Intertemporal Substitution and Hyperbolic Discounting*

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Abstract

Evidence from behavioral experiments suggests that intertemporal preferences reflect hyperbolic rather than exponential discounting. This paper shows that consumers tend to have a lower elasticity of intertemporal substitution under hyperbolic discounting. Furthermore, in contrast to the standard case of exponential discounting with iso-elastic utility, the elasticity of intertemporal substitution for hyperbolic consumers depends on the duration of the change in the intertemporal relative price. In particular, lasting changes in the real interest rate are likely to generate a smaller degree of intertemporal substitution in consumption than temporary changes. For plausible parameter values, the extent of intertemporal substitution is about 20% smaller for a permanent change than for a temporary change, so the effect is economically significant.

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JEL codes: D91, E21

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1 Introduction

Temptations are often irresistible. This inclination for immediate gratification reflects a bias in intertemporal preferences towards present rewards. Behavioral evidence indicates that intertemporal discount rates decline with the delay in rewards and are well-described by a hyperbolic discount function. This paper builds on the seminal contributions by Laibson (1996, 1997) and shows that hyperbolic discounting fundamentally affects intertemporal substitution. In contrast to the familiar result under exponential discounting with iso-elastic utility, the elasticity of intertemporal substitution for hyperbolic consumers depends on the duration of the change in the intertemporal relative price. This holds for both sophisticated consumers, who realize that they have dynamically inconsistent preferences and rationally anticipate their future behavior, and naive consumers, who do not foresee their future self-control problems and corresponding present bias. The result that the elasticity of intertemporal substitution is sensitive to the duration of the intertemporal price change is a novel theoretical finding that has important implications for the effects of macroeconomic policy.

Intuitively, the intertemporal substitution of consumption depends on the difference between the real interest rate and the (effective) discount rate. With hyperbolic discounting the discount rate declines as the time horizon increases and the effective discount rate is a consumption-weighted average of the high short-run and the low long-run discount rate. For a short change in the interest rate, future intertemporal trade-offs are not affected so the effective discount rate remains constant. But a lasting interest rate change generally influences the effective discount rate, which alters the effect of the interest rate on intertemporal substitution. In particular, when the income effect dominates the substitution effect of a permanent increase in the real interest rate, the consumption rate rises, which increases the effective discount rate towards the higher, short-run discount rate. This partially offsets the increase in the real interest rate and diminishes the degree of intertemporal substitution.

The theoretical literature has identified several ways in which hyperbolic and exponential consumers differ. Laibson (1998) provides a useful overview. One interesting distinction is that hyperbolic discounting helps to explain the empirical anomaly that the elasticity of intertemporal substitution is less than the inverse of the coefficient of relative risk aversion. This was first shown by Laibson (1996) for a permanent change in the real interest rate with sophisticated consumers in discrete time. The present paper establishes that hyperbolic discounting has a more profound effect on intertemporal substitution. In contrast to exponential discounting with iso-elastic utility, where the length of the change in the real interest rate is immaterial, the elasticity of intertemporal substitution under hyperbolic discounting depends on the duration of the intertemporal price change. For plausible levels of risk aversion, the elasticity of intertemporal substitution is smaller for more persistent changes. This result also applies to naive consumers and to exponential utility.

Another interesting new finding of this paper is the possibility of negative elasticity values for naive hyperbolic consumers with a low degree of self control. Together with the importance of persistence for intertemporal substitution, this provides a potential explanation for the wide range of estimates of the elasticity of intertemporal substitution in the empirical literature.

These effects of hyperbolic discounting already hold for the standard infinite-horizon model with one liquid asset and no financial market imperfections, for which the consumption behavior of hyperbolic and exponential agents is otherwise indistinguishable. The (quasi-) hyperbolic discrete-time model with sophisticated hyperbolic consumers with iso-elastic utility and a time-varying interest rate is presented in section 2. The main result of the paper, namely that the elasticity of intertemporal substitution under hyperbolic discounting is likely to be smaller than under exponential discounting and decreasing in the duration of the change in the interest rate, is established in section 3. Subsequently, section 4 shows that this result is robust: It also holds for naive consumers and for exponential utility. It is also relevant for more realistic 'buffer-stock' models. The empirical and policy implications are addressed in the concluding section 5.

2 Hyperbolic Discounting

Intertemporal discounting has been studied extensively in psychology. Experiments regarding human (and animal) behavior show that the rate of time preference depends on the time interval τ between the moment of choice and the actual events (e.g. Ainslie 1992). Imminent outcomes are discounted at a higher rate than payoffs in the distant future. This can be described by the generalized hyperbolic discount function $\phi_h(\tau) = (1 + \alpha\tau)^{-\gamma/\alpha}$ (Loewenstein and Prelec 1992). The corresponding discount rate $\gamma/(1 + \alpha\tau)$ decreases in the delay τ, which is consistent with behavioral data (e.g. Thaler 1981, Benzion, Rapoport and Yagil 1989).

Hyperbolic discounting gives rise to time-variant intertemporal preferences that feature a systematic bias towards immediate gratification.[1] Intertemporal choices in the distant future are evaluated at a lower discount rate than immediate choices, which gives rise to dynamic inconsistency. Since the currently optimal plan may no longer be optimal in the future, it is useful to model an individual as distinct 'temporal selves' who are each in control for one period. Generally, the optimal decision for the current self depends on the anticipated behavior of future selves. A 'sophisticated' person has rational expectations of future behavior, whereas a 'naive' person wrongly believes that future selves will act in the

[1]For a useful introduction to such time-variant preferences, see Rabin (1998, Section 4.D).

interest of the current self (Strotz 1956, Pollak 1968).

Laibson (1996) analyzes a standard consumption model with a 'quasi-hyperbolic' discount function that was first used by Phelps and Pollak (1968) to model imperfect intergenerational altruism and that mimics the hyperbolic shape of behavioral discount functions. In particular, it is assumed that each temporal self t maximizes life-time utility

$$U_t = u\left(C_t\right) + \beta \sum_{i=1}^{\infty} \delta^i u\left(C_{t+i}\right) \tag{1}$$

where $u\left(C\right)$ is the instantaneous utility from consumption C; β is the degree of self-control which reduces the 'present bias' in intertemporal preferences ($0 < \beta \leq 1$);[2] and δ is the intertemporal discount factor ($0 < \delta < 1$). Note that the quasi-hyperbolic specification conveniently nests exponential discounting as the special case in which the present bias parameter $\beta = 1$.

For analytical convenience utility is assumed to be iso-elastic with constant relative risk aversion (CRRA):

$$u\left(C\right) = \frac{C^{1-\rho} - 1}{1-\rho} \tag{2}$$

where ρ is the coefficient of relative risk aversion ($\rho > 0$). Each self s is endowed with life-time wealth W_s and is in control to choose the consumption level C_s. Each self s is able to invest in one (liquid) asset and faces no credit market imperfections, so $0 \leq C_s \leq W_s$. The subsequent period, self $s+1$ inherits the remaining wealth level

$$W_{s+1} = R_s\left(W_s - C_s\right) \tag{3}$$

where R_s is the gross real interest rate in period s. In contrast to Laibson (1996), who considers a constant interest rate ($R_s = \bar{R}$ for all s), this paper allows for a time-varying (yet deterministic) interest rate to analyze the effect of the duration of interest rate changes on intertemporal substitution. Finally, it is assumed that each self s is sophisticated and rationally anticipates the behavior of future selves. Extensions to this basic model are discussed in section 4.

Without loss of generality, let λ_s denote the fraction of life-time wealth W_s that is consumed by self s, so that $C_s = \lambda_s W_s$, where $0 \leq \lambda_s \leq 1$. Then, dynamic programming can be used to derive the intertemporal Euler equation for self s:[3]

$$u'\left(C_s\right) = R_s\left[\lambda_{s+1}\beta\delta + \left(1-\lambda_{s+1}\right)\delta\right] u'\left(C_{s+1}\right) \tag{4}$$

This resembles the Euler equation under exponential discounting, except that the discount factor δ is replaced by the effective discount factor $\delta_H \equiv \lambda_{s+1}\beta\delta + \left(1-\lambda_{s+1}\right)\delta$. The

[2]The term 'present-biased preferences' was first coined by O'Donoghue and Rabin (1999), who analyzed whether to do an activity now or later.

[3]The derivation is in appendix A.1.

standard exponential case is obtained for $\beta = 1$. The hyperbolic Euler equation shows that the intertemporal substitution of consumption depends on the real interest rate R and the effective discount rate δ_H. The latter is a weighted average of the short-run and long-run discount factors $\beta\delta$ and δ, where the weights are the next period consumption rate and saving rate, λ_{t+1} and $1 - \lambda_{t+1}$, respectively.

To find the optimal consumption rate, substitute (2), $C_s = \lambda_s W_s$ and (3) into (4), and rearrange to get the following recursion formula for λ_s:

$$\lambda_s = \frac{\lambda_{s+1}}{\left(R_s^{1-\rho}\delta\right)^{1/\rho}\left[1 - (1-\beta)\lambda_{s+1}\right]^{1/\rho} + \lambda_{s+1}} \tag{5}$$

When the horizon of the consumer is finite, the consumption rate λ_s can be computed recursively for any time pattern of the interest rate R_s using (5) and the fact that $\lambda_T = 1$ in the final period T. In the infinite-horizon model, (5) can be used to derive the effect of temporary changes in the interest rate. In particular, suppose there is a one-period change in $r_s \equiv \ln R_s$, which is the continuously compounded real interest rate. Then, the future consumption rate λ_{s+1} is not affected and the effect on the current consumption rate, $\partial\lambda_s/\partial r_s = R_s \partial\lambda_s/\partial R_s$, can be found by differentiating (5), which gives after simplifying

$$\frac{\partial\lambda_s}{\partial r_s} = \frac{\rho - 1}{\rho}\lambda_s\left(1 - \lambda_s\right) \tag{6}$$

The effect of the real interest rate on the consumption rate depends on the coefficient of relative risk aversion ρ. For $\rho > 1$, an increase in the interest rate raises the consumption rate ($\partial\lambda_s/\partial r_s > 0$) as the income effect outweighs the intertemporal substitution effect. For $\rho < 1$, an interest rate rise reduces the consumption rate ($\partial\lambda_s/\partial r_s < 0$) as the intertemporal substitution effect dominates. For $\rho = 0$, both effects offset each other and the consumption rate is independent of the interest rate ($\partial\lambda_s/\partial r_s = 0$). These results hold regardless of the degree of self-control β. Nevertheless, there is an important difference between exponential and hyperbolic consumers. It follows from (5) that the present-bias under hyperbolic discounting ($0 < \beta < 1$) causes a higher consumption rate λ_s for a given level of λ_{s+1}. As a result, the quantitative effect of an interest rate change on the consumption rate is different under hyperbolic discounting.

Before analyzing interest rate changes of various durations in the next section, it is useful to consider the special case in which the gross real interest rate remains constant: $R_s = \bar{R}$ for all s. Then, the model reduces to the one analyzed by Laibson (1996). With a constant interest rate, the consumer faces the same infinite-horizon problem for every period s, so the consumption ratio satisfies $\lambda_s = \bar{\lambda}$ for all s, where $0 < \bar{\lambda} < 1$. Substituting this into (5) and rearranging yields[4]

$$\bar{\lambda} = 1 - \left(\bar{R}^{1-\rho}\delta\right)^{1/\rho}\left[1 - (1-\beta)\bar{\lambda}\right]^{1/\rho} \tag{7}$$

[4]This expression corresponds to equation (9) in Laibson (1996).

This implicitly defines a unique optimal consumption rate $\bar{\lambda}$, but typically no closed-form solution exists.[5] For $\beta = 1$, the outcome under exponential discounting emerges:

$$\bar{\lambda}_E = 1 - \left(\bar{R}^{1-\rho}\delta\right)^{1/\rho} \tag{8}$$

Since hyperbolic discounters have a lower degree of self-control ($\beta < 1$), they consume at a higher rate than exponential discounters: $\bar{\lambda} > \lambda_E$.

Equipped with the expressions for the optimal consumption rate under hyperbolic discounting, the analysis now turns to intertemporal substitution.

3 Intertemporal Substitution

Intertemporal substitution by consumers depends on the intertemporal relative price of current consumption, R. The elasticity of intertemporal substitution measures how the intertemporal consumption ratio C_{t+1}/C_t is affected by the gross real interest rate R:

$$\sigma \equiv \frac{d\left(C_{t+1}/C_t\right)}{dR}\frac{R}{C_{t+1}/C_t} = \frac{d\ln\left(C_{t+1}/C_t\right)}{d\ln R}$$

In the case of exponential discounting ($\beta = 1$), it is straightforward to use (2) and (4) to show that $\sigma_E = 1/\rho$. This is the familiar result that for iso-elastic utility, the elasticity of intertemporal substitution equals the inverse of the coefficient of relative risk aversion ρ. This result holds regardless of the duration of the change in the interest rate R. Under exponential discounting, a one-period change and a permanent change in the intertemporal price R have exactly the same proportional effect on the intertemporal consumption ratio C_{t+1}/C_t. However, it turns out that this no longer holds when consumers are hyperbolic discounters. For hyperbolic consumers, the elasticity of intertemporal substitution generally depends on the duration of the change in the real interest rate.

Consider the effect of a change in the interest rate R_t for τ periods. Let R denote the changing gross real interest rate in periods $s \in \{t, t+1, ..., t+\tau-1\}$ and $\bar{R}$ the constant gross real interest rate in periods $s \in \{t+\tau, t+\tau+1, ...\}$. This means that starting in period $t+\tau$, the consumer faces an infinite-horizon problem with a constant interest rate $\bar{R}$ so that $\lambda_s = \bar{\lambda}$ for $s \in \{t+\tau, t+\tau+1, ...\}$, where $\bar{\lambda}$ is given by (7). For $s \in \{t, t+1, ..., t+\tau-1\}$, the optimal consumption rate λ_s is given by the recursion formula (5) with $R_s = R$ and $\lambda_{t+\tau} = \bar{\lambda}$. Using (2) and taking logs, the Euler equation (4) becomes

$$\ln\left(C_{t+1}/C_t\right) = \frac{1}{\rho}\left\{\ln R + \ln\delta + \ln\left[1 - (1-\beta)\lambda_{t+1}\right]\right\} \tag{9}$$

[5] An exception is logarithmic utility ($\rho = 1$), in which case $\bar{\lambda} = \frac{1-\delta}{1-(1-\beta)\delta}$.

So, the elasticity of intertemporal substitution of sophisticated hyperbolic consumers in response to a change in the gross real interest rate R is equal to

$$\sigma_S = \frac{d\ln\left(C_{t+1}/C_t\right)}{dr} = \frac{1}{\rho} - \frac{1}{\rho}\frac{1-\beta}{1-(1-\beta)\,\lambda_{t+1}}\frac{d\lambda_{t+1}}{dr} \tag{10}$$

where $r \equiv \ln R$ denotes the continuously compounded real interest rate. In the case of exponential discounting ($\beta = 1$), this expression reduces to $\sigma_E = 1/\rho$. However, under hyperbolic discounting ($\beta \neq 1$) the elasticity of intertemporal substitution σ_S depends on $d\lambda_{t+1}/dr$, which generally depends on the duration τ of the change in the real interest rate r. Let $\sigma_{S,\tau}$ denote the elasticity of intertemporal substitution of a sophisticated hyperbolic consumer in response to a change in the real interest rate r of τ periods. There is one special case in which the elasticity σ_S is independent of the duration τ. For $\rho = 1$, the consumption rate is independent of the interest rate (see (6)), so that $\sigma_S = 1/\rho = 1$, regardless of the duration of the interest rate change.

First, suppose the change in the interest rate R lasts one period ($\tau = 1$) so that $\bar{R}$ prevails from period $t+1$. Then $\lambda_{t+1} = \bar{\lambda}$, which is independent of R, so $d\lambda_{t+1}/dr = 0$. As a result, the elasticity of intertemporal substitution for sophisticated hyperbolic consumers in response to a one-period change in R is equal to

$$\sigma_{S,1} = \frac{1}{\rho}$$

This is identical to the outcome under exponential discounting. The reason is that for a one-period change in the interest rate, the intertemporal Euler equation (4) for hyperbolic consumers (with effective discount factor $\bar{\delta}_H \equiv \bar{\lambda}\beta\delta + \left(1-\bar{\lambda}\right)\delta$) is observationally equivalent to the one for exponential consumers (with discount factor δ), so that it implies the same degree of intertemporal substitution.

Now, consider a two-period change in the real interest rate ($\tau = 2$). This means that λ_t and λ_{t+1} are given by (5), where $\lambda_{t+2} = \bar{\lambda}$. So, $d\lambda_{t+2}/dr = 0$ and $d\lambda_{t+1}/dr$ is given by (6). Substituting this into (10) gives the elasticity of intertemporal substitution

$$\sigma_{S,2} = \frac{1}{\rho} - \frac{\rho-1}{\rho^2}\frac{1-\beta}{1-(1-\beta)\,\lambda_{t+1}}\lambda_{t+1}\left(1-\lambda_{t+1}\right)$$

This shows that under hyperbolic discounting ($0 < \beta < 1$), the elasticity for a two-period change $\sigma_{S,2}$ differs from the elasticity for one-period change $\sigma_{S,1}$, except when $\rho = 1$. In particular, $\sigma_{S,2} < \sigma_{S,1} = 1/\rho$ for $\rho > 1$, and $\sigma_{S,2} > \sigma_{S,1} = 1/\rho$ for $\rho < 1$. To understand the intuition behind this result, consider a two-period increase in the real interest rate R_t. If the income effect dominates the substitution effect ($\rho > 1$), the increase in R_{t+1} raises the consumption rate λ_{t+1}, which reduces the effective discount factor δ_H as it puts greater weight on the short run discount factor $\beta\delta$. This partially offsets the effect of the increase in

R_t and thereby diminishes the degree of intertemporal substitution. But, if the substitution effect dominates ($\rho < 1$), the consumption rate λ_{t+1} declines, which increases the effective discount factor δ_H and reinforces the effect of R_t on intertemporal substitution.

Now, suppose that the change in the real interest rate lasts three periods ($\tau = 3$). This means that λ_t, λ_{t+1} and λ_{t+2} are given by (5), where $\lambda_{t+3} = \bar{\lambda}$. Now, $d\lambda_{t+3}/dr = 0$, so $\frac{d\lambda_{t+1}}{dr} = \frac{\partial\lambda_{t+1}}{\partial r} + \frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}}\frac{\partial\lambda_{t+2}}{\partial r}$, where $\partial\lambda_s/\partial r$ is given by (6) for $s \in \{t+1, t+2\}$. Substituting this into (10) yields

$$\sigma_{S,3} = \frac{1}{\rho} - \frac{\rho-1}{\rho^2}\frac{1-\beta}{1-(1-\beta)\lambda_{t+1}}\left[\lambda_{t+1}(1-\lambda_{t+1}) + \lambda_{t+2}(1-\lambda_{t+2})\frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}}\right]$$

This shows that the elasticity of intertemporal substitution $\sigma_{S,3}$ is similar to $\sigma_{S,2}$, except for the extra term in square brackets. It can be shown that $\partial\lambda_{t+1}/\partial\lambda_{t+2} > 0$, so that this extra term is strictly positive (also for $\beta = 1$).[6] This reflects the fact that a longer change in the interest rate r has a bigger effect on the consumption rate λ_{t+1}. Under hyperbolic discounting, this induces a larger change in the effective discount factor δ_H. As a result, $\sigma_{S,3} < \sigma_{S,2} < \sigma_{S,1} = 1/\rho$ for $\rho > 1$, and $\sigma_{S,3} > \sigma_{S,2} > \sigma_{S,1} = 1/\rho$ for $\rho < 1$. In other words, the deviation of the hyperbolic elasticity $\sigma_{S,\tau}$ from the exponential elasticity $\sigma_E = 1/\rho$ is larger for a longer duration τ of the real interest rate change.

This result holds more generally. In fact, it is possible to derive an analytical expression for $\sigma_{S,\tau}$ and show that it is monotonic in τ for $\rho \neq 1$.

Proposition 1 *The elasticity of intertemporal substitution of a sophisticated hyperbolic consumer with CRRA utility (2) in response to a change in the real interest rate r of τ periods is equal to*

$$\sigma_{S,\tau} = \frac{1}{\rho} - \frac{\rho-1}{\rho^2}\frac{1-\beta}{1-(1-\beta)\lambda_{t+1}}\sum_{i=t+1}^{t+\tau-1}\lambda_i(1-\lambda_i)\prod_{s=t+1}^{i-1}\frac{\partial\lambda_s}{\partial\lambda_{s+1}} \tag{11}$$

for $\tau \in \{1, 2, 3, ...\}$, where λ_s is given by (5) for $s \in \{t+1, ..., t+\tau-1\}$ and $\lambda_{t+\tau} = \bar{\lambda}$, with $\bar{\lambda}$ determined by (7). The elasticity $\sigma_{S,\tau}$ is monotonically decreasing (increasing) in the duration τ if $\rho > 1$ ($\rho < 1$). For $\rho = 1$, $\sigma_{S,\tau} = 1$ regardless of τ.

The proof of this Proposition is in Appendix A.1. Intuitively, an increase in the real interest rate r raises the consumption rate λ when the income effect dominates the intertemporal substitution effect ($\rho > 1$). A longer interest rate increase causes a larger rise in the consumption rate. Under hyperbolic discounting, the higher consumption rate λ induces the consumer to put greater weight on the low, short-run discount factor $\beta\delta$ and less weight on the high, long-run discount factor δ, which reduces the effective hyperbolic discount

[6] Differentiating (5) and simplifying gives $\frac{\partial\lambda_s}{\partial\lambda_{s+1}} = \frac{1}{\rho}\frac{\lambda_s(1-\lambda_s)}{\lambda_{s+1}}\left(\rho + \frac{(1-\beta)\lambda_{s+1}}{1-(1-\beta)\lambda_{s+1}}\right) > 0$.

factor δ_H. This reduction in the effective discount factor partially offsets the effect of the interest rate increase, thereby diminishing intertemporal substitution of consumption. This effect is stronger for a more persistent increase in the interest rate. Hence, the elasticity of intertemporal substitution $\sigma_{S,\tau}$ is decreasing in the duration τ of the interest rate change for $\rho > 1$. Similarly, when the intertemporal substitution effect dominates ($\rho < 1$), an increase in the real interest rate r reduces the consumption rate λ, which raises the effective hyperbolic discount factor δ_H and reinforces the effect of the interest rate increase on the intertemporal substitution of consumption. Again, this effect is stronger for a more persistent increase in the interest rate, so that $\sigma_{S,\tau}$ is increasing in τ for $\rho < 1$. Since the elasticity of intertemporal substitution in response to a one-period change in the interest rate $\sigma_{S,1}$ is equal to the exponential outcome $\sigma_E = 1/\rho$, the monotonicy result in Proposition 1 implies that the deviation of the hyperbolic elasticity $\sigma_{S,\tau}$ from the exponential elasticity σ_E is increasing in the duration τ of the real interest rate change: $|\sigma_{S,\tau+1} - 1/\rho| \geq |\sigma_{S,\tau} - 1/\rho|$, with strict inequality for $\rho \neq 1$.

It is useful to consider the limiting case as $\tau \to \infty$, which means that the change in the interest rate is permanent. This case corresponds to the Laibson (1996) model, which assumes a constant interest rate. The elasticity of intertemporal substitution for a sophisticated hyperbolic consumer in response to a permanent change in the real interest rate equals[7]

$$\bar{\sigma}_S = \frac{1}{\rho} - \frac{\rho - 1}{\rho} \frac{(1-\beta)\left(1-\bar{\lambda}\right)}{\rho\left[1-(1-\beta)\bar{\lambda}\right] - (1-\beta)\left(1-\bar{\lambda}\right)} \tag{12}$$

It is possible to show that $\lim_{\tau\to\infty} \sigma_{S,\tau} = \bar{\sigma}_S$. Under exponential discounting ($\beta = 1$), $\bar{\sigma}_S$ reduces to $\sigma_E = 1/\rho$. But in a hyperbolic economy ($0 < \beta < 1$), $\bar{\sigma}_S < 1/\rho$ if $\rho > 1$.[8] So, the qualitative effect of a permanent change is the same as for temporary changes. But the deviation from the exponential outcome $\sigma_E = 1/\rho$ is larger for a permanent change. In particular, Proposition 1 implies that for $\rho > 1$, $\bar{\sigma}_S < ... < \sigma_{S,2} < \sigma_{S,1} = 1/\rho$. This means that the degree of intertemporal substitution becomes smaller as the persistence of the real interest rate increases. In other words, a one-period change has the largest effect on intertemporal substitution. For $\rho < 1$, the results are reversed and lasting changes are more effective.

To assess whether the difference between the 'permanent' elasticity $\bar{\sigma}_S$ and the 'one-period' elasticity $\sigma_{S,1}$ could be significant, suppose that the parameters are $\rho = 2$, $R = 1.028$, $\delta = 0.96$ and $\beta = 0.7$. These values are taken from Laibson, Repetto and Tobacman (2005), who estimate β and δ with the Method of Simulated Moments, assuming a struc-

[7]This expression, which corresponds to equation 15 in Laibson (1996), can be derived from (10) by using $\lambda_{t+1} = \bar{\lambda}$ and $\bar{R} = R$, and differentiating (7) to obtain $d\bar{\lambda}/d\bar{R}$.

[8]This is Proposition 5 in Laibson (1996). Or, rewrite (12) to get $\bar{\sigma}_S = \frac{1}{\rho+(\rho-1)\left(1-\bar{\lambda}\right)(1-\beta)/\beta}$, so for $\rho > 1$, $0 < \bar{\sigma}_S < 1/\rho$.

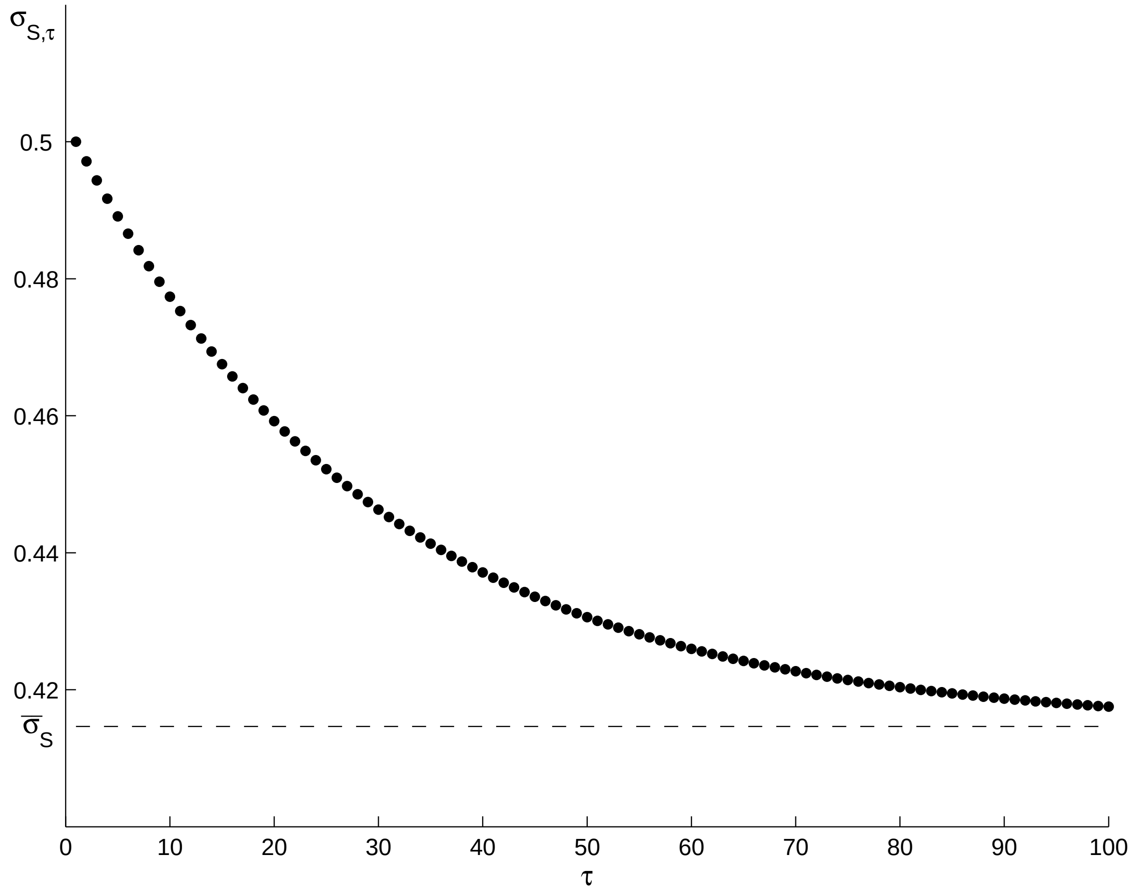

Figure 1: Elasticity of intertemporal substitution for sophisticated hyperbolic consumers.

tural 'buffer-stock' consumption model and using US data from the Survey of Consumer Finances (SCF) and the Panel Study of Income Dynamics (PSID). For these parameter values, a one-period change in the real interest rate gives $\sigma_{S,1} = 0.5$, whereas a permanent change yields $\bar{\sigma}_S = 0.415$. This implies that the effect on the intertemporal consumption ratio is 20.5% larger for a one-period change in the interest rate than for a permanent change. So, the difference between the effect of a temporary and a lasting change on intertemporal substitution could be economically significant.

An interesting question is how long the change in the interest rate needs to last to move away from the exponential outcome σ_E and get close to $\bar{\sigma}_S$. To investigate this, the effect of τ on $\sigma_{S,\tau}$ is analyzed numerically. In particular, $\sigma_{S,\tau}$ is computed using the expression in (11).[9] Again, the baseline parameters are $\rho = 2$, $R = 1.028$, $\delta = 0.96$

[9] Alternatively, $\sigma_{S,\tau}$ could be approximated by computing the numerical derivative $\Delta \ln \left(C_{t+1}/C_t \right) / \Delta r$ for small Δr using (9). For $\Delta r = 0.0001$ (i.e. one basis point), this gives virtually the same numerical results.

and $\beta = 0.7$, as estimated by Laibson et al. (2005). Figure 1 shows how the elasticity of intertemporal substution $\sigma_{S,\tau}$ for sophisticated hyperbolic consumers depends on the duration τ of the change in the real interest rate. For $\tau = 1$, the exponential outcome $\sigma_E = 1/2$ is obtained. As the duration τ of the interest rate change increases, the elasticity of intertemporal substitution $\sigma_{S,\tau}$ becomes smaller and gradually converges to $\bar{\sigma}_S = 0.415$. However, very high values of τ are required to get close to $\bar{\sigma}_S$. In particular, to achieve a value of $\sigma_{S,\tau}$ that bridges half of the gap between $\sigma_{S,1}$ and $\bar{\sigma}_S$, a duration τ_h of about 21 periods (years) is needed.

Using different parameter values gives qualitatively the same results, except for $\rho < 1$, when the elasticity $\sigma_{S,\tau}$ increases from $\sigma_E = 1/\rho$ to $\bar{\sigma}_S$ as τ rises, and for $\rho = 1$, when $\sigma_{S,\tau} = 1$ for all τ. But the 'speed' of the convergence of $\sigma_{S,\tau}$ to $\bar{\sigma}_S$ is sensitive to the specific parameter values. In particular, higher levels of ρ and δ, and lower levels of r increase the duration τ_h required to make up half the difference between $\sigma_{S,1}$ and $\bar{\sigma}_S$. For instance, increasing δ from 0.96 to 0.99 raises τ_h from 21 to 38 years. Even for the lowest plausible estimate for δ in the robustness check by Laibson et al. (2005, Table 5), namely $\delta = 0.94$, τ_h is still about 17 years. This suggests that a very long duration τ of the interest rate change is needed to obtain a significant difference between the exponential elasticity of intertemporal substitution σ_E and the hyperbolic elasticity $\sigma_{S,\tau}$.

4 Robustness

This section shows the robustness of the result that hyperbolic consumers exhibit an elasticity of intertemporal substitution that depends on the persistence of the interest rate and is smaller than for exponential consumers. Four variations on the baseline model in section 2 are considered. First, it is plausible that hyperbolic consumers may not be fully sophisticated, so section 4.1 analyzes the model with naive consumers that fail to anticipate their future self-control problems. Second, section 4.2 analyzes intertemporal substitution of sophisticated hyperbolic consumers with CARA utility. Third, section 4.3 discusses a richer, buffer-stock model with stochastic income and liquidity constraints. In each of these cases, the main results remain relevant.

4.1 Naive Consumers

Consider the basic consumption model in section 2 with the quasi-hyperbolic discount function, one liquid asset and no credit market imperfections, but now suppose the consumer is naive and incorrectly believes that future selves will act in the interest of the current self. More precisely, each self t maximizes life-time utility U_t (1) and thinks that future selves $s \in \{t+1, t+2, ...\}$ also maximize U_t (instead of U_s). Although the current self t knows that it is a hyperbolic discounter with an inclination for immediate gratification, it naively believes that future selves do not have present-biased preferences but behave as exponential discounters ($\beta = 1$). The formal analysis of the model with naive hyperbolic consumers is in Appendix A.2.

The naive self t believes that future selves $s \in \{t+1, t+2, ...\}$ have the same consumption rate as exponential consumers, so the *intended* future consumption rate equals

$$\tilde{\lambda}_s = \frac{\tilde{\lambda}_{s+1}}{\left(R_s^{1-\rho}\delta\right)^{1/\rho} + \tilde{\lambda}_{s+1}} \tag{13}$$

Believing that future selves set $\tilde{\lambda}_s$, the current naive self chooses

$$\lambda_{N,t} = \frac{\tilde{\lambda}_{t+1}}{\left(R_t^{1-\rho}\beta\delta\right)^{1/\rho} + \tilde{\lambda}_{t+1}} \tag{14}$$

This is the *actual* consumption rate of all naive hyperbolic selves. For $\beta = 1$, (14) is equal to the exponential outcome in (13). A lower degree of self-control ($0 < \beta < 1$) increases the naive consumption rate so that $\lambda_{N,t} > \tilde{\lambda}_t$. This means that the naive hyperbolic consumer is running down life-time wealth faster than an exponential consumer. Besides that the naive consumption rate $\lambda_{N,t}$ has the familiar property that it is increasing (decreasing) in the real interest rate R_t for $\rho > 1$ ($\rho < 1$), but independent of R_t for $\rho = 1$.

Suppose, as before, that the gross real interest rate equals R in periods $s \in \{t, t+1, ..., t+\tau-1\}$ and $\bar{R}$ in periods $s \in \{t+\tau, t+\tau+1, ...\}$. This means that starting in period $t+\tau$, the intended consumption rate equals the exponential outcome with a constant interest rate in (8). So, $\tilde{\lambda}_s = \bar{\lambda}_E$ for $s \in \{t+\tau, t+\tau+1, ...\}$. For $s \in \{t+1, ..., t+\tau-1\}$, $\tilde{\lambda}_s$ is given by the recursion formula in (13) with $R_s = R$. The actual naive consumption rate $\lambda_{N,s}$ is given by (14) for all selves s.

To analyze intertemporal substitution for naive hyperbolic consumers one can no longer rely on the intertemporal Euler equation for consumption. The reason is that it merely describes intended rather than actual intertemporal substitution for naive consumers. Instead, actual consumption based on (14) needs to be used. This leads to an analytical expression for the naive hyperbolic elasticity of intertemporal substitution $\sigma_{N,\tau}$, which has the same monotonicity properties as the sophisticated hyperbolic elasticity $\sigma_{S,\tau}$:

Proposition 2 *The elasticity of intertemporal substitution of a naive hyperbolic consumer with CRRA utility (2) in response to a change in the real interest rate r of τ periods is equal to*

$$\sigma_{N,\tau} = \frac{1}{\rho} - \frac{\rho-1}{\rho}\frac{\lambda_{N,t+1} - \tilde{\lambda}_{t+1}}{1-\tilde{\lambda}_{t+1}} \sum_{i=t+1}^{t+\tau-1} \prod_{s=t+1}^{i} \left(1 - \tilde{\lambda}_s\right) \tag{15}$$

for $\tau \in \{1, 2, 3, ...\}$, where $\lambda_{N,t+1}$ is given by (14), $\tilde{\lambda}_s$ is given by (13) for $s \in \{t+1, ..., t+\tau-1\}$, and $\tilde{\lambda}_{t+\tau} = \bar{\lambda}_E$ in (8). The elasticity $\sigma_{N,\tau}$ is monotonically decreasing (increasing) in the duration τ if $\rho > 1$ ($\rho < 1$). For $\rho = 1$, $\sigma_{N,\tau} = 1$ regardless of τ.

The proof of this Proposition appears in Appendix A.2. In the absence of a present bias ($\beta = 1$), the actual and intended naive consumption rates are equal ($\lambda_{N,s} = \tilde{\lambda}_s$), so (15) reduces to the exponential outcome $\sigma_E = 1/\rho$. But under hyperbolic discounting ($\beta < 1$), the naive consumption rate exceeds the exponential rate ($\lambda_{N,s} > \tilde{\lambda}_s$). So, $\sigma_{N,\tau} < 1/\rho$ for $\rho > 1$ and $\sigma_{N,\tau} > 1/\rho$ for $\rho < 1$, similar to the sophisticated case. In addition, the elasticity of intertemporal substitution $\sigma_{N,\tau}$ depends on the duration τ of the interest rate change. The deviation from the exponential elasticity $\sigma_E = 1/\rho$ is again increasing in the duration τ: $|\sigma_{N,\tau+1} - 1/\rho| \geq |\sigma_{N,\tau} - 1/\rho|$, with strict inequality for $\rho \neq 1$.

In the limiting case as $\tau \to \infty$, the change in the real interest rate is permanent. For a constant interest rate $\bar{R}$, the naive hyperbolic consumption rate follows from substituting λ_E in (8) for $\tilde{\lambda}_{t+1}$ in (14):

$$\bar{\lambda}_N = \frac{1 - \left(\bar{R}^{1-\rho}\delta\right)^{1/\rho}}{1 - \left(1 - \beta^{1/\rho}\right)\left(\bar{R}^{1-\rho}\delta\right)^{1/\rho}} \tag{16}$$

For $\beta = 1$ this reduces to the exponential outcome $\bar{\lambda}_E = 1 - \left(\bar{R}^{1-\rho}\delta\right)^{1/\rho}$, which is also the intended future consumption rate of the naive hyperbolic consumer. But the self-control problem ($\beta < 1$) causes the naive hyperbolic discounter to consume more than intended in every period ($\bar{\lambda}_N > \bar{\lambda}_E$).[10]

The elasticity of intertemporal substitution for a naive hyperbolic consumer in response to a permanent change in the real interest rate equals

$$\bar{\sigma}_N = \frac{1}{\rho} - \frac{\rho - 1}{\rho}\frac{\bar{\lambda}_N - \bar{\lambda}_E}{\bar{\lambda}_E} \tag{17}$$

This can be derived from (15) as $\lim_{\tau\to\infty} \sigma_{N,\tau} = \bar{\sigma}_N$.[11] For $\beta = 1$, $\bar{\lambda}_N = \bar{\lambda}_E$ so that $\bar{\sigma}_N$ reduces to $\sigma_E = 1/\rho$. But in a hyperbolic economy ($0 < \beta < 1$), $\bar{\lambda}_N > \bar{\lambda}_E$ so $\bar{\sigma}_N < 1/\rho$ for $\rho > 1$ and $\bar{\sigma}_N > 1/\rho$ for $\rho < 1$, just like $\sigma_{N,\tau}$ for temporary interest rate changes. The deviation from the exponential outcome $\sigma_E = 1/\rho$ is again largest for a permanent change. In particular, Proposition 2 implies that for $\rho > 1$, $\bar{\sigma}_N < ... < \sigma_{N,2} < \sigma_{N,1} = 1/\rho$, while

[10] The naive consumption rate $\bar{\lambda}_N$ is typically different from the sophisticated rate $\bar{\lambda}$. An exception is logarithmic utility ($\rho = 1$), when (7) and (16) yield $\bar{\lambda} = \frac{1-\delta}{1-(1-\beta)\delta} = \bar{\lambda}_N$, so that naive and sophisticated behavior coincide, as was first shown by Pollak (1968).

[11] Use the fact that $\lim_{\tau\to\infty} \tilde{\lambda}_s = \bar{\lambda}_E$ and $\lim_{\tau\to\infty} \lambda_{N,s} = \bar{\lambda}_N$. Alternatively, $C_s = \bar{\lambda}_N W_s$ and (3) imply $C_{t+1}/C_t = \bar{R}\left(1 - \bar{\lambda}_N\right)$, so $\bar{\sigma}_N = 1 - \frac{1}{1-\bar{\lambda}_N}\frac{d\bar{\lambda}_N}{d\bar{r}}$. Substituting $\frac{d\bar{\lambda}_N}{d\bar{r}} = \frac{\rho-1}{\rho}\frac{\bar{\lambda}_N}{\bar{\lambda}_E}\left(1 - \bar{\lambda}_N\right)$ from (16), and rearranging gives (17).

for $\rho < 1$, $\bar{\sigma}_N > ... > \sigma_{N,2} > \sigma_{N,1} = 1/\rho$. So, for $\rho > 1$ a one-period change has the strongest effect on intertemporal substitution, whereas for $\rho < 1$ a permanent change is most effective.

So far, the analysis suggests that the qualitative features of the sophisticated and naive hyperbolic elasticities $\sigma_{S,\tau}$ and $\sigma_{N,\tau}$ are exactly the same. However, there is one interesting difference. The naive elasticity of intertemporal substitution $\sigma_{N,\tau}$ could actually be negative for $\rho > 1$ and β sufficiently small. A lower degree of self-control β could increase $\bar{\lambda}_N$ so much that $\bar{\sigma}_N < 0$ for $\rho > 1$.[12] For example, $\rho = 3$, $\beta = 0.2$, $\delta = 0.99$ and $r = 3\%$ imply $\bar{\sigma}_N = -0.122$. Intuitively, when the income effect dominates the substitution effect and the degree of self-control is small enough, an increase in the interest rate could raise current consumption so much that the net effect on wealth is negative and the intertemporal consumption ratio C_{t+1}/C_t actually declines.

There could be a major difference between the one-period elasticity and the permanent elasticity for naive hyperbolic consumers. In the previous example, $\sigma_{N,1} = 0.333$ versus $\bar{\sigma}_N = -0.122$. This illustrates that the effect on intertemporal substitution could be both quantitatively and qualitatively different for a one-period and a permanent interest rate change when consumers are naive hyperbolic discounters. Using the baseline parameters $\rho = 2$, $R = 1.028$, $\delta = 0.96$ and $\beta = 0.7$, estimated by Laibson et al. (2005), the naive elasticities are positive and the difference is considerably smaller: $\sigma_{N,1} = 0.5$ versus $\bar{\sigma}_N = 0.406$. Nevertheless, this implies that the effect on the intertemporal consumption ratio for a one-period change in the interest rate is 23.2% larger than for a permanent change. So, for plausible parameter values, temporary and lasting interest rate changes have significantly different effects on the intertemporal substitution of naive hyperbolic consumers.

The effect of the duration τ of the interest rate change on the naive elasticity of intertemporal substitution $\sigma_{N,\tau}$ is very similar to the sophisticated case. For the baseline parameters $\rho = 2$, $R = 1.028$, $\delta = 0.96$ and $\beta = 0.7$, the profile of $\sigma_{N,\tau}$ is close to the one depicted in Figure 1. Just like in the sophisticated case, very long interest rate changes are needed to get close to the permanent elasticity $\bar{\sigma}_N$. In particular, to bridge the half the gap between $\sigma_{N,1}$ and $\bar{\sigma}_N$ again takes about 21 years for the baseline parameters.

This section has shown that the elasticities of intertemporal substitution for naive and sophisticated hyperbolic consumers display the same qualitative features, with one exception. Naive hyperbolic consumers could actually have a negative elasticity of intertemporal substitution when $\rho > 1$ and β is small. Besides that, intertemporal substitution by naive and sophisticated hyperbolic consumers is quite similar. Compared to exponential discounting, the elasticity of intertemporal substitution for hyperbolic discounting is generally

[12] More formally, using (16) and (17) gives $\lim_{\beta \to 0} \bar{\sigma}_N = 1 - (\rho - 1)/\rho\bar{\lambda}_E$, so $\bar{\sigma}_N < 0$ for $\rho > 1/\left(1 - \bar{\lambda}_E\right) > 1$ and β close to 0.

different, but there are two exceptions. First, for logarithmic utility ($\rho = 1$) the consumption rate λ is independent of the real interest rate R, so $\sigma_{N,\tau} = \sigma_{S,\tau} = \sigma_E = 1$. Second, for a one-period change in the real interest rate, $\sigma_{N,1} = \sigma_{S,1} = \sigma_E = 1/\rho$. However, for $\rho \neq 1$ and $\tau \neq 1$, Propositions 1 and 2 imply that $\sigma_{N,\tau}$ and $\sigma_{S,\tau}$ always differ from the exponential outcome $\sigma_E = 1/\rho$ and that they are monotonic in the duration τ of the interest rate change. For the plausible case in which $\rho > 1$, $\sigma_{N,\tau} < \sigma_E$ and $\sigma_{S,\tau} < \sigma_E$, which means that there is less intertemporal substitution with hyperbolic than with exponential discounters.

4.2 CARA Utility

The results so far have been derived for the constant relative risk aversion utility (2) and it is natural to wonder to what extent the results extend to the constant absolute risk aversion (CARA) utility function

$$u(C) = -\frac{1}{\theta} e^{-\theta C} \tag{18}$$

where θ is the coefficient of absolute risk aversion ($\theta > 0$). To derive optimal consumption with CARA utility, postulate that consumption by self s equals $C_s = \lambda_s W_s + \kappa_s$. Then it is easy to show that the Euler equation (4) continues to hold. To derive the optimal λ_s and κ_s, substitute (18), $C_s = \lambda_s W_s + \kappa_s$ and (3) into (4), and rearrange to get the following recursion formulas:

$$\lambda_s = \frac{\lambda_{s+1} R_s}{1 + \lambda_{s+1} R_s} \tag{19}$$

$$\kappa_s = \frac{1}{1 + \lambda_{s+1} R_s} \kappa_{s+1} - \frac{\ln R_s + \ln\left[\lambda_{s+1}\beta\delta + (1 - \lambda_{s+1})\,\delta\right]}{\theta\,(1 + \lambda_{s+1} R_s)} \tag{20}$$

This shows that a lower degree of self-control β increases autonomous consumption κ_s.

In the special case in which the real interest rate remains constant, $R_s = \bar{R}$ for all s, the consumer faces the same infinite-horizon problem for every period s, so $\lambda_s = \bar{\lambda}$ and $\kappa_s = \bar{\kappa}$ for all s. Substituting this into (19) and (20) and rearranging yields

$$\bar{\lambda} = \frac{\bar{R} - 1}{\bar{R}} \text{ and } \bar{\kappa} = -\frac{\ln\delta + \ln\left[\left(\bar{R} - 1\right)\beta + 1\right]}{\theta\left(\bar{R} - 1\right)} \tag{21}$$

Thus, with CARA utility there is a closed-form solution for optimal consumption under hyperbolic discounting, as first shown by Maliar and Maliar (2004) for a model with stochastic income shocks. For $\beta = 1$, the outcome under exponential discounting is obtained with $\bar{\kappa}_E = -\frac{\ln\delta + \ln\bar{R}}{\theta\left(\bar{R}-1\right)}$. Since hyperbolic discounters have a lower degree of self-control ($\beta < 1$), they have higher autonomous consumption ($\bar{\kappa} > \bar{\kappa}_E$), while their consumption rate is not affected ($\bar{\lambda} = \bar{\lambda}_E$).

As before, consider a change in the gross real interest rate R at time t for τ periods such that $R_s = R$ for $s \in \{t, t+1, ..., t+\tau-1\}$ and $R_s = \bar{R}$ for $s \in \{t+\tau, t+\tau+1, ...\}$. To determine the effect on intertemporal substitution of consumption, use (18) and rearrange the Euler equation (4) to get

$$\frac{C_{t+1}}{C_t} = \frac{1}{\theta C_t} \left\{ \ln R + \ln \delta + \ln \left[1 - (1 - \beta) \lambda_{t+1} \right] \right\} + 1 \tag{22}$$

This is very similar to (9), except that the constant coefficient of relative risk aversion ρ has been replaced by the relative risk aversion measure θC_t. Although the term in curly brackets is the same as for CRRA utility, interest rate changes also affect relative risk aversion θC_t under CARA utility, which could result in qualitatively different outcomes, even with exponential discounting ($\beta = 1$). Unfortunately, the fact that relative risk aversion is no longer constant greatly complicates the derivation of analytical results, so a numerical analysis is performed instead.

The level of consumption $C_t = \lambda_t W_t + \kappa_t$ can be obtained using the recursion formulas (19) and (20), with $\lambda_s = \bar{\lambda}$ and $\kappa_s = \bar{\kappa}$ for $s \in \{t + \tau, t + \tau + 1, ...\}$, where $\bar{\lambda}$ and $\bar{\kappa}$ are given by (21). The intertemporal consumption ratio C_{t+1}/C_t follows from (22) and the elasticity of intertemporal substitution $\sigma_{S,\tau}$ for a change in the real interest rate of duration τ is computed using the numerical derivative $\Delta \ln \left(C_{t+1}/C_t \right) / \Delta r$ for $\Delta r = 0.0001$.[13]

First, consider the baseline parameters $R = 1.028$, $\delta = 0.96$ and $\beta = 0.7$, use the normalization $W = 100$ and take $\theta = 0.45353$, which for a constant interest rate implies a level of relative risk aversion $\theta C_t = 2$, using (21). For these baseline settings, the elasticity of intertemporal substitution $\sigma_{S,\tau}$ under CARA utility looks very similar to the CRRA outcome in Figure 1, with $\sigma_{S,1} = 0.507$ but with an asymptotic minimum of 0.308.

However, the outcome for the elasticity of intertemporal substitution $\sigma_{E,\tau}$ for exponential discounters ($\beta = 1$) is quite different with CARA utility. In particular, $\sigma_{E,\tau}$ is generally no longer independent of the duration τ of the interest rate change. Intuitively, the duration τ generally affects consumption C_t, which determines relative risk aversion θC_t and thereby the elasticity of intertemporal substitution. For the baseline settings, $\sigma_{E,\tau}$ is non-monotonic with $\sigma_{E,1} = 0.591$, a maximum of $\sigma_{E,17} = 0.598$ and an asymptotic minimum of $\bar{\sigma}_E = 0.551$. Nevertheless, it is still the case that the elasticity of intertemporal substitution is smaller for sophisticated hyperbolic discounters than for exponential discounters: $\sigma_{E,\tau} > \sigma_{S,\tau}$.

So, for the baseline parameters $\sigma_{S,\tau}$ continues to be decreasing in τ, with a larger range than $\sigma_{E,\tau}$, while being less than $\sigma_{E,\tau}$. To establish whether these results continue to hold for other reasonable parameter values, a numerical analysis has been conducted for two different parameter spaces. The 'full' parameter space consists of $\beta \in [0.5, 0.9]$, $\delta \in [0.94, 0.98]$, $R \in [1.004, 1.052]$ and $\theta \in [0.001, 1.555]$. The latter amounts to a

[13]As pointed out in footnote 9, using the numerical derivative with $\Delta r = 0.0001$ to compute $\sigma_{S,\tau}$ gives very accurate results for CRRA utility.

range for relative risk aversion of $[0.77, 5]$ under the baseline parameters with a constant interest rate. The range for R implies a real interest rate between 0.40% and 5.1%, and it is based on a one standard error deviation from the baseline estimate in Laibson et al. (2005). The ranges for β and δ roughly correspond to the 95% confidence intervals based on the standard errors estimated by Laibson et al. (2005). The 'core' parameter space is defined by $\beta \in [0.6, 0.8]$, $\delta \in [0.95, 0.97]$, $R \in [1.014, 1.040]$ and $\theta \in [0.0864, 1.1878]$. The latter corresponds to a range for relative risk aversion of $[1, 4]$. Besides that, the core parameter space has a lower mean-preserving spread around the baseline parameters for β, δ and R compared to the full parameter space.

For each parameter space, 100,000 uniform random draws were made of β, δ, R and θ.[14] For each randomly drawn parameter configuration, the elasticities of intertemporal substitution $\sigma_{S,\tau}$ and $\sigma_{E,\tau}$ were computed numerically for $\tau \in \{1, 2, ..., 500\}$ and it was checked whether the following three properties hold:[15] (a) $\sigma_{S,\tau}$ is monotonically decreasing in τ; (b) $\sigma_{S,\tau}$ has a larger range over τ than $\sigma_{E,\tau}$ such that $\max_\tau \sigma_{S,\tau} - \min_\tau \sigma_{S,\tau} > \max_\tau \sigma_{E,\tau} - \min_\tau \sigma_{E,\tau}$; and (c) $\sigma_{S,\tau}$ is smaller than $\sigma_{E,\tau}$ (i.e. $\sigma_{E,\tau} > \sigma_{S,\tau}$). This gives rise to the following findings.

Numerical Result 1 With CARA utility, the elasticity of intertemporal substitution $\sigma_{S,\tau}$ in response to a change in the real interest rate r of τ periods for a sophisticated consumer with hyperbolic discounting
(a) is monotonically decreasing in τ for 73.3% of the full parameter space and 76.5% of the core parameter space.
(b) has a larger range over τ than for a consumer with exponential discounting ($\beta = 1$) for 70.4% of the full parameter space and 78.4% of the core parameter space.
(c) is smaller than for a consumer with exponential discounting ($\beta = 1$) for 99.0% of the full parameter space and 100% of the core parameter space.

This shows that the three properties frequently hold for the full parameter space and are even more likely to be satisfied for the core parameter space.

Another interesting finding is that the elasticity of intertemporal substitution could be negative with CARA utility, for both hyperbolic and exponential consumers. In fact, for sufficiently large τ, $\sigma_{S,\tau} < 0$ and $\sigma_{E,\tau} < 0$ for 30.1% and 26.2% of the full parameter space and 14.8% and 7.3% of the core parameter space, respectively. This outcome is

[14] Since the results in Laibson et al. (2005) suggest a strong negative correlation between the estimates of β and δ, the numerical analysis was also conducted for uniform random draws of β, δ, R and θ with a perfect negative correlation between β and δ, but the findings were quite similar.

[15] Since CARA utility could lead to negative levels of consumption, randomly drawn parameter configurations for which $C_s < 0$ for any s were discarded. This occurred for only 0.19% of the full and 0% of the core parameter space.

more common for lower values of R and θ. The fact that $\sigma_{S,\tau} < 0$ holds more frequently than $\sigma_{E,\tau} < 0$ is not surprising since generally, $\sigma_{E,\tau} > \sigma_{S,\tau}$ by Numerical Result 1(c).

To summarize the main findings, the elasticitity of intertemporal substitution $\sigma_{S,\tau}$ for sophisticated hyperbolic consumers with CARA utility is typically declining in the duration τ for reasonable parameter values. Although the elasticity $\sigma_{E,\tau}$ for consumers with exponential discounting is no longer constant for CARA utility, the range over τ remains larger for the hyperbolic elasticity $\sigma_{S,\tau}$ for a large majority of plausible parameter configurations. The result that the elasticity of intertemporal substitution $\sigma_{S,\tau}$ for hyperbolic discounters is less than for exponential discounters continues to hold for virtually all reasonable parameter values with CARA utility.

4.3 Buffer Stock Model

So far, the paper has considered a deterministic model in which consumers have access to perfect credit markets. In practice, income is stochastic and consumers face liquidity constraints. In particular, suppose that labor income Y_t is stochastic and that the consumer cannot borrow against uncertain future income so that $C_t \leq X_t$, where X_t is cash-on-hand in period t, which satisfies $X_t = R\left(X_{t-1} - C_{t-1}\right) + Y_t$. Harris and Laibson (2001) show that the hyperbolic Euler relation for sophisticated consumers in such a 'buffer-stock' model similar to Carroll (1997) equals:[16]

$$u'\left(c\left(X_t\right)\right) \geq \mathrm{E}_t\, R\left[c'\left(X_{t+1}\right)\beta\delta + \left(1 - c'\left(X_{t+1}\right)\delta\right)\right] u'\left(c\left(X_{t+1}\right)\right) \tag{23}$$

where $c\left(X_t\right)$ is the consumption function. For periods in which the liquidity constraint is non-binding so that $c\left(X_t\right) < X_t$, (23) holds with equality. This resembles the Euler equation (4), but the fraction of life-time wealth consumed λ is now replaced by the marginal propensity to consume out of cash-on-hand $c'\left(X_{t+1}\right)$ because of the borrowing constraint.

Intertemporal substitution in response to a permanent change in the real interest rate is given by

$$\frac{\partial \ln\left(C_{t+1}/C_t\right)}{\partial r} = \frac{1}{\rho} - \frac{1}{\rho}\frac{\left(1-\beta\right)}{1-\left(1-\beta\right)c'\left(X_{t+1}\right)}\frac{\partial c'\left(X_{t+1}\right)}{\partial r}$$

which is the buffer-stock equivalent of (10). For $\rho > 1$, the income effect dominates the substitution effect, so $\partial c'\left(X_{t+1}\right)/\partial r > 0$ and $\bar{\sigma}_S < 1/\rho$ (Laibson 1998, p. 867). Following the same approach as in section 3, (23) can be used to find that $\sigma_{S,1} = 1/\rho$ whenever the consumer is not liquidity constrained. As a result, the conclusions of section 3 hold more generally.

[16]To be precise, this is the 'strong' hyperbolic Euler relation formally derived by Harris and Laibson (2001) and it assumes that the consumption function $c\left(.\right)$ is Lipschitz continuous, which holds in a neighborhood of $\beta = 1$.

5 Conclusion

Intertemporal substitution plays a key role in macroeconomics. For instance, it affects the propagation mechanism in micro-founded business cycle models and it determines the effectiveness of tax policies. This paper establishes that the elasticity of intertemporal substitution exhibits novel features when consumers have a hyperbolic instead of an exponential discount function. It is well-known that under exponential discounting the elasticity of intertemporal substitution equals the inverse of the coefficient of relative risk aversion for iso-elastic utility. This holds regardless of the length of the change in the intertemporal price ratio. However, under hyperbolic discounting the intertemporal substitution elasticity typically depends on the duration of the intertemporal price change.

For a one-period change in the real interest rate, the elasticity of intertemporal substitution with iso-elastic utility equals the inverse of the coefficient of relative risk aversion for both exponential and hyperbolic discounters. Essentially, this is the structural preference parameter that measures the curvature of the intertemporal indifference curves. However, for a persistent change in the interest rate, the degree of intertemporal substitution is generally different for hyperbolic consumers because the effective discount rate is affected. The reason is that a persistent interest rate change typically influences the future consumption rate, which shifts the weight between the high short-run and the low long-run hyperbolic discount rate. This adjustment in the effective discount rate alters the effect of a lasting interest rate change on intertemporal substitution. For plausible values of risk aversion, the elasticity of intertemporal substitution for hyperbolic consumers is monotonically decreasing in the duration of the change in the real interest rate.

These results hold both for sophisticated hyperbolic discounters, who rationally anticipate the dynamic inconsistency of their preferences, and for naive consumers, who do not realize that the 'present bias' in their intertemporal preferences continues to exert itself in the future. It appears to be a fundamental property of hyperbolic discounting that already holds for a basic model with a single liquid asset and perfect credit markets. So, it does not rely on the presence of (partial) commitment devices, such as illiquid assets, that is usually required to distinguish (sophisticated) hyperbolic from exponential consumers. The result is also relevant in more realistic 'buffer-stock' models that feature stochastic income and liquidity constraints. Although the focus of the paper is on the intertemporal consumption decision, a similar argument applies to the intertemporal substitution of leisure. It appears to be a robust feature of hyperbolic discounting that the elasticity of intertemporal substitution depends on the duration of the intertemporal price change.

There is a large empirical literature on intertemporal substitution, including Mankiw, Rotemberg and Summers (1985), Hall (1988), Attanasio and Weber (1995) and Mulligan (2002). Such empirical studies have obtained a remarkably wide range of estimates for the elasticity of intertemporal substitution, with a typical parameter value of about 0.3. Although a large variety of parameter estimates is difficult to reconcile with exponential discounting, it is natural to get different estimates under hyperbolic discounting, depending on the persistence of the interest rate in the sample.

In addition, it is not unusual to find empirical elasticity estimates that are negative. This appears at odds with the standard model of exponential discounting. However, a negative elasticity of intertemporal substitution is consistent with the behavior of naive hyperbolic consumers with iso-elastic utility with plausible risk aversion, a sufficiently low degree of self control and a persistent interest rate change. For exponential utility, a negative elasticity of intertemporal substitution is more likely to occur with hyperbolic discounting than exponential discounting.

Thus, this paper shows that hyperbolic discounting could explain empirical findings on intertemporal substitution that are puzzling under exponential discounting. In addition, the result that the hyperbolic elasticity of intertemporal substitution depends on the persistence of the intertemporal price provides a new testable implication of hyperbolic discounting for iso-elastic utility. Although it appears interesting to pursue this further, calibrations indicate that a very long duration of the intertemporal price change is required to obtain a difference with the exponential elasticity of only 0.02, which is much smaller than the standard errors of typical empirical elasticity estimates. So, an empirical test that exploits the duration-dependence of the hyperbolic elasticity is probably not practicable.

However, this does not mean that the differences in intertemporal substitution between exponential and hyperbolic discounting are immaterial. Quite to the contrary. For plausible parameter values, the effect of a permanent price change on intertemporal consumption is about 20% larger for exponential discounters than for hyperbolic discounters. This is also the difference between the effect of a one-period and a permanent price change under hyperbolic discounting. Clearly, such a magnitude is economically significant.

This has important implications. First, models that assume exponential discounting overstate the relevance of intertemporal substitution effects when agents are in fact hyperbolic. For instance, predictions of the benefits of policy measures such as tax cuts are likely to be much rosier when they are based on policy models with exponential instead of hyperbolic discounting. Second, hyperbolic intertemporal substitution effects are significantly stronger for temporary policy measures than for permanent ones. This means that empirical estimates based on a temporary (or experimental) policy could seriously overstate the effectiveness of permanent implementation of the policy.

All in all, this paper finds interesting new results on intertemporal substitution under hyperbolic discounting.

A Appendix

This appendix contains the derivation of the basic hyperbolic model with sophisticated consumers presented in section 2. In addition, it derives the results for naive hyperbolic consumers in section 4.1.

A.1 Sophisticated Consumers

This section provides a derivation of the quasi-hyperbolic intertemporal Euler equation (4) for sophisticated consumers, and the proof of Proposition 1.

Derivation of (4):
Each self s faces a similar infinite-horizon optimization problem. Using (1), the optimal life-time utility of self s can be written as

$$U_s = u\left(C_s\right) + \beta\delta V\left(W_{s+1}; s+1\right) \tag{24}$$

where

$$V\left(W_{s+1}; s+1\right) = \sum_{i=s+1}^{\infty} \delta^{i-(s+1)} u\left(\lambda_i W_i\right)$$

Using (3), the continuation-value function for selves $s = \{t, t+1, ...\}$ satisfies

$$V\left(W_{s+1}; s+1\right) = u\left(\lambda_{s+1} W_{s+1}\right) + \delta V\left(R_{s+1}\left(1-\lambda_{s+1}\right) W_{s+1}; s+2\right) \tag{25}$$

Maximizing (24) with respect to C_s subject to (3) yields the first order condition

$$u'\left(C_s\right) = R_s \beta\delta V'\left(W_{s+1}; s+1\right) \tag{26}$$

Differentiate (25) and substitute for $V'\left(W_{s+2}; s+2\right)$ using (26) to get

$$\begin{aligned} u'\left(C_s\right) &= R_s\beta\delta\left[\lambda_{s+1} u'\left(C_{s+1}\right) + R_{s+1}\left(1-\lambda_{s+1}\right)\delta V'\left(W_{s+2}; s+2\right)\right] \\ &= R_s\left[\lambda_{s+1}\beta\delta + \left(1-\lambda_{s+1}\right)\delta\right] u'\left(C_{s+1}\right) \end{aligned}$$

This is the quasi-hyperbolic intertemporal Euler equation (4) for sophisticated consumers. ■

Proof of Proposition 1:
First, for a change in r of duration τ, $\frac{d\lambda_s}{dr} = \frac{\partial\lambda_s}{\partial r} + \frac{\partial\lambda_s}{\partial\lambda_{s+1}}\frac{d\lambda_{s+1}}{dr}$ for $s \in \{t, ..., t+\tau-1\}$, where $\frac{\partial\lambda_s}{\partial r_s}$ is given by (6) and $\frac{d\lambda_{t+\tau}}{dr} = 0$. So, by recursive substitution one can write

$$\frac{d\lambda_{t+1}}{dr} = \frac{\partial\lambda_{t+1}}{\partial r} + \frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}}\frac{\partial\lambda_{t+2}}{\partial r} + \frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}}\frac{\partial\lambda_{t+2}}{\partial\lambda_{t+3}}\frac{\partial\lambda_{t+3}}{\partial r} + ... + \left(\frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}} * ... * \frac{\partial\lambda_{t+\tau-2}}{\partial\lambda_{t+\tau-1}}\right)\frac{\partial\lambda_{t+\tau-1}}{\partial r}$$

$$
\begin{aligned}
&= \frac{\rho-1}{\rho}\left[\lambda_{t+1}\left(1-\lambda_{t+1}\right)+\lambda_{t+2}\left(1-\lambda_{t+2}\right)\frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}}+\lambda_{t+3}\left(1-\lambda_{t+3}\right)\frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}}\frac{\partial\lambda_{t+2}}{\partial\lambda_{t+3}}+...\right.\\
&\qquad \left. ...+\lambda_{t+\tau-1}\left(1-\lambda_{t+\tau-1}\right)\left(\frac{\partial\lambda_{t+1}}{\partial\lambda_{t+2}}*...*\frac{\partial\lambda_{t+\tau-2}}{\partial\lambda_{t+\tau-1}}\right)\right]\\
&= \frac{\rho-1}{\rho}\left[\sum_{i=t+1}^{t+\tau-1}\lambda_i\left(1-\lambda_i\right)\left(\prod_{s=t+1}^{i-1}\frac{\partial\lambda_s}{\partial\lambda_{s+1}}\right)\right]
\end{aligned}
$$

Substituting this into (10) yields (11).

To prove monotonicity, use (11) to write

$$
\sigma_{S,\tau+1}-\sigma_{S,\tau}=-\frac{\rho-1}{\rho^2}\frac{1-\beta}{1-(1-\beta)\lambda_{t+1}}\lambda_{t+\tau}\left(1-\lambda_{t+\tau}\right)\prod_{s=t+1}^{t+\tau-1}\frac{\partial\lambda_s}{\partial\lambda_{s+1}}
$$

Note that $0<\lambda_{t+\tau}<1$, and differentiate (5) and simplify to get

$$
\frac{\partial\lambda_s}{\partial\lambda_{s+1}}=\frac{1}{\rho}\frac{\lambda_s\left(1-\lambda_s\right)}{\lambda_{s+1}}\left[\rho+\frac{(1-\beta)\lambda_{s+1}}{1-(1-\beta)\lambda_{s+1}}\right]>0
$$

So, under hyperbolic discounting ($0<\beta<1$), $\sigma_{S,\tau+1}<\sigma_{S,\tau}$ if $\rho>1$, and $\sigma_{S,\tau+1}>\sigma_{S,\tau}$ if $\rho<1$, for any duration $\tau\in\{1,2,...\}$. For $\rho=1$, $\sigma_{S,\tau+1}=\sigma_{S,\tau}=1$ for all τ. ■

A.2 Naive Consumers

This section derives the results for naive consumers, which are discussed in section 4.1. In particular, it provides a derivation of (13) and (14), and the proof of Proposition 2.

Derivation of (13) and (14):

The naive hyperbolic consumer maximizes (1) believing that future selves are exponential discounters without present-biased preferences. So, the naive self t maximizes U_t in (24), where $V(W_{t+1})$ is now the anticipated continuation-value function for the future self $s=t+1$. All future selves $s\in\{t+1,t+2,...\}$ are believed to maximize U_s with $\beta=1$. Substituting $\beta=1$ into (5) gives the intended consumption rate $\tilde{\lambda}_s$ in (13) for future selves, which corresponds to the exponential outcome. So, the anticipated continuation-value function satisfies

$$
V\left(W_{s+1}\right)=u\left(\tilde{\lambda}_{s+1}W_{s+1}\right)+\delta V\left(R_{s+1}\left(1-\tilde{\lambda}_{s+1}\right)W_{s+1};s+2\right) \tag{27}
$$

for $s\in\{t,t+1,t+2,...\}$. The first order condition for the current self $s=t$ is still given by (26). However, for future selves $s\in\{t+1,t+2,...\}$, which are believed to set $\beta=1$, the anticipated first order condition is

$$
u'\left(C_s\right)=R_s\delta V'\left(W_{s+1};s+1\right) \tag{28}
$$

Differentiating (27) and substituting for $V'(W_{t+2}; s+2)$ using (28), (26) yields

$$\begin{aligned} u'(C_t) &= R_t\beta\delta\left[\tilde{\lambda}_{t+1}u'(C_{t+1}) + R_{t+1}\left(1-\tilde{\lambda}_{t+1}\right)\delta V'(W_{t+2}; t+2)\right] \\ &= R_t\beta\delta u'(C_{t+1}) \end{aligned} \tag{29}$$

To find the naive consumption rate, substitute (2), $C_{t+1} = \tilde{\lambda}_{t+1}W_{t+1}$ and (3) into (29), and rearrange to get the recursion formula (14). ■

Proof of Proposition 2:

To compute the intertemporal consumption ratio C_{t+1}/C_t it is no longer possible to rely on the Euler equation (29), because it only describes intended consumption. To obtain the actual intertemporal consumption ratio, use $C_s = \lambda_s W_s$ and (3) to get $C_{t+1}/C_t = \lambda_{t+1}R(1-\lambda_t)/\lambda_t$. Using (14) to get $(1-\lambda_t)/\lambda_t = (R^{1-\rho}\beta\delta)^{1/\rho}/\tilde{\lambda}_{t+1}$, and taking logs yields:

$$\ln(C_{t+1}/C_t) = \ln\lambda_{t+1} - \ln\tilde{\lambda}_{t+1} + \frac{1}{\rho}(\ln R + \ln\beta + \ln\delta) \tag{30}$$

Differentiating with respect to $r \equiv \ln R$ gives the elasticity of intertemporal substitution for naive hyperbolic discounters:

$$\sigma_N = \frac{d\ln(C_{t+1}/C_t)}{dr} = \frac{1}{\rho} + \frac{1}{\lambda_{t+1}}\frac{d\lambda_{t+1}}{dr} - \frac{1}{\tilde{\lambda}_{t+1}}\frac{d\tilde{\lambda}_{t+1}}{dr} \tag{31}$$

For a change in the real interest rate r of duration τ, $\tilde{\lambda}_{t+\tau} = \bar{\lambda}_E$, $\tilde{\lambda}_s$ is given by (13) for $s \in \{t+1, ..., t+\tau-1\}$, and λ_{t+1} is given by (14). To derive $\sigma_{N,\tau}$, expressions are needed for $\partial\lambda_{N,s}/dr$, $\partial\tilde{\lambda}_s/dr$, $\partial\lambda_{N,t+1}/\partial\tilde{\lambda}_{t+2}$ and $\partial\tilde{\lambda}_s/\partial\tilde{\lambda}_{s+1}$ for $s \in \{t+1, ..., t+\tau-1\}$. Using (14) and (13), $\partial\lambda_{N,s}/dr = \frac{\rho-1}{\rho}\lambda_{N,s}(1-\lambda_{N,s})$ and $\partial\tilde{\lambda}_s/dr = \frac{\rho-1}{\rho}\tilde{\lambda}_s\left(1-\tilde{\lambda}_s\right)$, similar to the sophisticated case in (6). Differentiating (14) and (13) with respect to $\tilde{\lambda}_{s+1}$ and simplifying gives

$$\begin{aligned} \frac{\partial\lambda_{N,t+1}}{\partial\tilde{\lambda}_{t+2}} &= (1-\lambda_{N,t+1})\frac{\lambda_{N,t+1}}{\tilde{\lambda}_{t+2}} \\ \frac{\partial\tilde{\lambda}_s}{\partial\tilde{\lambda}_{s+1}} &= \left(1-\tilde{\lambda}_s\right)\frac{\tilde{\lambda}_s}{\tilde{\lambda}_{s+1}} \end{aligned}$$

Substituting these results and simplifying yields:

$$\begin{aligned} \frac{1}{\tilde{\lambda}_{t+1}}\frac{d\tilde{\lambda}_{t+1}}{dr} &= \frac{1}{\tilde{\lambda}_{t+1}}\left[\frac{\partial\tilde{\lambda}_{t+1}}{\partial r} + \frac{\partial\tilde{\lambda}_{t+1}}{\partial\tilde{\lambda}_{t+2}}\frac{\partial\tilde{\lambda}_{t+2}}{\partial r} + \frac{\partial\tilde{\lambda}_{t+1}}{\partial\tilde{\lambda}_{t+2}}\frac{\partial\tilde{\lambda}_{t+2}}{\partial\tilde{\lambda}_{t+3}}\frac{\partial\tilde{\lambda}_{t+3}}{\partial r} + ...\right. \\ &\qquad \left. ... + \left(\frac{\partial\tilde{\lambda}_{t+1}}{\partial\tilde{\lambda}_{t+2}} * ... * \frac{\partial\tilde{\lambda}_{t+\tau-2}}{\partial\tilde{\lambda}_{t+\tau-1}}\right)\frac{\partial\tilde{\lambda}_{t+\tau-1}}{\partial r}\right] \\ &= \frac{\rho-1}{\rho}\left[\left(1-\tilde{\lambda}_{t+1}\right) + \left(1-\tilde{\lambda}_{t+1}\right)\left(1-\tilde{\lambda}_{t+2}\right) + \left(1-\tilde{\lambda}_{t+1}\right)\left(1-\tilde{\lambda}_{t+2}\right)\left(1-\tilde{\lambda}_{t+3}\right) + ...\right. \end{aligned}$$

$$
\begin{aligned}
& ... + \left(1-\tilde{\lambda}_{t+1}\right) * ... * \left(1-\tilde{\lambda}_{t+\tau-1}\right)\Big] \\
= \; & \frac{\rho-1}{\rho} \sum_{i=t+1}^{t+\tau-1} \left[\prod_{s=t+1}^{i} \left(1-\tilde{\lambda}_s\right) \right]
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{1}{\lambda_{N,t+1}} \frac{d\lambda_{N,t+1}}{dr} &= \frac{1}{\lambda_{N,t+1}} \left[\frac{\partial \lambda_{N,t+1}}{\partial r} + \frac{\partial \lambda_{N,t+1}}{\partial \tilde{\lambda}_{t+2}} \frac{\partial \tilde{\lambda}_{t+2}}{\partial r} + \frac{\partial \lambda_{N,t+1}}{\partial \tilde{\lambda}_{t+2}} \frac{\partial \tilde{\lambda}_{t+2}}{\partial \tilde{\lambda}_{t+3}} \frac{\partial \tilde{\lambda}_{t+3}}{\partial r} + ... \right. \\
& \left. ... + \frac{\partial \lambda_{N,t+1}}{\partial \tilde{\lambda}_{t+2}} \frac{\partial \tilde{\lambda}_{t+2}}{\partial \tilde{\lambda}_{t+3}} * ... * \frac{\partial \tilde{\lambda}_{t+\tau-2}}{\partial \tilde{\lambda}_{t+\tau-1}} \frac{\partial \tilde{\lambda}_{t+\tau-1}}{\partial r} \right] \\
&= \frac{\rho-1}{\rho} \Big[(1-\lambda_{N,t+1}) + (1-\lambda_{N,t+1}) \left(1-\tilde{\lambda}_{t+2}\right) + ... \\
& ... + (1-\lambda_{N,t+1}) * \left(1-\tilde{\lambda}_{t+2}\right) * ... * \left(1-\tilde{\lambda}_{t+\tau-1}\right) \Big] \\
&= \frac{\rho-1}{\rho} \frac{1-\lambda_{N,t+1}}{1-\tilde{\lambda}_{t+1}} \sum_{i=t+1}^{t+\tau-1} \left[\prod_{s=t+1}^{i} \left(1-\tilde{\lambda}_s\right) \right]
\end{aligned}
$$

Substituting this into (31) and rearranging:

$$
\sigma_{N,\tau} = \frac{1}{\rho} - \frac{\rho-1}{\rho} \left[1 - \frac{1-\lambda_{N,t+1}}{1-\tilde{\lambda}_{t+1}} \right] \sum_{i=t+1}^{t+\tau-1} \prod_{s=t+1}^{i} \left(1-\tilde{\lambda}_s\right)
$$

Simplifying gives the naive elasticity of intertemporal substitution for an interest rate change of τ periods in (15).

To prove monotonicity, use (15) to write

$$
\sigma_{S,\tau+1} - \sigma_{S,\tau} = -\frac{\rho-1}{\rho} \frac{\lambda_{N,t+1} - \tilde{\lambda}_{t+1}}{1-\tilde{\lambda}_{t+1}} \prod_{s=t+1}^{t+\tau} \left(1-\tilde{\lambda}_s\right)
$$

Note that $0 < \tilde{\lambda}_s < 1$ for $s \in \{t+1, ..., t+\tau\}$, and use (14) and (13) to see that $\lambda_{N,t+1} > \tilde{\lambda}_{t+1}$ under hyperbolic discounting ($0 < \beta < 1$). So, $\sigma_{S,\tau+1} < \sigma_{S,\tau}$ if $\rho > 1$, and $\sigma_{S,\tau+1} > \sigma_{S,\tau}$ if $\rho < 1$, for any duration $\tau \in \{1, 2, ...\}$. For $\rho = 1$, $\sigma_{S,\tau+1} = \sigma_{S,\tau} = 1$ for all τ. ■

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