Shadow Banking and Asset Pricing

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Abstract

A shadow banking system featuring collateral constraints is studied to investigate the joint determination of haircut and interest rate, as well as its interaction with collateral asset pricing. The banks with limited commitment serve the households' need for consumption smoothing by taking deposits with a risky asset used as collateral and pursue the maximal leverage returns. In a collateral equilibrium as in Geanakoplos (1997, 2003), agents' marginal rates of substitution are equalized only in non-default states, only the deposit contract with the highest liquidity value per unit of collateral is traded, and the risky asset price is boosted such that banks earn zero profit. Relative to the traditional banking with full commitment, banks are better off if they are endowed with the collateral asset while households are strictly worse off. I also find (i) higher households' risky asset endowment leads to a higher asset price because a stronger saving motive creates a scarcity of collateral, while higher banks' collateral endowment has the opposite effects; (ii) collateral use exhibits a diminishing return to scale in the amount of borrowing supported; (iii) for the quality of collateral, the higher asset price resulting from an upside improvement simply leads to a higher haircut with the interest rate unchanged since lenders do not care about upside risk; on the contrary, for lenders with a high risk aversion, a downside improvement of quality decreases the asset price because it alleviates the tension of imperfect risk sharing and, therefore, reduces the collateral value, but everything goes in opposite directions for a low risk aversion.

Keywords: Collateral Constraint, Haircut, Interest Rate, Asset Pricing, Quality of Collateral
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1 Introduction

Financial institutions have been intensively engaged in shadow banking activities such as repo. A common feature of these lines of business is the essential use of collateral. A repo transaction between a repo investor and a shadow bank is analogous to the demand deposit for the traditional banking, with the collateral serving as kind of deposit insurance (Gorton and Metrick, 2010). For this type of collateralized deposit, there are two dimensions in the pricing of a promised but potentially risky repayment: the interest rate and the haircut[1]. A bank can offer more favorable terms to a depositor, who are concerned about the risk of a fixed payment in the future, by either putting up more collateral, which means a higher haircut, or by reducing the initial deposit, which means a higher interest rate.[2] The main questions I want to ask here are: how the interest rate and haircut are jointly determined? What are the factors that may play a role in the mechanism and how? Unfortunately, these questions are not well answered in the literature.[3]

A companion question is on the pricing of the asset used as collateral. It is expected that its price would interact with the collateral constraint (Kiyotaki and Moore, 1997) and a collateral value might be incorporated when the constraint is binding. But the knowledge is limited on how the price moves together with its haircut and the interest rate of loans backed by it. This is particularly relevant during the financial crisis because when the cash flows of securitized assets deteriorated with increased uncertainty, their prices and haircuts and the interest rates of loans might change dramatically, although differential patterns were present depending on the asset quality (Gorton and Metrick, 2012; Copeland, Martin, and Walker, 2014; Krishnamurthy, Nagel, and Orlov, 2014). So a framework for analyzing the relation between asset price, haircut and interest rate and the role of quality of collateral is desirable.

[1]The term haircut is commonly used in the repo market. If a repo investor lends \$80 to a bank who puts up \$100 collateral, there is a haircut of 20% on the collateral. But haircut can be interpreted more broadly. If you buy a house of value \$1 million by a mortgage in which you pay a downpayment \$0.2 million and finance the rest with a bank, the bank indeed requires a haircut of 20% of the value of your house. If you start a business with a current value of \$1 million and finance the initial investment by a debt issuance of \$0.4 million, the bond holders are requiring a haircut of 60% from you, the equity holder.

[2]For example, in the programs of providing three-year loans to financial institutions as a Lender of Last Resort in December 2011 and February 2012, the European Central Bank charges a higher interest rate but a lower haircut on collateral than the primary market (Drechsler and Schnabl, 2014).

[3]Here is a quote from Gorton and Metrick (2012) who empirically investigate the behavior of the repo rate and haircut during the recent financial crisis (repo run in their terminology): "It could seem natural that repo spreads and repo haircuts should be jointly determined. Unfortunately, the theory is not sufficiently developed to provide much guidance here".

In this paper, I address these questions by investigating an exchange economy in which the risk averse households have consumption smoothing needs which could be met by the risk neutral banks who pursue the maximal leveraged return. But banks face a limited commitment problem in the sense that they cannot commit to pay back the deposits at the maturities.[4] However, there is a fixed supply of a risky asset in the economy which banks can put up as collateral to secure the deposits from households.

To be specific, banks and households are at the beginning endowed with a perishable consumption good and the risky asset which delivers a stochastic dividend next period. The choice of terms of deposit is modeled as buying bonds with face value 1 of a continuum of types indexed by the unit of risky asset, s, used as collateral in a competitive bond market in the spirit of Geanakoplos (1997, 2003). Agents choose optimally to hold the risky asset, buy bonds, or issue bonds (this requires collateral). In equilibrium, households find it optimal to only buy bonds, not holding the risky asset or issuing bonds because they are too risky. Moreover, banks only issue the single type of bond with the largest valuation difference, the liquidity value of a bond, per unit of collateral between households and banks. The risky asset price is boosted up from its fundamental value with banks as the marginal investor to reflect a collateral value which makes banks indifferent in issuing that bond. For any other type of bond not traded, households' valuation is less than the cost of the obligation for banks plus the collateral value paid for setting up the collateral required. There is always a possibility of default and agents' marginal rate of substitutions are equalized only for non-default states.

With the traditional banking with full commitment as a benchmark, although banks make the same zero profit in a shadow banking system, their welfare is enhanced if they are endowed with the collateral asset which earns a collateral value. Banks are taking advantage of the low deposit financing cost from households who are willing to pay more for the contract than what banks would ask for because of risk aversion. But it is costly to hold collateral because its price is boosted up offering a lower expected return than banks' time preference. In a completive equilibrium, these two factors offset exactly and banks earn an expected leveraged return equal to their time preference. Households' saving rate is adversely distorted except for a log-utility since there is a probability of default and the

[4]This limited commitment assumption, the key friction considered in this paper, is motivated by the empirical observation in Pozsar (2011) that prior to the recent financial crisis, the global non-financial corporations and institutional investors were holding large cash pools exceeding the insufficient supply of safe assets such as the partially insured deposits and short-term government securities. Then the secured lending featuring the shadow banking system became the natural alternative investment.

interest rate is risky. They save too little when their degree of risk aversion is low but too much when it is high. Even households' initial wealth is higher if endowed with the collateral asset, they are strictly worse off because of imperfect risk sharing.

The uniqueness of the type of bond actively traded, denoted as $\bar{s}$, gives rise to the joint determination of the interest rate and haircut in the shadow banking system. Factors that are relevant in the mechanism include the strengths of household and bank sectors in term of their endowments, agents' time preferences, households' risk aversion, and the quality of collateral. Table 1 lists all the factors and the predicted responses of variables of interest to shocks on them. The first several counterintuitive results illustrate the distinctive way this shadow banking system operates.

First, as households' endowment in the risky asset increases, the risky asset price turns out to increase as well. In this economy, as households become richer and thus are willing to save more to smooth consumption, it creates a scarcity of collateral with less units of asset backing each unit of promised repayment even though the total supply of asset is also higher which is of second order. This conservatism in using collateral is consistent with a higher collateral value and hence a higher risky asset price. So we have a rare case in which an excess supply generates an excess demand *outweighing* the excess supply.

Second, when households' initial endowment in either the consumption good or risky asset increases, even though households want to save more the interest rate goes up when households' relative risk aversion is less than 1. As discussed above, households' greater saving motive leads to less units of asset backing each unit of promised repayment. This implies a greater credit risk of the deposit contract. Although households' endogenous pricing kernel shifts upward because of risk aversion and a higher chance of default, the greater credit risk dominates when their relative risk aversion is less than 1. Households value the deposit contract less and a higher interest rate prevails. On the other hand, although the interest rate indeed decreases when the relative risk aversion is greater than 1, it is because households' increased valuation of payments in bad states dominates the greater credit risk.

Third, when banks are endowed with more risky asset, the effects are completely the opposite of the previous case. The reason is that when the household sector is held the same, more risky asset in the bank sector, where banks pursue leveraged returns, implies an affluence of collateral leading to more units of asset backing each unit of promised repayment. The collateral value of risky asset and hence its price decrease. The credit risk of the deposit contract declines driving the change of interest rate: decreases for low risk aversion while increases for high risk aversion.

Finally, although the households' risk aversion parameter plays both roles of risk aversion and elasticity of intertemporal substitution here, if we control for the effect of intertemporal substitution by assuming the same time preference for households and banks and let the risk aversion increase, one might expect households to be compensated by at least a higher interest rate or a higher haircut. However, both of them decrease if, for example, households are endowed with all the risky asset. This is because when households become more risk averse, they are more desperate in smoothing consumption and hence are less favored in the terms of deposit contract.

This paper also provides novel insights on the nature of haircut. Haircut can be interpreted as the equity-asset ratio if each collateralized deposit contract is considered as an establishment of a firm with the collateral, $\bar{s}$ units of risky asset, being the asset and the deposit being the debt. Hence, haircut is positively linked to the equity-debt ratio in which the equity is the banks' valuation of $\bar{s}$ call options on the risky asset with the $1/\bar{s}$ playing the role of a strike price, while the debt is the households' valuation of the standard debt-type payoff at the same strike price but reflecting their risk aversion. There are two channels for a shock to have an impact on the haircut. The first one is through the direct impact on the valuation of options contained in equity and debt which is relevant for shocks on the quality of collateral and households' risk aversion. The second one is through the indirect impact on the strike price, $1/\bar{s}$. In particular, we document the property that when the first channel is absent, the collateral use exhibits a diminishing return to scale in the amount of borrowing supported. That is, haircut is increasing in the unit of risky asset, $\bar{s}$, used as collateral, and therefore any shock, except the previous two, that increases $\bar{s}$ also increases haircut such as a negative shock on households' initial endowments, a positive shock on banks' risky asset endowment, a less patient household sector or a more patient bank sector.

The upside and downside quality of collateral[5] is shown to have asymmetric effects because households' valuation of the collateralized bonds depends only on its quality in states of default. When the upside quality improves, it does not change the type of deposit contract traded and hence the interest rate and collateral value remain. The asset price increases simply because of the improvement of quality and the haircut adjusts upward accordingly. But when the downside quality improves, the equilibrium contract traded

[5]In the sense of a first order or a second order stochastic improvement over the right or left tail of the asset future dividend distributions.

changes. It is interesting that households with low and high risk aversion respond differently because the contingent valuation of bond payment is increasing and concave (decreasing and convex, respectively) in the risky asset dividend when the risk aversion is low (high, respectively), leading to complete opposite predictions on changes of variables of interest. In particular, when the risk aversion is high, the asset price drops as its downside quality improves. This is because the collateral value falls as households' desire for consumption smoothing is dampened by the improvement in the downside asset quality. Also, although a first order and a second order stochastic dominance improvements of the quality have similar effects when they are local, asymmetry arises when they are global since the equity is always increasing and convex in the risky asset dividend.

Table 1: **Predictions on the Responses of Interest Rate, Haircut and Asset Price to Shocks**

		Predictions		
Shocks	Cases	Interest Rate R	Haircut H	Asset Price p_0
Household's endowments $e_0^h \uparrow$ or $s_0^h \uparrow$	-	$\uparrow$ for $\eta \in (0,1)$;$\downarrow$ for $\eta \in (1,\infty)$	$\downarrow$	$\uparrow$
Bank's endowment $s_0^b \uparrow$	-	$\downarrow$ for $\eta \in (0,1)$;$\uparrow$ for $\eta \in (1,\infty)$	$\uparrow$	$\downarrow$
Improving Collateral Quality $(s_0^h = 0)$				
FOSD/MI SOSD over upper tail $(1/\bar{s}_0, \infty)$	-	remains	$\uparrow$	$\uparrow$
MP SOSD over upper tail $(1/\bar{s}_0, \infty)$	-	remains	remains	remains
FOSD/SOSD over lower tail $(0, 1/\bar{s}_0)$	$\eta \in (0,1)$	$\downarrow$	$\downarrow$ iff $E[D_1 \mid D_1 > \frac{1}{\bar{s}_0}] > \left(\frac{\beta^h}{\beta^b}\right)^{\frac{1}{\eta}} \frac{e_0^h}{K}$	$\uparrow$
	$\eta \in (1,\infty)$	$\uparrow$	$\uparrow$ iff $E[D_1 \mid D_1 > \frac{1}{\bar{s}_0}] > \left(\frac{\beta^h}{\beta^b}\right)^{\frac{1}{\eta}} \frac{e_0^h}{K}$	$\downarrow$
Household's time preference $\beta^h \uparrow$	-	$\uparrow$ for $\eta \in (0,1)$;$\downarrow$ for $\eta \in (1,\infty)$	$\downarrow$	$\uparrow$
Bank's time preference $\beta^b \uparrow$	$s_0^h = 0$	$\downarrow$	$\uparrow$	$\uparrow$
	$s_0^h > 0, \eta \in (0,1)$	$\downarrow$	$\uparrow$	-
Household's risk aversion $\eta \uparrow (\beta^b = \beta^h)$	$s_0^h < K$	$\downarrow$	$\uparrow$ iff $E[D_1 \mid D_1 > \frac{1}{\bar{s}}] < \frac{e_0^h}{K}$	$\uparrow$
	$s_0^h = K$	$\downarrow$	$\downarrow$	$\uparrow$
Household's EIS $1/\eta \downarrow (s_0^h = K)$	$\beta^b > \beta^h$	$\downarrow$ for $\eta \in (1,\infty)$	$\downarrow$	$\uparrow$

Note: In this table, e_0^h and s_0^h are households' initial endowments in consumption good and risky asset, respectively. s_0^b is banks' initial endowment in risky asset. $K = s_0^h + s_0^b$ is the total supply of the risky asset. β^h and β^b are households' and banks' time preferences, respectively. η is households' degree of risk aversion. D_1 is the random dividend payment of the risky asset at time 1. $\bar{s}_0$ is the collateral level prior to shocks. FOSD and (MP, MI) SOSD stand for first and (mean preserving, mean increasing) second order stochastic dominance, respectively.

Related Literature. This paper is closely related to Simsek (2013) which also employs the collateral equilibrium concept of Geanakoplos (1997, 2003). Namely, we both treat contracts with different features as commodities in a competitive market and let the market select and price, fortunately, the unique contract traded. This helps to transform the non-price variables such as collateral level and haircut involved in collateralized borrowing, difficulties faced by other approaches, into market prices. But the questions we address are different. Simsek (2013) considers the speculative bet between risk-neutral optimists and pessimists of a risky asset and focuses on the asymmetric disciplining of the collateral constraint on the excess leverage taken by optimistic borrowers because of the lenders' pessimistic view of the probabilities of default. However, this paper considers trades driven by the consumption smoothing needs from risk averse households and the motive of maximizing expected return from risk-neutral banks and focuses on the roles of the relative strength of two sectors, the asset quality, and agents' risk and time preferences in determining the haircut and interest rate as well as the asset price. This paper is also connected to Fostel and Geanakoplos (2015) who shows that in a binomial economy any collateral equilibrium is equivalent (in real allocations and prices) to another one involving no default. This multiplicity and the no-default result come from the fact that all debt contracts involving partial default bear the same liquidity value as the safe contract with the maximum promise and, therefore, borrowers are indifferent between issuing any of them. If there is a continuum of states as in this paper, borrowers would optimally issue a unique contract with the maximum liquidity value which may involve default.

There is also a recent literature studying different aspects of the determination of repo haircut and interest rate. Eren (2014) focuses on the role of dealer banks as intermediaries of repo transactions in determining the repo haircut and repo rate. Although haircut and interest rate are both pinned down in his model[6], they are through separated mechanisms: the haircut is determined by the volume of lending by cash investors and the liquidity needs from dealer banks, and the interest rate is determined by the hedge funds' participation constraint. Moreover, collateral is assumed to be riskless there while in my model its quality is an important factor and its price is also endogenously determined. Dang, Gorton and

[6]In his model, the hedge funds have investment projects as well as collateral assets while the cash investors want to store cash but they cannot meet each other. The deal banks, who have current liquidity needs, do a repo (collateralized borrowing) with cash investors and a reverse repo (collateralized lending) with hedge funds simultaneously and charge a higher haircut on the collateral from the latter than they are charged by the former. Since the haircut and interest rate charged by cash investors are assumed to be zero, only the haircut and interest rate charged to the hedge funds need to be determined. The haircut is simply the ratio of dealer banks' liquidity needs over cash investors' lending volume.

Holmsrtöm (2013) also study the determination of haircuts. The tension in their model is that there is a chance that the borrower will be unable to pay off the debt in the future and, at the same time, the lender encounters a liquidity shock. Then the lender would like to sell the collateral to a third party at which stage he may face an adverse selection problem since the potential buyer may doubt the quality of the collateral. Therefore, the lender would suffer a loss which he takes into account up-front and requires a haircut.[7] Different from these models, this paper considers the determination of haircut and interest rate simply on the basis of endowments and preferences of borrowers and lenders and the quality of collateral without assuming any friction introduced by a third party. Hu, Pan, and Wang (2014) empirically study the behavior of haircut and repo rate in the tri-party repo market between dealer banks and money market funds.

This paper also relates to the literature on the role of financial intermediaries in transforming risky/illiquid assets into safe/liquid assets (Gorton, Lewellen, and Metrick, 2012). Here the shadow banking system essentially transforms the risky asset into safer collateralized bonds to facilitate the consumption smoothing of risk averse households. Krishnamurthy and Vissing-Jorgensen (2013) provide empirical evidence showing that the prevalence of short-term debt issued by the financial sector is driven by a large demand for safe and liquid asset for investment from the non-financial sector; that the financial sector supplies such debt by holding positions in other risky assets (loans, securities, etc.) that are funded by short-term debt. This evidence is consistent with the framework of this paper. Also, their evidence that investors are willing to pay a premium for holding such safe and liquid assets which incentivizes the financial sector is in line with the equilibrium feature of my model that the financing cost is so low that depositors are subsidizing banks[8]. From a theoretical point of view, Hébert (2015) also tries to answer the question why debt type contracts are so common in practice. In a securitized lending context, when the lender faces an agency problem from the borrower who can privately modify the quality of underlying assets, he shows that a debt type contract is optimal since it trades off the moral hazards of excessive risk-taking and lax effort from the borrower. This result provides a justification for limiting the contract space to collateralized bonds in this paper when dealing with the limited commitment problem of banks.

[7]The determinants of repo haircuts identified in their model include: (i) the information sensitivity of collateral, (ii) the default probability of borrower, (iii) the intermediate liquidity needs of lender, and (iv) his default probability in a subsequent repo transaction.

[8]This paper focuses on the private supply of safe assets. For theoretical treatments of the private supply of liquid assets and the liquidity premium, see Holmsrtöm and Tirole (1998, 2011).

The paper is organized as following. The next section describes the model and considers the equilibrium in the benchmark of a traditional banking with full commitment. Section 3 characterizes the equilibrium and analyzes the welfare implications. Section 4 conducts the comparative static analysis of how the haircut and interest rate as well as the risky asset price respond to shocks in the endowments of the household sector and bank sector. Section 5 investigates how a change in the quality of collateral impacts its haircut and the interest rate of collateralized debt and how these effects feed back into its price. Section 6 documents the effects of agents' time preferences and households' degree of risk aversion. Section 7 concludes.

2 The Model

Consider an economy with two dates: time 0 and time 1. There is a single perfectly divisible risky asset (a Lucas tree) circulated in the economy on both dates. The total supply of the asset is fixed at K. There is another perishable consumption good which is the dividend (the fruit) of the risky asset. For each unit of risky asset at time 0, it generates dividend $D_0 \in (0, \infty)$ at time 0, which belongs to the asset holders, and will generate dividend $D_1 \in (0, \infty)$ at time 1 which is stochastic with a common known distribution function $F(D_1)$ and a density function $f(D_1)$.

There is a continuum of agents of two types with equal mass one. The type 1 agents are households who consume at time 0 and time 1 with a time-0 expected utility

$$u(c_0^h) + \beta^h E(u(c_1^h)),$$

where c_0^h and c_1^h are households' consumption at time 0 and 1, respectively, $\beta^h \in (0, 1)$ is the households' time discount factor, and $u(c) = c^{1-\eta}/(1-\eta)$ is a CRRA utility function with a constant relative risk aversion coefficient $\eta > 0$. The type 2 agents are banks who are risk neutral and also consume at time 0 and time 1 with a time-0 expected utility[9]

$$c_0^b + \beta^b E(c_1^b),$$

where c_0^b and c_1^b are banks' consumption at time 0 and time 1, respectively, and $\beta^b \in (0, 1)$ is the banks' time discount factor. At time 0, the households and banks are endowed with

[9]We assume for convenience, as usual in the literature, that banks' time-0 consumption can be negative which can be interpreted as production of consumption good with a one-to-one labor cost. Alternatively, we can assume it is nonnegative but banks' endowment in consumption good is large relative to households' endowment in the risky asset such that this nonnegative constraint does not bind.

s_0^h and s_0^b units of risky asset, respectively, with $s_0^h + s_0^b = K$. The households and banks are also endowed with e_0^h and e_0^b units of consumption good, respectively, which come from the time-0 dividend of the risky asset with $e_0^h + e_0^b = KD_0$.

At time 0, agents can borrow and lend between each other but they have no commitment to pay off the debt at time 1. However, agents can borrow by setting the risky asset as collateral. Since for each unit of promised repayment we do not know what the collateral value, or haircut, should be for a given interest rate or vice versa, we choose to model this borrowing and lending with a competitive market for collateralized bonds. Thus we are able to transform the non-price variable, haircut, into the type of a commodity and let the market make a choice of it and determine the corresponding price at the same time. We normalize the face value of all bonds to 1.[10] There are potentially a continuum of bonds with different units of risky asset set as collateral. In particular, we index the collateralized bond backed by s units of risky asset by s and call it bond-s. Therefore, the set of bonds is $\{\text{bond-}s : s \in S \equiv (0, \infty)\}$. At time 1, for each unit of bond-s, if it does not default, the holder gets the face value 1; if it defaults, the holder seizes its collateral obtaining the dividend of the risky asset. So the payoff of bond-s is $min(1, sD_1)$. Let q^s be the price of bond-s in terms of time-0 consumption good. There is also a competitive market for the risky asset at time 0, which is a claim for D_1 units of consumption good at time 1, with a price p_0. Agents are free in trading the risky asset and buying collateralized bonds, but specific amount of collateral is required for issuing any bond.

From the perspective of corporate finance, each bond-s issuance is equivalent to establishing a firm with a capital structure interpreted as following: the firm's total asset is s shares of the risky asset with a value sp_0, the debt holder or the lender has a claim for \$1 next period with a current value q^s, and the equity holder or the borrower keeps the residual next period with a current value $sp_0 - q^s$. The leverage ratio (debt/equity) of the firm is $q^s/(sp_0 - q^s)$, which is negatively related to the equity/asset ratio, or the haircut in a repo contract. It is also helpful to keep in mind that the expected return on equity $r_e = (1 + leverage) * r_a - leverage * r_d$, where the expected return on asset is $r_a = E(D_1)/p_0$ and the expected return on debt is $r_d = E[min(1, sD_1)]/q^s$.

The households maximize their expected discounted utility. Although consumption smoothing is feasible by simply holding the risky asset, they are willing to hold some less

[10]For a simply debt contract in Simsek (2013), the unit of collateral asset is normalized to 1 and the contracts are indexed by their face values. For the convenience of working with bonds, I choose to normalize the face value to 1. Nevertheless, they are equivalent.

risky assets based on a risk-return tradeoff because of risk aversion. However, given that the banks are risk neutral, the risky asset would have the same expected return as the risk-free rate if any and, therefore, be dominated. In the absence of a risk-free asset in the economy, it turns out that a preferred way of deferring the current consumption good to time 1 is to buy collateralized bonds at time 0 in the market, which are "hybrid" assets in the sense that they inherit the downside risk of the risky asset but are free of the upside risk. Therefore, households are the natural buyers of bonds and, hence, borrowers in the economy. They have no interest in doing the reverse, i.e., buying more risky asset and issuing bonds, which means a leverage. We will analyze this more formally in the model. So far no assumption is made to restrict their portfolios.

At this point, we have no clue about which types of bonds would be traded. We model the bond portfolio of the household in the most general way as two distributions over the bond space $\mu^h, \mu^h_- : S \to \mathbb{R}$, corresponding to the long and short portfolios respectively, which are weakly increasing and right continuous with left limits[11]. Hence, for any Borel subset $S' \subseteq S$, $\mu^h(S')$ ($\mu^h_-(S')$) describes in the household's bond portfolio the mass of bonds with collateral level belonging to S' that the household bought (issued). Therefore, given the competitive bond prices $\{q^s\}_{s\in S}$ and the risky asset price p_0, the household's problem is[12]

$$\max_{\{c_0^h, c_1^h, s_+^h, \mu^h, \mu_-^h\}} u(c_0^h) + \beta^h E(u(c_1^h))$$

$$s.t. \quad c_0^h + s_+^h p_0 + \int_S q^s d\mu^h \leq e_0^h + s_0^h p_0 + \int_S q^s d\mu_-^h, \tag{1}$$

$$c_1^h + \int_S min(1, sD_1) d\mu_-^h \leq s_+^h D_1 + \int_S min(1, sD_1) d\mu^h, \tag{2}$$

$$\int_S s d\mu_-^h \leq s_+^h, \tag{3}$$

$$\mu^h(s) - \lim_{s' \to s^-} \mu^h(s') \geq 0, \tag{4}$$

$$\mu_-^h(s) - \lim_{s' \to s^-} \mu_-^h(s') \geq 0, \tag{5}$$

where s_+^h is the number of shares of risky asset the household purchases at time 0. Condition (1) says with the endowment at time 0 and the proceeds from issuing bonds, the household

[11]Mathematically, μ^h and μ^h_- are Borel measures on the set S. Note that they may not be absolutely continuous with respect to the Lebesgue measure and, therefore, do not have a density function. They are also mutually exclusive.

[12]Variables with subscripts + or -, which indicate long and short positions respectively, are irrelevant in equilibrium, as will be seen later. Namely, in the equilibrium the households do not hold trees or issue bonds. Similarly, the banks do not buy bonds in their problem described below.

can either consume, buy the risky asset or bonds. Condition (2) says the household's consumption at time 1 comes from the realized payoffs from the risky asset and long positions of bonds net of payments for short positions of bonds. The condition (3) is a collateral constraint which requires the household to hold enough risky asset as collateral for all bonds issued. The last two nonnegative constraints (4) and (5) are from the definitions of long and short portfolios of bonds.

The banks also maximize the expected discounted utility. However, being risk neutral, they try to hold the asset with maximum expected return. In particular, they are in favor of leverage which is what collateralized borrowing achieves. Put in another way, they can finance their purchase of risky asset by issuing bonds and putting up the risky asset as collateral for the borrowing. Analogous to the household's problem, let the bank's short and long positions of bond portfolio be distributions over the bond space $\mu^b, \mu^b_+ : S \to \mathbb{R}$, respectively. Hence, for any Borel subset $S' \subseteq S$, $\mu^b(S')$ ($\mu^b_+(S')$) describes among the bank's bond issuances (purchases) the mass of bonds with collateral level belonging to S'. Given the competitive bond prices $\{q^s\}_{s\in S}$ and the risky asset price p_0, the bank's problem is

$$\max_{\{c_0^b, c_1^b, s^b, \mu^b, \mu^b_+\}} c_0^b + \beta^b E(c_1^b)$$

$$s.t. \quad c_0^b + s^b p_0 + \int_S q^s d\mu^b_+ \leq e_0^b + s_0^b p_0 + \int_S q^s d\mu^b, \tag{6}$$

$$c_1^b \leq s^b D_1 + \int_S min(1, sD_1) d\mu^b_+ - \int_S min(1, sD_1) d\mu^b, \tag{7}$$

$$\int_S s d\mu^b \leq s^b, \tag{8}$$

$$\mu^b(s) - \lim_{s' \to s^-} \mu^b(s') \geq 0. \tag{9}$$

$$\mu^b_+(s) - \lim_{s' \to s^-} \mu^b_+(s') \geq 0. \tag{10}$$

where s^b is the number of shares of risky asset the bank purchases at time 0. Condition (6) says with the endowment at time 0 and the proceeds from issuing different types of bonds, the bank can either consume, buy the risky asset or bonds. Condition (7) says the bank's consumption at time 1 comes from the realized payoffs from the risky asset and long positions of bonds net of the payoffs of bonds to the lenders. The condition (8) is a collateral constraint which requires the bank to hold enough risky asset as collateral for all bonds issued. Again, the last two nonnegative constraints (9) and (10) are from the definitions of short and long bond portfolios of the bank.

Definition 1. *(Competitive Equilibrium with Homogenous Agents) A competitive equilibrium of the economy is a price system* $\{p_0, \{q^s\}_{s\in S}\}$ *and an allocation* $\{c_0^h, c_1^h, c_0^b, c_1^b\}$ *and asset holdings* $\{\mu^h, \mu_-^h, s_+^h, \mu_+^b, \mu^b, s^b\}$ *such that*

1. *Given the price system,* $\{c_0^h, c_1^h, \mu^h, \mu_-^h, s_+^h\}$ *solve the household's problem, and* $\{c_0^b, c_1^b, \mu_+^b, \mu^b, s^b\}$ *solve the bank's problem;*

2. *The bond market clears:* $\mu^h = \mu^b$, $\mu_-^h = \mu_+^b$ *;*

3. *The risky asset market clears:* $s_+^h + s^b = K$*;*

4. *The consumption good markets clear:* $c_0^h + c_0^b = KD_0$*, and* $c_1^h + c_1^b = KD_1$*.*

2.1 Benchmark: Traditional Banking with Full Commitment

When agents have full commitment, banks are taking deposit as in traditional banking with enough insurance, although the insurance is limited in reality.[13] A risk-free asset is available in the economy and all assets earn the same expected return, the risk-free rate, because of the risk neutrality of banks. For risk averse households, the risk-free asset dominates other assets and they smooth consumption by saving at a market risk-free rate. From the bank's preference, the market gross risk-free rate would be $R^* = 1/\beta^b$ and the price for the risky asset would be $p_0^* = \beta^b E(D_1)$.

The household's problem is

$$\max_{\{c_0^h, c_1^h\}} u(c_0^h) + \beta^h E(u(c_1^h))$$
$$s.t. \quad c_0^h + \beta^b c_1^h \leq e_0^h + s_0^h \beta^b E(D_1),$$

which implies the Euler equation

$$\beta^b = \beta^h \left(\frac{c_1^h}{e_0^h + s_0^h \beta^b E(D_1) - \beta^b c_1^h} \right)^{-\eta}, \tag{11}$$

i.e., the bank and household have the same intertemporal marginal rate of substitution, the inverse of risk-free rate. Let $w^* = e_0^h + s_0^h p_0^*$ be households' initial wealth. So households' optimal saving rate is

$$\kappa^* = \frac{\beta^b}{\beta^b + (\beta^b/\beta^h)^{1/\eta}},$$

[13]The current FDIC standard deposit insurance amount in 2015 is $250,000 per depositor, per insured bank, for each account ownership category.

and consumption plan is $((1-\kappa^*)w^*, \kappa^* w^*/\beta^b)$. For $\eta \neq 1$, their expected discounted utility can be expressed as[14]

$$U_h^* = \frac{w^{*1-\eta}}{1-\eta}(1-\kappa^*)^{-\eta}.$$

The banks' expected utility is their initial wealth

$$U_b^* = e_0^b + s_0^b p_0^*$$

with a consumption plan $(e_0^b - s_0^h p_0^* + \kappa^* w^*, KD_1 - \kappa^* w^*/\beta^b)$.

Although agents can buy or issue collateralized bonds and the households can also hold the risky asset, they have no incentive to do so. Moreover, the deposit opportunity improves the welfare of the households, while the banks' expected utility remains the same. Note that the equilibrium outcome is unchanged if the arrow securities are traded and the market gets complete. So this allocation is Pareto efficient.

3 Characterization of the Equilibrium

In this section, we characterize the equilibrium by first looking at the household's and bank's problems, respectively. As the previous analysis suggests and to save notation, we conjecture an equilibrium where only the banks hold the risky asset and issue bonds, which will be verified later. So currently we suppress the variables s_+^h, μ_-^h and μ_+^b.

For the household's problem, the budget constraints (1) and (2) must bind in equilibrium. Hence, $c_0^h = e_0^h + s_0^h p_0 - \int_S q^s d\mu^h$ and $c_1^h = \int_S min(1, sD_1) d\mu^h$. Let $\{\gamma^s \geq 0\}_{s\in S}$ be the Lagrangian multipliers associated with the nonnegative constraints (4) of bond positions. The first order conditions for the household's problem are bond pricing conditions

$$q^s = E\Big[\underbrace{\beta^h \frac{u'(c_1^h)}{u'(c_0^h)}}_{\text{pricing kernel } M^h(D_1)} min(1, sD_1)\Big] + \frac{\gamma^s}{u'(c_0^h)}, \forall s \in S,$$

and slackness conditions

$$\gamma^s \begin{cases} = 0, & \text{if bond-}s\text{ is actively traded;} \\ \geq 0, & \text{if bond-}s\text{ is not actively traded.} \end{cases}$$

[14] This is a feature specific to CRRA utility. To see this for $\eta \neq 1$

$$U_h^* = \frac{(c_0^h)^{-\eta}}{1-\eta}\left[c_0^h + \beta^h E\left(\left(\frac{c_1^h}{c_0^h}\right)^{-\eta} c_1^h\right)\right] = \frac{(c_0^h)^{-\eta}}{1-\eta}[c_0^h + \beta^b E(c_1^h)] = \frac{(c_0^h)^{-\eta}}{1-\eta} w^*,$$

which can be interpreted as the product of time-0 marginal utility and initial wealth adjusted by $1-\eta$, or the one time utility from the initial wealth adjusted by the saving rate.

In particular, for the actively traded bonds, their market prices equal the household's valuations $E\big[M^h(D_1)min(1, sD_1)\big]$ which can be considered as their bid prices. For any non-traded bond, it is too expensive relative to the household's valuation which is captured by the multiplier $\frac{\gamma^s}{u'(c_0^h)} \geq 0$.

For the bank's problem, the budget constraints (6) and (7) must bind in equilibrium too. Hence, $c_0^b = e_0^b + (s_0^b - s^b)p_0 + \int_S q^s d\mu^b$ and $c_1^b = s^b D_1 - \int_S min(1, sD_1)d\mu^b$. Let $\lambda^b \geq 0$ be the Lagrangian multiplier associated with the collateral constraint (8) and $\{\lambda^s \geq 0\}_{s\in S}$ be the Lagrangian multipliers associated with the nonnegative constraints (9) of bond positions. The first order conditions for the bank's problem are bond pricing conditions

$$q^s = E\big[\beta^b min(1, sD_1)\big] + \lambda^b s - \lambda^s, \forall s \in S,$$

the risky asset pricing condition

$$p_0 = E(\beta^b D_1) + \lambda^b, \tag{12}$$

and slackness conditions

$$\lambda^b\left(s^b - \int_S s d\mu^b\right) = 0,$$

$$\lambda^s \begin{cases} = 0, & \text{if bond-}s\text{ is actively traded;} \\ \geq 0, & \text{if bond-}s\text{ is not actively traded.} \end{cases}$$

λ^b reflects the scarcity of collateral. If the collateral constraint binds, the market price of the risky asset is its fundamental value $E(\beta^b D_1)$ for the bank plus a *collateral value* λ^b. For any actively traded bond, the bond market price equals the bank's cost of issuing that bond which we can think of as an ask price consisting of the bank's valuation of the obligation, $E\big[\beta^b min(1, sD_1)\big]$, and a premium, $\lambda^b s$, paid for the collateral required. The higher the collateral requirement s, the higher the premium is. For any non-traded bond, the bond market price is not enough to compensate the bank's cost of issuing the bond which is captured by the multiplier $\lambda^s \geq 0$. If the collateral constraint does not bind, the bank's valuation of the risky asset and the cost of issuing any bond are equal to their fundamental values.

If an equilibrium exists, then for any actively traded bond-s, the household's valuation of the bond must be equal to the bank's cost of issuing the bond, and they are both consistent with the bond market price q^s. For any non-traded bond-s, the household's valuation of the bond must be lower than or equal to the bank's cost of issuing the bond, and bond market price q^s is somewhere in between. Define $L(s)$ as the *liquidity value*[15] of bond-s, which is the difference between the bond fundamental values for the household and bank

$$L(s) = E[\underbrace{(M^h(D_1) - \beta^b)}_{\text{Difference in pricing kernels}} min(1, sD_1)]. \tag{13}$$

Let $S^* \subseteq S$ be the set of actively traded bonds in equilibrium. Then as analyzed above, we must have

$$L(s) = \lambda^b s, \forall s \in S^*; \quad L(s) \leq \lambda^b s, \forall s \notin S^*. \tag{14}$$

The liquidity value $L(s)$ depends on the household's bond portfolio $\mu^h|_{S^*}$, which fully pins down the household's consumption in both periods, as well as the distribution of dividend D_1. In equilibrium, given the bond holdings $\mu^h|_{S^*}$, the household's time-0 consumption is fixed and time-1 consumption $c_1^h(D_1)$ is a continuous, (weakly) increasing and (weakly) concave function of D_1 with $\lim_{D_1 \to 0} c_1^h(D_1) = 0$ and $\lim_{D_1 \to \infty} c_1^h(D_1) = \mu^h(S^*)$.[16] Hence the household's pricing kernel $M^h(D_1)$ is continuous and (weakly) decreasing with $\lim_{D_1 \to 0} M^h(D_1) = \infty$ and $\lim_{D_1 \to \infty} M^h(D_1) = M^h(\mu^h(S^*))$. The following properties for the function $L(s)$ are intuitive:

$$\lim_{s \to 0} L(s) = 0, \tag{15}$$

$$L'(s) = \int_0^{1/s} (M^h(D_1) - \beta^b) D_1 dF(D_1), \tag{16}$$

$$L''(s) = -\frac{1}{s^3}\left(M^h\left(\frac{1}{s}\right) - \beta^b\right) f\left(\frac{1}{s}\right). \tag{17}$$

If the level of collateral is low enough, the bank and household agree that the bond value is close to zero. When the level of collateral increases, the marginal increase in the bond value for the bank is the expected value of losing a tree upon a default, while it is the expected value of obtaining a tree upon a default for the household. With different pricing kernels, they have different valuations of a marginal increase in collateral, which drives the shape of the liquidity value function. If the household's pricing kernel is higher at the default threshold of dividend, the difference in the valuation of a marginal increase in collateral decreases since the bond gets safer at the margin.

We can go one step further to have the following insights on the equilibrium. The household's pricing kernel is decreasing but in equilibrium it cannot be greater than the bank's

[15] The terminology introduced by Fostel and Geanakoplos (2008).

[16] It is possible that $\lim_{D_1 \to \infty} c_1^h(D_1) = \mu^b(S^*) = \infty$ and $\lim_{D_1 \to \infty} M^h(D_1) = M^h(\mu^h(S^*)) = 0$.

for all levels of dividend. Otherwise, an equilibrium satisfying (14) cannot be obtained even with a binding collateral constraint which adds a linear issuance cost for the bank.[17] See Case I in Figure 1. Hence, we are left with possibilities in equilibrium: (i) the household's pricing kernel intersects the bank's at some dividend level $\hat{D}_1 = 1/\hat{s}$,[18] see Case II in Figure 1; or (ii) the household's pricing kernel touches the bank's at some dividend level $\tilde{D}_1 = 1/\tilde{s}$ and remains the same beyond that level,[19] see Case III in Figure 1.

Based on these properties of the liquidity value function, we have the following result regarding the set of actively traded bonds in equilibrium.

Lemma 1. *If an equilibrium exists, the set of actively traded bonds $S^* \subseteq S$ satisfying (14) is a singleton, i.e., $S^* = \{\bar{s}\}$. Moreover, the collateral constraint binds.*

Proof. See Appendix. □

Now we are ready to show the existence and characterize an equilibrium. Suppose in equilibrium the bond-$\bar{s}$ is traded and the collateral constraint binds, then the trading volume is $\frac{K}{\bar{s}}$. From the household's problem, the price for bond-$\bar{s}$ is

$$q^{\bar{s}} = E\left[\beta^h \left(\frac{min(1, \bar{s}D_1)K/\bar{s}}{e_0^h + s_0^h p_0 - q^{\bar{s}}K/\bar{s}}\right)^{-\eta} min(1, \bar{s}D_1)\right]. \tag{18}$$

The households and the banks agree on this price, that is, $L(\bar{s}) - \lambda^b \bar{s} = 0$, which implies that the risky asset's collateral value is given by

$$\lambda^b = \frac{q^{\bar{s}} - \beta^b E[min(1, \bar{s}D_1)]}{\bar{s}}. \tag{19}$$

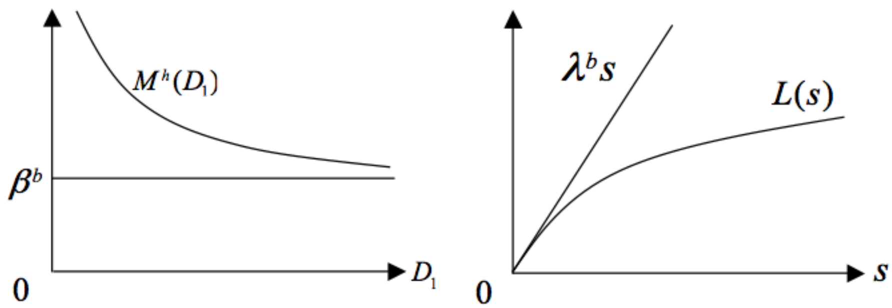
So the price of collateral asset is boosted up by a collateral value such that banks are indifferent in issuing the type of bond traded. The first order condition $L'(\bar{s}) - \lambda^b = 0$

[17] Note that the bank's valuation is then lower than the household's for any bond and the bank's valuation of a marginal increase in collateral is also always lower (the function $L(s)$ is strictly increasing) and, in particular, the marginal difference is strictly decreasing to zero as the level of collateral approaches infinity (the function $L(s)$ is strictly concave).

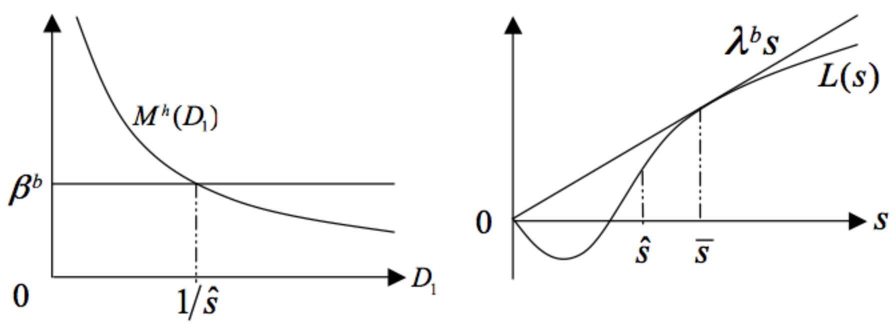
[18] In this case, $M^h(\hat{D}_1) = \beta^h$, $M^h(D_1) > \beta^h, \forall D_1 \in (0, \hat{D}_1)$, and $M^h(D_1) < \beta^h, \forall D_1 \in (\hat{D}_1, \infty)$. Moreover, $L'(\hat{s}) > 0$ and $L''(\hat{s}) = 0$; $\forall s \in (0, \hat{s})$, $L''(s) > 0$, and $\lim_{s\to 0} L'(s) = E[(M^h(D_1) - \beta^b)D_1]$; $\forall s \in (\hat{s}, \infty)$, $L''(s) < 0$, and $\lim_{s\to\infty} L'(s) = 0$.

[19] In this case, $M^h(D_1) > \beta^h, \forall D_1 \in (0, \tilde{D}_1)$ and $M^h(D_1) = \beta^h, \forall D_1 \in [\tilde{D}_1, \infty)$, and any bond-$s$ with $s < \tilde{s}$ is not traded and at least bond-$\tilde{s}$ is traded.

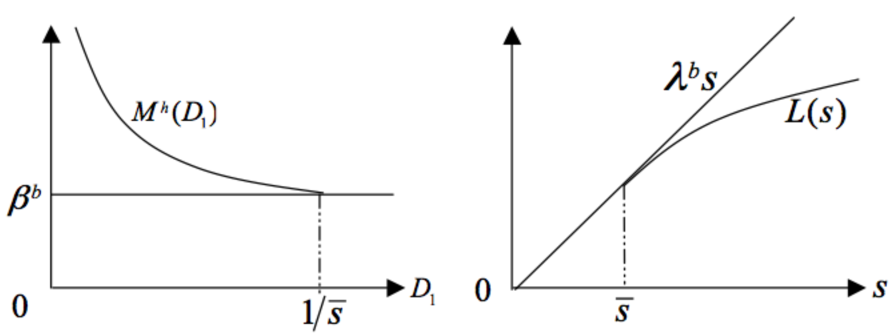
Figure 1: Pricing Kernels (left panel) and Liquidity Value of Bonds (right panel)



(a) Case I



(b) Case II



(c) Case III

implies[20]

$$\beta^b = \beta^h \left(\frac{K/\bar{s}}{e_0^h + s_0^h p_0 - q^{\bar{s}} K/\bar{s}} \right)^{-\eta}, \tag{20}$$

which says the household's and bank's pricing kernels are the same when the dividend exceeds the default threshold and the household gets the maximum time-1 consumption. So the Case III in Figure 1 turns out to be true in equilibrium. Substituting (20) into (18), we get the bond price as a function of collateral level $\bar{s}$

$$q^{\bar{s}} = \beta^b E[min(1, \bar{s}D_1)^{1-\eta}], \tag{21}$$

and substituting it into (19)

$$\lambda^b = \frac{\beta^b E[min(1, \bar{s}D_1)^{1-\eta}] - \beta^b E[min(1, \bar{s}D_1)]}{\bar{s}} > 0. \tag{22}$$

For actively traded bond-$\bar{s}$, the household's valuation is higher than the bank's. Hence the banks would like to issue as many bonds-$\bar{s}$ as possible. But this drives up the price of the risky asset which is required for issuing any amount of bond. In equilibrium, the price of risky asset is such that the banks are indifferent between issuing or not issuing the actively traded bond, which is unique, and prefer not issuing any other bond.

The risky asset pricing condition (12) together with (20), (21) and (22) form a system of equations for $\bar{s}$, $q^{\bar{s}}$, λ^b and p_0. Indeed, there exists a unique solution.

Theorem 1. *There exists a unique equilibrium in which the collateral constraint binds and a single bond-$\bar{s}$ is actively traded, i.e., $S^* = \{\bar{s}\}$, with $\bar{s} \in S$ determined by*

$$\beta^b = \beta^h \left(\frac{K/\bar{s}}{e_0^h + s_0^h E(\beta^b D_1) - \{(K - s_0^h)\beta^b E[min(1, \bar{s}D_1)^{1-\eta}] + s_0^h \beta^b E[min(1, \bar{s}D_1)]\}/\bar{s}} \right)^{-\eta}. \tag{23}$$

[20] This can be seen from

$$\begin{aligned} L'(s)|_{s=\bar{s}} - \lambda^b &= \int_0^{1/\bar{s}} \left[\beta^h \left(\frac{min(1, \bar{s}D_1)K/\bar{s}}{e_0^h + s_0^h p_0 - q^{\bar{s}} K/\bar{s}} \right)^{-\eta} - \beta^b \right] D_1 dF(D_1) - \frac{q^{\bar{s}} - \beta^b E[min(1, \bar{s}D_1)]}{\bar{s}} \\ &= \frac{1}{\bar{s}} \left[\int_{1/\bar{s}}^{\infty} \beta^b dF(D_1) - \int_{1/\bar{s}}^{\infty} \beta^h \left(\frac{min(1, \bar{s}D_1)K/\bar{s}}{e_0^h + s_0^h p_0 - q^{\bar{s}} K/\bar{s}} \right)^{-\eta} dF(D_1) \right] \\ &= \frac{1}{\bar{s}} \left(1 - F\left(\frac{1}{\bar{s}}\right) \right) \left[\beta^b - \beta^h \left(\frac{K/\bar{s}}{e_0^h + s_0^h p_0 - q^{\bar{s}} K/\bar{s}} \right)^{-\eta} \right] \\ &= 0. \end{aligned}$$

The collateral value for the risky asset is given by (22) and the price for the risky asset is given by (12). The price for bond-$\bar{s}$ is given by (21). For any bond-s, $s \in S \setminus \{\bar{s}\}$, the price q^s is indeterminate and can be anything in the bid-ask interval

$$\left[\beta^b E[min(1, \bar{s}D_1)^{-\eta} min(1, sD_1)], \beta^b E[min(1, sD_1)] + s\lambda^b\right]. \tag{24}$$

The allocations are

$$(c_0^h, c_1^h) = \left(e_0^h + s_0^h p_0 - \frac{K}{\bar{s}} q^{\bar{s}}, \frac{K}{\bar{s}} min(1, \bar{s}D_1)\right),$$
$$(c_0^b, c_1^b) = \left(e_0^b - s_0^h p_0 + \frac{K}{\bar{s}} q^{\bar{s}}, \frac{K}{\bar{s}} max(\bar{s}D_1 - 1, 0)\right),$$

with bond portfolio $\mu^h = \mu^b$ being a Dirac measure with mass $\frac{K}{\bar{s}}$ at $\bar{s} \in S$.

Proof. See Appendix. □

Recall that we conjectured the equilibrium in which only banks hold the risky asset and issue bonds. It is the time to verify the households indeed have no incentive to do the same thing in the equilibrium characterized above. It is clear that it is not in the household's interest to issue collateralized bonds because the fundamental values of bonds for them are already higher than those for the banks, $L(s) > 0$, let alone the issuer of bonds needs to pay a premium due to the collateral required. For the risky asset, its value for the households turns out to be the same as the market price, making them indifferent in holding a marginal share of it. To see this,

$$E[M^h(D_1)D_1] - p_0 = \int_0^{1/\bar{s}} (M^h(D_1) - \beta^b) D_1 dF(D_1) - \lambda^b = L'(\bar{s}) - \lambda^b = 0. \tag{25}$$

A dividend payment higher than $1/\bar{s}$ has the same value for the household and bank. However, the household's excess valuation of dividend payments lower than $1/\bar{s}$ over the bank's exactly matches the collateral value of the risky asset. This completes the characterization. In Part B of the Appendix, it is shown that this equilibrium is robust to a heterogeneous wealth distribution.

Several remarks are helpful to clarify the equilibrium obtained. First, one of the key insights in the characterization is that market selects the type of bond $\bar{s}$ with the maximum liquidity value per unit of collateral *given* the households' pricing kernel in equilibrium. This does not mean $\bar{s}$ maximizes the unconditional liquidity value per unit of collateral or the collateral value in (22). Indeed, we will see later that $\partial\lambda^b/\partial\bar{s} < 0$ so a maximum does not exist.

Second, banks choose to issue bond-$\bar{s}$ because prices of other bonds are too low. This can certainly be interpreted as maximizing the expected leveraged return given the equilibrium prices $\{q^s\}_{s\in S}$, for example, at the lower bound of the bid-ask interval (24). Simsek (2013) shows that the collateral constraint provides a discipline against the excess leverage taken by the borrower since the financing cost is increasing in the leverage.[21] But here the expected cost of deposit financing in equilibrium is

$$r_d = \frac{E[min(1, \bar{s}D_1)]}{q^{\bar{s}}} = \frac{E[min(1, \bar{s}D_1)]}{\beta^b E[min(1, \bar{s}D_1)^{1-\eta}]}, \tag{28}$$

which is not necessarily decreasing in $\bar{s}$ so as to provide a force against leverage.[22]

Third, the market's *unconditional* selection of the type of bond traded is based on an equality of marginal rate of substitution (20) between banks and households in non-default states. Analogous to the Euler equation (11) in the benchmark, it is as if households treat the nominal interest rate $1/q^{\bar{s}}$ as risk free and save the amount of $K/\bar{s}$. This condition comes from the observation that only one type of bond would be traded, the characterization of the set of bonds actively traded (14) and the implied first order condition.

3.1 Welfare Analysis

Let us first look at the expected returns on the contingent claims from the banks' perspective. The expected cost of deposit financing is given in (28) with $r_d < 1/\beta^b$ which is cheap relative to their time preference. But they cannot make profit directly from it because of the collateral requirement. The holding of collateral indeed incurs a cost for banks as can be seen by the expected return on the risky asset

$$r_a = \frac{E[\bar{s}D_1]}{\bar{s}p_0} = \frac{E(D_1)}{\beta^b E(D_1) + \lambda^b} < \frac{1}{\beta^b}, \tag{29}$$

[21] In the model of collateralized lending between the pessimist and the optimist in Simsek (2013), the market selection of the contract traded is equivalent to a principal-agent problem in which, subject to lender's participation constraint, the optimist (borrower) maximizes an expected return analogous to (no discounting)

$$r'_e = \frac{E_o[\bar{s}D_1] - E_o[min(1, \bar{s}D_1)]}{\bar{s}p_0 - E_p[min(1, \bar{s}D_1)]} \tag{26}$$

with E_o and E_p denoting the expectation operators by the risk-neutral optimist and pessimist (lender), respectively. The optimist wants to leverage the asset return $r'_a = E_o(D_1)/p_0$ by financing with a perceived gross interest rate cost of

$$r'_d = \frac{E_o[min(1, \bar{s}D_1)]}{E_p[min(1, \bar{s}D_1)]} \geq 1, \tag{27}$$

which is decreasing in the collateral level $\bar{s}$ providing a counterforce against the optimist's desire for leverage.

[22] For example, when $\eta \in (1, \infty)$.

which would be too low for banks to hold it in the absence of the purpose of using it for collateral. Although banks are the marginal investors of the risky asset in both settings with or without commitment, in this shadow banking system, the price is more than the simple expected discounted dividend because of a collateral value on top of it. Put in another way, we can say the banks try to leverage the low expected return on holdings of the risky asset by financing at an even lower cost of the deposit since $r_a > r_d$. In either way, the banks' true return on capital is the equity return

$$r_e = \frac{E[max(\bar{s}D_1 - 1, 0)]}{\bar{s}p_0 - q^{\bar{s}}} = \frac{1}{\beta^b}. \tag{30}$$

So the low return on the collateral asset and the low cost of deposit financing exactly cancel out and banks actually do not make any profit from this business. This is not surprising since banks are indifferent in issuing bond-$\bar{s}$ in equilibrium. Therefore, banks' expected utility is their initial wealth $U_b = e_0^b + s_0^b p_0$. If banks are not endowed with any risky asset, they would have the same expected utility as in the benchmark of traditional banking. However, if banks are endowed with an amount of the risky asset, they would enjoy welfare gains relative to the traditional banking because the risky asset endowment is earning a collateral value, $p_0 > p_0^*$.

Now we look at the welfare of households. Their initial wealth is $w = e_0^h + s_0^h p_0$ and saving rate is

$$\kappa = \frac{q^{\bar{s}}}{q^{\bar{s}} + (\beta^b/\beta^h)^{1/\eta}}.$$

It turns out that for $\eta \neq 1$ their expected discounted utility can again be expressed as[23]

$$U_h = \frac{w^{1-\eta}}{1-\eta}(1-\kappa)^{-\eta} \tag{32}$$

i.e., the one time utility from the wealth adjusted by the saving rate.

If households are not endowed with the risky asset, then $w = w^*$. For $\eta \in (0, 1)$, the nominal interest rate in shadow banking is greater than the risk-free rate in traditional

[23] Note that when $\eta \neq 1$

$$\begin{aligned} U_h &= \frac{(c_0^h)^{-\eta}}{1-\eta}\left[c_0^h + \frac{K}{\bar{s}}\beta^h E\left(\left(\frac{K/\bar{s}}{c_0^h}\right)^{-\eta} min(1, \bar{s}D_1)^{1-\eta}\right)\right] \\ &= \frac{(c_0^h)^{-\eta}}{1-\eta}\left[w - \frac{K}{\bar{s}}q^{\bar{s}} + \frac{K}{\bar{s}}\beta^b E[min(1, \bar{s}D_1)^{1-\eta}]\right] \\ &= \frac{(c_0^h)^{-\eta}}{1-\eta}w. \end{aligned} \tag{31}$$

banking $1/q^{\bar{s}} > 1/\beta^b$ and the saving rate is distorted downward $\kappa < \kappa^*$. So the marginal utility at time 0 is lower and the expected utility is lower. For $\eta \in (1, \infty)$, the nominal interest rate in shadow banking is less than the risk-free rate in traditional banking $1/q^{\bar{s}} < 1/\beta^b$ and the saving rate is distorted upward $\kappa > \kappa^*$. So the marginal utility at time 0 is larger and the expected utility (negative) is again lower. Households save too little when their risk aversion is low and too much when it is high.

For the special case of a log-utility with $\eta = 1$, the nominal interest rate in the shadow banking is the same as the risk-free rate in the traditional banking $1/q^{\bar{s}} = 1/\beta^b$, so is the saving rate $\kappa = \kappa^*$. Although the nominal interest rate in the shadow banking is risky, the low payment in a bad future state is offset by households' high valuation of it at the same time. Nevertheless, their expected utility is still lower since their consumption at time 1 is the same as that in the benchmark when banks do not default but is lower otherwise.

When households are endowed with some risky asset, $s_0^h > 0$, they would experience a positive wealth effect of a higher risky asset price, $w = w^* + s_0^h \lambda^b$. But this is not enough to reverse the fact they are choosing an affordable but not optimal consumption plan in the benchmark.[24] The analysis above can be summarized as following:

Theorem 2. *The banks' expected utility in the shadow banking is no less than that in the traditional banking and is strictly higher if their endowment in the risky asset is positive,* $s_0^b > 0$. *The households are strictly worse off in the shadow banking compared to in the traditional banking.*

In this shadow banking, the collateral constraint resulting from the limited commitment problem prevents agents from perfect risk sharing. The trading of collateralized bonds only facilitates the upside risk sharing leaving households to bear the downside risk which creates the inefficiency relative to the traditional banking. Because of the concern of downside risk, households are willing to pay an extra liquidity value to banks when buying bonds. Although agents are not profiting directly from deposits with banks being indifferent in issuing bond-$\bar{s}$ and households paying the fundamental value for bonds,[25] the excess valuation of the bond by households over banks generates a collateral value for the initial holders of the risky asset. Although banks and households can both benefit from this positive wealth effect, households are still worse off overall. Note that agents have homogenous beliefs and

[24] Note that the cost of households' consumption plan here at the prices in the benchmark is $w^* - s_0^b \lambda^b$.

[25] This is in contrast to Simsek (2013) in which the optimist/borrower has all the bargaining power and extracts all the surplus while the lender earns the same expected return as the storage technology.

this collateral value are not from a speculative bet between optimists and pessimists.[26] Instead, it comes from the difference in the endogenously determined pricing kernels.

4 Scarcity of Collateral

The uniqueness of the type of bond traded in the equilibrium characterized above pins down the interest rate and haircut simultaneously in the shadow banking system. The framework also makes it feasible to analyze the response of these two variables of interest to varieties of shocks to the economy. Given the equilibrium prices, define the haircut H as

$$H = 1 - \frac{q^{\bar{s}}}{\bar{s}p_0} \propto \frac{\bar{s}p_0}{q^{\bar{s}}} = 1 + \frac{E[max(\bar{s}D_1 - 1, 0)]}{E[min(1, \bar{s}D_1)^{1-\eta}]}, \tag{33}$$

which is as a percentage of the collateral value, the difference between the collateral value and the money raised (bond price). For example, if a borrower puts up 2 shares of a stock with price \$0.5 and issues one unit of bond with face value \$1 at a price \$0.8, the haircut is 20%, i.e., \$0.2. If we interpret the collateral value as the asset of a "firm" and the bond price as its debt, haircut is positively related to the asset/debt ratio or the equity/debt ratio at the most right hand side of the equation. Clearly, equity is the value of a call option with $1/\bar{s}$ playing the role of a strike price, and debt is households' valuation of a standard debt-type payoff. Define the interest rate R as

$$R = \frac{1}{q^{\bar{s}}} - 1,$$

so in the example above, the interest rate would be 25%.

We first look at the comparative statics with respect to the strength of the household sector, i.e., their initial endowments in consumption good and risky asset. Since the bank sector endowment in consumption good is irrelevant in determining the equilibrium prices (only matters for bank's time-0 consumption) in this economy, the shock to the household sector endowment in consumption good can be simply interpreted as a shock to the dividend per share of asset at time 0, keeping the asset endowments fixed. But for a positive shock to the household sector endowment in asset, that could be due to an increase in the total supply, keeping the bank sector strength fixed, or due to a reallocation from the bank to the household sector, keeping the total supply fixed. In either way, however, we have the

[26] Speculative bet driven by heterogenous beliefs is the case in Simsek (2013) and Brunnermeier, Simsek and Xiong (2014).

following unambiguous predictions on the changes in the asset price, the interest rate and haircut.

Proposition 1. *When the household sector time-0 dividend endowment e_0^h or asset endowment s_0^h increases, keeping either the total supply of the asset K fixed or the bank's asset endowment s_0^b fixed, the collateral level $\bar{s}$ drops, the collateral value λ^b and the price of risky asset p_0 increase; the haircut H decreases. For $\eta \in (0,1)$, the bond price $q^{\bar{s}}$ decreases and the interest rate R increases; for $\eta \in (1,\infty)$, the bond price $q^{\bar{s}}$ increases and the interest rate R decreases.*

Proof. See Appendix. □

The possible increase in interest rate is quite surprising since when the households become rich, they would like to smooth consumption and save more and, therefore, buy more bonds, which would typically push the interest rate down for traditional banking based on a simple demand/supply analysis. However, for the shadow banking, given the limited supply of risky asset[27], the higher demand for bonds first drives down the collateral level $\bar{s}$ per unit of bond. The bond becomes riskier but households' valuations of payments in bad states also increase. For $\eta \in (0,1)$, the effect of an increased risk dominates, bond-$\bar{s}$ becomes cheaper implying a higher interest rate; for $\eta \in (1,\infty)$, the effect of higher valuations in bad states dominates, bond-$\bar{s}$ becomes more expensive implying a lower interest rate. The increase in the risky asset price is also counterintuitive when its total supply becomes higher in the economy. The reason is that in this economy, a low level of collateral coincides with a scarcity of collateral which implies a higher collateral value λ^b and price p_0 for the risky asset. Namely, we have $\partial\lambda^b/\partial\bar{s} < 0$ in general[28]. Furthermore, everything else being equal[29], the haircut is increasing in $\bar{s}$, that is, the number of shares put up as collateral exhibits a diminishing return to scale in terms of the amount of borrowing they can support. To see this, note that $\partial(\bar{s}p_0/q^{\bar{s}})/\partial\bar{s} > 0$. So the haircut decreases in the economy which, in together with a greater risk of the bond, facilitates the higher demand of saving. Note that, for $\eta \in (0,1)$, an increase in the risky asset price is outweighed by a lower collateral level, so the collateral value $\bar{s}p_0$ declines.

On the other hand, if the bank sector gets stronger in a way of shifting the risky asset from the household to the bank sector, we simply have the results above reversed. The

[27]The increase in the total supply of asset in the second case is of second order.
[28]After controlling for banks' time preference β^b, the cdf F and the relative risk aversion coefficient η.
[29]After controlling for the cdf F and the relative risk aversion coefficient η.

more interesting case is what would happen if the bank sector has more collateral available while the strength of the household sector remains the same.

Proposition 2. *When banks' risky asset endowment s_0^b increases (K increases accordingly), the collateral level $\bar{s}$ increases, the collateral value λ^b and the price of risky asset p_0 decrease; the haircut H increases. For $\eta \in (0, 1)$, the bond price $q^{\bar{s}}$ increases and the interest rate R decreases; for $\eta \in (1, \infty)$, the bond price $q^{\bar{s}}$ decreases and the interest rate R increases.*

Proof. See Appendix. □

The banks in this economy always pursue an expected return as high as possible. With more assets in hand, they want to leverage up. This creates an excess supply of bonds relative to households' demand at the equilibrium prices. This tension is solved by banks' increasing the collateral level, $\bar{s}$, for each unit of bond. The interest rate declines for $\eta \in (0, 1)$ because the bond is safer, while it increases for $\eta \in (1, \infty)$ because households value payments in bad states less. The property of a diminishing return to scale in the use of collateral implies a higher haircut. Remember the haircut is the equity/asset ratio of a firm in the language of corporate finance, so a higher haircut means a lower leverage. Different from an increase in the supply of risky asset in the household sector, an increase in the bank sector is now of first order since the strength of the household sector is unchanged. The affluence of collateral implies a lower collateral value and, therefore, a lower asset price. We might expect banks to earn a lower expected return in a competitive market, but the return on asset increases while the cost of deposit financing may either increase or decrease, and the lower leverage is such that banks earn the same expected return $1/\beta^b$ as before.

Remark 1. *The prices in the economy, including the collateral level, the bond price and interest rate, the collateral value and risky asset price, and the haircut, are homogenous of degree 0 with respect to the total supply of risky asset, keeping the dividend per share and the proportion allocation of risky asset fixed, while the quantities of consumption and bond trading volume are homogenous of degree 1.*

5 Quality of Collateral

The quality of collateral in this economy is described by the distribution of future dividend of the risky asset at time 1. During the recent financial crisis, the housing market went down which deteriorated the quality of the mortgage-related collateral assets and led further to the turmoil in the shadow banking system including markets such as repo and asset backed commercial paper. Now we consider the theoretical predictions of the model on how changes in the quality of collateral affect its price and haircut as well as the interest rate of deposits.

To simplify the analysis and to focus on the role of quality of collateral in the equilibrium deposit contract, we shut down the channel of a wealth effect of collateral quality through households' endowment by assuming they are endowed with zero risky asset but positive consumption good at time 0. Then note that the households care about the asset quality only because of the credit risk of their deposit and in an asymmetric way. If a default would not happen, no matter how promising the asset is, the households enjoy no benefit; but the downside risk is relevant because they would keep the collateral upon default. On the other hand, the banks care about the upward potential of the risky asset because the downside risk in the asset holdings is hedged by the short positions in bonds, or put differently, they are shareholders of "firms". If a default is triggered, how bad the situation is has nothing to do with them. However, there is a caveat here. The downside risk still matters for banks through the collateral constraint because the bond price reflects the households' valuation of the downside risk.

This role played by the collateral constraint asymmetrically for the downside risk is similar to its asymmetric disciplining of the downside optimism in the Simsek's (2013) setting with belief disagreements, but the mechanisms are different. In Simsek (2013), the optimist (borrower) maximizes an expected return analogous to (26) by leveraging the asset return with a perceived financing cost given by (27). The more pessimistic is the lender relative to the borrower over the downside risk, the higher the perceived interest cost which disciplines the leverage taken and limits the impact of belief disagreements on the asset price. In the current setting, agents have the same belief and the perceived deposit financing cost for banks as given by (28) is even lower than their time preference $r_d < 1/\beta^b$ because households value payments in bad future states more. So the deposit financing is even a subsidy without taking the collateral requirement into account and the asymmetric disciplining story is absent here. Nevertheless, the downside risk plays an asymmetric role here relative to the upside risk. That being said, it is clear to see the impact of an improvement of asset quality in the upper tail in a sense of first order stochastic dominance (FOSD) or second order stochastic dominance (SOSD). Suppose the collateral level is $\bar{s}_0$ before the shock.

Proposition 3. *Assume $s_0^h = 0$. If there is a FOSD improvement of the quality of collateral* *over the upper tail $(\bar{s}_0, \infty)$, keeping the quality unchanged in the lower tail $(0, \bar{s}_0)$, then the collateral level $\bar{s}$, the bond price $q^{\bar{s}}$, the interest rate R, and the collateral value λ^b remain the same; the price of risky asset p_0, the collateral asset value $\bar{s}p_0$, and the haircut H increase.*

However, a mean preserving (mean increasing, respectively) SOSD improvement of the quality of collateral over the upper tail $(\bar{s}_0, \infty)$, keeping the quality unchanged in the lower tail $(0, \bar{s}_0)$, has no effect (the same effect as FOSD, respectively) in the economy.

Proof. See Appendix. □

If the quality of collateral in the lower tail $(0, \bar{s}_0)$ does not change, for either a FOSD or a SOSD improvement in the upper tail, households's valuations of bonds and banks' costs of issuing bonds do not change. So market selects the same of type of bond traded $\bar{s}$ as before, and the same bond price $q^{\bar{s}}$, interest rate R and collateral value prevail. For a FOSD improvement in the upper tail, the asset price increases because its fundamental value $\beta^b E[D_1]$ is higher, so is the collateral value $\bar{s}p_0$. Then the haircut increases because the money raised $q^{\bar{s}}$ is the same as before. For a mean preserving SOSD improvement in the upper tail, the fundamental value of the asset does not change, so does the haircut. The message here is that households only care about the downside risk, and any shock to the upper tail of the quality of asset and, hence, asset price will be reflected in the haircut without changing the interest rate. For banks, although the expected return on asset may increase while the cost of deposit financing does not change, their expected return on equity remains because of deleveraging.

Now we consider a change in the quality of collateral in the lower tail.

Proposition 4. *Assume $s_0^h = 0$. If there is a FOSD improvement of the quality of collateral over the lower tail $(0, \bar{s}_0)$, keeping the quality unchanged in the upper tail $(\bar{s}_0, \infty)$, then for $\eta \in (1, \infty)$, the collateral level $\bar{s}$ and the bond price $q^{\bar{s}}$ decrease, and the interest rate R increases; the asset price p_0 decreases; the haircut increases if and only if*

$$E[D_1|D_1 > 1/\bar{s}_0] > (\beta^h/\beta^b)^{1/\eta} e_0^h/K. \tag{34}$$

For $\eta \in (0, 1)$, the collateral level $\bar{s}$ and the bond price $q^{\bar{s}}$ increase, and the interest rate R drops; the asset price p_0 increases; the haircut decreases if and only if (34) holds.

Moreover, these predictions are the same for a SOSD improvement of the quality of collateral over the lower tail $(0, \bar{s}_0)$, keeping the quality unchanged in the upper tail $(\bar{s}_0, \infty)$.

Proof. See Appendix. □

Take the case $\eta \in (1, \infty)$ for illustration. Since the value of contingent bond payment $\beta^b min(1, \bar{s}D_1)^{1-\eta}$ for households is a strictly decreasing and strictly convex function of D_1 in the lower tail, a FOSD or SOSD improvement of the quality of collateral in this region decreases the households' valuation of the bond. The increased interest rate has a positive income effect on households who will save more pushing the collateral level $\bar{s}$ down.[30]

Although the improvement in the downside quality increases the fundamental value (except for mean preserving SOSD) and the decrease in $\bar{s}$ tends to increase the collateral value, these two effects are dominated by the decline in households' valuation of the bond which reduces the collateral value and, therefore, the risky asset price. To see it more clearly, recall that, in the capital structure terminology, the risky asset price is the firm value per unit of collateral and the firm value can be interpreted as the sum of equity and debt. Then we have[31]

$$p_0 = \frac{\text{equity} + \text{debt}}{\bar{s}} = \beta^b E\left[max\left(D_1 - \frac{1}{\bar{s}}, 0\right)\right] + \frac{e_0^h}{K} - \frac{1}{\bar{s}}\left(\frac{\beta^b}{\beta^h}\right)^{1/\eta}, \tag{35}$$

in which the equity per unit of collateral decreases because the strike price increases. In this economy with a binding collateral constraint, a lower $\bar{s}$ implies a higher consumption for households in non-default states. To have agents' intertemporal substitution equalized in non-default states, households must have saved less which indicates a lower bond price per unit of collateral. So the price of risky asset decreases unambiguously as its downside quality improves. The intuition is that when the risk aversion is high, households are desperate in smoothing consumption and willing to pay a higher bond price. The improvement in the downside quality of asset alleviates this tension and the bond price falls accordingly, so does the collateral value.

Although a lower $\bar{s}$ tends to reduce the haircut by the diminishing return to scale property, the improvement of quality of collateral decreases the debt value and thus does the opposite. This tradeoff can be reduced to a comparative statics with respect to only $\bar{s}$

$$H \propto \frac{\text{equity}}{\text{debt}} = \frac{\beta^b E[max(\bar{s}D_1 - 1, 0)]}{\bar{s}e_0^h/K - (\beta^b/\beta^h)^{1/\eta}} \tag{36}$$

which is decreasing in $\bar{s}$ if and only if the inequality (34) holds. The intuition behind is that both equity and debt are increasing in $\bar{s}$ and locally have negative intercepts, the ratio

[30] Although a lower $\bar{s}$ tends to increase the bond price in this case, it is of second order.

[31] When $s_0^h = 0$, the bond price can be expressed as $\bar{s}e_0^h/K - (\beta^b/\beta^h)^{1/\eta}$ as shown in the proof.

is decreasing in $\bar{s}$ if and only if the local slope-intercept (absolute value) ratio is larger for the equity. That is, if the conditional expected value of the upper tail of the risky asset is large relative to households' consumption good endowment per unit of total supply of collateral, the haircut increases.

For $\eta \in (0,1)$, since the value of contingent bond payment $\beta^b min(1, \bar{s}D_1)^{1-\eta}$ for households is a strictly increasing and strictly concave function of D_1 in the lower tail, it is interesting that everything above goes in the opposite direction.

If the quality of collateral changes involving both the lower and upper tails, some of the results above can be generalized. For the interest rate, since only the downside quality matters for the bond payoff and therefore for the strike price $1/\bar{s}$ in equilibrium, the argument in Proposition 4 goes through and for either a FOSD or SOSD over the full support, the interest rate decreases for $\eta \in (0,1)$ but increases for $\eta \in (1,\infty)$.

For the price of risky asset, first consider a FOSD improvement for $\eta \in (0,1)$, it increases $\bar{s}$ as in Proposition (4) as well as the equity per unit of collateral in (35), so the price of risky asset increases. However, for $\eta \in (1,\infty)$, it decreases $\bar{s}$ but still increases the equity per unit of collateral, so the prediction is not clear. Now consider a SOSD, it increases $\bar{s}$ for $\eta \in (0,1)$ while decreases $\bar{s}$ for $\eta \in (1,\infty)$; however, since a call option is increasing in the volatility, it decreases the equity per unit of collateral. So the price of risky asset decreases for $\eta \in (1,\infty)$ but it depends for $\eta \in (0,1)$. These asymmetries are due to the different responses of equity and bond (under different risk aversions) valuations to changes of future distribution of dividend.

Similarly for the haircut, in the expression (36), the downside quality affects the number of call option $\bar{s}$ and the strike price $1/\bar{s}$ as before through its impact on the bond price, while the upside quality affects the equity. Since the prediction of first effect depends on the inequality (34), for a FOSD, the second effect strengths (loosens) the inequality for a $\eta \in (0,1)$ ($\eta \in (1,\infty)$). However, for a SOSD, the second effect loosens (strengths) the inequality for a $\eta \in (0,1)$ ($\eta \in (1,\infty)$).

6 Time and Risk Preferences

We now look at the effects of agents' time preferences.

Proposition 5. *When the household time preference β^h increases, the collateral level $\bar{s}$ drops, the collateral value λ^b and the price of risky asset p_0 increase; the haircut H decreases.*

For $\eta \in (0, 1)$, the bond price $q^{\bar{s}}$ decreases and the interest rate R increases; for $\eta \in (1, \infty)$, the bond price $q^{\bar{s}}$ increases and the interest rate R decreases.

Proof. See Appendix. □

When the households become more patient, they would like to transfer more consumption to time 1 and, therefore, buy more bonds. This will have the same effects as a stronger household sector as in Proposition 1.

On the other hand, if the banks are more patient, there are two direct effects. First, the interest rate for bond-$\bar{s}$ tends to be lower which has a substitution effect inducing households to consume more and save less. Second, the lower interest rate has a negative income effect for households, while the higher price of the risky asset at given $\bar{s}$ has a positive wealth effect. Nevertheless, if households are not endowed with any risky asset or for large elasticity of intertemporal substitution, $\eta \in (0, 1)$, the substitution effect dominates.

Proposition 6. *Assume banks are endowed with all the risky asset, $s_0^h = 0$. When the bank time preference β^b increases, the collateral level $\bar{s}$ increases, the bond price $q^{\bar{s}}$ increases, and the interest rate R drops; the price of risky asset p_0 increases; the collateral asset value $\bar{s}p_0$ and the haircut H increase.*

If $s_0^h > 0$, as long as the elasticity of intertemporal substitution is large $\eta \in (0, 1)$, then when the bank time preference β^b increases, the collateral level $\bar{s}$ increases, the bond price $q^{\bar{s}}$ increases, and the interest rate R drops; the collateral asset value $\bar{s}p_0$ and the haircut H increase.

Proof. See Appendix. □

When banks get more patient, the current business bringing them an expected return of $1/\beta^b$ becomes profitable, and they would like to expand the leveraged investment until the expected return is pushed down to a new zero profit level. When the wealth effect through the risky asset endowment is absent, $s_0^h = 0$, the higher intertemporal rate of substitution of banks gives rise to a substitution effect inducing households to save less. This results in a higher collateral level $\bar{s}$ for each unit of bond and a higher haircut follows from the decreasing return to scale property. Although a higher collateral level $\bar{s}$ tends to decrease the collateral value, banks' expanded investment due to improved patience increases the asset price. Even a higher collateral level $\bar{s}$ tends to decrease the bond price for $\eta \in (1, \infty)$, the upward shift of households' intertemporal rate of substitution dominates leading to a higher bond price and a lower interest rate. Note that this lower interest rate has a negative income effect for households which also induces them to save less. During the adjustment, banks experience a lower expected return on asset and a process of deleveraging.

We now look at the comparative statics with respective to η which plays two roles here governing both risk aversion and elasticity of intertemporal substitution of households. The rise aversion is reflected in the decreasing marginal utility and the pricing of contingent claims. We decompose the analysis in these two respects. To consider the effect of a higher risk aversion, assume households and banks have the same time preference.

Proposition 7. *Assume $\beta^h = \beta^b$. If $s_0^h \in [0, K)$, when the household's risk aversion η increases, the collateral level $\bar{s}$ and the bond price $q^{\bar{s}}$ increase, and the interest rate R declines; the collateral value λ^b and the price of risky asset p_0 increase; the collateral asset value $\bar{s}p_0$ increases, and the haircut increases if and only if $E[D_1|D_1 > 1/\bar{s}] < e_0^h/K$.*

If $s_0^h = K$, when the household's risk aversion η increases, the collateral level $\bar{s}$ is unchanged, the bond price $q^{\bar{s}}$ increases, and the interest rate R declines; the collateral value λ^b and the price of risky asset p_0 increase; the collateral asset value $\bar{s}p_0$ increases and the haircut H decreases.

Proof. See Appendix. □

An increase in households' risk aversion increases the market price of the bond. As long as the households do not own the entire supply of risky asset, it has a negative income effect and decreases their saving demand implying a higher collateral level $\bar{s}$ for each unit of bond. Even an increase in $\bar{s}$ tends to decrease the bond price for $\eta \in (1, \infty)$, the bond price increases overall and the interest rate declines. At the same time, the increase in the bond price lowers the cost of deposit financing for banks. Their pursuit of leveraged investment would push up the asset price. While an increase in $\bar{s}$ tends to increase the haircut, the improved willingness of households to lend money has the opposite effect. A critical condition for a higher haircut is that the conditional mean quality of collateral in non-default states is lower than households' consumption good endowment per unit of supply of collateral asset in the economy.

When the wealth effect due to the impact the risk averse has on the prices, as well as the substitution effect, is absent, the type of bond traded remains the same. The bond price increases and interest rate drops since a higher risk aversion increases the value of payments in bad states. The risky asset price increases as it has a greater collateral value for banks who want to take advantage of the lower financing cost. The haircut drops simply because the households are more willing to contribute debt capital.

To consider the effect of a lower elasticity of intertemporal substitution, we assume households are endowed with all the risky asset so that the effects of risk aversion on the bond price and collateral value are offset by each other.

Proposition 8. *Assume $s_0^h = K$. If $\beta^h < \beta^b$, when the household's elasticity of intertemporal substitution $1/\eta$ decreases, the collateral level $\bar{s}$ decreases, the collateral value λ^b and the price of risky asset p_0 increase; the haircut H decreases. For $\eta \in (1, \infty)$, the bond price $q^{\bar{s}}$ increases, and the interest rate R declines.*

Proof. See Appendix. □

Since households are less patient, they maintain a negative consumption growth pattern. When the elasticity of intertemporal substitution decreases, the optimal consumption pattern becomes flatter. So households save more which pushes down the collateral level $\bar{s}$. Furthermore, a higher liquidity value for any bond-s resulting from a higher risk aversion attracts more business from banks in issuing bonds. Overall, this creates a scarcity of collateral increasing the collateral value and price of risky asset. The haircut decreases because of both a diminishing return to scale in using collateral and an enhanced willingness of household to purchase bonds as a result of higher valuations of payments in bad states. Although the bond price tends to increase for a higher risk aversion, households' low consumption in bad states from a riskier bond traded makes the change of its price depend on the degree of risk aversion. For $\eta \in (1, \infty)$, a higher risk aversion and a higher risk both increase the price of bond traded.

7 Conclusion

This paper provides a framework to analyze the joint determination of haircut and interest rate and the pricing of collateral assets in a shadow banking system. The key mechanism is that banks issue the type of bond with the highest liquidity value per unit of collateral, the asset price increases to bear a collateral value such that banks are indifferent in issuing that bond at the margin, and agents' marginal rates of substitution are equalized only in non-default states. In terms of welfare, banks are no worse than in the traditional banking but are strictly better off if endowed with the risky asset while households are strictly worse off.

The determinants of the variables of interest and their effects are fully documented. An increase in households' endowment increases the asset price and reduces the haircut with the change of interest rate depending on the tradeoff between state prices and credit risk governed by the level of risk aversion. An increase in banks' asset endowment has the opposite effects. The households' more patient time preference has the same effects as an increase in their endowment through the enhanced saving motive. The banks' more patient time preference (controlling for the wealth effect) reduces the interest rate and increases the haircut and asset price. Households' increased risk aversion (controlling for the substitution effect) reduces the interest rate and increases the asset price.

The upside and downside quality of collateral have asymmetric effects on the variables of interest. The improvement of upside quality only increases asset price with the haircut adjusting upward accordingly and the credit risk and interest rate unchanged. But the downside quality improvement has an impact on all variables. In particular, the effects are the opposite for households with a high versus a low risk aversion. This has empirical relevance since researchers typically use measures of asset overall quality such as variance of returns. This paper shows that the predictions regarding the effects of the overall quality on the haircut and interest rate are quite ambiguous and it is important to distinguish the upside and downside quality.

Several interesting properties in an economy with collateral are found including a diminishing return in scale in collateral use and a decreasing collateral value in the units of collateral, everything else being equal. Predictions on the changes in the asset price and interest rate may be far different from those made based on a typical demand/supply analysis. An increase in the supply of assets may raise their prices and an improvement of the quality of assets may bring their prices down. There is not necessarily a tradeoff between the haircut and interest rate and they can move in the same or the opposite directions depending on the type of shocks.

Appendix

A. Proofs.

Proof of Lemma 1: For any $s \in S^*$, s is a local maximum of $L(s) - \lambda^b s$. Hence it must be that $L''(s) \leq 0$ and $L'(s) - \lambda^b = 0$.

For the case there exists a unique $\hat{s}$ such that $M^h(1/\hat{s}) = \beta^h$, the second order condition implies $s \in [\hat{s}, \infty)$. Since $L'(s)$ is strictly decreasing in $(\hat{s}, \infty)$, there is at most one s satisfying the first order condition. Moreover, $L'(s) > 0$ in $(\hat{s}, \infty)$, hence $\lambda^b > 0$.

For the case $M^h(D_1) = \beta^h, \forall D_1 \in [1/\tilde{s}, \infty)$ for some maximum $\tilde{s}$, the liquidity value function is linear in $(0, \tilde{s}]$ and strictly concave in $[\tilde{s}, \infty)$. Then bond-$\tilde{s}$ is the only bond that could be traded and $\lambda^b > 0$.

Proof of Theorem 1: Substituting the bond price, the collateral value and the risky asset price into the pricing kernel equality (20), we have (23). To show the existence and uniqueness, rearrange (23)

$$\underbrace{\bar{s}[e_0^h + s_0^h E(\beta^b D_1)]}_{\text{LHS}(\bar{s})} = \underbrace{(K - s_0^h)\beta^b E[min(1, \bar{s}D_1)^{1-\eta}] + s_0^h \beta^b E[min(1, \bar{s}D_1)] + K\left(\frac{\beta^b}{\beta^h}\right)^{1/\eta}}_{\text{RHS}(\bar{s})}. \tag{37}$$

Note that $\lim_{\bar{s}\to 0} RHS(\bar{s}) > 0$ and $\lim_{\bar{s}\to\infty} RHS(\bar{s}) < \infty$. This guarantees the existence of an equilibrium. For the uniqueness, first consider the case $\eta \in (0, 1]$. Note that

$$LHS'(\bar{s}) = e_0^h + \frac{\beta^b}{\bar{s}} s_0^h \int_0^\infty \bar{s}D_1 dF(D_1),$$

$$RHS'(\bar{s}) = \frac{\beta^b}{\bar{s}}\left[(1-\eta)(K - s_0^h)\int_0^{1/\bar{s}} (\bar{s}D_1)^{1-\eta} dF(D_1) + s_0^h \int_0^{1/\bar{s}} \bar{s}D_1 dF(D_1)\right],$$

$$RHS''(\bar{s}) = -\frac{\beta^b}{\bar{s}^2}\left[\eta(1-\eta)(K - s_0^h)\int_0^{1/\bar{s}} (\bar{s}D_1)^{1-\eta} dF(D_1) + \frac{1}{\bar{s}} f\left(\frac{1}{\bar{s}}\right)[(1-\eta)(K - s_0^h) + s_0^h]\right].$$

Hence

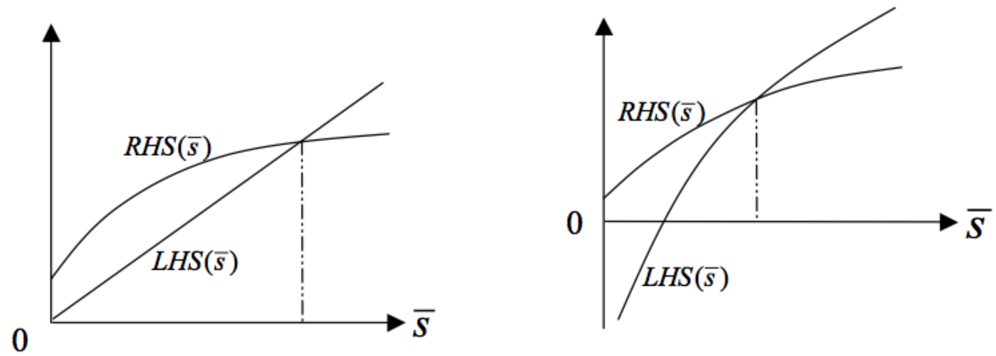
$$RHS'(\bar{s}) > 0, RHS''(\bar{s}) < 0,$$

so the solution is unique. See the left panel in Figure 2.

For $\eta \in (1, \infty)$, consider

$$\underbrace{\bar{s}[e_0^h + s_0^h E(\beta^b D_1)] - (K - s_0^h)\beta^b E[min(1, \bar{s}D_1)^{1-\eta}]}_{\text{LHS}(\bar{s})} = \underbrace{s_0^h \beta^b E[min(1, \bar{s}D_1)] + K\left(\frac{\beta^b}{\beta^h}\right)^{1/\eta}}_{\text{RHS}(\bar{s})}. \tag{38}$$

Figure 2: Existence and Uniqueness of Equilibrium



(a) $\eta \in (0,1)$ (b) $\eta \in (1,\infty), s_0^h \in (0,K)$

Note that for $s_0^h \in (0,K)$, the $LHS(\bar{s})$ is strictly increasing and strictly concave with $\lim_{\bar{s}\to 0} LHS(\bar{s}) = -\infty$ and $\lim_{\bar{s}\to\infty} LHS(\bar{s}) = \infty$; the RHS is strictly increasing and strictly concave with $\lim_{\bar{s}\to 0} RHS(\bar{s}) = K(\beta^b/\beta^h)^{1/\eta}$ and $\lim_{\bar{s}\to\infty} RHS(\bar{s}) = s_0^h\beta^b + K(\beta^b/\beta^h)^{1/\eta}$. Moreover, $RHS'(\bar{s}) < LRH'(\bar{s})$ for any $\bar{s} \in S$. Hence there exists a unique solution. It is also true for $s_0^h = 0$ or $s_0^h = K$. See the right panel in Figure 2.

Proof of Proposition 1: When e_0^h increases, for either (37) with $\eta \in (0,1)$ or (38) with $\eta \in (1,\infty)$, the LHS shifts upward, while the RHS is unchanged. This implies a drop in $\bar{s}$.

When s_0^h increases, total differentiating (37), we have in general

$$\frac{\partial \bar{s}}{\partial s_0^h} = \frac{\beta^b E[min(1,\bar{s}D_1)] - \beta^b E(\bar{s}D_1) + \frac{\partial (K-s_0^h)}{\partial s_0^h}\beta^b E[min(1,\bar{s}D_1)^{1-\eta}]}{LHS'(\bar{s}) - RHS'(\bar{s})},$$

where $\frac{\partial (K-s_0^h)}{\partial s_0^h}$ equals -1 if K is fixed or 0 if s_0^b is fixed. Note that the denominator is positive from the proof of Theorem 1. In the nominator, the first two terms already result in a negative value, so it is negative regardless of the assumption made on fixing K or s_0^b. Overall, $\bar{s}$ declines.

For the collateral value, note that

$$\frac{\partial \lambda^b}{\partial \bar{s}} = -\frac{\eta\beta^b}{\bar{s}^2}\int_0^{1/\bar{s}} (\bar{s}D_1)^{1-\eta} dF(D_1) < 0,$$

For the haircut, consider

$$\frac{\bar{s}p_0}{q^{\bar{s}}} = 1 + \frac{\int_{1/\bar{s}}^{\infty}(\bar{s}D_1 - 1)dF(D_1)}{E[min(1,\bar{s}D_1)^{1-\eta}]}$$

with

$$\frac{\partial(\bar{s}p_0/q^{\bar{s}})}{\partial\bar{s}} = \frac{\int_{1/\bar{s}}^{\infty} D_1 dF(D_1) E[min(1,\bar{s}D_1)^{1-\eta}] - (1-\eta)\int_{1/\bar{s}}^{\infty}(D_1-\frac{1}{\bar{s}})dF(D_1)\int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)}{E[min(1,\bar{s}D_1)^{1-\eta}]^2} > 0. \tag{39}$$

The results for $q^{\bar{s}}$ and R can be seen from (21) with $\partial q^{\bar{s}}/\partial\bar{s} > 0$ for $\eta \in (0,1)$ and $\partial q^{\bar{s}}/\partial\bar{s} < 0$ for $\eta \in (1,\infty)$. The result for p_0 can be seen from (12).

Proof of Proposition 2: For (37) with $\eta \in (0,1)$, when s_0^b increases, the LHS does not change while the RHS shifts upward. So $\bar{s}$ increases. For (38) with $\eta \in (1,\infty)$, when s_0^b increases, the LHS shifts downward while the RHS does not change. So $\bar{s}$ also increases. The changes in other variables follow from the proof in Proposition 1.

Proof of Proposition 3: For the case $s_0^h = 0$, (37) becomes

$$\bar{s}e_0^h = K\beta^b E[min(1,\bar{s}D_1)^{1-\eta}] + K\left(\frac{\beta^b}{\beta^h}\right)^{1/\eta}. \tag{40}$$

Since the quality of collateral in the lower tail $(0,\bar{s}_0)$ remains the same for either of the two shocks, the collateral level $\bar{s}$ is the same as before. So the bond price $q^{\bar{s}}$, by (21), and its interest rate R do not change. By (22), the collateral value λ^b is unaffected. By (12), the asset price increases for a FOSD improvement in the upper tail, so does the collateral value $\bar{s}p_0$. Then the haircut increases by (33). For a mean preserving SOSD improvement in the upper tail, the asset price and haircut do not change. A mean increasing SOSD improvement is a combination of the previous two cases.

Proof of Proposition 4: First consider $\eta \in (0,1)$. In (40), note that $min(1,\bar{s}_0 D_1)^{1-\eta}$ is strictly increasing in D_1 over $(0,1/\bar{s}_0)$. So a FOSD improvement increases $E[min(1,\bar{s}D_1)^{1-\eta}]$, leading to a greater collateral level $\bar{s}$ which further increases the bond price $q^{\bar{s}}$ in (21), and a lower interest rate R follows. Substituting the expression of the bond price in (40) into the asset price (12), we have (35) which increases with $\bar{s}$. For the haircut, substituting the same expression into (33), we have (36) with $\partial(\text{equity/debt})/\partial\bar{s} < 0$ if and only if

$$\frac{\int_{1/\bar{s}}^{\infty} D_1 dF(D_1)}{\bar{s}\int_{1/\bar{s}}^{\infty} D_1 dF(D_1) - \int_{1/\bar{s}}^{\infty} dF(D_1)} < \frac{e_0^h/K}{\bar{s}e_0^h/K - (\beta^b/\beta^h)^{1/\eta}}. \tag{41}$$

Since the function $a/(\bar{s}a-1)$ is decreasing in a in $(1/\bar{s},\infty)$, the haircut decreases if and only if the inequality (34) holds.

For a SOSD improvement, note that $min(1,\bar{s}_0 D_1)^{1-\eta}$ is strictly concave over $D_1 \in (0,1/\bar{s}_0)$. So $E[min(1,\bar{s}D_1)^{1-\eta}]$ increases and the rest of results are the same as above.

For $\eta \in (1, \infty)$, note that $min(1, \bar{s}_0 D_1)^{1-\eta}$ is now strictly decreasing and strictly convex over $D_1 \in (0, 1/\bar{s}_0)$ and, therefore, the bond price $\beta^b E[min(1, \bar{s}D_1)^{1-\eta}]$ tends to decrease for either a FOSD or a SOSD improvement. So $\bar{s}$ decreases. Although this tends to increase the bond price, it is of second order as can be seen from (40) in which the bond price changes in the same direction as $\bar{s}$. So the interest rate increases. The asset price decreases by (35) and haircut decreases if the opposite of (41) is true.

Proof of Proposition 5: When β^h increases, the LHS of either (37) or (38) is unchanged, while the RHS shifts downward. The rest of the proof is the same as above.

Proof of Proposition 6: When $s_0^h = 0$, total differentiating (37) with respect to β^b, we have

$$\frac{\partial \bar{s}}{\partial \beta^b} = \frac{KE[min(1, \bar{s}D_1)^{1-\eta}] + \frac{K}{\eta \beta^b}\left(\frac{\beta^b}{\beta^h}\right)^{1/\eta}}{LHS'(\bar{s}) - RHS'(\bar{s})} > 0,$$

so $\bar{s}$ increases. For the bond price $q^{\bar{s}}$,[32] we have

$$\begin{aligned} \frac{\partial q^{\bar{s}}}{\partial \beta^b} &= E[min(1, \bar{s}D_1)^{1-\eta}] + \frac{\partial \beta^b E[min(1, \bar{s}D_1)^{1-\eta}]}{\partial \bar{s}} \frac{\partial \bar{s}}{\partial \beta^b} \\ &= \frac{e_0^h E[min(1, \bar{s}D_1)^{1-\eta}] + \frac{\partial \beta^b E[min(1, \bar{s}D_1)^{1-\eta}]}{\partial \bar{s}} \frac{K}{\eta \beta^b}\left(\frac{\beta^b}{\beta^h}\right)^{1/\eta}}{LHS'(\bar{s}) - RHS'(\bar{s})} \\ &= \frac{\bar{s}e_0^h E[min(1, \bar{s}D_1)^{1-\eta}] + \left(\frac{1}{\eta} - 1\right) \int_0^{1/\bar{s}} (\bar{s}D_1)^{1-\eta} dF(D_1) K \left(\frac{\beta^b}{\beta^h}\right)^{1/\eta}}{\bar{s}[LHS'(\bar{s}) - RHS'(\bar{s})]}. \end{aligned} \tag{42}$$

Since by (37) $\bar{s}e_0^h - K(\beta^b/\beta^h)^{1/\eta} > 0$, $\frac{\partial q^{\bar{s}}}{\partial \beta^b} > 0$, so $q^{\bar{s}}$ increases and R declines. For the asset

[32] It is straightforward for $\eta \in (0, 1)$ since $q^{\bar{s}}$ is increasing in both β^b and $\bar{s}$.

price

$$\begin{aligned}
\frac{\partial p_0}{\partial \beta^b} &= E[D_1] + \frac{E[min(1,\bar{s}D_1)^{1-\eta}] - E[min(1,\bar{s}D_1)]}{\bar{s}} - \frac{\eta\beta^b}{\bar{s}^2}\int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)\frac{\partial \bar{s}}{\partial \beta^b} \\
&= \frac{1}{\bar{s}[LHS'(\bar{s}) - RHS'(\bar{s})]}\Bigg[\bigg(\int_{1/\bar{s}}^{\infty}(\bar{s}D_1 - 1)dF(D_1) + E[min(1,\bar{s}D_1)^{1-\eta}]\bigg)\bigg(e_0^h - \\
&\quad \frac{(1-\eta)K\beta^b}{\bar{s}}\int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)\bigg) - \\
&\quad \frac{\eta\beta^b}{\bar{s}}\int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)\bigg(KE[min(1,\bar{s}D_1)^{1-\eta}] + \frac{K}{\eta\beta^b}\bigg(\frac{\beta^b}{\beta^h}\bigg)^{1/\eta}\bigg)\Bigg] \\
&> \frac{1}{\bar{s}[LHS'(\bar{s}) - RHS'(\bar{s})]}\Bigg[\bigg(\int_{1/\bar{s}}^{\infty}\bar{s}D_1 dF(D_1) + \int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)\bigg)\bigg(e_0^h - \\
&\quad \frac{K\beta^b}{\bar{s}}\int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)\bigg) - \frac{K}{\bar{s}}\bigg(\frac{\beta^b}{\beta^h}\bigg)^{1/\eta}\int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)\Bigg] \\
&> \frac{\int_{1/\bar{s}}^{\infty}\bar{s}D_1 dF(D_1)\big(e_0^h - \frac{K\beta^b}{\bar{s}}\int_0^{1/\bar{s}}(\bar{s}D_1)^{1-\eta}dF(D_1)\big)}{\bar{s}[LHS'(\bar{s}) - RHS'(\bar{s})]} \\
&> 0.
\end{aligned} \tag{43}$$

The last two inequalities use the equilibrium condition (37), $\bar{s}e_0^h = K\beta^b E[min(1,\bar{s}D_1)^{1-\eta}] + K(\beta^b/\beta^h)^{1/\eta}$, when $s_0^h = 0$. So p_0 and $\bar{s}p_0$ increase. The haircut increases by (33).

For $s_0^h \in (0, K]$ and $\eta \in (0,1)$, total differentiating (37) and substituting it into the nominator, we have

$$\frac{\partial \bar{s}}{\partial \beta^b} = \frac{1}{\beta^b}\frac{\bar{s}e_0^h - K(\beta^b/\beta^h)^{1/\eta} + \frac{K}{\eta}(\beta^b/\beta^h)^{1/\eta}}{LHS'(\bar{s}) - RHS'(\bar{s})} > 0.$$

$q^{\bar{s}}$ increases since it is increasing in $\bar{s}$ for $\eta \in (0,1)$, as well as β^b. The haircut increases by (33). Given that $q^{\bar{s}}$ and H increase, $\bar{s}p_0$ must increase too.

Proof of Proposition 7:

When $\beta^b = \beta^h$, for the case $s_0^h \in [0, K)$, as η increases, the LHS of (37) and the RHS of (38) do not change, while the RHS of (37) increases and the LHS of (38) decreases, so $\bar{s}$ increases for any η. Then note that both of the two equations can be written as

$$q^{\bar{s}} = \frac{\bar{s}e_0^h + s_0^h\beta^b E[max(\bar{s}D_1 - 1, 0)] - K}{K - s_0^h}, \tag{44}$$

so $q^{\bar{s}}$ increases as the right hand is increasing in $\bar{s}$, and R declines. For the asset price, substituting (44) into (12)

$$p_0 = \frac{K}{K - s_0^h}\left(\beta^b E[max(D_1 - 1/\bar{s}, 0)] - \frac{1}{\bar{s}} + \frac{e_0^h}{K}\right), \tag{45}$$

so p_0 increases as the right hand is also increasing in $\bar{s}$, which implies λ^b increases. For the haircut, substituting (44) into (33), we have

$$\frac{\bar{s}p_0}{q^{\bar{s}}} = 1 + \beta^b(K - s_0^h)\frac{E[max(\bar{s}D_1 - 1, 0)]}{\bar{s}e_0^h - K + \beta^b s_0^h E[max(\bar{s}D_1 - 1, 0)]}$$

with $\frac{\partial(\bar{s}p_0/q^{\bar{s}})}{\partial \bar{s}} > 0$ if and only if

$$K\int_{1/\bar{s}}^{\infty} D_1 dF(D_1) < e_0^h \int_{1/\bar{s}}^{\infty} dF(D_1)$$

or $E[D_1|D_1 > 1/\bar{s}] < e_0^h/K$. So the haircut increases if and only if this inequality holds.

For the case $s_0^h = K$, η has no effect on either (37) or (38). So $\bar{s}$ remains the same. The bond price $q^{\bar{s}}$ increases by (21) driving down R, which also leads to a higher collateral value λ^b, by (22), and p_0, and a lower haircut by (33).

Proof of Proposition 8: The RHS of either (37) or (38) shifts downward, while the LHS is unaffected. So $\bar{s}$ decreases. Since $\frac{\bar{s}p_0}{q^{\bar{s}}}$ is increasing in $\bar{s}$ but decreasing in η, the haircut decreases. Since λ^b is decreasing in $\bar{s}$ but increasing in η, the collateral value and asset price increase. When $\eta \in (1, \infty)$, the bond price $q^{\bar{s}}$ is decreasing in $\bar{s}$ but increasing in η, the interest rate drops.

B. Robustness of the Equilibrium to the Wealth Distribution.

In this section, we show that the equilibrium established in the main context is robust to a heterogenous endowment distribution. Since banks have linear utility function, it does not matter for the equilibrium if they have different endowments at time 0 except for the banks' consumption flow. So we assume the banks still have the same endowments. However, if the households have different endowments in either the dividend or the risky asset at time 0, we would like to know how this may affect the equilibrium obtained before with homogenous agents.

Assume there are I types of household and each type i , $i = 1, \cdots, I$, is of mass m_i with $\sum_i m_i = 1$. Each type i household is endowed with e_{i0}^h units of dividend and s_{i0}^h units of risky asset at time 0 with $e_0^h \equiv \sum_i m_i e_{i0}^h = KD_0 - e_0^b$ and $s_0^h \equiv \sum_i m_i s_{i0}^h = K - s_0^b$. [33] It turns out that the equilibrium is largely independent of the endowment distribution across

[33] This can be extended to a continuum of types.

the households. Each household with whatever endowments will trade only one type of bond by Lemma 1. Moreover, all types of household with different endowments will trade the same type of bond-$\bar{s}$ because the collateral value in (22) is consistent with only a single collateral level $\bar{s}$. Each type i household will buy quantity $K_i/\bar{s}$ of bond-$\bar{s}$ such that all households have the same pricing kernel. Households with higher wealth buy more bonds. On the bank side, the collateral is allocated for trading with different types of household such that $K = \sum_i m_i K_i$.

Remark 2. *There exists a unique equilibrium in the economy $(I, \{m_i, e^h_{i0}, s^h_{i0}\}_{i=1}^I, e^b_0, s^b_0, K)$ in which the collateral constraint binds and a single bond-$\bar{s}$ is actively traded, i.e., $S^* = \{\bar{s}\}$, with $\bar{s} \in S$ determined by (23) where e^h_0 and s^h_0 are the aggregate dividend and risky asset endowments of households.*

The collateral value for the risky asset, the price for the risky asset, and price for bond-$\bar{s}$ are still given by (22), (12), and (21), respectively. For any bond-s, $s \in S \setminus \{\bar{s}\}$, the price q^s is undetermined and can be anything in the interval given by (24).

The allocations are

$$(c^h_{i0}, c^h_{i1}) = \left(e^h_{i0} + s^h_{i0}p_0 - \frac{K_i}{\bar{s}}q^{\bar{s}}, \frac{K_i}{\bar{s}}min(1, \bar{s}D_1)\right), \quad i = 1, \cdots, I$$
$$(c^b_0, c^b_1) = K(D_0, D_1) - \sum_i m_i(c^h_{i0}, c^h_{i1}),$$

where

$$\frac{K_i}{\bar{s}} = \frac{e^h_{i0} + s^h_{i0}p_0}{q^{\bar{s}} + (\beta^b/\beta^h)^{1/\eta}}.$$

The bond portfolio μ^h_i is a Dirac measure with mass $\frac{K_i}{\bar{s}}$ at $\bar{s} \in S$ and μ^b is a Dirac measure with mass $\frac{K}{\bar{s}}$ at $\bar{s} \in S$.

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