# Ambiguity Aversion, Disagreement, and the Theory of the Firm*

David L. Dicks
Kenan-Flagler Business School
University of North Carolina

Paolo Fulghieri
Kenan-Flagler Business School
University of North Carolina
CEPR and ECGI

December 20, 2014

**Abstract**

We present a novel source of disagreement grounded in decision theory: ambiguity aversion. We show that ambiguity aversion generates endogenous disagreement between a firm's insider and outside shareholders, creating a new rationale for corporate governance systems. In our paper, optimal corporate governance depends on both firm characteristics and the composition of the outsiders' overall portfolio. A strong governance system is desirable when the value of the firm's assets in place, relative to the growth opportunity, is sufficiently small or is sufficiently large, suggesting a corporate governance life cycle. In addition, more diversified outsiders (such as generalist mutual funds) prefer stronger governance, while outsiders with a portfolio heavily invested in the same asset class as the firm (such as venture capitalists or private equity investors) are more willing to tolerate a weak governance system, where the portfolio companies' insiders have more leeway in determining corporate policies. Finally, we find that ambiguity aversion introduces a direct link between the strength of the corporate governance system and firm transparency, whereby firms with weaker governance should also optimally be more opaque.

*We would like to thank Jonathan Berk, Diego Garcia, Lorenzo Garlappi, Ron Giammarino, Christopher Hennessy, Ali Lazrak, Doron Levit, Kevin Murphy, Jacob Sagi, Merih Sevilir, Anjan Thakor, Geoff Tate, and seminar participants at the UNC Brown Bag Seminar, UBC Winter Finance Conference, the University of Houston, the Finance, Organizations, and Markets Conference, and Miami Behavioral Finance Conference for very helpful comments. All errors are only our own. We can be reached at David_Dicks@kenan-flagler.unc.edu and Paolo_Fulghieri@kenan-flagler.unc.edu

Disagreement adds spice to life. Differences of opinions among economic agents are an important driver of trades in financial markets, where agents hold different "beliefs" on the future performance of securities, and the design of corporate financial structures, where agents respond to differences in valuation of firms' investment opportunities. While financial economists have long recognized the importance of disagreement, there is lack of consensus on the source of such disagreement. In this paper, we derive endogenous disagreement from ambiguity aversion and apply our finding to the question of allocation of control within a firm.

Differences of opinion may reflect differences of information among economic agents (e.g., Grossman, 1976). They may also reflect the fact that economic agents hold heterogeneous priors, or that they have a different interpretation of the same facts (see Harris and Raviv, 1993, Morris, 1994 and 1995, Kandel and Pearson, 1995, and more recently Van den Steen 2009). A common feature of the second group of papers is that differences of opinion are determined exogenously, by endowing agents either with heterogeneous priors or with different updating rules (i.e., different likelihood functions) used to form posteriors beliefs. As a result, agents hold different "views of the world," which leads them to differences of opinion.[1]

In this paper, we present a novel source of disagreement that is grounded in decision theory: ambiguity aversion. We argue that ambiguity aversion generates *endogenous* differences of beliefs among agents.[2] Our economy is populated by agents who are endowed with the same set of "core beliefs," but are heterogeneous in other dimensions, such as endowments.[3] We show that agents' heterogeneity generates differences of beliefs as the outcome of their different exposure to risk factors in the economy. We assume that this difference in risk exposure cannot be resolved contractually due to contractual frictions or incompleteness.[4] The key benefit of our approach is that ambiguity aversion creates a direct link between differences of beliefs and economic fundamentals, allowing us to study the effect of changes in fundamentals on disagreement among agents.[5]

[1] Models with heterogenous prisors are also common in the asset pricing literature. See, for example, Harrison and Kreps (1978), Detemple and Murthy (1994), Basak (2000), and David (2008), among many others.

[2] In our model, economic agents disagree in the sense that they have different de Finetti probabilities; see, for example, de Finetti (1974).

[3] Alternatively, agents' heterogeneity may derive from differences in preferences, such as risk attitudes, or skills (ability), a route that we do not follow in this paper.

[4] An example of such contractual incompleteness is the limited ability to insure insiders' human capital.

[5] Our paper has also implications for the question of the persistence of disagreement over time in cases where agents observe (infinitely many) signals on uncertain payoff-relevant parameters (see, for example, Cripps et al., 2008, and Acemoglu, Chernozukov, and Yildiz, 2009). Specifically, our model suggests that if agents' heterogeneity persists

We develop our theory of disagreement in the context of corporate governance and the allocation of control.[6] Diversity of opinions makes decision-making process in organizations important and creates the necessity of corporate governance systems.[7] A corporate governance system represents a set of rules for the allocation of control and, thus, of decision making. Allocation of decision making is critical when agents have different beliefs that cannot be reconciled contractually. In our paper, by design, we explicitly ignore contracts and, rather, we focus on agents' differences of beliefs. Since in our model differences of opinion are endogenous, we can determine how disagreement among a firm's stakeholders evolves as the firm's fundamentals change. In this way, we are able to study the forces that contribute to shape a firm's corporate governance system over a firm's life cycle.

Our paper allows us to take a new look at classic problems in the theory of the firm. Specifically, we ask the following questions: is a strong corporate governance system, from an investors' perspective, always preferable to weak governance? Is greater firm transparency always preferable to less transparency? More generally, is there a relationship between firm transparency and corporate governance? Can deliberate opacity be potentially desirable for shareholders? Our stylized model is able to explain the optimality of several commonly observed features in corporate organizations, such as the separation of ownership and control, low level of transparency at weakly governed firms, and why firms do not report more information than required by the regulatory framework they must adhere to.

Our economy is endowed with two classes of risky assets (in addition to the riskless assets) which have a different exposure to the source of uncertainty in the economy. Firm outsiders and the insider are heterogenous in that they hold a different portfolio of the two risky assets. Such portfolio heterogeneity may reflect the insider's undiversifiable human capital or the presence of an (optimal) incentive contract. This portfolio heterogeneity leads the insider and outsiders to have endogenous differences of opinion and, thus, to a different "sentiment" on firm investment

over time, beliefs will not necessarily converge.

[6] Our approach could be easily applied to other economic environments, such as trading with heterogeneous beliefs.

[7] Disagreements among shareholders, boards, and directors are common events in corporate life. Examples include the ousting of Carly Fiorina from Hewlett and Packard, allegedly "after she and directors disagreed on how to carry out Hewlett's corporate strategy," (see "Hewlett-Packard's Chief Forced Out, Ending Rocky Tenure," New York Times, February 9, 2005). Similarly, Christopher Galvin was ousted from his position of Chairman and Chief Executive of Motorola because of "disagreements over pace, strategy and progress," especially concerning the company's strategy on semiconductors, one of Motorola's traditional strong products (see "For Motorola, Chief's Ouster Seen Bringing Strategy Shift," New York Times, September 22, 2003).

opportunities.

The firm is endowed with one type of assets plus a growth opportunity. The growth opportunity can either be an expansion of its current assets in place (that is, a "focused" project), or an investment in the other type of asset (that is, a "diversifying" project). The insider is undiversified and holds only the firm's equity, while outsiders are well-diversified. At the outset, outsiders choose the corporate governance system of their firm, which affects investment policy. Outsiders can either select a strong corporate governance system, where they retain the control their firm, or they can select a weak governance system, where investment decisions are delegated to the insider.

We show that the strength of the corporate governance system depends on both firm characteristics and the composition of the outside shareholders' overall portfolio. The main benefit of a weak corporate governance system is that, by delegating decision-making authority to the insider, the outsiders reduce their exposure to information revelation, which is harmful to ambiguity-averse agents.[8] This effect always makes delegation (and corporate opacity) attractive to the outsiders.[9] The second effect results from endogeneity of beliefs (due to ambiguity aversion) and its effect on the level of investment, which is potentially harmful to outsiders. Specifically, delegation of decision making to insiders has the additional advantage of allowing outsiders to avoid their ex-post suboptimal investment, a feature that is due to a time-inconsistency problem caused by ambiguity aversion, but it has the disadvantage of creating a disagreement with the insiders on the firm's investment decisions (with respect to level of investment preferred ex-ante by outsiders).[10]

We find that a strong corporate governance system is optimal when the value of the firm's assets in place, relative to the growth opportunity, is either sufficiently small or sufficiently large. This property suggests a corporate governance life cycle, whereby stronger governance is optimal for young and mature firms, while weaker governance is optimal for firms at the intermediate stage of their development. In addition, a weaker corporate governance system is also optimal (all else

[8] Note that in our model delegation of control to insiders reduces (in fact, eliminates) outsiders' access to relevant information on states of nature that affect the fundamental value of the firm. The link between information and control is quite subtle and it has been examined by several papers in the literature, such as Aghion and Tirole (1997), Dessein (2002), Harris and Raviv (2005), (2008), and (2010), Adams and Ferreira (2007) and Chakraborty and Yilmaz (2013).

[9] There is empirical evidence that investors are willing to pay a premium for opaque assets: see Coval, Jurek, and Stafford (2009), Henderson and Pearson (2011), and Célérier and Vallée (2013), and Sato (2014) for a theoretical treatment.

[10] As we discuss later, this time-inconsistency originates from the lack of "rectangularity" of beliefs.

equal) for more productive firms, while stronger corporate governance is optimal for less productive ones. These results suggest that private and public companies will be characterized by different governance structures and investment policies, and predict a role for "going private" transactions such as LBOs.

An additional implication of our model is that ambiguity aversion introduces a direct link between the strength of the corporate governance system and firm transparency. Because of ambiguity aversion, outsiders prefer (all else equal) to have less, rather than more, information on the true state of the firm. This means that outsiders prefer the firm to be less transparent, unless they can benefit from the greater transparency by exerting control. Thus, firms with weaker governance should also optimally be more opaque.

Finally, our paper has also implications for the governance structure of private equity and venture capital funds and of their portfolio companies. We find that more diversified outsiders prefer stronger governance, while outsiders with a portfolio more heavily invested in the same asset class as the firm's tolerate weaker governance systems. This happens because more diversified outsiders are likely to be in greater disagreement with the insider on the firm's investment policy and, thus, to prefer stronger governance. If outsiders are institutional investors such as a venture capital or a private equity fund, this property implies that generalist funds should impose relatively stronger governance systems on their portfolio companies. In contrast, specialized funds are more willing to tolerate weaker governance systems, where the portfolio companies' insiders have more leeway in determining corporate investment policies, for portfolio companies falling in their specialty, while imposing strong governance on firms outside their specialty.

The persistence of organizational forms characterized by division of ownership and control is a classic puzzle in corporate finance. Both Adam Smith[11] and Berle and Means[12] passionately

[11] "The directors of such companies, however, being the managers rather of other people's money than of their own, it cannot well be expected that they should watch over it with the same anxious vigilance with which the partners in a private copartnery frequently watch over their own. Like the stewards of a rich man, they are apt to consider attention to small matters as not for their master's honour, and very easily give themselves a dispensation from having it. Negligence and profusion, therefore, must always prevail, more or less, in the management of the affairs of such a company." Book 5, Chapter 1, Part 3, Article 1, v1.107, Smith (1776)

[12] "In its new aspect the corporation is a means whereby the wealth of innumerable individuals has been concentrated into huge aggregates and whereby control over this wealth has been surrendered to a unified direction. The power attendant upon such concentration has brought forth princes of industry, whose position in the community is yet to be defined. The surrender of control over their wealth by investors has effectively broken the old property relationships and has raised the problem of defining these relationship anew. The direction of industry by persons other than those who have ventured their wealth has raised the question of the motive force back of such direction and the effective

warned about the negative implications of the separation of ownership and control. In a deliberately provocative paper, Jensen (1989) advocates the "eclipse of the public company" and proposes the "LBO association" (and the avoidance of the separation of ownership and control) as a superior organizational structure. Rappaport (1990), while acknowledging that separation of ownership and control was suboptimal, countered that LBOs may be optimal organization structures temporarily, and defended the "staying power" of public companies. Corporate finance models typically examine the costs and benefits of alternative governance systems as mechanisms to impose discipline on a firm's insiders, where typically delegation of decision making to insiders is costly (see, for example, Harris and Raviv, 2008 and 2010, and extensive surveys in Shleifer and Vishny, 1997, and Becht, Bolton, and Roell, 2003). More recent literature examines the impact of disagreement on incentives and organization design (Van den Steen, 2004, 2009, 2010a and especially 2010b, and Boot and Thakor, 2011).

This paper is related to several current strands of literature. The first one is the emerging literature in corporate finance focusing on the effect of disagreement on firms' corporate governance and financing strategy. Van den Steen (2004) and (2010b) examine the impact of disagreement on incentives and organization design. Boot, Gopalan, and Thakor (2006) and Boot, Gopalan, and Thakor (2008) examine a firm's choice between private and public ownership as a trade-off between managerial autonomy and liquidity. Boot and Thakor (2011) argue that potential disagreement with new investors causes the initial shareholders to prefer weak corporate governance (that is, "soft claims" that allows managerial discretion). Huang and Thakor (2013) suggest that heterogeneous beliefs affect firms' decision to do share repurchases. Bayar, Chemmanur, and Liu (2011) examine the impact of heterogeneous beliefs on equity carve-outs. Harris and Raviv (1993) examine the impact of disagreement on the volume of trading and the reaction to public announcements.

A second related strand of literature focuses on the determinants of a firm's disclosure policy, that is, its transparency, with and without disagreement. This includes Boot and Thakor (2001), Fishman and Hagerty (2003), Ferreira and Rezende (2007), and Kogan et al. (2009) among many others. Closer to our work, Thakor (2013) examines the optimal disclosure policy in the presence of disagreement, and suggests that firms may prefer to disclose less information, and thus to remain

distribution of the returns from business enterprise." p. 4, Berle & Means (1967)

more opaque, when information disclosure increase disagreement. In contrast, in our paper, the benefit of opacity is an outcome of the information aversion that characterizes ambiguity. In a different setting, Levit and Malenko (2014) also argue that transparency may be harmful and, in fact, weaken corporate governance quality when expression of dissent by board members affect adversely their reputation and, thus, the value of their outside opportunities. In all these papers disagreement is exogenously given and it derives from heterogeneous priors among a firm's stakeholders (that is, it is a "primitive" of the model). Our paper provides the heterogenous priors approach with an explicit decision-theoretic foundation that is based on ambiguity aversion.

An exception to the literature mentioned above is Garlappi, Giammarino, and Lazrak (2013). They consider a model with ambiguity-averse agents, where disagreement is driven by (exogenous) differences in the agents' "core beliefs" sets. In our paper, rather, agents share at the outset the same "core beliefs"; we derive agents' disagreement endogenously as the outcome of differences in their exposure to uncertainty (due to heterogeneous portfolios), and we study its implications on corporate governance and corporate disclosure policy.

More generally, our paper contributes to the impact of ambiguity aversion on financial decision making and asset pricing. For a thorough literature review, see Epstein and Schneider (2008) and (2011). Maenhout (2004) solves a dynamic portfolio and consumption problem of an ambiguity-averse investor, and shows that ambiguity aversion produces a larger equity premium and a smaller risk-free rate. Easley and O'Hara (2009) shows ambiguity aversion can explain nonparticipation. Easley and O'Hara (2013) examine the impact of microstructure design on market liquidity in the presence of ambiguity-averse traders. Colacito and Croce (2012) examine disagreements in an international context.

Our paper has two main limitations that can provide fruitful avenues for future research. First, we take the insider's lack of diversification as given. Insiders may be undiversified for a number of important reasons. For example, lack of diversification may be due to the presence of firm-specific human capital or it may be the outcome of an incentive contract due to moral hazard. A second limitation is that, in our model, agents are risk neutral. The presence of risk aversion would provide an additional source of disagreement between insiders and outsiders that could be addressed

with optimal contracts (see Ross, 1973).[13] Unlike ambiguity-aversion, however, risk-aversion would make a weak corporate governance system (i.e., delegation of control) always dominated by strong governance. Thus, a more general model that explicitly considers ambiguity aversion with either moral hazard or risk aversion, will have to incorporate the effects discussed in this paper as drivers of optimal contracts. For example, our model suggests that, because of ambiguity aversion, insiders should work under contracts that optimally generate some exposure to industry wide (or even economy wide) shocks, in order to reduce the extent of the disagreement with the outsiders.[14]

The paper is organized as follows. In Section 1 we introduce ambiguity aversion and outline its main features. In Section 2, we describe the basic model. In Section 3, we derive the paper's main results. In Section 4, we present the paper's empirical implications. Section 5 concludes. All proofs are in the Online Appendix.

# 1 Ambiguity Aversion

Traditional economic models assume that agents know the distribution of possible outcomes: that is, economic agents maximize their Subjective Expected Utility (SEU). Given a von-Neumann Morgenstern (vNM) utility function $u$ and a wealth distribution $\mu$, each agent maximizes $U^e = E_\mu [u(w)]$. However, the Ellsberg paradox shows that the assumption of a single prior is not warranted.[15] Under ambiguity aversion, a player does not know the true prior, but only knows that the prior is from a certain set.

A common way of modeling ambiguity aversion is minimum expected utility (MEU), promoted

[13] For example, if the insider is more risk averse than outsiders, then her optimal investment will be lower than the outsider.

[14] See Gopalan, Milbourn, and Song (2010).

[15] A good illustration of the Ellsberg paradox is from Keynes (1921). There are two urns. Urn K has 50 red balls and 50 blue balls. Urn U has 100 balls, but the subject is not told how many of them are red (all balls are either red or blue). The subject will be given \$100 if the color of their choice is drawn, and the subject can choose which Urn is drawn from. Subjects typically prefer urn K, revealing aversion to ambiguity (this preference is shown to be strict if the subject receives \$101 from selecting Urn U but \$100 from Urn K being drawn). To see this, suppose the subject believes that the probability of drawing Blue is p. If $p < \frac{1}{2}$, the subject prefers to draw Red from Urn U. If $p > \frac{1}{2}$, the subject prefers to draw Blue from Urn U. If $p = \frac{1}{2}$, the subject is indifferent. Because subjects strictly prefer to draw from Urn K, such behavior cannot be consistent with a single prior on Urn U. This paradox motivates the use of multiple priors.

in Epstein and Schneider (2011). In this frameworks, economic agents maximize

$$U^a = \min_{\mu \in \mathcal{M}} E_\mu \left[ u\left( w \right) \right]. \tag{1}$$

As shown in Gilboa and Schmeidler (1989), the MEU approach is a consequence of replacing the Sure-Thing Principle of Anscombe and Aumann (1963),[16] with the Uncertainty Aversion Axiom. This assumption captures the intuition that agents prefer risk to ambiguity – they prefer known probabilities to unknown. MEU has the intuitive feature that a player first calculates expected utility with respect to each prior, and then takes the worst-case scenario over all priors. In other words, agents follow the maxim "Average over what you know, then worry about what you don't know."[17]

In this paper, we use the MEU approach with recursively defined utilities, as described in Epstein and Schnieder (2011). Formally, we model *sophisticated ambiguity-averse* economic agents with consistent planning. In this setting, agents are sophisticated in that they correctly anticipate their future ambiguity aversion. Consistent planning accounts for the fact that agents take into account how they will actually behave in the future.[18] In the context of our model, there will be an initial contracting phase (when control is allocated) at $t = 0$. Information is revealed and an action is taken at $t = 1$. All payoffs are determined at $t = 2$. Players at the initial contracting phase, $t = 0$, correctly anticipate behavior at the interim stage, $t = 1$. Our results are smooth (a.e.) because we explore a setting where we can apply a minimax theorem.

A second critical feature of our model is that we do not impose rectangularity of beliefs (as in Epstein and Schneider 2003). Rectangularity of beliefs effectively implies that prior beliefs in the set of admissible priors can be chosen independently from each other. In our model, we assume that the agent faces a restriction on the set of the core beliefs $\mathcal{M}$ over which the minimization problem (1) is taking place. These restrictions are justified by the observation that the nature of

[16] Anscombe and Aumann (1963) is an extension of the Savage (1972) framework: the Anscombe and Aumann framework has both objective and subjective probabilities, while the Savage framework has only subjective probabilities.

[17] Another approach is the smooth ambiguity model developed by Klibanoff, Marinacci, and Mukerji (2005), which we will refer to as KMM. In their model, agents maximize expected felicity of expected utility. Agents are ambiguity averse if the felicity function is concave.

[18] Siniscalchi (2011) describes this framework as preferences over trees.

the economic problem imposes certain consistency requirements in the set of the core beliefs $\mathcal{M}$ that an ambiguity-averse agent must satisfy. In other words, we recognize that the "fundamentals" of the economic problem faced by the ambiguity-averse agent generates a loss of degree of freedom in the selection of prior beliefs.[19] An important implication of abandoning rectangularity of beliefs is that agents' preferences are not necessarily time-consistent, a feature that plays a role in our analysis.[20]

We conclude by pointing out that an important property of this class of models is that information can harm an ambiguity-averse agent if the agent does not use the information. This property is a direct consequence of the minimization operator, and can be stated as follows. Suppose that each prior, $\mu(\cdot)$, is a joint distribution of wealth, $w$, and a signal of wealth, $s \in S$. Further, following Epstein and Schneider (2011), let $\mu(\cdot; s)$ be the condition distribution of $w$ given $s$. Note, by the Law of Iterated Expectations, for all $\mu$: $U(\mu) = E_s E_{\mu(\cdot;s)}[u(w)]$. We have the following:

**Lemma 1** *Information harms an ambiguity averse agent if the agent does not use it:*

$$\min_{\mu \in \mathcal{M}} E_s E_{\mu(\cdot;s)}[u(w)] \geq E_s \min_{\mu(\cdot;s) \in \mathcal{M}_s} E_{\mu(\cdot;s)}[u(w)]. \tag{2}$$

The property (2) implies that information harms ambiguity-averse agents because the minimization can be more fine-tuned when agents anticipate having more information.[21] A potential offsetting advantage of learning the signal $s$ may come in cases where the agent's utility depends also on a specific action (possibly chosen by the agent) which affects the distribution of the agent's final payoff, $w$. In this case, the agent may find it desirable to choose the action only after learning the realization of the signal $s$ so as to be able to condition the choice of the action to the observed signal.[22] This property creates an endogenous cost of disclosure, and provides the benefit

[19] For example, an ambiguity-averse producer may face uncertainty on the future consumption demand exerted by her customers. The beliefs held by the ambiguity-averse agent on consumer demand must be consistent with basic restrictions, such as the fact that the consumer choices must satisfy an appropriate budget constraint.

[20] Because agents take long positions in ambiguous assets in our setting, if we were to impose rectangular beliefs, the worst-case scenario would not depend on the portfolio chosen by the agent, so agents would just behave as if they were pessimistic. In this case, if the insider and outsiders were equivalently ambiguity averse (that is, if they held the same set of core of beliefs), there would no disagreement between them, making the allocation of control irrelevant.

[21] Note that property (2) mirrors the well-known feature that a portfolio of options is worth more than an option on a portfolio and, thus, that writing a portfolio of options is more costly than writing an option on a portfolio. By similar intuition, more information harms an ambiguity-averse agent since the minimization process can be more fine-tuned.

[22] Though the proof of Lemma 1 assumes the minimum expected utility framework, similar intuition applies in

of separation of ownership and control in our model.

## 2 The Model

We consider the problem of the allocation of control between the outsiders of a firm (the "shareholders"), denoted by $S$, and its insider ("the manager"), denoted by $M$. We assume that at the outset the firm outsiders own a fixed fraction $1-\alpha$ of the firm equity, and the insider retains the residual fraction $\alpha$ for herself. For example, the insider may be an entrepreneur who, after founding the firm, has divested a fraction $1-\alpha$ of its equity to outside investors to raise capital in earlier financing rounds. In turn, the outsiders could be private equity investors, such as VCs, or a group of dispersed shareholders. We assume that the outside investors behave as a single block and, for brevity, they will be referred to as the "outsiders."

We study a simple two-periods model with three dates, $t \in \{0,1,2\}$. At the beginning of the first period, $t=0$, the firm outsiders must choose the governance structure, $\delta$, of the firm. Specifically, the outsiders must decide whether to retain control of the firm's decisions for themselves, denoted by $\delta = r$ ("retention") or to delegate control to the firm's insider, the management, denoted by $\delta = d$ ("delegation"). Note that the outsiders can implement retention of control in a number of ways, for example, by having a management-independent board of directors that responds to them, rather than one dominated by the insider. More generally, outsiders' retention of control can be implemented by setting up a strong corporate governance system. Thus, we can interpret retention as a "strong" corporate governance system, and delegation as a "weak" corporate governance system. The outsiders allocate control to maximize their ex-ante utility, as described below.

We assume that the economy is endowed by three (classes of) assets: a riskless asset which will serve as our numeraire, and two types of risky assets: type-$A$ and type-$B$ assets. Type-$\tau$ assets, with $\tau \in \{A,B\}$, are risky in that they generate at the end of the second period, $t=2$, a random payoff denominated in terms of the riskless asset. Specifically, a unit of type-$\tau$ asset produces at $t=2$ a payoff $H$ (success) with probability $p_\tau$, and a payoff $L$ (failure) with probability $1-p_\tau$. For

the KMM framework (this works by the same intuition that a mean-preserving spread harms a risk-averse economic agent). Indeed, Caskey (2009) shows that an ambiguity averse agent will optimally ignore ambiguous information. Caskey (2009) interprets the unambiguous information as aggregate information and ambiguous information as firm-specific information.

notational simplicity, we normalize these payoffs to $H = 1$ and $L = 0$.

In our most general model, we assume both outsiders and the insider are ambiguity averse, that is, they are MEU agents. In particular, we assume that ambiguity-averse agents hold multiple prior beliefs on the success probability of the risky assets. Following Epstein and Schneider (2011), we model ambiguity aversion by assuming that the success probability of an asset of type-$\tau$ depends on the value of an underlying parameter $\theta$, and is denoted by $p_\tau(\theta)$. Ambiguity averse agents treat the parameter $\theta$ as ambiguous, and believe that $\theta \in C \equiv \left[\hat{\theta}_0, \hat{\theta}_1\right] \subset [\theta_0, \theta_1]$, where $C$ represents the set of "*core beliefs.*" In contrast, a SEU agent has a unique prior on the success probability $p_\tau$ of the risky assets. For SEU agents the set of core beliefs $C$ is a singleton, which we denote as $C^e = \{\theta^e\}$. We also assume that $\theta^e = \frac{1}{2}(\theta_0 + \theta_1) \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$. As will become apparent below, this assumption ensures that an SEU agent has the same beliefs (defined below) as a well-diversified MEU agent. For expositional simplicity, we also assume that the core of beliefs is symmetric, so that $\theta_1 - \hat{\theta}_1 = \hat{\theta}_0 - \theta_0$.

The parameter $\theta$ affects the two assets in our economy differently. For analytical tractability, we assume that $p_A(\theta) = e^{\theta - \theta_1}$ and $p_B(\theta) = e^{\theta_0 - \theta}$.[23] In this specification, increasing the value of the parameter $\theta$ increases the success probability of type $A$ assets and decreases the success probability of type $B$ assets. This means that a greater $\theta$ is "favorable" for asset A and "unfavorable" for asset B.[24] Also, for a given value of the parameter $\theta$, the distributions $p_\tau(\theta)$, $\tau \in \{A, B\}$, are independent.[25]

We assume that at the beginning of the first period, $t = 0$, the firm is endowed with $V_A$ units of type-$A$ assets and $V_0$ units of the riskless asset.[26] At the interim date, $t = 1$, the firm has access to a new investment opportunity. The type of investment opportunity which becomes available to the

[23] Note that this assumption allows us to dispense with rectangularity of beliefs in a tractable way, but is not necessary. For example, our paper's main results will go through for $p_\tau(\theta_\tau)$, with $\theta_\tau \in [\theta_0, \theta_1]$, as long as the core belief set $C$ is a strictly convex, compact subset of $[\theta_0, \theta_1]^2$ with a smooth boundary, such that $\{\theta_A, \theta_B\} \in C$. Because the worst-case scenario will be on the lower bound of $C$, the ambiguous parameter must affect the two asset types differently.

[24] A simple example of our economy is one with two consumption goods, $A, B$. Consumers' preferences over the two consumption goods (that is, their relative valuation) is random and is characterized by the parameter $\theta$. In this case, a higher (respectively, lower) value of $\theta$ represents a stronger consumer preference for good $A$ (respectively, $B$) with respect to the other good.

[25] Our model can easily be extended to the case where, given $\theta$, the realization of the asset payoffs at the end of the period are correlated.

[26] More generally, we could assume that a certain fraction of firms in the economy is endowed with type-$A$ assets, and the remaining firms with type-$B$ assets.

firm is random, and is not known at the outset to both outsiders and the insider. We assume that the investment opportunity can either be a project in the same type of assets currently owned by the firm, type-$A$ assets, or an investment in type-$B$ assets. We denote these investment opportunities respectively as the *focused* and the *diversified* project.

The characteristics of the investment opportunity depend on the state of the world, $\omega_\tau$, realized at $t = 1$, with $\tau \in \{A, B\}$. Specifically, in state $\omega_\tau$ the firm can acquire $I_\tau$ units of type $\tau$ assets at the cost of $c(I_\tau)$ units of the riskless asset. We assume that the firm is not cash-constrained in that it has a sufficient amount of riskless assets to be able to implement the desired investment $I_\tau$ in the risky asset $\tau \in \{A, B\}$.[27] We also assume that $c(0) = 0$, $0 \leq c'(0) < e^{\theta_0 - \theta_1}$, $c'(I_\tau) > 0$ for $I_\tau > 0$, and $c''(I_\tau) > 0$.[28] The cost function $c(\cdot)$ is the same for type-$A$ and type-$B$ projects. To derive comparative statics, for analytical tractability we assume $c(I) = \frac{1}{Z(1+\gamma)} I^{1+\gamma}$ where $\gamma > 0$.[29] The parameter $Z$ affects the cost of acquiring the risky assets and, thus, the value of the investment project; it will be interpreted as characterizing the firm's "productivity." For simplicity, we assume that both agents believe that these states are equally likely, i.e., $\Pr\{\omega_\tau\} = 1/2$. We also assume that there is no ambiguity on the states $\omega_\tau$: both the insider and the outsiders have a single common prior on the probability of the intermediate states $\omega_\tau$.[30]

Allocation of control is important because the party in control chooses the level of investment $I_\tau$ to maximize his/her expected utility. The choice of the investment level, $I_\tau$, is made by the party in control after observation of the realization of $\omega_\tau$, that is, after the nature of the investment project available to the firm becomes known (i.e., whether the firm has a type-$A$ or type-$B$ project). The realization of the state of the world $\omega_\tau$ is always observable by the insider, and it is observable by the outsiders if they are in control at the time the investment is made.[31]

[27] We leave for future research the important question of raising capital under ambiguity aversion.

[28] This restriction on the cost function $c(\cdot)$ captures the notion that the the investment project is characterized by diminishing marginal returns, and that a small positive investment is always optimal.

[29] Our paper's main results do not depend on this specification of the firm's cost function.

[30] Note that our result will go through even if there is ambiguity about which type of project is drawn. Ambiguity on project type is easily modeled by letting $\Pr\{w_A\} = q$ and $\Pr\{w_B\} = 1 - q$ where $q \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ and $\varepsilon > 0$. The effect of adding ambiguity on project type is that outsiders will overweigh the outcome which is worse to them. In equilibrium, however, they will be indifferent whether the project is focused or diversifying in all cases except when both insiders and outsiders are ambiguity averse. In the latter case, it can be shown that the region that the outsider prefers delegation will shrink.

[31] We assume that the outsiders must learn the type of the project if they are going to exert control. Outsiders would never find it optimal to exert control without information if the project is one of many potential projects, and the firm learns which one is profitable at $t = 1$. Conversely, by Lemma 1, outsiders will always select an opaque information environment if they delegate control to the insider.

Finally, we assume that at the end of the period, $t = 2$, agents consume their holdings of the riskless asset. We assume that agents are endowed with vNM utility functions, $u(\cdot)$, which are linear in the riskless asset. Following Epstein and Schneider (2011), this means that they are risk-neutral MEU or SEU agents.

### 2.1 Endogenous Beliefs

A critical feature of our model is that ambiguity aversion endogenously generates differences of opinion in an economy populated by heterogeneous agents, even when agents have identical core prior beliefs. This happens because, within a MEU framework, an agent's beliefs are determined by the solution to that agent's expected utility *minimization* problem. This means that agent heterogeneity (for example, in their endowments) generates different solutions to the minimization problem and, thus, different beliefs. Therefore, differences of beliefs emerge endogenously. As we show in Section 3, these differences are meaningful and impact the firm's investment decision, making the ex-ante allocation of control meaningful.

Consider an agent endowed with a portfolio $\Pi \equiv \{\bar{w}_A, \bar{w}_B, \bar{w}_0\}$, where $\bar{w}_\tau$, $\tau \in \{A, B\}$ represents the overall units of the risky asset type $\tau$ owned by the agent, and $\bar{w}_0$ represents the units of riskless asset in the agent's portfolio. For a given value of the parameter $\theta$, this portfolio provides the agent with an expected utility of

$$\mathbb{E}\left[u\left(\bar{w}_A, \bar{w}_B, \bar{w}_0\right); \theta\right] = e^{\theta-\theta_1}\bar{w}_A + e^{\theta_0-\theta}\bar{w}_B + \bar{w}_0 \tag{3}$$

An ambiguity-averse agent fears the worst possible outcome of $\theta$:

$$U\left(\bar{w}_A, \bar{w}_B, \bar{w}_0\right) \equiv \min_{\theta \in C} \mathbb{E}\left[u\left(\bar{w}_A, \bar{w}_B, \bar{w}_0\right); \theta\right] \tag{4}$$

Thus, an ambiguity-averse agent's beliefs, denoted as $\theta^a$, are determined by the value that minimizes the agent's expected utility, that is by

$$\theta^a\left(\Pi\right) \equiv \arg\min_{\theta \in C} \mathbb{E}\left[u\left(\bar{w}_A, \bar{w}_B, \bar{w}_0\right); \theta\right]. \tag{5}$$

It is clear from (3) and (5) that the agent's belief $\theta^a$ depends on the amount of asset $A$ and asset $B$ in his overall portfolio $\Pi$. The solution to problem (4) is characterized in the following lemma.

**Lemma 2** *Let*

$$\tilde{\theta}^a(\Pi) \equiv \frac{1}{2}(\theta_0 + \theta_1) + \frac{1}{2}\ln\frac{\bar{w}_B}{\bar{w}_A}. \tag{6}$$

*The beliefs held by an ambiguity averse agent, $\theta^a(\Pi)$, are given by*

$$\theta^a(\Pi) = \begin{cases} \hat{\theta}_0 & \tilde{\theta}^a(\Pi) \leq \hat{\theta}_0 \\ \tilde{\theta}^a(\Pi) & \tilde{\theta}^a(\Pi) \in \left(\hat{\theta}_0, \hat{\theta}_1\right) \\ \hat{\theta}_1 & \tilde{\theta}^a(\Pi) \geq \hat{\theta}_1 \end{cases}. \tag{7}$$

We refer to $\tilde{\theta}^a(\Pi)$ as the "portfolio-distorted" beliefs. We say that the agent has *interior beliefs* when $\theta^a \in \left(\hat{\theta}_0, \hat{\theta}_1\right)$. In this case, the agent's beliefs are equal to the portfolio-distorted beliefs, that is $\theta^a(\Pi) = \tilde{\theta}^a(\Pi)$. It is important to note that the beliefs of an ambiguity-averse agent depend essentially on the composition of his portfolio $\Pi$, that is, on his endowment. In particular, in our specification, the portfolio-distorted beliefs $\tilde{\theta}^a(\Pi)$ differ from the SEU beliefs $\theta^e \equiv \frac{1}{2}(\theta_0 + \theta_1)$ by a term that depends on the degree of heterogeneity of the agent's endowment, $\bar{w}_B/\bar{w}_A$**.** The following corollary can be immediately verified.

**Corollary 1** *Holding type $B$ assets constant, when the agent has a larger position in type $A$ assets, the agent is more pessimistic about type $A$ assets and more optimistic about type $B$ assets. Furthermore, the agent has scale-invariant beliefs that depend only on the ratio $\bar{w}_B/\bar{w}_A$.*

Corollary 1 shows that when an agent has relatively greater endowment of asset $A$, the agent will be relatively more concerned about the priors that are less favorable to that asset. Thus, the agent shifts his beliefs toward the lower end of the core beliefs set $C$, and he will give more weight to the states of nature that are less favorable for asset $A$. In other words, the agent will be more "pessimistic" about the future value of (or the return on) that asset. Correspondingly, the agent will become more "optimistic" with respect to the other asset, asset $B$.[32] Note also that portfolio-distorted beliefs $\tilde{\theta}^a$ remain the same when the ratio of the endowment in the two types of

[32] Corollary 1 is similar to hedging demand in Epstein and Schneider (2007).

assets is constant, that is, $\tilde{\theta}^a$ is scale invariant (formally, $\tilde{\theta}^a$ is homogeneous of degree zero in the endowments $\bar{w}_B$ and $\bar{w}_A$). Note that, from (6), the assumption that $\theta^e = \frac{1}{2}(\theta_0 + \theta_1)$ guarantees that the beliefs held by a SEU agent coincide with the beliefs of a well-diversified MEU agent, for whom $\bar{w}_A = \bar{w}_B$.

An important property of the MEU approach is that even if agents are endowed with vNM utility functions that are linear in wealth, they display decreasing marginal utility in the value of any single asset in their portfolio, when the amount of all other assets remain constant. This happens because of the negative impact on an agent's beliefs that is due to the increase in the endowment of any specific asset, when the endowment of all other assets remains the same.

**Lemma 3** *Holding the position in the other asset type constant, an agent has decreasing marginal utility from a particular type of asset,* $\frac{d^2U}{d(\bar{w}_\tau)^2} \leq 0$, *for* $\tau \in \{A, B\}$. *For interior beliefs, this inequality is strict.*

This property plays an important role in the investment problem below.

# 3 Ambiguity and Allocation of Control

We now consider the allocation of control, $\delta \in \{r, d\}$, faced by outsiders at the beginning of the game. Because the optimal level of investment chosen by an ambiguity-averse agent depends on her portfolio composition, the allocation of control becomes critical.

As a reference point, we start our discussion by considering the benchmark case where there is no separation of ownership and control: the agent making the decision has full ownership of the firm. We assume that, in addition to the full ownership of the firm, the owner is also endowed with other resources (outside the firm) denoted by $\{w_A, w_B, w_0\}$. Thus, in state $\omega_A$, that is for investment projects involving type-$A$ assets, the overall owner's portfolio after the investment is made becomes $\Pi(I_A, 0) \equiv \{w_A + V_A + I_A, w_B, w_0 + V_0 - c(I_A)\}$. Similarly, in state $\omega_B$, that is for investment projects involving type-$B$ assets, after the investment is made the owner's portfolio becomes $\Pi(0, I_B) \equiv \{w_A + V_A, w_B + I_B, w_0 + V_0 - c(I_B)\}$.

In state $\omega_A$, an ambiguity-averse owner chooses at $t = 1$ the optimal investment in a type-$A$

project by maximizing the minimum expected utility function, $U_1$, given by

$$U_1\left(\Pi\left(I_A,0\right)\right) \equiv \min_{\theta \in C \equiv \left[\hat{\theta}_0, \hat{\theta}_1\right]} \mathbb{E}\left[u\left(\Pi\left(I_A,0\right)\right);\theta\right], \tag{8}$$

where

$$\mathbb{E}\left[u\left(\Pi\left(I_A,0\right)\right);\theta\right] = e^{\theta-\theta_1}(w_A+V_A+I_A) + e^{\theta_0-\theta}w_B + w_0 + V_0 - c(I_A). \tag{9}$$

Note that the owner's beliefs, $\theta^a\left(\Pi\left(I_A,0\right)\right)$, are given by the solution to the minimum expected utility problem (8) as

$$\theta^a\left(\Pi\left(I_A,0\right)\right) = \underset{\theta \in C \equiv \left[\hat{\theta}_0, \hat{\theta}_1\right]}{\arg\min} \left\{\mathbb{E}\left[u\left(\Pi\left(I_A,0\right)\right);\theta\right]\right\}.$$

By Lemma 2, the insider's beliefs, $\theta^a\left(\Pi\left(I_A,0\right)\right)$, are given by

$$\theta^a\left(\Pi\left(I_A,0\right)\right) = \begin{cases} \hat{\theta}_0 & \tilde{\theta}\left(\Pi^a\left(I_A,0\right)\right) \leq \hat{\theta}_0 \\ \tilde{\theta}^a\left(\Pi\left(I_A,0\right)\right) & \tilde{\theta}^a\left(\Pi^a\left(I_A,0\right)\right) \in \left(\hat{\theta}_0, \hat{\theta}_1\right) \\ \hat{\theta}_1 & \tilde{\theta}^a\left(\Pi(I_A,0\right) \geq \hat{\theta}_1 \end{cases}. \tag{10}$$

where the portfolio-distorted beliefs are

$$\tilde{\theta}^a\left(\Pi\left(I_A,0\right)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B}{w_A+V_A+I_A}\right], \tag{11}$$

since $\theta^e \equiv \frac{1}{2}\left(\theta_0+\theta_1\right)$. The corresponding investment is

$$I_A^a \equiv \underset{I_A}{\arg\max}\, U_1\left(\Pi^a\left(I_A,0\right)\right). \tag{12}$$

where $U_1\left(\Pi^a\left(I_A,0\right)\right) = \mathbb{E}[u\left(\Pi^a\left(I_A,0\right)\right);\theta^a\left(\Pi^a\left(I_A,0\right)\right)]$. Similarly, in state $\omega_B$, the availability of an investment project involving type-$B$ assets leads the owner-manager to an investment level of $I_B^a$, where the portfolio-distorted beliefs are now given by

$$\tilde{\theta}^a\left(\Pi(0,I_B\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B+I_B}{w_A+V_A}\right]. \tag{13}$$

Note that when deciding the optimal level of investment, $I_\tau^a$, the MEU owner is sophisticated in that he anticipates the impact of his investment choice on his own beliefs, $\theta^a$. This implies that the agent has "no regret" in the sense that the agent will not change beliefs after the investment $I_\tau^a$ is made (and, thus, it remains optimal after it is implemented). It also means that the optimal level of investment is determined by two effects. The first effect is the traditional "marginal cost" effect that is due to the convexity of the cost function, $c(I_\tau)$. The second effect is a "pessimism" effect due to ambiguity aversion: by increasing investment in the type-$\tau$ asset, from (11) and (13) the owner changes his beliefs in a way that he becomes more pessimistic about that asset. Thus, the owner limits the investment in those assets. These considerations lead to the following theorem.

**Theorem 1** *The optimal levels of investment* $\{I_A^a, I_B^a\}$ *depend on the owner's pre-existing portfolio. An increase of* $w_\tau$ *leads to a decrease of* $I_\tau^a$ *and an increase of* $I_{\tau'}^{S,a}$, $\tau' \neq \tau$.

Theorem 1 implies that, under ambiguity aversion, the optimal investment in a project depends on the decision-maker's overall endowment. Specifically, the optimal investment in any given project is a decreasing function of the owner's initial endowment in the same asset, and an increasing function of the endowment of the other asset. This property makes investment projects effectively complementary. We denote this complementarity as the "spillover effect" of ambiguity. This spillover effect happens because we know that an increase in the endowment of asset $A$ makes an ambiguity-averse agent relatively more pessimistic about asset $A$ and more optimistic about asset $B$, resulting in a decrease in $I_A^a$ and a increase in $I_B^a$. Symmetric results hold for an increase of type-$B$ assets. Theorem 1 also implies that beliefs depend on the composition of the owner's overall portfolio, generating disagreement on the firm's optimal investment policy.

For the remainder of the paper, we consider the case where outsiders own fraction $(1-\alpha)$ of the firm, in addition to an endowment external to the firm, $\{w_A, w_B, w_0\}$, while the remaining fraction $\alpha$ is owned by the insider. Thus, the outsiders' initial portfolio is

$$\Pi^S \equiv \{w_A + (1-\alpha) V_A, w_B, w_0 + (1-\alpha) V_0\}.$$

Further, we discuss the special case in which the outsiders are truly diversified in that their overall endowment of assets of type-$A$ and type-$B$ are the same; in this case $w_B = w_A + (1-\alpha) V_A \equiv K$,

where $K$ characterizes the size of outsiders' "external" portfolio. In contrast, the insider is not well diversified: her entire wealth consists of the fraction $\alpha$ of the firm's stock. Thus, the insider's portfolio is given by $\Pi^M \equiv \{\alpha V_A, 0, \alpha V_0\}$.

We now study four possible scenarios in which the insider and outsiders can, in turn, be MEU or SEU maximizers.

### 3.1 Expected Utility Outsiders, Expected Utility Insider

As a benchmark, we begin with the simplest (and least interesting) case in which both parties are SEU maximizers and share a common belief $C = \{\theta^e\}$, with $\theta^e = \frac{1}{2}(\theta_0 + \theta_1)$. In this case, both parties chooses the same investment levels for either a focused or a diversifying project and, thus, the allocation of control is irrelevant. In addition, as we show in Theorem 4 of Section 3.3, the assumption that $\theta^e \equiv \frac{1}{2}(\theta_0 + \theta_1)$ and that the outsiders' portfolio is well diversified together imply that the investment preferred ex-ante by ambiguity averse outsiders is equal to the optimal level of investment for a SEU agent.

A focused project allows the firm to make an investment in type-$A$ assets, $I_A$. Thus, outsiders' portfolio is $\Pi^S(I_A, 0) \equiv \{w_A + (1-\alpha)[V_A + I_A], w_B, (1-\alpha)[V_0 - c(I_A)]\}$, while the insider's is $\Pi^M(I_A, 0) \equiv \{\alpha[V_A + I_A], 0, \alpha[V_0 - c(I_A)]\}$. Thus, with an investment level of $I_A$, the outsiders' expected utility is

$$\begin{aligned}\mathbb{E}\left[u\left(\Pi^S(I_A, 0)\right); \theta^e\right] &= e^{\theta^e - \theta_1}[w_A + (1-\alpha)[V_A + I_A]] \\ &\quad + e^{\theta_0 - \theta^e} w_B + (1-\alpha)[V_0 - c(I_A)],\end{aligned}$$

while the insider's expected utility is

$$\mathbb{E}\left[u\left(\Pi^M(I_A, 0)\right); \theta^e\right] = e^{\theta^e - \theta_1}\alpha[V_A + I_A] + \alpha[V_0 - c(I_A)].$$

A diversified project allows the firm to invest in type-$B$ assets, $I_B$. Thus, the outsiders' portfolio is $\Pi^S(0, I_B) = \{w_A + (1-\alpha)V_A, w_B + (1-\alpha)I_B, (1-\alpha)[V_0 - c(I_B)]\}$, while the insider's is $\Pi^M(0, I_B) = \{\alpha V_A, \alpha I_B, \alpha[V_0 - c(I_B)]\}$. Thus, with an investment level of $I_B$, the outsiders'

expected utility is

$$\begin{aligned}\mathbb{E}\left[u\left(\Pi^{S}\left(0,I_{B}\right)\right);\theta^{e}\right] &= e^{\theta^{e}-\theta_{1}}\left[w_{A}+\left(1-\alpha\right)V_{A}\right] \\ &\quad +e^{\theta_{0}-\theta^{e}}\left[w_{B}+\left(1-\alpha\right)I_{B}\right]+\left(1-\alpha\right)\left[V_{0}-c\left(I_{B}\right)\right],\end{aligned}$$

while the insider's expected utility is

$$\mathbb{E}\left[u\left(\Pi^{M}\left(0,I_{B}\right)\right);\theta^{e}\right]=e^{\theta^{e}-\theta_{1}}\alpha V_{A}+e^{\theta_{0}-\theta^{e}}\alpha I_{B}+\alpha\left[V_{0}-c\left(I_{B}\right)\right].$$

**Theorem 2** *In the absence of ambiguity aversion, the insider and the outsiders choose the same investment level $I_{\tau}^{e}=I^{e}$ for both projects, determined by*

$$c'\left(I_{\tau}^{e}\right)=e^{\frac{1}{2}\left(\theta_{0}-\theta_{1}\right)}.$$

*Thus, the initial allocation of control is irrelevant.*

When neither party is ambiguity averse, both the insider and outsiders share the same beliefs, $\theta^{e}$, and they agree on the optimal level of investment, $I_{\tau}^{e}$, for both the focused and the diversified project. Therefore, in the absence of ambiguity aversion, control rights are irrelevant.[33]

### 3.2 Expected Utility Outsiders, Ambiguity Averse Insider

We consider now the case in which the outsiders are SEU maximizers, while the insider is a MEU agent. Because the outsiders are SEU, they choose an investment level equal to $I_{\tau}^{e}$ if they have control (Theorem 2). The MEU insider, however, behaves differently.

If the firm has a focused project, that is in state $\omega_{A}$, by investing $I_{A}$ the insider obtains a portfolio $\Pi^{M}\left(I_{A},0\right)=\left\{\alpha\left(V_{A}+I_{A}\right),0,\alpha\left[V_{0}-c\left(I_{A}\right)\right]\right\}$. Given the portfolio $\Pi^{M}\left(I_{A},0\right)$, at $t=1$

[33] Note that Theorem 2 assumes a common core of beliefs. Since, under Subjective Expected Utility, the core of beliefs is a singleton, this means that Theorem 2 effectively assumes that agents have common beliefs. With exogenous difference of opinion, outsiders believe that the insider will make an investment decision they perceive as inefficient, and they will always retain control.

the insider's minimum expected utility is

$$U_1^M\left(\Pi^M\left(I_A,0\right)\right) \equiv \min_{\theta\in C} \mathbb{E}\left[u\left(\Pi^M\left(I_A,0\right)\right);\theta\right],$$

where $\mathbb{E}\left[u\left(\Pi^M\left(I_A,0\right)\right);\theta\right] = e^{\theta-\theta_1}\alpha\left(V_A+I_A\right)+\alpha\left(V_0-c\left(I_A\right)\right)$. Thus, under ambiguity aversion the insider's beliefs, $\theta^{M,a}\left(\Pi^M\left(I_A,0\right)\right)$, are determined by minimization of her expected utility, that is

$$\theta^{M,a}\left(\Pi^M\left(I_A,0\right)\right) = \arg\min_{\theta\in C}\left\{e^{\theta-\theta_1}\alpha\left(V_A+I_A\right)+\alpha\left[V_0-c\left(I_A\right)\right]\right\}.$$

Because the insider holds only type $A$ assets, the beliefs held by the ambiguity-averse insider, $\theta^{M,a}\left(\Pi^M\left(I_A,0\right)\right)$, are at the lower bound of $C$, the set of core beliefs:

$$\theta^{M,a}\left(\Pi^M\left(I_A,0\right)\right) = \hat{\theta}_0. \tag{14}$$

Since the insider's portfolio is not well diversified, the insider's beliefs give maximum weight to the priors that are least favorable to the only risky asset in which they have a long position, asset $A$. Given the insider's beliefs, $\theta^{M,a}\left(\Pi^M\left(I_A,0\right)\right) = \hat{\theta}_0$, the optimal investment $I_A^{M,a}$ is determined by maximizing the insider's minimum expected utility

$$I_A^{M,a} = \arg\max_{I_A} U_1^M\left(\Pi^M\left(I_A,0\right)\right), \tag{15}$$

where the insider's MEU is equal to $U_1^M\left(\Pi^M\left(I_A,0\right)\right) = \mathbb{E}\left[u\left(\Pi^M\left(I_A,0\right)\right);\hat{\theta}_0\right]$. The optimal level of investment, $I_A^{M,a}$, is set by the insider under the "worst-case scenario" belief that the parameter $\theta$ is at the lowest possible level $\hat{\theta}_0$. This property makes the insider very conservative when making focused investments.

Similarly, if the firm has a diversifying project, that is in state $\omega_B$, by investing $I_B$ the insider obtains a portfolio $\Pi^M\left(0,I_B\right) = \{\alpha V_A, \alpha I_B, \alpha\left(V_0-c\left(I_B\right)\right)\}$. Thus, the insider's minimum expected utility is

$$U_1^M\left(\Pi^M\left(0,I_B\right)\right) = \min_{\theta} \mathbb{E}\left[u\left(\Pi^M\left(0,I_B\right)\right);\theta\right],$$

where $\mathbb{E}\left[u\left(\Pi^M\left(0,I_B\right)\right);\theta\right] = e^{\theta-\theta_1}\alpha V_A + e^{\theta_0-\theta}\alpha I_B + \alpha\left[V_0-c\left(I_B\right)\right]$. Insider's beliefs are determined

by minimization of her expected utility:

$$\theta^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)=\underset{\theta\in C}{\arg\min}\left\{e^{\theta-\theta_{1}}\alpha V_{A}+e^{\theta_{0}-\theta}\alpha I_{B}+\alpha\left[V_{0}-c\left(I_{B}\right)\right]\right\}.$$

By Lemma 2, beliefs held by an ambiguity-averse insider, $\theta^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)$, are

$$\theta^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)=\begin{cases}\hat{\theta}_{0} & \tilde{\theta}^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)\leq\hat{\theta}_{0}\\ \tilde{\theta}^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right) & \tilde{\theta}^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)\in\left(\hat{\theta}_{0},\hat{\theta}_{1}\right)\\ \hat{\theta}_{1} & \tilde{\theta}^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)\geq\hat{\theta}_{1}\end{cases}, \tag{16}$$

where the insider's portfolio-distorted beliefs, $\tilde{\theta}^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)$, are

$$\tilde{\theta}^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)=\theta^{e}+\frac{1}{2}\ln\left(\frac{I_{B}}{V_{A}}\right). \tag{17}$$

Given the insider's beliefs, $\theta^{M,a}(\Pi^{M}\left(0,I_{B}\right))$, the optimal investment $I_{B}^{M,a}$ chosen by the insider is determined by maximizing the insider's minimum expected utility,

$$I_{B}^{M,a}=\underset{I_{B}}{\arg\max}\ U_{1}^{M}\left(\Pi^{M}\left(0,I_{B}\right)\right), \tag{18}$$

where insider's MEU is $U_{1}^{M}\left(\Pi^{M}\left(0,I_{B}\right)\right)=\mathbb{E}\left[u\left(\Pi^{M}\left(0,I_{B}\right)\right);\theta^{M,a}\left(\Pi^{M}\left(0,I_{B}\right)\right)\right]$. The optimal investment policy of a MEU insider is characterized in the following.

**Theorem 3** *If in control, the ambiguity-averse insider underinvests in focused projects relative to the investment desired by the SEU outsiders. Her investment in diversifying projects depends on firm characteristics, and is an increasing function of the value of assets in place, $V_A$: if assets in place are sufficiently large, $V_A > I^e$, the insider overinvests in diversifying projects; otherwise, if $V_A < I^e$, she underinvests in diversifying projects. Thus, SEU outsiders will not delegate control to an ambiguity-averse insider.*

Because the insider holds only type-$A$ assets, a priori, she places a lower value on type-$A$ assets than an SEU investor. Thus, the insider underinvests in focused projects (i.e., in type-$A$ assets) relative to the investment that is optimal for SEU outsiders, $I^e$. This means that the extent of

underinvestment in the focused project becomes more severe when the lower bound of the core belief set, $C \equiv \left[\hat{\theta}_0, \hat{\theta}_1\right]$, is smaller, that is when the worst case scenario $\hat{\theta}_0$ becomes "even worse."[34]

Investment in type-$B$ projects depends on the size of the assets in place, $V_A$, relative to the size of type-$B$ assets that firm will have after the investment is made. When the firm has a sufficiently large endowment of assets in place, that is, if $V_A > I^e$, the insider finds it desirable to invest relatively more in the diversifying project than SEU outsiders, leading to overinvestment. In contrast, if the size of assets is relatively small, $V_A \leq I^e$, the insider prefers to limit exposure to type-$B$ assets, and she underinvests. In either case, the insider's investment policy will differ from the one preferred by the SEU outsiders, who always retain control.

**Corollary 2** *Outsiders' loss of welfare from delegating control to the insider is an inverted U-shaped function of $V_A$.*

From Theorem 3 we know that it is never optimal to grant control to the insider. This implies that if control is delegated to the insider, it will always have a negative impact on firm value. In addition, the impact is an inverted U-shaped function of $V_A$. This means that the loss of value due to delegation of decision making is greater at the extreme cases, either for very young firms where investment is considerably larger that assets in place, $V_A < I^e$, or for mature firm, where investment is substantially smaller that assets in place, $V_A > I^e$. The intuition for this is simple: the insider invests in focused projects according to the worst-case scenario, $\theta^M = \hat{\theta}_0$, so the negative impact on firm value from a focused project is independent of $V_A$. However, the investment in the diversifying project, $I_B^{M,a}$, is increasing in $V_A$. When $V_A = I^e$, $I_B^{M,a} = I^e$, so the insider chooses the diversifying investment optimally from the outsider's perspective. Any departure from $V_A = I^e$ results in a greater loss from entrenchment.

### 3.3 Ambiguity Averse Outsiders, Expected Utility Insider

We show that ambiguity-averse outsiders find it optimal to delegate authority to a SEU insider. There are two reasons why ambiguity-averse outsiders prefer to grant control to an SEU insider. First, well-diversified ambiguity-averse outsiders ex-post underinvest relative to what they would

[34] In general, distortions become (weakly) larger as the measure of the core beliefs become more "dispersed," that is, as the measure of set $C$ increases.

have wanted to invest ex-ante. This effect is due to the impact of the arrival of the new project on the outsiders' ex-post beliefs. Second, ambiguity-averse outsiders would prefer not to learn the realization at $t = 1$ of the state of the world $\omega_\tau$, that is, to learn the kind of projects that becomes available to the firm in the intermediate date. This effect is due to the harmful effect of the arrival of new information on ambiguity-averse agents described in Lemma 1. These two effects make delegation of decision making to the insider is ex-ante desirable to the outsiders.

Consider first delegation: $\delta = d$. If outsiders delegate control to an insider who will select investment levels $\{I_A, I_B\}$, the outsiders ex-ante expected utility is determined as follows. While outsiders anticipate that the insider will implement investment of $I_\tau$ in state $\omega_\tau$, they will not know which state of the world is realized, and thus, which type of project the firm has actually drawn. In addition, since the outsiders do not display (by our simplifying assumption)[35] ambiguity aversion with respect to this random variable, the outsiders' expected utility at $t = 1$ is given by

$$\begin{aligned} \mathbb{E}\left[u\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right); \theta\right] &= e^{\theta-\theta_1}\left[w_A + (1-\alpha)\left(V_A + \frac{1}{2}I_A\right)\right] \\ &+ e^{\theta_0-\theta}\left[w_B + (1-\alpha)\frac{1}{2}I_B\right] \quad + (1-\alpha)\left[V_0 - \frac{1}{2}c\left(I_A\right) - \frac{1}{2}c\left(I_B\right)\right] \end{aligned} \tag{19}$$

where $1_{\tau=A}$ is the indicator variable for the state $\omega_A$ (it equals 1 if the project if focused, and 0 if it is diversifying). Thus, the outsiders' minimum expected utility is given by

$$U_1^S\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right) = \min_{\theta \in C} \mathbb{E}\left[u\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right); \theta\right].$$

Because the outsider learns nothing at $t = 1$, $U_0^{S,d} = U_1^{S,d}$ with probability 1. Thus, the outsider's payoff under delegation, $\delta = d$, is $U_0^{S,d}\left(I_A, I_B\right)$. What investment policy would outsiders prefer the insider implement? The optimal levels of investment, $I_\tau^{S*}$, are given by

$$\left\{I_A^{S*}, I_B^{S*}\right\} = \underset{I_A, I_B}{\arg\max}\ U_1^S\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right)$$

and are characterized in the following lemma.

[35] Because the insider will execute balanced investment, $I_A = I_B = I^e$, outsiders are indifferent between a focused project and a diversifying project. Thus, the results of this section follow even if outsiders treat the randomization over project type as ambiguous.

**Lemma 4** *If ambiguity-averse outsiders do not have control, their ex-ante optimal investment levels are* $I_A^{S*} = I_B^{S*} = I^e$.

Lemma 4 shows that ambiguity-averse outsiders would like to commit to the level of investment chosen by SEU agents (because the insider is SEU, she will select $I_A = I_B = I^e$ by Theorem 2). This result depends on our assumptions that outsiders have a balanced portfolio, $w_B = w_A + (1-\alpha) V_A$, and that the SEU beliefs are $\theta^e = \frac{1}{2}(\theta_0 + \theta_1)$. Therefore, by delegating control to an expected utility insider, outsiders earn the payoff $U_0^{S,d}(I^e, I^e)$.

Alternatively, if outsiders maintain control, the optimal levels of investment $I_\tau$ in state $\omega_\tau$ is determined in a way similar to Theorem 1. With a focused project, that is, in state $\omega_A$, an investment level of $I_A$ gives the outsiders the portfolio $\Pi^S(I_A, 0)$. Thus, the beliefs held by ambiguity-averse outsiders, $\theta^{S,a}\left(\Pi^S(I_A,0)\right)$, are given by (10) where the portfolio-distorted beliefs, $\tilde{\theta}^{S,a}\left(\Pi^S(I_A,0)\right)$, are

$$\tilde{\theta}^{S,a}\left(\Pi^S(I_A,0)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B}{w_A + (1-\alpha)(V_A + I_A)}\right]. \tag{20}$$

The optimal investment $I_A^{S,a}$ is determined by maximizing the outsiders' minimum expected utility

$$I_A^{S,a} = \arg\max_{I_A} U_1^S\left(\Pi^S(I_A,0)\right), \tag{21}$$

where the outsiders' MEU is equal to

$$U_1^S\left(\Pi^S(I_A,0)\right) = \mathbb{E}\left[u\left(\Pi^S(I_A,0)\right); \theta^{S,a}\left(\Pi^S(I_A,0)\right)\right], \text{ and} \tag{22}$$

$\mathbb{E}\left[u\left(\Pi^S(I_A,0)\right);\theta\right] = e^{\theta-\theta_1}\left[w_A + (1-\alpha)\left[V_A + I_A\right]\right] + e^{\theta_0-\theta} w_B + (1-\alpha)\left[V_0 - c(I_A)\right]$. The optimal level of investment for the ambiguity-averse outsiders is determined by the combination of the "marginal cost" and the "pessimism" effects we discussed above. Note that the impact of "pessimism" depends on the outsiders' overall portfolio. Because outsiders are well-diversified, $w_A + (1-\alpha)\left[V_A + I_A\right] > w_B$ for $I_A > 0$. This implies $\tilde{\theta}^{S,a}\left(\Pi^S(I_A,0)\right) < \theta^e$ and, thus, that the ambiguity-averse outsiders are pessimistic ex-post on type-$A$ assets relative to an SEU agent. This implies that $I_A^{S,a} < I_A^e$, or equivalently, outsiders underinvest in focused projects, relative to what they would like to commit to a prior from Lemma 4.

Similarly, when the firm has a diversifying project, that is in state $\omega_B$, an investment level of $I_B$ gives the outsiders the portfolio $\Pi^S(0, I_B)$. The outsiders' beliefs, $\theta^{S,a}\left(\Pi^S(0, I_B)\right)$, are given by (10) where portfolio-distorted beliefs, $\tilde{\theta}^{S,a}\left(\Pi^S(0, I_B)\right)$, now are equal to

$$\tilde{\theta}^{S,a}\left(\Pi^S(0, I_B)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B + (1-\alpha) I_B}{w_A + (1-\alpha) V_A}\right]. \tag{23}$$

Thus the optimal investment level, $I_B^{S,a}$, is chosen by the outsiders by maximizing their minimum expected utility

$$I_B^{S,a} = \arg\max_{I_B} U_1^S\left(\Pi^S(0, I_B)\right), \tag{24}$$

where the outsiders' MEU is now equal to

$$U_1^S\left(\Pi^S(0, I_B)\right) = \mathbb{E}\left[u\left(\Pi^S(0, I_B)\right); \theta^{S,a}\left(\Pi^S(0, I_B)\right)\right]. \tag{25}$$

Note

$$\begin{aligned}\mathbb{E}\left[u\left(\Pi^S(0, I_B)\right); \theta\right] &= e^{\theta-\theta_1}\left[w_A + (1-\alpha) V_A\right] + e^{\theta_0-\theta}\left[w_B + (1-\alpha) I_B\right] \\ &\quad + (1-\alpha)\left[V_0 - c(I_B)\right].\end{aligned}$$

Because outsiders are diversified a priori, $w_B + (1-\alpha) I_B > w_A + (1-\alpha) V_A$ for $I_B > 0$. This implies $\tilde{\theta}^{S,a}\left(\Pi^S(0, I_B)\right) > \theta^e$ and, thus, that the ambiguity averse outsiders are pessimistic ex-post on type-$B$ assets relative to an SEU agent. This implies that $I_B^{S,a} < I_B^e$ and that the outsiders underinvest in the diversifying project as well.

The above discussion implies that well-diversified ambiguity-averse outsiders underinvest in both focused and diversifying projects, relative to what they would like to commit to ex ante from Lemma 4, leading to the following Lemma.

**Lemma 5** *If they have control, well-diversified ambiguity-averse outsiders make the same investment in focused and diversified projects, and underinvest with respect to a SEU agent,* $I_A^{S,a} = I_B^{S,a} < I^e$. *In addition, underinvestment is more severe when the firm is large (relative to the outsiders' overall portfolio).*

By combining (22) and (25) we obtain that if the outsiders maintain control, that is, $\delta = r$, the ex-ante expected utility that ambiguity-averse outsiders can obtain is

$$U_0^{S,r}\left(I_A^{S,a}, I_B^{S,a}\right) = \frac{1}{2}U_1^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right) + \frac{1}{2}U_1^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right) \tag{26}$$

Note that (26) assumes that the MEU outsiders do not show ambiguity aversion with respect to the state of the world $\omega_\tau$, and that they are again sophisticated in that they correctly anticipate they future beliefs due their ambiguity aversion.

Lemma 4 and Lemma 5 lead us to the following theorem, which is the main result for this section.

**Theorem 4** *Ambiguity-averse outsiders delegate control to an expected-utility insider.*

In summary, ambiguity-averse outsiders have two motivations to grant control to an expected-utility insider. First, outsiders would like to commit ex-ante to a level of investment that is not optimal ex-post if they maintain control. This happens because of the impact of ambiguity aversion on posterior beliefs, outsiders would underinvest ex-post in both types of projects. Second, outsiders would prefer to be blind to the realization of the interim state of the world $\omega_\tau$ (i.e., the type of project available to the firm), because knowledge of the project type exposes outsiders to additional ambiguity. Granting control to the insider allows outsiders to not see this information and to increase ex-ante expected utility.

The desirability to outsiders of delegation depends on firm characteristics:[36]

**Corollary 3** *Delegation is more desirable to outsiders when $Z$ is greater, that is, when the growth options are more valuable.*

Corollary 3 follows for two reasons. First, as growth options increase in value, that is for greater values of $Z$, outsiders' underinvestment worsens. This happens because an increase of the productivity of growth options $Z$, increases the values of both the investment level of SEU insider, $I^e$, and the investment level of MEU outsiders, $I_\tau^{S,a}$. However, the positive impact of $Z$ on investment

[36] Remember that in our comparative statics results, we will assume that $c(I) = \frac{1}{Z(1+\gamma)}I^{1+\gamma}$ for analytical tractability.

is greater in the case of $I^e$ than $I_\tau^{S,a}$, making the underinvestment problem of outsiders' retention of control more severe. The second effect is the adverse impact of information revelation on ambiguity-averse outsiders. Firms with more valuable growth options invest more (greater $I_\tau^{S,a}$), and the new investment becomes a larger portion of the outsiders' portfolio. From (20) and (23) it easy to see that greater investment levels leads to greater dispersion of the posteriors, $\tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right)$ and $\tilde{\theta}^{S,a}\left(\Pi^S\left(0,I_B\right)\right)$, which in turn exacerbates the outsiders' welfare loss due to ambiguity aversion. Together, these properties imply that while MEU outsiders always grant control to a SEU insider, the value creation from delegation is an increasing function of the value of firm's growth options.

We conclude this section by noting that while the SEU insider and well-diversified MEU outsiders agree ex-ante on the optimal level of investment in both projects, $I^e$, they disagree ex-post when they learn the state of the world $\omega_\tau$. In addition, MEU outsiders will be ex-post more "pessimistic" than the SEU insider. The ex-post disagreement between insiders and outsiders derives endogenously from the effect of ambiguity aversion on ex-post beliefs. In this way, this section mirrors results obtained in models with heterogenous priors.[37] However, in our model, outsiders are better off by delegating authority an SEU insider, even in face of ex-post disagreement. The value of delegation derives from the combination of time-inconsistency of desired investment levels and the welfare loss of information arrival that characterizes MEU agents.

### 3.4 Ambiguity Averse Outsiders, Ambiguity Averse Insider

The more interesting case is when both insider and outsiders are ambiguity averse, which provides the core results of our paper. The outsiders' choice of whether to implement a strong corporate governance system, and thus retain control, or to allow for a weak governance system and to delegate decision making to the firm's insider is based on the trade-off of two distinct effects.

First is the effect of control on investment. If outsiders retain control, they ex-post underinvest in both types of projects with respect to the level of investment that they would prefer ex-ante, $I^e$. From Corollary 3, we know that this effect is more severe when the growth options are more valuable. In contrast, if given control, the insider always underinvests in focused projects, but either overinvests or underinvests in diversifying projects, depending on the relative size of the existing

[37] For example, Boot, Gopalan, and Thakor (2006), Boot, Gopalan, and Thakor (2008), and Boot and Thakor (2011) all share these characteristics.

assets and the value of growth options (Theorem 3). In addition, from Corollary 2, the loss of value due to delegation is more severe when assets in place are very small (small $V_A$ relative to $I^e$) or very large (large $V_A$ relative to $I^e$).

Second is the negative impact of information resolution on ambiguity-averse agents. This effect always makes outsiders prefer to delegate control to the insider, all else equal. This effect is more severe when the outsiders' posterior beliefs differ substantially from their prior beliefs. From (20) and (23), it is easy to see that this happens when the level of investment is large relative to relative to the owners' outside endowment, that is when outsiders have a "small" portfolio. We now characterize these trade-offs explicitly, and derive comparative statics.

If outsiders retain control, $\delta = r$, they behave as described in Lemma 5. Thus, their payoff is equal to $U_0^{S,r}\left(I_A^{S,a}, I_B^{S,a}\right)$, defined in eq. (26). If given control, $\delta = d$, from Theorem 3, the insider chooses a level of investment $\{I_A^{M,a}, I_B^{M,a}\}$ as described in eq. (15) and (18), respectively. In this case, the outsiders expected utility is

$$\begin{aligned}\mathbb{E}\left[u\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right);\theta\right] &= e^{\theta-\theta_1}\left[w_A + (1-\alpha)\left(V_A + \frac{1}{2}I_A^{M,a}\right)\right] \\ +e^{\theta_0-\theta}\left[w_B + (1-\alpha)\frac{1}{2}I_B^{M,a}\right] &\quad +(1-\alpha)\left[V_0 - \frac{1}{2}c\left(I_A^{M,a}\right) - \frac{1}{2}c\left(I_B^{M,a}\right)\right]. \qquad (27)\end{aligned}$$

The outsiders' ex-ante minimum expected utility and their payoff under delegation of control, $\delta = d$, is given by

$$U_0^{S,d}\left(I_A^{M,a}, I_B^{M,a}\right) = \min_{\theta\in C}\mathbb{E}\left[u\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right);\theta\right].$$

The optimal allocation of control – that is, the strength of the firm's governance system – is then determined as follows. When the insider and the outsiders choose the same (or similar) levels of investment, outsiders are strictly better off delegating control to the insider than retaining control, i.e., to have a weak rather than a strong governance system. By delegating control to the insider, outsiders remain blind to the realization of the interim uncertainty, which increases their ex-ante payoff. In contrast a strong governance system (that is, retention of control) is optimal when the insider chooses investment levels that are very inefficient with respect to the investment that the outsiders would choose is they retained control.

The optimal allocation of control depends on both firm characteristics, $V_A$, $Z$, and the outsiders' overall portfolio size, $K$, as follows.

**Theorem 5** *There are critical values* $\{\underline{V}_A, \overline{V}_A, \underline{Z}, \bar{Z}, \bar{K}\}$,*with* $\underline{V}_A < \overline{V}_A$, $\underline{Z} \leq \bar{Z}$, *and* $\bar{K} > 0$, *such that the outsiders:*

1. *retain control for all* $V_A < \underline{V}_A$ *and for all* $V_A > \overline{V}_A$, *and delegate for all* $V_A \in \left(\underline{V}_A, \overline{V}_A\right)$;
2. *retain control for all* $Z < \underline{Z}$ *and delegate control for all* $Z > \bar{Z}$, *where* $\bar{Z}\left(V_A\right)$ *is U-shaped;*
3. *retain control if* $K \geq \bar{K}$.

A strong governance system is optimal when the value of assets in place, $V_A$, is either sufficiently small, $V_A \leq \underline{V}_A$, or sufficiently large, $V_A \geq \overline{V}_A$. In these cases, the insider and outsiders strongly disagree on the optimal levels of investment. Specifically, relative to the outsiders' desired investment levels, the insider underinvests (i.e., is more pessimistic) always in focused projects and in diversifying projects when $V_A \leq \underline{V}_A$, yet the insider overinvests in diversifying projects (i.e., is more optimistic) when $V_A \geq \overline{V}_A$. Thus, in both cases outsiders prefers to retain control in order to select a level of investment better aligned with their ex-ante objectives, even at the cost of being exposed to the adverse effect of information revelation. In the intermediate range, where $V_A \in \left(\underline{V}_A, \overline{V}_A\right)$, insider's and outsiders' optimal investment policies are more closely aligned, limiting disagreement. Thus, weak governance, where the insider has more freedom to decide the firm's investment policy, is optimal.

A strong governance system (retention) is also optimal for less productive firms (low values of $Z$) or when outsiders have a sufficiently large portfolio (a large value of $K$). This happens because in both cases the realization of the project type (the state $\omega_\tau$) has a small impact on outsiders' wealth levels. In this case, the adverse effect of information revelation on the outsiders and the efficiency losses due to underinvestment are both small. Thus, outsiders are better-off by establishing a strong governance system and retaining control. Conversely, in more productive firms (large $Z$) or when the firm is sufficiently large component of the outsiders' portfolio, outsiders optimally delegate control to the insider by implementing a weak governance system. Finally, note that the value of assets in place has a non-monotonic effect on the threshold $\bar{Z}\left(V_A\right)$.

The effect of the corporate governance system on firm investment policy is examined in the following corollary.

**Corollary 4** *Under retention, investment in diversified and focused projects are balanced. Under delegation, investment in diversified projects is larger than investment in focused projects.*

Corollary 4 has the interesting result that if control is retained by outsiders, the firm has balanced investment ($I_A^{S,a} = I_B^{S,a}$), while the insider, granted control, overinvests in diversifying projects ($I_B^{M,a} > I_A^{M,a}$). This means that firms endowed with a strong governance system follow a more balanced investment policy than firms endowed with a weak governance system, which overinvest in diversifying projects. These results hold even though governance is optimally chosen.

For the discussion in the remainder of this section, it is helpful to define the value of delegation as the difference in firm value under delegation and retention: $U_0^{S,d} - U_0^{S,r}$. Note that this difference can also be interpreted as the differential value of firms with weak and strong corporate governance systems, and is characterized in the following.

**Corollary 5** *The value of delegation,* $U_0^{S,d} - U_0^{S,r}$*, is*

1. *decreasing in outside portfolio size $K$ for well diversified portfolios if $\gamma \geq \underline{\gamma} \equiv \frac{\hat{\theta}_1 - \hat{\theta}_0}{\ln 2}$;*
2. *decreasing in the outsiders' endowment in $w_B$;*
3. *increasing in the productivity of the growth options, $Z$, if $Z$ is large enough.*

Point 1 of Corollary 5 follows by a similar intuition to Theorem 5: increasing the size of the outside portfolio diminishes the impact on outsiders of the adverse effect of information revelation due to ambiguity, reducing the benefits of delegation. The intuition for Point 2 is as follows. First, the value of delegation decreases in the size of the outside portfolio, as shown in Point 1. In addition, increasing $w_B$ also increases the ex-ante disagreement between outsiders and the insider, aggravating the difference of opinion on desired investment. Both of these effects make retention more attractive.[38] Finally, Point 3 derives from the fact that an increase of the productivity, $Z$, increases as the value of the growth options and the adverse effect of ambiguity aversion on outsiders, making delegation more attractive. This happens only when the productivity parameter $Z$ is sufficiently large, because an increase of productivity, $Z$, has an indeterminate effect on the disagreement between the insider and outsiders on investment, which drives the costs of delegation.

[38] Note that these effects work in opposite directions for $w_A$, so we cannot derive comparative statics for $w_A$. Numerical simulations, reported below, suggest the comparative statics are nonmonotonic in $w_A$.

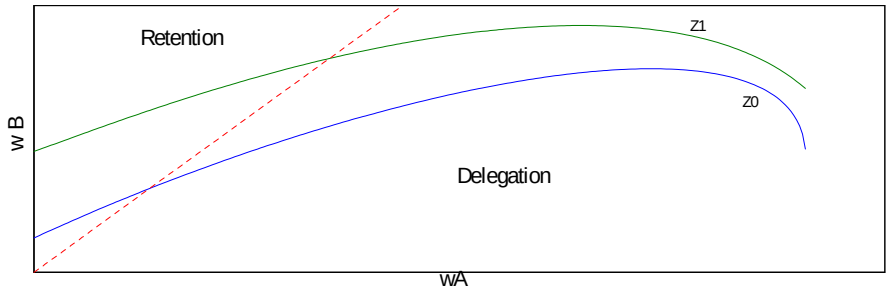


Figure 1: Indifference Curve for Outside Portfolio. The solid lines plot the indifference curve between delegation and retention for different levels of productivity, $Z$. The dotted line plots $w_B = w_A + (1-\alpha)\, V_A$. An increase in $w_B$ causes retention to be more attractive, while an increase in $w_A$ is non-monotonic, favoring first delegation and then retention. When $Z$ is larger, this cutoff increases: $Z1 > Z0$.

In the proof of Theorem 5, however, we have shown that when $Z$ is sufficiently large, both insider's and outsiders' beliefs converge to the lower end of the core-belief set (i.e., to the "worst-case" scenario), progressively eliminating this disagreement.

We now present several numerical comparative statics results that correspond to Theorem 5 and Corollary 5. Figure 1 shows the indifference curve between retention and delegation as a function of the outsiders' endowment, $w_A$ and $w_B$. First note that, as shown in Point 1 of Corollary 5, when the outsiders have a diversified portfolio (that is, along the dotted line) weak governance is optimal when the outsiders' portfolio is relatively small, that is, it is closer to the origin, and strong governance is optimal for larger outsiders' portfolios, that is for values of $w_A$ and $w_B$ further away from the origin on the dotted line. Second, as shown in Point 2 of Corollary 5, retention is more attractive when $w_B$ is larger. Finally, as anticipated, the effect of $w_A$ is nonmonotonic. Increasing $w_A$ decreases disagreement between the insider and outsiders, making delegation more attractive. This happens because insider's and outsiders' portfolios become more similar, decreasing disagreement. Increasing $w_A$ also decreases the importance of new investment to outsiders, reducing the adverse welfare effect of ambiguity and, thus, making retention more attractive. This happens because the dispersion of ex post beliefs decreases as $w_A$ increases. The disagreement effect dominates for small values of $w_A$, while the portfolio effect dominates for large values of $w_A$, resulting in the inverted U-shaped relationship. Finally, as shown in Point 3 of Corollary 5, delegation is more attractive when $Z$, the value of growth options, is larger.

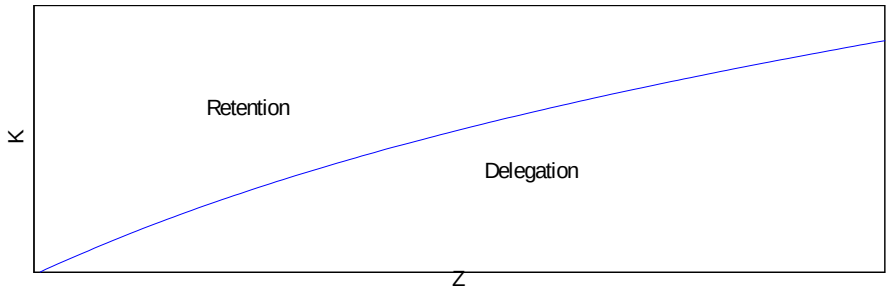


Figure 2: Indifference Curve for Productivity, $Z$, and Outside Portfolio, $K$. Increasing the outside portfolio of outsiders, $K$ (where $w_A = K-(1-\alpha)\,V_A$ and $w_B = K$), makes retention more attractive, but increasing the growth options, $Z$, makes delegation more attractive.

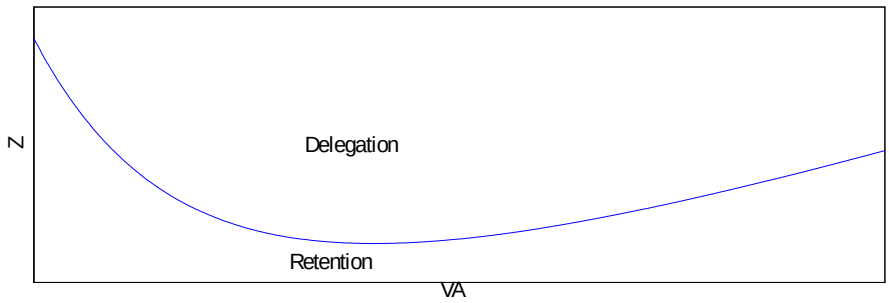


Figure 3: Indifference Curve for Assets in Place, $V_A$, and Productivity, $Z$. The line plots the indifference curve between delegation and retention against productivity, $Z$, and assets in place, $V_A$. An increase in $Z$ causes delegation to be more attractive, while the relationship with $V_A$ is nonmonotonic.

Figure 2 displays the indifference curve between strong governance (retention) and weak governance (delegation) as a function of the size of the outside portfolio, $K$, and the value of growth options, $Z$. The outsiders' portfolio is assumed to be diversified, that is, $w_A+(1-\alpha)\,V_A \;=\; w_B = K$. As shown in Point 1 of Corollary 5, strong governance (retention) is more attractive as the outside portfolio becomes larger. As shown in Point 3 of Corollary 5, weak governance (delegation) becomes more attractive as the value of growth options increases.

Finally, Figure 3 displays the indifference curve between strong governance (retention) and weak governance (delegation) as a function of productivity of growth options, $Z$, and the value of assets in place, $V_A$. First, note that, as suggested in Point 3 of Corollary 5, as productivity $Z$ increases, delegation becomes more attractive. Second, note that, as stated in Point 2 of Theorem 5, the relationship between $Z$ and $V_A$ is a U-shaped function. This feature reflects the fact that, as shown

in Theorem 5, for a given level of $Z$, for small values of $V_A$, retention is optimal, for intermediate values of $V_A$, delegation is optimal, while for larger values of $V_A$, retention is once again optimal.

# 4 Empirical Implications

We determine the optimal governance structure of firms in the presence of disagreement between firm insiders and outsiders. In our model, disagreement emerges endogenously among ambiguity-averse agents with heterogenous portfolios. The strength of a firm's corporate governance system and the allocation of control is made by the firm's outsiders, who maximize their ex-ante MEU. The optimal allocation of control depends on both firm characteristics and the overall portfolio composition of the firm's outsiders. In addition, ambiguity aversion endogenously creates a direct link between corporate governance and firm transparency.

The model has the following empirical and policy implications.

*1. Corporate governance life-cycle:* If the value of a firm's assets in place increases over a firm's life cycle (relative to the value of its growth opportunities), Point 2 of Theorem 5 suggests that firms should follow a governance structure life cycle. In particular, in the earlier stages of development, a young, high-growth firm should have a strong governance system; as the firm ages, it should move to a weak corporate governance system where firm insiders have discretion over investment decisions. Finally, as the firm matures, it should revert back to strong governance system. Because we expect it is easy to give control to a CEO but difficult to take back control, this suggests a role for LBOs as a mechanism for outside investors to regain control.

*2. Corporate governance and market-to-book ratio:* Point 2 of Theorem 5 also suggests that a non-monotonic relationship between the strength of a firm's corporate governance system and its market-to-book ratio. This property may be seen as follows. Taking $V_A$ as a proxy for book value, and the value of the firm to outsiders as market value, Point 2 of Theorem 5 demonstrates that we should observe strong governance systems at the two extremes of growth firms and value firms, but weak governance in the middle of the spectrum.

*3. Well-diversified outsiders, where the firm represents a small fraction of their overall portfolio, prefer strong governance system.* This result follows from Theorem 5 and suggests that, all else

equal, small firms, that are more likely to represent a smaller proportion of the owners' portfolio, should have a strong governance system. Conversely, larger firms, which are more likely to represent a greater proportion of their owners' portfolio, should have weak governance. In addition, firms with well-diversified owners, such as a mutual fund, are more likely to have strong governance.

Corollary 5 also suggests that outsiders whose portfolio is focused in sectors different from the firm's core business prefer a strong governance system, while outsiders whose portfolio has the same focus as the firm's core business are more likely to prefer a weak governance system. This means that generalist venture capital or private equity funds should impose strong governance systems on their portfolio companies, while specialized funds are more willing to tolerate weak governance systems, where the management of their portfolio companies have more leeway in determining company corporate policies. In addition, Corollary 5 also implies that diversified outsiders implement a strong governance system in firms with less productive growth options, but implement a weak governance system in firms with more productive growth options. This result suggests that, all else equal, more valuable firms and firms with more productive growth options (higher $Z$) should have weak governance. Firms with less productive growth options (smaller $Z$) should have strong governance.

*4. A decline in firm productivity leads to stronger corporate governance system.* Point 3 in Corollary 5 shows that a weak governance system is more valuable to the outsiders when the firm is more productive, and that the firm should switch to a strong corporate governance system when productivity decreases. This suggests that a weakening of a firm productivity leads to a strengthening of its corporate governance system. As suggested above, a stronger governance system may be obtained by having the outsiders take over the firm through a LBO. This means that weaker firm performance may lead to going-private transactions.

5. *Weak corporate governance systems should also be less transparent.* Firms with weak corporate governance systems should also be more opaque. If the outsiders retain control, the firm discloses relevant information (in our model, the project type) so that outsiders can make an informed decision. If outsiders delegate control, they require that the insider do not disclose the type of project. Thus, the model predicts that outsider controlled firms with a strong governance system will be more transparent, while insider controlled firms with a weak governance system will

be more opaque.

These observations have implications for the regulation of corporate disclosure. If the government were to implement mandatory disclosure regulation, firms would, in general, be harmed. Mandatory disclosure regulation destroys the benefit of delegation, insulation from ambiguity, so outsiders at all firms would find it optimal to retain control. However, firms that would have found it optimal to delegate control to the insider would be harmed. The harmful effect of mandatory disclosure regulation would be worse if control rights have already been delegated to the insider.

*6. Weak-governance firms overinvest in diversifying projects relative to their investment in focused projects. Strong-governance firms implement balanced investment in focused and diversifying projects.* This result, which follows directly from Corollary 4, implies that firms with weak corporate governance systems tend to be more diversified than comparable firms with a stronger governance system. In addition, weak-governance firms diversifying projects underperform ex post focused projects, while in strong-governance firms, ex post performance is similar for focused and diversifying projects. This observation can be seen as follows. A measure of ex-post performance can be obtained by defining $R(I) \equiv I/c(I)$ as the return on investment for a given project. It is easy to verify that $R(I)$ is strictly decreasing in $I$ (from convexity of $c(I)$). From Corollary 4, this implies that firms with weak governance systems underperform in their diversifying investments, $R\left(I_B^{M,a}\right) < R\left(I_A^{M,a}\right)$, while firms with strong governance systems have a more uniform performance across divisions.

*7. Private firms have stronger governance and greater transparency with their shareholders than public firms and will follow different investment policies.* The results of our paper suggest that public and private firms are characterized by different corporate governance structures. In particular, young private firms that are at the earlier stage of development have access to valuable growth opportunities and have large inside ownership. From Theorem 5, our paper suggests that these firms should have strong governance and, therefore, be transparent with their shareholders. In contrast, more mature public firms have a greater proportion of assets in place relative to their growth opportunities. Our model predicts that these firms should optimally have a weaker and less transparent governance system, where insiders have more control on their firm's decision. In addition, from Corollary 4, our model predicts that public firms will invest relatively more in

diversifying projects than equivalent privately held firms, who in contrast will be more focused.

# 5 Conclusions

We study a model where agents' ambiguity aversion generates endogenously differences of opinion between a firm's insider and its outsiders. We show that the allocation of control and, thus, the strength of the corporate governance system, depends on firm characteristics and the portfolio composition of both the insider and outsiders. We predict that small firms and less productive firms should have stronger governance, while larger firms and more productive firms should have weaker governance systems. Even with optimally chosen governance, firms with weak corporate governance will overinvest in diversifying projects relative to their investment in focused projects, and the diversifying projects underperform the focused projects ex post. In addition, we predict that firms should display a corporate governance life cycle, where both younger and more mature firms should be characterized by a stronger corporate governance system, while firms at their intermediate deployment stage have weaker governance, where the firm insiders have more discretion over corporate investment decisions. Finally, we argue that weaker governance systems are optimally less transparent.

# References


[1] Acemoglu, D., Chernozhukov, V., and Yildiz, M. (2009) "Fragility of Asymptotic Agreement under Bayesian Learning," MIT Working Paper.

[2] Adams, R. and Ferreira, D. (2007) "A Theory of Friendly Boards," *Journal of Finance*, **62**: 217-250.

[3] Aghion, P. and Tirole, J. (1997) "Formal and Real Authority in Organizations," *Journal of Political Economy*, **105**: 1-29.

[4] Anscombe, F. and Aumann, R. (1963) "A Definition of Subjective Probability," *The Annals of Mathematical Statistics*, **34**: 199-205.

[5] Basak, S. (2000), "A Model of Dynamic Equilibrium Asset Pricing with Heterogeneous Beliefs and Extraneous Risk," *Journal of Economic Dynamics and Control*, **24**: 63-95.

[6] Bayar, O., Chemmanur, T., and Liu, M. (2011) "A Theory of Equity Carve-Outs and Negative Stub Values under Heterogeneous Beliefs," *Journal of Financial Economics*, **100**: 616-638.

[7] Becht, M., Bolton, P., and Roell, A. (2003) "Corporate Governance and Control," in: G.M. Constantinides, M. Harris, & R. M. Stulz (ed.), *Handbook of the Economics of Finance*, edition 1, volume 1, chapter 1, pages 1-109, Elsevier.

[8] Berle, A. and Means, G. (1967) *The Modern Corporation and Private Property* (revised edition). New York: Harcourt, Brace and World, Inc.

[9] Boot, A., Gopalan, R. and Thakor, A. (2006) "The Entrepreneur's Choice between Private and Public Ownership," *Journal of Finance*, **61**: 803–836.

[10] Boot, A., Gopalan, R. and Thakor, A. (2008) "Market Liquidity, Investor Participation and Managerial Autonomy: Why do Firms Go Private?" *Journal of Finance*, **63**: 2013-2059.

[11] Boot, A. and Thakor, A. (2001) "The Many Faces of Information Disclosure," *Review of Financial Studies*, **14**: 1021-1057.

[12] Boot, A. and Thakor, A. (2011) "Managerial Autonomy, Allocation of Control Rights, and Optimal Capital Structure," *Review of Financial Studies*, **24**: 3434-3485.

[13] Caskey, J. (2009) "Information in Equity Markets with Ambiguity-Averse Investors," *Review of Financial Studies*, **22**: 3595-3627.

[14] Célérier, C. and Vallée, B. (2014) "What Drives Financial Complexity? A Look into the Retail Market for Structured Products" HEC Paris Research Paper No. FIN-2013-1013, SSRN Working Paper 2289890.

[15] Chakraborty A. and Yilmaz, B. (2013) "Authority, Consensus, and Governance," Wharton Working Paper.

[16] Colacito, R. and Croce, M. (2012) "International Robust Disagreement," *American Economic Review*, **102**: 152-155.

[17] Coval, J., Jurek, J., and Stafford, E. (2009) "Economic Catastrophe Bonds," *American Economic Review*, **99**: 628-666.

[18] Cripps, M., Ely, J., Mailath, G. and Samuelson, L. (2008) "Common Learning," *Econometrica*, **76**: 909-933.

[19] David, A. (2008) "Heterogeneous Beliefs, Speculation, and the Equity Premium," *Journal of Finance,* **63**: 41-83.

[20] de Finetti, B. (1974) *Theory of Probability: A Critical Introductory Treatment.* John Wiley & Sons Ltd.

[21] Dessein, W. (2002) "Authority and Communication in Organizations," *Review of Economic Studies*, **69**: 811-838.

[22] Detemple, J. and Murthy, S. (1994) "Intertemporal asset pricing with heterogeneous beliefs," *Journal of Economic Theory*, **62**: 294–320.

[23] Easley, D. and O'Hara, M. (2009) "Ambiguity and Nonparticipation: The Role of Regulation," *Review of Financial Studies*, **22**: 1817-1843.

[24] Easley, D. and O'Hara, M. (2013) "Microstructure and Ambiguity," *Journal of Finance*, **65**: 1817-1846.

[25] Epstein, L. and Schneider, M. (2003) "Recursive Multiple Priors," *Journal of Economic Theory*, **113**: 1-31.

[26] Epstein, L. and Schneider, M. (2007) "Learning Under Ambiguity," *Review of Economic Studies*, **74**: 1275-1303.

[27] Epstein, L. and Schneider, M. (2008) "Ambiguity, Information Quality and Asset Pricing," *Journal of Finance*, **63**, 197–228.

[28] Epstein, L. and Schneider, M. (2011) "Ambiguity and Asset Markets," *Annual Review of Financial Economics*, **2**: 315-346.

[29] Ferreira, D. and Rezende, M. (2007) "Corporate Strategy and Information Disclosure," *Rand Journal of Economics*, **38**: 164-184.

[30] Fishman, M. and Hagerty, K. (2003) "Mandatory Versus Voluntary Disclosure in Markets with Informed and Uninformed Customers," *Journal of Law, Economics, and Organization*, **19**: 45-63.

[31] Garlappi, L., Giammarino, R., and Lazrak, A. (2013) "Ambiguity in Corporate Finance: Real Investment Dynamics," SSRN Working Paper 2214557.

[32] Gilboa, I. and Schmeidler, D. (1989) "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics*, **18**: 141-153.

[33] Gopalan, R., Milbourn, T., and Song, F. (2010) "Strategic Flexibility and the Optimality of Pay for Sector Performance," *Review of Financial Studies*, **23**: 2060-2098.

[34] Grossman, S. (1976) "On the Effciency of Competitive Stock Markets Where Traders Have Diverse Information," *Journal of Finance* **31**: 573-585.

[35] Harris, M. and Raviv, A. (1993) "Differences of Opinion Make a Horse Race," *Review of Financial Studies*, **6**: 473-506.

[36] Harris, M. and Raviv A. (2005) "Allocation of Decision-Making Authority," *Review of Finance*, **9**: 353-383.

[37] Harris, M. and Raviv, A. (2008) "A Theory of Board Control and Size," *Review of Financial Studies*, **21**: 1797-1832.

[38] Harris, M. and Raviv, A. (2010) "Control of Corporate Decisions: Shareholders vs. Management." *Review of Financial Studies*, **23**: 4115-4147.

[39] Harrison, M. and Kreps, D. (1978) "Speculative Investor Behavior in a Stock Market with Heterogenous Expectations," *Quarterly Journal of Economics*, **92**: 323-336.

[40] Henderson, B. and Pearson, N. (2011) "The Dark Side of Financial Innovation: A Case Study of the Pricing of a Retail Financial Product," *Journal of Financial Economics*, **100**: 227-47.

[41] Huang, S. and Thakor A. (2013) "Investor Heterogeneity, Investor-Management Disagreement and Share Repurchases," *Review of Financial Studies*, forthcoming.

[42] Jensen, M. (1989) "Eclipse of the Public Corporation," *Harvard Business Review*, **67**:5, 61-74.

[43] Kandel, E. and Pearson, N. (1995) "Differential Interpretation of Public Signals and Trade in Speculative Markets," *Journal of Political Economy,* **104**: 831-872.

[44] Keynes, J. (1921) *A Treatise on Probability*. London: Macmillan.

[45] Klibanoff, P., Marinacci, M. and Mukerji, S. (2005) "A Smooth Model of Decision Making under Ambiguity," *Econometrica*, **73**: 1849–1892.

[46] Kogan, S., Routledge, B., Sagi, J., and Smith, N. (2010) "Information Content of Public Firm Disclosures and the Sarbanes-Oxley Act," SSRN Working Paper 1584763.

[47] Levit, D. and Malenko, N. (2014) "The Labor Market for Directors and Externalities in Corporate Governance," SSRN Working Paper 2101874.

[48] Maenhout, P. (2004) "Robust Portfolio Rules and Asset Pricing," *Review of Financial Studies*, **17**: 951-983.

[49] Morris, S. (1994) "Trade with Heterogeneous Prior Beliefs and Asymmetric Information," *Econometrica*, **62**: 1327-1347.

[50] Morris, S. (1995) "The Common Prior Assumption in Economic Theory," *Economics and Philosophy*, **11**: 227-253.

[51] Rappaport, A. (1990) "The Staying Power of the Public Corporation," *Harvard Business Review*, **68**:1, 96-104.

[52] Ross, S. (1973) "The Economic Theory of Agency: The Principal's Problem," *American Economic Review*, **63**: 134-139.

[53] Satu, Y. (2014) "Opacity in Financial Markets," *Review of Financial Studies*, 27: 3502-3546.

[54] Savage, L. J. (1972) *The Foundations of Statistics* (2nd revised edition). New York: Dover Publications.

[55] Shleifer, A. and Vishny R. (1997) "A Survey of Corporate Governance," *Journal of Finance*, **52**: 737-783.

[56] Siniscalchi, M. (2011) "Dynamic Choice under Ambiguity," *Theoretical Economics*, **6**: 379–421.

[57] Smith, A. (1776) *An Inquiry into the Nature and Causes of the Wealth of Nations*. Edwin Cannan, ed. 1904. Library of Economics and Liberty. 20 September 2013. <http://www.econlib.org/library/Smith/smWN.html>.

[58] Thakor, A. (2013) "Strategic Information Disclosure When There Is Fundamental Disagreement," *Journal of Financial Intermediation*, forthcoming.

[59] Van den Steen, E. (2004) "Rational Overoptimism (and Other Biases)," *American Economic Review*, **94**: 1141-1151.

[60] Van den Steen, E. (2009) "Authority versus Persuasion," *American Economic Review: Papers and Proceedings* **99**: 448–453.

[61] Van den Steen, E. (2010a) "Disagreement and the Allocation of Control." *Journal of Law, Economics & Organization* **26**: 385–426.

[62] Van den Steen, E. (2010b) "Interpersonal Authority in a Theory of the Firm," *American Economic Review*, **100**: 466–490.

# A Appendix: Proofs

**Proof of Lemma 1.** The claim holds due to a property of the minimum. Suppose we have a set of priors $\mu(w,s) \in M$, so that each prior gives the joint distribution of wealth, $w$, and the signal, $s \in S$, and all priors share a common support of $s$. For a given $s$, for each $\mu$, define $\mu(\cdot;s)$ as the conditional distribution of wealth given $s$ (each conditional distribution exists and is a.s. unique by the standard arguments).[39] For each $s$, define $M_s = \{\mu(\cdot;s) \,|\, \mu(w,s) \in \mathcal{M}\}$. By the definition of $\mu(\cdot;s)$, for all $\mu$ and for all $s$, $E_\mu[u(w)] = E_s E_{\mu(\cdot;s)}[u(w)]$. Define $\underline{\nu} = \arg\min_{\mu\in\mathcal{M}} E_\mu[u(w)]$, so that $\underline{\nu}$ is the worst-case scenario if the agent does not learn $s$. For each $s$, define $\underline{\nu}(\cdot;s)$ as the conditional distribution of $\underline{\nu}$ given $s$ is observed. Similarly, define $\nu(s) = \arg\min_{\mu(\cdot;s)\in\mathcal{M}_s} E_{\mu(\cdot;s)}[u(w)]$ as the worst-case scenario after the agent has learned $s$. By definition of $\nu(s)$, and because $\underline{\nu}(\cdot;s) \in M_s$, $E_{\underline{\nu}(\cdot;s)}[u(w)] \geq E_{\nu(s)}[u(w)]$. By the monotonicity of integration, this implies that $E_s E_{\underline{\nu}(\cdot;s)}[u(w)] \geq E_s E_{\nu(s)}[u(w)]$. Because $E_s E_{\underline{\nu}(\cdot;s)}[u(w)] = E_{\underline{\nu}}[u(w)] = \min_{\mu\in\mathcal{M}} E_s E_{\mu(\cdot;s)}[u(w)]$, and $E_s E_{\nu(s)}[u(w)] = E_s \min_{\mu(\cdot;s)\in\mathcal{M}_s} E_{\mu|s}[u(w)]$, the claim is shown. ■

**Proof of Lemma 2.** The minimization problem is $\min_{\theta\in C} E[u(\bar{w}_A, \bar{w}_B, \bar{w}_0)\,|\theta]$, where

$$E[u(\bar{w}_A, \bar{w}_B, \bar{w}_0);\theta] = e^{\theta-\theta_1}\bar{w}_A + e^{\theta_0-\theta}\bar{w}_B + \bar{w}_0.$$

Note that $\frac{\partial \mathbb{E}[u|\theta]}{\partial\theta} = e^{\theta-\theta_1}\bar{w}_A - e^{\theta_0-\theta}\bar{w}_B$, and $\frac{\partial^2 \mathbb{E}[u|\theta]}{\partial\theta^2} = e^{\theta-\theta_1}\bar{w}_A + e^{\theta_0-\theta}\bar{w}_B$. $\frac{\partial^2 \mathbb{E}[u|\theta]}{\partial\theta^2} \geq 0$ because $\bar{w}_A \geq 0$ and $\bar{w}_B \geq 0$ (usually, one or both of these inequalities will be strict, so $\frac{\partial^2 \mathbb{E}[u|\theta]}{\partial\theta^2} > 0$). Because the objective is convex, and $C$ is closed and connected, the solution is unique and continuous. Interior solutions to the inner problem satisfy $\frac{\partial \mathbb{E}[u|\theta]}{\partial\theta} = 0$. Let

$$\tilde{\theta}^a(\Pi) = \frac{1}{2}(\theta_0+\theta_1) + \frac{1}{2}\ln\left[\frac{\bar{w}_B}{\bar{w}_A}\right].$$

If $\tilde{\theta}^a(\Pi) < \hat{\theta}_0$, $\frac{\partial \mathbb{E}[u|\theta]}{\partial\theta} > 0$ for all $\theta \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$, so $\theta^a(\Pi) = \hat{\theta}_0$. Similarly, if $\tilde{\theta}^a(\Pi) > \hat{\theta}_1$, $\frac{\partial \mathbb{E}[u|\theta]}{\partial\theta} < 0$ for all $\theta \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$, so $\theta^a(\Pi) = \hat{\theta}_1$. Therefore, the endogenous beliefs are given by (7). ■

**Proof of Lemma 3.** This property may be seen from the minimax theorem, as follows. From (4) we have that $\frac{dU}{d\bar{w}_A} = \frac{\partial \mathbb{E}(u)}{\partial \bar{w}_A} + \frac{\partial \mathbb{E}(u)}{\partial\theta}\frac{d\theta^a(\Pi)}{d\bar{w}_A}$. The second term is uniformly zero, since for interior solutions we have that $\frac{\partial \mathbb{E}(u)}{\partial\theta} = 0$, and for corner solutions we have that $\frac{d\theta^a(\Pi)}{d\bar{w}_A} = 0$. Thus, $\frac{dU}{d\bar{w}_A} = \frac{\partial \mathbb{E}(u)}{\partial \bar{w}_A} = e^{\theta^a(\Pi)-\theta_1} > 0$, and $\frac{d^2U}{d(\bar{w}_A)^2} = e^{\theta^a(\Pi)-\theta_1}\frac{d\tilde{\theta}^a(\Pi)}{d\bar{w}_A} \leq 0$, because $\frac{d\tilde{\theta}^a(\Pi)}{d\bar{w}_A} \leq 0$. These inequalities are strict for interior $\tilde{\theta}^a$. Similarly, $\frac{dU}{d\bar{w}_B} = e^{\theta_0-\theta^a(\Pi)} > 0$ and $\frac{d^2U}{d(\bar{w}_B)^2} = -e^{\theta_0-\theta^a(\Pi)}\frac{d\tilde{\theta}^a(\Pi)}{d\bar{w}_B} \leq 0$, because $\frac{d\tilde{\theta}^a(\Pi)}{d\bar{w}_B} \geq 0$, with strict inequalities for interior $\tilde{\theta}^a(\Pi)$. ■

**Proof of Theorem 1.** Consider type $A$ projects. Investment is chosen to maximize

$$U_1(\Pi(I_A,0)) \equiv \min_{\theta\in C} \mathbb{E}[u(\Pi(I_A,0));\theta],$$

where $\Pi(I_A,0) = \{w_A + V_A + I_A, w_B, w_0 + V_0 - c(I_A)\}$. Applying the envelope theorem (either $\frac{\partial}{\partial\theta}E[u;\theta^a] = 0$ or $\frac{d\theta^a}{dI_A} = 0$)[40], the benefit of increasing investment is

$$\frac{d}{dI_A}U_1(\Pi(I_A,0)) = e^{\theta^a-\theta_1} - c'(I_A),$$

because $\frac{\partial E[u;\theta^a]}{\partial w_A} = e^{\theta^a-\theta_1}$ and $\frac{\partial E[u;\theta^a]}{\partial w_0} = 1$. However, the equilibrium beliefs depend on the level of investment,

[39] In our setting, $s$ is discrete, so we can express $\mu(w;s) = \frac{\mu(w,s)}{\sum_s \mu(w,s)}$, though our proof still applies with general distributions.

[40] With a focused project, $U_1$ and $\theta^a$ are understood to be functions of the portfolio $\Pi^a(I_A,0) = \{w_A + V_A + I_A, w_B, w_0 + V_0 - c(I_A)\}$. For ease of notation, we will not always write out $U_1(\Pi^a(I_A,0))$ and $\theta^a(\Pi^a(I_A,0))$, but that is how $U_1$ and $\theta^a$ should be interpreted. Similarly, for a diversified project, $U_1$ and $\theta^a$ understood to be functions of the portfolio $\Pi^a(0,I_B) = \{w_A + V_A, w_B + I_B, w_0 + V_0 - c(I_B)\}$.

similar to Lemma 2: $\tilde{\theta}^a\left(\Pi\left(I_A,0\right)\right)=\theta^e+\frac{1}{2}\ln\left[\frac{w_B}{w_A+I_A}\right]$ and

$$\theta^a\left(\Pi\left(I_A,0\right)\right)=\begin{cases}\hat{\theta}_0 & \tilde{\theta}^a\left(\Pi\left(I_A,0\right)\right)\leq\hat{\theta}_0\\ \tilde{\theta}^a\left(\Pi\left(I_A,0\right)\right) & \tilde{\theta}^a\left(\Pi\left(I_A,0\right)\right)\in\left(\hat{\theta}_0,\hat{\theta}_1\right)\\ \hat{\theta}_1 & \tilde{\theta}^a\left(\Pi\left(I_A,0\right)\right)\geq\hat{\theta}_1\end{cases}.$$

Note that $\frac{d^2}{dI_A^2}U_1=e^{\theta^a-\theta_1}\frac{d\theta^a}{dI_A}-c''\left(I_A\right)$. Because $\frac{d\theta^a}{dI_A}\leq 0$ and $c''>0$, $\frac{d^2}{dI_A^2}U_1\left(\Pi^a\left(I_A,0\right)\right)<0$, so FOCs are sufficient for a maximum.

Similarly, for a type $B$ project, investment is chosen to maximize

$$U_1\left(\Pi\left(0,I_B\right)\right)\equiv\min_{\theta\in C}\mathbb{E}[u(\Pi\left(0,I_B\right));\theta],$$

where $\Pi\left(0,I_B\right)=\left\{w_A+V_A,w_B+I_B,w_0+V_0-c\left(I_B\right)\right\}$. Applying the envelope theorem (either $\frac{\partial E[u;\theta^a]}{\partial\theta}=0$ or $\frac{d\theta^a}{dI_B}=0$),

$$\frac{d}{dI_B}U_1\left(\Pi\left(0,I_B\right)\right)=e^{\theta_0-\theta^a}-c'\left(I_B\right)$$

because $\frac{\partial E[u;\theta^a]}{\partial w_B}=e^{\theta_0-\theta^a}$ and $\frac{\partial E[u;\theta^a]}{\partial w_0}=1$, where $\tilde{\theta}^a\left(\Pi\left(0,I_B\right)\right)=\theta^e+\frac{1}{2}\ln\left[\frac{w_B+I_B}{w_A}\right]$ and

$$\theta^a\left(\Pi\left(0,I_B\right)\right)=\begin{cases}\hat{\theta}_0 & \tilde{\theta}^a\left(\Pi\left(0,I_B\right)\right)\leq\hat{\theta}_0\\ \tilde{\theta}^a\left(\Pi\left(0,I_B\right)\right) & \tilde{\theta}^a\left(\Pi\left(0,I_B\right)\right)\in\left(\hat{\theta}_0,\hat{\theta}_1\right)\\ \hat{\theta}_1 & \tilde{\theta}^a\left(\Pi\left(0,I_B\right)\right)\geq\hat{\theta}_1\end{cases}.$$

Also, $\frac{d^2}{dI_B^2}U_1=-e^{\theta_0-\theta^a}\frac{d\theta^a}{dI_B}-c''\left(I_B\right)$. Because $\frac{d\theta^a}{I_B}\geq 0$ and $c''>0$, $\frac{d^2}{dI_B^2}U_1<0$, so FOCs are sufficient for a maximum.

For comparative statics on $I_A$, note that

$$\frac{\partial}{\partial w_A}\left[\frac{d}{dI_A}U_1\right]=e^{\theta^a-\theta_1}\frac{\partial\theta^a}{\partial w_A}.$$

Because $\frac{\partial\theta^a}{\partial w_A}\leq 0$, with strict inequality for interior $\theta^a$, $\frac{\partial}{\partial w_A}\left[\frac{d}{dI_A}U_1\right]\leq 0$, with strict inequality for interior $\theta^a$. Because $\frac{d^2}{dI_A^2}U_1<0$, it follows that $\frac{dI_A}{dw_A}\leq 0$, with strict inequality for interior $\theta^a$. Therefore, optimal investment in a type $A$ project is decreasing in the portfolio position the player has in type $A$ assets. Similarly, $\frac{\partial}{\partial w_B}\left[\frac{d}{dI_A}U_1\right]=e^{\theta^a-\theta_1}\frac{\partial\theta^a}{\partial w_B}$. Because $\frac{\partial\theta^a}{\partial w_B}\geq 0$, $\frac{\partial}{\partial w_B}\left[\frac{d}{dI_A}U_1\right]\geq 0$. Because $\frac{d^2}{dI_A^2}U_1<0$, $\frac{dI_A}{dw_B}\geq 0$ (similarly, strict inequality for interior $\theta^a$). Therefore, optimal investment in a type $A$ project is increasing in the portfolio position the player has in type $B$ assets.

For comparative statics on $I_B$, note $\frac{\partial}{\partial w_A}\left[\frac{d}{dI_B}U_1\right]=-e^{\theta_0-\theta^a}\frac{\partial\theta^a}{\partial w_A}$. Because $\frac{\partial\theta^a}{\partial w_A}\leq 0$, $\frac{\partial}{\partial w_A}\left[\frac{d}{dI_B}U_1\right]\geq 0$. Because $\frac{d^2}{dI_B^2}U_1<0$, $\frac{dI_B}{dw_A}\geq 0$ (strict inequality for interior $\theta^a$). Therefore, optimal investment in a type $B$ project is increasing in the portfolio position the agent has in type $A$ assets. Also, $\frac{\partial}{\partial w_B}\left[\frac{d}{dI_B}U_1\right]=-e^{\theta_0-\theta^a}\frac{\partial\theta^a}{\partial w_B}$. Because $\frac{\partial\theta^a}{\partial w_B}\geq 0$, $\frac{\partial}{\partial w_B}\left[\frac{d}{dI_B}U_1\right]\leq 0$. Because $\frac{d^2}{dI_B^2}U_1<0$, $\frac{\partial I_B}{\partial w_B}\leq 0$ (strict inequality for interior $\theta^a$). ■

**Proof of Theorem 2.** First, consider a focused project. If outsiders have control, they choose $I_A$ to maximize

$$\mathbb{E}\left[u\left(\Pi^S\left(I_A,0\right)\right);\theta^e\right]=e^{\theta^e-\theta_1}\left[w_A+\left(1-\alpha\right)\left[V_A+I_A\right]\right]+e^{\theta_0-\theta^e}w_B+\left(1-\alpha\right)\left[V_0-c\left(I_A\right)\right],$$

so $\frac{d}{dI_A}E\left[u\left(\Pi^S\left(I_A,0\right)\right);\theta^e\right]=\left(1-\alpha\right)\left[e^{\theta^e-\theta_1}-c'\left(I_A\right)\right]$. Because $\theta^e=\frac{1}{2}\left(\theta_0+\theta_1\right)$, $e^{\theta^e-\theta_1}=e^{\frac{1}{2}(\theta_0-\theta_1)}$. Thus, outsiders set $I_A^{S,e}$ so that $c'\left(I_A^{S,e}\right)=e^{\frac{1}{2}(\theta_0-\theta_1)}$ ($\frac{d^2}{dI_A^2}E\left[u\left(\Pi^S\left(I_A,0\right)\right);\theta^e\right]=-\left(1-\alpha\right)c''\left(I_A\right)$ and $c$ is convex, so SOCs are satisfied). If the insider has control, she chooses $I_A$ to maximize

$$\mathbb{E}\left[u\left(\Pi^M\left(I_A,0\right)\right);\theta^e\right]=e^{\theta^e-\theta_1}\alpha\left[V_A+I_A\right]+\alpha\left[V_0-c(I_A)\right],$$

so $\frac{d}{dI_A}E\left[u\left(\Pi^M\left(I_A,0\right)\right);\theta^e\right]=\alpha\left[e^{\theta^e-\theta_1}-c'\left(I_A\right)\right]$. Thus, the insider chooses $I_A^{M,e}$ so that $c'\left(I_A^{M,e}\right)=e^{\frac{1}{2}(\theta_0-\theta_1)}$. Therefore, outsiders and the insider will choose the same level of investment for a focused project: $I_A^e\equiv I_A^{S,e}=I_A^{M,e}$.

Second, consider a diversified project. If outsiders have control, they choose $I_B$ to maximize

$$\mathbb{E}\left[u\left(\Pi^S\left(0,I_B\right)\right);\theta^e\right]=e^{\theta^e-\theta_1}\left[w_A+\left(1-\alpha\right)V_A\right]+e^{\theta_0-\theta^e}\left[w_B+\left(1-\alpha\right)I_B\right]+\left(1-\alpha\right)\left[V_0-c(I_B)\right],$$

so $\frac{d}{dI_B}E\left[u\left(\Pi^S\left(0,I_B\right)\right);\theta^e\right]=\left(1-\alpha\right)\left[e^{\theta_0-\theta^e}-c'\left(I_B\right)\right]$. Thus, outsiders choose $I_B^{S,e}$ so that $c'\left(I_B^{S,e}\right)=e^{\frac{1}{2}(\theta_0-\theta_1)}$. If the insider has control, she chooses $I_B$ to maximize

$$\mathbb{E}\left[u\left(\Pi^M\left(0,I_B\right)\right);\theta^e\right]=e^{\theta^e-\theta_1}\alpha V_A+e^{\theta_0-\theta^e}\alpha I_B+\alpha\left[V_0-c(I_B)\right],$$

so $\frac{d}{dI_B}E\left[u\left(\Pi^M\left(0,I_B\right)\right);\theta^e\right]=\alpha\left[e^{\theta_0-\theta^e}-c'\left(I_B\right)\right]$. Thus, the insider chooses $I_B^{M,e}$ so that $c'\left(I_B^{M,e}\right)=e^{\frac{1}{2}(\theta_0-\theta_1)}$. Therefore, the insider and outsiders will choose the same level of investment for a diversified project: $I_B^e\equiv I_B^{S,e}=I_B^{M,e}$. Because the same level of investment results independent of who is given control or which project is chosen, $I^e\equiv I_A^e=I_B^e$. Thus, the allocation of control does not matter. ■

**Proof of Theorem 3.** In this proof, we will consider optimal behavior by the insider. Because outsiders are not averse to ambiguity, they will behave as in Theorem 2 if they retains control. Further, they will retain control iff the insider acts suboptimally from their perspective.

For focused projects, the insider's minimum expected utility is

$$U_1^M\left(\Pi^M(I_A,0)\right)=\min_\theta\mathbb{E}\left[u\left(\Pi^M\left(I_A,0\right)\right);\theta\right],$$

where $E\left[u\left(\Pi^M\left(I_A,0\right)\right);\theta\right]=e^{\theta-\theta_1}\alpha\left(V_A+I_A\right)+\alpha\left(V_0-c\left(I_A\right)\right)$. Because she is exposed only to type $A$ assets, her worst-case scenario is $\theta^{M,a}\left(\Pi^M\left(I_A,0\right)\right)=\hat{\theta}_0$ (Lemma 2). Thus, her objective becomes

$$U_1^M\left(\Pi^M\left(I_A,0\right)\right)=e^{\hat{\theta}_0-\theta_1}\alpha\left(V_A+I_A\right)+\alpha\left(V_0-c\left(I_A\right)\right),$$

which implies $\frac{d}{dI_A}U_1^M\left(\Pi^M\left(I_A,0\right)\right)=\alpha\left[e^{\hat{\theta}_0-\theta_1}-c'\left(I_A\right)\right]$. Therefore, the insider chooses $I_A^{M,a}$ so that

$$c'\left(I_A^{M,a}\right)=e^{\hat{\theta}_0-\theta_1}.$$

Because $\hat{\theta}_0<\theta^e$, $I_A^{M,a}<I^e$, so the insider underinvests in focused projects.

For diversifying projects, the insider's objective is

$$U_1^M\left(\Pi^M\left(0,I_B\right)\right)=\min_\theta\mathbb{E}\left[u\left(\Pi^M\left(0,I_B\right)\right);\theta\right]$$

where $E\left[u\left(\Pi^M\left(0,I_B\right)\right);\theta\right]=e^{\theta-\theta_1}\alpha V_A+e^{\theta_0-\theta}\alpha I_B+\alpha\left[V_0-c\left(I_B\right)\right]$. For a given choice of $I_B$, she has the portfolio $\Pi^M\left(0,I_B\right)=\{\alpha V_A,\alpha I_B,\alpha\left(V_0-c\left(I_B\right)\right)\}$: her beliefs follow from Lemma 2 for a given level of investment $I_B$. Thus, her endogenous beliefs are given by $\theta^{M,a}$:

$$\theta^{M,a}\left(\Pi^M\left(0,I_B\right)\right)=\begin{cases}\hat{\theta}_0 & \tilde{\theta}^{M,a}\left(\Pi^M\left(0,I_B\right)\right)\leq\hat{\theta}_0\\ \tilde{\theta}^{M,a}\left(\Pi^M\left(0,I_B\right)\right) & \tilde{\theta}^{M,a}\left(\Pi^M\left(0,I_B\right)\right)\in\left(\hat{\theta}_0,\hat{\theta}_1\right)\\ \hat{\theta}_1 & \tilde{\theta}^{M,a}\left(\Pi^M\left(0,I_B\right)\right)\geq\hat{\theta}_1\end{cases},$$

where $\tilde{\theta}^{M,a}\left(\Pi^M\left(0,I_B\right)\right)=\theta^e+\frac{1}{2}\ln\left[\frac{I_B}{V_A}\right]$. Applying the minimax theorem, either $\frac{\partial\mathbb{E}u}{\partial\theta}=0$ or $\frac{d\theta^{M,a}}{dI_B}=0$, so $\frac{d}{dI_B}U_1^M\left(\Pi^M\left(0,I_B\right)\right)=\frac{\partial\mathbb{E}u}{\partial I_B}$. Because $\frac{\partial\mathbb{E}u}{\partial w_B}=e^{\theta_0-\theta^{M,a}}$ and $\frac{\partial\mathbb{E}u}{\partial w_0}=1$, $\frac{\partial\mathbb{E}u}{\partial I_B}=e^{\theta_0-\theta^{M,a}}-c'\left(I_B\right)$. Thus, the insider chooses investment $I_B^{M,a}$ so that

$$c'\left(I_B^{M,a}\right)=e^{\theta_0-\theta^{M,a}\left(\Pi^M\left(0,I_B^{M,a}\right)\right)}.$$

She may underinvest or overinvest in this situation. Totally differentiating with respect to $V_A$,

$$\left[c''\left(I_B^{M,a}\right)+e^{\theta_0-\theta^{M,a}\left(\Pi^M(0,I_B)\right)}\frac{\partial\theta^{M,a}}{\partial I_B}\right]\frac{dI_B^{M,a}}{dV_A}=-e^{\theta_0-\theta^{M,a}\left(\Pi^M(0,I_B)\right)}\frac{\partial\theta^{M,a}}{\partial V_A}.$$

For corner $\theta^{M,a}$, $\frac{\partial\theta^{M,a}}{\partial I_B}=\frac{\partial\theta^{M,a}}{\partial V_A}=0$, so $\frac{dI_B^{M,a}}{dV_A}=0$. For interior $\theta^{M,a}$, $\theta^{M,a}=\tilde{\theta}^{M,a}\left(\Pi^M\left(0,I_B\right)\right)$, so $\frac{\partial\theta^{M,a}}{\partial I_B}=\frac{1}{2I_B}$ and $\frac{\partial\theta^{M,a}}{\partial V_A}=-\frac{1}{2V_A}$, which implies $\frac{dI_B^{M,a}}{dV_A}>0$.

The optimal investment under expected utility, $I^e$, satisfies $c'\left(I^e\right)=e^{\frac{1}{2}(\theta_0-\theta_1)}$ (Theorem 2). If $V_A=I^e$, it follows that $I_B^{M,a}=I^e$, because $\tilde{\theta}^{M,a}\left(\Pi^M\left(0,I^e\right)\right)=\theta^e$, so $c'\left(I^e\right)=e^{\theta_0-\theta^e}=e^{\frac{1}{2}(\theta_0-\theta_1)}$. Because $\frac{dI_B^{M,a}}{dV_A}>0$, $I_B^{M,a}>I^e$ when $V_A>I^e$ and $I_B^{M,a}<I^e$ when $V_A<I^e$. Therefore, the insider overinvests in diversifying projects if $V_A>I^e$ but underinvests if $V_A<I^e$. Because the insider always underinvests in focused projects, and invests with distortions a.s. in diversifying projects, the SEU outsiders refuse to delegate control to her. ■

**Proof of Corollary 2.** If outsiders delegate control to the insider, their payoff is

$$\begin{aligned}U_0^{S,d} &= e^{\theta^e-\theta_1}\left[w_A+(1-\alpha)\left(V_A+\frac{1}{2}I_A^{M,a}\right)\right]+e^{\theta_0-\theta^e}\left[w_B+(1-\alpha)\frac{1}{2}I_B^{M,a}\right]\\&\quad+(1-\alpha)\left[V_0-\frac{1}{2}c\left(I_A^{M,a}\right)-\frac{1}{2}c\left(I_B^{M,a}\right)\right].\end{aligned}$$

If they retain control, however, their payoff is

$$\begin{aligned}U_0^{S,r} &= e^{\theta^e-\theta_1}\left[w_A+(1-\alpha)\left(V_A+\frac{1}{2}I^e\right)\right]+e^{\theta_0-\theta^e}\left[w_B+(1-\alpha)\frac{1}{2}I^e\right]\\&\quad+(1-\alpha)\left[V_0-\frac{1}{2}c\left(I^e\right)-\frac{1}{2}c\left(I^e\right)\right].\end{aligned}$$

If the insider is exogenously granted control, the impact on outsider's utility is $\Delta=U_0^{S,d}-U_0^{S,r}$, which simplifies to

$$\Delta=\frac{1}{2}(1-\alpha)\left[\left(\rho\left(I_A^{M,a}\right)-\rho\left(I^e\right)\right)+\left(\rho\left(I_B^{M,a}\right)-\rho\left(I^e\right)\right)\right]$$

where $\rho(I)=e^{\frac{1}{2}(\theta_0-\theta_1)}I-c(I)$, the outsiders' payoff from investing $I$ in either project. Note $\rho'(I)=e^{\frac{1}{2}(\theta_0-\theta_1)}-c'(I)$ and $\rho''(I)=-c''(I)<0$. $I^e$ maximizes $\rho$ because $c'\left(I^e\right)=e^{\frac{1}{2}(\theta_0-\theta_1)}$, so $\Delta$ is strictly negative. Neither $I^e$ nor $I_A^{M,a}$ ($I_A^{M,a}$ satisfies $c'\left(I_A^{M,a}\right)=e^{\hat{\theta}_0-\theta_1}$) depend on $V_A$, so neither $\rho\left(I_A^{M,a}\right)$ nor $\rho\left(I^e\right)$ depend on $V_A$. Theorem 3 showed that $I_B^{M,a}$ is increasing in $V_A$, and that $I_B^{M,a}=I^e$ when $V_A=I^e$. Thus, an increase in $V_A$ increases $\Delta$ when $V_A<I^e$ but decreases $\Delta$ when $V_A>I^e$, resulting in the inverted U-shaped relationship. ■

**Proof of Lemma 4.** Suppose outsiders know they will not know which type of project the firm draws, but they anticipate that investment of $I_A$ and $I_B$ will be implemented. Thus, their MEU is

$$U_1^S\left(\Pi^S\left(I_A1_{\tau=A},I_B1_{\tau=B}\right)\right)=\min_{\theta\in C}\mathbb{E}\left[u\left(\Pi^S\left(I_A1_{\tau=A},I_B1_{\tau=B}\right)\right);\theta\right],$$

where

$$\begin{aligned}\mathbb{E}\left[u\left(\Pi^S\left(I_A1_{\tau=A},I_B1_{\tau=B}\right)\right);\theta\right] &= e^{\theta-\theta_1}\left[w_A+(1-\alpha)\left(V_A+\frac{1}{2}I_A\right)\right]\\&\quad+e^{\theta_0-\theta}\left[w_B+(1-\alpha)\frac{1}{2}I_B\right]+(1-\alpha)\left[V_0-\frac{1}{2}c\left(I_A\right)-\frac{1}{2}c\left(I_B\right)\right].\end{aligned}$$

Define $\theta^{S,a}\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right) = \underset{\theta\in C}{\arg\min} E\left[u\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right);\theta\right]$. As shown in Lemma 2,

$$\theta^{S,a}\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right) = \begin{cases} \hat{\theta}_0 & \tilde{\theta}^{S,a}\left(\Pi^S\right) \leq \hat{\theta}_0 \\ \tilde{\theta}^{S,a}\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right) & \tilde{\theta}^{S,a}\left(\Pi^S\right) \in \left(\hat{\theta}_0, \hat{\theta}_1\right) \\ \hat{\theta}_1 & \tilde{\theta}^{S,a}\left(\Pi^S\right) \geq \hat{\theta}_1 \end{cases}$$

where $\tilde{\theta}^{S,a}\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B + (1-\alpha)\frac{1}{2} I_B}{w_A + (1-\alpha)\left(V_A + \frac{1}{2} I_A\right)}\right]$. Applying the minimax theorem, the FOCs are $\frac{\partial U_1^S}{\partial I_A} = \frac{1}{2}(1-\alpha)\left[e^{\theta^{S,a}-\theta_1} - c'(I_A)\right]$ and $\frac{\partial U_1^S}{\partial I_B} = \frac{1}{2}(1-\alpha)\left[e^{\theta_0-\theta^{S,a}} - c'(I_B)\right]$, so optimal investment satisfies $c'(I_A) = e^{\theta^{S,a}-\theta_1}$ and $c'(I_B) = e^{\theta_0-\theta^{S,a}}$. Suppose $I_A > I_B$. Because outsiders are diversified ex ante, this implies $\theta^{S,a}\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right) < \theta^e$, so $e^{\theta^{S,a}-\theta_1} < e^{\theta_0-\theta^{S,a}}$. This implies, however, that $c'(I_A) < c'(I_B)$, which requires $I_A < I_B$. Contradiction. Therefore, $I_A \leq I_B$. We can show similarly that $I_B \leq I_A$. Thus, $I_A = I_B$, which implies $\theta^{S,a}\left(\Pi^S\left(I_A 1_{\tau=A}, I_B 1_{\tau=B}\right)\right) = \theta^e$. Therefore, outsiders would like to commit, a priori, to efficient levels of investment: $I_A = I^e$ and $I_B = I^e$. ■

**Proof of Lemma 5.** With a focused project, an investment level of $I_A$ provides outsiders with utility

$$U_1^S\left(\Pi^S\left(I_A, 0\right)\right) = \min_{\theta\in C} \mathbb{E}\left[u\left(\Pi^S\left(I_A, 0\right)\right);\theta\right],$$

where $E\left[u\left(\Pi^S\left(I_A,0\right)\right);\theta\right] = e^{\theta-\theta_1}\left[w_A + (1-\alpha)\left(V_A + I_A\right)\right] + e^{\theta_0-\theta} w_B + (1-\alpha)\left[V_0 - c\left(I_A\right)\right]$. Beliefs are given by Lemma 2:

$$\theta^{S,a}\left(\Pi^S\left(I_A,0\right)\right) = \begin{cases} \hat{\theta}_0 & \tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right) \leq \hat{\theta}_0 \\ \tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right) & \tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right) \in \left(\hat{\theta}_0, \hat{\theta}_1\right) \\ \hat{\theta}_1 & \tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right) \geq \hat{\theta}_1 \end{cases}$$

where $\tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B}{w_A + (1-\alpha)(V_A+I_A)}\right]$. Applying the minimax theorem,

$$\frac{d}{dI_A} U_1^S\left(\Pi^S\left(I_A,0\right)\right) = (1-\alpha)\left[e^{\theta^{S,a}-\theta_1} - c'\left(I_A\right)\right].$$

Thus, $I_A^{S,a}$ is chosen so that $c'\left(I_A^{S,a}\right) = e^{\theta^{S,a}\left(\Pi^S(I_A,0)\right)-\theta_1}$. Because $w_A + (1-\alpha)V_A = w_B$, for all $I_A^{S,a} > 0$, $\theta^{S,a}\left(\Pi^S\left(I_A,0\right)\right) < \theta^e$, which implies $I_A^{S,a} < I^e$.

With a diversifying project, an investment level of $I_B$ provides outsiders with utility

$$U_1^S\left(\Pi^S\left(0, I_B\right)\right) = \min_{\theta\in C} \mathbb{E}\left[u\left(\Pi^S\left(0, I_B\right)\right);\theta\right],$$

where $E\left[u\left(\Pi^S\left(0,I_B\right)\right);\theta\right] = e^{\theta-\theta_1}\left[w_A + (1-\alpha)V_A\right] + e^{\theta_0-\theta}\left[w_B + (1-\alpha)I_B\right] + (1-\alpha)\left[V_0 - c\left(I_B\right)\right]$. Beliefs are given by Lemma 2:

$$\theta^{S,a}\left(\Pi^S\left(0,I_B\right)\right) = \begin{cases} \hat{\theta}_0 & \tilde{\theta}^{S,a}\left(\Pi^S\left(0,I_B\right)\right) \leq \hat{\theta}_0 \\ \tilde{\theta}^{S,a}\left(\Pi^S\left(0,I_B\right)\right) & \tilde{\theta}^{S,a}\left(\Pi^S\left(0,I_B\right)\right) \in \left(\hat{\theta}_0, \hat{\theta}_1\right) \\ \hat{\theta}_1 & \tilde{\theta}^{S,a}\left(\Pi^S\left(0,I_B\right)\right) \geq \hat{\theta}_1 \end{cases}$$

where $\tilde{\theta}^{S,a}\left(\Pi^S\left(0,I_B\right)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B + (1-\alpha)I_B}{w_A + (1-\alpha)V_A}\right]$. Applying the minimax theorem,

$$\frac{dU_1^S}{dI_B} = (1-\alpha)\left[e^{\theta_0-\theta^{S,a}\left(\Pi^S(0,I_B)\right)} - c'\left(I_B\right)\right].$$

Thus, $I_B^{S,a}$ is chosen so that $c'\left(I_B\right) = e^{\theta_0-\theta^{S,a}\left(\Pi^S(0,I_B)\right)}$. Because $w_B = w_A + (1-\alpha)V_A$, for all $I_B^{S,a} > 0$, $\theta^{S,a}\left(\Pi^S\left(0,I_B\right)\right) > \theta^e$, so $I_B^{S,a} < I^e$.

To show that $I_A^{S,a} = I_B^{S,a}$, note that $\theta^{S,a}\left(\Pi^S\left(I,0\right)\right) - \theta_1 = \theta_0 - \theta^{S,a}\left(\Pi^S\left(0,I\right)\right)$ for all $I$ because the outsider is diversified a priori, $w_B = w_A + (1-\alpha)V_A$. Thus, the pessimism effect is identical for focused and diversifying projects.

Finally, we will show that underinvestment is more severe at large firms (relative to outsiders' portfolio) by showing the equivalent claim – underinvestment is less severe when outsiders' portfolio is larger. Let $K = w_B = w_A + (1-\alpha)V_A$. Suppose ambiguity-averse outsiders are faced with a focused project: their portfolio-distorted beliefs are given by $\tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{K}{K+(1-\alpha)I_A}\right]$. Focused investment by ambiguity-averse outsiders satisfies $c'\left(I_A^{S,a}\right) = e^{\theta^{S,a}-\theta_1}$. Totally differentiating with respect to $K$ and rearranging,

$$\left[c''\left(I_A^{S,a}\right) - e^{\theta^{S,a}-\theta_1}\frac{\partial\theta^{S,a}}{\partial I_A^{S,a}}\right]\frac{dI_A^{S,a}}{dK} = e^{\theta^{S,a}-\theta_1}\frac{\partial\theta^{S,a}}{\partial K}.$$

If $\theta^{S,a}$ is a corner solution, then $\frac{\partial\theta^{S,a}}{\partial K} = \frac{\partial\theta^{S,a}}{\partial I_A^{S,a}} = 0$, so $\frac{dI_A^{S,a}}{dK} = 0$. If $\theta^{S,a}$ is interior, then $\theta^{S,a} = \tilde{\theta}^{S,a}\left(\Pi^S\left(I_A,0\right)\right)$, so $\frac{\partial\theta^{S,a}}{\partial I_A^{S,a}} = -\frac{1-\alpha}{2[K+(1-\alpha)I_A]}$ and $\frac{\partial\theta^{S,a}}{\partial K} = \frac{(1-\alpha)I_A}{2K[K+(1-\alpha)I_A]}$. Because $\frac{\partial\theta^{S,a}}{\partial I_A^{S,a}} < 0 < \frac{\partial\theta^{S,a}}{\partial K}$, this implies that $\frac{dI_A^{S,a}}{dK} > 0$. Recall $I_A^{S,a} < I^e$ ($I^e$ does not depend on $K$). Thus, underinvestment is less severe when $K$ is larger, and underinvestment is more severe when $K$ is smaller. This is equivalent to the firm size result, because a large firm will be more important to the portfolio of its owners (the diversifying portfolio will be smaller). Identical results hold for diversifying projects, $\frac{dI_B^{S,a}}{dK} > 0$, by similar proof. ■

**Proof of Theorem 4.** Lemma 1 shows that exposure to information harms ambiguity-averse outsiders. Lemma 5 demonstrates that ambiguity-averse outsiders underinvest, both relative to first best and to what they would like to commit to ex ante by Lemma 4. By Theorem 2, a SEU insider chooses investment optimally, setting $I_A = I_B = I^e$. Thus, outsiders protects themselves from ambiguity and achieve efficient investment by delegating control to the insider. ■

**Proof of Corollary 3.** Outsiders' payoff from retention is, from equation 26,

$$U_0^{S,r}\left(I_A^{S,a}, I_B^{S,a}\right) = \frac{1}{2}U_1^S\left(\Pi^S\left(I_A^{S,a},0\right)\right) + \frac{1}{2}U_1^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right).$$

where $U_1^S\left(\Pi^S\left(I_A^{S,a},0\right)\right)$ and $U_1^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)$ are defined in (22) and (25). Applying the minimax theorem and envelope theorem, the only effect of a change in $Z$ is the direct effect. By direct differentiation, and from $c(I) = \frac{1}{Z(1+\gamma)}I^{1+\gamma}$, we have that

$$\frac{dU_0^{S,r}\left(I_A^{S,a}, I_B^{S,a}\right)}{dZ} = \frac{1}{2}(1-\alpha)\frac{1}{Z}c\left(I_A^{S,a}\right) + \frac{1}{2}(1-\alpha)\frac{1}{Z}c\left(I_B^{S,a}\right).$$

Under delegation, the outsiders' payoff is given by

$$U_0^{S,d}\left(I_A^{M,a}, I_B^{M,a}\right) = \min_{\theta\in C}\mathbb{E}\left[u\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right);\theta\right]$$

where

$$\begin{aligned}\mathbb{E}\left[u\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right);\theta\right] &= e^{\theta-\theta_1}\left[w_A + (1-\alpha)\left(V_A + \frac{1}{2}I_A^{M,a}\right)\right] \\ +e^{\theta_0-\theta}\left[w_B + (1-\alpha)\frac{1}{2}I_B^{M,a}\right] &\quad + (1-\alpha)\left[V_0 - \frac{1}{2}c\left(I_A^{M,a}\right) - \frac{1}{2}c\left(I_B^{M,a}\right)\right].\end{aligned}$$

By the minimax theorem, $\frac{\partial U_0^{S,d}}{\partial\theta}\frac{d\theta^S}{dZ} = 0$. Note that we cannot apply the Envelope Theorem under delegation, because

the insider chooses investment optimally for herself. Thus,

$$\frac{dU_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{dZ}=\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial Z}+\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial I_A}\frac{dI_A^{M,a}}{dZ}+\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial I_B}\frac{dI_B^{M,a}}{dZ}.$$

where $\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial Z}=\frac{1-\alpha}{2Z}\left[c\left(I_A^{M,a}\right)+c\left(I_B^{M,a}\right)\right]$, $\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial I_A}=\frac{1-\alpha}{2}\left(e^{\theta^S-\theta_1}-c'\left(I_A^{M,a}\right)\right)$, and $\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial I_B}=\frac{1-\alpha}{2}\left(e^{\theta_0-\theta^S}-c'\left(I_B^{M,a}\right)\right)$. Because the insider is SEU, she sets investment optimally: $c'\left(I_A^{M,a}\right)=e^{\theta^e-\theta_1}$ and $c'\left(I_B^{M,a}\right)=e^{\theta_0-\theta^e}$ (Theorem 2), and her investment is balanced, $I_A^{M,a}=I_B^{M,a}$, so $\theta^S=\theta^e$. Thus, $\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial I_A}=\frac{\partial U_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{\partial I_B}=0$. Thus, $\frac{dU_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{dZ}=\frac{1-\alpha}{2Z}\left[c\left(I_A^{M,a}\right)+c\left(I_B^{M,a}\right)\right]$ and $\frac{dU_0^{S,r}\left(I_A^{S,a},I_B^{S,a}\right)}{dZ}=\frac{1-\alpha}{2Z}\left[c\left(I_A^{S,a}\right)+c\left(I_B^{S,a}\right)\right]$. The insider is SEU, $I_A^{M,a}=I_B^{M,a}=I^e$, but outsiders underinvest ex post (Lemma 5), so $I_A^{S,a}=I_B^{S,a}<I^e$. Thus, $\frac{dU_0^{S,d}\left(I_A^{M,a},I_B^{M,a}\right)}{dZ}>\frac{dU_0^{S,r}\left(I_A^{S,a},I_B^{S,a}\right)}{dZ}$. Therefore, as growth options improve ($Z$ increases), delegation becomes more valuable. ■

**Proof of Theorem 5.** To prove Point (1), we show that the benefit of delegation, $U_0^{S,d}-U_0^{S,r}$, is inverted U-shaped in $V_A$ (holding $w_A+(1-\alpha)V_A$ constant).[41] It is helpful to define $I_B^{M,a}(V_A)$ as the diversifying investment by the insider when the value of the assets in place is $V_A$. When $V_A$ is small, $V_A\leq V_A^1\equiv Z^{\frac{1}{\gamma}}e^{2\theta^e+\frac{1}{\gamma}\theta_0-\left(2+\frac{1}{\gamma}\right)\hat{\theta}_1}$, the insider sets $I_B^{M,a}$ so that $c'\left(I_B^{M,a}\right)=e^{\theta_0-\hat{\theta}_1}$, or equivalently, $I_B^{M,a}=\left[Ze^{\theta_0-\hat{\theta}_1}\right]^{\frac{1}{\gamma}}$. When $V_A$ is large, $V_A\geq V_A^2\equiv Z^{\frac{1}{\gamma}}e^{2\theta^e+\frac{1}{\gamma}\theta_0-\left(2+\frac{1}{\gamma}\right)\hat{\theta}_0}$, the insider sets $I_B^{M,a}$ so that $c'\left(I_B^{M,a}\right)=e^{\theta_0-\hat{\theta}_0}$, or equivalently, $I_B^{M,a}=\left[Ze^{\theta_0-\hat{\theta}_0}\right]^{\frac{1}{\gamma}}$. Note $I_B^{M,a}(V_A)$ is constant for $V_A\leq V_A^1$ and for $V_A\geq V_A^2$. For $V_A\in\left(V_A^1,V_A^2\right)$, $I_B^{M,a}$ is chosen to that $c'\left(I_B^{M,a}\right)=e^{\theta_0-\tilde{\theta}^M}$, where $\tilde{\theta}^M=\theta^e+\frac{1}{2}\ln\left[\frac{I_B^{M,a}}{V_A}\right]$, which implies $I_B^{M,a}=Z^{\frac{2}{2\gamma+1}}e^{\frac{2}{2\gamma+1}(\theta_0-\theta^e)}V_A^{\frac{1}{2\gamma+1}}$. Thus, $I_B^{M,a}(V_A)$ is strictly increasing in $V_A$ for $V_A\in\left(V_A^1,V_A^2\right)$.

For this result, we increase $V_A$ and decrease $w_A$ so that $w_A+(1-\alpha)V_A$ remains constant.[42] Thus, define $\tilde{w}_A=w_A-(1-\alpha)\varepsilon$ and $\tilde{V}_A=V_A+\varepsilon$. By construction, $\frac{\partial U_0^{S,r}}{\partial\varepsilon}=\frac{\partial U_0^{S,d}}{\partial\varepsilon}=0$. Similar to the proof of Corollary 3, this implies $\frac{dU_0^{S,r}}{d\varepsilon}=0$, while

$$\frac{dU_0^{S,d}}{d\varepsilon}=\frac{\partial U_0^{S,d}}{\partial I_A}\frac{dI_A^{M,a}}{d\varepsilon}+\frac{\partial U_0^{S,d}}{\partial I_B}\frac{dI_B^{M,a}}{d\varepsilon}.$$

Also, $\frac{dI_A^{M,a}}{d\varepsilon}=0$ because $I_A^{M,a}=\left[Ze^{\hat{\theta}_0-\theta_1}\right]^{\frac{1}{\gamma}}$. As shown above, $\frac{dI_B^{M,a}}{d\varepsilon}=0$ for $V_A\leq V_A^1$ and for $V_A\geq V_A^2$, but $\frac{dI_B^{M,a}}{d\varepsilon}>0$ for $V_A\in\left(V_A^1,V_A^2\right)$. Thus, $\frac{dU_0^{S,d}}{d\varepsilon}>\frac{dU_0^{S,r}}{d\varepsilon}$ iff $\frac{\partial U_0^{S,d}}{\partial I_B}\frac{dI_B^{M,a}}{d\varepsilon}>0$. $\frac{\partial U_0^{S,d}}{\partial I_B}=\frac{1}{2}(1-\alpha)\left[e^{\theta_0-\theta^S}-e^{\theta_0-\theta^M}\right]$, because $c'\left(I_B^{M,a}\right)=e^{\theta_0-\theta^M}$. For $V_A\in\left(V_A^1,V_A^2\right)$, $I_B^{M,a}(V_A)$ is strictly increasing in $V_A$. Outsiders believe

$$\theta^S=\theta^e+\frac{1}{2}\ln\left[\frac{w_B+\frac{1}{2}(1-\alpha)I_B^{M,a}(V_A)}{w_A+(1-\alpha)V_A+\frac{1}{2}(1-\alpha)I_A^{M,a}}\right].$$

Note $\theta^S$ is increasing in $V_A$ (because $\frac{d}{d\varepsilon}\left[w_A+(1-\alpha)V_A\right]=0$, $\frac{dI_A^{M,a}}{d\varepsilon}=0$, and $\frac{dI_B^{M,a}}{d\varepsilon}\geq 0$). The insider believes $\theta^M=\theta^e+\frac{1}{2}\ln\left[\frac{I_B^{M,a}}{V_A}\right]$ where

$$\frac{I_B^{M,a}}{V_A}=Z^{\frac{2}{2\gamma+1}}e^{\frac{2}{2\gamma+1}(\theta_0-\theta^e)}V_A^{-\frac{2\gamma}{2\gamma+1}},$$

[41] The proof does not require that delegation and retention are both optimal for some values of $V_A$. For example, if other parameters are such that retention is optimal for all $V_A$ (for example, very large $K$ or very small $Z$), the result holds by setting $\underline{V_A}=\overline{V_A}$. Alternatively, if other parameters are such that delegation is optimal for all $V_A$ (for example, very large $Z$), the result holds by setting $\underline{V_A}=0$ and $\overline{V_A}=\infty$.

[42] We show numerically that the value of delegation is nonmonotonic in $w_A$. See Figure 1.

so $\theta^M$ is decreasing in $V_A$. Because $\frac{\partial U_0^{S,d}}{\partial I_B} = \frac{1}{2}(1-\alpha)\left[e^{\theta_0-\theta^S} - e^{\theta_0-\theta^M}\right]$, $\frac{\partial}{\partial V_A}\frac{\partial U_0^{S,d}}{\partial I_B} < 0$. It is easily shown that $\frac{\partial U_0^{S,d}}{\partial I_B} > 0$ for $V_A \leq V_A^1$ and $\frac{\partial U_0^{S,d}}{\partial I_B} < 0$ for $V_A \geq V_A^2$. Thus, there exists a unique $\tilde{V}_A$ such that $\frac{\partial U_0^{S,d}}{\partial I_B} > 0$ for $V_A < \tilde{V}_A$ and $\frac{\partial U_0^{S,d}}{\partial I_B} < 0$ for $V_A > \tilde{V}_A$. Therefore, $\frac{dU_0^{S,d}}{d\varepsilon} > \frac{dU_0^{S,r}}{d\varepsilon}$ for $V_A \in \left(V_A^1, \tilde{V}_A\right)$ and $\frac{dU_0^{S,d}}{d\varepsilon} < \frac{dU_0^{S,r}}{d\varepsilon}$ for $V_A \in \left(\tilde{V}_A, V_A^2\right)$.[43] If $U_0^{S,d}|_{V_A=\tilde{V}_A} > U_0^{S,r}|_{V=\tilde{V}_A}$, then define $\underline{V_A} < \tilde{V} < \overline{V_A}$ such that $U_0^{S,d}|_{V_A=\underline{V_A}} = U_0^{S,r}|_{V_A=\underline{V_A}}$ and $U_0^{S,d}|_{V_A=\overline{V_A}} = U_0^{S,r}|_{V_A=\overline{V_A}}$. If $U_0^{S,d}|_{V_A=\tilde{V}_A} \leq U_0^{S,r}|_{V=\tilde{V}_A}$, define $\underline{V_A} = \overline{V_A} = \tilde{V}_A$ and the claim trivially holds.

Point (2) claims that retention is optimal for $Z < \underline{Z}$, retention is optimal for $Z > \bar{Z}$, and that $\bar{Z}(V_A)$ is U-Shaped. We will prove these separately. When the project is small ($Z$ small), the pessimism effect disappears, but the insider invests inefficiently, so outsiders retain control. Because $U_0^{S,r}(0,0) = U_0^{S,d}(0,0)$, to show $U_0^{S,r} > U_0^{S,d}$ for all $Z \in (0, \underline{Z})$, it is sufficient to show that $\frac{dU_0^{S,r}}{dZ}|_{Z=\varepsilon} > \frac{dU_0^{S,d}}{dZ}|_{Z=\varepsilon}$ for small positive $\varepsilon$. From the proof of Corollary 3, $\frac{dU_0^{S,r}}{dZ} = \frac{\partial U_0^{S,r}}{\partial Z}$, and $\frac{\partial U_0^{S,r}}{\partial Z} = \frac{1-\alpha}{2Z}\left[c\left(I_A^{S,a}\right) + c\left(I_B^{S,a}\right)\right]$. Because $c(I) = \frac{1}{1+\gamma}Z^{\frac{1}{\gamma}}\left[c'(I)\right]^{\frac{1+\gamma}{\gamma}}$ and $c'\left(I_A^{S,a}\right) = e^{\theta^S-\theta_1}$ and $c'\left(I_B^{S,a}\right) = e^{\theta_0-\theta^S}$, and because $Z \to 0$, $\theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right) \to \theta^e$ and $\theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right) \to \theta^e$, so for sufficiently small $\varepsilon$,

$$\frac{\partial U_0^{S,r}}{\partial Z}|_{Z=\varepsilon} = \frac{1-\alpha}{2(1+\gamma)}\varepsilon^{\frac{1}{\gamma}-1}\left[e^{\frac{1+\gamma}{\gamma}(\theta^e-\theta_1)} + e^{\frac{1+\gamma}{\gamma}(\theta_0-\theta^e)}\right].$$

Similarly, the proof of Corollary 3 shows

$$\frac{dU_0^{S,d}}{dZ} = \frac{\partial U_0^{S,d}}{\partial Z} + \frac{\partial U_0^{S,d}}{\partial I_A}|_{I_A=I_A^{M,a}}\frac{dI_A^{M,a}}{dZ} + \frac{\partial U_0^{S,d}}{\partial I_B}|_{I_B=I_B^{M,a}}\frac{dI_B^{M,a}}{dZ}$$

and $\frac{\partial U_0^{S,d}}{\partial Z} = \frac{1}{2}(1-\alpha)\left[\frac{1}{Z}c\left(I_A^{M,a}\right) + \frac{1}{Z}c\left(I_B^{M,a}\right)\right]$. Because the insider is not diversified, $c'\left(I_A^{M,a}\right) = e^{\hat{\theta}_0-\theta_1}$ and, for sufficiently small $Z$, $c'\left(I_B^{M,a}\right) = e^{\theta_0-\hat{\theta}_0}$. Thus, for sufficiently small $\varepsilon$,

$$\frac{\partial U_0^{S,d}}{\partial Z}|_{Z=\varepsilon} = \frac{1-\alpha}{2(1+\gamma)}\varepsilon^{\frac{1}{\gamma}-1}\left[e^{\frac{1+\gamma}{\gamma}\left(\hat{\theta}_0-\theta_1\right)} + e^{\frac{1+\gamma}{\gamma}\left(\theta_0-\hat{\theta}_0\right)}\right].$$

For indirect focused-investment effects, $\frac{\partial U_0^{S,d}}{\partial I_A}|_{I_A=I_A^{M,a}} = \frac{1-\alpha}{2}\left[e^{\theta^S-\theta_1} - c'\left(I_A^{M,a}\right)\right]$. Because $c'\left(I_A^{M,a}\right) = e^{\hat{\theta}_0-\theta_1}$, $I_A^{M,a} = Z^{\frac{1}{\gamma}}e^{\frac{1}{\gamma}\left(\hat{\theta}_0-\theta_1\right)}$, so $\frac{dI_A^{M,a}}{dZ} = \frac{1}{\gamma}Z^{\frac{1}{\gamma}-1}e^{\frac{1}{\gamma}\left(\hat{\theta}_0-\theta_1\right)}$. As $Z$ gets small (sufficiently small $\varepsilon$), $\theta^S$ approaches $\theta^e$, so

$$\frac{\partial U_0^{S,d}}{\partial I_A}|_{I_A=I_A^{M,a}}\frac{dI_A^{M,a}}{dZ}|_{Z=\varepsilon} = \frac{1-\alpha}{2\gamma}\varepsilon^{\frac{1}{\gamma}-1}\left[e^{\theta^e-\theta_1} - e^{\hat{\theta}_0-\theta_1}\right]e^{\frac{1}{\gamma}\left(\hat{\theta}_0-\theta_1\right)}.$$

For indirect diversified-investment effects, $\frac{\partial U_0^{S,d}}{\partial I_B}|_{I_B=I_B^{M,a}} = \frac{1-\alpha}{2}\left[e^{\theta_0-\theta^S} - c'\left(I_B^{M,a}\right)\right]$. For sufficiently small $Z$, $\theta^M = \hat{\theta}_0$, so $\frac{dI_B^{M,a}}{dZ} = \frac{1}{\gamma}Z^{\frac{1}{\gamma}-1}e^{\frac{1}{\gamma}\left(\theta_0-\hat{\theta}_0\right)}$. As $Z$ gets small (sufficiently small $\varepsilon$), $\theta^S$ approaches $\theta^e$, so

$$\frac{\partial U_0^{S,d}}{\partial I_B}|_{I_B=I_B^{M,a}}\frac{dI_B^{M,a}}{dZ}|_{Z=\varepsilon} = \frac{1-\alpha}{2\gamma}\varepsilon^{\frac{1}{\gamma}-1}\left[e^{\theta_0-\theta^e} - e^{\theta_0-\hat{\theta}_0}\right]e^{\frac{1}{\gamma}\left(\theta_0-\hat{\theta}_0\right)}.$$

Define $\phi(\theta)$ so that

$$\phi(\theta) = \frac{1}{1+\gamma}\left[e^{\frac{1+\gamma}{\gamma}(\theta-\theta_1)} + e^{\frac{1+\gamma}{\gamma}(\theta_0-\theta)}\right] + \frac{1}{\gamma}\left[e^{\theta^e-\theta_1} - e^{\theta-\theta_1}\right]e^{\frac{1}{\gamma}(\theta-\theta_1)} + \frac{1}{\gamma}\left[e^{\theta_0-\theta^e} - e^{\theta_0-\theta}\right]e^{\frac{1}{\gamma}(\theta_0-\theta)}.$$

Note $\frac{dU_0^{S,r}}{dZ}|_{Z=\varepsilon} = \frac{1-\alpha}{2}\varepsilon^{\frac{1}{\gamma}-1}\phi(\theta^e)$ and $\frac{dU_0^{S,d}}{dZ}|_{Z=\varepsilon} = \frac{1-\alpha}{2}\varepsilon^{\frac{1}{\gamma}-1}\phi\left(\hat{\theta}_0\right)$. It is sufficient to show $\phi(\theta^e) > \phi\left(\hat{\theta}_0\right)$. With a little rearranging, it follows that

$$\phi'(\theta) = \frac{1}{\gamma^2}\left[e^{\theta^e-\theta_1} - e^{\theta-\theta_1}\right]e^{\frac{1}{\gamma}(\theta-\theta_1)} - \frac{1}{\gamma^2}\left[e^{\theta_0-\theta^e} - e^{\theta_0-\theta}\right]e^{\frac{1}{\gamma}(\theta_0-\theta)}.$$

[43] For $V_A \leq V_A^1$ and $V_A \geq V_A^2$, $\frac{dU_0^{S,d}}{d\varepsilon} = \frac{dU_0^{S,r}}{d\varepsilon}$.

Thus, $\phi'(\theta) > 0$ for all $\theta \in \left(\hat{\theta}_0, \theta^e\right)$, so $\phi(\theta^e) > \phi\left(\hat{\theta}_0\right)$. Therefore, $\frac{dU_0^{S,r}}{dZ}|_{Z=\varepsilon} > \frac{dU_0^{S,d}}{dZ}|_{Z=\varepsilon}$ for sufficiently small positive $\varepsilon$. Thus, $U_0^{S,r} > U_0^{S,d}$ for $Z$ close to zero: equivalently, there exists $\underline{Z}$ such that $U_0^{S,r} > U_0^{S,d}$ for all $Z < \underline{Z}$.

For the second claim of Point (2), we will show that, when growth options are sufficiently large, equilibrium beliefs of outsiders and the insider coincide, so equilibrium investment will be the same. By Lemma 1, outsiders delegate control to the insider. Consider the equilibrium beliefs of the outsider who retains control and is faced with a focused project: $\tilde{\theta}^S\left(\Pi^S(I_A, 0)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B}{w_A+(1-\alpha)(V_A+I_A)}\right]$. Note that $\theta^S = \hat{\theta}_0$ iff $\tilde{\theta}^S \leq \hat{\theta}_0$ iff $I_A \geq \bar{I}_A^S$ where

$$\bar{I}_A^S \triangleq \frac{1}{1-\alpha}\left[e^{2\left(\theta^e - \hat{\theta}_0\right)} w_B - w_A - (1-\alpha) V_A\right].$$

Thus, if outsiders invest sufficiently, they will agree with the insider (because $\theta^M = \hat{\theta}_0$ as shown in Theorem 3). This is optimal if $c'\left(\bar{I}_A\right) \leq e^{\hat{\theta}_0 - \theta_1}$, or equivalently, if

$$Z \geq e^{\theta_1 - \hat{\theta}_0}\left\{\frac{1}{1-\alpha}\left[e^{2\left(\theta^e - \hat{\theta}_0\right)} w_B - w_A - (1-\alpha) V_A\right]\right\}^{\gamma}.$$

Similarly, with a diversifying project, $\tilde{\theta}^S\left(\Pi^S(0, I_B)\right) = \theta^e + \frac{1}{2}\ln\left[\frac{w_B+(1-\alpha)I_B}{w_A+(1-\alpha)V_A}\right]$. Note that $\theta^S = \hat{\theta}_1$ iff $\tilde{\theta}^S \geq \hat{\theta}_1$ iff $I_B \geq \bar{I}_B^S$ where

$$\bar{I}_B^S \triangleq \frac{1}{1-\alpha}\left\{e^{2\left(\hat{\theta}_1 - \theta^e\right)}\left[w_A + (1-\alpha) V_A\right] - w_B\right\}.$$

This is optimal if $c'\left(\bar{I}_B\right) \leq e^{\theta_0 - \hat{\theta}_1}$, or equivalently, if

$$Z \geq e^{\hat{\theta}_1 - \theta_0}\left\{\frac{1}{1-\alpha}\left[e^{2\left(\hat{\theta}_1 - \theta^e\right)}\left[w_A + (1-\alpha) V_A\right] - w_B\right]\right\}^{\gamma}.$$

Because $w_B = w_A + (1-\alpha) V_A$ and $\hat{\theta}_1 - \theta^e = \theta^e - \hat{\theta}_0$, $\bar{I}_A^S = \bar{I}_B^S$ (so are the cutoffs for $Z$). When growth options are sufficiently large, outsiders invest according to the worst-case scenario for the type of project drawn: $\theta^S\left(\Pi^S(I_A, 0)\right) = \hat{\theta}_0$ and $\theta^S\left(\Pi^S(0, I_B)\right) = \hat{\theta}_1$ for $I_A \geq \bar{I}_A^S$ and $I_B \geq \bar{I}_B^S$.

The insider always invests in focused projects according to the worst-case scenario: $\theta^M\left(\Pi^M(I_A, 0)\right) = \hat{\theta}_0$. Her portfolio-distorted beliefs for the diversifying project are given by $\tilde{\theta}^M\left(\Pi^M(0, I_B)\right) = \theta^e + \frac{1}{2}\ln\frac{I_B}{V_A}$. $\theta^M = \hat{\theta}_1$ iff $I_B \geq e^{2\left(\hat{\theta}_1 - \theta^e\right)} V_A$. It is optimal for her to set $I_B \geq e^{2\left(\hat{\theta}_1 - \theta^e\right)} V_A$ iff $c'\left(e^{2\left(\hat{\theta}_1 - \theta^e\right)} V_A\right) \leq e^{\theta_0 - \hat{\theta}_1}$, which holds iff $Z \geq e^{\hat{\theta}_1 - \theta_0}\left(e^{2\left(\hat{\theta}_1 - \theta^e\right)} V_A\right)^{\gamma}$. Therefore, when growth options are sufficiently profitable, the insider will invest the same as the outsider. By Lemma 1, delegation is strictly preferred.

For the third part of Point (2), define $\Delta = U_0^{S,d} - U_0^{S,r}$ as the value of delegation, and define $\bar{Z}(V_A)$ as the value of $\bar{Z}$ for a given $V_A$, holding everything else constant. Thus, $\Delta > 0$ for all $Z > \bar{Z}(V_A)$ and $\Delta < 0$ for $Z = \bar{Z}(V_A) - \varepsilon$ for small positive $\varepsilon$. This implies that $\frac{d\Delta}{d\bar{Z}}|_{Z=\bar{Z}(V_A)} > 0$. By definition of $\bar{Z}(V_A)$, $\Delta\left(\bar{Z}(V_A)\right) = 0$ for all $V_A$. Totally differentiating $\Delta$ with respect to $V_A$, $\frac{d\Delta}{dV_A} = \frac{\partial\Delta}{\partial V_A} + \frac{\partial\Delta}{\partial Z}\frac{d\bar{Z}(V_A)}{dV_A}$. As shown in the proof of Point (1), $\frac{\partial\Delta}{\partial V_A} = 0$ for $V_A < V_A^1$, $\frac{\partial\Delta}{\partial V_A} > 0$ for $V_A \in \left(V_A^1, \tilde{V}_A\right)$, $\frac{\partial\Delta}{\partial V_A} < 0$ for $V_A \in \left(\tilde{V}_A, V_A^2\right)$, and $\frac{\partial\Delta}{\partial V_A} = 0$ for $V_A > V_A^2$. This implies that $\frac{d\bar{Z}(V_A)}{dV_A} = 0$ for $V_A < V_A^1$, $\frac{d\bar{Z}(V_A)}{dV_A} < 0$ for $V_A \in \left(V_A^1, \tilde{V}_A\right)$, $\frac{d\bar{Z}(V_A)}{dV_A} > 0$ for $V_A \in \left(V_A^2, \tilde{V}_A\right)$, and $\frac{d\bar{Z}(V_A)}{dV_A} = 0$ for $V_A > V_A^2$. Thus, $\bar{Z}(V_A)$ is U-Shaped in $V_A$.

Finally, for Point (3), when diversified outsiders have a sufficiently large portfolio, they will always want control. Let $w_A = K - (1-\alpha) V_A$ and $w_B = K$, so

$$\begin{aligned}
\tilde{\theta}^{S,a}\left(\Pi^S(I_A, 0)\right) &= \theta^e + \frac{1}{2}\ln\left[\frac{K}{K + (1-\alpha) I_A}\right], \\
\tilde{\theta}^{S,a}\left(\Pi^S(0, I_B)\right) &= \theta^e + \frac{1}{2}\ln\left[\frac{K + (1-\alpha) I_B}{K}\right], \\
\tilde{\theta}^{S,a}\left(\Pi^S(I_A 1_{\tau=A}, I_B 1_{\tau=A})\right) &= \theta^e + \frac{1}{2}\ln\left[\frac{K + \frac{1}{2}(1-\alpha) I_B}{K + \frac{1}{2}(1-\alpha) I_A}\right].
\end{aligned}$$

As $K \to \infty$, all of these converge to $\theta^e$: that is, the outsiders' worst case scenario converges to $\theta^e$ for either project. Because the worst-case scenario is not moving around, they do not fear the ambiguity in the limit (Lemma 1 implies strict preference only when the worst-case scenario is not constant). Thus, when the outsiders' outside portfolio is sufficiently large, they exert control. ■

**Proof of Corollary 4.** The result for outsiders follow from Lemma 5, while the results for the insider follows from Theorem 3. ■

**Proof of Corollary 5.** The value of delegation is $U_0^{S,d} - U_0^{S,r}$. To show that Point 1 holds, consider increasing both $w_A$ and $w_B$ by a small amount. Similar to the proof of Corollary 3,

$$\frac{dU_0^{S,d}}{dw_A} = \frac{\partial U_0^{S,d}}{\partial w_A} + \frac{\partial U_0^{S,d}}{\partial I_A}\frac{dI_A^{M,a}}{dw_A} + \frac{\partial U_0^{S,d}}{\partial I_B}\frac{dI_B^{M,a}}{dw_A}.$$

The outside portfolio of the outsider does not affect the investment decisions of the insider, so $\frac{dI_A^{M,a}}{dw_A} = \frac{dI_B^{M,a}}{dw_A} = 0$. Therefore, $\frac{dU_0^{S,d}}{dw_A} = \frac{\partial U_0^{S,d}}{\partial w_A}$. Further, $\frac{\partial U_0^{S,d}}{\partial w_A} = e^{\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right)-\theta_1}$. Similarly, $\frac{dU_0^{S,d}}{dw_B} = e^{\theta_0-\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right)}$. Thus, the impact of an increase in outside portfolio on $U_0^{S,d}$ is

$$\frac{dU_0^{S,d}}{dw_A} + \frac{dU_0^{S,d}}{dw_B} = e^{\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right)-\theta_1} + e^{\theta_0-\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right)}.$$

Under retention, because utility is defined recursively,

$$U_0^{S,r} = \frac{1}{2}\left(U_1^{S,a}\left(\Pi^S\left(I_A^{S,a}, 0\right)\right) + U_1^{S,a}\left(\Pi^S\left(0, I_B^{S,a}\right)\right)\right).$$

Similar to the proof of Corollary 3, $\frac{dU_1^{S,a}}{dw_A} = \frac{\partial U_1^{S,a}}{\partial w_A}$

$$\begin{aligned}
\frac{\partial U_1^{S,a}\left(\Pi^S\left(I_A^{S,a}, 0\right)\right)}{\partial w_A} &= e^{\theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right)-\theta_1}, \\
\frac{\partial U_1^{S,a}\left(\Pi^S\left(0, I_B^{S,a}\right)\right)}{\partial w_A} &= e^{\theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right)-\theta_1},
\end{aligned}$$

so

$$\frac{dU_0^{S,r}}{dw_A} = \frac{1}{2}\left[e^{\theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right)-\theta_1} + e^{\theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right)-\theta_1}\right].$$

Similarly,

$$\frac{dU_0^{S,r}}{dw_B} = \frac{1}{2}\left[e^{\theta_0-\theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right)} + e^{\theta_0-\theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right)}\right].$$

Thus, the impact of increasing both $w_A$ and $w_B$ is

$$\begin{aligned}
\frac{dU_0^{S,r}}{dw_A} + \frac{dU_0^{S,r}}{dw_B} &= \frac{1}{2}\left[e^{\theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right)-\theta_1} + e^{\theta_0-\theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right)}\right] \\
&\quad + \frac{1}{2}\left[e^{\theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right)-\theta_1} + e^{\theta_0-\theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right)}\right].
\end{aligned}$$

Define $g(\theta) = e^{\theta-\theta_1} + e^{\theta_0-\theta}$. Note $\frac{dU_0^{S,d}}{dw_A} + \frac{dU_0^{S,d}}{dw_B} = g\left(\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right)\right)$ and

$$\frac{dU_0^{S,r}}{dw_A} + \frac{dU_0^{S,r}}{dw_B} = \frac{1}{2}\left[g\left(\theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right)\right) + g\left(\theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right)\right)\right].$$

It can be easily verified that $g$ is convex, achieves its minimum at $\theta = \theta^e$, and is symmetric around $\theta^e$. By Corollary 4, $I_A^{S,a} = I_B^{S,a}$, so it can quickly be verified that

$$\theta^e - \theta^S\left(\Pi^S\left(I_A^{S,a}, 0\right)\right) = \theta^S\left(\Pi^S\left(0, I_B^{S,a}\right)\right) - \theta^e,$$

so

$$g\left(\theta^S\left(\Pi^S\left(I_A^{S,a},0\right)\right)\right)=g\left(\theta^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)\right).$$

Therefore, $\frac{dU_0^{S,d}}{dw_A}+\frac{dU_0^{S,d}}{dw_B}=g\left(\theta^S\left(\Pi^S\left(I_A^{M,a}1_A,I_B^{M,a}1_B\right)\right)\right)$ and $\frac{dU_0^{S,r}}{dw_A}+\frac{dU_0^{S,r}}{dw_B}=g\left(\theta^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)\right)$. By Corollary 4, $I_B^{M,a}\geq I_A^{M,a}$, so $\theta^S\left(\Pi^S\left(I_A^{M,a}1_A,I_B^{M,a}1_B\right)\right)\geq\theta^e$. Further, $I_B^{S,a}>0$ so $\theta^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)>\theta^e$. Because $g$ achieves its minimum at $\theta^e$, $g$ is increasing in $\theta$ for $\theta\geq\theta^e$. Therefore, $\frac{dU_0^{S,r}}{dw_A}+\frac{dU_0^{S,r}}{dw_B}\geq\frac{dU_0^{S,d}}{dw_A}+\frac{dU_0^{S,d}}{dw_B}$ iff $\tilde{\theta}^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)\geq\tilde{\theta}^S\left(\Pi^S\left(I_A^{M,a}1_A,I_B^{M,a}1_B\right)\right)$. Because

$$\tilde{\theta}^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)=\theta^e+\frac{1}{2}\ln\left[\frac{w_B+(1-\alpha)\,I_B^{S,a}}{w_A+(1-\alpha)\,V_A}\right]$$

and

$$\tilde{\theta}^S\left(\Pi^S\left(I_A^{M,a}1_A,I_B^{M,a}1_B\right)\right)=\theta^e+\frac{1}{2}\ln\left[\frac{w_B+(1-\alpha)\,\frac{1}{2}I_B^{M,a}}{w_A+(1-\alpha)\left(V_A+\frac{1}{2}I_A^{M,a}\right)}\right],$$

it can be shown (after some messy algebra) that $\frac{dU_0^{S,r}}{dw_A}+\frac{dU_0^{S,r}}{dw_B}\geq\frac{dU_0^{S,d}}{dw_A}+\frac{dU_0^{S,d}}{dw_B}$ iff

$$I_B^{M,a}\leq 2I_B^{S,a}+\frac{w_B+(1-\alpha)\,I_B^{S,a}}{w_A+(1-\alpha)\,V_A}I_A^{M,a}. \tag{28}$$

In numerical simulations, we have never found an occasion when (28) failed to hold. Note that (28) is satisfied if $I_B^{M,a}\leq 2I_B^{S,a}$, which is guaranteed to hold if $\gamma\geq\underline{\gamma}\equiv\frac{\hat{\theta}_1-\hat{\theta}_0}{\ln 2}$, where $\gamma$ controls the curvature of the cost function: $c(I)=\frac{1}{Z(1+\gamma)}I^{1+\gamma}$. Thus, Point 1 is proven: increasing the outside portfolio size makes retention more attractive. By having a larger outside portfolio, ambiguity is lessened.

For Point 2, the claim is that the value of delegation, $U_0^{S,d}-U_0^{S,r}$, is decreasing in $w_B$, or equivalently, that $\frac{dU_0^{S,r}}{dw_B}>\frac{dU_0^{S,d}}{dw_B}$. The claim is shown by proving that $\frac{dU_0^{S,r}}{dw_B}>e^{\theta_0-\theta^e}\geq\frac{dU_0^{S,d}}{dw_B}$. The impact of $w_B$ under retention is

$$\frac{dU_0^{S,r}}{dw_B}=\frac{1}{2}\left[e^{\theta_0-\theta^S\left(\Pi^S\left(I_A^{S,a},0\right)\right)}+e^{\theta_0-\theta^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)}\right].$$

Because $e^{\theta_0-\theta}$ is convex in $\theta$, and because $\frac{1}{2}\left[\theta^S\left(\Pi^S\left(I_A^{S,a},0\right)\right)+\theta^S\left(\Pi^S\left(0,I_B^{S,a}\right)\right)\right]=\theta^e$,

$$\frac{dU_0^{S,r}}{dw_B}>e^{\theta_0-\theta^e}.$$

The impact of $w_B$ on the payoff under delegation is $\frac{dU_0^{S,d}}{dw_B}=e^{\theta_0-\theta^S\left(\Pi^S\left(I_A^{M,a}1_A,I_B^{M,a}1_B\right)\right)}$ (from above). By the Corollary 4, $I_B^{M,a}\geq I_A^{M,a}$, which implies that $\theta^S\left(\Pi^S\left(I_A^{M,a}1_A,I_B^{M,a}1_B\right)\right)\geq\theta^e$. Because $e^{\theta_0-\theta}$ is decreasing in $\theta$, $\frac{dU_0^{S,d}}{dw_B}\leq e^{\theta_0-\theta^e}$. Therefore, $\frac{dU_0^{S,r}}{dw_B}>\frac{dU_0^{S,d}}{dw_B}$.

For Point 3, under retention, by the proof of Corollary 3, $\frac{dU_0^{S,r}}{dZ}=\frac{1-\alpha}{2Z}\left[c\left(I_A^{S,a}\right)+c\left(I_B^{S,a}\right)\right]$. Under delegation, by the proof of Corollary 3,

$$\frac{dU_0^{S,d}}{dZ}=\frac{\partial U_0^{S,d}}{\partial Z}+\frac{\partial U_0^{S,d}}{\partial I_A}\frac{dI_A^{M,a}}{dZ}+\frac{\partial U_0^{S,d}}{\partial I_B}\frac{dI_B^{M,a}}{dZ}.$$

The insider chooses investment, $c'\left(I_A^{M,a}\right)=e^{\hat{\theta}_0-\theta_1}$, so $\frac{\partial U_0^{S,d}}{\partial I_A}|_{I_A=I_A^{M,a}}=\frac{1}{2}(1-\alpha)\left[e^{\theta^S-\theta_1}-e^{\hat{\theta}_0-\theta_1}\right]$. Because $\theta^S\geq\hat{\theta}_0$ (usually with strict inequality), $\frac{\partial U_0^{S,d}}{\partial I_A}|_{I_A=I_A^{M,a}}>0$. Further, $\frac{dI_A^{M,a}}{dZ}>0$, so $\frac{\partial U_0^{S,d}}{\partial I_A}\frac{dI_A^{M,a}}{dZ}>0$. Similarly, $\frac{\partial U_0^{S,d}}{\partial I_B}|_{I_B=I_B^{M,a}}=\frac{1-\alpha}{2}\left[e^{\theta_0-\theta^S}-e^{\theta_0-\theta^M}\right]$. We cannot sign $\frac{\partial U_0^{S,d}}{\partial I_B}$, but we can say that for $Z$ big enough, it is strictly positive (see below).

It can be quickly verified that $I_B^{M,a}$ is increasing in $Z$,[44] which implies $\theta^M$ is increasing in $Z$. As $Z$ goes to zero, investment goes to zero, so $\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right)$ approaches $\theta^e$. For small $Z$, $\theta^M\left(\Pi^M\left(0, I_B^{M,a}\right)\right) = \hat{\theta}_0$ (type $B$ assets are an insignificant portion of her portfolio). Thus, for small values of $Z$, $e^{\theta_0-\theta^S} < e^{\theta_0-\theta^M}$, so $\frac{\partial U_0^{S,d}}{\partial I_B} < 0$. When $Z$ gets large, $Z \geq e^{\hat{\theta}_1(2\gamma+1)-\theta_0(\gamma+1)-\theta_1\gamma}V_A^{\gamma}$, $I_B^{M,a} \geq e^{2\left(\hat{\theta}_1-\theta^e\right)}V_A$, so $\theta^M\left(\Pi^M\left(0, I_B^{M,a}\right)\right) = \hat{\theta}_1$, so $I_B^{M,a} = I_A^{M,a}$, which implies $\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right) = \theta^e$. Thus, $\theta^S\left(\Pi^S\left(I_A^{M,a}1_A, I_B^{M,a}1_B\right)\right) = \theta^e$ but $\theta^M\left(\Pi^M\left(0, I_B^{M,a}\right)\right) = \hat{\theta}_1$ for $Z \geq e^{\hat{\theta}_1(2\gamma+1)-\theta_0(\gamma+1)-\theta_1\gamma}V_A^{\gamma}$, so $e^{\theta_0-\theta^S} > e^{\theta_0-\theta^M}$ and $\frac{\partial U_0^{S,d}}{\partial I_B} > 0$.[45]

Therefore, $\frac{\partial U_0^{S,d}}{\partial I_A} > 0$ and, for sufficiently large $Z$, $\frac{\partial U_0^{S,d}}{\partial I_B} > 0$. Further, $\frac{\partial U_0^{S,d}}{\partial Z} = \frac{1-\alpha}{2Z}\left[c\left(I_A^{M,a}\right) + c\left(I_B^{M,a}\right)\right]$, so the total impact of an increase in $Z$ on outsiders' utility under delegation is

$$\frac{dU_0^{S,d}}{dZ} = \frac{1-\alpha}{2Z}\left[c\left(I_A^{M,a}\right) + c\left(I_B^{M,a}\right)\right] + \frac{\partial U_0^{S,d}}{\partial I_A}\frac{dI_A^{M,a}}{dZ} + \frac{\partial U_0^{S,d}}{\partial I_B}\frac{dI_B^{M,a}}{dZ}.$$

$\frac{\partial U_0^{S,d}}{\partial I_A} > 0$ and $\frac{\partial U_0^{S,d}}{\partial I_B} > 0$ for $Z$ large enough. The total impact of an increase in $Z$ on outsider's utility under retention is

$$\frac{dU_0^{S,r}}{dZ} = \frac{1-\alpha}{2Z}\left[c\left(I_A^{S,a}\right) + c\left(I_B^{S,a}\right)\right].$$

$I_j^{k,a}$ is increasing in $Z$ for all $j \in \{A, B\}$ and $k \in \{S, M\}$, yet as $Z$ gets big, any party in control would invest $I_j^{k,a} = I_{\min}$ where $c'(I_{\min}) = e^{\hat{\theta}_0-\theta_1}$. Thus, for $Z$ very large, $\frac{dU_0^{S,r}}{dZ}$ equals the first term of $\frac{dU_0^{S,d}}{dZ}$, and the other two terms are strictly positive, so there exists a $\tilde{Z}$ such that $\frac{dU_0^{S,d}}{dZ} > \frac{dU_0^{S,r}}{dZ}$ for all $Z > \tilde{Z}$. ■

[44] Note $I_B^{M,a}$ satisfies $\phi\left(I_B^{M,a}\right) = 0$ where $\phi(I, \theta, Z) = e^{\theta_0-\theta^M} - c'\left(I_B^{M,a}\right)$. The result follows by totally differentiating $\phi$ w.r.t. $Z$.

[45] We do not need $Z$ to be so big that $I_B^{M,a} \geq e^{2\left(\hat{\theta}_1-\theta^e\right)}V_A$. For intermediate values of $Z$, $I_B^{M,a} \geq I_A^{M,a}$, so $\theta^S \geq \theta^e$. Depending on the value of $Z$, $\theta^M \in \left(\hat{\theta}_0, \hat{\theta}_1\right)$, but $\theta^M$ is strictly increasing in $Z$ on this range. Thus, there is a $\tilde{Z}$ such that $\frac{\partial U_0^{S,d}}{\partial I_B} > 0$ for all $Z > \tilde{Z}$. Because $\theta^S$ is inverted U-shaped in $Z$, we cannot say that there is a unique $Z$ such that $\frac{\partial U_0^{S,d}}{\partial I_B} = 0$.