Value or Growth?

Pricing of Idiosyncratic Cash-Flow Risk with Heterogeneous Beliefs

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ABSTRACT

We study a continuous-time pure exchange economy where idiosyncratic cash flow risks are priced via investors' heterogeneous beliefs. Investors perceive idiosyncratic cash flow risks differently through heterogeneous subjective mean growth rates on a firm's cash flow. This impacts equilibrium quantities. Our model shows that idiosyncratic cash flow shocks priced through belief differences can explain cross-sectional variations in stock returns and cash flows. Quantitative results show that a value premium arises, as value stocks have higher idiosyncratic cash-flow volatilities, lower average cash flows, and higher belief differences, which is empirically supported. A growth premium prevails without belief differences.

Keywords: idiosyncratic cash-flow risk, heterogeneous beliefs, general equilibrium, cross-section of stock returns, habit formation, the value premium.
JEL classification: G00, G02, G10, G11, G12

I. Introduction

Explaining the cross section of stock returns is one of the most important topics in asset pricing, and the value premium anomaly is a key issue in this enterprise. Many theoretical and empirical studies attempted to account for the cause and nature of this phenomenon. Recently the role of cash flow risk, defined as the covariance between a firm's cash flow and the aggregate cash flow, has been emphasized in the literature. Abel (1999), Bansal and Yaron (2004), Da (2009), and others study theoretical aspects. Bansal, Dittmar, and Lundblad (2005), Santos and Veronesi (2006), Yang (2007), Cohen, Polk, and Vuolteenaho (2009), Campbell, Polk, and Vuolteenaho (2010), and many others study empirical aspects of cash flow risks in the cross section of stock returns. Notably several papers have tried to link cash flow risk and cash flow duration to cross-sectional return variation. For instance, Campbell and Vuolteenaho (2004), Bansal, Dittmar, and Lundblad (2005), Kiku (2007), Hansen, Heaton, and Li (2008), Zhang (2005), Lettau and Wachter (2007), Da (2009), Santos and Veronesi (2010), and Choi, Johnson, Kim, and Nam (2013) developed structural models that directly associate cash flow risk or cash flow duration with book-to-market and expected stock returns to this end.

When a prototypical asset pricing model produces a cross-sectional variation associated with cash flows, one puzzling feature arises. Value (growth) stocks have shorter (longer) durations, therefore; value stocks have a smaller risk premium in light of discount risk than growth stocks contrary to the empirical evidence. Thus, economic models that explain the time-series properties of asset prices have difficulty in matching their cross sectional variations and vice versa. Little attention was paid to this issue until recently. Lettau and Wachter (2007, 2011) state that this problem can disappear if the time-varying price of discount-rate risk is uncorrelated with aggregate dividends or consumption in their reduced-form model. However, significant empirical evidence exists that time-varying equity risk premia are countercyclical and closely associated with aggregate consumption or dividends. Further, Santos and Veronesi (2010) show that, in equilibrium, when the stochastic discount factor generates a time-varying risk premia correlated with aggregate cash flows and simultaneously accounts for the aggregate moments of macroeconomic and stock market variables, a counterfactual growth premium arises. In their model, a value premium can prevail only when aggregate cash flows are counterfactually volatile so that a cash flow puzzle appears.

In this paper, we tackle this issue in an exchange economy setting by investigating the effect of idiosyncratic cash flow fluctuations on the cross-section of stock returns. The main departure of our paper is to incorporate belief differences of investors into cash flow dynamics of individual firms with the following features.

First, we model both aggregate and individual cash flow processes consistently by using an exogenous model that is impacted by aggregate risk only. However, an individual firm's

cash flow process is subject to idiosyncratic risk in addition to aggregate risk. The dynamics of the cash flow processes (including idiosyncratic risk exposure) implies that our model is constructed such that in the aggregate, idiosyncratic cash flow risk cancels out.

Second, we introduce investors' heterogeneous beliefs into cash flow processes. The key assumption regarding investors' belief heterogeneity is that investors have different opinions on the long-run mean of the share process with respect to firm- or asset-specific risk.[1] This differs from most of the heterogeneous beliefs literature where investors update their perceptions of the drift of underlying processes through aggregate risk. Since the market is equipped with a sufficient number of assets to make the asset market complete, idiosyncratic cash flow risk is priced in equilibrium through belief differences.

One of our main theoretical findings is that individual expected stock returns are *positively* affected by idiosyncratic cash flow risk through belief differences.[2] The higher the belief difference, the stronger the effect of idiosyncratic cash flow risks on equilibrium stock returns. Individual equilibrium returns are positively affected by the cash flow share ratio ($\bar{s}/s_t$), negatively by the habit ratio ($\bar{H}/H_t$), and negatively by the interaction between the share ratio and the habit ratio.[3] Cross-sectional return variations result from differences in those variables in addition to idiosyncratic cash flow risk via belief differences. It turns out that value stocks have higher values in the share ratio and belief differences. Thus, the theory connects these firm characteristics and related investor behavior to the cross section of stock returns. Furthermore, our quantitative study reveals that the cross-sectional return variation is largely attributed to the pricing of idiosyncratic cash flow risk in equilibrium. Specifically, our simulation results state that a growth premium arises with a model where idiosyncratic cash flow risk is ignored because of the absence of belief differences, which is consistent with Santos and Veronesi (2010). Our results imply that sorting stocks based on price-to-fundamental ratios endogenously picks up stocks with higher (idiosyncratic) cash flow risk and *higher degrees of belief differences* in the cross-section, so that the value premium arises.

In this light, the main contribution of our paper is to show that idiosyncratic cash flow

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[1]This assumption is similar to Basak (2000) where investors have different beliefs about the cash flow growth rate through non-fundamental risk.

[2]Babenko, Boguth, and Tserlukevich (2013) show that in an option theoretic setting, idiosyncratic cash flow risk negatively affects equilibrium stock returns. High idiosyncratic cash flow shocks positively affect a firm's profit, which in turn increases the firm size. When the firm size increases, the price of risk, measured by the CAPM beta, decreases so that the expected excess return decreases. However, Cochrane, Longstaff, and Santa-Clara (2008), Martin (2013), Choi, Johnson, Kim, and Nam (2013), and Choi and Kim (2014) show that in an equilibrium setting, larger idiosyncratic cash flows can increase the market risk, because of under-diversification. Because their model does not consider the stochastic discount factor derived in equilibrium, the overall effect is unclear. In fact, later, we show that firms with higher idiosyncratic cash flow risks happen to be value firms.

[3]Cash flow share, s_t, is defined as the ratio of firm cash flow to the market cash flow, and $\bar{s}$ is the long-run mean. Following Menzley, Santos, and Veronesi (2004), we use the share process to represent individual cash flows.

risk, in conjunction with investors' heterogeneous beliefs, can explain the cross section of stock returns and the related cash flow dynamics. Another contribution of the paper is to shed light on the characteristics of value and growth stocks. Empirically, we estimate individual cash flow share using data on stock returns, firm characteristics, and analyst forecasts. From our empirical results, value stocks tend to have the lower long-run mean of the share, the higher share ratio of the long-run mean to the current share, and a slower mean reversion of the share than growth stocks.

Lower long-run mean of the share can be interpreted that value firms have lower growth potential compared to growth firms. In addition, higher share ratios of value stocks imply that value firms, despite their lower long-run mean of the share, have even lower current shares, implying that the value firms currently suffer from lower profitability.[4] A slower mean reversion of the share process also indicates that value firms may grow more slowly. In addition, value stocks have higher idiosyncratic volatilities of the cash flow share and a higher degree of belief differences. Interestingly, aggregate cash flow volatilities of the two types of equities do not differ, reinforcing our argument on the importance of idiosyncratic cash flows.

The importance of the pricing of idiosyncratic cash flow risk via belief differences sheds light on the challenge that existing asset pricing models such as Campbell and Cochrane (1999), Menzley, Santos, and Veronesi (2004), Lettau and Wachter (2007), Santos and Veronesi (2010), and Lettau and Wachter (2011) face. In particular, Lettau and Wachter (2007) and Santos and Veronesi (2010) show that if there is a negative correlation between shocks to aggregate cash flows and shocks to the stochastic discount factor, as in the model of Campbell and Cochrane (1999), then a growth premium will prevail in the cross-section that is opposite to the data. Lettau and Wachter (2007) assume that the two shocks above have zero correlation. With this structure, shocks to state variables that drive the stochastic discount factor increases the impact of cash flows relative to discount rate leading to a value premium. In our model, while the aggregate shock to the stochastic discount factor is negatively correlated with the shock to aggregate cash flows, investor belief differences, related to idiosyncratic shocks appearing in the stochastic discount factor, are uncorrelated with shocks to aggregate cash flows. The idiosyncratic cash flow risk in equilibrium via belief differences increases the overall cash flow risk effect. The increased cash flow risk component can suppress the effect of the discount-rate risk in the cross-section so that the value premium arises. Therefore, our model provides an economic rationale for the partial equilibrium set up of the stochastic discount factor in Lettau and Wachter (2007), as well as for the counterfactual magnification of cash flow risk in the cross-section in Santos and Veronesi (2010).

Finally, this paper is related to the heterogeneous beliefs literature with the following

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[4]The relation between book-to-market ratio and share ratio is consistent with Avramov, Cederburg, and Hore (2012) and Chen (2013).

additions. First, we impose that investors' beliefs work through idiosyncratic risk, unlike the existing work on belief differences that focuses on aggregate risk. Second, our study is related to the literature studying the risk premium relation from belief disagreement. There is conflicting evidence that investors' belief differences lead to either a positive or a negative risk premium. Positive risk premium has been shown in Varian (1985), Varian (1989), Abel (1989), David (2008), Qu, Starks, and Yan (2004), and Carlin, Longstaff, and Matoba (2013) for example. Negative premium has been shown in Miller (1977), Harrison and Kreps (1978), Diether, Malloy, and Scherbina (2005), Chen, Hong, and Stein (2002), Goetzmann and Massa (2005), Johnson (2004), Park (2005), Zhang (2006), and others. Anderson, Ghysels, and Juergens (2005) find both negative and positive risk premium depending on the frequency of belief dispersion. Our equilibrium result shows that a positive risk premium exists, both in time series and cross section and is closely linked to the value premium anomaly.

This paper is organized starting with Section 2 that introduces our model. In Section 3, we derive equilibrium quantities and discuss the equilibrium impact of idiosyncratic cash flow risk both qualitatively and quantitatively. Section 4 studies quantitative implications of the model and provides a detailed discussion on how the value premium arises in our model. Proofs are in the appendix and additional equilibrium results are found in the internet appendix.

II.   The Economy

In this section, we develop our model. We begin with modeling the cash flows of assets, followed by describing how belief differences are incorporated. Then, we define investor preferences and securities, along with equilibrium conditions. In so doing, we label major assumptions related to cash flows and belief differences to highlight the main features of the model.

A.   Cash Flow Modeling

We consider a continuous-time, pure-exchange equilibrium model with two trees. For the specification of the tree process, we follow Menzley, Santos, and Veronesi (2004) by taking the cash flow share process as exogenous to describe the relative movement of individual cash flow processes in the economy.

Assumption 1. *Individual cash flow processes follow mean-reverting share processes. The share, s_t, is defined as individual cash flow ($\equiv D_s(t)$) divided by aggregate cash flow ($\equiv D(t)$).*

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s_t follows

$$ds_t = \phi_s(\bar{s} - s_t)dt + \tilde{\boldsymbol{\sigma}}(s_t)d\boldsymbol{B}'_t, \tag{II.1}$$

where

$$
\begin{aligned}
\tilde{\boldsymbol{\sigma}}(s_t) &\equiv s_t \cdot \boldsymbol{\sigma}(s_t), \\
\boldsymbol{\sigma}(s_t) &\equiv (\sigma_{s,A}(t), \sigma_{s,I}(t)), \\
\sigma_{s,j} &\equiv v_{s,j} - s_t v_{s,j} - (1 - s_t)v_{(1-s),j}, \quad j = A, I \\
d\boldsymbol{B}_t &\equiv (dB_A(t), dB_I(t)), \\
\phi_s &> 0,
\end{aligned}
\tag{II.2}
$$

where $\bar{s}$ is the long-run mean of the share of the asset under consideration, $B_A(t)$ and $B_I(t)$ represent the aggregate Brownian risk and the idiosyncratic Brownian risk respectively, and $v_{s,j}$ and $v_{(1-s),j}$ are the diffusion coefficients of individual assets with the share s_t and the share $(1 - s_t)$.

Assumption 1 states that a single asset cannot dominate the entire market in the long-run as was shown in Menzley, Santos, and Veronesi (2004). This stationarity enables us to analyze the cross-section of stock returns in the long-run. Also note that this specification is a simpler version of Menzley, Santos, and Veronesi (2004) and Santos and Veronesi (2010). The reason that we have this specification is closely related to the specification of individual cash flow process that can be derived from the share process. (II.6) shows that individual cash flow process is influenced from its own idiosyncratic risk but not from other idiosyncratic risks. As we want that individual cash flow process has two risk exposures, i.e., aggregate risk and its own idiosyncratic risk, our specification in (II.1) fits our purpose.

Our share process is also flexible and tractable in modeling risks in an equilibrium setting. Gabaix (2009) shows that this share process belongs to the family of linearity-generating processes such that a closed-form solution can be derived. Appendix A provides more details of the individual cash flow share process.

B.   Belief Difference

Now we embed heterogeneous beliefs into the cash flow share process. The importance of modeling investors' heterogeneous beliefs is emphasized early by Lintner (1965), Miller (1977), and Harrison and Kreps (1978). Later work studied the impact of economic agents' different beliefs about underlying fundamental economic processes on equilibrium quantities. Detemple and Murthy (1994) study the effect of belief differences in a production economy. For exchange economies, key contributions include Zapatero (1998), Basak (2000), Basak (2005), Buraschi

and Jiltsov (2006), Jouini and Napp (2007), Gallmeyer and Hollifield (2008), David (2008), Dumas, Kurshev, and Uppal (2009), Weinbaum (2009) and Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2012). In particular, we assume that investors have different beliefs about the long-run mean of the share ($\bar{s}$), which can be viewed as a measure of growth potentials. The following assumption spells this out.

Assumption 2. *Investors face the same information about the underlying cash flow processes including both aggregate and individual cash flow processes, but agree to disagree about the long-run mean of individual cash flow share processes. Specifically, investors have different beliefs through idiosyncratic risk (or firm-specific information). By including the belief difference into the share process, we write investors' perceived share process as follows:*

$$\frac{ds_t}{s_t} = \phi_s \left(\frac{\bar{s}^{(k)}}{s_t} - 1 \right) dt + \sigma_{s,A}(t) dB_A(t) + \sigma_{s,I}(t) dB_I^{(k)}(t), \tag{II.3}$$

where $k = 1, 2$ refers to the individual investors. From the optimal filtering theory (Liptser and Shiryaev (2001)), the innovation process $B_I(t)$ is given as

$$dB_I^{(k)}(t) \equiv \eta_t^{(k)} dt + dB_I(t), \tag{II.4}$$

where $\eta_t^{(k)} \equiv \frac{\phi_s(\bar{s} - \bar{s}^{(k)})}{\sigma_{s,I}(t) s_t}$. Note that $\eta_t^{(k)}$ measures the difference between the true long-run mean of the share and k-th investor's perceived long-run mean of the share.

The aggregate cash flow process is given by

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_{D,A} dB_A(t). \tag{II.5}$$

Theoretical foundation for the "agree-to-disagree" assumption is found in Varian (1985), Harris and Raviv (1993), Morris (1994), and Morris (1996). Although Assumption 2 is similar to the assumptions in the previously mentioned papers, a major difference of our model is that investors hold their different beliefs on the long-run mean of the cash flow through idiosyncratic risk, and they agree about the aggregate cash flow dynamics and the impact of aggregate risk on individual cash flow processes.

This assumption seems plausible as investors are more likely to disagree with each other in interpreting individual firm-specific information due to their differences in educational backgrounds, cultural views, expertise, or even cognitive capabilities. In addition, this particular assumption shares a common feature with the rational inattention theory as seen in Sims (2003). Rational inattention theory suggests that economic agents process important information first. If they still have information processing capacity, then they process the remaining information. This assumption comes from the notion that agents' informational capacity is a

scarce resource.

If we view aggregate risk as the important information and firm-specific risk as residual information where different investors have different capacities of processing information, our assumption can be thought of as a special form of rational inattention. For example, all investors process the aggregate information first in the same manner so that they all agree about how aggregate risk affects underlying economic processes. After that, they process the remaining individual firm-specific information, but with differences due to limited resources in information processing (Sims (2006) and Xiong and Peng (2006) for details).

Although individual cash flow processes are subject to both aggregate and idiosyncratic risks, the aggregate cash flow process has exposure only to aggregate risk. Thus, equation (II.5) implies that in the aggregate, the idiosyncratic risks are diversified away such that both individual and aggregate cash flows are modeled consistently. According to the definition of the share process s_t, an individual cash flow process is defined as the product of the share process and the aggregate dividend. By applying Ito's lemma to the product of s_t and D_t, we can write perceived individual cash flow process as:

$$\frac{dD_s(t)}{D_s(t)} = \mu_{D_s}^{(k)}(t)dt + \sigma_{D_s,A}(t)dB_A(t) + \sigma_{D_s,I}(t)dB_I^{(k)}(t) \quad k = 1, 2, \qquad (\text{II.6})$$

where

$$
\begin{aligned}
\mu_{D_s}^{(k)}(t) &\equiv \mu_D + \phi_s\left(\frac{\bar{s}^{(k)}}{s_t} - 1\right) + \theta_s^{CF} - s_t\theta_s^{CF} - (1 - s_t)\theta_{(1-s)}^{CF}, \\
\sigma_{D_s,A}(t) &\equiv \sigma_{D,A} + \sigma_{s,A}(t), \\
\sigma_{D_s,I}(t) &\equiv \sigma_{s,I}(t),
\end{aligned}
\qquad (\text{II.7})
$$

where $\theta_s^{CF} \equiv v_{s,A} \cdot \sigma_{DA}$. θ_s^{CF} is the unconditional covariance between the share process and the aggregate cash flow process. We define θ_s^{CF} as a fundamental cash flow risk parameter following Menzley, Santos, and Veronesi (2004). θ_s^{CF} plays an important role in quantitative study later because it enables us to estimate individual cash flow parameters $v_{s,A}$ and $v_{s,I}$.[5]

C. *Investor Preference*

Investor preferences are represented by a constant relative risk aversion utility function with an external habit formation, such as "catching-up-with-Joneses". Risk aversion parameters are

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[5]Identification of individual cash flow risk parameters such as $v_{s,A}$, $v_{s,I}$, $v_{(1-s),A}$ and $v_{(1-s),I}$ is explained in the appendix. The roles of the parameters are discussed in the quantitative results.

set to be the same across investors for simplicity. Investor k's utility function is given by:

$$u(c_k(t)) = \frac{1}{1-\gamma}\left(\frac{c_k(t)}{X(t)}\right)^{1-\gamma}, \quad k = 1, 2. \tag{II.8}$$

where $X(t)$ represents a ratio habit as in Abel (1989). For tractability, we assume the economy is populated by two investors. The habit process X is defined following Constantinides (1990), Detemple and Zapatero (1991), and Santos and Veronesi (2010):

$$X_t \equiv \delta \int_0^t e^{-\delta(t-\tau)} D_\tau d\tau. \tag{II.9}$$

In particular, we follow Santos and Veronesi (2010) to define the process of $H_t \equiv (D_t/X_t)^{(1-\gamma)}$ and assume:

$$dH_t = h_1(\bar{H} - H_t)dt + h_2 H_t dB_A(t), \tag{II.10}$$

where $h_1 > 0$, $h_2 > 0$.

D. Equilibrium

In our economy, there are two risky assets and one riskless asset. Without loss of generality, we consider the market portfolio and an asset with the share process s_t for the risky assets. The asset with share s is referenced as asset s, while the market portfolio is referenced as M. The price process of the market portfolio is computed as:

$$\frac{dP_{M,t} + D_t}{P_{M,t}} = \mu_{P_M}(t)dt + \sigma_{P_M,A}(t)dB_A(t). \tag{II.11}$$

Accordingly, the perceived price of the asset s, denoted as P_s is given by

$$\frac{dP_{s,t} + D_{s,t}}{P_{s,t}} = \mu_{P_s}^{(k)}(t)dt + \sigma_{P_s,A}(t)dB_A(t) + \sigma_{P_s,I}(t)dB_I^{(k)}(t) \quad \text{for } k = 1, 2, \tag{II.12}$$

where

$$\mu_{P_s}^{(k)}(t) \equiv \mu_{P_s}(t) - \frac{\sigma_{P_s,I}(t)\phi_s(\bar{s} - \bar{s}^{(k)})}{\sigma_{s,I}(t)s(t)}. \tag{II.13}$$

Further, r_t refers to the rate of return for the riskless asset.

Turning to the consumption-portfolio problem of the individual investor, investor k's wealth

$W^{(k)}(t)$ evolves as:

$$
\begin{aligned}
dW_t^{(k)} =&\, W_t^{(k)} \left[r_t - \tilde{c}_{k,t} + \pi_M^{(k)}(t)(\mu_P(t) - r_t) + \pi_s^{(k)}(t)(\mu_{P_s}^{(k)}(t) - r_t) \right] dt \\
&+ W_t^{(k)} \left[\pi_M^{(k)}(t)\sigma_{P,A}(t) + \pi_s^{(k)}(t)\sigma_{P_s,A}(t) \right] dB_A(t) \\
&+ W_t^{(k)} \left[\pi_s^{(k)}(t)\sigma_{P_s,I}(t) \right] dB_I^{(k)}(t),
\end{aligned}
\tag{II.14}
$$

where $\tilde{c}_{k,t}$ is the consumption fraction of the k-th investor, $c_{k,t}/W_t^{(k)}$. The quantities $\pi_M^{(k)}$ and $\pi_s^{(k)}$ are the k-th investor's risky investment fractions of wealth in the market portfolio and the asset that corresponds to the share process, s_t. The riskless investment is defined as $b_k(t) \equiv 1 - \pi_M^{(k)}(t) - \pi_s^{(k)}(t)$. Following Dybvig and Huang (1988), we impose a non-negativity condition on the wealth process in order to rule out arbitrage strategies.

We now specify state price densities across investors as follows:

$$
d\xi_t^{(k)} = -\xi_t^{(k)} \left[r_t dt + \theta_A(t)dB_A(t) + \theta_I^{(k)}(t)dB_I^{(k)}(t) \right] \quad \text{for } k = 1, 2,
\tag{II.15}
$$

where θ_A is the market price of aggregate risk and $\theta_I^{(k)}$s is the perceived market price of idiosyncratic risk for investor k. Market prices of risks are:

$$
\begin{aligned}
\theta_A(t) &\equiv \frac{\mu_P(t) - r_t}{\sigma_{P,A}}, \\
\theta_I^{(k)}(t) &\equiv \left[-\frac{\sigma_{P_s,A}}{\sigma_{P_s,I}} \theta_A(t) + \frac{1}{\sigma_{P_s,I}} \left(\mu_{P_s} - r \right) - \eta_t^{(k)} \right].
\end{aligned}
\tag{II.16}
$$

Thus, the following link exists between the two idiosyncratic market prices of risks:

$$
\theta_I^{(1)}(t) - \theta_I^{(2)}(t) = \eta_t^{(2)} - \eta_t^{(1)} = \bar{\eta}_t.
\tag{II.17}
$$

For simplicity, we assume the second investor is always the more optimistic investor such that $\bar{s}^{(2)}$ is bigger than $\bar{s}^{(1)}$ which brings us to our next assumption.

Assumption 3.

$$
\bar{\eta}_t \equiv \eta_t^{(2)} - \eta_t^{(1)} = \frac{\phi_s\big(\bar{s}^{(1)} - \bar{s}^{(2)}\big)}{\sigma_{s,I}(t)s_t} < 0.
$$

Assumption 3 simply states that a belief difference exists in this economy and is represented by the $\bar{\eta}$ term which is negative.

Investors are assumed to be infinitely lived and the market is complete in our economy. Thus, we can formulate an individual optimization problem using martingale methods as fol-

lows:

$$\max_{c_k} E^{(k)} \left[\int_0^\infty u_k(c_k(t)) \mathrm{d}t \right]$$

$$\text{subject to} \qquad\qquad\qquad\qquad (\text{II}.18)$$

$$E^{(k)} \left[\int_0^\infty \xi^{(k)}(t) c_k(t) \mathrm{d}t \right] \leq W^{(k)}(0) \equiv w_k P(0),$$

where $P(t)$ is the total wealth held by both investors at time t since it is the value of the market portfolio. Also note that $W^{(1)}(t) + W^{(2)}(t)$ is the total wealth in the economy such that it equals $P(t)$. The quantity w_k is the initial fraction of wealth held by investor k of the market portfolio. From the maximization problem in (II.18), the optimality condition for investor k's consumption is given by:

$$c_k(t) = I_k \left(\frac{\xi^{(k)}(t)}{\lambda_k} \right)$$

$$= \left(\frac{1}{X_t} \right)^{\frac{1-\gamma}{\gamma}} \left[\frac{\xi^{(k)}(t)}{\lambda_k} \right]^{-\frac{1}{\gamma}}, \qquad\qquad (\text{II}.19)$$

where $1/\lambda_k$ is the Lagrange multiplier for investor k's optimal consumption-portfolio choice problem and $I_k(\cdot)$ is the inverse of investor k's utility function. From the static budget constraint of investor k's problem, we have:

$$\lambda_k = \left[\frac{E^{(k)} \left[\int_0^\infty \left\{ \xi^{(k)}(t) X_t \right\}^{\frac{\gamma-1}{\gamma}} dt \right]}{w_k P_M(0)} \right]^{-\gamma}. \qquad\qquad (\text{II}.20)$$

Then, equilibrium in this economy is defined as follows:

Definition 1. *Given preferences, endowments, and beliefs structures, an equilibrium in this economy is a collection of allocations* $\left(\overset{*}{c}_k, \overset{*}{\pi}_M^{(k)}, \overset{*}{\pi}_s^{(k)}, \overset{*}{b}_k \right)_{k=1,2}$ *and a supporting price system* $\left(r, \mu_P, \mu_{P_s}^{(k)}, \sigma_P, \sigma_{P_s} \right)$ *such that* $\left(\overset{*}{c}_k, \overset{*}{\pi}_M^{(k)}, \overset{*}{\pi}_s^{(k)}, \overset{*}{b}_k \right)$ *optimally solves investor k's consumption-portfolio choice problem given his/her perceived price processes, security prices are consistent*

across investors, and all markets clear for $t \in [0, T]$:

$$\sum_{k=1}^{2} \overset{*}{c}_k(t) = D(t),$$

$$\sum_{k=1}^{2} \overset{*}{\pi}_M^{(k)}(t) = 1,$$

$$\sum_{k=1}^{2} \overset{*}{\pi}_s^{(k)}(t) = s(t),$$

$$\sum_{k=1}^{2} \overset{*}{b}_k(t) = 0.$$

(II.21)

To derive the equilibrium prices, we find two stochastic discount factors that clear the consumption good market:

$$\overset{*}{c}_1(\xi^{(1)}(t)/\lambda_1, t) + \overset{*}{c}_2(\xi^{(2)}(t)/\lambda_2, t) = D(t).$$

(II.22)

For computational purpose, we define the stochastic weighting process λ_t as follows:

$$\lambda_t \equiv \frac{\lambda_1 \xi_t^{(2)}}{\lambda_2 \xi_t^{(1)}},$$

(II.23)

where $\lambda_0 = \frac{\lambda_1}{\lambda_2}$, since $\xi^{(k)}(0) = 1$ for $k = 1, 2$. As discussed in Basak (2000), λ_t provides information about the differences in the investors' opportunity sets given heterogeneous beliefs. To solve for the equilibrium, we construct a representative investor's utility function. For this, we follow Huang (1987) and Cuoco and He (1994). This method has been applied in many equilibrium studies such as Basak and Cuoco (1998), Basak (2000), Detemple and Serrat (2003), Basak and Gallmeyer (2003), and Gallmeyer and Hollifield (2008). The λ_t process is a stochastic weight in the representative investor's utility function for computing the equilibrium as follows:

$$U(C, \lambda) = \max_{c_1 + c_2 \leq D} \frac{\lambda_t}{\lambda_1} \frac{(c_1/X)^{1-\gamma}}{1-\gamma} + \frac{1}{\lambda_2} \frac{(c_2/X)^{1-\gamma}}{1-\gamma},$$

(II.24)

where C is the aggregate consumption; therefore, $C \equiv D$. By applying Itô's lemma to λ_t, we obtain the diffusion process of $\lambda(t)$ as:

$$\frac{d\lambda_t}{\lambda_t} = \bar{\eta}_t dB_I^{(2)}(t).$$

(II.25)

Thus the process of λ_t is fully described by investors' disagreements, $\bar{\eta}_t \equiv \left[\eta_t^{(2)} - \eta_t^{(1)} \right]$, and

the second (optimistic) investor's perceived idiosyncratic Brownian risk $B_I^{(2)}(t)$. Using this stochastic weight process, we can write the consumption goods clearing condition as:

$$\overset{*}{c}_1(\xi^{(2)}(t)/[\lambda_2\lambda(t)], t) + \overset{*}{c}_2(\xi^{(2)}(t)/\lambda_2, t) = D(t). \tag{II.26}$$

Since the risk aversion coefficient, γ, is the same across two investors, the stochastic discount factor for each investor is obtained as follows:

$$\begin{aligned}
\frac{\xi_t^{(2)}}{\lambda_2} &= D_t^{-\gamma}\left(\frac{1}{X_t}\right)^{1-\gamma}\left[1+\left(\frac{1}{\lambda_t}\right)^{-(1/\gamma)}\right]^{\gamma}, \\
\frac{\xi_t^{(1)}}{\lambda_1} &= \frac{\xi_t^{(2)}}{\lambda_2}\frac{1}{\lambda_t}.
\end{aligned} \tag{II.27}$$

III.   Theoretical Results

We now derive equilibrium prices and quantities of the economy. Because $\xi^{(1)}$ and $\xi^{(2)}$ are linked through the λ_t process, which is the Radon-Nikodym derivative between the two investors' perceived probability measures, it is sufficient to compute the price of an asset with the share, s_t, using the second investor's state price density. The equilibrium price-dividend ratio of asset s is derived as follows:

Proposition 1. *The equilibrium stock price with the share process s_t is given by:*

$$\frac{P_s(t)}{D_s(t)} = \left[\beta_{s,0} + \beta_{s,1}\left(\frac{\bar{H}}{H_t}\right) + \beta_{s,2}\left(\frac{\bar{s}^{(2)}}{s_t}\right) + \beta_{s,3}\left(\frac{\bar{s}^{(2)}}{s_t}\frac{\bar{H}}{H_t}\right)\right], \tag{III.1}$$

where coefficients $\beta_{s,k}$ are functions of average of investors' belief difference η_t, and parameters determining cash flow characteristics as follows:

$$\beta_{s,k} \equiv f_k(v_{s,I}, v_{s,A}, \overline{\overline{\eta}}_t, \gamma, h_1, h_2, \phi_s, \bar{s}^{(2)}), \tag{III.2}$$

for $k = 0, 1, 2, 3$ and $\overline{\overline{\eta}}_t$ is the time-series average of belief difference measure for the stock with the share, s_t. Details of the coefficients $\beta_{s,k}$'s are given in Appendix C. The quantities $\bar{s}/s_t$ and $\bar{H}/H_t$ are referred to the share ratio and the habit ratio, respectively.

Proof: See Appendix C.

Equation (III.1) states that the equilibrium price-dividend ratio of a stock with the share s_t depends on three main variables: the share ratio $(\bar{s}^{(2)}/s_t)$, the habit ratio $(\bar{H}/H_t)$, and the

interaction between the share ratio and the habit ratio. As coefficients $\beta_{s,k}$'s are different across different assets (depending on characteristics of the share s_t), we might have strong equilibrium cross sectional effects from the coefficients. In addition, the $\beta_{s,k}$ coefficients are nonlinear functions of the habit parameters (h_1 and h_2), long-run mean of the cash flow share, and the average value of belief differences. This raises an important issue to assess the sensitivity of the price-dividend ratio of a stock with respect to the key state variables, such as the share ratio. For instance, if $\beta_{s,2}$ and $\beta_{s,3}$ were positive constants, then as the share ratio increases, its price-dividend ratio will increase. However, in our case, the effect is more complicated as the price-dividend ratio is affected by the interaction between the share ratio and the habit ratio as well as the habit ratio.

[Insert Figure 1]

To verify this, we plot the time-series of the share ratio and the price-dividend ratio of value and growth U.S. stocks during 1983-2011 period. Figure 1 suggests that the relation between the share ratio and the price-dividend ratio is nonlinear. Although the overall trend of the two variable appears to have a weakly positive correlation, negative correlations often prevail at business cycle frequencies and this pattern is stronger for value stocks. In addition, the share ratios of value stocks are much higher that those of growth stocks. This suggests that simple models linking the cash flow shares and the price-dividend ratios face difficulties in explaining time-series and cross-sectional features of stock returns.

Similarly, we compute the expected excess return of an asset with share s_t.

Proposition 2. *Equilibrium return process for a stock with the share process, s_t is given by*

$$dR_s = \mu_{R_s}^{(2)}dt + \sigma_{R_s,A}dB_A + \sigma_{R_s,I}dB_I^{(2)}, \tag{III.3}$$

where $\mu_{R_s}^{(2)}$ is the expected excess return, i.e., $E_t^{(2)}[dR_{s,t}]$, and it is given by:

$$E_t^{(2)}[dR_{s,t}] = \left[\frac{D_s(t)}{P_s(t)}\right]\left[\mu_{s,t}^{A,I} + \mu_{s,t}^I\right], \tag{III.4}$$

where

$$\mu_{s,t}^{A,I} \equiv \beta_{s,0}\left(\sigma_{D,A} + \sigma_{s,A}(t)\right)\left(\sigma_{D,A} - h_2\right) + \beta_{s,1}\left(\sigma_{D,A} + \sigma_{s,A}(t) - h_2\right)\left(\sigma_{D,A} - h_2\right)\left[\frac{\bar{H}}{H_t}\right]$$

$$+ \beta_{s,2}\sigma_{D,A}\left(\sigma_{D,A} - h_2\right)\left[\frac{\bar{s}^{(2)}}{s_t}\right] + \beta_{s,3}\left(\sigma_{D,A} - h_2\right)^2\left[\frac{\bar{s}^{(2)}}{s_t}\frac{\bar{H}}{H_t}\right], \tag{III.5}$$

$$\mu_{s,t}^I \equiv -\frac{1}{2}\sigma_{s,I}(t)\bar{\eta}_t\left(\beta_{s,0} + \beta_{s,1}\frac{\bar{H}}{H_t}\right),$$

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and $\left(\beta_{s,0} + \beta_{s,1}\frac{\bar{H}}{H_t}\right) > 0.$

Proof: See Appendix C

Similar to Proposition 1, the individual equilibrium expected excess return depends on three variables. As was shown previously, we have equilibrium cross sectional effects from $\beta_{s,k}$ coefficients, idiosyncratic cash flow risk $\sigma_{s,I}(t)$, and the share ratio $\bar{s}^{(2)}/s_t$. By carefully grouping components, we can decompose the equilibrium expected excess return into two parts. The first one is mixed with both aggregate and idiosyncratic cash flow risks, $\mu_{s,t}^{A,I}$, and the other is purely an idiosyncratic cash flow risk part, $\mu_{s,t}^{I}$ which depends on the interaction between the idiosyncratic cash flow risk and investors' belief differences. We show that $\mu_{s,t}^{I}$ is positive because $-\sigma_{s,I}(t)\bar{\eta}_t$ is positive. This implies that *the idiosyncratic cash flow risk can positively affect the equilibrium individual expected excess return.* The reason this happens is that the idiosyncratic cash flow risk is priced in equilibrium through the investors' belief differences. This is captured by the covariance between the idiosyncratic shock to the stochastic discount factor and the idiosyncratic shock to the share process, which is $\sigma_{s,I}(t)\bar{\eta}_t$. In equilibrium, this covariance includes $-(1/2)\sigma_{s,I}(t)\bar{\eta}_t$. Thus, in our model, belief differences can induce a positive risk premium, particularly, in light of idiosyncratic cash flow risks. Nevertheless, the overall effect of the share ratio on the expected excess return from the other terms is uncertain, according to the equations (III.4) and (III.5). Before we quantitatively assess the model, the following figure empirically shows how the share ratios are associated with future stock returns sorted by the book-to-market ratio.

[Insert Figure 2]

Figure 2 shows that there is a positive relation between share ratios and the returns of portfolios in value deciles. Because Figure 1 shows a negative relation between the share ratio and the price-dividend ratio at business cycle frequencies, these two empirical observations suggest that the share ratio measures a source of risk in the short run. Previous studies such as Menzley, Santos, and Veronesi (2004) and Avramov, Cederburg, and Hore (2012) also report that the share ratio is positively related to the expected excess return. In these studies, the share ratio is viewed as the expected future dividend growth rate to interpret that higher expected returns result from higher discount risk. However, we note that the difference in the share ratio comes from both the difference in the long-run mean of the share and the difference in the current share. If the long-run mean is lower and the current share is even lower, this makes the share ratio high, but this stock is risky and distressed in terms of current and future cash flows. Furthermore, in this case, if the speed of catching up to its long-run mean is slow, a high share ratio can reflect a serious cash flow risk.

Recently, many asset pricing studies have investigated the relative contribution of the discount rate risk component and the cash flow risk component to the cross sectional variation of stock returns.[6] Following this fashion, we attempt to decompose the equilibrium individual expected excess return into the discount rate risk component and the cash flow risk component.

We first compute the cash flow risk premium and define the discount rate risk premium as the difference between the equilibrium expected excess return and the return driven by the cash flow risk. In order to extract the cash flow risk component, we pay attention to the effect of changes in individual cash flows on the expected excess return. The cash flow risk component in the equilibrium expected excess return is directly affected by changes to individual cash flows, $D_s(t)$, and indirectly affected by changes in s_t.[7] A direct cash flow effect can be obtained by investigating the price elasticity with respect to the cash flow D_s:

$$\frac{\partial P_s(t)/P_s(t)}{\partial D_s(t)/D_s(t)} = \left[\beta_{s,0} + \beta_{s,1}\left(\frac{\bar{H}}{H_t}\right)\right]\left(\frac{D_s(t)}{P_s(t)}\right). \tag{III.6}$$

The indirect effect caused by s_t is embedded in the share ratio, $\bar{s}/s_t$. Thus, the expected return that is associated with the share ratio:

$$\beta_{s,2}\sigma_{D,A}\left(\sigma_{D,A} - h_2\right)\left(\frac{\bar{s}^{(2)}}{s_t}\right)\left(\frac{D_s(t)}{P_s(t)}\right)$$

is regarded as a part of the cash flow risk premium. Putting these together, we define the cash flow risk component, $\mu_{s,t}^{CF}$ as:

$$
\begin{aligned}
\mu_{s,t}^{CF} \equiv \mu_{s,t}^{I} &+ \left[\beta_{s,0} + \beta_{s,1}\left(\frac{\bar{H}}{H_t}\right)\right]\left(\frac{D_s(t)}{P_s(t)}\right)\left(\sigma_{D,A} + \sigma_{s,A}(t)\right)\left(\sigma_{D,A} - h_2\right) \\
&+ \beta_{s,2}\sigma_{D,A}\left(\sigma_{D,A} - h_2\right)\left(\frac{\bar{s}^{(2)}}{s_t}\right)\left(\frac{D_s(t)}{P_s(t)}\right).
\end{aligned} \tag{III.7}
$$

As a result, the discount rate risk component, $\mu_{s,t}^{DR}$ is defined as the residual as follows:

$$\mu_{s,t}^{DR} \equiv E_t^{(2)}[dR_{s,t}] - \mu_{s,t}^{CF}. \tag{III.8}$$

The following proposition summarizes the above result.

Proposition 3. *The equilibrium expected excess return of a stock with the share, s_t, is decom-*

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[6]See Campbell and Vuolteenaho (2004), Campbell, Polk, and Vuolteenaho (2010), Lettau and Wachter (2007), Santos and Veronesi (2010), as well as many others for the study of the relative importance of the discount rate risk component and the cash flow risk component in explaining the cross sectional return variation.

[7]When the share, s_t, changes, the individual cash flow, D_s, also changes.

posed into the discount rate risk return and the cash flow risk return:

$$E^{(2)}[dR_{s,t}] = \mu_{s,t}^{DR} + \mu_{s,t}^{CF}. \tag{III.9}$$

As a special case of an individual equilibrium quantity when the share is one, $s_t \equiv 1$, without investor belief differences, the equilibrium expected return of the market portfolio is given as follows:

Proposition 4. *The aggregate cash flow corresponds to the case of $s_t \equiv 1$. By applying $s_t \equiv 1$ and no individual belief differences, we get the approximate equilibrium expected excess return of the market portfolio as:*

$$E_t[dR_{M,t}] = (\sigma_{D,A} - h_2)^2 + \frac{h_2}{h_1}(\sigma_{D,A} - h_2)\frac{D_t}{P_t}. \tag{III.10}$$

Proof: See Appendix C.

Because there is no belief difference about the aggregate risk, the aggregate equilibrium quantities have no exposure to idiosyncratic cash flow risks. The aggregate equilibrium return of the market portfolio depends only on aggregate risk parameter ($\sigma_{D,A}$) and the parameters in the habit process, h_1 and h_2.

In summary, when idiosyncratic cash flow risk is priced in equilibrium through investor belief differences, it *positively* affects the individual equilibrium expected excess return. However, since all the coefficients in the equilibrium price-dividend ratio and expected excess return are (nonlinear) functions of the share s_t, investors' belief difference, η_t, as well as parameters determining firms' cash flow characteristics, it is hard to predetermine the effects of the main variables. We investigate the effects of the main variables on the equilibrium expected excess return both in the cross-section and in the time-series in the next section.

IV. Quantitative Analysis

A. Data

In this subsection, we construct cash flow data of individual assets, measures of investor belief differences, and corresponding individual asset returns covering the period 1983-2011. We begin with investor belief differences. We use earnings forecasts (EPS forecasts) from Institutional Brokers' Estimate System (I/B/E/S) to extract investors belief differences. In particular, we use monthly EPS forecasts with the most recent forecasts from the I/B/E/S

STATSUM data set. Each month, we find the latest analysts' forecasts available. If missing values exist in EPS forecasts, we replace them with the previously available forecast values. At the end of each forecast period, we take the standard deviation of EPS forecasts. If there is more than one report on the standard deviation, we average them. We then compute the coefficient of variation of EPS forecasts, i.e., the standard deviation of EPS forecasts divided by the mean EPS forecasts.[8] We use this quantity as an empirical measure for belief differences, denoted as BD. Our measure is similar to Diether, Malloy, and Scherbina (2005) and Yu (2011).[9] Note that at each point in time, the belief difference measure is determined prior to the actual report date of an asset's return. Thus, we have a stock's belief difference measure whose value is determined in the previous month to make sure that the belief difference measure is formed only using the information available to investors in each period.

Recall that the belief difference measure proposed in the theory, $\bar{\eta}_t$ is defined as the difference between the pessimistic investor's long-run mean of the cash flow share and that of the optimistic investor and hence its sign is negative. Therefore, to be consistent with the empirical measure based on dispersion, we put a minus in front of $\bar{\eta}$. In addition, the empirical measure uses the earnings per share and the theoretical measure is based upon the cash flow share. We construct $-\bar{\eta}_t$ for the purpose of simulation, by projecting the share ratio onto the EPS and using the estimated coefficients times the EPS forecast as the predicted share.

For stock price and return data, we use the data from the Center for Research in Security Prices (CRSP) and the COMPUSTAT. For the benchmark, we use both the CRSP-COMPUSTAT merged data set and the CRSP-COMPUSTAT-I/B/E/S merged data set from January, 1983 to December, 2011. The latter data set is the one incorporating the belief difference measure. One caveat of the CRSP-COMPUSTAT-I/B/E/S merged data set is the coverage of stocks and due to a low coverage of stocks in I/B/E/S data set, we have about 43% of the whole universe of CRSP stocks.

Regarding the cash flows of individual stocks, we follow Stephens and Weisbach (1998) and Grullon and Michaely (2002) to use accounting information in the COMPUSTAT data set. We define the cash flow as the sum of dividends and stock repurchases. We construct time-series of cash flows of each stock on a monthly basis. For dividends, we use returns with and without dividends to compute the dividend yield. Then, multiplying the dividend yield by the market capitalization of a stock is defined as the dividend. For market capitalization, we take the

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[8] We take the coefficient of variation of EPS forecasts to get rid of a size effect embedded in standard deviation as it is well known that the bigger the market size of a stock, the larger the standard deviation. In addition, this measure is invariant in its direction in portfolio deciles sorted by book-to-market when we replace the mean EPS forecasts with the book-equity price. This invariance has been explored in Diether, Malloy, and Scherbina (2005).

[9] This method is also similar to using a confidence interval as a measure of belief difference. See David (2008).

product between the mid-point price of a stock within a month and the outstanding number of shares of a stock. For stock repurchases, we first compute the decrease in numbers of the outstanding shares of a stock every month by using accounting information in COMPUSTAT. If this decrease is positive (number of shares decreases), then we multiply the decrease in number of shares with the mid-price of a stock in a month. If decreases in the outstanding number of shares are negative (numbers of shares increase), then we take zero as a stock repurchase. This is the same procedure as the one used in Stephens and Weisbach (1998). Finally, we take the moving sum of current and past two months of dividends with share repurchases as the monthly cash flow to mitigate the missing value problem which frequently occurs at the firm level. The aggregate cash flow is computed by summing individual cash flows across all individual firms. After constructing individual cash flows, we assign each stock its book-to-market ratio decile using the NYSE breakpoints. This sorting procedure can be found on Kenneth French's website.

[Insert Table I]
[Insert Figure 3]

Table I shows summary statistics of the market portfolio and portfolios in value deciles. Panel A shows summary statistics of the market portfolio from 1983 to 2011. Panels B and C show the cross-sectional properties of average returns, average book-to-market ratio, average price-dividend ratio, and the Sharpe ratio of decile portfolios sorted by book-to-market from the CRSP-COMPUSTAT merged data and the CRSP-COMPUSTAT-I/B/E/S merged data, respectively.

Figure 3 displays the cross-section of stock returns from the CRSP-COMPUSTAT merged data set in Panel B of Table I. According to the table and figure above, the value premium is clearly pronounced from 1983 to 2011. When stocks are sorted based on the book-to-market ratio, stocks with high book-to-market ratio (value stocks) earn about 0.5% extra return per month over stocks with low book-to-market ratio (growth stocks) on average. A very similar pattern prevails in average returns across stocks sorted on the price-dividend ratio where stocks with high book-to-market ratios correspond to stocks with low price-dividend ratios. Given that our model considers a simple exchange economy, we regard firms with low (high) price-dividend ratios as value (growth) firms.

B. Calibrated Parameters

Aggregate parameters representing the investor preferences (γ, δ, h_1, and h_2) and the aggregate cash flows (μ_D and $\sigma_{D,A}$) are calibrated by matching the first and second aggregate moments to their data counterparts.

Tables II and III report the matched moments and the resulting calibrated values of the parameters. In Table II, Panel A refers to the moments at the monthly frequency and Panel B at the quarterly frequency. Both cases show nearly identical results. We use the monthly version as the baseline parameters. Details of the calibration method are described in Appendix B.

C. Estimation Results of Cash Flow Processes

In this subsection, we report the parameter estimates that characterize the cash flows of value and growth stocks in equations II.1 to II.4. We also compute the covariance between the share and the aggregate cash flow (θ^{CF}), average share ratio ($Avg(\bar{s}/s_t)$), and the ratio of the standard deviation of the share to its mean ($CV(s_t)$) to discuss economic implications. All the results are displayed in Tables IV and V. Panel A shows the result without merging the IBES data, whereas Panel B is the table incorporating the IBES data to show the degree of belief difference in portfolios with the book-to-market ratio deciles.

Table IV reveals several characteristics of cash flow risk associated with the value premium. First, the long-run mean of the share, $\bar{s}$, is higher in growth firms than in value firms. This is intuitively plausible because the long-run mean of the share represents a firm's long-run growth; and therefore, measures how well a firm is expected to perform in terms of total payout. Thus, consistent with the labels of value and growth firms, a growth firm is likely to have a higher long-run mean of the cash flow share $\bar{s}$ than a value firm.

How about the conditional expected growth in cash flow share or $E_t(ds_t/s_t)$? This is also a popular measure of firm growth and from the equation (II.1) $E_t(ds_t/s_t) = \phi_s(\bar{s}/s_t - 1)dt$ holds. Thus, both the share ratio ($\bar{s}/s_t$) and the mean reversion (ϕ_s) matter to measure the expected growth of payout.

Given this information, Table IV illustrates the estimate of the share ratio, $\bar{s}/s_t$, is *higher* in value firms than growth firms. To check the robustness of this observation, we measure several different estimates of the share ratio in value deciles: the average of the share ratio over time, the quantile values of the share ratio, and the average of the share ratio in each quantile group. Table V shows that the share ratio of value stocks is still higher than that of growth stocks in all cases. This appears to be counter-intuitive in that the value stocks are associated with a longer distance between $\bar{s}$ and s_t, suggesting a higher share growth. However, because the long-run mean of the share ($\bar{s}$) is lower for the value stock, higher share ratios of

the value stocks mean that the value stocks' current shares (s_t) are quite low, despite the low value of the long-run mean of shares. Thus, value firms are distressed and likely to suffer from lower profitability than growth firms, as reported in previous studies such as Fama and French (1992), Zhang (2005) and Choi, Johnson, Kim, and Nam (2013).

This leaves the mean reversion coefficient ϕ_s to account for the different expected growth in the cash flow share between value and growth firms. Table IV reports that the speed of mean reversion is clearly higher for the growth stocks, implying higher expected growths. Thus, value stocks have lower growth potentials ($\bar{s}$), even lower current cash flow shares (s_t), and slower mean reversion (ϕ_s). In addition, the share ratio ($\bar{s}/s_t$) can proxy for the degree of cash flow risk in the short run because mean reversion is going to occur rather slowly, especially for the value firms. This cross-sectional result is consistent with our time-series empirical finding in the previous section. The relation between the share ratio and the price-dividend ratio is negative at business cycle frequencies, though the long-run trend is weakly positive suggesting that the share ratio measures a short-term risk source.

Related, we find that a higher share ratio is associated with bigger cash flow fluctuations. The bottom of Panel A in Table IV displays the coefficient of variation of the share in the value decile. The coefficient of variation of a value firm is higher than that of growth firms. This implies that value firms have larger exposure to current cash flow fluctuations. Interestingly, the higher cash flow fluctuation in value stocks is mostly captured by higher idiosyncratic cash flow risk ($v_{s,I}$) of the value stocks, as aggregate cash flow risks ($v_{s,A}$) do not significantly differ in value decile.

Lastly, we can infer an important aspect of consumption risk from belief differences in the data. The last row of Panel B in Table IV shows that belief differences (BD) are positively associated with the share ratio or the belief difference is higher in case of the value stocks. In our theory, equilibrium consumption sharing rules are exposed to both aggregate and idiosyncratic risk.[10] Aggregate risk is the same as the risk of the aggregate cash flow process, but the idiosyncratic consumption risk depends on investors' belief differences. This implies that value firms induce larger idiosyncratic consumption risk because they are exposed to higher investors' belief differences through which idiosyncratic cash flow risk translates into equilibrium consumption. Therefore, investors can be subject to larger idiosyncratic consumption risks when they invest in value firms which result from higher idiosyncratic cash flow risk as well as larger belief dispersion.

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[10]See the internet appendix for additional equilibrium results.

D. Simulation Results

In this subsection, we describe the simulation method and results. We first compute the unconditional covariance between the share process and the aggregate cash flow process for 10 book-to-market sorted portfolios. This covariance yields the fundamental aggregate cash flow risk, θ_s^{CF}. Dividing this by the diffusion coefficient of the aggregate cash flow process, $\sigma_{D,A}$, enables us to pin down individual aggregate cash flow risk parameter, $v_{s,A}$, as shown in Appendix A.[11] With this estimated parameter, we proceed to compute the total variability of the individual cash flow process that is defined by the product of the share process and the aggregate cash flow process. The total variability of the individual cash flow process gives us a restriction by which we can compute the individual idiosyncratic cash flow risk parameter, $v_{s,I}$. Specifically, as shown in Appendix A, if we evaluate the total variability of an individual cash flow at the long-run mean of the share, then we can extract the idiosyncratic cash flow risk parameter, $v_{s,I}$. Finally, we use the identification condition of the share process from Appendix A to compute $v_{(1-s),A}$ and $v_{(1-s),I}$. Once these parameters are retrieved, we estimate the mean reversion coefficient of the share process, using the generalized least square method using the fact that $\sigma_{s,A}$ and $\sigma_{s,I}$ are computed from their parametric restrictions.

We simulate 200 firms for 5,000 months and report results with the last 2,000 months. In so doing, we put the first set of values of ϕ_s, $v_{s,A}$, and $v_{s,I}$ in Panel A of Table IV on the first 20 firms (growth firms) and repeat the procedure for the next set of firms. .

[Insert Figure 4]
[Insert Figure 5]

Figures 4 and 5 show simulated sample paths of the equilibrium price-dividend ratios, equilibrium expected excess returns, and the share ratios on value and growth firms respectively. In particular, Figures 4 and 5 correspond to their empirical counterparts Figure 1 and Figure2. The nonlinear relation between the share ratio and the price-dividend ratio is clear with a negative correlation in the short run and a positive link between the share ratio and future returns. Thus, our model can produce data compatible with the actual times series data.

Now, we investigate cross-sectional properties of the simulation results. Table VI and Figure 6 show the cross-sectional results of the simulated data. The equilibrium expected excess returns sorted by price-dividend ratios show a sizable value premium and resemble the empirical cross-sectional pattern in Figure 3. Sharpe ratios are increasing from growth to value firms, which is also consistent with the empirical evidence in Table I.

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[11]The diffusion coefficient of the aggregate cash flow process, $\sigma_{D,A}$, is estimated by using a maximum likelihood method.

[Insert Table VI]
[Insert Figure 6]

Additionally, the decomposition of the simulated average excess returns reveals that the returns of growth firms are mostly explained by the discount risk (DR), yet those of value firms are heavily dependent upon the cash flow risks (CF). Consistent with this observation, the coefficient of variation $(CV(s_t))$ is greater for the value firms and the sensitivity of stock price with respect to cash flow changes also increases along the value decile. Thus, our model produces a growth premium that results from discount risks, but incorporating cash flow risks generates a value premium consistent with the data. In a similar setting, Santos and Veronesi (2010) show that the value premium prevails only by making the aggregate cash-flow volatility abnormally high and the cash-flow volatility puzzle arises. On the other hand, our model emphasizes the role of belief differences related to idiosyncratic cash flows and the quantitative result shows the reasonable size of the value premium.

E. Value or Growth? Role of Belief Differences

If the belief differences explain the cross-sectional variations of stock returns, the simulated data without the belief differences on idiosyncratic cash flows should produce a growth premium. To verify this, we simulate the model by turning off the belief differences channel. Figure 7 shows the mean cross-sectional returns across the logarithm of the price-dividend ratios.

[Insert Figure 7]

The top panel of Figure 7 illustrates our baseline model and the bottom panel shows the case without belief differences.[12] Consistent with our prediction, the model generates a strong growth premium when there are no belief differences. In the case of no belief differences, the idiosyncratic cash flow risk can still be priced through the share process in equilibrium, as pointed out by Cochrane, Longstaff, and Santa-Clara (2008). However, as quantitatively shown by Santos and Veronesi (2010), the discount rate risk plays a major role in the cross-section to produce a counterfactual growth premium.

Thus, the comparison between our model and the one without belief differences highlights their importance. If the idiosyncratic cash flow risk is positively associated with belief differences in the cross-section, its effect can be magnified by the belief differences mechanism

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[12]The simulation with no belief differences is carried out using the same cash flow risk parameters in the value decile, but with the fixed mean-reversion coefficient of the share process at 0.09. This method of simulation is very similar to the one in Santos and Veronesi (2010).

even if fluctuations of idiosyncratic cash flows are relatively moderate. Recall that the value firms have higher idiosyncratic cash flow risk and higher belief differences empirically. Thus, they are exposed to more (idiosyncratic) cash flow risk than growth firms. According to our computations, when the idiosyncratic cash flow risk is priced in equilibrium, increased cash flow risk component dominates the discount rate risk component hence the value premium arises.

Lettau and Wachter (2007) study the value premium puzzle in this context using an exogenous stochastic discount factor that generates a time-varying risk premium. They consider value stocks as those that pay currently and growth stocks as those that will pay in the distant future. They assume that shocks to the state variables in the stochastic discount factor are uncorrelated with shocks to the aggregate dividend. As a result, shocks to the stochastic discount factor suppress the effect of the discount rate risk in determining the cross-section while matching aggregate moments. Since value stocks covary more with front-loaded cash flows and growth stocks covary more with cash flows far in the future, a value premium arises if investors fear fluctuations of front-loaded cash flows. Under a typical asset pricing model, such as an external habit formation of Campbell and Cochrane (1999), the two aforementioned shocks are perfectly negatively correlated which is the key to explain the aggregate equity premium. However, Lettau and Wachter (2007) show that given the negative correlation between the two shocks, a growth premium arises because the effect of the discount rate risk is strongly pronounced. This result is in line with Santos and Veronesi (2010) in that they generate the value premium by incorporating excessively high aggregate cash flow volatilities. Thus, both Lettau and Wachter (2007) and Santos and Veronesi (2010) raise an important issue faced by equilibrium asset pricing models.

Our model complements both Lettau and Wachter (2007) and Santos and Veronesi (2010). The most crucial assumption in Lettau and Wachter (2007) is that shocks to taste are uncorrelated with shocks to the aggregate cash flows. This differs from asset pricing models, such as Campbell and Cochrane (1999), Menzley, Santos, and Veronesi (2004), and other models with external habit formation, since these models produce time-varying risk premiums via current and past aggregate cash flows. In our model, the measure of investors' belief differences showing up in the stochastic discount factor has a zero correlation with shocks to aggregate cash flows. Hence, it is similar to shocks in stochastic discount factor of Lettau and Wachter (2007). Priced idiosyncratic cash flow risk increases the overall cash flow risk component which reduces the effect of discount rate risk such that the value premium arises. Therefore, our model provides potential microeconomic foundations to the reduced form of the stochastic discount factor proposed by Lettau and Wachter (2007). Additionally, our model explains the puzzle of magnified cash flow risk in Santos and Veronesi (2010). Their results show that individual aggregate cash flow risk does not differ much in the value decile so that a growth premium

arises as discount rate risk dominates. Our model offers an equilibrium pricing of idiosyncratic cash flow risk that increases with the overall cash flow risk component without challenging the empirical facts on stock returns.

V.   Conclusions

In this paper, we show that differences in investors' beliefs on firms' cash flows play an instrumental role in explaining the return of portfolios sorted by the book-to-market ratio. In the data, we find that aggregate cash flow risks do not differ much in value deciles, yet the fluctuation of cash flow shares increases from the growth stocks to the value stocks. This implies that idiosyncratic cash flow risk should be high for the value stocks. Our model with belief heterogeneity allows the idiosyncratic cash flow risk to be priced in equilibrium. Furthermore, our model states that idiosyncratic cash flow risk and belief difference are positively associated. The effect of idiosyncratic cash flow risk can be magnified along the value deciles, if the value stocks are more prone to divergence of opinions. That is, value stocks can have higher expected returns than the growth stocks due to the (idiosyncratic) cash flow risk along with higher belief difference. Our empirical result shows that the value stocks indeed have higher belief differences. We can quantitatively produce the size of the value premium observed in the data.

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Appendix A : Basics of the Share Process

For the model of individual cash flow share process, we follow Menzley, Santos, and Veronesi (2004). Thus the regularity conditions such as $s_t \geq 0$ and $\sum s_t = 1$ hold. Specifications of diffusion coefficients are similar to Menzley, Santos, and Veronesi (2004) but ours is simpler version. As mentioned before, this simpler specification allow us to deal with only one idiosyncratic risk in individual dividend processes while keeping the flexibility of modeling individual cash flow processes.

Following Menzley, Santos, and Veronesi (2004), restrictions are imposed to guarantee that dividends are positive and the share is positive. Since $\sigma_{s,j}$ and $\sigma_{(1-s),j}$ for $j = A, I$ are invariant to adding the same vector to each v_s or $v_{(1-s)}$, we can normalize v_s and $v_{(1-s)}$ so that , as in Menzley, Santos, and Veronesi (2004), we have the following equation (identification condition):

$$\bar{s}\mathbf{v}_s + (1 - \bar{s})\mathbf{v}_{(1-s)} = 0, \tag{1}$$

where $\mathbf{v}_i$ is the row vector of $v_{i,A}$ and $v_{i,I}$ for $i = s, (1 - s)$.

We now derive the individual cash flow process. Given the share process, an individual cash flow $D_s(t)$ is defined as $D_s(t) \equiv s_t D_t$. By applying Itô lemma to $s_t D_t$, we can derive the diffusion process of an individual cash flow $D_s(t)$:

$$\frac{dD_s(t)}{D_s(t)} = \mu_{D_s}(t)dt + \sigma_{D_s,A}(t)dB_A(t) + \sigma_{D_s,I}(t)dB_I(t), \tag{2}$$

where

$$
\begin{aligned}
\mu_{D_s}(t) &\equiv \mu_D + \phi_s\left(\frac{\bar{s}}{s_t} - 1\right) + \theta_s^{CF} - s_t\theta_s^{CF} - (1 - s_t)\theta_{(1-s)}^{CF}, \\
\sigma_{D_s,A}(t) &\equiv \sigma_{D,A} + \sigma_{s,A}(t), \\
\sigma_{D_s,I}(t) &\equiv \sigma_{s,I}(t),
\end{aligned}
\tag{3}
$$

with $\theta_s^{CF} \equiv \sigma_{D,A}v_{s,A}$ and $\theta_{(1-s)}^{CF} \equiv \sigma_{D,A}v_{s,A}$. The covariance between share and the aggregate dividend(consumption) growth is given by

$$Cov_t\left(\frac{ds_t}{s_t}, \frac{dD_t}{D_t}\right) = \theta_s^{CF} - \left[\theta_s^{CF}s_t + \theta_{(1-s)}^{CF}(1 - s_t)\right]. \tag{4}$$

By computing the unconditional covariance from the data, we obtain:

$$E\left[Cov_t\left(\frac{ds_t}{s_t}, \frac{dD_t}{D_t}\right)\right] = v_{s,A} \cdot \sigma_{D,A} \equiv \theta_s^{CF}, \tag{5}$$

due to the identification condition of the share process (1). Thus we can pin down $v_{s,A}$ by computing unconditional covariance between the share process and aggregate cash flow process in the data. Note that the conditional variance of individual cash flow process is given by

$$var_t\left(\frac{dD_s(t)}{D_s(t)}\right) = \left[\sigma_{D,A} + \sigma_{s,A}(t)\right]^2 + \left[\sigma_{s,I}(t)\right]^2. \tag{6}$$

When the conditional variance is evaluated at $s_t = \bar{s}$, we get

$$\left(\sigma_{D,A} + v_{s,A}\right)^2 + \left(v_{s,I}\right)^2. \tag{7}$$

Since $v_{s,A}$ is already pinned down from unconditional covariance between the share process and the aggregate

cash flow process, one can recover $v_{s,I}$ based on a individual total cash flow volatility. Finally by using (1), we have:

$$v_{(1-s),j} = -\frac{\bar{s}v_{s,j}}{1-\bar{s}}, \quad j = A, I,$$ (8)

for both aggregate and idiosyncratic terms.

Appendix B : Calibration

In this appendix, we explain the calibration of our model. The calibration is applied to the aggregate moments which include: the mean aggregate market excess return, aggregate market volatility, the mean aggregate price-to-dividend ratio, the Sharpe ratio, the mean riskless rate, and the volatility of riskless rate. After the computing the aggregate price-dividend ratio, we obtain the diffusion process of aggregate excess market return as follows:

$$dR_{M,t} = \mu_{R_M}dt + \sigma_{R_M,A}dB_A,$$ (1)

where

$$
\begin{aligned}
\mu_{R_M} &\equiv (\sigma_{DA} - h_2)^2 + \frac{h_2}{h_1}(\sigma_{DA} - h_2)\frac{D_t}{P_t}, \\
\sigma_{R_M,A} &\equiv (\sigma_{DA} - h_2) + 2\frac{h_2}{h_1}\frac{D_t}{P_t}.
\end{aligned}
$$ (2)

Note that the stochastic discount factor is also represented by H_t. Thus, when we match the unconditional theoretical aggregate moments to their sample counterpart, we need the probability density function of H_t. The stationary density function of H_t is given by:

$$f(H) = \frac{\exp\left[-2b\left(\frac{\bar{H}}{H}\right)\right] \times H^{-2b-2}}{\int_0^\infty \exp\left[-2b\left(\frac{\bar{H}}{h}\right)\right] \times h^{-2b-2}},$$ (3)

where $b \equiv h_1/h_2^2$. By using the stationary density $f(H)$, we can compute $E[dR_t]$, $E[r_f]$, $E\left[\frac{P_t}{D_t}\right]$, $E\left[\sigma_{r_f}^2\right]$, and $\frac{E[dR_{M,t}]}{E[dR_{M,t}^2]}$. We proceed by matching the mean aggregate price-dividend ratio first and then matching other aggregate moments to their sample counterpart; such as the average of the aggregate excess return, the average riskless rate, the volatility of riskless rate, and the aggregate Sharpe ratio. In matching first and second moments of the aggregate excess return, we focus on matching the Sharpe ratio.

Appendix C : Proofs

Derivation of λ_t process. By applying Itô's lemma to the definition of λ_t process, we get:

$$\frac{d\lambda(t)}{\lambda(t)} = \left[-r_t + \mu_{\xi^{(1)-1}}(t) - \theta_A^2(t) - \theta_I^{(1)}(t)\theta_I^{(2)}(t)\right]dt + \theta_I^{(1)}(t)dB_I^{(1)}(t) - \theta_I^{(2)}(t)dB_I^{(2)}(t),$$ (1)

where

$$\mu_{\xi^{(1)}-1}(t) \equiv r_t + \theta_A^2(t) + \theta_I^{(1)^2}(t). \tag{2}$$

Diffusion terms are expressed as $\bar{\eta}_t dB_I^{(2)}(t) - \bar{\eta}_t \theta_I^{(1)}(t) dt$. And the drift term in (1) and $-\bar{\eta}_t \theta_I^{(1)}(t) dt$ are summed up to zero since $\bar{\eta}_t = \eta_t^{(2)} - \eta_t^{(1)} = \theta_I^{(1)}(t) - \theta_I^{(2)}(t)$; therefore:

$$\frac{d\lambda_t}{\lambda_t} = \bar{\eta}_t dB_I^{(2)}(t). \tag{3}$$

$\square$

Derivation of market prices of risks. Since the financial market is dynamically complete, we can use price processes of the market portfolio and the asset with the share process, $s_{t,}$, for determining the market prices of risks; thus:

$$
\begin{aligned}
\begin{pmatrix} \theta_A \\ \theta_I^{(k)} \end{pmatrix} &= \begin{pmatrix} \sigma_{P_M,A} & 0 \\ \sigma_{P_s,A} & \sigma_{P_s,I} \end{pmatrix}^{-1} \begin{pmatrix} \mu_{P_M} - r \\ \mu_{P_s}^{(k)} - r \end{pmatrix} \begin{pmatrix} dB_A \\ dB_I^{(k)} \end{pmatrix} \\
&= \frac{1}{\sigma_{P_M,A}\sigma_{P_s,I}} \begin{pmatrix} \sigma_{P_s,I}(\mu_P - r) \\ -\sigma_{P_s,A}(\mu_P - r) + \sigma_{P_M,A}\left(\mu_{P_s}^{(k)} - r\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\mu_{P_M}-r}{\sigma_{P,A}} \\ -\frac{\sigma_{P_s,A}}{\sigma_{P_s,I}}\frac{1}{\sigma_{P,A}}(\mu_{P_M} - r) + \frac{1}{\sigma_{P_s,I}}\left(\mu_{P_s}^{(k)} - r\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\mu_{P_M}-r}{\sigma_{P_M,A}} \\ -\frac{\sigma_{P_s,A}}{\sigma_{P_s,I}}\theta_A + \frac{1}{\sigma_{P_s,I}}(\mu_{P_s} - r) - \eta^{(k)} \end{pmatrix}.
\end{aligned} \tag{4}
$$

$\square$

Proof of Proposition 1. Note that the equilibrium stock price can be represented by using either one of the state price densities across investors due to the existence of the λ_t process. The equilibrium price of an asset with the share process s_t is represented as:

$$
\begin{aligned}
P_s(t) &= E_t^{(2)}\left[\int_t^\infty \frac{\xi_\tau^{(2)}}{\xi_t^{(2)}} s_\tau D_t d\tau\right] \\
&= \frac{1}{\left(1 + \lambda_t^{1/\gamma}\right)^\gamma \left(\frac{1}{X_t}\right)^{1-\gamma} D_t^{-\gamma}} E_t^{(2)}\left[\int_t^\infty \left(1 + \lambda_\tau^{1/\gamma}\right)^\gamma \left(\frac{1}{X_\tau}\right)^{1-\gamma} D_\tau^{-\gamma} s_\tau D_\tau d\tau\right] \\
&= \frac{s_t D_t}{s_t \left(1 + \lambda_t^{1/\gamma}\right)^\gamma \left(\frac{D_t}{X_t}\right)^{1-\gamma}} E_t^{(2)}\left[\int_t^\infty s_\tau \left(1 + \lambda_\tau^{1/\gamma}\right)^\gamma \left(\frac{D_\tau}{X_\tau}\right)^{1-\gamma} d\tau\right] \\
&= \frac{s_t D_t}{q_t} E_t^{(2)}\left[\int_t^\infty q_\tau d\tau\right] \\
&= \frac{D_s(t)}{q_t} E_t^{(2)}\left[\int_t^\infty q_\tau d\tau\right],
\end{aligned}
$$

where

$$q_t \equiv s_t z_t H_t,$$
$$z_t \equiv \left(1 + \lambda_t^{1/\gamma}\right)^\gamma, \tag{4}$$
$$H_t \equiv \left(\frac{D_t}{X_t}\right)^{1-\gamma}.$$

Now we define the diffusion process H_t. We name $\bar{H}/H_t$ as a "Habit Ratio". In Campbell and Cochrane (1999) and Santos and Veronesi (2010), the consumption surplus ratio, S_t^γ, is the proxy for the shock to aggregate discount rate as an element in stochastic discount factor. However, in our model, a similar variable H_t does not directly proxy the aggregate discount rate since it does not induce the time-varying risk preference. It represents the aggregate shock to stochastic discount factor, as well as an indicator of economic conditions. When this ratio is high, the economy is in a good condition and vice versa. By applying Itô's lemma to the process $H_t \equiv (D_t/X_t)^{1-\gamma}$, we get:

$$d\left(\frac{D_t}{X_t}\right)^{1-\gamma} = (1-\gamma)\left(\frac{D_t}{X_t}\right)^{1-\gamma}\left\{\left[\mu_D - \lambda\left(\frac{D_t}{X_t} - 1\right) - \frac{1}{2}\gamma\sigma_{DA}^2\right]dt + \sigma_{DA}dB_A\right\}. \tag{5}$$

Following Campbell and Cochrane (1999), Menzley, Santos, and Veronesi (2004), and Santos and Veronesi (2010), we assume a simpler process of H_t as follows[13]:

$$dH_t = h_1(\bar{H} - H_t)dt + h_2 H_t dB_A(t). \tag{6}$$

Note that the diffusion process of $\lambda_t^{1/\gamma}$ is given by:

$$\frac{d\lambda_t^{1/\gamma}}{\lambda_t^{1/\gamma}} = \alpha_1(t)dt + \alpha_2(t)dB_I^{(2)}, \tag{7}$$

where

$$\alpha_1(t) \equiv \frac{1}{2}\frac{1}{\gamma}\left(\frac{1}{\gamma} - 1\right)\bar{\eta}_t^2, \quad \alpha_2(t) \equiv \frac{1}{\gamma}\bar{\eta}_t. \tag{8}$$

Using this we get the diffusion process of $z_t \equiv \left(1 + \lambda_t^{1/\gamma}\right)^\gamma$, as follows:

$$\frac{dz_t}{z_t} = \left[\frac{1}{2}\gamma(\gamma-1)\left(\frac{x_t}{1+x_t}\right)^2\alpha_1(t) + \gamma\left(\frac{x_t}{1+x_t}\right)\alpha_2^2(t)\right]dt + \gamma\left(\frac{x_t}{1+x_t}\right)\alpha_2(t)dB_I^{(2)}, \tag{9}$$

where $x_t \equiv \lambda_t^{1/\gamma}$. For mathematical tractability, we approximate this process by simplifying $x_t/(1+x_t)$. Belief differences, η_t, determines the process of Radon-Nikodym derivative λ_t. In our quantitative, study we use 0.02 to 0.2 of η_t for firms in value decile. Simulation shows that $x_t/(1+x_t)$ is very similar to 0.5. By plugging $\alpha_1(t)$ and $\alpha_2(t)$ into the equation above and using the approximation that $x_t/(1+x_t) \approx 1/2$ (see Figure below), we have an approximate process of z_t as follows:
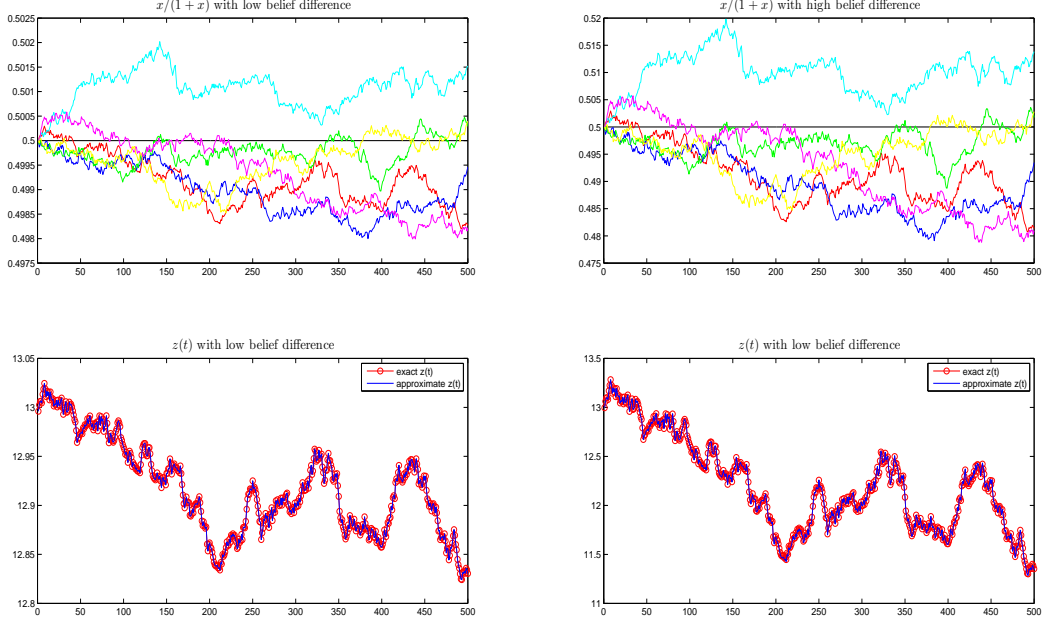
$$\frac{dz_t}{z_t} \approx \tilde{\alpha}_1(t)dt + \tilde{\alpha}_2(t)dB_I^{(2)}, \tag{10}$$

---

[13]The assumption on the process H_t is very similar to the one in Santos and Veronesi (2010).

where

$$\tilde{\alpha}_1(t) \equiv -\frac{1}{8}\left(1 - \frac{1}{\gamma}\right)\bar{\eta}_t^2,$$

$$\tilde{\alpha}_2(t) \equiv \frac{1}{2}\bar{\eta}_t. \tag{11}$$

Accordingly, we use this approximate z_t process for the rest of the proof. In order to get the diffusion process of q_t, we use the diffusion process of $z_t H_t$:

$$\frac{d(z_t H_t)}{z_t H_t} = \mu_{zH}dt + h_2 dB_A + \tilde{\alpha}_2 dB_I^{(2)}, \tag{12}$$

where $\mu_{zH} \equiv \tilde{\alpha}_1 + h_1\left[\bar{H}/H_t - 1\right]$. By using this process, we get the diffusion process of q_t as follows:

$$\frac{dq_t}{q_t} = \mu_q(t)dt + \left(\sigma_{s,A}(t) + h_2\right)dB_A + \left(\tilde{\alpha}_2(t) + \sigma_{s,I}(t)\right)dB_I^{(2)}, \tag{13}$$

where

$$\mu_q(t) \equiv \tilde{\alpha}_1(t) + h_1\left(\frac{\bar{H}}{H_t} - 1\right) + \phi_s\left(\frac{\bar{s}^{(2)}}{s_t} - 1\right) + h_2\sigma_{s,A}(t) + \tilde{\alpha}_2(t)\sigma_{s,I}(t)$$

. In drift term of dq_t, $q_t\mu_q(t)$, we have terms of q_t, $s_t z_t$ and $z_t H_t$ as:

$$q_t\mu_q(t) \equiv \left(\tilde{\alpha}_1(t) + \tilde{\alpha}_2(t)\sigma_{s,I}(t) + h_2\sigma_{s,A}(t) - h_1 - \phi_s\right)[q_t] + h_1\bar{H}\left[s_t z_t\right] + \phi_s\bar{s}^{(2)}\left[z_t H_t\right]. \tag{14}$$

Note that:

$$d(s_t z_t) = \left\{\left(\tilde{\alpha}_1(t) - \phi_s + \tilde{\alpha}_2(t)\sigma_{s,I}(t)\right)[s_t z_t] + \phi_s\bar{s}^{(2)}[z_t]\right\}dt + [\cdots]dB_A + [\cdots]dB_I^{(2)},$$

$$d(z_t H_t) = \left\{\left(\tilde{\alpha}_1(t) - h_1\right)[z_t H_t] + h_1\bar{H}[z_t]\right\}dt + [\cdots]dB_A + [\cdots]dB_I^{(2)}. \tag{15}$$

We have z_t, q_t, $s_t z_t$, and $z_t H_t$ variables in the diffusion processes of variables in the drift of dq_t. Thus, we take a vector process $y_t \equiv [z_t, q_t, s_t z_t, z_t H_t]'$ for computing equilibrium price-dividend ratio. y_t follows a diffusion process as follows:

$$dy_t = Y_1 y_t dt + \Sigma(y_t) d\mathbf{B'}^{(2)}, \tag{16}$$

where $\Sigma(y_t)$ is the appropriate matrix of diffusion coefficients, $Y_1 \equiv [y_{ij}]_{4\times 4}$ is the matrix of drift coefficients:

$$\begin{pmatrix} \tilde{\alpha}_1(t) & 0 & 0 & 0 \\ 0 & \tilde{\alpha}_1(t) + \tilde{\alpha}_2(t)\sigma_{s,I}(t) + h_2\sigma_{sA}(t) - h_1 - \phi_s & h_1\bar{H} & \phi_s\bar{s}^{(2)} \\ \phi_s\bar{s}^{(2)} & 0 & \tilde{\alpha}_1(t) + \tilde{\alpha}_2(t)\sigma_{s,I}(t) - \phi_s & 0 \\ h_1\bar{H} & 0 & 0 & \tilde{\alpha}_1(t) - h_1 \end{pmatrix}. \tag{17}$$

To avoid notational abuse, we denote Y_1 as:

$$\begin{pmatrix} y_{11} & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & y_{24} \\ y_{24} & 0 & y_{33} & 0 \\ y_{23} & 0 & 0 & y_{44} \end{pmatrix}, \tag{18}$$

where $y_{11} = y_{11}(\bar{\eta}_t)$, $y_{22} = y_{22}(\bar{\eta}_t, s_t)$, $y_{33} = y_{33}(\bar{\eta}_t, s_t)$, and $y_{44} = y_{44}(\bar{\eta}_t)$ since $\sigma_{s,A}(t)$ and $\sigma_{s,I}(t)$ are functions of s_t. For the feasibility of the computation of the expectation in the right hand side of above equation, we approximate the time-varying terms in y_{ij}'s so that the expectation can be done without technical difficulty. In order to approximate y_{ij}'s, especially y_{22} and y_{33}, we follow MSV (2004). In their approximation, MSV (2004) uses the normalization condition for parameters $v_{s,A}, v_{(1-s),A}, v_{s,I}$ and $v_{(1-s),I}$ in $\sigma_{s,A}$ and $\sigma_{s,I}$;

$$\bar{s}v_{s,A} + (1-\bar{s})v_{(1-s),A} = 0,$$
$$\bar{s}v_{s,I} + (1-\bar{s})v_{(1-s),I} = 0. \tag{19}$$

Associated with the condition (19), the parametric definition of $\sigma_{s,A}$ and $\sigma_{s,I}$ that are evaluated at $s_t = \bar{s}$, yields that both $\sigma_{s,A}$ and $\sigma_{s,I}$ are approximated by $v_{s,A}$ and $v_{s,I}$. Thus we have

$$y_{22} \approx \tilde{\alpha}_1(t) + h_2 v_{s,A} + \tilde{\alpha}_2(t)v_{s,I} - h_1 - \phi_s,$$
$$y_{33} \approx \tilde{\alpha}_1(t) + v_{s,I} + \tilde{\alpha}_2(t) - \phi_s. \tag{20}$$

Also $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ can be substituted by fixed mean value of $\bar{\eta}_t$, $\bar{\bar{\eta}}_t$. By using approximate value of y_{ij}, expected value of $E_t^{(2)}[q_\tau]$ can now be computed as follows.

$$E_t^{(2)}[q_\tau] = E_t^{(2)}[q_{t+\tau}] = e_2 E_t^{(2)}[y_{t+\tau}]$$
$$= e_2 \Psi(\tau) y_t, \tag{21}$$

where $e_2 \equiv (0,1,0,0)$,

$$\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1}, \tag{22}$$

Λ is the diagonal matrix with eigenvalues of Y_1 as its elements and U is the eigenvector matrix of Y_1.

Eigenvalues of the matrix, Y_1, are given by diagonal elements of Y_1. For mathematical tractability, we assume that all eigenvalues are negative following Menzley, Santos, and Veronesi (2004). Later our simulation study shows that all diagonal elements are indeed negative. Note that the conditional expectation $E_t[q_{t+\tau}]$ is

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given by:

$$E_t^{(2)}[q_{t+\tau}] = e_2 E_t^{(2)}[y_{t+\tau}]$$
$$= e_2 \Psi(\tau) y_t, \tag{23}$$

where $e_2 \equiv (0, 1, 0, 0)$,

$$\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1}, \tag{24}$$

Λ is the diagonal matrix with its elements being eigenvalues of Y_1 and U is the matrix of eigenvectors of Y_1. Note that eigenvalues of Y_1 are diagonal elements of Y_1. Thus U is given by:

$$U = \begin{pmatrix} u_{11} & 0 & 0 & 0 \\ u_{21} & 1 & u_{23} & u_{24} \\ u_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

where

$$u_{11} = \frac{y_{11} - y_{44}}{y_{23}},$$
$$u_{21} = \frac{y_{24}(2y_{11} - y_{33} - y_{44})}{(y_{11} - y_{22})(y_{11} - y_{33})},$$
$$u_{31} = \frac{y_{31}(y_{11} - y_{44})}{y_{23}(y_{11} - y_{33})}, \tag{26}$$
$$u_{23} = \frac{y_{23}}{y_{33} - y_{22}},$$
$$u_{24} = \frac{y_{24}}{y_{44} - y_{22}}.$$

The inverse matrix $U^{-1} \equiv V = [v_{ij}]$ is given by:

$$\begin{pmatrix} v_{11} & 0 & 0 & 0 \\ v_{21} & 1 & v_{23} & v_{24} \\ v_{31} & 0 & 1 & 0 \\ v_{41} & 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

where

$$v_{11} = 1/u_{11},$$
$$v_{21} = \frac{-u_{21} + u_{24} + u_{23}u_{31}}{u_{11}},$$
$$v_{23} = -u_{23},$$
$$v_{24} = -u_{24}, \tag{28}$$
$$v_{31} = \frac{-u_{31}}{u_{11}},$$
$$v_{34} = -u_{34},$$
$$v_{41} = -1/u_{11}.$$

Using these quantities, we can compute $\Psi(\tau)$ so that we get $E_t[q_{t+\tau}]$ as following:

$$E_t[q_{t+\tau}] = e_2\Psi(\tau)y_t = \Psi_1(\tau)z_t + \Psi_2(\tau)q_t + \Psi_3(\tau)s_t z_t + \Psi_4(\tau)z_t H_t, \tag{29}$$

where

$$
\begin{aligned}
\Psi_1(\tau) &= v_{11}u_{21}e^{y_{11}\tau} + v_{21}e^{y_{22}\tau} + v_{31}u_{23}e^{y_{33}\tau} + v_{41}u_{24}e^{y_{44}\tau}, \\
\Psi_2(\tau) &= e^{y_{22}\tau}, \\
\Psi_3(\tau) &= v_{23}e^{y_{22}\tau} + u_{23}e^{y_{33}\tau}, \\
\Psi_4(\tau) &= v_{24}e^{y_{22}\tau} + u_{24}e^{y_{44}\tau}.
\end{aligned} \tag{30}
$$

Therefore:

$$
\begin{aligned}
E_t^{(2)}\left[\int_0^\infty q_{t+\tau}d\tau\right] &= \int_0^\infty E_t^{(2)}[q_{t+\tau}]\,d\tau \\
&= \int_0^\infty e_2\Psi(\tau)y_t d\tau \\
&= \sum_{k=1}^4 \left[\int_0^\infty \Psi_k(\tau)d\tau\right]y_k(t),
\end{aligned} \tag{31}
$$

where $y_k(t)$ is the k-th row vector y_t. $\int_0^\infty \Psi_k(\tau)$'s are given by:

$$
\begin{aligned}
\int_0^\infty \Psi_1(\tau)d\tau &= \left[-\frac{v_{11}u_{21}}{y_{11}} - \frac{v_{21}}{y_{22}} - \frac{v_{31}u_{23}}{y_{33}} - \frac{v_{41}u_{24}}{y_{44}}\right], \\
\int_0^\infty \Psi_2(\tau)d\tau &= -\frac{1}{y_{22}}, \\
\int_0^\infty \Psi_3(\tau)d\tau &= -\frac{v_{23}}{y_{22}} - \frac{u_{23}}{y_{33}}, \\
\int_0^\infty \Psi_4(\tau)d\tau &= -\frac{v_{24}}{y_{22}} - \frac{u_{24}}{y_{44}}.
\end{aligned} \tag{32}
$$

The integrations above were conducted under the continuing assumption that all the eigenvalues are negative. Thus, the approximated equilibrium stock price with the share process s_t is given by:

$$
\begin{aligned}
P_s(t) &\approx \frac{D_s(t)}{q_t}\sum_{k=1}^4 \left[\int_0^\infty \Psi_k(\tau)d\tau\right]y_k(t) \\
&= D_s(t)\sum_{k=1}^4 \left[\int_0^\infty \Psi_k(\tau)d\tau\right]\left[\frac{y_k(t)}{q_t}\right] \\
&= D_s(t)\left\{\left[\int_0^\infty \Psi_1(\tau)d\tau\right]\frac{1}{s_t H_t} + \left[\int_0^\infty \Psi_2(\tau)d\tau\right] + \left[\int_0^\infty \Psi_3(\tau)d\tau\right]\frac{1}{H_t} + \left[\int_0^\infty \Psi_4(\tau)d\tau\right]\frac{1}{s_t}\right\} \\
&= D_s(t)\left\{\left[\int_0^\infty \Psi_2(\tau)d\tau\right] + \left[\int_0^\infty \frac{\Psi_1(\tau)}{\bar{s}^{(2)}}d\tau\right]\frac{\bar{s}^{(2)}}{s_t}H_t^{-1} + \left[\int_0^\infty \Psi_3(\tau)d\tau\right]H_t^{-1} + \left[\int_0^\infty \frac{\Psi_4(\tau)}{\bar{s}^{(2)}}d\tau\right]\frac{\bar{s}^{(2)}}{s_t}\right\} \\
&= D_s(t)\left[\beta_{0,t} + \beta_{1,t}\left(\frac{\bar{H}}{H_t}\right) + \beta_{2,t}\left(\frac{\bar{s}^{(2)}}{s_t}\right) + \beta_{3,t}\left(\frac{\bar{s}^{(2)}}{s_t}\frac{\bar{H}}{H_t}\right)\right],
\end{aligned} \tag{33}
$$

where β_j's are

$$\beta_{s,0} \equiv \int_0^\infty \Psi_2(\tau)d\tau, \; \beta_{s,1} \equiv \int_0^\infty \frac{\Psi_3(\tau)}{\bar{H}}d\tau, \; \beta_{s,2} \equiv \int_0^\infty \frac{\Psi_4(\tau)}{\bar{s}^{(2)}}d\tau, \; \beta_{s,3} \equiv \int_0^\infty \frac{\Psi_1(\tau)}{\bar{s}^{(2)}\bar{H}}d\tau. \tag{34}$$

The equilibrium price-dividend ratio of the shared stock is, hence, given by:

$$\frac{P_s(t)}{D_s(t)} = \left[\beta_{s,0} + \beta_{s,1}\left(\frac{\bar{H}}{H_t}\right) + \beta_{s,2}\left(\frac{\bar{s}^{(2)}}{s_t}\right) + \beta_{s,3}\left(\frac{\bar{s}^{(2)}}{s_t}\frac{\bar{H}}{H_t}\right) \right]. \tag{35}$$

$\square$

Proof of Proposition 2. First we find diffusion coefficients of $\frac{dP_{s_t}}{P_{s_t}}$ for investor 2. Applying Itô's lemma to $P_s(t)$ that was derived in the previous Proposition, we get diffusion coefficients of $\frac{dP_{s_t}}{P_{s_t}}$ as follows.

$$dB_A : \quad \left(\frac{D_s}{P_s}\right)\beta_{s,0}\left(\sigma_{D,A} + \sigma_{s,A}(t)\right) + \left(\frac{D_s}{P_s}\right)\beta_{s,1}\left(\sigma_{D,A} + \sigma_{s,A}(t) - h_2\right)\frac{\bar{H}}{H_t}$$
$$+ \left(\frac{D_s}{P_s}\right)\beta_{s,2}\sigma_{D,A}\left(\frac{\bar{s}^{(2)}}{s_t}\right) + \left(\frac{D_s}{P_s}\right)\beta_{s,3}\left(\sigma_{D,A} - h_2\right)\left(\frac{\bar{s}^{(2)}}{s_t}\frac{1}{H_t}\right),$$
$$dB_I^{(2)} : \quad \left(\frac{D_s}{P_s}\right)\sigma_{s,I}(t)\left(\beta_{s,0} + \beta_{s,1}\frac{\bar{H}}{H_t}\right).$$

The diffusion coefficients of a shared asset's excess return (defined as R_s and $dR_s \equiv \frac{dP_{s_t}+D_s(t)}{P_{s_t}} - r_t dt$) is the same as ones in $\frac{dP_{s_t}}{P_{s_t}}$. Note that in equilibrium the expected excess stock return $E_t[dR_s]$ is given by the negative of the inner product of the diffusion coefficient vector of dR_s and the diffusion coefficient vector of the state price density $\xi_t^{(2)}$ since the equilibrium return is defined by the covariance between the aforementioned two quantities. Applying Itô's lemma to $\xi_t^{(2)} = (1 + \lambda_t^{1/\gamma})^\gamma(1/X_t)^{1-\gamma}D_t^{-\gamma} = z_t H_t D_t^{-1}$ using the approximate diffusion process of z_t, gives:

$$d\xi_t^{(2)}/\xi_t^{(2)} = \mu_{\xi^{(2)}}dt + (h_2 - \sigma_{D,A})dB_A + \tilde{\alpha}_2(t)dB_I^{(2)}, \tag{36}$$

where $\mu_{\xi^{(2)}} = \tilde{\alpha}_1(t) + h_1(\bar{H}/H_t - 1) + \sigma_{D,A}^2 - \mu_D - h_2\sigma_{D,A}$. Since the expected excess return is determined by the negative of the sum of multiplications of diffusion coefficients given in the equation (36) with $\sigma_{D,A} - h_2$ and $-\tilde{\alpha}_2(t)$, respectively:

$$E_t^{(2)}[dR_{s,t}] \approx \left[\frac{D_s(t)}{P_s(t)}\right]\left[\mu_{s,t}^{A,I} + \mu_{s,t}^I\right], \tag{37}$$

where

$$\mu_{s,t}^{A,I} \equiv \beta_{s,0}\left(\sigma_{D,A} + \sigma_{s,A}(t)\right)\left(\sigma_{D,A} - h_2\right) + \beta_{s,1}\left(\sigma_{D,A} + \sigma_{s,A}(t) - h_2\right)\left(\sigma_{D,A} - h_2\right)\frac{1}{H_t}$$
$$+ \beta_{s,2}\sigma_{D,A}\left(\sigma_{D,A} - h_2\right)\frac{\bar{s}^{(2)}}{s_t} + \beta_{s,3}\left(\sigma_{D,A} - h_2\right)^2\frac{\bar{s}^{(2)}}{s_t}\frac{\bar{H}}{H_t}, \tag{38}$$
$$\mu_{s,t}^I \equiv -\sigma_{s,I}(t)\tilde{\alpha}_2(t)\left(\beta_{s,0} + \beta_{s,1}\frac{\bar{H}}{H_t}\right).$$

As was already shown above, the diffusion process of the return of an shared asset, $R_s(t)$, is given by:

$$dR_s = \mu_{R_s}^{(2)}dt + \sigma_{R_s,A}dB_A + \sigma_{R_s,I}dB_I^{(2)}, \tag{39}$$

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where $\mu_{R_s}^{(2)}$ is the expected excess return given above and both $\sigma_{R_s,A}$ and $\sigma_{R_s,I}$ are diffusion coefficients of dP_s/P_s given above. $\square$

Proof of Proposition 3. The equilibrium price dividend ratio of the market portfolio can be obtained as a special case of the equilibrium price of the individual shared asset. Applying Itô's lemma to the equilibrium aggregate price dividend ratio, we get the diffusion coefficient of dP_t/P_t as follows:

$$\left(\sigma_{D,A} - h_2\right) + \frac{h_2}{h_1}\left(\frac{D_t}{P_t}\right). \tag{40}$$

Aggregate expected excess return is given by the negative of product of the diffusion coefficient above and the aggregate market price of risk.

$$E_t\left[dR_{M,t}\right] = \left(\sigma_{D,A} - h_2\right)^2 + \frac{h_2}{h_1}\left(\sigma_{D,A} - h_2\right)\frac{D_t}{P_t}. \tag{41}$$

And the diffusion coefficient of the aggregate excess return R_t is given by:

$$dR_M = \mu_{R_M}dt + \sigma_{R_M,A}dB_A, \tag{42}$$

where μ_{R_M} is the expected excess return given above and $\sigma_{R_M,A}$ is the diffusion coefficient given above.

$\square$

Table I Summary Statistics of the Data – Panel A summarizes basic statistics for the market portfolio from 1983 to 2011 on a monthly basis. Mean return of the market, $\overline{R_M^e}$, is the average of excess returns on the market portfolio. r_f is the riskless rate of return. Return and volatility are expressed in percentage. Panel B and C summarize key cross sectional moments for book-to-market decile portfolios for CCM (CRSP-COMPUSTAT merged) data and CCIM (CRSP-COMPUSTAT-I/B/E/S merged) data respectively. Returns and volatilities are expressed in percentage.

Panel A: Summary statistics on the market portfolio

$\overline{R_M^e}$	$\overline{\sigma_{R_M^e}}$	Sharpe ratio	$\overline{r_f}$	$\overline{\sigma_{r_f}}$
0.57	4.57	0.126	0.36	0.22

Panel B: Statistics on decile portfolio CCM

Growth										Value
Portfolio	1	2	3	4	5	6	7	8	9	10
Mean Return	0.89	0.99	1.04	0.99	0.96	1.10	1.07	1.06	1.07	1.41
Mean B/M	0.15	0.309	0.415	0.514	0.614	0.72	0.84	0.99	1.22	2.43
Mean ln(P/D)	5.07	4.81	4.75	4.61	4.53	4.52	4.43	4.42	4.49	4.52
Sharpe Ratio	0.103	0.13	0.139	0.127	0.121	0.162	0.14	0.146	0.123	0.152

Panel C: Statistics on decile portfolio CCIM

Growth										Value
Portfolio	1	2	3	4	5	6	7	8	9	10
Mean Return	0.953	1.00	1.02	0.97	0.91	1.08	1.1	1.01	1.04	1.53
Mean B/M	0.167	0.307	0.416	0.513	0.615	0.721	0.841	0.986	1.218	2.08
Mean ln(P/D)	5.80	5.67	5.50	5.37	5.32	5.29	5.09	5.35	5.19	5.63
Sharpe Ratio	0.1139	0.1271	0.1307	0.1193	0.1075	0.1431	0.1356	0.1228	0.1126	0.1477

Table II Calibration – Panel A shows the calibration of aggregate moments using monthly data from 1983 to 2011. Panel B shows the calibration of aggregate moments using quarterly data from 1946 to 2011. Both calibrations use the same aggregate equilibrium equations in matching expected moments to their sample counterparts. Returns and volatilities are expressed in percentage. ∗ indicates the matched moments.

Panel A: Monthly data from 1983 to 2011

	Data	Calibration
Average Excess Return∗	0.57%	0.6%
Volatility of Excess Return	4.61%	9.0%
Sharpe Ratio	0.123	0.067
Average P/D ∗	93.36	93.36
Average Riskless Rate ∗	0.036%	0.036%
Volatility of Riskless Rate	0.22%	0.55%

Panel B: Quarterly data from 1946 to 2011

	Data	Calibration
Average Excess Return ∗	1.9%	1.9%
Volatility of Excess Return	8.3%	12.96%
Sharpe Ratio	0.229	0.1466
Average P/D ∗	93.36	93.36
Average Riskless Rate ∗	0.035%	0.035%
Volatility of Riskless Rate	0.22%	0.25%

Table III Calibrated Parameters – This table shows calibrated parameters that correspond to Panel A in table II.

μ_D	$\sigma_{D,A}$	γ	δ	h_1	h_2
0.02	0.13	3.7	0.9	0.0107	0.08851

Table IV Characteristics of decile portfolio – This table shows basic statistics of characteristics of decile portfolios sorted by book-to-market from 1983 to 2011 for CRSP-COMPUSTAT(CCM) and CRSP-COMPUSTAT-I/B/E/S(CCIM) respectively. θ^{CF} is an unconditional covariance between the share process and the aggregate cash flow process. $v_{s,A}$ is pinned down by the relation $\theta^{CF} = v_{s,A}\sigma_{D,A}$. $v_{s,I}$ can be computed from the identification condition imposed on the share process. Mean-reverting coefficients, ϕ_s, are estimated using generalized least square estimation using $\sigma_{s,A}$ and $\sigma_{s,I}$. Coefficient of variation of s_t, the ratio of the standard deviation of s_t to the mean of the share s_t for each portfolio is defined as $CV(s_t)$.

	Panel A: CCM, 1983 to 2011									
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
θ^{CF}	-0.028	-0.025	-0.030	-0.032	-0.028	-0.035	-0.027	-0.031	-0.031	-0.025
$v_{s,A}$	-0.184	-0.166	-0.195	-0.205	-0.181	-0.229	-0.177	-0.204	-0.201	-0.163
$v_{s,I}$	0.205	0.234	0.222	0.214	0.240	0.206	0.276	0.283	0.300	0.458
ϕ_s	0.068	0.077	0.120	0.078	0.118	0.123	0.123	0.133	0.096	0.039
$\bar{s}$	0.131	0.108	0.080	0.077	0.078	0.064	0.067	0.059	0.051	0.036
$\text{Avg}(\frac{\bar{s}}{s_t})$	1.249	1.214	1.140	1.209	1.224	1.208	1.315	1.290	1.444	2.140
$CV(s_t)$	0.415	0.40	0.31	0.39	0.431	0.40	0.614	0.499	0.713	1.08

	Panel B: CCIM, 1983 to 2011									
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
θ^{CF}	-0.027	-0.017	-0.029	-0.033	-0.030	-0.031	-0.026	-0.029	-0.028	-0.033
$v_{s,A}$	-0.176	-0.111	-0.186	-0.213	-0.197	-0.201	-0.166	-0.190	-0.184	-0.215
$v_{s,I}$	0.354	0.434	0.437	0.442	0.365	0.451	0.368	0.579	0.657	0.624
ϕ_s	0.14	0.08	0.041	0.033	0.022	0.038	0.025	0.016	0.02	0.011
$\bar{s}$	0.0039	0.0039	0.0032	0.0031	0.0037	0.0022	0.0026	0.0021	0.0025	0.0017
$\text{Avg}(\frac{\bar{s}}{s_t})$	1.3647	1.5656	2.0512	2.1637	2.4594	2.0390	2.5351	3.2580	3.5944	6.2508
$BD \ (\propto -\bar{\eta})$	0.045	0.053	0.081	0.085	0.102	0.126	0.174	0.160	0.273	0.419

Table V Share ratio of decile portfolio – This table shows the quantile values, the average values in each quantile group, and the percentage of share ratios less than 1 of share ratios of decile portfolios sorted by book-to-market from 1983 to 2011 in CRSP-COMPUSTAT(CCM) data set.

	CRSP-COMPUSTAT, 1983 to 2011									
	Growth									Value
Portfolio	1	2	3	4	5	6	7	8	9	10
Quantile value	0.821	0.829	0.867	0.867	0.798	0.819	0.900	0.814	0.791	0.817
	1.037	1.016	1.011	1.056	1.079	1.074	1.129	1.075	1.304	1.183
	1.476	1.442	1.228	1.326	1.450	1.373	1.760	1.594	1.794	2.779
	11.107	10.886	8.654	11.020	6.640	6.434	3.912	4.684	5.800	12.115
Average in Quantile	0.660	0.696	0.743	0.694	0.639	0.658	0.614	0.624	0.551	0.566
	0.922	0.909	0.942	0.961	0.960	0.961	1.007	0.952	1.084	0.989
	1.232	1.218	1.109	1.170	1.250	1.192	1.399	1.279	1.54	1.832
	2.184	2.032	1.765	2.012	2.048	2.023	2.239	2.306	2.601	5.172
% of share ratio < 1	45.4	49.1	46.8	42.2	40.8	40.2	36.8	42.0	32.5	37.4

Table VI Model Simulation – This table shows simulation results of the model. 200 firms for 5,000 months were simulated. Individual expected excess returns are generated according to the equation (III.4). Individual assets are sorted into decile portfolios based on simulated price-dividend ratios following (III.1). Other return quantities follow equations (III.7) and (III.8). For the simulation, we assign parameter values for firms following Table IV. Average individual aggregate cash flow risk parameters, v_{sA}, are from -0.184 to -0.163 and the average individual idiosyncratic cash flow risk parameters, v_{sI}, are from 0.205 to 0.458 across decile portfolios. Average mean-reverting coefficients, ϕ_s, are taken from 0.039 to 0.133 with the order in Table IV. Coefficient of variation of the share, $CV(s_t)$, is the ratio of the standard deviation of shares to the mean of shares in each portfolio.

| Portfolio	Growth									Value
	1	2	3	4	5	6	7	8	9	10
Average excess return	0.47%	0.48%	0.53%	0.57%	0.59%	0.67%	0.7%	0.74%	0.85%	1.6%
Average ln(P/D)	8.59	8.16	7.38	7.08	6.82	6.38	6.23	6.10	5.87	5.19
CF component	0.05%	0.07%	0.13%	0.17%	0.20%	0.29%	0.33%	0.37%	0.50%	1.28%
DR component	0.42%	0.42%	0.41%	0.40%	0.39%	0.37%	0.37%	0.36%	0.35%	0.32%
Average Share Ratio	1.012	1.013	1.013	1.013	1.012	1.015	1.017	1.017	1.019	1.044
Price elasticity w.r.t. CF	0.013	0.014	0.014	0.015	0.016	0.018	0.021	0.027	0.064	0.082
Sharpe ratio	0.07	0.072	0.077	0.081	0.083	0.089	0.091	0.095	0.108	0.187
$CV(s_t)$	0.113	0.122	0.12	0.122	0.121	0.118	0.133	0.137	0.14	0.20
Belief Difference $(-\eta)$	0.042	0.052	0.078	0.1	0.115	0.13	0.14	0.14	0.16	0.20
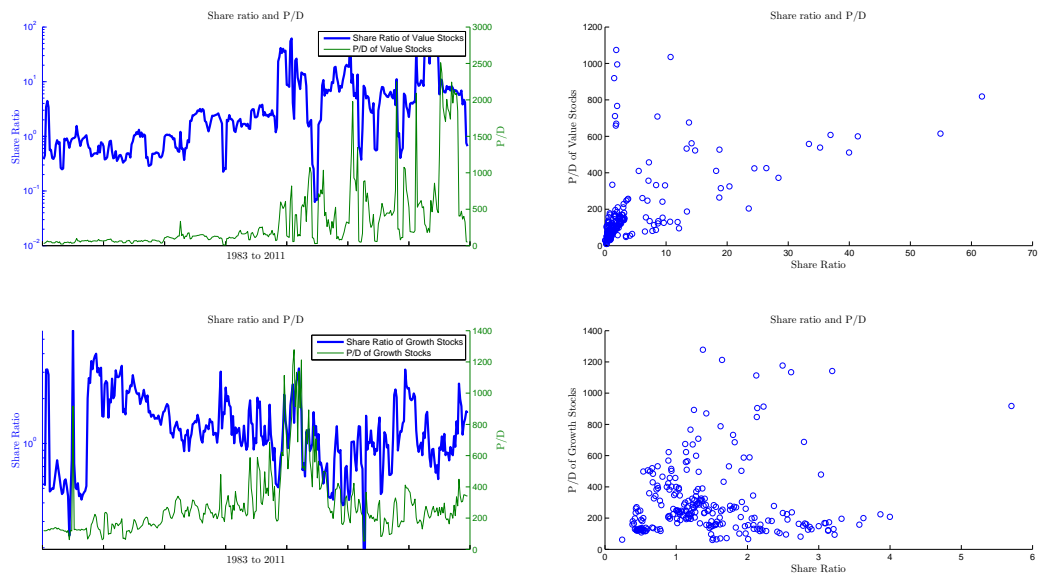
Figure 1. The Relation between Price-Dividend Ratio and Share Ratios between Value and Growth Stocks – Figures of time-series relation between price-dividend ratios and share ratios of value and growth stocks. First row is for the value portfolio and the second row is the case for the growth portfolio. The left column is the time-series relation between price-dividend ratios and share ratios and the right column is the scatter plot of share ratios and price-dividend ratios.
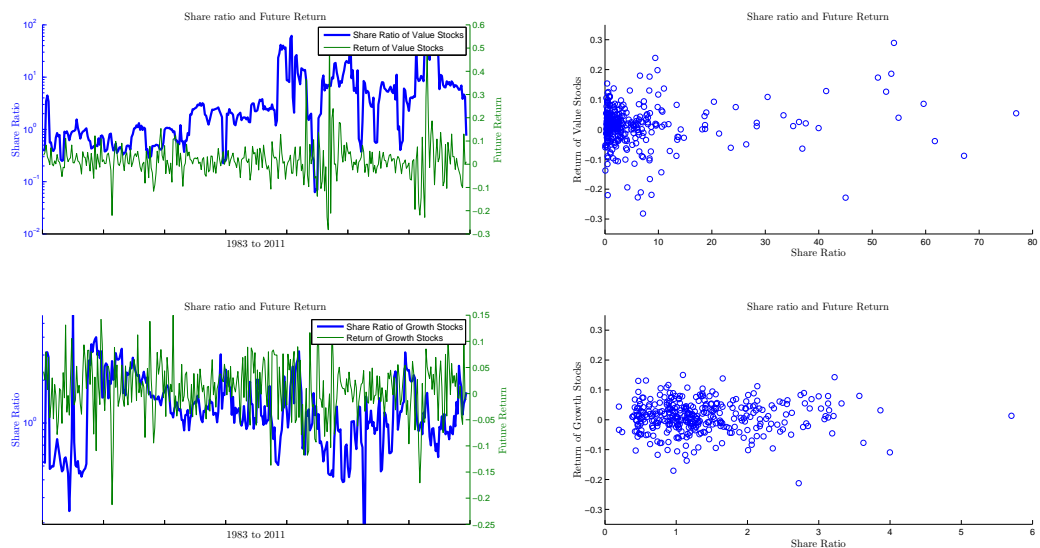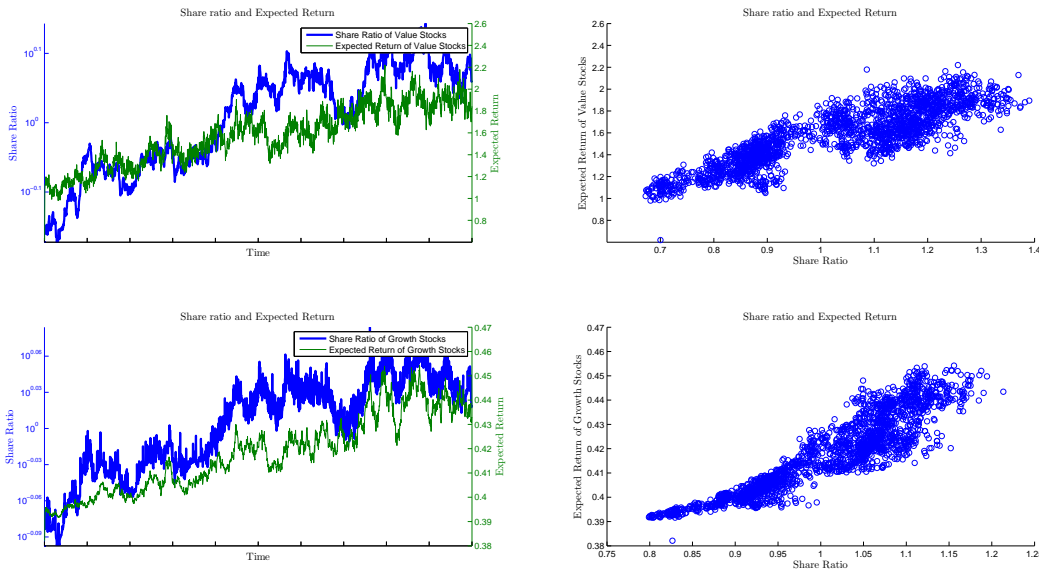

Figure 2. The Relation between Returns and Share Ratios between Value and Growth Stocks – Figures of time-series relation between returns and share ratios of value and growth stocks. First row is for the value portfolio and the second row is the case for the growth portfolio. The left column is the time-series relation between 1-month ahead returns and share ratios and the right column is the scatter of 1-month ahead returns and share ratios.
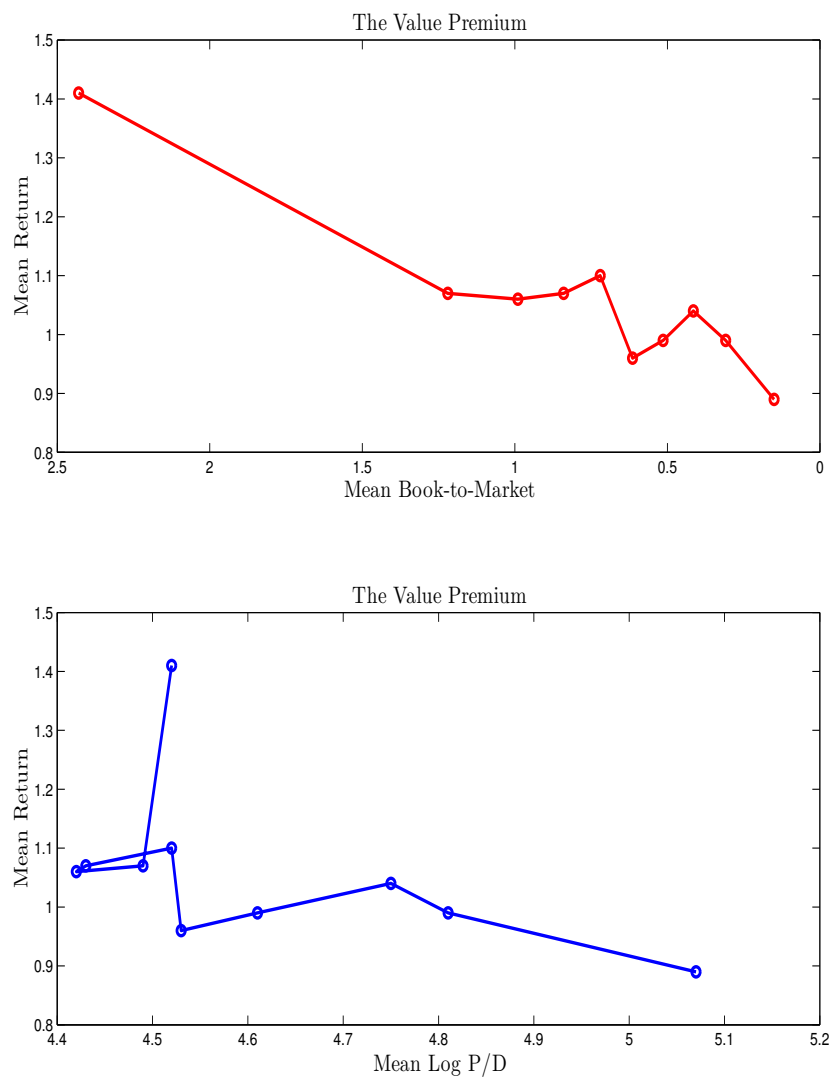
Figure 3. Value Premium in the Data – January, 1983 to December, 2011. The data is adopted from CRSP-COMPUSTAT merged data. Stocks are sorted based on both Book-to-Market ratios and Price-to-Dividend ratios whose breakpoints are given in Kenneth French's Data Library.
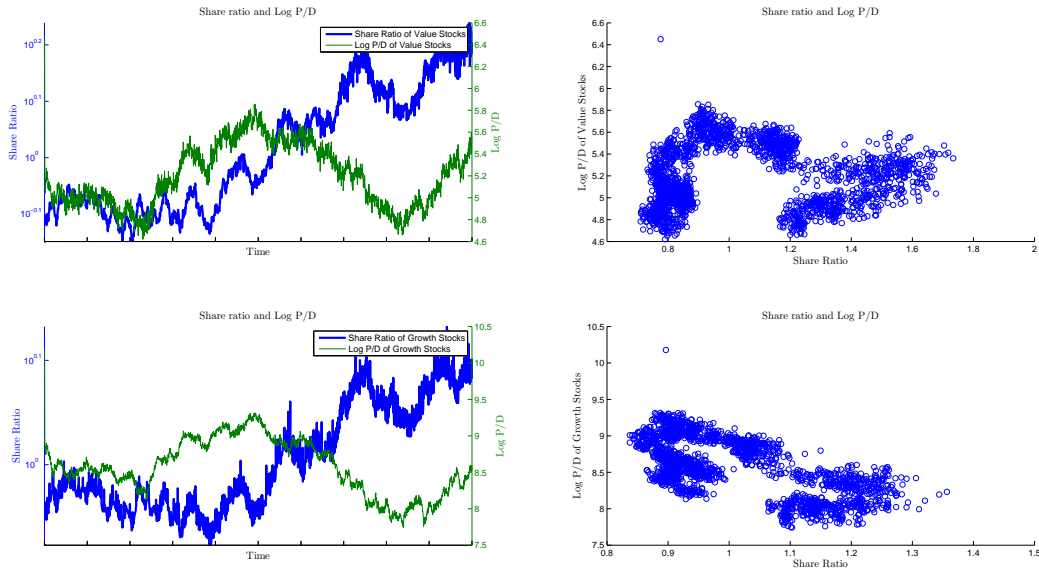
Figure 4. The Relation between the Equilibrium Price-Dividend Ratio and the Share Ratio – The model is simulated with $(\gamma, \delta) = (3.7, 1.9)$ and $(h_1, h_2) = (0.0107, 0.089)$. The upper row shows the relation between equilibrium price-dividend ratios and the share ratios for value firms and lower row shows the same relation for growth firms.


Figure 5. The Relation between Equilibrium Expected Excess Return and the Share Ratio – The model is simulated with $(\gamma, \delta) = (3.7, 1.9)$ and $(h_1, h_2) = (0.0107, 0.089)$. Upper row shows the relation between equilibrium expected excess returns and the share ratios for value firms and lower row shows the same relation for growth firms.
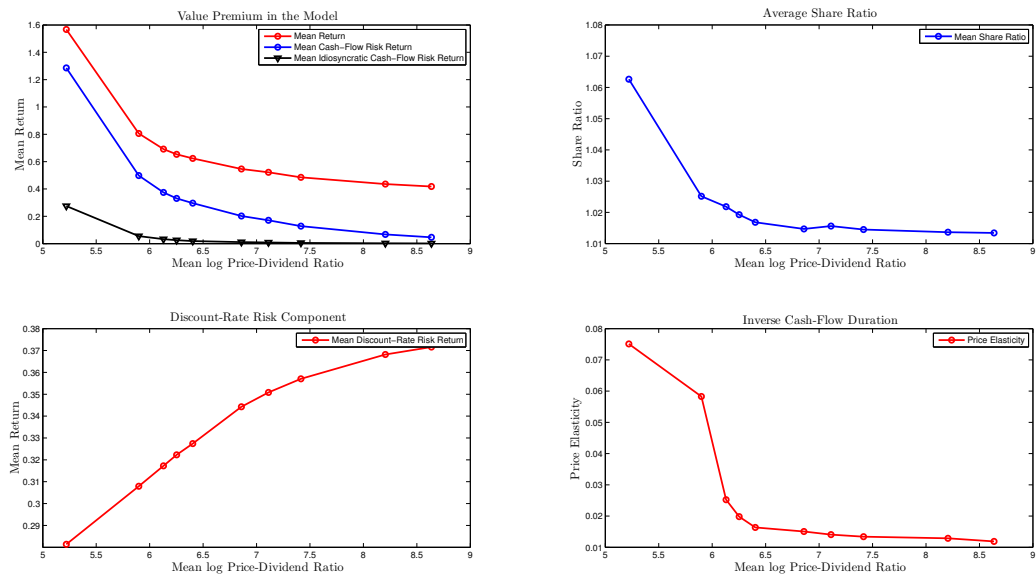
Figure 6. The Value Premium and Return Decomposition – The model is simulated with $(\gamma, \delta) = (3.7, 1.9)$ and $(h_1, h_2) = (0.0107, 0.089)$. The top figure shows simulated cross-sectional average excess returns and cash flow risk returns. The bottom figure shows the discount-rate risk return. The simulation and return decompositions follow equilibrium equation in Proposition 2 and equations (III.7) and (III.8). Returns are expressed in percentage.
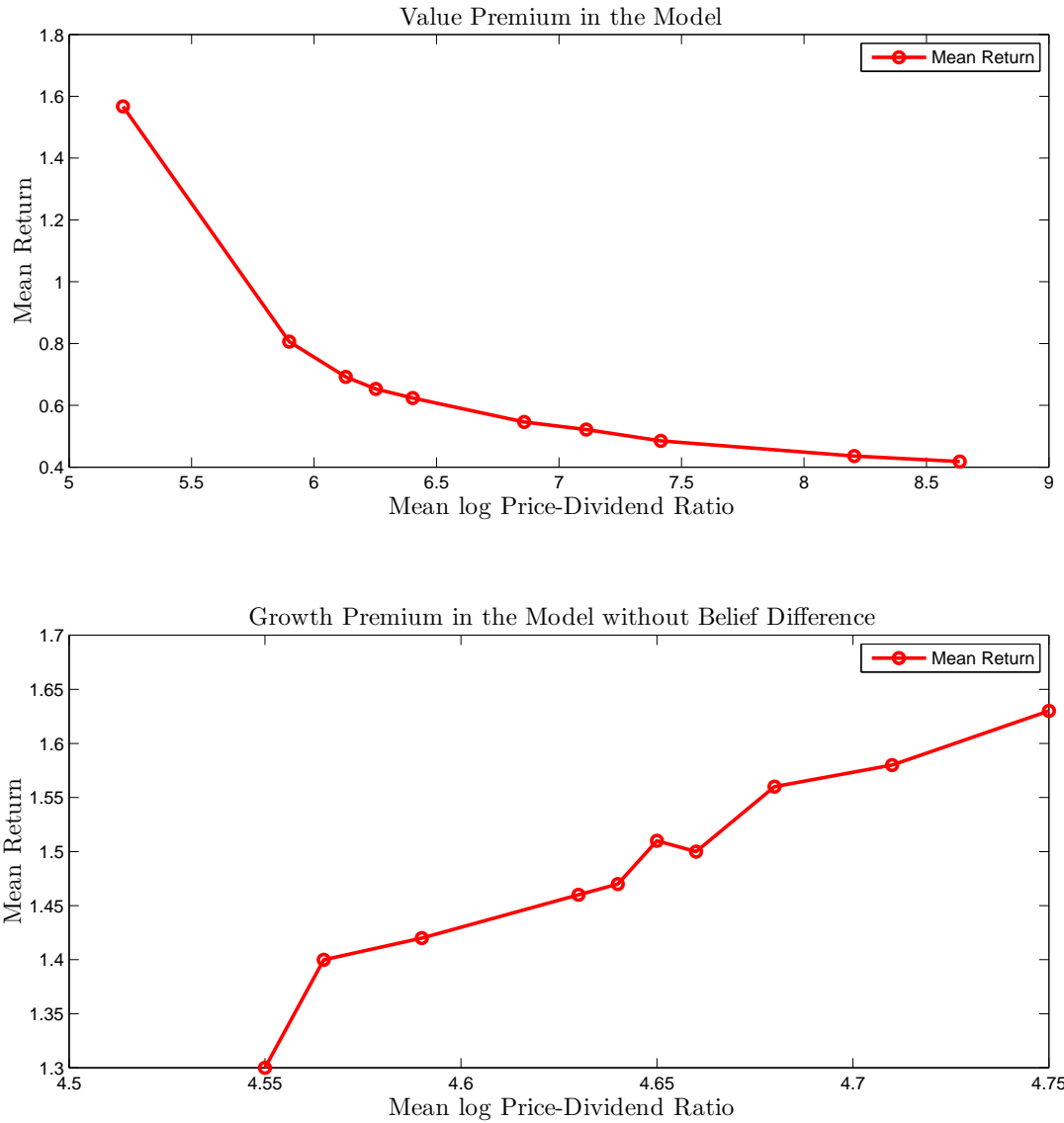
Figure 7. Results with and without belief difference – The first figure shows simulated cross-sectional average excess returns sorted by price-dividend ratio from our model. The Second figure is the simulated cross-sectional average excess returns sorted by price-dividend ratio from the benchmark model where investors' belief differences do not exist. In this case, $v_{s,I}$ does not exist in equilibrium, neither does $\sigma_{s,A}$. Benchmark result roughly corresponds to the cross-sectional return simulation of Santos and Veronesi (2010) in the sense that we fix the mean-reverting coefficient, ϕ_s, for all assets at 0.09. The simulation method is the same as the description in Table VI. Mean returns are expressed in percentage.