{us'} + \Pr\left(\tau > \Delta, \tau = \tau^J | a_t\right) W_r^{es} \right]\end{aligned}$$

or equivalently:

$$\begin{aligned}\max_{a\in A} \quad & E_0\left[\rho \int_0^{\Delta\wedge\tau} e^{-\rho t} u(\bar{c}, a) dt\right] \\ & + e^{-\rho(\Delta\wedge\tau)} \left[e^{-(\lambda_s + q_s(a_t))\Delta} W_r^{us} + \left(1 - e^{-(\lambda_s + q_s(a_t))\Delta}\right) \left(\frac{\lambda_s}{\lambda_s + q_s(a_t)} W_r^{us'} + \frac{q_s(a_t)}{\lambda_s + q_s(a_t)} W_r^{es} \right) \right]\end{aligned}$$

The objective value is W_r^{us}. So for any $\hat{a} \in [0, \bar{a}]$ and $\Delta > 0$ we have:

$$\begin{aligned}W_r^{us} \geq \; & E_0\left[\rho \int_0^{\Delta\wedge\tau} e^{-\rho t} u(\bar{c}, \hat{a}) dt\right] \\ & + e^{-\rho(\Delta\wedge\tau)} \left[e^{-(\lambda_s + q_s(\hat{a}))\Delta} W_r^{Us} + \left(1 - e^{-(\lambda_s + q_s(\hat{a}))\Delta}\right) \left(\frac{\lambda_s}{\lambda_s + q_s(\hat{a})} W_r^{us'} + \frac{q_s(\hat{a})}{\lambda_s + q_s(\hat{a})} W_r^{es} \right) \right]\end{aligned}$$

and

$$\begin{aligned}0 \geq \; & \frac{1}{\Delta} E_0\left[\rho \int_0^{\Delta\wedge\tau} e^{-\rho t} u(\bar{c}, \hat{a}) dt\right] + \frac{1}{\Delta}\left(e^{-\rho(\Delta\wedge\tau)} e^{-(\lambda_s + q_s(a_t))\Delta} W_r^{Us} - W_r^{Us} \right) \\ & + \frac{1}{\Delta} e^{-\rho(\Delta\wedge\tau)} \left(1 - e^{-(\lambda_s + q_s(\hat{a}))\Delta}\right) \left(\frac{\lambda_s}{\lambda_s + q_s(\hat{a})} W_r^{us'} + \frac{q_s(\hat{a})}{\lambda_s + q_s(\hat{a})} W_r^{es} \right).\end{aligned}$$

Let $\Delta \to 0$, then we have

$$0 \geq \rho u(\bar{c}, \hat{a}) - W_r^{us} + \lambda_s \left(W_r^{us'} - W_r^{us} \right) + q_s(\hat{a}) \left(W_r^{es} - W_r^{us} \right)$$

and the equality holds only if $\hat{a}$ is optimal. So we have (21).

If the worker is employed, for any $\Delta > 0$, we have

$$\begin{aligned}W_r^{eEs} = \; & E_0[\rho \int_0^{\Delta\wedge\tau} e^{-\rho t} u(\bar{c} + \omega, \hat{a}) dt] \\ & + e^{-\rho(\Delta\wedge\tau)} \left[e^{-(\lambda_s + p_s)\Delta} W_r^{es} + e^{-\rho(\Delta\wedge\tau)} \left(1 - e^{-(\lambda_s + p_s)\Delta}\right) \left(\frac{\lambda_s}{\lambda_s + p_s} W_r^{es'} + \frac{p_s}{\lambda_s + p_s} W_r^{us} \right) \right]\end{aligned}$$

or

$$\begin{aligned}0 = \; & \frac{1}{\Delta} E_0\left[\rho \int_0^{\Delta\wedge\tau} e^{-\rho t} u(\bar{c} + \omega, \hat{a}) dt\right] + \frac{1}{\Delta}\left(e^{-\rho(\Delta\wedge\tau)} e^{-(\lambda_s + p_s)\Delta} W_r^{es} - W_r^{es} \right) \\ & + \frac{1}{\Delta} e^{-\rho(\Delta\wedge\tau)} (1 - e^{-(\lambda_s + p_s)\Delta}) (\frac{\lambda_s}{\lambda_s + p_s} W_r^{es} + \frac{p_s}{\lambda_s + p_s} W_r^{us})\end{aligned}$$

and let $\Delta \to 0$ we have (22).

A.4 Proof of Proposition 3

We suppose that the economy is in boom and the worker is unemployed (other cases can be proved by a similar procedure). At time t, we assume that the promised utility of the worker is $W_t = W \in (W_l^{uG}, W_r^{uG})$. Let c be a consumption process. Let W_t^J and W_t^S be the adjusted continuation utility levels at the arrival of a job offer and the economic state change respectively. $\{W_t^J\}_{t\in[0,\infty)}$ induces effort process a. Let $\tau = \tau^S \wedge \tau^J$ with τ^S being the next recession and τ^J be that of the next job offer. At time t we have

$$\begin{aligned} V(W_t, u, G) \quad \geq \quad & -E_t \left[\rho \int_t^{(t+\Delta)\wedge\tau} e^{-\rho\hat{t}} v(c_{\hat{t}}) d\hat{t} | \mathcal{F}_{t-} \right] \\ & + e^{-\rho((t+\Delta)\wedge\tau)} \Pr(\tau > t+\Delta | a) V(W_{t+\Delta}, u, G) \\ & + e^{-\rho((t+\Delta)\wedge\tau)} \Pr\left(\tau > t+\Delta, \tau = \tau^J | a\right) V(W_{t+\Delta}^J, e, G) \\ & + e^{-\rho((t+\Delta)\wedge\tau)} \Pr\left(\tau > t+\Delta, \tau = \tau^S | a\right) V(W_{t+\Delta}^S, u, B). \end{aligned}$$

In particular, we let Δ be a small positive number and $W_{\tilde{t}}^J = W_{\tilde{t}} + \xi$ such that $W_{\tilde{t}} + \xi \in [W_l^{Ge}, W_r^{Ge}]$ for some ξ and $\tilde{t} \in [t, t+\Delta]$. Then the effort is constant in $[t, t+\Delta]$ and equal to $a^G(W_t, W_t^J)$and

$$\begin{aligned} 0 \quad \geq \quad & -\frac{1}{\Delta} E_t \left[\rho \int_t^{(t+\Delta)\wedge\tau} e^{-\rho\hat{t}} v(c_{\hat{t}}) d\hat{t} | \mathcal{F}_{t-} \right] \\ & + \frac{1}{\Delta} \left(e^{-\rho((t+\Delta)\wedge\tau)} e^{-(\lambda_G + q_G(a^G(W_t, W_t^J)))\Delta} V(W_{t+\Delta}, u, G) - V(W_t, u, G) \right) \\ & + \frac{1}{\Delta} e^{-\rho((t+\Delta)\wedge\tau)} \left(1 - e^{-(\lambda_G + q_G(a^G(W_t, W_t^J)))\Delta} \right) \frac{q_G(a^G(W_t, W_t^J))}{\lambda_G + q_G(a^G(W_t, W_t^J))} V(W_{t+\Delta}^J, e, G) \\ & + \frac{1}{\Delta} e^{-\rho((t+\Delta)\wedge\tau)} \left(1 - e^{-(\lambda_G + q_G(a^G(W_t, W_t^J)))\Delta} \right) \frac{\lambda_G}{\lambda_G + q_G(a^G(W_t, W_t^J))} V(W_{t+\Delta}^S, u, B). \end{aligned}$$

Let $\Delta \to 0$ and we have

$$\begin{aligned} 0 \quad \geq \quad & -\rho\left(v(c_t) + V(W_t, u, G)\right) + V_W(W_t, u, G)\rho\left(W_t - u(c_t, a_t)\right) \\ & - q_G a^G(W_t, W_t^J) \left(\frac{W_t^J - W_t}{\rho} - \lambda_G \frac{W_t^S - W_t}{\rho} \right) \\ & + q_G a^G(W_t, W_t^J) \left(V(W_t^J, e, G) - V(W_t, u, G) \right) + \lambda_G \left(V(W_t^S, u, B) - V(W_t, u, G) \right). \end{aligned}$$

The equality holds if c_t, W_t^J and W_t^S are optimal, so we have (8).

B Calculations for the Solvable Special Case

B.1 Employed Workers

It is easy to see that the value function is independent of s and we guess that it is of the form:

$$V(W, e, s) = V(W, e) = -V_e(-W)^{-\frac{\theta_P}{\theta_A}}.$$

One can verify this directly by using the fact that c is constant under employment, but here we use the guess and verify approach because many similar calculations will be repeated below. Note that this implies:

$$V_W(W, e) = -V_e \frac{\theta_P}{\theta_A} (-W)^{-\frac{\theta_P}{\theta_A} - 1}.$$

Substituting this expression into the optimality condition for c implies:

$$\begin{aligned}
-\theta_P \exp(\theta_P(c-\omega)) &= -V_e \frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P}{\theta_A}-1}\theta_A \exp(-\theta_A c) \\
\Rightarrow \exp((\theta_P+\theta_A)c) &= V_e \frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}}\theta_A \exp(\theta_P \omega) \\
\Rightarrow c &= \frac{1}{\theta_P+\theta_A}\log(V_e) - \frac{1}{\theta_A}\log(-W) + \frac{\theta_P}{\theta_P+\theta_A}\omega
\end{aligned}$$

Therefore we have:

$$v(c-\omega) = V_e^{\frac{\theta_P}{\theta_P+\theta_A}} \exp(-\frac{\theta_A\theta_P}{\theta_A+\theta_P}\omega)(-W)^{-\frac{\theta_P}{\theta_A}},$$

and:

$$u(c) = -V_e^{-\frac{\theta_A}{\theta_P+\theta_A}} \exp(-\frac{\theta_A\theta_P}{\theta_A+\theta_P}\omega)(-W).$$

Thus we can write the HJB equation as:

$$\begin{aligned}
-V_e(-W)^{-\frac{\theta_P}{\theta_A}} &= -V_e^{\frac{\theta_P}{\theta_P+\theta_A}} \exp(-\frac{\theta_A\theta_P}{\theta_A+\theta_P}\omega)(-W)^{-\frac{\theta_P}{\theta_A}} \\
&\quad -V_e\frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}}[-1+V_e^{-\frac{\theta_A}{\theta_P+\theta_A}}\exp(-\frac{\theta_A\theta_P}{\theta_A+\theta_P}\omega)](-W)
\end{aligned}$$

The terms in W cancel and we see that V_e solves:

$$\begin{aligned}
-V_e &= -V_e^{\frac{\theta_P}{\theta_P+\theta_A}} \exp(-\frac{\theta_A\theta_P}{\theta_A+\theta_P}\omega) - V_e\frac{\theta_P}{\theta_A}[-1+V_e^{-\frac{\theta_A}{\theta_P+\theta_A}}\exp(-\frac{\theta_A\theta_P}{\theta_A+\theta_P}\omega)] \\
\Rightarrow -(1+\frac{\theta_P}{\theta_A}) &= -V_e^{-\frac{\theta_A}{\theta_P+\theta_A}}\exp(-\frac{\theta_A\theta_P}{\theta_A+\theta_P}\omega)(1+\frac{\theta_P}{\theta_A}) \\
\Rightarrow V_e &= \exp(-\theta_P\omega)
\end{aligned}$$

Substituting this into the expression for c gives the other result.

B.2 Full Information

Using the conjectured form of the value function in the first slope matching condition gives:

$$-V_u(s)\frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}} = -\exp(-\theta_P\omega)(-W^J)^{-\frac{\theta_P+\theta_A}{\theta_A}},$$

so therefore we have:

$$W^J = \left(\frac{V_u(s)}{\exp(-\theta_P\omega)}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} W.$$

In turn this implies:

$$V(W^J,e) - V(W,u,s) = [-V_u(s)^{\frac{\theta_P}{\theta_P+\theta_A}} V_e^{\frac{\theta_A}{\theta_A+\theta_P}} + V_u(s)](-W)^{-\frac{\theta_P}{\theta_A}}$$

Similarly, for the second slope matching condition we get

$$-V_u(s)\frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}} = -V_u(s')\frac{\theta_P}{\theta_A}(-W^S)^{-\frac{\theta_P+\theta_A}{\theta_A}}$$

and thus we have:

$$W^S = \left(\frac{V_u(s)}{V_u(s')}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} W,$$

and therefore:

$$V(W^S,u,s') - V(W,u,s) = [-V_u(s')\left(\frac{V_u(s)}{V_u(s')}\right)^{\frac{\theta_P}{\theta_P+\theta_A}} + V_u(s)](-W)^{-\frac{\theta_P}{\theta_A}}.$$

From the optimality condition for c we then have:

$$c = \frac{1}{\theta_P+\theta_A}\log(V_u(s)) - \frac{1}{\theta_A}\log(-W) + \frac{\theta_A}{\theta_P+\theta_A}h(\bar{a}(s))$$

Therefore we have in addition:

$$\begin{aligned}
u(c,\bar{a}(s)) &= -V_u(s)^{-\frac{\theta_A}{\theta_P+\theta_A}}\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s)))(-W) \\
u_a(c,\bar{a}(s)) &= -\theta_A h'(\bar{a}(s))V_u(s)^{-\frac{\theta_A}{\theta_P+\theta_A}}\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s)))(-W) \\
v(c) &= V_u(s)^{\frac{\theta_P}{\theta_P+\theta_A}}\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s)))(-W)^{-\frac{\theta_P}{\theta_A}}
\end{aligned}$$

Then we can write the optimality condition for a as:

$$\begin{aligned}
&-\rho V_u(s)\frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}}\left(-\theta_A h'(\bar{a}(s))V_u(s)^{-\frac{\theta_A}{\theta_P+\theta_A}}\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s)))(-W)\right) \\
&+\rho V_u(s)\frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}}\frac{q_s}{\rho}\left(\left(\frac{V_u(s)}{\exp(-\theta_P\omega)}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} - 1\right)(-W) \\
&= q_s[-V_u(s)^{\frac{\theta_P}{\theta_P+\theta_A}}V_e^{\frac{\theta_A}{\theta_A+\theta_P}} + V_u(s)](-W)^{-\frac{\theta_P}{\theta_A}}
\end{aligned}$$

Thus we see that W cancels out and we get:

$$V_u(s)^{-\frac{\theta_A}{\theta_P+\theta_A}}\left(\rho\theta_P h'(\bar{a}(s))\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s))) + (1+\frac{\theta_P}{\theta_A})q_s V_e^{\frac{\theta_A}{\theta_A+\theta_P}}\right) = q_s(1+\frac{\theta_P}{\theta_A}) \tag{23}$$

Substituting all of the above results into the HJB equation gives:

$$\begin{aligned}
&-\rho V_u(s)(-W)^{-\frac{\theta_P}{\theta_A}} = -\rho V_u(s)^{\frac{\theta_P}{\theta_P+\theta_A}}\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s)))(-W)^{-\frac{\theta_P}{\theta_A}} \\
&-\rho V_u(s)\frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}}\left(W - V_u(s)^{-\frac{\theta_A}{\theta_P+\theta_A}}\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s)))(-W)\right) \\
&-\rho V_u(s)\frac{\theta_P}{\theta_A}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}}\left[\frac{q_s}{\rho}\bar{a}(s)\left(\left(\frac{V_u(s)}{\exp(-\theta_P\omega)}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} - 1\right) + \frac{\lambda_s}{\rho}\left(\left(\frac{V_u(s)}{V_u(s')}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} - 1\right)\right](-W) \\
&+q_s\bar{a}(s)[-V_u(s)^{\frac{\theta_P}{\theta_P+\theta_A}}V_e^{\frac{\theta_A}{\theta_A+\theta_P}} + V_u(s)](-W)^{-\frac{\theta_P}{\theta_A}} + \lambda_s[-V_u(s')\left(\frac{V_u(s)}{V_u(s')}\right)^{\frac{\theta_P}{\theta_P+\theta_A}} + V(s)](-W)^{-\frac{\theta_P}{\theta_A}}
\end{aligned}$$

Therefore again the terms in W cancel. After simplifying and again canceling a common $(1+\frac{\theta_P}{\theta_A})$ factor we get:

$$V_u(s)^{-\frac{\theta_A}{\theta_P+\theta_A}}\left(\rho\exp(\frac{\theta_A\theta_P}{\theta_A+\theta_P}h(\bar{a}(s))) + q_s\bar{a}(s)V_e^{\frac{\theta_A}{\theta_A+\theta_P}} + \lambda_s V_u(s')^{\frac{\theta_A}{\theta_P+\theta_A}}\right) = \rho + q_s\bar{a}(s) + \lambda_s \tag{24}$$

Therefore we have 4 equations, (23) and (24) which each must hold for $s = G, B$, in the 4 unknowns $(\bar{a}(G), V_u(G), \bar{a}(B), V_u(B))$.

B.3 Moral Hazard

To ease some of the derivations, define the constant in the consumption function as $c^*(s) + h(a^*(s))$. Then given the forms of the guesses for c and a we can evaluate:

$$
\begin{aligned}
v(c) &= \exp(\theta_P(c^*(s) + h(a^*(s))))(-W)^{-\frac{\theta_P}{\theta_A}} \\
u(c, a^*(s)) &= -\exp(-\theta_A c^*(s))(-W) \\
u_c(c, a^*(s)) &= \theta_A \exp(-\theta_A c^*(s))(-W) \\
u_a(c, a^*(s)) &= -\theta_A h'(a^*(s)) \exp(-\theta_A c^*(s))(-W) \\
u_{aa}(c, a^*(s)) &= -[\theta_A^2 h'(a^*(s))^2 + \theta_A h''(a^*(s))] \exp(-\theta_A c^*(s))(-W) \\
u_{ac}(c, a^*(s)) &= \theta_A^2 h'(a^*(s)) \exp(-\theta_A c^*(s))(-W)
\end{aligned}
$$

In addition, we can evaluate:

$$W^J = W - \frac{\rho}{q_s}(-\theta_A h'(a^*(s)) \exp(-\theta_A c^*(s)))(-W) = [1 - \frac{\rho}{q_s}\theta_A h'(a^*(s)) \exp(-\theta_A c^*(s))]W.$$

So therefore we have:

$$
\begin{aligned}
V(W^J, e) &= -V_e[1 - \frac{\rho}{q_s}\theta_A h'(a^*(s)) \exp(-\theta_A c^*(s))]^{-\frac{\theta_P}{\theta_A}}(-W)^{-\frac{\theta_P}{\theta_A}} \\
V_W(W^J, e) &= -\frac{\theta_P}{\theta_A} V_e[1 - \frac{\rho}{q_s}\theta_A h'(a^*(s)) \exp(-\theta_A c^*(s))]^{-\frac{\theta_A+\theta_P}{\theta_A}}(-W)^{-\frac{\theta_P+\theta_A}{\theta_A}}
\end{aligned}
$$

The first order conditions and solution for W^S are of the same form as in the full information case, and thus we can write:

$$W^S = \left(\frac{V^*(s)}{V^*(s')}\right)^{-\frac{\theta_A}{\theta_P+\theta_A}} W,$$

and therefore:

$$V(W^S, u, s') - V(W, u, s) = [-V^*(s')\left(\frac{V^*(s)}{V^*(s')}\right)^{\frac{\theta_P}{\theta_P+\theta_A}} + V^*(s)](-W)^{-\frac{\theta_P}{\theta_A}}.$$

The optimality condition for c then can be written, canceling the terms in W:

$$
\begin{aligned}
&\theta_P \exp(\theta_P(c^*(s) + \theta_P h(a^*(s)))) = -V^*(s)\frac{\theta_P}{\theta_A}\left[-\theta_A \exp(-\theta_A c^*(s)) + a^*(s)\theta_A^2 h'(a^*(s)) \exp(-\theta_A c^*(s))\right] \\
&+a^*(s)\theta_A^2 h'(a^*(s)) \exp(-\theta_A c^*(s))\frac{\theta_P}{\theta_A} V_e[1 - \frac{\rho}{q_s}\theta_A h'(a^*(s)) \exp(-\theta_A c^*(s))]^{-\frac{\theta_A+\theta_P}{\theta_A}}.
\end{aligned}
$$

Simplifying this gives:

$$
\begin{aligned}
&\exp((\theta_P + \theta_A)c^*(s) + \theta_P h(a^*(s))) = V^*(s)\left[1 - a^*(s)\theta_A h'(a^*(s))\right] \\
&+a^*(s)\theta_A h'(a^*(s)) V_e\left[1 - \frac{\rho}{q_s}\theta_A h'(a^*(s)) \exp(-\theta_A c^*(s))\right]^{-\frac{\theta_A+\theta_P}{\theta_A}}
\end{aligned}
\tag{25}
$$

The optimality condition for a can be written, again canceling the terms in W:

$$
\begin{aligned}
&a^*(s)\left[\theta_A^2 h'\left(a^*(s)\right)^2 + \theta_A h''\left(a^*(s)\right)\right] \exp\left(-\theta_A c^*(s)\right) \\
&\frac{\theta_P}{\theta_A}\left[-V^*(s) + V_e\left(1 - \frac{\rho}{q_s}\theta_A \exp\left(-\theta_A c^*(s)\right) h'\left(a^*(s)\right)\right)^{-\frac{\theta_A+\theta_P}{\theta_A}}\right] \\
&= \frac{q_s}{\rho}\left[-V_e\left(1 - \frac{\rho}{q_s}\theta_A \exp\left(-\theta_A c^*(s)\right) h'\left(a^*(s)\right)\right)^{-\frac{\theta_P}{\theta_A}} + V^*(s)\right]
\end{aligned}
$$

Simplifying yields:

$$a^*(s)\theta_P\left(\theta_A h'\left(a^*(s)\right)^2+h''\left(a^*(s)\right)\right)\left[V^*(s)-V_e\left(1-\frac{\rho}{q_s}\theta_A\exp\left(-\theta_A c^*(s)\right)h'\left(a^*(s)\right)\right)^{-\frac{\theta_A+\theta_P}{\theta_A}}\right]$$
$$=\exp\left(\theta_A c^*(s)\right)\frac{q_s}{\rho}\left[V^*(s)-V_e\left(1-\frac{\rho}{q_s}\theta_A\exp\left(-\theta_A c^*(s)\right)h'\left(a^*(s)\right)\right)^{-\frac{\theta_P}{\theta_A}}\right] \tag{26}$$

The final equation comes from substituting the results into the HJB equation and simplifying, which gives:

$$\begin{aligned}0 \;=\;& (\rho+q_s a^*(s)+\lambda_s)V^*(s)-\rho\exp\left(\theta_P\left(c^*(s)+h\left(a^*(s)\right)\right)\right) \qquad (27)\\ &+\rho\frac{\theta_P}{\theta_A}V^*(s)\left[1-\exp\left(-\theta_A c^*(s)\right)+a^*(s)\theta_A\exp\left(-\theta_A c^*(s)\right)h'\left(a^*(s)\right)-\frac{\lambda_s}{\rho}\left(\left(\frac{V^*(s')}{V^*(s)}\right)^{\frac{\theta_A}{\theta_A+\theta_P}}-1\right)\right]\\ &-q_s a^*(s)V_e\left(1-\frac{\rho}{q_s}\theta_A\exp\left(-\theta_A c^*(s)\right)h'\left(a^*(s)\right)\right)^{-\frac{\theta_P}{\theta_A}}-\lambda_s V^*(s')^{\frac{\theta_A}{\theta_A+\theta_P}}V^*(s)^{\frac{\theta_P}{\theta_A+\theta_P}}\end{aligned}$$

Thus for $s=G,B$ the equations (25) - (27) determine the six constants $(V^*(s),a^*(s),c^*(s))$.

C Calculations for the Implementation

C.1 An Employed Worker

Using the guess in the first order condition we can find c as:

$$c(x,e)=-\frac{1}{\theta_A}\log\frac{r^e J_e}{\rho}+r^e x,$$

and so we have:

$$u(c)=-\frac{r^e J_e}{\rho}\exp(-r^e\theta_A x).$$

Substituting this into the HJB equation, and canceling the terms in $J_e\exp(-r^e\theta_A x)$ gives:

$$-\rho=-r^e+r^e\log\frac{r^e J_e}{\rho}+r^e\theta_A b^e$$

Solving for J_e gives the result in the text. Substituting the resulting J_e into the consumption function gives the other result.

C.2 An Unemployed Worker

Using the guess of the value function and the structure of A we have:

$$\begin{aligned}J(x+B(s),e)-J(x,u,s) &= [1-\exp(-r(s)\theta_A B(s))]\exp(-r(s)\theta_A x)\\ J(x+A(s,x),u,s')-J(x,u,s) &= \exp(-r(s)\theta_A x)-\exp(-r(s')\theta_A(x+A(s,x)))\\ &= [1-\exp(-r\theta_A\hat{A}(s))]\exp(-r(s)\theta_A x)\end{aligned}$$

The first order condition for c then implies the consumption function in the text. The first order condition for a gives:

$$\rho\theta_A h'(a)\exp(-\theta_A(c-h(a)))=q_s[1-\exp(-r(s)\theta_A B(s))]\exp(-r(s)\theta_A x).$$

Using the result for c and canceling terms in x gives:

$$r(s)\theta_A h'(a(s)) = q_s[1 - \exp(-r(s)\theta_A B(s))],$$

which verifies that effort is independent of x.

Using the previous results, and canceling the terms in x, the HJB equation then becomes:

$$\begin{aligned}
-\rho &= -r(s) + r(s)\theta_A[\frac{1}{\theta_A}\log\frac{r(s)}{\rho} - h(a(s)) + b^u(s)] \\
&\quad + q_s a(s)[1 - \exp(-r(s)\theta_A B(s))] + \lambda_s[1 - \exp(-r(s)\theta_A \hat{A}(s))].
\end{aligned}$$

We now substitute in the policies for $b_u(s)$, $B(s)$ and $\hat{A}(s)$ from (14)-(16), and evaluate at c^*, a^*:

$$-\rho = -r(s) - \mu_W(s) + q_s a^*(s)(1 - w_J(s)) + \lambda_s(1 - w_S(s)). \tag{28}$$

But then we recall that:

$$\mu_W(s) = \rho + \rho u(c^*(s)) - q_s a^*(s)(w_J(s) - 1) - \lambda_s(w_S(s) - 1).$$

Using this in (28) leaves us with (13), the interest rate policy. Thus the HJB equation is satisfied, which verifies our guess of the form of the value function and shows that the policy implements the optimal contract.

D The Benchmark Contract

We now show how to calculate the worker's utility levels and search effort under the benchmark contract. First, let W_s^e be the expected utility of the worker under the benchmark contract when he is employed in state s. We suppose these levels are given, and now show how to determine the evolution of promised utility over an unemployment spell. First, we consider the period after T_B, in which all unemployment benefits are expired in both states, and denote the worker's promised utility by W_s^{u3} in this region. Suppose at some date $\bar{t} > T_B$, the economy is in a boom and let τ be the first date of a switch in either job status or the aggregate state. That is $\tau = \tau^S \wedge \tau^J$ with τ^S the time at which the economy enters a recession and τ^J the time at which the worker finds a job. So the worker chooses his optimal effort to solve the following problem:

$$W_G^{u3} = \max_{a \in A} E_{\bar{t}}^a[\rho \int_{\bar{t}}^{\tau} e^{-\rho(t-\bar{t})}(u(\alpha) - h(a_t))dt + e^{-\rho(\tau-\bar{t})}(\mathbf{1}(\{\tau = \tau^J\})W_G^e + \mathbf{1}(\{\tau = \tau^S\})W_B^{u3})].$$

Following the same steps as in deriving the HJB equations above, we see that the values W_s^{u3} solve the HJB equations for $s = G, B$:

$$\rho W_s^{u3} = \max_{\hat{a} \in [0,\bar{a}]} \rho(u(\alpha) - h(\hat{a})) + q_s(\hat{a})(W_s^e - W_s^{u3}) + \lambda_s(W_{s'}^{u3} - W_s^{u3}).$$

Now, we turn to the time interval $[T_G, T_B]$ when the worker only receives benefits in a recession, but not in a boom. Denote the continuation value of the worker during this period by $W_s^{u2}(t)$, where the values are now time dependent due to the impending termination of benefits. Similar to the previous case, these values satisfy the following pair of HJB equations:

$$\begin{aligned}
\rho W_G^{u2}(t) - \frac{d}{dt}W_G^{u2}(t) &= \max_{\hat{a} \in [0,\bar{a}]} \rho(u(\alpha) - h(\hat{a})) + q_G(\hat{a})(W_G^e - W_G^{u2}(t)) + \lambda_G(W_B^{u2}(t) - W_G^{u2}(t)) \\
\rho W_B^{u2}(t) - \frac{d}{dt}W_B^{u2}(t) &= \max_{\hat{a} \in [0,\bar{a}]} \rho(u(c^B + \alpha) - h(\hat{a})) + q_B(\hat{a})(W_B^e - W_B^{u2}(t)) + \lambda_B(W_G^{u2}(t) - W_B^{u2}(t))
\end{aligned}$$

with boundary conditions $W_G^{u2}(T_B) = W_G^{u3}$ and $W_B^{u2}(T_B) = W_B^{u3}$.

Finally, in the time interval $[0, T_G]$, the worker receives unemployment benefits in both booms and recessions. Again, promised utility which we denote by $W_s^{u1}(t)$ is time dependent and satisfies the HJB equation for $s = G, B$:

$$\rho W_s^{u1}(t) - \frac{d}{dt} W_s^{u1}(t) = \max_{\hat{a} \in [0,\bar{a}]} \rho(u(c^B + \alpha) - h(\hat{a})) + q_s(\hat{a})(W_s^e - W_s^{u1}(t)) + \lambda_s(W_{s'}^{u1}(t) - W_s^{u1}(t))$$

with boundary conditions $W_s^{u1}(T_G) = W_s^{u2}(T_G)$.

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