The Piketty Transition*

Daniel R. Carroll[†]
Research Department
Federal Reserve Bank of Cleveland

Eric R. Young[‡]
Department of Economics
University of Virginia

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Abstract

We study the effects on inequality of a "Piketty transition" to zero growth. In a model with a worker-capitalist dichotomy, we show that the relationship between inequality (measured as a ratio of incomes for the two types) and growth is complicated; zero growth can raise or lower inequality, depending on parameters. In particular, the elasticity of substitution between capital and labor in production needs to be considerably greater than 1 in order for income inequality be higher with zero growth; furthermore, the inequality effects operate through labor supply when the elasticity is high, rather than through $r - g$. Extending our model to include idiosyncratic wage risk we show that growth has quantitatively negligible effects on inequality, and furthermore the effect is negative rather than positive. Modifications designed to mitigate the decline in returns (such as financial development or capital-biased technical change) along the transition either strengthen or do not affect our conclusions.

1 Introduction

Thomas Piketty's *Capital in the Twenty-First Century* attacks the question of wealth inequality from two perspectives. The first is a monumental study of historical data, going back hundreds

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[†]Email: `Daniel.Carroll@clev.frb.org`.

[‡]Corresponding author. Email: `ey2d@virginia.edu`.

of years, that documents the dynamics of wealth inequality across several countries. There is no doubt that this data will be a fruitful source of material, and Piketty has graciously made the entire set publicly available for researchers to mine.

The second part is a rough sketch of an economic model that details the disastrous effects (as Piketty sees them) of low economic growth, in terms of ever-expanding inequality. Roughly speaking, Piketty's model has two groups of households, workers and capitalists, who derive all of their income from a single source (labor and capital, respectively); in this environment a natural measure of inequality is therefore capital's share of national income. Under some unusual assumptions about the form of the production function and the savings behavior of capitalists (see Krusell and Smith 2014), Piketty arrives at the conclusion that inequality will increase explosively as growth falls to zero. As noted in Krusell and Smith (2014), this result is sensitive to the unusual assumptions made, particularly to the behavioral assumption about constant net savings as a function of net output.

Our goal is to shed light on this prediction using (fairly) standard macroeconomic tools. The basic model of macroeconomic inequality is Aiyagari (1994) (with predecessors Bewley 1986, İmrohoroğlu 1989, and Huggett 1993), where *ex ante* identical households experience different realizations of their labor productivity and, as a result, accumulate differing amounts of wealth. This model has been successful at matching a large number of facts about US inequality, at least when extended in appropriate ways (Krusell and Smith 1998, Carroll 2001, Castañeda, Díaz-Giménez, and Ríos-Rull 2003). We use a variant of this model, extended to include a capitalist-worker dichotomy, to study how inequality would be expected to respond in the presence of declining growth.

Our basic model has the following ingredients. Some households, called capitalists, own claims to the productive technology while other ones, called laborers, do not; both types have an endowment of time that can be rented to firms in return for labor income. We first study a version of this model where workers do not use financial markets at all and idiosyncratic risk is absent. In this model, we can analytically characterize many features of the relationship between growth and inequality (measured as the ratio of capitalist income to laborer income). We use numerical examples to uncover the behavior of income inequality as the long run growth rate of the economy goes to zero.

We find that, contrary to Piketty's assertion, steady states with zero growth generally have lower inequality than steady states with positive growth rates. When growth is low, capitalists discount the future at a lower rate, and thus accumulate more capital, just as Piketty asserts; however, this accumulation leads to an abundance of capital relative to labor and results in a higher equilibrium wages, both absolutely and relative to capital's return. In terms of inequality, the movement in relative factor prices away from capital and toward labor mitigates the effects of increasing wealth. Generally, the factor price effect dominates the rise the wealth. Zero growth steady states are associated with higher inequality only when the elasticity of substitution in production between capital and labor is considerably greater than 1 (i.e., strong substitutes in production). A high elasticity of substitution mutes the factor price effect by preventing wages from becoming "too large" relative to the return to capital. The elasticity of substitution cannot be arbitrarily high, however. Conditional on a particular capital share in production, balanced growth places a limit on the elasticity of substitution: the higher is capital's share the lower is the upper limit on the elasticity of substitution. We find that the explosive increase in long run inequality described by Piketty only occurs when the elasticity of substitution is very close to its upper bound. Moreover, the values required for this explosion in inequality to obtain are starkly at odds with empirical measurements of the elasticity of substitution as well as estimates of capital's share of income; to be specific, if capital's share of income is 0.36 (as documented by Gollin 2002 for a large sample of countries) the required elasticity is 1.33, which exceeds nearly every estimate surveyed by Chirinko (2008).

We then turn to a risky environment in which (i) workers save via a return-dominated asset (money or stored consumption) and (ii) labor productivity is stochastic for both types. Thus, our model is a two-type version of the one discussed briefly in Krusell and Smith (2014), and we think it captures better the features that Piketty seems to have in mind. In this model we are able to measure inequality using standard concepts (in particular, Lorenz curves and Gini coefficients), and find that low growth involves small *decreases* in inequality. The mechanism is the same one identified in the simpler environment, because the asset supply is curve is nearly perfectly-elastic in the neighborhood of the equilibrium return. The reasons that the asset supply is very elastic are well known and elaborated formally in Aiyagari (1994). Roughly put, agents who are wealthy face little risk of becoming constrained in the future and are therefore "well-insured", leading to a decline in their precautionary savings. As r rises, more agents become well-insured and the economy approaches the perfectly-elastic complete market case. Elastic labor supply enhances this effect by insulating sufficiently rich agents from wage productivity variations.

The transition our model produces implies that returns to capital will fall over time as the capital-labor ratio rises with falling growth. Piketty maintains in his model that returns do not fall, so to assess his transition's effects we need a mechanism for preventing a decline in returns. As noted already, the basic model has little "room" for preventing declines in returns, because the wedge between the discount factor and the return to savings is small whenever idiosyncratic risk is unimportant (which is the case in our model if capitalists are very wealthy and do not work). We consider two possibilities here, namely financial innovation that eliminates idiosyncratic return risk for capitalists and capital-biased technical change. The first feature opens a wider gap between the return to saving and the time rate of preference by increasing the amount of precautionary savings. The second feature shifts the demand for capital to the right, counteracting the decline in returns associated with rising capital. The combination could generate a muted decline in returns, but does not.

2 Model

The model economy is populated by two groups, called capitalists and workers, who are situated in dynasties that live forever and value the utility of descendents; the size of the two groups are $\mu \in (0, 1)$ and $1-\mu$. Both groups face uninsurable random movements in the productivity of their labor effort; we remain agnostic as to the sources of these fluctuations (losing one job and finding one that pays less money, promotions, changes in ability across generations, etc.). Both groups also have identical preferences over consumption and leisure (non-work time), so that capitalists are not just "patient" people who got rich because they were thrifty. Our main assumption is that there is no mobility across groups – at some point in the infinite past, some dynasties were lucky enough to get granted access to a productive asset called capital, and some were not, and that situation has persisted.

We can represent the dynamic problem of a typical capitalist as

$$v(k,e) = \max_{k',h,c} \left\{ \frac{\left(c(1-h)^{\theta}\right)^{1-\sigma}}{1-\sigma} + \beta(1+g)^{1-\sigma} E\left[v\left(k',e'\right)\right] \right\} \tag{1}$$

$$c + (1+g)k' \leq (1+r)k + weh \tag{2}$$

$$k' \geq 0$$

$$h \geq 0$$

$$c \geq 0.$$

That is, the capitalist chooses consumption c, work effort h, and capital holdings k' to maximize lifetime utility; we have already incorporated growth in labor productivity g in the usual method to ensure the (normalized) wealth of the capitalist remains bounded over time (see King, Plosser, and Rebelo 1988 for details on how this normalization is done). Note the absence of insurance claims against e, the productivity of labor. As a result, there is a "precautionary saving" motive that leads capitalists to accumulate more capital than they normally would; however, this motive can disappear as the capitalist can choose to completely eliminate the risk by setting $h = 0$, and sufficiently wealthy capitalists will do so. We require that the coefficient of relative risk aversion/inverse of the intertemporal elasticity of substitution satisfies $\sigma > 0$, and will in general further assume $\sigma \geq 1$.[1]

[1] This assumption is uncontroversial for the risk aversion coefficient, but recently there has been an argument made that the IES is actually substantially above one as well (so that $\sigma < 1$ is appropriate). We could extend our model to permit Epstein-Zin preferences to allow both aspects of preferences to be separately matched; in our risk-free model this distinction does not exist, and quantities in models with EZ preferences generally are quite similar to expected utility.

The dynamic problem of a typical worker is

$$V(m,s) = \max_{m',l,x} \left\{ \frac{\left(x(1-l)^{\theta}\right)^{1-\sigma}}{1-\sigma} + \beta (1+g)^{1-\sigma} E\left[V\left(m',s'\right)\right] \right\} \tag{3}$$

$$x + (1+g)\, m' \leq \frac{m}{1+\pi} + wsl \tag{4}$$

$$m' \geq 0$$

$$l \geq 0$$

$$x \geq 0.$$

Note here the key difference: the return to the worker saving is, on net, negative (we suppose $\pi \geq 0$); while capitalists can earn a positive return by renting capital to firms, workers can only "store" their savings as money and thus lose purchasing power over time via inflation. One could just as easily imagine the workers saving in the form of inventories of goods that rot slowly over time. As above, there are no insurance claims against variations in s available.

We can obtain the aggregate capital stock and labor input by summing over all individuals. Let $\Gamma(k,e)$ be the density of capitalists across different levels of capital and productivity, and $\Upsilon(m,s)$ be the density of workers over money and productivity.

$$K = \int_k \int_e k\Gamma(k,e) \tag{5}$$

$$N = \int_k \int_e eh(k,e)\, \Gamma(k,e) + \int_m \int_s sl(m,s)\, \Upsilon(m,s)\,. \tag{6}$$

Note the asymmetry – capitalists supply all the capital, but labor is (at least in principle) supplied by both; note also that aggregate labor input is in terms of "effective" units of labor (hours weighted by productivity). The wage index w is then to be interpreted in the same way – one effective unit of labor earns w units of wage as compensation.

The supply side of our economy consists of a single firm employing a constant returns to scale production technology (nothing would change if we had a large number of identical firms, except notation would be more tedious):

$$Y = \left(\alpha K^{\nu} + (1-\alpha)\, N^{\nu}\right)^{\frac{1}{\nu}}, \tag{7}$$

where $\alpha \in (0,1)$ is the "share" of capital in production and $\nu \leq 1$ governs the elasticity of substitution. If $\nu = 1$, capital and labor are perfectly substitutable, so that the firm will employ only the cheaper factor. If $\nu = -\infty$, capital and labor are perfect complements, and therefore will be employed in fixed ratios (given by $\frac{\alpha}{1-\alpha}$). If $\nu = 0$, we get the Cobb-Douglas case where the shares of capital and labor income in total income will be fixed at α and $1 - \alpha$. Profit maximization yields

$$r = \alpha \left(\alpha + (1 - \alpha) \left(\frac{K}{N} \right)^{-\nu} \right)^{\frac{1-\nu}{\nu}} - \delta \tag{8}$$

$$w = (1 - \alpha) \left(\alpha \left(\frac{K}{N} \right)^{\nu} + 1 - \alpha \right)^{\frac{1-\nu}{\nu}} . \tag{9}$$

Note that both the rental rate and the wage rate are related to the capital-labor ratio, but not to the levels of capital and labor.[2]

Finally, there are aggregate conditions that relate supply and demand in each of three markets – the markets for capital, labor, and "goods". First, the firm must hire all the capital and labor supplied by households (these conditions are ensured by variations in r and w). Second, the supply of goods must be sufficient to cover the consumption of capitalists, the consumption of workers, and the investment by capitalists into new capital:

$$\int_k \int_e c(k,e)\, \Gamma(k,e) + \int_m \int_s x(m,s)\, \Upsilon(m,s) + G + \int_k \int_e \left(k'(k,e) - (1-\delta)\, k \right) \Gamma(k,e) = Y. \tag{10}$$

The term G denotes the loss of resources associated with worker saving (since $\pi \geq 0$, $G \geq 0$ can be interpreted as government consumption that is financed by seigniorage or as inventory adjustments depending on how one views the worker savings instrument). Walras's law ensures that the goods market condition will be satisfied provided both the labor and capital markets clear.

3 The Model without Idiosyncratic Risk

We first discuss a simplified version of the main model that can be analyzed without using numerical methods, in order to make the forces at play as transparent as possible and to highlight the role of key parameters. The model environment will be just like that from the main model

[2] The assumption that firms operate in perfectly competitive goods markets is not restrictive; assuming monopolistic competition does not change our results, it only changes the market clearing condition for capital by subtracting the value of profits from the savings of capitalists.

with one exception – there are no shocks to labor productivity (similar to Moll 2014). Denote by e and s the respective constant labor productivities for capitalists and laborers; we maintain that $e \geq s$, and generally will assume $e = s$.

3.1 Household Problems

3.1.1 Laborers

In the absence of risk, a laborer has no incentive to hold an asset which pays a rate of return strictly below the rate of time preference. Therefore money will not be held, and in each period the laborers will consume their earnings. A typical laborer's problem then is reduced to a static choice of how many hours, l, to supply at a given wage w:

$$V(K) = \max_{l} \left\{ \frac{\left[wls(1-l)^{\theta} \right]^{1-\sigma}}{1-\sigma} \right\}. \tag{11}$$

The solution is

$$l^{*} = \frac{1}{1+\theta} \tag{12}$$

$$x^{*} = \frac{ws}{1+\theta}. \tag{13}$$

3.1.2 Capitalists

A typical capitalist chooses consumption, hours, and savings to solve the dynamic program

$$v(k,K) = \max_{k',h,c} \left\{ \frac{\left[c(1-h)^{\theta} \right]^{1-\sigma}}{1-\sigma} + \beta (1+g)^{1-\sigma} v\left(k', K'\right) \right\} \tag{14}$$

subject to

$$c + (1+g)\,k' \leq whe + (1+r)\,k \tag{15}$$

$$k' \geq 0, c \geq 0, h \in [0,1).$$

Since the first two boundary conditions will never bind, we ignore them from now on. Taking the first-order conditions and applying the envelope condition produces three conditions:

$$\left[c(1-h)^{\theta} \right]^{-\sigma} = \beta (1+g)^{-\sigma} \left[c'\left(1-h'\right)^{\theta} \right]^{-\sigma} \left(1+r'\right) \tag{16}$$

$$\left[c\left(1-h\right)^{\theta}\right]^{-\sigma}\left[we\left(1-h\right)^{\theta}-\theta c\left(1-h\right)^{\theta-1}\right]\leq 0 \tag{17}$$

$$c+\left(1+g\right)k'=whe+\left(1+r\right)k; \tag{18}$$

the second condition holds with equality if $h>0$.

3.1.3 General Equilibrium

A recursive competitive equilibrium is a set of household functions

$$\left\{V\left(K\right),v\left(k,K\right),h\left(k,K\right),c\left(k,K\right),k'\left(k,K\right),l\left(k,K\right),x\left(k,K\right)\right\},$$

price functions $r\left(K\right)$ and $w\left(K\right)$, and aggregate labor $N\left(K\right)$ such that

1. Given pricing functions, the household functions solve the capitalist and laborer problems;

2. Given pricing functions the firm maximizes profit by demanding K and $N\left(K\right)$;

3. Markets clear:

$$k=\mu K$$

$$N\left(K\right)=\mu h\left(K,K\right)e+\left(1-\mu\right)l\left(K,K\right)s$$

$$Y\left(K\right)=\mu c\left(K,K\right)+\left(1-\mu\right)x\left(K,K\right)+\mu\left(1+g\right)k'\left(K,K\right)-\left(1-\delta\right)\mu K.$$

3.2 Steady State

The balanced growth path is characterized by the system of equations

$$1=\beta\left(1+g\right)^{-\sigma}\left(1+r\right) \tag{19}$$

$$h=\max\left\{\frac{we-\theta\left(r-g\right)k}{\left(1+\theta\right)we},0\right\} \tag{20}$$

$$c=whe+\left(r-g\right)k \tag{21}$$

$$l=\frac{1}{1+\theta} \tag{22}$$

$$x=w\left(r\right)ls \tag{23}$$

$$w\left(r\right)=\left(1-\alpha\right)^{\frac{1}{\nu}}\frac{r+\delta}{\alpha}\left[\left(\frac{r+\delta}{\alpha}\right)^{\frac{\nu}{1-\nu}}-\alpha\right]^{\frac{\nu-1}{\nu}}. \tag{24}$$

The steady state Euler equation pins down r,

$$r = \frac{(1+g)^{\sigma} - \beta}{\beta}, \tag{25}$$

as well as the steady state wage

$$w = (1-\alpha)^{\frac{1}{\nu}} \frac{r+\delta}{\alpha} \left[\left(\frac{r+\delta}{\alpha} \right)^{\frac{\nu}{1-\nu}} - \alpha \right]^{\frac{\nu-1}{\nu}}. \tag{26}$$

Notice that for $\sigma > 0$, the steady state interest rate is increasing in g. If we restrict attention to non-negative growth rates, the interest rate attains its minimum and the wage rate its maximum when $g = 0$, where the interest rate is

$$r_{\min} = \frac{1-\beta}{\beta}$$

and the wage is

$$w_{\max} = (1-\alpha)^{\frac{1}{\nu}} \frac{r_{\min}+\delta}{\alpha} \left[\left(\frac{r_{\min}+\delta}{\alpha} \right)^{\frac{\nu}{1-\nu}} - \alpha \right]^{\frac{\nu-1}{\nu}}. \tag{27}$$

Notice that while $r_{\min}$ is determined only by the discount factor, $w_{\max}$ also depends upon capital share in production, α, and the elasticity of substitution parameter, ν. Figure 1 plots the steady state wage when $g = 0$. For higher values of ν, the steady state wage increases exponentially, and the slope is increasing in α. Not all combinations of α and ν are permissible since

$$\alpha^{\frac{1}{1-\nu}} < (r_{\min} + \delta)^{\frac{\nu}{1-\nu}} \tag{28}$$

must hold for wages to be real numbers. Given (α, β, δ), the upper bound on ν is $\nu_{\max} = \frac{\log(\alpha)}{\log(r_{\min}+\delta)}$. In order to allow for capital and labor to be either complements or substitutes in production, we impose that $\alpha > \frac{1-\beta}{\beta} + \delta$, which implies $\nu_{\max} > 0$. Under the baseline calibration (see below), $r_{\min} \approx 0.0101$, implying that $\nu_{\max} \approx 0.305$. Thus, balanced growth puts a restriction on the degree to which capital and labor are substitutable; letting $\xi = \frac{1}{1-\nu}$ denote the elasticity, we find $\xi_{\max} = \frac{1}{1-0.305} = 1.438$.

Under appropriate conditions for α and ν, we can find K by imposing the the capital market clearing condition at r:

$$K = \left\{ \frac{\left[\frac{r+\delta}{\alpha}\right]^{\frac{\nu}{1-\nu}} - \alpha}{1-\alpha} \right\}^{-\frac{1}{\nu}} N$$
$$= \varphi N \tag{29}$$

where

$$N = \mu he + (1-\mu)\frac{1}{1+\theta}s. \tag{30}$$

Note here that φ is the the capital-to-labor ratio. Since under the restrictions on α and ν

$$\frac{d\varphi}{dr} = -\frac{1}{\alpha\ (1-\alpha)\ (1-\nu)} \left(\frac{\left(\frac{r+\delta}{\alpha}\right)^{\frac{\nu}{1-\nu}} - \alpha}{1-\alpha} \right)^{-\frac{1+\nu}{\nu}} \left(\frac{\alpha}{r+\delta}\right)^{\frac{1-2\nu}{1-\nu}} < 0 \tag{31}$$

and $\frac{dr}{dg} > 0$, φ rises as g falls. Thus, the capital-labor ratio will be higher in a low-growth economy. Finally, the steady state relative return on capital (as compared to human capital) $r - g$ falls if and only if

$$\sigma (1+g)^{\sigma-1} > \beta,$$

which, near $g = 0$, requires $\sigma \geq 1$. We have already noted that there is a case for $\sigma < 1$ (in this situation the IES interpretation is the only relevant one since risk is absent); estimates run from nearly zero (Hall 1988, Campbell 1999) to close to one for the subgroup of stock market participants (Guvenen 2004) to significantly above one (Vissing-Jorgensen 2002); given the appropriate estimates would apply to stockholders, the latter two estimates are likely the right ones, but in our quantitative work below we choose a compromise value of 0.5 because we do not wish to introduce preference heterogeneity.

Aggregate effective labor is a function of K because the capitalist hours decision depends upon wealth. If wealth is sufficiently high, then the non-negativity constraint on hours binds. We consider both the binding and non-binding cases below.

3.2.1 Case 1: Capitalists Work

Under the assumption that capitalists supply positive hours, we can obtain aggregate capital by substituting the definition of h into the market clearing condition for capital,

$$\begin{aligned} K &= \varphi N \\ &= \frac{\varphi\left(\frac{\mu e+(1-\mu)s}{1+\theta}\right)}{1+\varphi\frac{\theta}{1+\theta}\frac{(r-g)}{w}}. \end{aligned} \tag{32}$$

It can be shown that

$$\frac{w}{\varphi} = \left(\frac{r+\delta}{\alpha}\right)^{\xi} - (r+\delta) > 0, \tag{33}$$

where $\xi = \frac{1}{1-\nu}$ is the elasticity of substitution between capital and labor. The strict inequality results from imposing the restriction $\nu < \nu_{\max}$. Multiplying the numerator and denominator by $\frac{w}{\varphi}$, we arrive at

$$K = \frac{w\left[\mu e + (1-\mu)s\right]}{(1+\theta)\left[\left(\frac{r+\delta}{\alpha}\right)^{\xi} - (r+\delta)\right] + \theta(r-g)}.$$

Individual capitalist wealth is

$$\begin{aligned} k &= \frac{K}{\mu} \\ &= \frac{\left[\mu e + (1-\mu)s\right]}{\mu}\frac{w}{r-g}\frac{1}{(1+\theta)\frac{r}{r-g}z(g)+\theta}, \end{aligned}$$

where

$$z(g) = \frac{\left(\frac{r+\delta}{\alpha}\right)^{\xi} - (r+\delta)}{r} = \frac{\frac{w}{r}}{\frac{K}{N}} > 0. \tag{34}$$

Two of three factors in determining the behavior of steady state inequality, the factor price ratio and the capital-to-labor ratio, are expressed in the function $z(g)$. The sensitivity of the factor price ratio relative to the capital-to-labor ratio affects how inequality responds near zero growth; and this sensitivity depends directly upon the elasticity of substitution between capital and labor.

Substituting k into h,

$$h = \frac{1}{1+\theta}\left[1 - \frac{\mu e + (1-\mu)s}{\mu e}\frac{\frac{\theta}{1+\theta}}{\frac{r}{r-g}z(g) + \frac{\theta}{1+\theta}}\right]. \tag{35}$$

We can now derive steady state income inequality, ζ, measured by the ratio of capitalist's income, $y = whe + rk$, to laborer's income, $q = \frac{ws}{1+\theta}$:

$$\begin{aligned}\zeta &= \frac{y}{q} \\ &= \frac{e}{s} + \frac{\mu e + (1-\mu)\, s}{\mu s} Q\left(z\left(g\right);\theta\right)\end{aligned}$$

where

$$Q\left(z\left(g\right);\theta\right) = \frac{\frac{r}{r-g} - \frac{\theta}{1+\theta}}{\frac{r}{r-g} z\left(g\right) + \frac{\theta}{1+\theta}}.$$

Inequality increases as the measure of capitalists, μ, decreases. Holding all other parameters constant, the steady state Euler equation implies a unique capital-to-effective labor input ratio, and consequently w and r are invariant to μ. Because laborer's hours are constant, a lower μ necessarily implies higher effective labor supply. Therefore, K must rise in proportion to N, and so capitalists' wealth, $k = \frac{K}{\mu}$, also increases. Because factor prices do not change, laborer's income is constant, but capitalists' income increases.

Given population share, labor productivities, and preferences, the behavior of inequality fundamentally depends on the term $Q\left(z\left(g\right);\theta\right)$. Notice that $Q\left(z\left(g\right);\theta\right)$ is continuous in θ. We now state a series of propositions which characterize steady state income inequality conditional on g. In the interest of space, the proofs are assigned to an appendix.

Proposition 1. *Inequality is bounded from below by the productivity ratio* $\frac{e}{s}$.

It follows immediately from the non-negativity of Q, that inequality is increases with the capitalist-laborer productivity ratio, $\frac{e}{s}$. We assume that capitalists are at least as productive as laborers, and since $\frac{e}{s}$ is the lower bound, we can without loss of generality assume that $e = s = 1$. We do this in the remainder of the paper.

Proposition 2. *Holding* $z\left(g\right)$ *constant, inequality is increasing in* $\frac{r}{r-g}$.

As $g \to 0$, $\frac{r}{r-g}$ declines monotonically to 1 so it acts to reduce inequality in a zero growth steady state. Note that since $\frac{r}{r-g}$ can be written as

$$\frac{1}{1-\left(\frac{r}{g}\right)^{-1}},$$

the above proposition is the same as saying that inequality rises in $\frac{r}{g}$ (again ignoring $z(g)$), which is consistent with Piketty's claim that as the gap between r and g increases so does inequality. Of course, $\frac{r}{r-g}$ cannot move without changing $z(g)$ as well, since z is a function of both r and g. It is useful though to understand that if inequality *increases* as g goes to zero, it must result entirely from a decline in $z(g)$. The sign of $\frac{dz}{dg}$ is ambiguous, and so then is the sign of $\frac{dQ}{dg}$. In a later section, we use numerical methods to get a clearer picture of exactly how g affects ζ.

Proposition 3. *Steady state inequality is greater when the preference for leisure is weak (i.e., θ is small).*

This statement is proven by signing the derivative of Q with respect to θ and highlights the importance of the capitalists' hours decision for determining long run income inequality.

3.2.2 Case 2: Capitalists Do Not Work

The expressions are simpler when capitalists do not work. When $h = 0$, N is fixed at $\frac{1-\mu}{1+\theta}s$, and $K = \varphi\left(\frac{1-\mu}{1+\theta}s\right)$:

$$k = \frac{K}{\mu} = \frac{1-\mu}{\mu}\varphi\frac{s}{1+\theta},$$

so

$$\zeta = \frac{1-\mu}{\mu}\frac{\varphi}{w}r.$$

Substituting in (33), inequality can be written as a function of the rental rate and the model parameters,

$$\begin{aligned}\zeta &= \left[\frac{1}{\mu}-1\right]\frac{r}{\left(\frac{r+\delta}{\alpha}\right)^{\xi}-(r+\delta)} \\ &= \left[\frac{1}{\mu}-1\right]\frac{1}{z(g)} \\ &= \left[\frac{1}{\mu}-1\right]Q(z(g);0)\end{aligned}$$

Having solved for inequality in terms of Q, we can, for a fixed growth rate, bound steady state inequality and order it over θ.

Proposition 4. *For a given growth rate, g, $\zeta \in \left[\left(\frac{1}{\mu}-1\right)Q(z(g);0), 1+\frac{1}{\mu}Q(z(g);0)\right]$.*

Because $\varphi = \frac{K}{N}$, we can re-write $Q(z(g,0))$ as the capital-to-labor ratio divided by the factor price ratio.

$$Q(z(g,0)) = \frac{1}{z(g)} = \frac{\frac{K}{N}}{\frac{w}{r}} = \frac{rK}{wN}.$$

The market clearing conditions for capital and labor imply

$$\frac{r+\delta}{w} = \frac{MPK}{MPN} = \frac{1-\alpha}{\alpha}\left(\frac{K}{N}\right)^{\frac{1}{\xi}}.$$

This enables us to express the bounds on inequality in terms of the capital-to-labor ratio and the steady state interest rate,

$$\zeta \in \left[\frac{1-\mu}{\mu}\frac{\alpha}{1-\alpha}\chi\left(\frac{K}{N}\right)^{\left(1-\frac{1}{\varepsilon}\right)}, 1+\frac{1}{\mu}\frac{\alpha}{1-\alpha}\chi\left(\frac{K}{N}\right)^{\left(1-\frac{1}{\varepsilon}\right)}\right],$$

where $\chi = \frac{r}{r+\delta}$.

Proposition 5. *Steady state inequality is an increasing function of α and ξ.*

Finally, we state a necessary condition for steady inequality to be higher in a zero growth steady state than it is in a near zero growth steady state.

Proposition 6. *For $e = s$, if $\zeta(g) < \zeta(0)$, then the elasticity of substitution is greater than* 1*.*

Intuitively, this proposition says that in order for zero growth to lead to a steady state with greater inequality, factor prices $\frac{w}{r}$ must rise by less than $\frac{K}{N}$, which only happens if the elasticity of substitution between labor and capital is above 1. Additionally, positive depreciation increases the required degree of substitutibility. The tradeoff can be seen most easily when $z(g)$ is written as the ratio

$$z(g) = \frac{\frac{w}{r}}{\frac{K}{N}} = \frac{\frac{w}{r+\delta}}{\chi\frac{K}{N}}.$$

and $\xi = 1$ (i.e., the production function is Cobb-Douglas). In that case,

$$z(g) = \frac{(1-\alpha)}{\alpha}\frac{1}{\chi(g)}.$$

and

$$z(0) > z(g)$$

so $\zeta(g) > \zeta(0)$. Note that without depreciation, χ would be 1, and $z(g)$ would be constant which again would imply that $\zeta(g) > \zeta(0)$.

3.2.3 How Income Inequality Changes with Growth

In the steady state income inequality can be decomposed into the sum of two ratios,

$$\zeta \propto \frac{rk}{wls} + \frac{he}{ls}.$$

The first term is the ratio of capital income to laborer's income. If capitalists do not work, then income inequality is proportional the ratio of capital income to labor income in the economy. When capitalists supply positive hours, some algebra shows that

$$\begin{aligned}\zeta &\propto \frac{we - \theta\,(r-g)\,k + (1+\theta)\,rk}{ws} \\ &= \frac{1}{s}\left[e + \frac{r+\theta g}{w}k\right].\end{aligned}$$

Regardless of capitalists' hours, whether inequality rises or falls depends solely upon the product of wealth, k, and $\frac{r+\theta g}{w}$, which behaves in the same way as the factor price ratio $\frac{r}{w}$.

Because the closed-form expressions for steady state inequality do not yield unambiguous results for the effect of g on inequality, we use a computer to evaluate the expressions and plot the results for long run growth rates between 0 and 10 percent. To analyze the model numerically, we need to assign values to the structural parameters of the model. Here, we pick a reasonable set of values for some parameters, where reasonable means "gives rise to aggregates roughly consistent with US post-war averages." These numbers are $\beta = 0.99$, $\sigma = 2$, $\alpha = 0.36$, $\delta = 0.025$, $\theta = 1.25$, $g = 0.02$, and $e = s = 1$. Finally, to give Piketty's argument a stronger case, we set $\nu = 0.1$ ($\xi \approx 1.11$) so that capital and labor are more substitutable than the usual Cobb-Douglas case ($\xi = 1$); this value of ν satisfies the restrictions needed to have a steady state growth path.[3] Figures 2-3 show the steady state ratio of capitalist income to laborer income for growth rates between 0 and 10 percent for the baseline capital share of income in production and for a higher value.

In most cases, within a neighborhood of zero growth, inequality is falling rather than rising, even if capital and labor are substitutes. High inequality only obtains whenever the elasticity of substitution is close to the maximum conditional on α.

[3] Thus, our productivity growth should be interpreted as purely labor-augmenting or as exogenously-accumulating human capital (see King, Plosser, and Rebelo 1988); with Cobb-Douglas it does not matter whether the productivity growth affects capital, labor, or both.

For all of the parameter values, wealth decreases with growth. Figures 4-5 plot $k(g)$ for several values of ξ and of α. As the elasticity of substitution between capital and labor increases, the level of capital in the zero-growth steady state becomes very large, especially when capital's share, α, is high. If we ignored the general equilibrium effect on prices, savings behavior alone would suggest extreme inequality; however, the factor price ratio also responds to g. Because $w(r)$ is decreasing in r and r is increasing in g,

$$\frac{dw}{dg} < 0.$$

Therefore as the long-run growth rate declines so does the factor price ratio. Numerical results show that unless ξ is close to $\xi_{\max}(\alpha) = \frac{1}{1-\nu_{\max}}$, the declining factor price ratio more than offsets wealth near $g = 0$, so capital income to laborer income declines in the neighborhood of zero growth.

To make the key point clear, Figure (8) displays ζ in the steady state as a function of g; the kink occurs when the capitalist's labor supply hits zero. At higher levels of g, where, under the baseline calibration, capitalists supply positive hours, inequality is decreasing in g, but the effects are not very large. In this region, the rise in $\frac{r}{r-g}$ discussed in Proposition 2 is offset by a rise in $z(g)$. Remember that $z(g) = \frac{w}{r}/\frac{K}{N}$. By (31), we know that the denominator of $z(g)$ is falling. As capital becomes scarce relative to labor, the numerator falls as well. Because ξ is not sufficiently greater than 1 in the baseline, the factor price ratio declines faster than the capital-labor ratio. The dashed line below inequality, labeled $\zeta(h=0)$ in the figure, shows the behavior of $z(g)$ more clearly. It plots $\left[\frac{1}{\mu} - 1\right] Q(z(g);0)$, the appropriate measure when capitalists do not work, which only depends on $z(g)$. Notice that this line declines much faster in g than does actual inequality.

Continuing toward $g = 0$, we see that steady state income inequality rises slightly after the kink before descending sharply. To the left of the kink, capitalists do not work, so $\frac{r}{r-g}$ has no direct effect ζ. Initially, $z(g)$ is still falling and without a corresponding decline in $\frac{r}{r-g}$ income inequality rises. Eventually, as g moves closer to 0 and $\frac{K}{N}$ climbs steeply, the aggressive adjustment for the factor price takes over and $z(g)$ rises. As before, the dashed line below inequality shows a counterfactual measure, this time using the measure for when capitalists work. Notice that it falls more sharply than ζ, highlighting the combined effects of $z(g)$ and $\frac{r}{r-g}$ falling.

We also plot in Figure (9) the dynamics of ζ starting from the steady state with $g = 0.02$; inequality initially jumps up due to changes in labor supply, then declines monotonically as capital accumulates. In the first period, income inequality jumps for two reasons. First, because only capitalists supply labor elastically, the increase in N_t is due entirely to capitalists. The wage falls in response, but not in the same proportion as N_t rises, so capitalist's labor income increases. Second, although K_t is inelastic, r_t increases, pushing up capital income. Therefore, both sources of capitalist's income rise while laborer's income falls slightly because of lower wages. After the initial surge in inequality, capitalists accumulate wealth over time, so wages rise and the return on capital falls. In the new steady state, income inequality is well *below* its original level.

3.2.4 Piketty's Predictions in a Zero Growth Steady State

Piketty and Zucman (2014) predicts that a zero growth steady state will be associated with very high levels of capital relative to income, capital share of income, and income inequality. These predictions are based largely on strong assumptions about the elasticity of substitution between capital and labor and the capital share in production. As shown above, in order for extreme levels of income inequality, capital relative to output, and capital share of output, to appear as growth declines, the capital share and the elasticity of substitution in the production function must be high. In fact, given a capital share, α, these quantities only become extreme when ξ approaches the upper bound placed upon it by balanced growth. Figures 6-7 plot contour maps $\frac{K}{Y}$ and ζ, respectively, for (α, ξ) combinations when $g = 0$.[4] Consistent with Piketty and Zucman (2014), locations farther to the north and east (i.e., higher capital share in production and higher elasticity of substitution) are associated with both greater $\frac{K}{Y}$ (and, consequently, $\frac{(r+\delta)K}{Y}$). This pattern persists until the (α, ξ) combination violates the condition imposed by balanced growth, that is $\nu > \frac{\log(\alpha)}{\log(r_{\min}+\delta)}$, where $\xi > \xi_{\max}(\alpha)$. For instance, at $\alpha = 0.36$, the maximum elasticity of substitution is roughly 1.44.[5] At this maximum, $\frac{K}{Y}$ is 2.8 times larger than it is when the elasticity of substitution is 1. Likewise, capital share of income goes from 0.36 to 0.99. Notice, however, that for most of the permissible region, $\frac{K}{Y}$ is much lower and $\frac{(r+\delta)K}{Y}$ is not so dramatically high. At the same capital share, an elasticity of substitution of 1.2 produces a $\frac{K}{Y}$ only 60 percent

[4]Because when $g = 0$, $r = r_{\min}$, a plot of $\frac{(r+\delta)K}{Y}$ looks the same as a plot of $\frac{K}{Y}$.

[5]Gollin (2002) finds that capital's share of income is roughly one-third, once one takes careful account of self-employment income; see also Gomme and Rupert (2007).

larger than the ratio with unit elasticity, and $\frac{(r+\delta)K}{Y}$ is 0.57.[6] Piketty and Zucman (2014) suggest $\alpha = 0.21$, substantially smaller than conventional estimates. To reach extreme values of $\frac{K}{Y}$ and $\frac{(r+\delta)K}{Y}$ with this lower capital share of production, the elasticity of substitution must be 1.87! Likewise extreme levels of income inequality only occur in a region very close to the balanced growth boundary.

A natural question to ask is whether an elasticity of substitution of 1.44 is even "reasonable", let alone 1.87. Chirinko (2008) conducts a survey of estimates; of the 31 studies, only two support an elasticity above 1.5, and only three additional studies find evidence for an elasticity above one, while the median is significantly below one.[7] More recently, Karabarbounis and Neiman (2014), Rognlie (2015), and Semieniuk (2014) argue that Piketty overstates the aggregate elasticity of substitution and that the true value is likely less than one.[8]

3.3 Takeaways

This simplified two-household model has shown that the parameters that primarily govern the behavior of inequality in a zero growth steady state are related to production. The capital share and the elasticity of substitution between capital and labor control how quickly both the steady state wage rate and wealth rise as g nears zero. In addition, they also change the response of hours. In general, steady state hours are higher when $g = 0$, but if both α and ν are sufficiently high, hours are lower (perhaps zero) in low growth steady states and rise as g increases.

In all cases, steady state inequality is higher when $g = 0$ than it is under a positive growth rate. The cause for the rise, however, depends upon the parameters. Generally, it is the result of capitalists supplying more hours. In fact, capitalist's income from wealth relative to laborer's income declines as growth nears zero. Only when capital's share of production and the elasticity

[6] For reference, in the 2013 wave of the Survey of Consumer Finances, the ratio of average real income of the top quintile to the median is about 5.84. Income in this calculation is measured by the SCF variable "INCOME."

[7] Palivos (2008) provides two additional references that find elasticities above one, based on abandoning the CES structure in favor of a production function with a variable elasticity. These elasticities are just barely above one, however. The most extreme estimate in Chirinko (2008) is 3.4, based on Mexican data and long-run tax reforms; it is unclear whether such an estimate should be applied to the questions at hand.

[8] Rognlie (2015) focuses on the distinction between gross and net elasticities, showing that net elasticities are always smaller; this result implies that Piketty is overstating the elasticity that applies to his model (which is the net elasticity). This distinction is not important for our purposes.

of substitution are high does a rise in capital income relative to laborer's income account for high inequality. Finally, steady states with extreme values of capital to income, capital share of income, and income inequality only arise when the elasticity of substitution between capital and labor and capital share in production are jointly very near to values which violate balanced growth.

In our view, though, this model provides a view of inequality that is somewhat misleading – within group inequality is also important, particularly for discussions of the so-called 1 percent. For evidence, we point to the fact that capital income as a share of total income varies substantially across individuals and capital and labor income are positively correlated (see Table 1 in Carroll and Young 2009 or Tables 4 and 5 in Budría Rodríguez *et al.* 2002). Furthermore, there is substantial mobility in earnings and wealth (see Tables 15 and 16 Budría Rodriguez *et al.* 2002 or Tables 2 and 3 in Carroll, Dolmas, and Young 2014). Thus, there an empirical argument for introducing idiosyncratic risk back into the model.

There is also a theoretical argument for the introduction of idiosyncratic risk. As we showed above, without risk r and g are inextricably linked through Equation (25), so that changes in g simply cannot involve constant r (except in the unreasonable case of $\sigma = 0$). Idiosyncratic risk will introduce a wedge between discounting and returns operating through precautionary savings, which potentially allows us to consider the possibility that returns will remain high along the transition.

4 The Model with Idiosyncratic Risk

We now suppose that e and s follow identical, highly persistent AR(1) processes in logs:

$$\log\left(e'\right) = 0.95\log\left(e\right) + 0.1\eta'$$

where η' is a standard normal random variable. The definition of equilibrium for this model is a straightforward extension of the model without idiosyncratic risk and is omitted.[9] We set $\pi = 0.02$, equal to the average inflation rate in the postwar US.

[9] There is significant debate over the appropriate form for the labor productivity process, as well as the value of the coefficients. Our results are not sensitive to changes in the parameters of the process for e (and s). They are also not sensitive to allowing capitalists and workers to have different mean productivities.

Due to the special relationship between r and w we can solve this model by finding a single number, namely the return r, such that at that given rate the household's supply of capital and labor, if hired entirely by the firm, lead to a marginal product of capital (net of depreciation) equal to r itself (that is simply Equation 8). We can draw a picture of the steady state as the intersection of the "demand curve" corresponding to the right-hand-side of Equation 8 and a "supply curve" that links the aggregate capital/labor ratio (as chosen by households) to the return; see Figure 10.[10]

We focus on measuring inequality using Lorenz curves and the Gini coefficient. We present in Figure 11 the Lorenz curves both for our model and for the recent US, using the Survey of Consumer Finances 2007 sample. Our model does a reasonable job of fitting the US Lorenz curve (see Figure 11); the middle part would be matched better if we allowed occasional transits between capitalists and workers. Our model is also consistent with mean wealth – the ratio of K/Y is 6 in the benchmark, consistent with the measurement in McGrattan and Prescott (2013) as well as the number used by Piketty (2014).

Making comparisons across Lorenz curves is difficult, since they could cross multiple times. When making comparisons we will use the Gini coefficient, which is obtained by integrating the area between the perfect-equality line and the actual Lorenz curve. Larger Gini coefficients translate into more unequal distributions. In terms of Gini coefficients, our model does a reasonable job reproducing the extreme inequality observed in the US – our Gini coefficient is even slightly larger, at 0.84, than the US at 0.8. Comparing the two model curves we see that inequality actually drops as g goes to zero, at least in the long run, but not much – the dashed-line curve lies everywhere above the solid one but they are quite close together (the new Gini coefficient is 0.83). Thus, Piketty's prediction of explosive inequality, at least if measured in the conventional way, is not consistent with our model. However, the fact that r drops significantly is also not consistent with Piketty's maintained hypothesis. r drops because the capital-output ratio roughly doubles while aggregate labor input remains roughly constant, leading to a large increase in the capital/labor ratio and a concomitant decline in returns.[11]

[10]We use standard numerical methods to solve for the steady state and the transitional dynamics; a technical appendix outlines the details and is available upon request.

[11]Krusell and Smith (2014) also find small effects of g on inequality, although their model features only one type of household and shocks to household discount factors drive much of the inequality. The underlying reasons are

4.1 Transition to Zero Growth

Because the long run comparisons could be misleading if inequality temporarily rises or r takes a long time to fall, we explicitly compute the transition path as the economy moves from the initial growth path with $g = 0.02$ to the one with $g = 0$. We focus on four variables because these are the ones Piketty highlights, namely the capital/output ratio, the return to capital, capital's share of income, and the Gini coefficient on wealth. As seen in Figure 12, the transition takes over 100 years to complete for the mean capital stock (which is all that matters for r and $\frac{K}{Y}$), and takes even longer for the Gini coefficient (these two results are manifestations of an approximate aggregation property of this model, as discussed in Krusell and Smith 1998, namely that higher moments of the distribution of wealth do not materially affect prices). The Gini coefficient first drops a bit, then recovers, but the quantitative size of the movements are small, meaning that the steady state is not hiding substantial inequality dynamics.

5 Preventing Declines in r

Clearly, our model does not reproduce the transition that Piketty envisions – while $\frac{K}{Y}$ rises significantly, r falls and therefore $\frac{rK}{Y}$ increases but not substantially. In contrast, Piketty maintains that r will not fall, meaning that the increase in $\frac{K}{Y}$ will translate directly into an increase in $\frac{rK}{Y}$. It is clear from inspecting Figure 10 that the model cannot reconcile a decline in g with a constant r, since there is no "room" between r and the effective discount factor of the capitalists $(\beta (1 + g)^{-\sigma})$ – if g drops by a nontrivial amount (say, from the postwar average of 2 percent to zero), r must fall in the new steady state. We therefore consider some changes to the benchmark model that either (i) open a wider gap between time rates of preference and returns or (ii) shift the demand for capital outward.

5.1 Financial Innovation/Capital-Biased Technical Change

Piketty suggests a number of possible mechanisms that would prevent r from falling (or even cause it to increase). We pick one of these proposed mechanisms here – financial innovation:

> In addition, capital markets may become more and more sophisticated and more and more

the same as ours, though.

"perfect" in the sense used by economists (meaning that the return on capital will become increasingly disconnected from the individual characteristics of the owner). (Piketty 2014, page 376).

We explore this idea by assuming that, in the initial steady state, the capitalist is exposed to an iid idiosyncratic shock to end-of-period wealth u, changing his program to

$$v(k,e) = \max_{k',h,c} \left\{ \frac{\left(c(1-h)^{\theta}\right)^{1-\sigma}}{1-\sigma} + \beta(1+g)^{1-\sigma} E\left[v\left(u'k',e'\right)\right] \right\}$$

$$\begin{aligned} c + (1+g)\,k' &\leq (r+1-\delta)\,k + weh \\ k' &\geq 0 \\ h &\geq 0 \\ c &\geq 0. \end{aligned}$$

We set the standard deviation of u to be 0.1, equal to the innovation variance of the wage shock. Figure 13 shows that the initial steady state now has a substantially lower r and a larger gap between the discount factor and the return (a symptom of market incompleteness). As discussed in Mendoza, Quadrini, and Ríos-Rull (2009), a decline in idiosyncratic risk will make the asset supply curve shift to the left as precautionary motives are blunted, a force which will increase r and work against the decline in g.[12]

The transition with financial innovation looks very similar qualitatively to the benchmark model, with one clear exception – there is now a substantial *decrease* in the Gini coefficient on wealth. Thus, inequality as measured by Piketty ($\frac{rK}{Y}$) and by standard measures (Gini) move in opposite directions. Figures 14 and 15 show the Lorenz curves and transitional dynamics; the decrease in inequality occurs because there is less inequality in wealth within the capitalist population.[13]

[12]This setup is not isomorphic to an entrepreneurial economy with persistent productivity shocks, such as the model used by Cagetti and DeNardi (2008) to study wealth inequality, but it captures enough of the critical details to make our point here.

[13]The "blip" in the dynamic path for the Gini coefficient is not a numerical artifact – it is an upward jump followed by a quick but continuous decline back to the previous transition path. We have not been able to figure out where this blip comes from, but it clearly does not affect the results we emphasize.

We also consider an increase in α, a form of capital-biased technical change; specifically, we consider what happens if α increases to 0.45 at the same time the variance of u goes to zero and g goes to zero.[14] Figure 16 shows the rightward shift in the demand curve that a rise in α generates. Thus, a combination of the two forces – financial innovation and capital-biased technical change – could result in a small (or even zero) decline in r; however, as noted earlier, the near-infinite elasticity of asset supply near the equilibrium return makes a decline in r inevitable. The result is that changes in α have little effect.

Piketty suggests a number of other alternatives. A "race to the bottom" in capital taxation would lead to the effective return to capital rising. An increasing returns to scale technology for financial management would lead to wealthy agents receiving higher returns on their saving, as would a risk-return tradeoff in a model with multiple risky assets. Finally, inflation increases would raise the relative return to capital by reducing the return on worker saving. We do not examine these changes explicitly here. A reduction in a flat capital income tax has little effect on inequality (Carroll, Dolmas, and Young 2014). The second and third alternatives would lead to transitory changes only, since the long-run return would still be determined by a version of Figure 10.[15] Finally, changing π has no quantitative effect in our model, although it does operate in the right direction – while a rise in π leads to increasing labor supply by workers and therefore a higher marginal product of capital the effect is very small because workers already hold very little assets.

Rather than elaborate formally some additional alternatives that did not work, we will simply note for the reader that we also considered a two-sector economy where workers saved through capital in a sector with low capital efficiency (giving them low returns) and a model where workers simply own inefficient capital. Neither change made any change worth devoting space to; results can be obtained upon request of the corresponding author.

[14] Remember that the size of α is restricted, so 0.45 is almost as large as we can make it, given the value of ν, without destroying the balanced-growth property of the model.

[15] McKay (2013) develops a model of search and returns that leads to small inequality effects in a life-cycle model when calibrated to match the median fee for financial management and the average time spent by households on financial management.

6 Conclusion

We have not engaged Piketty's policy suggestions in this paper. He suggests that a tax on wealth, particularly inherited wealth, will be needed to defend society against the corrupting influence of the explosion in inequality. Recently optimal tax theorists have studied the nature of optimal taxation when redistribution is a concern and growth is small (see the references in Farhi and Werning 2014), finding that the case for a progressive wealth tax relies on very specific assumptions about how individuals value their children. We leave (as Farhi and Werning themselves do) to others careful scrutiny of those assumptions empirically.

One could easily think about optimal allocations for our model. In our setup, the reasons that households cannot borrow and cannot buy contingent claims are not specified explicitly. If we assume that, whatever these factors are, they apply also to the government, then the government cannot transfer resources across individuals or over time (no insurance and no debt), as in Dávila *et al.* (2012). In that paper, which does not feature either labor effort or differentiated access to capital markets, ever-increasing inequality and an enormous increase in $\frac{K}{Y}$ turns out to be optimal, because the increase in capital supports the wages of the poor. Clearly, this setup does not capture Piketty's policy prescriptions, which involve large transfers to the workers financed by progressive capital taxation. Whether such a policy is optimal in this model is something we are currently exploring.

We also do not address the political implications of inequality, which arguably is the main concern Piketty raises. It is feasible in our model to study political equilibria, using the machinery developed in Carroll, Dolmas, and Young (2014), Corbae, D'Erasmo, and Kuruşçu (2009), and Bachmann and Bai (2014). If wealth raises the voting power of individuals, extreme inequality could lead to a heavily distorted tax system that reinforces the inequality. While we find some questions interesting, they lie well beyond our goals for this paper.

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7 Appendix: Proofs

Proof of Proposition 1

Proof. Because $e, s > 0$ and $\mu \in (0,1)$, we only need to show that

$$Q\left(z\left(g\right);\theta\right) > 0.$$

Notice first, that $\frac{\theta}{1+\theta} \in [0,1)$. If $\sigma \geq 1$, then the $\min \frac{r}{r-g} = 1$ for non-negative growth rates, since the $\lim_{g\to\infty} \frac{r}{r-g} = 1$. Finally, $z\left(g\right) > 0$, so for finite θ, $Q\left(z\left(0\right);\theta\right) > 0$, and $\min \zeta = \frac{e}{s}$. □

Proof of Proposition 2

Proof. It is immediate that

$$\begin{aligned}
\frac{dQ}{d\left[\frac{r}{r-g}\right]} &= \frac{\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta} - z\left(g\right)\left(\frac{r}{r-g} - \frac{\theta}{1+\theta}\right)}{\left[\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}\right]^2} \\
&= \frac{\frac{\theta}{1+\theta} + z\left(g\right)\frac{\theta}{1+\theta}}{\left[\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}\right]^2} \\
&> 0.
\end{aligned}$$

□

Proof of Proposition 3

Proof. A larger θ implies less willingness to work on the part of households. The steady state interest rate and $z\left(g\right)$ are independent of θ. Then

$$\begin{aligned}
\frac{dQ}{d\theta} &= \frac{dQ}{d\frac{\theta}{1+\theta}}\frac{d\frac{\theta}{1+\theta}}{d\theta} \\
&= \left[-\frac{\frac{r}{r-g}z\left(g\right) + \frac{r}{r-g}}{\left(\frac{r}{r-g}z\left(g\right) + \frac{\theta}{1+\theta}\right)^2}\right]\frac{1}{\left(1+\theta\right)^2} \\
&< 0
\end{aligned}$$

Inequality rises as θ falls. □

Proof of Proposition 4

Proof. Consider $\bar{\theta}$ such that for all $\theta > \bar{\theta}$, $h = 0$, and all $\theta \leq \bar{\theta}$, $h > 0$. Let $0 < \theta_1 < \bar{\theta} < \theta_2$. Then since $\frac{\theta}{1+\theta} \in [0, 1)$ is continuous, strictly monotonic, and decreasing in θ, and since Q is decreasing in θ,

$$\begin{aligned}
\zeta_{\theta_2} &= \left[\frac{1-\mu}{\mu}\right] Q\left(z\left(g\right);0\right) \\
&\leq \zeta_{\bar{\theta}} \\
&= 1 + \frac{1}{\mu} Q\left(z\left(g\right);\bar{\theta}\right) \\
&< 1 + \frac{1}{\mu} Q\left(z\left(g\right);\theta_1\right) \\
&< 1 + \frac{1}{\mu} Q\left(z\left(g\right);0\right) \\
&= 1 + \frac{1}{1-\mu} \zeta_{\theta_2},
\end{aligned}$$

where the first equality holds because $h = 0$ at θ_2, and the second line follows from the continuity of Q in θ. Because capitalists do not work in a steady state with $\theta > \bar{\theta}$, inequality will be unaffected by increasing θ beyond $\bar{\theta}$. Therefore, for any $\theta \geq 0$, ζ_θ is bounded below by the $\left[\frac{1-\mu}{\mu}\right] \frac{1}{z(g)}$ and above by $1 + \frac{1}{\mu}\frac{1}{z(g)}$. □

Proof of Proposition 5

Proof. Fix θ. Then

$$Q\left(z\left(g\right);\theta\right) = \frac{\frac{r}{r-g} - \frac{\theta}{1+\theta}}{\frac{r}{r-g} z\left(g\right) + \frac{\theta}{1+\theta}}$$

Because steady state r is invariant to changes in α and ξ, all that matters for inequality is $z\left(g\right)$.

$$\begin{aligned}
z\left(g\right) &= \frac{\left[\left(\frac{r(g)+\delta}{\alpha}\right)^{\xi} - \left(r\left(g\right)+\delta\right)\right]}{r\left(g\right)} \\
&= \frac{1}{\chi} \frac{\left[\alpha^{-\xi}\left(r\left(g\right)+\delta\right)^{\xi} - \left(r\left(g\right)+\delta\right)\right]}{r\left(g\right)+\delta} \\
&= \frac{1}{\chi}\left[\alpha^{-\xi}\left(r\left(g\right)+\delta\right)^{\xi-1} - 1\right].
\end{aligned}$$

Because $\xi \geq 0$, a greater capital's share decreases z and increases steady state inequality. Likewise steady state inequality will be greater for larger elasticities of substitution since

$$\frac{dz\left(g\right)}{d\xi} = \frac{1}{\chi}\left\{-\alpha^{-\xi}\left(r\left(g\right)+\delta\right)^{\xi-1}\log\left(r\left(g\right)+\delta\right)\left[1 - \frac{\log\left(\alpha\right)}{\log\left(r\left(g\right)+\delta\right)}\right]\right\} < 0$$

where the strict inequality holds since $r+\delta<\alpha<1$. $\square$

Proof of Proposition 6

Proof. First, in order for $\zeta(g)<\zeta(0)$, $z(0)<z(g)$. There are four cases to consider: (1) $h(g)=0,\ h(0)=0$; (2) $h(g)>0,\ h(0)>0$; (3) $h(g)>0,\ h(0)=0$; and (4) $h(g)=0,\ h(0)>0$. It will be shown that the fourth case cannot obtain. The first case is obvious since $\zeta(g)<\zeta(0)$ implies

$$\frac{1}{z(g)}<\frac{1}{z(0)}$$

since z is non-negative. For the second case, $\zeta(g)<\zeta(0)$ implies

$$\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g)+\frac{\theta}{1+\theta}}<\frac{1-\frac{\theta}{1+\theta}}{z(0)+\frac{\theta}{1+\theta}}.$$

By Proposition 2,

$$\frac{1-\frac{\theta}{1+\theta}}{z(g)+\frac{\theta}{1+\theta}}<\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g)+\frac{\theta}{1+\theta}}$$

so

$$\frac{1-\frac{\theta}{1+\theta}}{z(g)+\frac{\theta}{1+\theta}}<\frac{1-\frac{\theta}{1+\theta}}{z(0)+\frac{\theta}{1+\theta}}.$$

For the third case,

$$1+\frac{1}{\mu}\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g)+\frac{\theta}{1+\theta}}<\left(\frac{1-\mu}{\mu}\right)\frac{1}{z(0)}.$$

If this is true, then by Proposition 2 and Proposition 3

$$1+\frac{1}{\mu}\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g)+\frac{\theta}{1+\theta}}<1+\frac{1}{\mu}\frac{1-\frac{\theta}{1+\theta}}{z(0)+\frac{\theta}{1+\theta}}<1+\frac{1}{\mu}\frac{\frac{r(g)}{r(g)-g}-\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(0)+\frac{\theta}{1+\theta}}.$$

Now for the final case, which we will show implies a contradiction. From (35),

$$h=\max\left\{0,\frac{1}{1+\theta}\left[1-\frac{1}{\mu}\frac{\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g)+\frac{\theta}{1+\theta}}\right]\right\}.$$

Because $h(g)=0$ and $h(0)>0$

$$\frac{1}{\mu}\frac{\frac{\theta}{1+\theta}}{\frac{r(g)}{r(g)-g}z(g)+\frac{\theta}{1+\theta}}>1>\frac{1}{\mu}\frac{\frac{\theta}{1+\theta}}{z(0)+\frac{\theta}{1+\theta}}$$

so

$$z(0) > \frac{r(g)}{r(g) - g} z(g) > z(g).$$

We can see from the (35), for each growth rate, there is a $\bar{\theta}(g)$ such that

$$\begin{aligned} h(g) &= 0 \ \forall\ \theta > \bar{\theta}(g) \\ &> 0 \ \forall\ \theta < \bar{\theta}(g). \end{aligned}$$

In order for $h(0) > h(g) = 0$ at θ, it would have to be the case that $\bar{\theta}(g) < \bar{\theta}(0)$, and $\theta \in \left[\bar{\theta}(g), \bar{\theta}(0)\right]$. Additionally, $\zeta(g)$ must be at its minimum

$$\zeta(g) = \left(\frac{\mu - 1}{\mu}\right) \frac{1}{z(g)}$$

while

$$\zeta(0) > \left(\frac{\mu - 1}{\mu}\right) \frac{1}{z(0)}.$$

We know from case (1), that $\min[\zeta(g)] < \min[\zeta(0)]$, which implies that $z(g) < z(0)$. Because z is invariant to θ, it cannot be the case that $z(0) > z(g)$, so case (4) cannot obtain. Simply put, $\bar{\theta}(0) < \bar{\theta}(g)$. Therefore whenever $\zeta(g) < \zeta(0)$, $z(g) > z(0)$. Now to close the proof

$$z(g) = \frac{1}{\chi(g)} \left[\alpha^{-\xi} (r(g) + \delta)^{\xi - 1} - 1\right]$$

where again $\chi = \frac{r(g)}{r(g)+\delta}$. If $\zeta(0) > \zeta(g)$, then

$$\frac{1}{\chi(0)} \left[\alpha^{-\xi} (r(0) + \delta)^{\xi - 1} - 1\right] < \frac{1}{\chi(g)} \left[\alpha^{-\xi} (r(g) + \delta)^{\xi - 1} - 1\right]$$

Because χ is increasing in g,

$$(r(0) + \delta)^{\xi - 1} < (r(g) + \delta)^{\xi - 1}$$

Now since $r(g) > r(0) = r_{\min}$, this condition can only hold if $\xi > 1$. $\square$

Figure 1: Equilibrium Wage in $g = 0$ Steady State

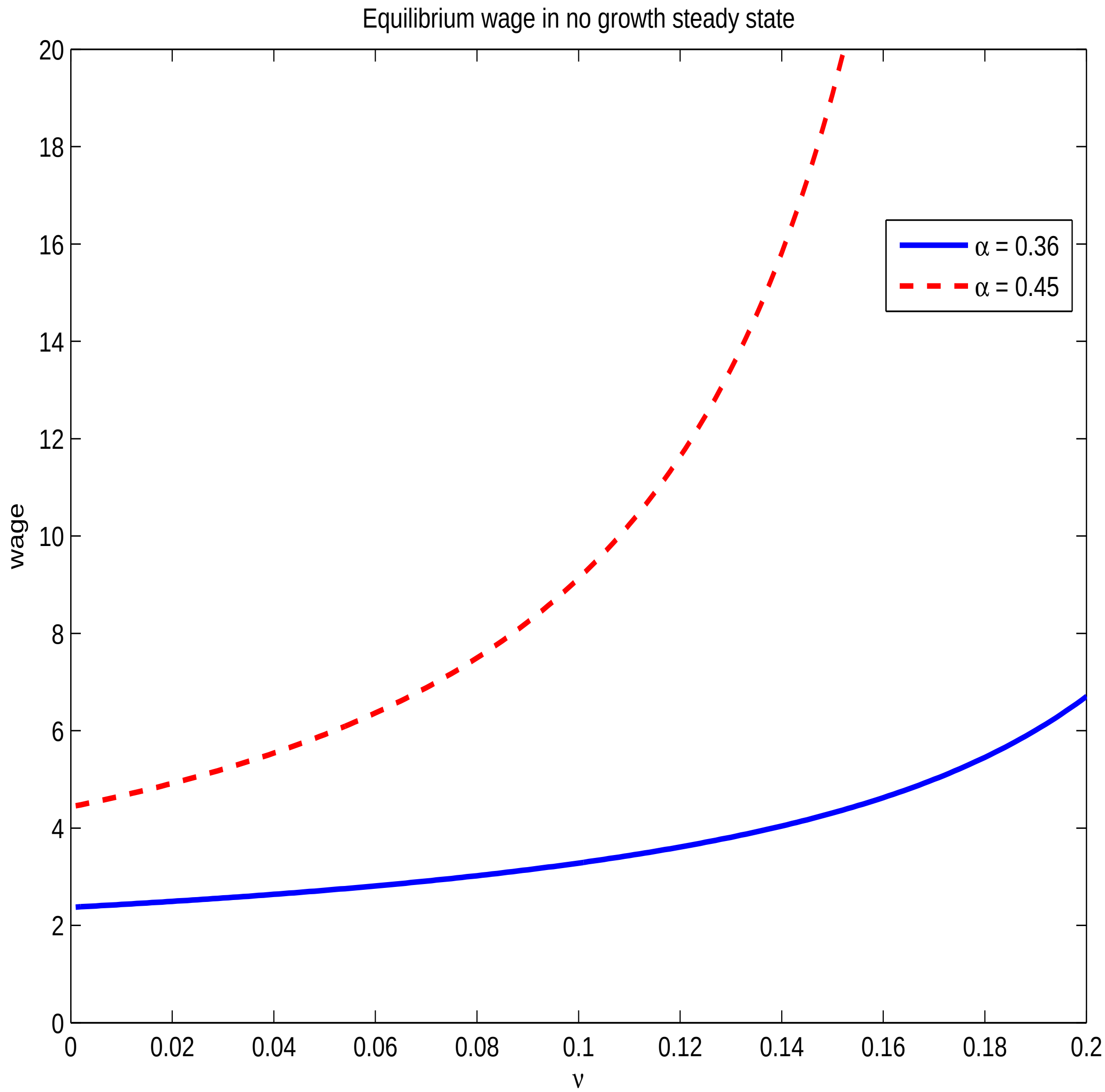

Figure 2: Steady State Inequality

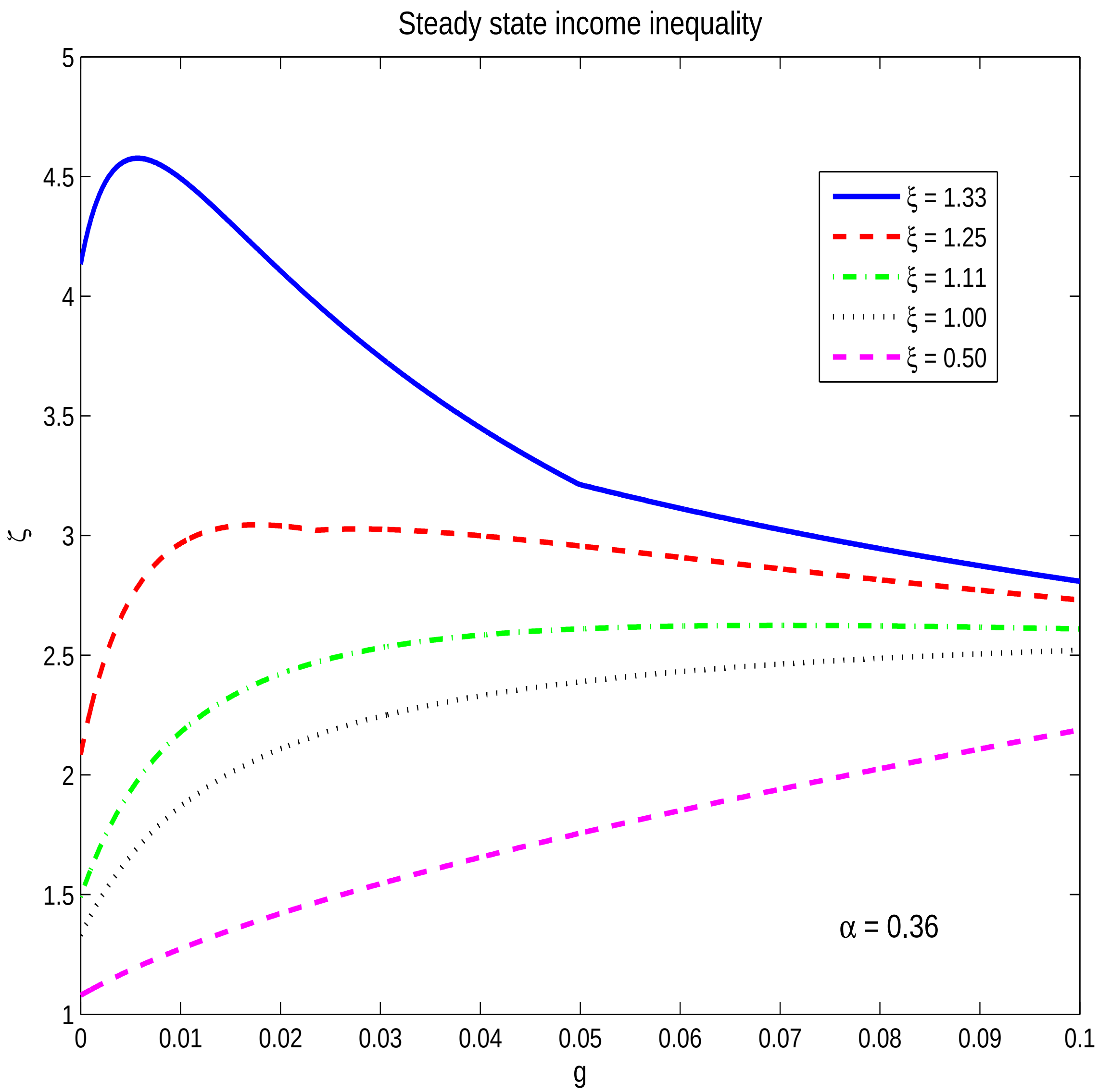

Figure 3: Steady State Inequality

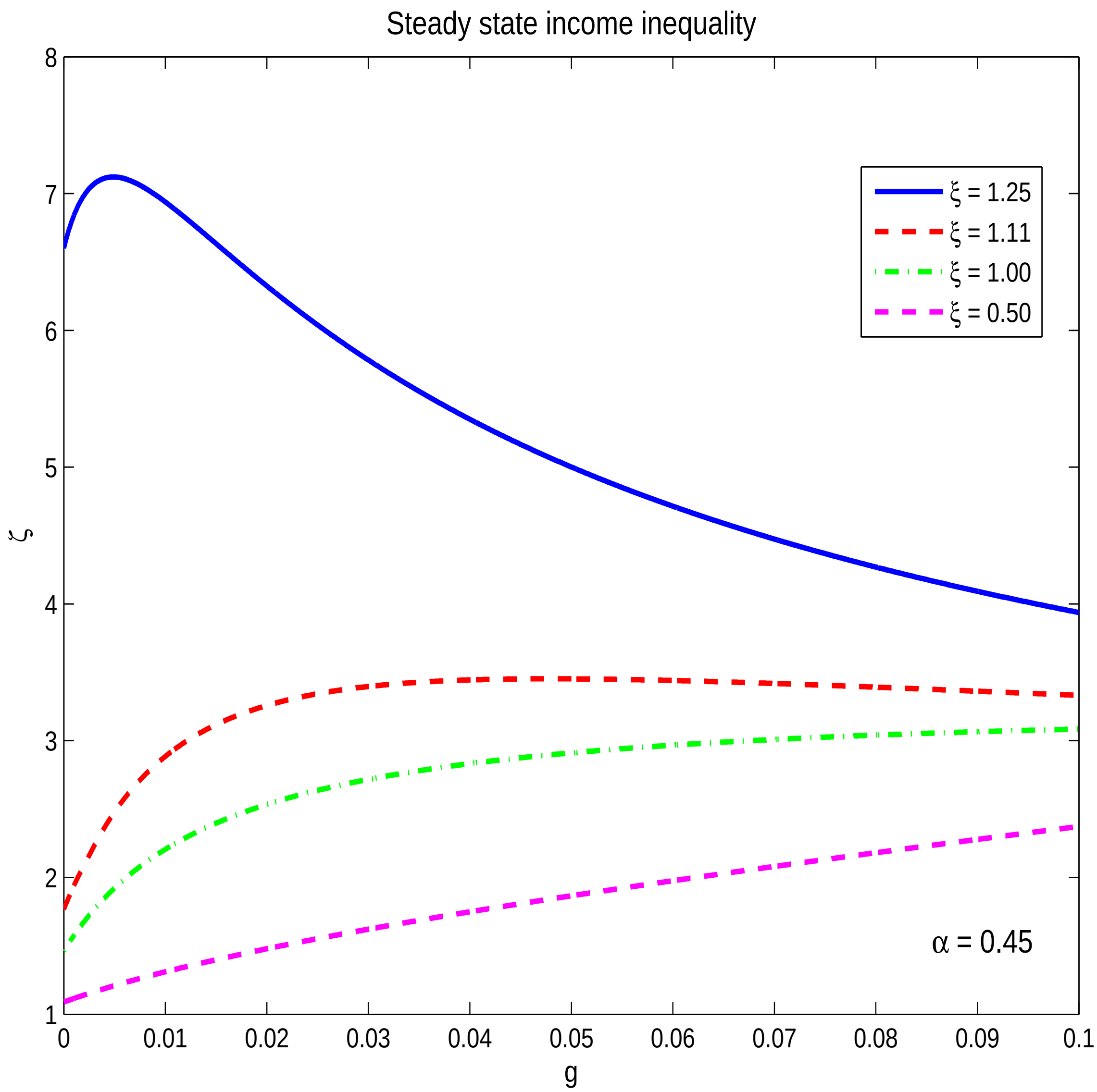

Figure 4: Steady State Wealth

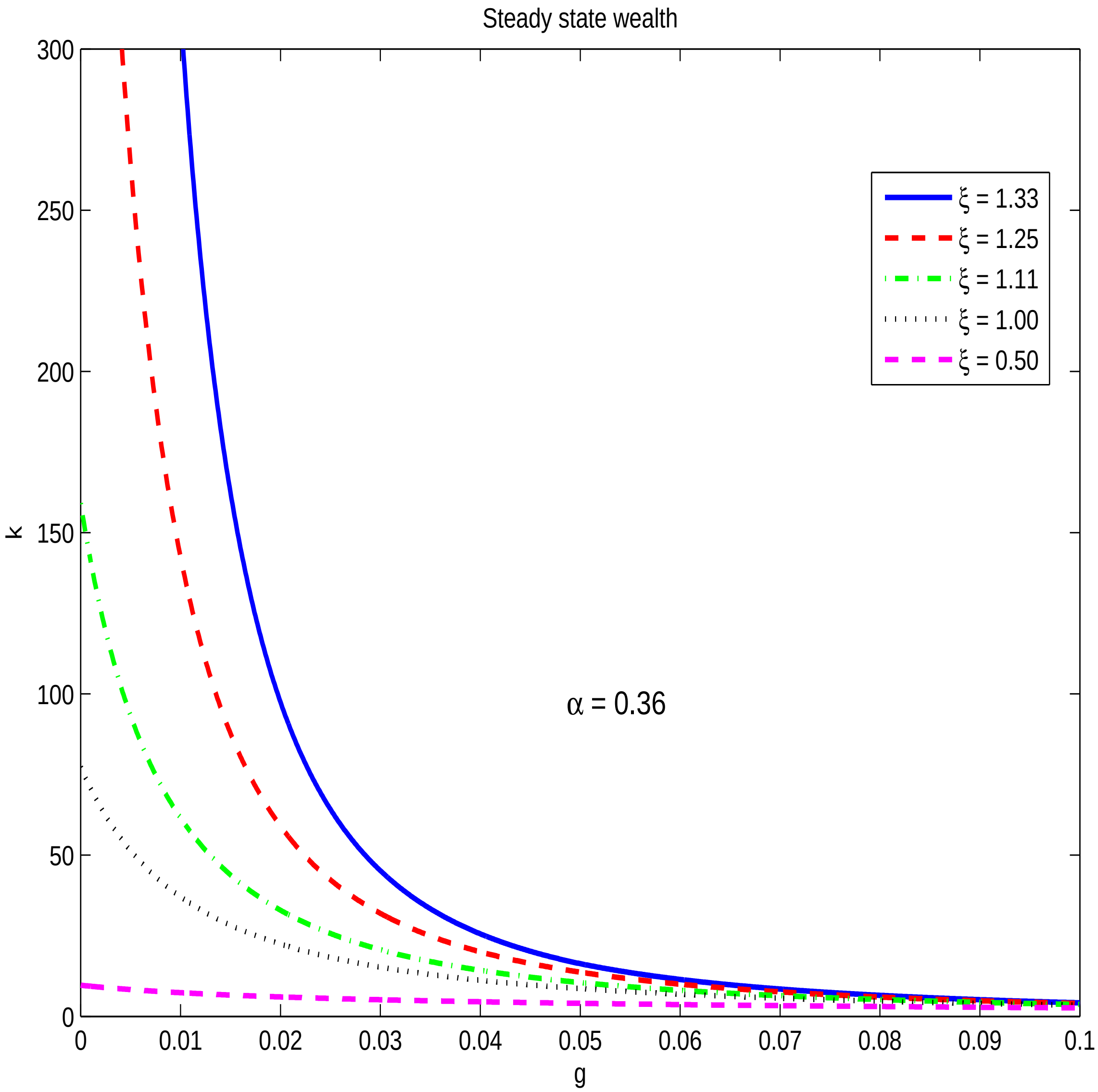

Figure 5: Steady State Wealth

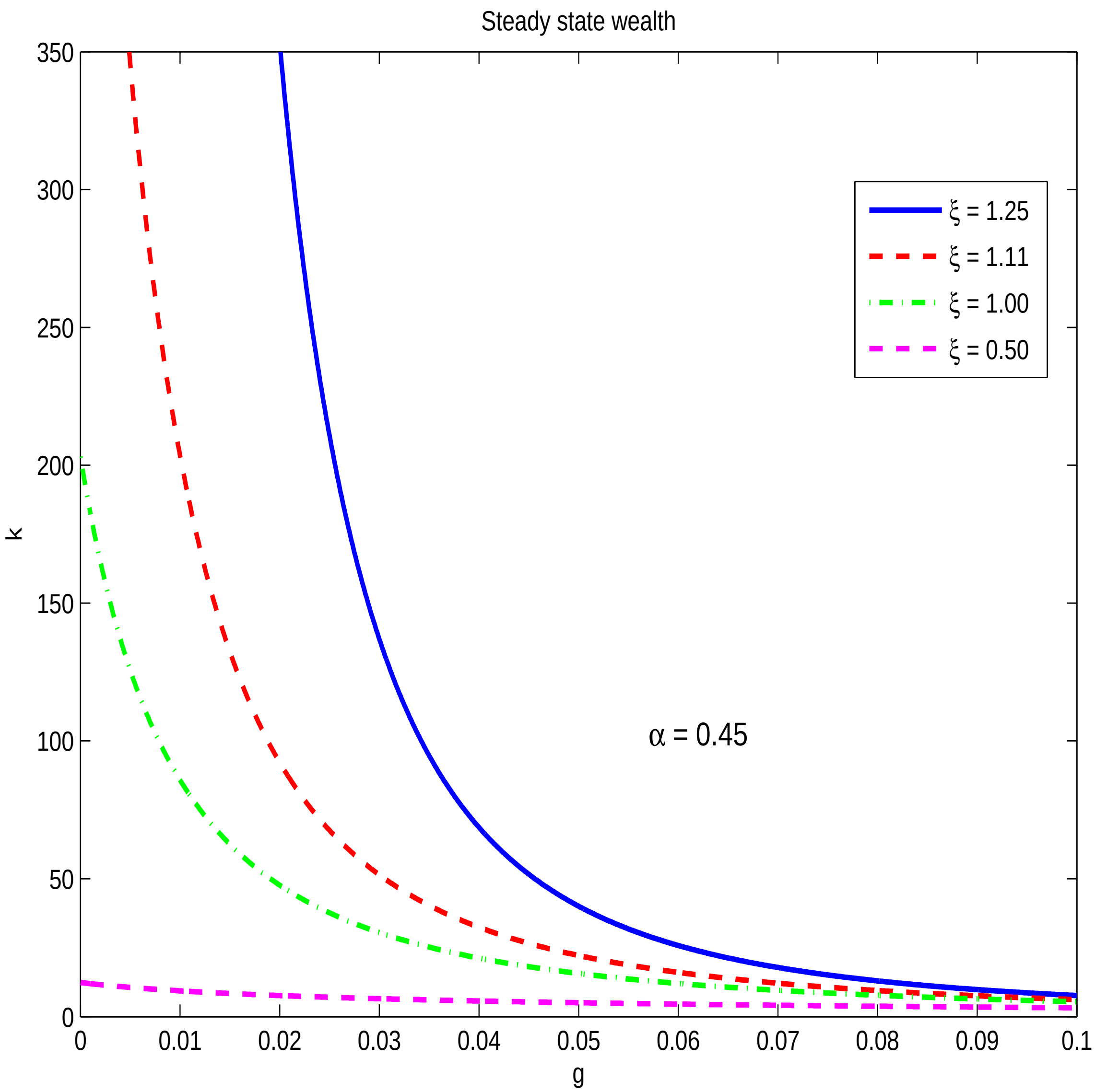

Figure 6: Capital to Income Ratio in a Zero Growth Steady State

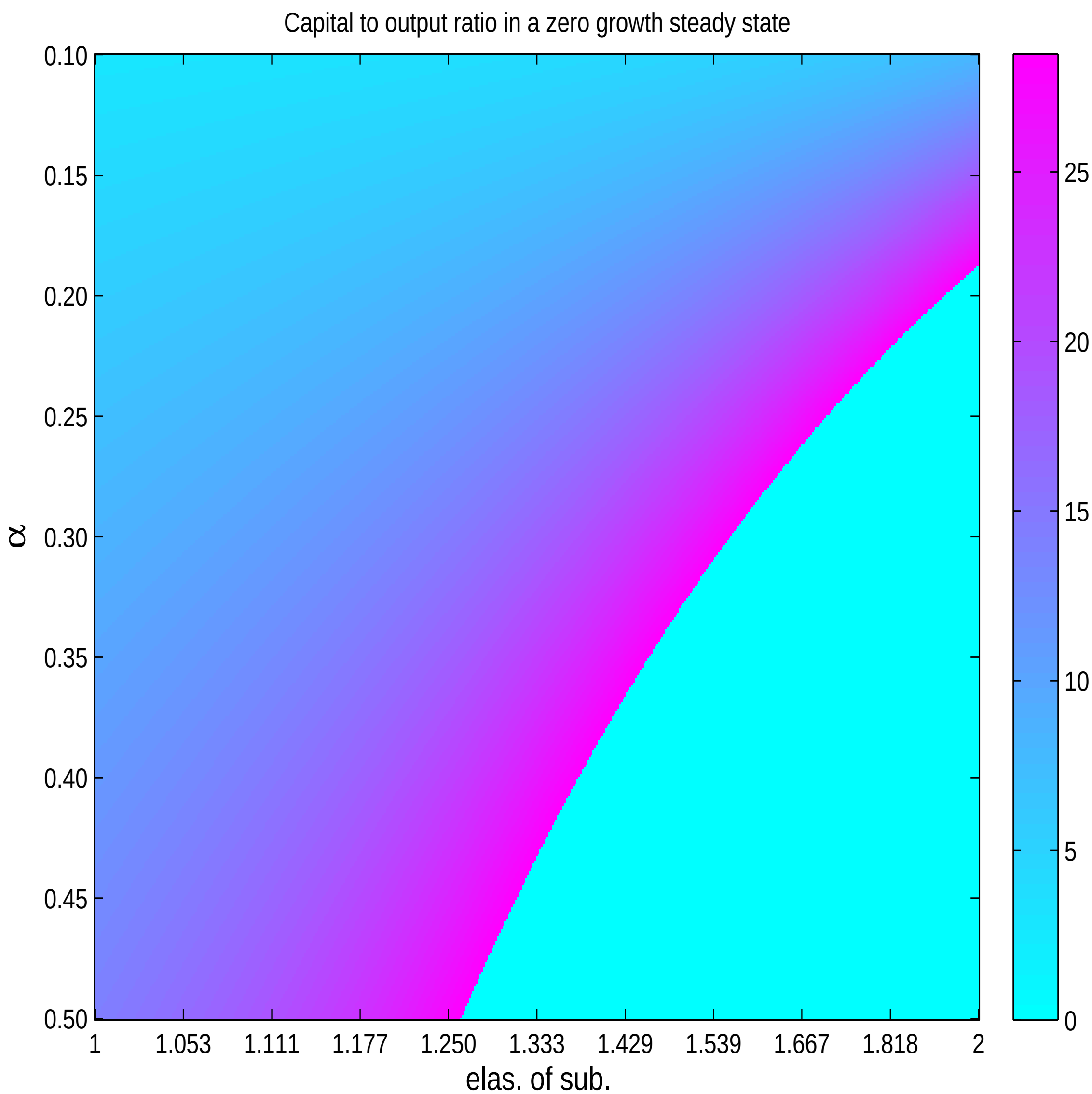

Figure 7: Capital Share of Income in a Zero Growth Steady State

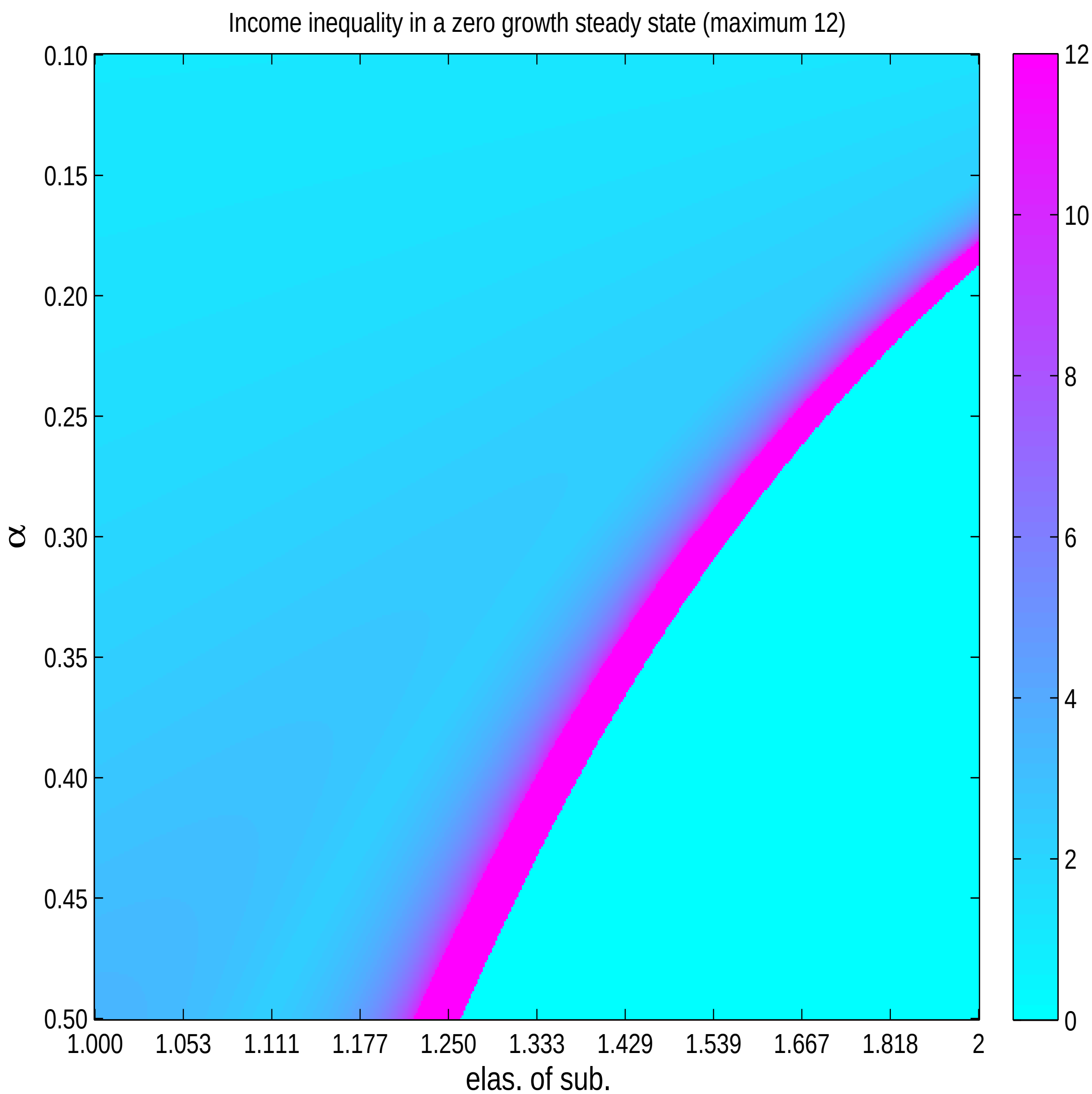

Figure 8: Steady State Inequality

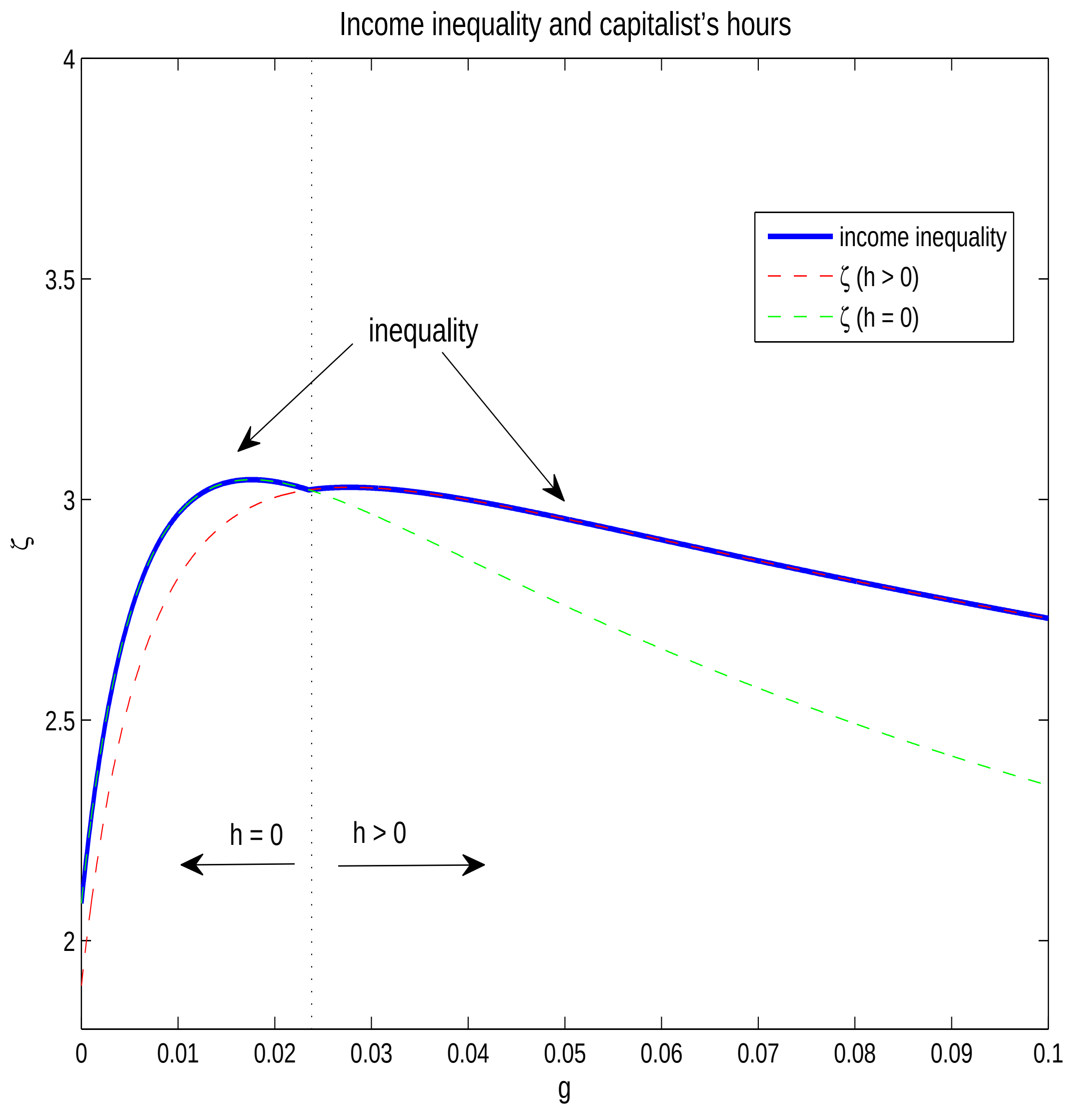

Figure 9: Dynamics of Inequality

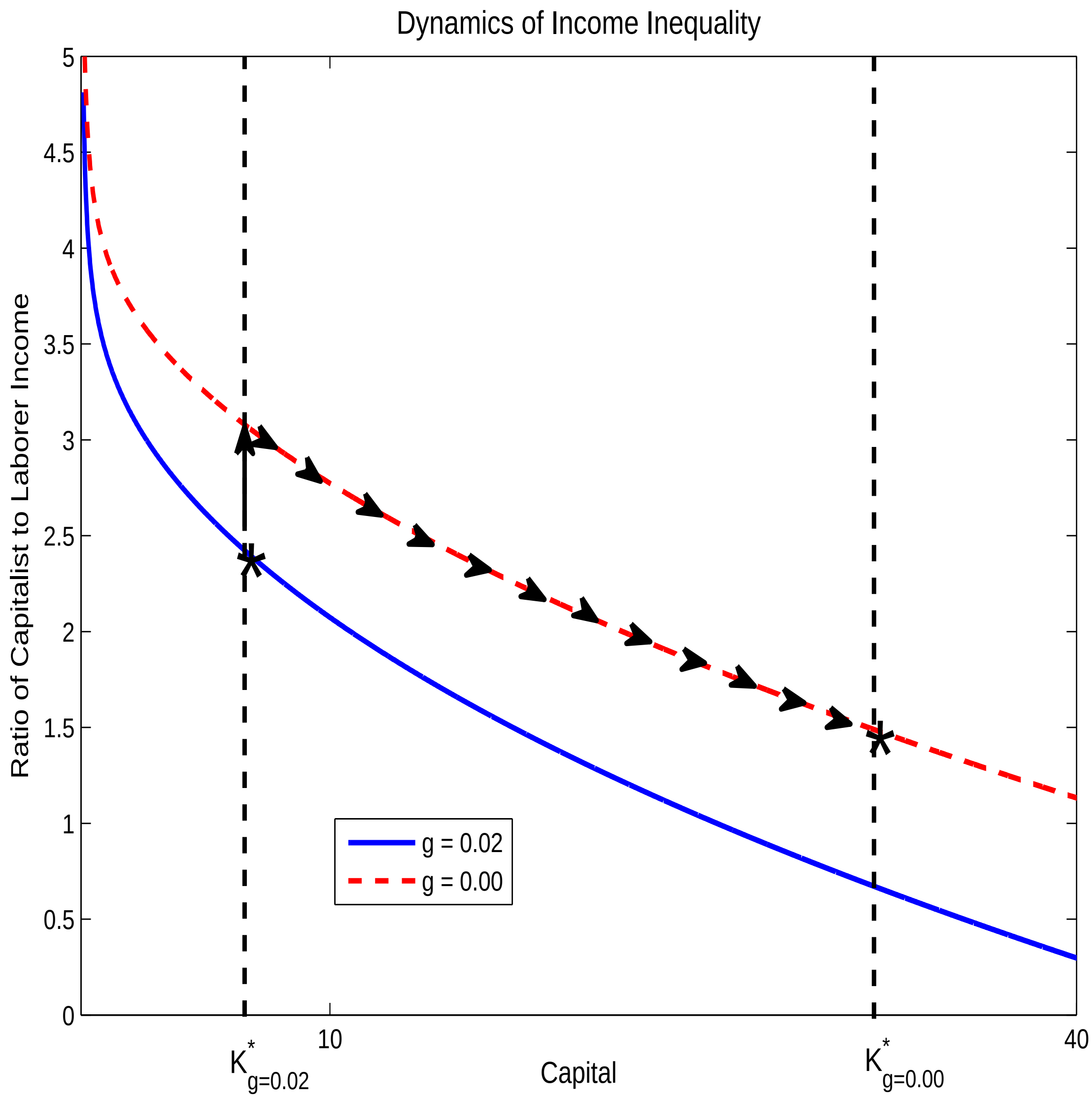

Figure 10: Steady State with Labor Productivity Risk

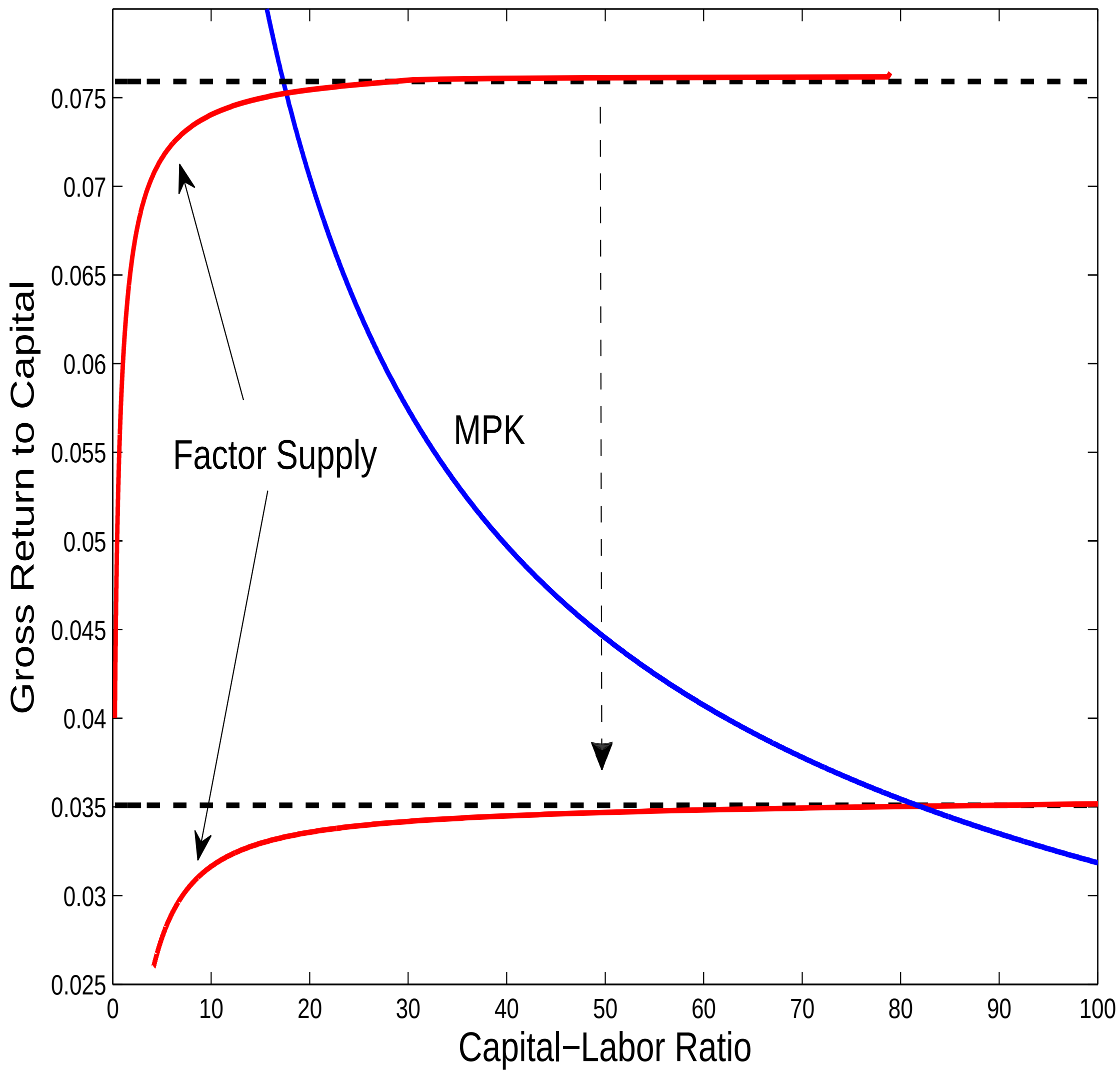

Figure 11: Lorenz Curves

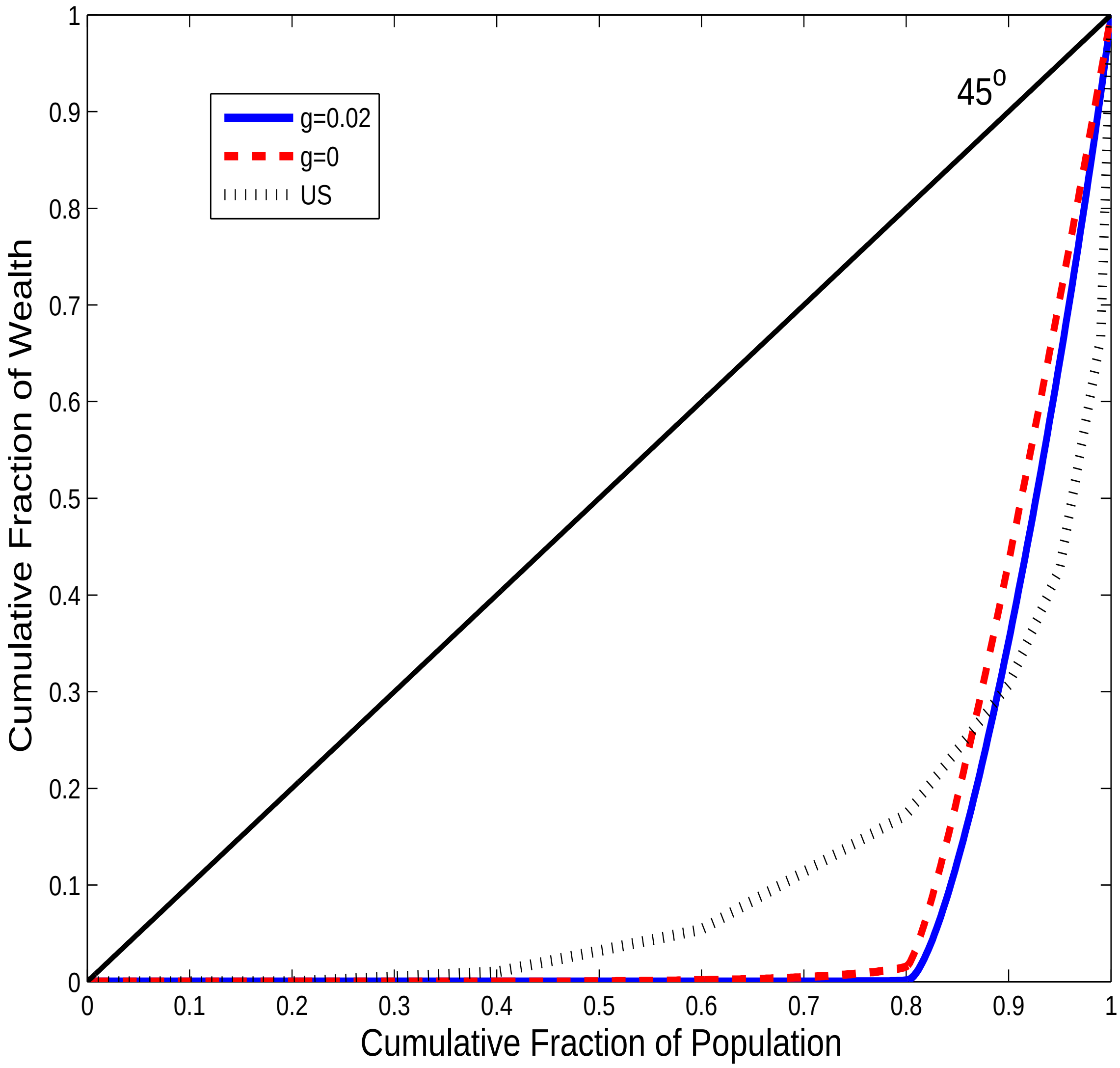

Figure 12: Transitional Dynamics

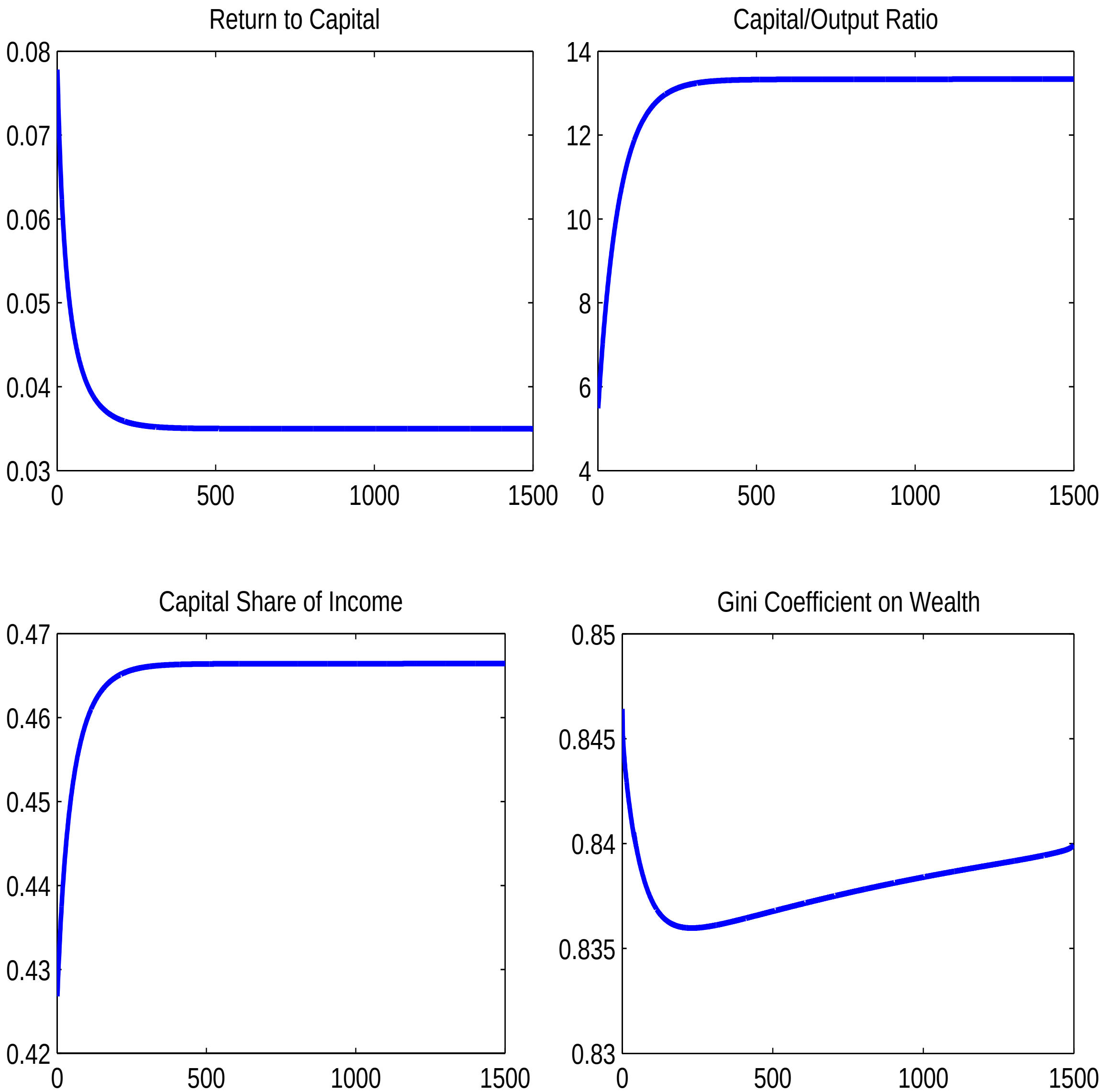

Figure 13: Steady State with Return Risk

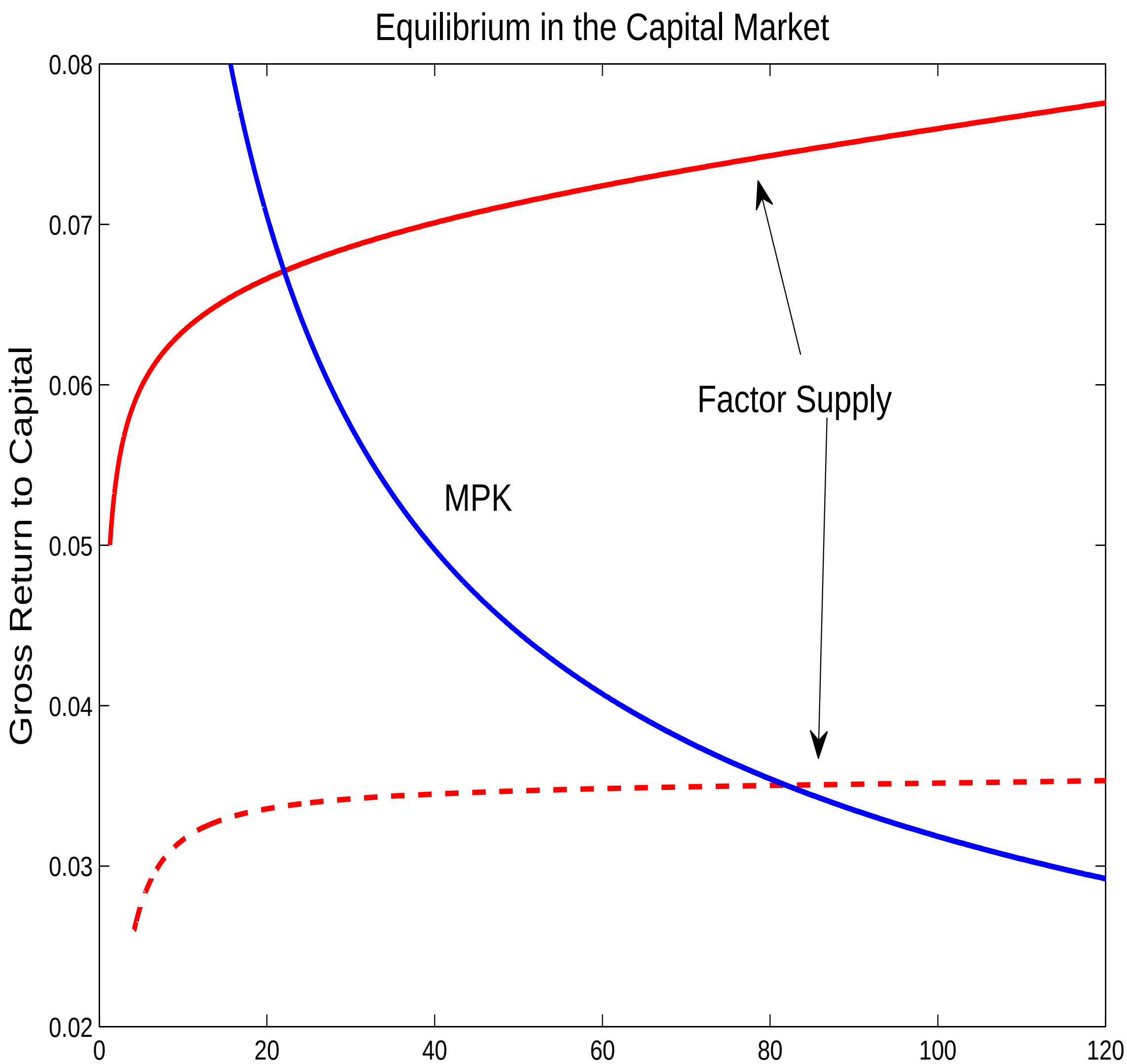

Figure 14: Lorenz Curves

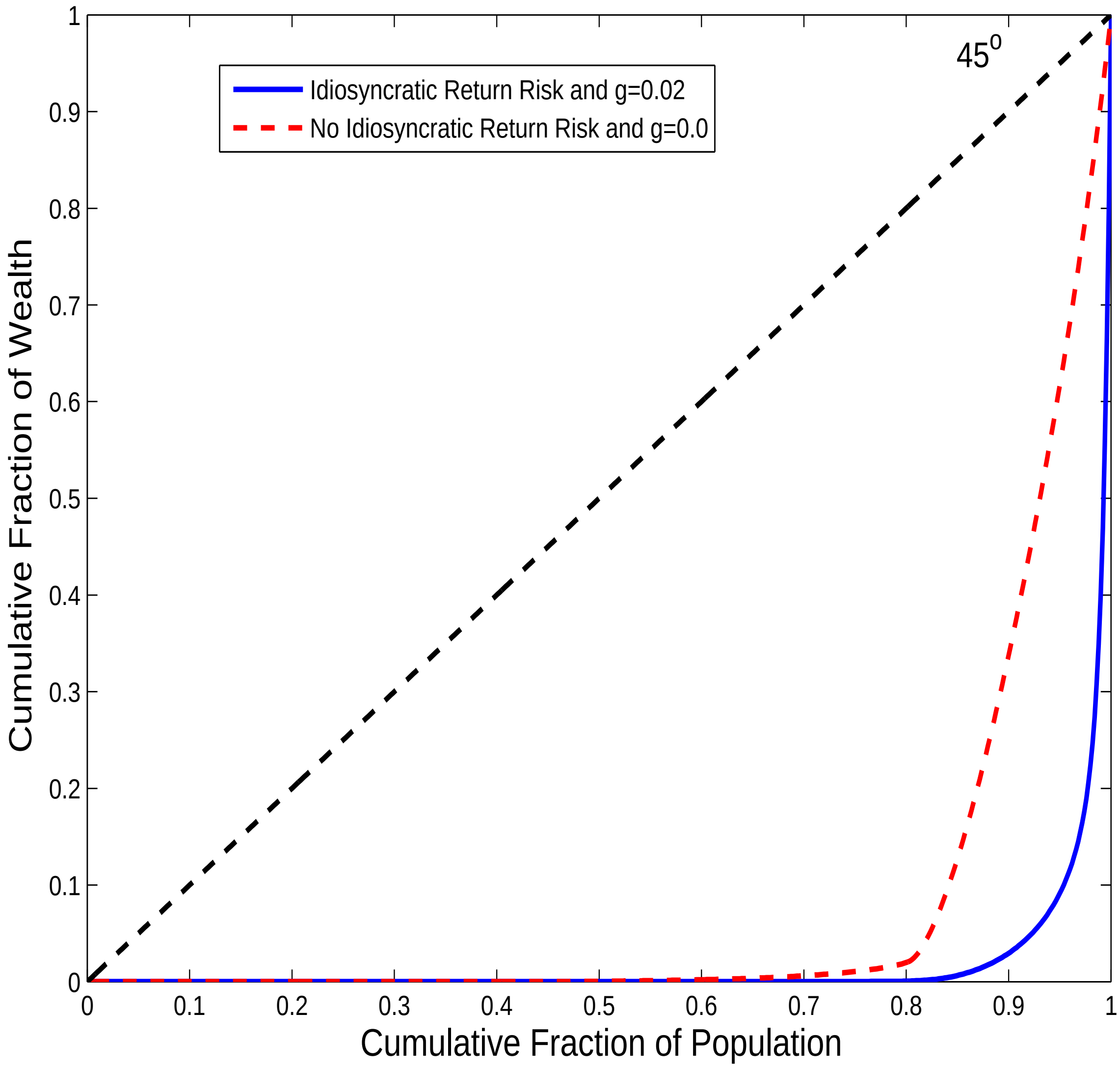

Figure 15: Transitional Dynamics

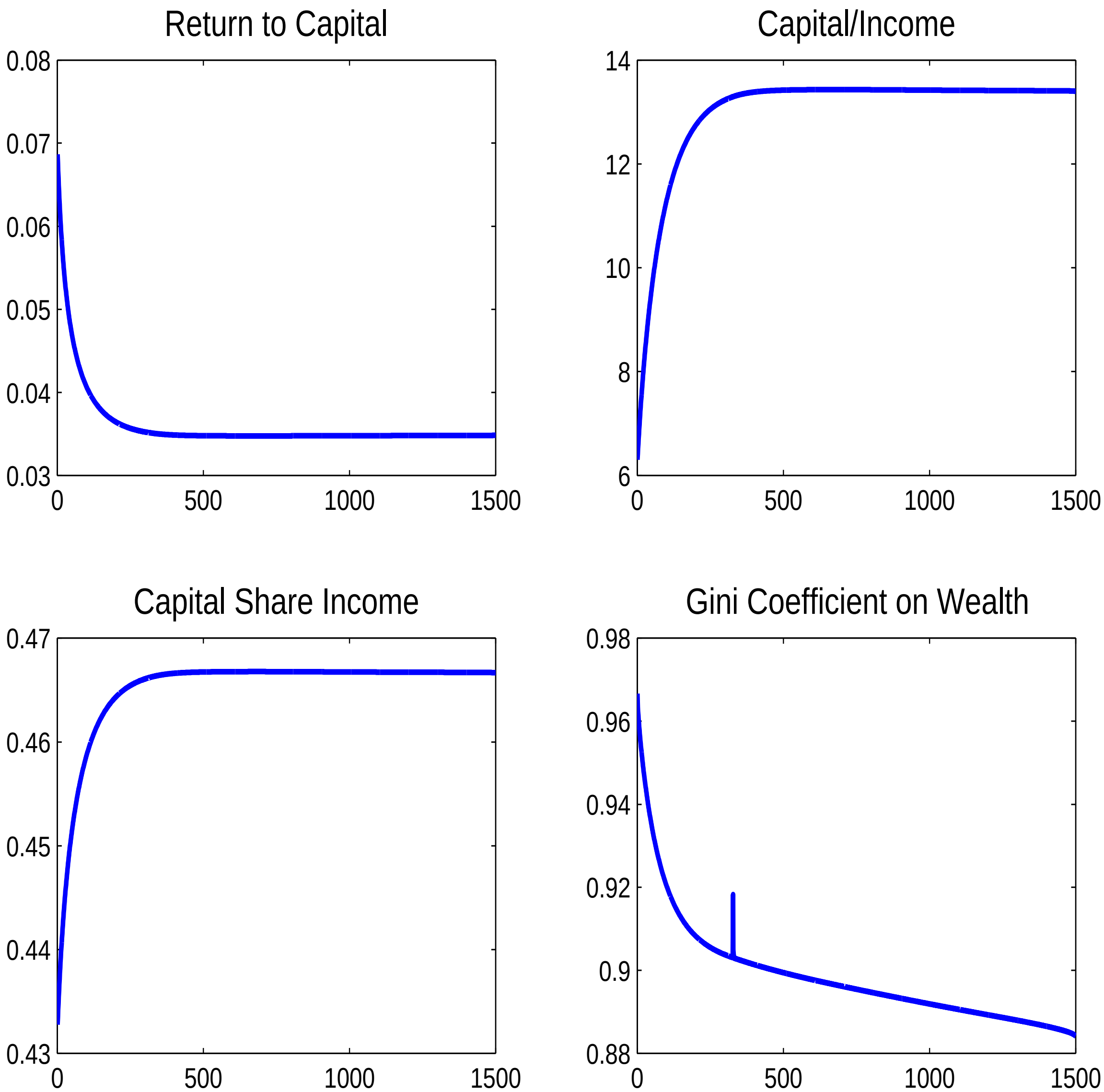

Figure 16: Steady State with Technological Progress