Random Serial Dictatorship: The One and Only.

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Abstract

Fix a Pareto optimal, strategy proof and non-bossy deterministic matching mechanism and define a random matching mechanism by assigning agents to the roles in the mechanism via a uniform lottery. Given a profile of preferences, the lottery over outcomes that arises under the random matching mechanism is identical to the lottery that arises under random serial dictatorship, where the order of dictators is uniformly distributed. This result extends the celebrated equivalence between the core from random endowments and random serial dictatorship to the grand set of all Pareto optimal, strategy proof and non-bossy matching mechanisms.

1 Introduction

Matching mechanisms map profiles of preferences over some indivisible objects, henceforth called houses, to matchings. Under a matching each agent obtains (at most) one house. A matching mechanism is Pareto optimal if it maps any profile of preferences to a Pareto optimum; it is strategy proof if no agent can benefit from the report of a false preference at any profile of

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preferences. A mechanism is non-bossy if no agent can alter someone else's match without also altering his own. Call any mechanism that satisfies these three properties *good*. No good mechanism treats equals equally: when two different agents submit the same preference to a good mechanism they end up with different houses. Randomization fixes this flaw.

Take serial dictatorship as an example. Serial dictatorship is a good mechanism which gives one agent - the first dictator - the right to choose a house from the grand set. A next agent - the second dictator - gets to choose from the remainder, and so forth. Equals are not treated equally: if the first and second dictator submit the same preferences the first gets the house ranked highest by both. Under *random serial dictatorship* (also know as *random priority*) agents are randomly assigned to the roles of 1st, 2nd, ..., nth dictator. Any two agents who submit the same preferences face the same lottery over houses. So random serial dictatorship treats equals equally. It is moreover ex post Pareto optimal, ordinally strategy proof and non-bossy.[1]

The same method can be used to *symmetrize* any good mechanism. Instead of assigning one agent, say Karl, to assume the role of agent 1 a mechanism, and assigning Betty to the role of agent 2, and so forth, one could use a uniform lottery over all possible assignments to generate a random matching mechanism. Since the set of all good mechanisms is diverse, this method would seem to generate many different random matching mechanisms, viewed as mappings from profiles of preferences to lotteries over matchings. This is not the case: Theorem 1 shows that the symmetrization of any good mechanism coincides with random serial dictatorship.

The first observations of this sort, by Abdulkadiroglu and Sönmez [1] and Knuth [7], relate to Gale's top trading cycle mechanism (GTTC). A GTTC starts with an initial matching called the endowment and requires that there are equally many agents and houses. In a first round all agents point to their

[1] A random matching mechanism is ex post Pareto optimal if it maps any profile of preferences to a lottery over Pareto optima. It is ordinally strategy proof if there exists no profile of preferences, house h^* and agent i such that the agent i's probability obtaining a house better than h^* increases when he reports a false preference. A random matching mechanism is non-bossy if no agent can alter someone else's lottery over matches without altering his own. For a proof that non-bossiness is robust to randomization when the base mechanism is Pareto optimal and strategy proof, see Bade [3].

most preferred house and each house points to its owner according to the endowment. At least one pointing cycle forms and each agent in such a cycle is matched with the house he points to and exits. The same procedure is then repeated until all agents exit. The symmetrization of GTTC is called the *core from random endowments*.

Abdulkadiroglu and Sönmez [1] and Knuth [7] independently proved the identity of random serial dictatorship and the core from random endowments. Both their proofs start by fixing an arbitrary profile of preferences. They then construct a bijection between the set of all orders of agents as dictators and the set of initial endowments such that the outcome of the serial dictatorship under a given order equals the outcome of GTTC under the image of this given order. The bijection ensures that (at the fixed profile of preferences) the number of orders under which serial dictatorship yields some fixed matching equals the number of initial endowments under which GTTC yields the same matching. So the probability of that matching under serial dictatorship (share of the orders under which serial dictatorship yields the matching) equals the probability of the matching under the core from random endowments (share of the initial endowments under which GTTC yields the matching). This result has been extended to increasingly larger sets of good mechanisms by Pathak and Sethuraman [10], Caroll [5], and Lee an Sethuraman [8].

There are three major differences between the present paper and the preceding results. First, I show the equivalence for *all* good mechanisms. Second, my proof directly applies to sets of agents and houses of any sizes; no additional steps are required to cover the case when there are different numbers of agents and houses. These two differences are made possible by the third innovation of this paper: a new simple strategy of proof.

This strategy relies on the construction of a sequence of mechanisms $M^0, M^1, \cdots, M^K$ from an arbitrary good mechanism M^0 to a serial dictatorship M^K, such that any two consecutive mechanisms have identical symmetrizations. To show the identity of the symmetrizations of any two consecutive mechanisms M^k, M^{k+1}, I use the bijective strategy pioneered by Abdulkadiroglu and Sönmez [1] and Knuth [7]. The notable feature of my approach is that any two mechanisms in the sequence differ only marginally, implying that simple bijections show that their symmetrizations are identical. So, instead of constructing one grand (and complicated) bijection, I construct many simple bijections, each of which swaps the roles of at most three agents.

I use the fact that any good mechanism can be represented as a trading and braiding mechanism following Bade [2]. Like GTTC, trading and braiding uses rounds of trade in pointing cycles to determine outcomes. In a round, houses point to their owners and owners point to their most preferred houses. Trading and braiding generalizes three aspects of GTTC. One agent might own multiple houses, a feature introduced by Papai [9]. There is a second form of control called brokerage, introduced by Pycia and Unver [11]. Finally, if there are only three houses left a trading and braiding mechanism might terminate in a *braid*. Braids are good mechanisms for exactly three houses, which sums up all that needs to be known about braids to understand the upcoming proof.

To calculate the outcome of a trading and braiding mechanism at some profile of preferences, a brokered house points to its broker - just like an owned house points to its owner. There is at most one brokered house and its broker points to the house he most likes among the owned houses. At least one cycle forms. The houses and agents in a cycle are matched and exit the mechanism. A new trading round ensues. If there are exactly three houses left, the process either continues with another round of pointing-cycles or as a braid. The process terminates once a matching is reached.

For a sketch of the proof, consider the classic case of GTTC. My strategy is to build a sequence that starts with M^0, the GTTC such that each agent i is endowed with house h_i, and ends with M^K a serial dictatorship. Each mechanism in this sequence is derived from its predecessor by consolidating the ownership of exactly two agents. The sequence terminates with a serial dictatorship when all ownership has been maximally consolidated. M^1 is identical to GTTC except that agent 1 owns h_1 and h_2. Once agent 1 exits, agent 2 inherits the unmatched house in $\{h_1, h_2\}$.

To see that the symmetrizations of M^0 (GTTC) and M^1 are identical, fix some profile of preferences. Fix an assignment of agents to roles (initial endowment). To keep things simple, let this endowment be the original one where agent i owns house h_i. Suppose that some cycle at the start of M^0 involves agent 1 but not agent 2. In this case, agent 1 is part of the same cycle at the start of M^1; the only difference is that agent 1 additionally owns house h_2 under M^1. But for the formation of cycles under the given preferences this difference does not matter. Once the cycle involving agent 1 is matched, M^0 and M^1 continue identically (given that agent 2 inherits house h_2 under M^1). So M^0 and M^1 yield the same outcome under the given profile of preferences and assignment of agents to roles.

Now assume that in the first round of M^0 there is a single pointing cycle. Assume that this cycle involves agent 2: $2 \to h \to i \to h^* \cdots \to h_2 \to 2$, but not agent 1: $1 \neq i$ points to h^*. Every agent in the pointing cycle is matched to the house he points to, in particular agent i is matched with house h^*. Under M^1, 1 owns h_2, the cycle $1 \to h^* \cdots \to h_2 \to 1$ forms in the first round, and 1 is matched to h^*. A new assignment of agents to roles under which M^1 yields the same matching as does M^0 is easily found: switch the roles of agents 1 and 2. Under this new assignment, 2 owns h_1 and h_2 at the start of M^1. The logic of the preceding paragraph applies: the cycle $2 \to h \to i \to h^* \cdots \to h_2 \to 2$ which forms in the first round of M^0 also forms in the first round of M^1 under the new assignment. The fact that 2 additionally owns h_1 is irrelevant. Once the agents and houses in this cycle exit, M^1 under the new assignment is identical to M^0 under the original assignment (given that agent 1 inherits house h_1).

The preceding two paragraphs illustrate the bijections used to prove the identity of symmetrizations. In the first case M^0 and M^1 map the given profile of preferences to the same outcome. The bijection then maps the original assignment of roles onto itself. In the second case, M^0 and M^1 map the profile of preferences to the different outcomes. In this case the original assignment is mapped to a new assignment in which agents 1 and 2 swap roles. It turns out that the above reasoning also applies when neither agent 1 or agent 2 is matched in the first round and when both take part in the same cycle under M^0. So there exists a bijection between the assignments of roles such that M^0 under the original assignment and M^1 under the image of that assignment map the given profile of preferences to the same matching. The existence of such a bijection implies that the symmetrizations of M^0 and

M^1 are identical. The remaining mechanisms in the sequence $M^2, \cdots, M^K$ are constructed via the further consolidation of ownership. At the start of M^2 agent 1 owns h_1, h_2 and h_3, at the start of M^4 he also owns h_4. The process of consolidation terminates when M^0 has been transformed into a serial dictatorship M^K.

To apply the same strategy with any good mechanism M^0 as the start of the sequence the above arguments have to be extended to more complicated cases. The proof needs to address more ownership structures other than the one in GTTC. A somewhat different approach is needed to absorb brokers and to replace braids with serial dictatorships. All these cases share one basic feature with the two cases discussed above: the bijections between assignments of roles which show that the symmetrization of two consecutive mechanisms M^k and M^{k+1} are identical, either map an assignment onto itself or they switch the roles of at most three agents.

My paper is concerned with a fully symmetric treatment of all agents. It does not speak to existing results on random matching mechanisms that treat different agents differently. Ekici [6], for example, considers two different matching mechanisms which both respect initial (private) allocations. He shows that a uniform randomization of any additional social endowment of houses in the two mechanisms yields the same random matching mechanism. Similarly Carroll [5] considers the case of matching mechanisms in which agents are partitioned into groups and provides an equivalence result for the case in which agents are treated symmetrically only within groups. Lee and Sethuraman's [8] equivalence results also cover the case where agents are partitioned into groups which are treated asymmetrically.

2 Definitions

A housing problem consists of a set of agents $N\colon = \{1, \cdots, n\}$, a finite set of houses H and a profile of preferences $R = (R_i)_{i=1}^n$. The option to stay homeless $\emptyset$ is always available: $\emptyset \in H$. An agents' preference R_i is a linear order[2] on H and each agent prefers any house to homelessness, so $hR_i\emptyset$ holds

[2] So hR_ih' and $h'R_ih$ together imply $h = h'$.

for all $i \in N, h \in H$. The notation hR_iH' means that agent i prefers h to each house in H'. The set of all profiles R is denoted by $\mathcal{R}$. A profile of preferences $\overline{R}$ is a restriction of R to some subsets $N' \subset N$ and $H' \subset H$ if $h\overline{R}_ig \Leftrightarrow hR_ig$ holds for all $h, g \in H'$ and all $i \in N'$.

Submatchings match subsets of agents to at most one house each. A **submatching** is a function $\nu : N \to H$ such that $\nu(i) = \nu(j)$ and $i \neq j$ imply $\nu(i) = \emptyset$. The sets of agents and houses matched under ν are $N_\nu :\ = N \setminus \nu^{-1}(\emptyset)$ and $H_\nu :\ = \nu(N_\nu)$. When $\nu(i) \neq \emptyset$ then i is matched to the house $\nu(i)$; $\overline{N}_\nu :\ = N \setminus N_\nu$ and $\overline{H}_\nu :\ = H \setminus H_\nu$ are the sets of agents and houses that remain unmatched under ν. When convenient I interpret a submatching ν as the set of agent-house pairs $\{(i, h) : \nu(i) = h \neq \emptyset\}$. For two submatchings ν and ν' with $N_\nu \cap N_{\nu'} = \emptyset = H_\nu \cap H_{\nu'}$ the submatching $\nu \cup \nu' : N_\nu \cup N_{\nu'} \to H_\nu \cup H_{\nu'}$ is defined by $(\nu \cup \nu')(i) = \nu(i)$ if $i \in N_\nu$ and $(\nu \cup \nu')(i) = \nu'(i)$ otherwise. A submatching ν is considered **maximal** (**minimal**) in a set of submatchings if there exists no ν' in the set such that $\nu \subsetneq \nu'$ ($\nu' \subsetneq \nu$). Any maximal submatching (in the set of all submatchings) is a **matching**. Note that μ is a matching if and only if $H_\mu = H$ or $N_\mu = N$ (or both) hold. The sets of all matchings and of all lotteries over matchings are denoted $\mathcal{M}$ and $\Delta\mathcal{M}$ respectively.

A (deterministic) **mechanism** is a function $M : \mathcal{R} \to \mathcal{M}$. The outcome of M at R, $M(R)$, matches agent i to house $M(R)(i)$. A mechanism M is **Pareto optimal** if for no R there exists a matching $\mu' \neq M(R)$ such that $\mu'(i)R_iM(R)(i)$ for all i.[3] A mechanism M is **strategy proof** if $M(R)(i)R_iM(R'_i, R_{-i})(i)$ holds for all triples R, R'_i, i: declaring one's true preference is a weakly dominant strategy. A mechanism M is **non-bossy** if $M(R)(i) = M(R'_i, R_{-i})(i)$ implies $M(R) = M(R'_i, R_{-i})$ for all triples R, R'_i, i, so an agent can only change someone else's match if he also changes his own match. A mechanism M is **good** if it is Pareto optimal, strategy proof and non-bossy.

Let P be the set of all permutations on $p : N \to N$. A **transposition** (j, j') is a permutation involving only two agents, formally, $(j, j')(j') = j$ and $(j, j')(i) = i$ for $i \neq j, j'$. For any mechanism M and any permutation p

[3]Since all R_i are linear at least one agent must strictly prefer $\mu'(i)$ to $M(R)(i)$ if $\mu' \neq M(R)$.

define the **permuted mechanism** $p \odot M : \mathcal{R} \to \mathcal{M}$ via $(p \odot M)(R)(i) = M(R_{p(1)}, \cdots, R_{p(m)})(p^{-1}(i))$. The permutation p assigns each agent in N to a "role" in the mechanism, such that the agent $p(i)$ under $p \odot M$ assumes the role that agent i plays under M. The symbol $\odot$ is chosen as a reminder that $p \odot M$ arises out of a non-standard composition of the permutation p and the mapping M: $\odot$ is similar to but different from $\circ$, the standard operator for compositions.

Let $S : \mathcal{R} \to \mathcal{M}$ be the serial dictatorship in which agent i is the ith dictator. So $p(i)$ is the i-th dictator under $p \odot S$. To calculate $(p \odot S)(R)$ we need to substitute agent $p(i)$'s preference for agent i's preference to obtain the new profile of preferences $(R_{p(1)}, \cdots, R_{p(n)})$. Under $S(R_{p(1)}, \cdots, R_{p(n)})$ agent 1 is matched with the most preferred house of agent $p(1)$. Under $(p \odot S)(R)$ agent $p(1)$ is matched with this house: $(p \odot S)(R)(p(1)) = S(R_{p(1)}, \cdots, R_{p(n)})(p^{-1}(p(1))) = S(R_{p(1)}, \cdots, R_{p(n)})(1)$.

The **symmetrization** of a mechanism $M : \mathcal{R} \to \mathcal{M}$ is a random matching mechanism $\Delta M : \mathcal{R} \to \Delta\mathcal{M}$ that calculates the probability of matching μ at the profile R as the probability of a permutation p with $\mu = (p \odot M)(R)$ under the uniform distribution on P. So we have

$$\Delta M(R)(\mu) := \frac{|\{p : (p \odot M)(R) = \mu\}|}{n!}.$$

Abdulkadiroglu and Sönmez [1] call ΔM a **random serial dictatorship** if M is a serial dictatorship and the **core from random endowments** if M is GTTC.

Definition 1 *Two (deterministic) mechanisms M and M' are called* ***s-equivalent***[4] *if $\Delta M = \Delta M'$.*

3 The Result

Theorem 1 *Any good mechanism is s-equivalent to serial dictatorship.*

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[4]The letter "s" in s-equivalent is a reminder that we are looking at symmetrizations to establish this equivalence.

For my proof I fix an arbitrary good mechanism M^0 and construct a sequence of of mechanisms $M^1, \cdots, M^K$ such that $\Delta M^k = \Delta M^{k+1}$ holds for $0 \leq k < K$ and M^K is a serial dictatorship. The s-equivalence of any two adjacent mechanisms M^k, M^{k+1} in this sequence implies the s-equivalence of M^0 and M^K, the start and end of this sequence. The crux of my proof is to let M^{k+1} differ from M^k by so little that it is easy to identify a bijection $f : P \to P$ such that $(p \odot M^k)(R) = (f(p) \odot M^{k+1})(R)$ holds for all $p \in P$. Such a bijection exists if and only if $|\{p : (p \odot M^k)(R) = \mu\}| = |\{p : (p \odot M^{k+1})(R) = \mu\}|$ holds for all R and all matchings μ. The construction of the sequence $M^1, \cdots, M^K$ relies on the representation of good mechanisms as trading and braiding mechanisms following Bade [2].

4 Trading and braiding mechanisms

A **control rights function** at some submatching ν $c_\nu : \overline{H}_\nu \to \overline{N}_\nu \times \{o, b\}$ assigns control rights over any unmatched house to some unmatched agent and specifies a type of control. If $c_\nu(h) = (i, x)$, then agent i **controls** house h at ν. If $x = o$, then i **owns** h; if $x = b$ he **brokers** h. Control rights functions satisfy the following three criteria:

(C1) If more than one house is brokered, then there are exactly three houses and they are brokered by three different agents.
(C2) If exactly one house is brokered then there are at least two owners.
(C3) No broker owns a house.

A **general control rights structure** c maps a set of submatchings ν to control rights functions c_ν. For now just assume that c is defined for sufficiently many submatchings to ensure that the following algorithm is well defined for any fixed R.

Initialize with $r = 1$, $\nu_1 = \emptyset$

Round r: only consider the remaining houses and agents $\overline{H}_{\nu_r}$ and $\overline{N}_{\nu_r}$.

Braiding: If more than one house is brokered under c_{ν_r} let B be the braid defined (below) by the avoidance matching ω with $c_{\nu_r}(\omega(i)) = (i, b)$. Terminate the process with $M(R) = \nu_r \cup B(\overline{R})$ where $\overline{R}$ is the restriction of R to $\overline{H}_{\nu_r}$ and $\overline{N}_{\nu_r}$. If not, go on to the next step.

Pointing: Each house points to the agent who controls it, so $h \in \overline{H}_{\nu_r}$ points to $i \in \overline{N}_{\nu_r}$ with $c_{\nu_r}(h) = (i, \cdot)$. Each owner points to his most preferred house, so owner $i \in \overline{N}_{\nu_r}$ points to house $h \in \overline{H}_{\nu_r}$ if $hR_i\overline{H}_{\nu_r}$. Each broker points to his most preferred owned house, so broker $j \in \overline{N}_{\nu_r}$ with $c_{\nu_r}(h_b) = (j, b)$ points to house $h \in \overline{H}_{\nu_r} \setminus \{h_b\}$ if $hR_j\overline{H}_{\nu_r} \setminus \{h_b\}$.

Cycles: Select at least one cycle. Define ν° such that $\nu^\circ(i) := h$ if i points to h in one of the selected cycles.

Continuation: Define $\nu_{r+1} := \nu_r \cup \nu^\circ$. If ν_{r+1} is a matching terminate the process with $M(R) = \nu_{r+1}$. If not, continue with round $r + 1$.

A submatching ν is **reachable under** c at R if some round of a trading and braiding process can start with ν. A submatching ν is c**-relevant** if it is reachable under under c at some R.[5] A submatching ν' is a **direct** c**-successor of** of some c-relevant ν if there exists a profile of preferences R such that ν is reachable under $c(R)$ and ν' arises out of matching a single cycle at ν. A **control rights structure** c maps any c-relevant submatching ν to a control rights function c_ν and satisfies requirements (C4), (C5), and (C6).

Fix a c-relevant submatching ν° together with a direct c-successor ν.

(C4) If $i \notin N_\nu$ owns h at ν° then i owns h at ν.
(C5) If at least two owners at ν° remain unmatched at ν and if $i \notin N_\nu$ brokers h at ν° then i brokers h at ν.
(C6) If i owns h at ν° and ν and if $i' \notin N_\nu$ brokers h' at ν° but not at ν, then i owns h' at ν and i' owns h at $\nu \cup \{(i, h')\}$.

[5]Consider a control rights structure c with three agents $\{1, 2, 3\}$ and 4 houses $\{h, g, k, h'\}$, where agent 1 starts out owning house h and g and agent 2 starts out owning the remaining houses. Suppose the profile of preferences R is such that 1 most prefers h, and 2 most prefers h'. Then the following submatchings involving agents 1 and 2 are reachable under $c(R)$: $\{(1, h)\}, \{(2, h')\}$ and $\{(1, h), (2, h')\}$. The submatching $\{(1, g)\}$ is c-relevant since agent 1 could appropriate house g, but not it is not reachable under $c(R)$ given that 1 prefers h to g. The submatching $\{(3, h)\}$ is not c-relevant since 3 does not own any house at the start of the mechanism.

The **braid** $B : \mathcal{R} \to \mathcal{M}$ is a good mechanism for a problem with exactly three houses and at least as many agents. It is fully defined through an **avoidance matching** ω. Matchings $B(R)$ are chosen to avoid matching i to $\omega(i)$ while keeping the set of matched agents equal to the set of agents matched under ω. For any R let $\overline{PO}(R)$ be the set of Pareto optima μ with $N_\omega = N_\mu$. If $\min_{\mu \in \overline{PO}(R)} \mid \{i : \mu(i) = \omega(i)\} \mid$ is attained at a unique μ^* then let $B(R) = \mu^*$. If not, at least two agents in N_ω must rank some house $h^* = \omega(i^*)$ at the top and the pair (i^*, h^*) is decisive in the following sense. If only one agent $j \neq i^*$ ranks h^* at the top then $B(R)$ is the unique minimizer that matches j to h^*. If both agents $i \neq i^*$ rank h^* at the top, then $B(R)$ is the unique minimizer preferred by i^*.

The trading and braiding mechanism defined by the control rights structure c is also denoted c and $c(R)$ is the outcome of the trading and braiding mechanism c at the profile of preferences R. Bade [2] has shown that any good mechanism has a unique representation as a trading and braiding mechanism and that any trading and braiding mechanism is good. Trading and braiding cycles have a long family tree. With GTTC they share the the iterative process of trade in cycles. Unlike GTTC but like Papai's [9] hierarchical exchange mechanisms, agents might own multiple houses in trading and braiding mechanisms. A trading and braiding mechanism c is a **hierarchical exchange mechanism** following Papai [9] if $c_\nu(h) = (\cdot, o)$ holds for all c-relevant submatchings ν. Trading and braiding mechanisms share most with their direct predecessor: Pycia and Unver's [11] trading cycles mechanisms. Pycia and Unver [11] introduced the notion of brokerage into the matching literature. They moreover developed the formalism of control rights structures, which turns out to be very convenient for the upcoming proof.

Any c-relevant submatching ν defines a **submechanism** $c[\nu]$ that maps restrictions $\overline{R}$ (of $R \in \mathcal{R}$ to $\overline{N}_\nu$ and $\overline{H}_\nu$) to submatchings μ' with the feature that $\nu \cup \mu'$ is a matching in the original problem. The control rights structure defining $c[\nu]$ is such that $\nu^\circ = \nu \cup \nu'$ is c-relevant if and only if ν' is $c[\nu]$-relevant. For any such pair ν°, ν' we have $c[\nu]_{\nu'} = c_{\nu^\circ}$. It is easy to check that $c[\nu]$ also defines a trading and braiding mechanism. Fixing any c, R, and ν that is reachable under $c(R)$, the definition of the trading-cycles process implies $c(R) = \nu \cup c[\nu](\overline{R})$, where $\overline{R}$ is the restriction of the profile of preferences R to the set of agents $\overline{N}_\nu$ and the houses $\overline{H}_\nu$.

A c-relevant submatching ν is c-**dictatorial** if a single agent owns all houses $\overline{H}_\nu$ according to c_ν. The owner of all houses $\overline{H}_\nu$ is the **dictator at** ν. A c-relevant submatching that is not dictatorial is c-**nondictatorial**. A trading and braiding mechanism c is a **path dependent serial dictatorship** if any c-relevant submatching is c-dictatorial. A path dependent serial dictatorship c is a serial dictatorship if the dictator at any c-dictatorial submatching ν only depends on the number of agents matched under ν.

4.1 Permuting trading and braiding mechanisms

A submatching ν is $(p \odot c)$-relevant if and only if $\nu \circ p$ is c-relevant.[6] If ν is $(p \odot c)$-relevant then $(p \odot c)_\nu(h) = (p(i), o)$ holds if $c_{\nu \circ p}(h) = (i, o)$ and $(p \odot c)_\nu(h) = (p(i), b)$ holds if $c_{\nu \circ p}(h) = (i, b)$. So if agent i controls house h at $\nu \circ p$ under c, then agent $p(i)$ controls h at ν under $p \odot c$; the type of control stays the same.

Consider two mechanisms c, c' that only differ on c-relevant submatchings ν with $\nu^* \subset \nu$. So $c_\nu = c'_\nu$ holds for any c-relevant ν with $\nu^* \not\subset \nu$ implying also that any ν with $\nu^* \not\subset \nu$ is c-relevant if and only if it is c'-relevant. If $\nu^* \not\subset c(R)$ then c and c' prescribe the same control rights function for any reachable ν under $c(R)$ and we obtain $c(R) = c'(R)$. In the context of permuted mechanisms $(p \odot c')(R) = (p \odot c)(R)$ holds for any p for which $\nu^* \circ p^{-1} \not\subset (p \odot c)(R)$.

4.2 An auxiliary concept

The consolidation of ownership step in the sequence $c^0, \cdots, c^K$ works as follows. When a set of houses is owned by two different agents at some ν under c^k the same set of houses is owned by just one of these two agents

[6]Here and later in the text I abuse notation and let p or p^{-1} stand for the restriction of p or p^{-1} under which the given composition is well-defined. In the present case p stands for the restriction under which under which $\nu \circ p$ is welldefined. The expression $\nu \circ p$ could be replaced by the more precise and cumbersome $\nu \circ \overline{p}$ where $\overline{p}$ is the restriction of p that has N_ν as its image.

under c^{k+1} at the same ν. Such consolidation is problematic when under c^k at ν there are exactly two owners and a broker. If we define c^{k+1} at ν by transferring all ownership to one agent while keeping the brokerage relation intact, then (C2) is violated at that ν. To circumvent this problem, I define **new structures** by weakening (C2) in the definition of control rights structures to (C2)' which requires that there is at least one owner if there is a broker, keeping all other features of the definition intact.

I show next that a mechanism is good if and only if it can be represented by a new structure. Given Bade's [2] result that a mechanism is good if and only if it can be represented as a trading and braiding mechanism (that satisfies the stronger (C2) instead of the weaker (C2)'), any good mechanism can be represented as a new structure. To see the converse consider a new structure $\overline{c}$ that violates (C2) at exactly one $\overline{c}$-relevant submatching ν^*. So $\overline{c}_{\nu^*}(h_b) = (1, b)$ holds for some $h_b \in \overline{H}_{\nu^*}$ while $\overline{c}_{\nu^*}(h) = (2, o)$ holds for all other $h \in \overline{H}_{\nu^*}$. Let c coincide with $\overline{c}$ on all $\overline{c}$-relevant submatchings other than ν^*, let $c_{\nu^*}(h) = (2, o)$ for all $h \in \overline{H}_{\nu^*}$ and $c_{\nu'}(h) = (1, o)$ for all $h \in \overline{H}_{\nu'}$ where $\nu' = \nu^* \cup \{(2, h_b)\}$. To see that c is a control rights structure with $c(R) = \overline{c}(R)$ for any R, fix some R.

If ν^* is not reachable under $\overline{c}(R)$ then $\overline{c}(R) = c(R)$ since in that case c and $\overline{c}$ prescribe the same control rights function at every ν, that is reachable under either $\overline{c}(R)$ or $c(R)$. So fix R such that ν^* is reachable under c at R and let $h^* R_1 \overline{H}_{\nu^*} \setminus \{h_b\}$. First consider $h_b R_2 \overline{H}_{\nu^*}$, so 2 prefers h_b to all remaining houses at ν^*. Under $\overline{c}(R)$ agents 1 and 2 form a pointing cycle at ν^* and the submatching $\nu^* \cup \{(1, h^*), (2, h_b)\}$ is reached. Under $c(R)$ agent 2 owns all houses $\overline{H}_{\nu^*}$ at ν^*, he appropriates h_b and $\nu^* \cup \{(2, h_b)\}$ is reached. Under c agent 1 gets to own all remaining houses at $\nu^* \cup \{(2, h_b)\}$. Given R he appropriates house h^* and $\nu^* \cup \{(1, h^*), (2, h_b)\}$ is also reached under $c(R)$. If $h_b R_2 \overline{H}_{\nu^*}$ does not hold then agent 2 owns his most preferred house at ν^*, say h^*, under $\overline{c}(R)$ and $c(R)$ and the submatching $\nu^* \cup \{(2, h^*)\}$ is reached under $\overline{c}(R)$ and $c(R)$. The submechanisms following $\nu^* \cup \{(2, h_b)(1, h^*)\}$ and $\nu^* \cup \{(2, h^*)\}$ under $\overline{c}$ and c are identical so $\overline{c}(R) = c(R)$ holds in the either case. Since $\overline{c}$ violates (C2) only at ν^*, c satisfies (C2) at all c-relevant submatchings. So c is a trading and braiding mechanism. For any new structure $\overline{c}$ that violates (C2) at multiple c-relevant submatchings, inductively apply the above arguments to obtain a trading and braiding mechanism c that represents the same good mechanism.[7]

5 The proof

Fix any trading and braiding mechanism c^0. In Section 5.1 I construct a sequence between the fixed mechanism c^0 and a serial dictatorship c^K. Step α chooses one out of three types of transformations to derive c^{k+1} from c^k. Step β consolidates ownership. Step γ replaces a braid with a serial dictatorship. Step δ reorders dictators in a path dependent serial dictatorship. In Section 5.2 I show that the sequence is well-defined and terminates indeed with a serial dictatorship. In Section 5.3 I show that any two mechanisms c^k, c^{k+1} s-equivalent.

5.1 Construction of a sequence $c = c^0, c^1, \cdots, S$

Let $c = c^0$ and go to Step $(\alpha, 0)$.

Step (α, k): If c^k is a serial dictatorship let $k = K$ and terminate the process. If not, go to Step (δ, k) if c^k is a path dependent serial dictatorship. If neither case applies fix a minimal c^k-nondictatorial submatching ν^*. If $c^k_{\nu^*}(h) = (\cdot, b)$ holds for at most one $h \in \overline{H}_{\nu^*}$ go to Step (β, k) if not go Step (γ, k).

Step (β, k): Let $c^k_\nu = c^{k+1}_\nu$ for any c^k-relevant ν with $\nu^* \not\subset \nu$. Assume w.l.o.g. that 1 and 2 own houses under $c^k_{\nu^*}$. Define a new structure $\bar{c}$ for $\overline{N}_{\nu^*}$ and $\overline{H}_{\nu^*}$. For any c^k-relevant $\nu^* \cup \nu$ with $1, 2 \notin N_\nu$ let $\bar{c}_\nu(h) = (1, o)$ if $c^k_\nu(h) = (2, o)$ and $\bar{c}_\nu(h) = c^k_\nu(h)$ for all other $h \in \overline{H}_\nu$. For any direct $\bar{c}$-successor ν' to such a ν with $1 \in N_{\nu'}$, let $h^* \in H_{\nu'}$ be such that $\bar{c}_\nu(h^*) = (1, o)$. If $c^k_\nu(h^*) = (1, o)$ then let $\bar{c}[\nu'] = c^k[\nu^* \cup \nu']$, if $c^k_\nu(h^*) = (2, o)$ then let $\bar{c}[\nu'] = ((1, 2) \odot c^k)[\nu^* \cup \nu']$.[8] Let $c^{k+1}[\nu^*]$ be the trading and braiding mechanism that represents the same mechanism as the new structure $\bar{c}$. Go to Step $(\alpha, k + 1)$.

[7] Bade [2] shows that any good mechanism has a *unique* representation as a trading and braiding mechanism c. Multiple new structures may represent the same good mechanism.

[8] For any ν' there is a unique house h^* since agent 1 can trade only one house in a cycle. Moreover any house h^* owned by 1 at ν under $\bar{c}$ must be owned by either 1 or 2 at the same ν under c^k.

Step (γ, k): Let $c^k_\nu = c^{k+1}_\nu$ for any c^k-relevant ν with $\nu^* \not\subset \nu$. Let $c^{k+1}[\nu^*]$ be a serial dictatorship. Go to Step $(\alpha, k+1)$.

Step (δ, k): Since c^k is not a serial dictatorship, there exist three c^k-relevant submatchings $\nu^* \subset \nu' \subset \nu''$, such that $c^k[\nu^*]$ is a serial dictatorship with i the dictator at ν' and $j < i$ the dictator at ν''. Say $i = 2$ and $j = 1$. Let $c^k_\nu = c^{k+1}_\nu$ for any c^k-relevant ν with $\nu^* \not\subset \nu$ and $c^{k+1}[\nu^*] = (1,2) \odot c^k[\nu^*]$. Go to Step $(\alpha, k+1)$.

5.2 All transformations from c^k to c^{k+1} are welldefined

First consider only c^k-relevant submatchings ν with $\nu^* \not\subset \nu$. Since $c^k_\nu = c^{k+1}_\nu$ holds for any such ν and since c^k is a trading and braiding mechanism (C1), (C2) and (C3) are satisfied by c^{k+1}_ν for any such ν. By the same reasoning (C4), (C5) and (C6) are satisfied for any c^{k+1}-relevant ν° with a direct c^{k+1}-successor ν if $\nu^* \not\subset \nu$. Moreover any ν with $\nu^* \not\subset \nu$ is c^k-relevant if and only if it is c^{k+1}-relevant. Now fix ν°, such that ν^* is a direct c^k-successor of ν°. Since ν° is c^k-dictatorial,[9] conditions (C4)-(C6) do not restrict the definition of c^{k+1} at ν^*. In sum c^{k+1} is a trading and braiding mechanisms if $c^{k+1}[\nu^*]$ is a trading and braiding mechanism.

If c^{k+1} is constructed via Step (γ, k) or (δ, k) then $c^{k+1}[\nu^*]$ is a path dependent serial dictatorship and we are done. To see that $c^{k+1}[\nu^*]$ is a trading and braiding mechanism when c^{k+1} is derived from c^k via Step (β, k) it suffices to show that $\bar{c}$ is a new structure.

Let $\nu^* \cup \nu^\circ$ be c^k-relevant and let ν' be a direct $\bar{c}$-successor of ν° with $1 \notin N_{\nu'}$. For any $1 \neq j \in \overline{N}_{\nu'}$ with $\bar{c}_{\nu^\circ} h = (j, \cdot)$ we have $\bar{c}_{\nu^\circ}(h) = c^k_{\nu^* \cup \nu^\circ}(h) = c^k_{\nu^* \cup \nu'}(h) = \bar{c}_{\nu'}(h)$, where the first and third equality follow from the definition of $\bar{c}$. The middle equality follows from (C4) and (C5) - given that under c^k agents 1 and 2 both own houses at $\nu^* \cup \nu^\circ$ and at $\nu^* \cup \nu'$. (C5) and (C6) are satisfied at ν°, ν', given that under $\bar{c}$ any agent $i \in N_{\nu'}$ who brokers at ν° continues to do so at ν'. To see that (C4) is satisfied we also have to check

[9] If c^{k+1} is constructed following Steps (β, k) or (γ, k) then ν^* is a minimal c^k-nondictatorial submatching and ν° is therefore c^k-dictatorial. If c^{k+1} is constructed via (δ, k) then c^k is a path dependent serial dictatorship and any c^k-relevant submatching is dictatorial. Since $\nu^* \not\subset \nu^\circ$ we have $c^k_{\nu^\circ} = c^{k+1}_{\nu^\circ}$ and consequently ν° is c^{k+1}-dictatorial.

that ownership of agent 1 continues from ν° to ν' under $\bar{c}$. If $\bar{c}_{\nu^\circ}(h) = (1, o)$ then $c^k_{\nu^* \cup \nu^\circ}(h) = (i, o)$ holds for $i \in \{1, 2\}$. Since c^k satisfies (C4) $c^k_{\nu^* \cup \nu^\circ}(h) = c^k_{\nu^* \cup \nu'}(h)$ and by the definition of $\bar{c}$: $\bar{c}_{\nu'}(h) = (1, o)$. In sum we have that (C4) is satisfied at ν°, ν'.

By the definition of $\bar{c}$ there is one less owner under $\bar{c}_\nu$ than under $c^k_{\nu^* \cup \nu}$ for any $\bar{c}$-relevant ν with $1 \notin N_\nu$. Since $c^k_{\nu^* \cup \nu}$ satisfies (C2) (at least two owners), $\bar{c}_\nu$ satisfies (C2)' (at least one owner). Since $\bar{c}_\nu(h) = (i, b)$ only holds if $c^k_{\nu^* \cup \nu}(h) = (i, b)$ holds and since $c^k_{\nu^* \cup \nu}$ satisfies (C1) and (C3), $\bar{c}_\nu$ does too.

Now replace the assumption that $1 \notin N_{\nu'}$ with $1 \in N_{\nu'}$ and $1 \notin N_{\nu^\circ}$. Let $\bar{c}_{\nu^\circ}(h) = (i, b)$, $i \notin N_{\nu'}$ and $\bar{c}_{\nu'}(h) \neq (i, b)$. So we have $c^k_{\nu^* \cup \nu^\circ}(h) = (i, b)$. First consider the case that $\bar{c}[\nu'] = c^k[\nu^* \cup \nu']$. In that case $\bar{c}_{\nu'}(h) \neq (i, b)$ implies $c^k_{\nu^* \cup \nu'}(h) \neq (i, b)$. Since 2 is neither matched at $\nu^* \cup \nu^\circ$ nor at $\nu^* \cup \nu'$ and since 2 is an owner under c^k at $\nu^* \cup \nu^\circ$, 2 must also be an owner under c^k at $\nu^* \cup \nu'$ (given that c^k satisfies (C4)). Since c^k satisfies (C5) and since i looses brokerage in the round from $\nu^* \cup \nu^\circ$ to $\nu^* \cup \nu'$ 2 is the only owner under c^k at both these matchings. The derivation of $\bar{c}$ from c^k implies that there is no agent who owns houses at ν° and ν' under $\bar{c}$. So $\bar{c}$ satisfies (C5) and (C6) at ν°, ν' if $\bar{c}[\nu'] = c^k[\nu^* \cup \nu']$. In the alternative case that $\bar{c}[\nu'] = ((1, 2) \odot c^k)[\nu^* \cup \nu']$ agents 1 and 2 have to be switched in the above arguments to show that $\bar{c}$ satisfies (C5) and (C6) at ν°, ν'. Now let $\bar{c}_{\nu^\circ}(h) = (i, o)$ and $i \notin N_{\nu'}$. By the definition of $\bar{c}$ we have $c^k_{\nu^* \cup \nu^\circ}(h) = (i, o)$. Since $i \neq 1, 2$[10], and since c^k satisfies (C4), $c^k_{\nu^* \cup \nu'}(h) = (i, o)$ and $c^k_{(\nu^* \cup \nu') \circ (1,2)}(h) = (i, o)$ both hold. Using the definition of $\bar{c}$ once again we obtain $\bar{c}_{\nu'}(h) = (i, o)$ and $\bar{c}$ satisfies (C4) at ν°, ν'.

Since (β, k) defines $\bar{c}[\nu']$ as a trading and braiding mechanism, (C1), (C2) and (C3) are satisfied at any $\bar{c}$-relevant ν with $\nu' \subset \nu$ under $\bar{c}$. By the same token (C4), (C5) and (C6) are satisfied in this submechanism. In sum $\bar{c}$ is a new structure, and c^{k+1} is well-defined no whether it was derived from c^k via Step β, γ, or δ.

If ν is c^{k+1}-relevant whereas neither ν nor $\nu \circ (1, 2)$ is c^k-relevant then c^{k+1} either arises out of the replacement of a braid with a serial dictatorship

[10] The assumption $\bar{c}_{\nu^\circ}(h) = (i, o)$ implies $i \neq 2$; the assumption that $i \notin N_{\nu'}$ together with $1 \in N_{\nu'}$ imply that $i \neq 1$.

via (γ, k) or c^{k+1} is constructed via (β, k) and $\nu = \nu' \cup \{(i, h_b)\}$ holds for a house h_b that is brokered at ν' under c^k but owned by i at ν' under c^{k+1}. In either case one agent owns all houses at ν according to c^{k+1}. It follows that the number of c^k-nondictatorial submatchings does not increase in k. When there exists at least one c^k-nondictatorial submatching, Step (α, k) prescribes a to follow Step (β, k) or (γ, k). Since (β, k) reduces the number of owners at at least one c^k-relevant submatching and since (γ, k) replaces a braid with a serial dictatorship, the process of transformations eventually reaches a trading and braiding mechanism $c^{k'}$ such that any $c^{k'}$-relevant submatching is $c^{k'}$-dictatorial. But such a $c^{k'}$ is a path dependent serial dictatorship. Finally Step (δ, k) is iteratively applied to reorder agents as dictators such that an earlier dictator i swaps roles with a later dictator j if $i > j$. Such reordering occurs until the sequence $c^0, c^1, \cdots$ terminates with a serial dictatorship c^K.

5.3 $\Delta c^k = \Delta c^{k+1}$ holds for all k

Fix two mechanisms c^k and c^{k+1} and a profile of preferences R. The s-equivalence of c^k and c^{k+1} follows from the existence of a bijection $f : P \to P$ such that $(p \odot c^k)(R) = (f(p) \odot c^{k+1})(R)$ holds for all $p \in P$. Since $c^k_\nu = c^{k+1}_\nu$ holds for any c^k- and c^{k+1}-relevant ν with $\nu^* \not\subset \nu$, $(p \odot c^k)(R) = (p \odot c^{k+1})(R)$ holds for any p such that $\nu^* \circ p^{-1} \not\subset (p \odot c^k)(R)$. For any such p let $f(p) \colon = p$.

Since ν^* is a minimal c^k-nondictatorial submatching, $\nu^* \circ p^{-1}$ is a minimal $p \odot c^k$-nondictatorial submatching. For $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ to hold, each of the dictators on the path to reach $\nu^* \circ p^{-1}$ must choose in accordance with $\nu^* \circ p^{-1}$. If $\nu^* \neq \emptyset$, the dictator at $\emptyset$ under $p \odot c^k$, say agent i, must choose $\nu^*(p^{-1}(i))$ for $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ to hold and so forth. So $\nu^* \circ p^{-1}$ is reachable under $(p \odot c^k)(R)$ if and only if $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$. In the following three sections 5.3.1, 5.3.2, and 5.3.3 I define f for permutations p such that $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$.

5.3.1 When Step (β, k) is used to construct c^{k+1}

Fix any p with $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ and let $\overline{\nu}$ be the maximal reachable submatching under $(p \odot c^k)(R)$ such that $p(1), p(2) \notin N_{\overline{\nu}}$. If $p(1)$ is part of a cycle at $\overline{\nu}$ let $f(p) \colon = p$ if not let $f(p) \colon = p \circ (1, 2)$.

In the first case let ν' be the direct c^k-successor of $\overline{\nu}$ with $p(1) \in N_{\nu'}$. If also $p(2) \in N_{\nu'}$ holds then some cycle $p(1) \to \mu(1) \to \cdots \to h_2 \to p(2) \to \mu(2) \to \cdots \to h_1 \to p(1)$ forms at $\overline{\nu}$ under $(p \odot c^k)(R)$. Under $(p \odot \overline{c})(R)$ the cycle $p(1) \to \mu(1) \to \cdots \to h_2 \to p(1)$ forms at $\overline{\nu}$, since $p(1)$ owns h_2 under $p \odot \overline{c}$ at $\overline{\nu}$. Since $(p \odot c^k)_{\overline{\nu}}(h_2) = (p(2), o)$ holds for the house h_2 that $p(1)$ trades in the cycle $p(1) \to \mu(1) \to \cdots \to h_2 \to p(1)$, agent $p(2)$ inherits h_1 under $p \odot \overline{c}$ once this cycle is matched. The cycle $p(2) \to \mu(2) \to \cdots \to h_1 \to p(2)$ forms and ν' is also reachable under $(p \odot c^{k+1})(R)$. If $p(2) \notin N_{\nu'}$ then $p(1)$ is part of the same cycle at $\overline{\nu}$ under $(p \odot c^k)(R)$ and $(p \odot \overline{c})(R)$, so also in this case ν' is reachable under $(p \odot \overline{c})(R)$. In either case the definition of $\overline{c}$ implies that the submechanisms of $p \odot c^k$ and $p \odot \overline{c}$ following ν' are identical: $(p \odot c^k)[\nu'] = (p \odot \overline{c})[\nu']$. In sum we have $(p \odot c^k)(R) = (p \odot \overline{c})(R) = (p \odot c^{k+1})(R) = (f(p) \odot c^{k+1})(R)$. We can conclude that $(p \odot c^k)(R) = (f(p) \odot c^{k+1})(R)$ holds when $p(1)$ is part of a cycle at $\overline{\nu}$ under c^k.

In the alternative case when $p(1)$ does not take part in any cycle at $\overline{\nu}$ under $p \odot \overline{c}$, agent $p(2)$ must be part of the unique such cycle $p(2) \to \mu(2) \to \cdots \to h_2 \to p(2)$.[11] Let ν' the direct c^k-successor of $\overline{\nu}$ that arises out of matching this cycle. Under $(f(p) \odot \overline{c})(R)$ at $\overline{\nu}$ agent $p(2)$ is the owner of all houses owned by agents $p(1)$ and $p(2)$ under $(p \odot c^k)(R)$ at $\overline{\nu}$ which implies that $p(2) \to \mu(2) \to \cdots \to h_2 \to p(2)$ also forms under $(f(p) \odot c^{k+1})(R)$ at $\overline{\nu}$. Once this cycle is eliminated the mechanism continues under $f(p) \odot \overline{c}$ as if agent $p(1)$ had always played the role of agent 1 in c^k, formally $(f(p) \odot \overline{c})[\nu'] = (p \odot c^k)[\nu']$, since the transposition $(1, 2)$ used to define $\overline{c}$ is inverted by the transposition $(1, 2)$ used to transform p under f: $f(p) = p \circ (1, 2)$. In sum we have that that $(p \odot c^k)(R) = (f(p) \odot \overline{c})(R)$ holds for all p.

To see that f is a bijection note that there exists a set P^0 such that $f(p) = p$ holds for all $p \in P^0$ while $f(p) = p \circ (1, 2)$ holds for $p \in \overline{P} := P \setminus P^0$. Restricted to P^0 f is a bijection, restricted to $\overline{P}$ f is one-to-one. To see that f is a bijection it therefore suffices to show that $f(\overline{P})$ is a subset of $\overline{P}$. Note that $p \in \overline{P}$ if $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ and $p(1)$ is not part of a cycle at $\overline{\nu}$

[11] The existence and uniqueness of this cycle follows from $\overline{\nu}$ being maximal in the set of submatchings ν with $p(1), p(2) \notin N_\nu$ that are reachable under $(p \odot c^k)(R)$.

under $(p \odot c^k)(R)$. Fix any $p \in \overline{P}$ and let $p(2) \to \mu(2) \to \cdots \to h_2 \to p(2)$ be the unique cycle at $\overline{\nu}$. The uniqueness of this cycle implies that $p(1)$ is part of some chain $p(1) \to h^* \to \cdots \to h' \to i'$ under $(p \odot c^k)(R)$ at $\overline{\nu}$ that terminates with an agent i' in the cycle $p(2) \to \mu(2) \to \cdots \to h_2 \to p(2)$. Under $(f(p) \odot c^k)(R)$ agent $p(1)$ owns house h_2 at $\overline{\nu}$ so the cycle $p(1) \to h^* \to \cdots \to h' \to i' \to \cdots \to h_2 \to p(1)$ forms at $\overline{\nu}$ under $(f(p) \odot c^k)(R)$. In that case $p(2)$ is part of the chain $p(2) \to \mu(2) \to \cdots \to i''$ that terminates with an agent i'' in the cycle $p(1) \to h^* \to \cdots \to h' \to i' \to \cdots \to h_2 \to p(1)$. There cannot be another cycle at $\overline{\nu}$ under $(f(p) \odot c^k)(R)$ since this cycle would only involve agents $i \neq p(1), p(2)$ and would therefore also form under $(p \odot c^k)(R)$ - contradicting the assumption that $\overline{\nu}$ is the maximal reachable submatching under $(p \odot c^k)(R)$ which neither involves $p(1)$ nor $p(2)$. So there is only one cycle at $\overline{\nu}$ under $(f(p) \odot c^k)(R)$ and this cycle involves $f(p)(2) = p(1)$ implying that $f(p) \in \overline{P}$ as required.

5.3.2 When the transformation is constructed using Step (γ, k)

The construction of f for the set of permutations p with $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ relies on Lemma 1 on random matching mechanisms with just three agents and three houses which in turn requires some more concepts. For any random matching mechanism $\mathfrak{M} : \mathcal{R} \to \Delta\mathcal{M}$ let $\mathfrak{M}(R)[i,h]$ be the probability that agent i is matched with house h when the agents announce the profile of preferences R. A random matching mechanism $\mathfrak{M} : \mathcal{R} \to \Delta\mathcal{M}$ is **ex post Pareto optimal** if any μ in the support of $\mathfrak{M}(R)$ is Pareto optimal at R. The mechanism $\mathfrak{M}$ is **ordinally strategy** proof if $\sum_{hR_ih^*} \mathfrak{M}(R)[i,h] \geq \sum_{hR_ih^*} \mathfrak{M}(R'_i, R_{-i})[i,h]$ holds for all R, R'_i, i, h^*. So under an ordinally strategy proof mechanism no agent can misrepresent his preferences to increase his probability to get a house he prefers to some fixed h^*. A mechanism $\mathfrak{M}$ satisfies **equal treatment of equals** if for any R with $R_i = R_j$ agents i and j face the same distribution over matches under $\mathfrak{M}(R)$, $R_i = R_j \Rightarrow \mathfrak{M}(R)[i,h] = \mathfrak{M}(R)[j,h]$ for all $h \in H$. The symmetrization of any good mechanism is ex post Pareto optimal, ordinally strategy proof and satisfies equal treatment of equals.

Lemma 1 *Let $H = \{a, b, c\}$ and $N = \{1, 2, 3\}$. Let $\mathfrak{M} : \mathcal{R} \to \Delta\mathcal{M}$ be ex* *post Pareto optimal, ordinally strategy proof and satisfy equal treatment of equals. Then $\mathfrak{M}$ is a random serial dictatorship.*

Proof Fix an arbitrary profile of preferences R. Let $aR_i^*bR_i^*c$, $aR_i^\circ cR_i^\circ b$, $bR_i'\{a,c\}$, and $cR_i''\{a,b\}$. For any R I identify a set of 9 linearly independent linear equations that uniquely determine the 9 values $\mathfrak{M}(R)[i,h]$. These equations arise out of ex post Pareto optimality, equal treatment of equals, ordinal strategy proofness, and the fact that $\mathfrak{M}(R)$ is a probability distribution over matchings. Since random serial dictatorship is ex post Pareto optimal, ordinally strategy proof and satisfies equal treatment of equals it equals $\mathfrak{M}$.

Case (I) there is a unique Pareto optimal matching at R. Ex post Pareto optimality requires that $\mathfrak{M}(R)$ assigns probability 1 to this matching. Case (II) $R_1 = R_2 = R_3$. Equal treatment of equals requires that $\mathfrak{M}(R)[i,h] = \frac{1}{3}$ for all i,h. Case (III) (R_1'', R_{-1}^*). Ex post Pareto optimality implies $\mathfrak{M}(R_1'', R_{-1}^*)[1,c] = 1$. Equal treatment of equals implies $\mathfrak{M}(R_1'', R_{-1}^*)[2,\cdot] = M(R_1'', R_{-1}^*)[3,\cdot]$, and therefore $\mathfrak{M}(R_1'', R_{-1}^*)[i,h] = \frac{1}{2}$ for $i = 2,3$ and $h = a,b$. Case (IV) (R_1°, R_{-1}^*). Ex post Pareto optimality implies $\mathfrak{M}(R_1^\circ, R_{-1}^*)[1,b] = 0$; ordinal strategyproofness and (II) imply that $\mathfrak{M}(R_1^\circ, R_{-1}^*)[1,a] = \frac{1}{3}$. Equal treatment of equals implies $\mathfrak{M}(R_1^\circ, R_{-1}^*)[2,\cdot] = \mathfrak{M}(R_1^\circ, R_{-1}^*)[3,\cdot]$. This system of linear equations has a unique solution with $\mathfrak{M}(R_1^\circ, R_{-1}^*)[1,c] = \frac{2}{3}$, $\mathfrak{M}(R_1^\circ, R_{-1}^*)[i,a] = \frac{1}{3}$ $\mathfrak{M}(R_1^\circ, R_{-1}^*)[i,b] = \frac{1}{2}$ and $\mathfrak{M}(R_1^\circ, R_{-1}^*)[i,c] = \frac{1}{6}$ for $i = 2,3$.

Case (V) (R_1', R_{-1}^*). Ex post Pareto optimality implies $\mathfrak{M}(R_1', R_{-1}^*)[1,a] = 0$. Equal treatment of equals implies $\mathfrak{M}(R_1', R_{-1}^*)[2,\cdot] = \mathfrak{M}(R_1', R_{-1}^*)[3,\cdot]$. Ordinal strategy-proofness and (II) imply $\mathfrak{M}(R_1', R_{-1}^*)[1,a] + \mathfrak{M}(R_1', R_{-1}^*)[1,b] = \mathfrak{M}(R^*)[1,a] + \mathfrak{M}(R^*)[1,b] = \frac{2}{3}$. The system of linear equations has a unique solution with $\mathfrak{M}(R_1', R_{-1}^*)[i,a] = \frac{1}{2}, \mathfrak{M}R_1', R_{-1}^*)[i,b] = \frac{1}{6}, \mathfrak{M}(R_1', R_{-1}^*)[i,c] = \frac{1}{3}$ for $i = 2,3$, $\mathfrak{M}(R_1', R_{-1}^*)[1,b] = \frac{2}{3}$ and $\mathfrak{M}(R_1', R_{-1}^*)[1,c] = \frac{1}{3}$.

Case (VI) (R_1', R_2°, R_3^*). Ex post Pareto optimality implies $\mathfrak{M}(R_1', R_2^\circ, R_3^*)[1,a] = \mathfrak{M}(R_1', R_2^\circ, R_3^*)[2,b] = 0$. Ordinal strategyproofness and $\mathfrak{M}(R_2^\circ, R_{-1}^*)[1,a] + \mathfrak{M}(R_2^\circ, R_{-1}^*)[1,b] = \frac{5}{6}$ as implied by Case (III) yield $\mathfrak{M}(R_1', R_2^\circ, R_3^*)[1,a] + \mathfrak{M}(R_1', R_2^\circ, R_3^*)[1,b] = \frac{5}{6}$. Ordinal strategyproofness and $\mathfrak{M}(R_1', R_{-1}^*)[2,a] = \frac{1}{2}$ as implied by Case (V) yield $\mathfrak{M}(R_1', R_2^\circ, R_3^*)[2,a] = \frac{1}{2}$. This system of linear equations has a unique solution with $\mathfrak{M}R_1', R_2^\circ, R_3^*)[1,b] = \frac{5}{6}$, $\mathfrak{M}(R_1', R_2^\circ, R_3^*)[1,c] =$

$\frac{1}{6}$, $\mathfrak{M}R'_1, R^\circ_2, R^*_3)[2,c] = \frac{1}{2}$, $\mathfrak{M}(R'_1, R^\circ_2, R^*_3)[3,a] = \frac{1}{2}$, $\mathfrak{M}(R'_1, R^\circ_2, R^*_3)[3,b] = \frac{1}{6}$ and $\mathfrak{M}(R'_1, R^\circ_2, R^*_3)[3,c] = \frac{1}{3}$. Mutatis mutandis all profiles of preferences are covered by Cases (I) through (VI). $\square$

Let $\mathcal{I} = \{I \subset N :\mid I \mid= 3\}$ be the set of all three agent subsets of N. Let P^I be the set of permutations p such that on the one hand $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ and on the other hand $I = \{i : (p \odot c^k)_{\nu^* \circ p^{-1}}(h) = (i,b)$ for some $h \in H\}$.[12] Let Q^I be the set of permutations p such that on the one hand $\nu^* \circ p^{-1} \subset (p \odot c^{k+1})(R)$ and on the other hand I is the set of the first three dictators in the serial dictatorship prescribed by $p \odot c^{k+1}$ at $\nu^* \circ p^{-1}$. Since c^k and c^{k+1} prescribe the same control rights functions for any ν with $\nu^* \not\subset \nu$, $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ holds if $\nu^* \circ p^{-1} \subset (p \odot c^{k+1})(R)$ holds. So $\{P^I\}_{I \in \mathcal{I}}$ and $\{Q^I\}_{I \in \mathcal{I}}$ both partition the set of all p such that $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$. By Lemma 1 the symmetrization of any braid with a fixed set of three houses H and a fixed set of three agents I is identical to random serial dictatorship. So for any I there exists a bijection $f^I : P^I \to Q^I$ such that $(p \odot c^k)(R) = (f^I(p) \odot c^{k+1})(R)$.

Given that $\{P^I\}_{I \in \mathcal{I}}$ and $\{Q^I\}_{I \in \mathcal{I}}$ both partition the set of all p such that $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ the function $f : P \to P$ with $f(p) := p$ for $\nu^* \circ p^{-1} \not\subset (p \odot c^k)(R)$ and $f(p) := f^I(p)$ for any $p \in P^I$ is a bijection. By construction $(p \odot c^k)(R) = (f(p) \odot c^{k+1})(R)$ holds for all p.

5.3.3 When the transformation is constructed using Step (δ, k)

Define $f(p) = p$ for any p with $\nu^* \circ p^{-1} \not\subset (p \odot c^k)(R)$ and $f(p) = p \circ (1,2)$ otherwise. If $f(p) = p$ then $(f(p) \odot c^{k+1})[\nu^*] = (p \odot c^k)[\nu^*]$ holds by the arguments given above. If $f(p) = p \circ (1,2)$ then $(f(p) \odot c^{k+1})[\nu^*] = (p \odot c^k)[\nu^*]$ holds, since the transposition $(1,2)$ used to define c^{k+1} is inverted by the transposition $(1,2)$ used to transform p under f. Moreover f is a bijection since $\nu^* \circ p^{-1} \subset (p \odot c^k)(R)$ holds if and only if $\nu^* \circ f(p)^{-1} \subset (f(p) \odot c^k)(R)$ given that $f(p)(i) = p(i)$ holds for all $i \in N_{\nu^* \circ p^{-1}}$.

[12] So I consists of the three agents in the avoidance matching of the braid at $\nu^* \circ p^{-1}$ under $p \odot c^k$.

6 Conclusion

Two approaches had so far been used to establish the equivalence between symmetrizations of different good mechanisms: Abdulkadiroglu and Sönmez [1] as well as Knuth [7] constructed bijections to show the s-equivalence of GTTC and serial dictatorship. Carroll [5] constructed yet more involved bijections to show the s-equivalence of serial dictatorship and any top trading cycles mechanism. Pathak and Sethuraman [10] and Lee and Sethuraman [8] used an inductive strategy over the number of agents in a mechanism to prove that any hierarchical exchange mechanism following Papai [9] with equally many houses and agents is *s*-equivalent to serial dictatorship.

Could one use either one of these strategies to extend the s-equivalence result to differently many agents and houses or to good mechanisms that are not hierarchical exchange mechanisms? Existing equivalence results can easily be extended to the case of there being more agents than houses. To see this fix a mechanism M with equally many agents and houses that is s-equivalent to serial dictatorship. Considering a problem with more agents than houses, create $\mid N \mid - \mid H \mid$ dummy houses. For any profile of preferences R on the original set of houses H define an auxiliary profile of preferences R' such that R_i and R'_i coincide on H for all i and such that any agent ranks all houses in H above all dummy-houses. Derive $\Delta M(R) = \Delta S(R)$ by equating the probability that an agent obtains a dummy house under $\Delta M(R')$ or $\Delta S(R')$ with the probability that the agent obtains no house under $\Delta M(R)$ and $\Delta S(R)$.

This trick does not work when there are more houses than agents. In this case we would not only have to create dummy agents, these dummy agents would have to be endowed with "dummy preferences". When dummy a agent is matched with a house that some real agent prefers to his match, the exclusion of the dummy-house match leads to a Pareto inferior match. To avoid this problem, one could use results on partial symmetrizations by Carroll [5] and Lee and Sethuraman [8] that treat the agents in some sets symmetrically while maintaining their relative place with respect to other sets of agents.

Could we use one of the existing strategies of proof for the case of a good mechanisms that is not a hierarchical exchange mechanism? The task of directly constructing a bijection to prove Theorem 1 seems out of the question. Carroll's work [5] probably hits the limit in this dimension. The inductive strategy or Pathak and Sethuraman [10] and Lee and Sethuraman [8] relies on mechanisms being representable as trading mechanisms in which each agent points to their most preferred house. Given that brokers may not do so and given that braids are not representable as such trading mechanisms, this strategy of proof does not extend to the grand set of mechanisms.

Since my strategy of proof relies on the construction of sequences of marginally different good mechanisms, it can only be used on a sufficiently rich set of mechanisms. To apply the present proof-strategy to show that GTTC is s-equivalent to serial dictatorship, these two mechanisms need to be embedded set a rich set of mechanisms. Historically speaking, it would have been difficult to apply the present strategy of proof in 1996 or 1998, when Knuth [7] and Abdulkadiroglu and Sönmez [1] respectively published their proofs, given that Papai's [9] hierarchial exchange mechanisms came out in 2000.

The inductive consolidation of ownership works best with hierarchical exchange mechanisms. The length of the present paper is owed to the fact that further arguments are required to deal with brokers and braids. To see this reconsider this paper without brokers or braids. In other words consider the set of hierarchical exchange mechanisms. All that remains in Section 4 on the characterization of mechanisms, are control rights structures c that map any c-relevant submatching to a control rights function in which all houses are owned. In this setup (C1), (C2), (C3), (C5) and (C6) are trivially satisfied, only (C4) is relevant. Without brokers the definition of new structures becomes obsolete. Given that there are no braids Step α assigns any c^k to one of *two* possible transformations: c^{k+1} is either constructed through the consolidation of ownership in Step β or through a reordering of dictators in Step δ. The proof that the sequence is well-defined becomes notably easier, since we only need to check that (C4) remains valid in the transformations. The case of hierarchical exchange mechanisms could be dealt with in less than half the pages necessary to cover the grand set of good mechanisms. Differently from the predecessors in the literature this proof that any hierarchical exchange mechanism is s-equivalent to serial dictatorship, does not involve any combinatorial arguments.

To deal with braids I showed Lemma 1: random serial dictatorship is the unique ex post Pareto optimal and ordinally strategy proof random matching mechanism for three agents and three houses that satisfies equal treatment of equals. This Lemma yields a vastly more general conjecture than Theorem 1. Could random serial dictatorship be the unique ex post Pareto optimal and ordinally strategy proof random matching mechanism that satisfies equal treatment of equals? Unfortunately, the method used in my proof of Lemma 1 becomes cumbersome with more houses and agents. The probability distribution $\mathfrak{M}(R)$ would have to be identified via $\mid H \mid \times \mid N \mid$ linearly independent linear equations on the probabilities $\mid H \mid \times \mid N \mid$ $\mathfrak{M}(R)[i,h]$. One example of these equations is $\mathfrak{M}(R)[i,h] = 0$ if there exists no Pareto optimum at R under which the pair i, h. The problem is that Saban and Sethuraman [12] have shown that the identification of these equations is an NP-complete problem. With more houses and agents it becomes increasingly difficult to find all house-agent pairs that are not matched under any Pareto optimum. So a proof of the more general conjecture requires a very different attack.

Some papers, such as Bogomolnaia and Moulin [4] have presented possible tradeoffs between Pareto optimality and strategy proofness while maintaining equal treatment of equals and non-bossiness. In this context, random serial dictatorship is typically used as the benchmark of a mechanism that is best in terms of its incentive properties (ordinally strategy proof) and worst in terms of its welfare properties (only ex post Pareto optimal). This paper strengthens the case for using random serial dictatorship as the benchmark. While initially one could have criticized the choice of a particular good mechanism as the base of the symmetrization, I have shown that this choice does not matter: the symmtrization of any good mechanism leads to random serial dictatorship.

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