Monetary Policy Expectations at the Zero Lower Bound*

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Abstract

We show that conventional dynamic term structure models (DTSMs) estimated on recent U.S. data severely violate the zero lower bound (ZLB) on nominal interest rates and deliver poor forecasts of future short rates. In contrast, shadow-rate DTSMs account for the ZLB by construction, capture the resulting distributional asymmetry of future short rates, and achieve good forecast performance. These models provide more accurate estimates of the most likely path for future monetary policy—including the timing of policy liftoff from the ZLB and the pace of subsequent policy tightening. We also demonstrate the benefits of including macroeconomic factors in a shadow-rate DTSM when yields are constrained near the ZLB.

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1 Introduction

Divining the path of future monetary policy has been of special interest during the Great Recession and its aftermath. Expectations of future monetary policy actions are commonly obtained from the term structure of interest rates, which captures financial market participants' views regarding the prospective path of the short-term interest rate—the policy instrument of central banks. Gaussian affine dynamic term structure models (DTSMs) are the standard representation in finance used to extract such short-rate expectations (e.g., Piazzesi, 2010). However, while these models have provided good empirical representations of yield curves in the past, they are ill-suited to represent the dynamics of recent near-zero interest rates that have prevailed in many countries. In particular, they do not recognize that in the real world, with currency available as an alternative asset, interest rates are bounded below by zero because negative nominal interest rates would lead to riskless arbitrage opportunities.[1]

The fact that Gaussian affine DTSMs ignore the zero lower bound (ZLB) was of little consequence when interest rates were well above zero. However, as nominal interest rates have fallen to near zero, the lack of an appropriate nonnegativity restriction in conventional models has become a conspicuous theoretical deficiency. This paper presents evidence showing that the theoretical failure of standard Gaussian affine DTSMs to account for the ZLB has been an important practical deficiency in recent years in terms of fit, point and distributional forecasting ability, and accuracy of estimated monetary policy expectations. Our benchmark for comparison is an alternative model based on the shadow-rate concept proposed by Black (1995). The shadow-rate representation replaces the affine short-rate specification of standard DTSMs with an identical affine process for an unobserved shadow short rate. The observed short rate is set equal to this shadow short rate when it is positive; otherwise, it is set to zero (or some other near-zero minimum value). Shadow-rate models are therefore able to account for the ZLB constraint, however, this comes at the cost of losing the convenient analytical bond pricing of affine models, so that numerical solutions are required.

The focus of our paper is the estimation of monetary policy expectations at the ZLB. To this end, we consider both yields-only and macro-finance shadow-rate models, where the latter includes measures of economic activity and inflation as risk factors. There is now a sizable literature arguing that a joint macro-finance approach is a very productive research avenue for term structure modeling (e.g., Rudebusch, 2010), but this paper is the first to

[1]The value of the lower bound on nominal interest rates is not precisely zero due to institutional factors including the size of costs associated with storing, transferring, and spending large amounts of currency. For convenience, we will describe this constraint as a *zero* lower bound even though our model in principle allows for a non-zero lower bound.

include macroeconomic factors into a shadow-rate model. We show that when the nominal term structure is constrained by the ZLB, the addition of macroeconomic variables to the DTSM information set is useful for inference about the future evolution of the yield curve. Intuitively, the ZLB limits the information content of the yield curve because its short end is pinned at zero. In such a situation, macro variables provide important additional information for forecasting future yields, particularly for predicting how long the policy rate will remain near zero.[2]

We begin our analysis with an evaluation of affine and shadow-rate models during the past near-decade of very low interest rates in the United States. Given the close proximity of interest rates to the ZLB during this period, we find that shadow-rate DTSMs provide a statistically significant and economically relevant improvement in fit and forecasting performance compared with standard Gaussian affine DTSMs. Affine models frequently violate the ZLB, produce substantial estimated probabilities of negative future short rates, and consequently produce quite inaccurate short-rate forecasts at the ZLB. In contrast, we document that shadow-rate models can accurately forecast prolonged near-zero policy rates in an out-of-sample forecast exercise.

Shadow-rate models account for the substantial asymmetry in the distribution of future short rates during periods of near-zero policy rates. This feature is especially valuable for assessing monetary policy expectations embedded in the yield curve at the ZLB. For example, one key question is how to estimate the anticipated timing of the liftoff of the policy rate from the ZLB. A common approach among financial market researchers and investors is to use the horizon at which forward rates cross a given threshold, say 25 basis points, as an estimate of the expected date of liftoff. But forward rates correspond to (risk-neutral) expectations of future short rates, and using this *mean path* to estimate liftoff is problematic because it ignores the asymmetry of the distribution of future short rates near the ZLB. A more useful measure of policy expectations is the *modal path*—the most-likely path for future short rate rates—which is readily available from a shadow-rate model. Importantly, it appropriately accounts for the distributional asymmetry of future short rates. The difference between the mean and modal paths, which we term the "ZLB wedge," reflects the asymmetry induced by the ZLB on the distribution of future short rates, and hence reveals how tightly the ZLB constraint is binding. We use the ZLB wedge between the ten-year yield and the corresponding shadow yield as a measure of the tightness of the ZLB constraint, and document that it increased substantially over the period from 2009 to 2012, and then gradually decreased over 2013 and 2014, a period

[2] The value of a macro-finance approach is also consistent with the many central bank statements that have stressed that the timing of liftoff from the ZLB is dependent on the flow of incoming macroeconomic data.

when macroeconomic conditions improved notably.

To measure monetary policy expectations at the ZLB, we focus on two key metrics: the time until liftoff and the subsequent pace of tightening. We show that the date at which the modal path escapes from near zero provides a forecast of the time until liftoff that is approximately optimal under an absolute-error loss function. We compute the full forecast distribution of the liftoff horizon in order to verify the modal-path-based liftoff estimate and to obtain interval forecasts for liftoff. Model-based liftoff estimates based on a macro-finance yield curve model closely accord with private-sector forecasts of the timing of monetary policy liftoff, and are consistent with the Federal Open Market Committee's (FOMC) calendar-based forward guidance. Overall, the liftoff horizon can therefore serve as a useful univariate summary of monetary policy at the ZLB.[3] Our second metric, the initial pace of policy tightening is calculated as the cumulative increase in the modal short rate path during the first two years after liftoff. Our macro-finance term structure model forecasts a much more gradual increase in the policy rate than in previous policy tightening cycles, which is consistent with statements by Federal Reserve policymakers.

Overall, our analysis documents the empirical relevance of the ZLB constraint and the importance of accounting for it when carrying out inference about interest rates and monetary policy near the ZLB. Our paper is related to a number of recent studies that have also used shadow-rate DTSMs. Bomfim (2003) employs a two-factor shadow-rate model to estimate the probability of the future policy rate hitting the ZLB in the U.S. during the 2002–2003 period. Using Japanese yield curve data, Kim and Singleton (2012) estimate two-factor models and demonstrate the good performance of shadow-rate models compared to alternatives, and Christensen and Rudebusch (2015) document the sensitivity of shadow-rate estimates to model specification in estimated one-, two- and three-factor models.[4] Several other studies have considered the recent U.S. experience, including Ichiue and Ueno (2013), Christensen and Rudebusch (forthcoming), and Krippner (2015). Our study goes beyond these works in several ways. In particular, we demonstrate how to capture various aspects of monetary policy expectations at the ZLB using the modal path, incorporate information both from the yield curve and from macroeconomic variables, and provide novel estimates and results for the ZLB period in the United States.

[3] In contrast, model-implied shadow short rates, which some have advocated as measures of the policy stance near the ZLB (Krippner, 2013; Wu and Xia, 2014), are highly sensitive to model specification and the exact data at the short end of the yield curve. Their lack of robustness raises a warning flag about using shadow short rates as a measure of monetary policy.

[4] Ueno et al. (2006) and Ichiue and Ueno (2007) also study Japanese yields, using one-factor and two-factor models, respectively.

2 Dynamic term structure models

In this section, we describe our model specifications, the role of the ZLB constraint in these models, and our empirical implementation, which uses monthly U.S. data.

2.1 Affine models

The canonical affine Gaussian DTSM is based on three assumptions. First, the short-term interest rate—the one-month rate in our context—is affine in the N risk factors X_t, i.e.,

$$r_t = \delta_0 + \delta_1' X_t. \tag{1}$$

Second, it is assumed that there exists a risk-neutral probability measure $\mathbb{Q}$ which prices all financial assets—hence, there are no arbitrage opportunities—and that under $\mathbb{Q}$ the risk factors follow a Gaussian vector autoregression (VAR),

$$X_t = \mu^{\mathbb{Q}} + \phi^{\mathbb{Q}} X_{t-1} + \Sigma \varepsilon_t^{\mathbb{Q}}, \tag{2}$$

where Σ is lower triangular and $\varepsilon_t^{\mathbb{Q}}$ is an *i.i.d.* standard normal random vector under $\mathbb{Q}$. Third, under the real-world probability measure $\mathbb{P}$, X_t also follows a Gaussian VAR,

$$X_t = \mu + \phi X_{t-1} + \Sigma \varepsilon_t, \tag{3}$$

where ε_t is an *i.i.d.* standard normal random vector under $\mathbb{P}$.[5] Note that these assumptions imply the existence of a stochastic discount factor which is essentially-affine as in Duffee (2002). The price of a bond with a maturity of m periods is determined by

$$P_t^m = E_t^{\mathbb{Q}} \left[\exp\left(- \sum_{i=0}^{m-1} r_{t+i} \right) \right]. \tag{4}$$

In an affine model, this expectation can be found analytically, and it is exponentially affine in the risk factors. Model-implied yields therefore are affine functions of the factors. The details are well-known, see Bauer and Rudebusch (2015a). Importantly, a Gaussian model implies that interest rates can turn negative with non-zero probability.

[5] That is, as is standard, forecasts for the state variables can be calculated under two different probability measures: the real-world $\mathbb{P}$ measure (also know as the physical or historical or objective measure) and the risk-neutral $\mathbb{Q}$ measure that investors use to value assets because of their risk aversion. Specifically, investors value assets just as a risk-neutral agent would if that agent believed that the dynamics of state variables were characterized by the $\mathbb{Q}$ measure.

2.2 Shadow-rate models

Following Black (1995), our shadow-rate DTSMs are closely similar to our affine models except that the affine short-rate equation (1) is replaced by a shadow-rate specification:

$$r_t = \max(s_t, r_{min}), \quad s_t = \delta_0 + \delta_1' X_t. \tag{5}$$

The shadow short rate, s_t, is modeled as affine Gaussian, exactly as the short rate in affine models. Equation (5) ensures that the short rate and all other model-implied interest rates cannot go below r_{min}. Black (1995) set $r_{min} = 0$, and this is our choice as well. This ZLB on nominal interest rates is typically motivated by the presence of physical currency. Since the storage and use of large amounts of physical currency can incur significant transaction costs, the ZLB has been violated at times in the past when interest rates have dipped into slightly negative territory, which could justify a small negative value for r_{min}. On the other hand, the federal funds rate, the key short-term interest rate managed by the Federal Reserve, in practice typically remains above zero, which would be an argument in favor of a slightly positive value for r_{min}. Different authors have made alternative choices, e.g., Wu and Xia (2014) set $r_{min} = 25$ basis points, and Kim and Priebsch (2013) treat r_{min} as a parameter and estimate it. We have found that our main results about policy expectations at the ZLB remain essentially unaffected by the choice of r_{min}.

How useful are estimates of the shadow short rate s_t? Some have interpreted the shadow short rate as an alternative indicator of the stance of monetary policy—see, in particular, Krippner (2014), Ichiue and Ueno (2013), and Wu and Xia (2014). However, estimated shadow short rates are highly sensitive to different choices of r_{min} and to the model specification.[6] This sensitivity raises a warning flag and suggests that the use of shadow rates as indicators of monetary policy at the ZLB is problematic.[7] In this paper, we will generally focus on expected future instead of current shadow short rates.

In addition to accounting for the ZLB, a key advantage of shadow-rate models is that away from the ZLB, they behave exactly as the corresponding affine DTSM. Another advantage is that in contrast to other tractable non-Gaussian models that respect the ZLB constraint, such as square-root diffusion (Cox-Ingersoll-Ross) models and quadratic models, the probability of a zero future short rate is non-zero. This becomes crucial when addressing the issue of the duration of near-zero policy rates and the time of future liftoff, as we do in this paper.

[6] See Bauer and Rudebusch (2015a) and the references therein.

[7] More promising approaches have recently been suggested by Lombardi and Zhu (2014), who infer a shadow short rate that is consistent with other observed indicators of monetary policy and financial conditions, and Krippner (2015), who considers the area between shadow rates and their long-term level.

The advantages of a shadow-rate model come at the cost that it does not lead to closed-form solutions for yields and bond prices. Hence the need arises for approximative solution methods. Priebsch (2013) compares different approaches in this context, and proposes a new method based on second-order approximations that is fast and highly accurate. Here we use a discrete-time adaption of the Priebsch method, which we describe in detail in Bauer and Rudebusch (2015a).[8]

2.3 Macroeconomic variables as risk factors

A key modeling choice is which risk factors to include in the DTSM. We estimate both "yields-only" models, where X_t reflects only information in the yield curve, and "macro-finance" models, where X_t also includes macroeconomic variables.

We use yields-only affine and shadow-rate models with three risk factors, denoting the affine model by YA(3) and the shadow-rate model by YZ(3). We use the canonical form of Joslin et al. (2011). The risk factors are linear combinations of yields, with the weights corresponding to the loadings of the first N principal components of observed yields. In the affine model, the risk factors are linear combinations of model-implied yields—they correspond to level, slope, and curvature of the yield curve.[9] In the shadow-rate model, the yield factors are linear combinations of *shadow yields*—the yields that obtain when the shadow short rate is used for discounting payoffs—so that they can be interpreted as shadow level, shadow slope, and shadow curvature.[10]

Macroeconomic variables are likely to be particularly informative when the yield curve is constrained by the ZLB. To investigate this, we estimate macro-finance DTSMs that include measures of inflation and economic activity in addition to the yield factors. Here, we use the canonical form of Joslin et al. (2013b). We use affine and shadow-rate models with two ($L = 2$) yield factors in addition to the two macro factors, and denote our models by $MA(2)$ and $MZ(2)$.[11] As in the case of yields-only models, the yield factors are linear combinations of (model-implied/shadow) yields, with weights corresponding to principal components of

[8] An even simpler but somewhat less accurate approach was proposed by Krippner (2014), which is based on an approximation of forward rates. Christensen and Rudebusch (2015) perform the necessary derivations for the Krippner-method in an affine Nelson-Siegel model. In a discrete-time model, Wu and Xia (2014) independently derive a bond-price approximation that is equivalent to the Krippner-method, as shown in Krippner (2015).

[9] Our affine yields-only models correspond to the RKF model specification in Joslin et al. (2011).

[10] Shadow yields can be calculated by using the risk factors of a shadow-rate model in combination with affine loadings.

[11] We have also considered models with two macro factors and only one yields factor, as in Joslin et al. (2013b). We found that these models are not able to accurately fit observed yields, and hence focus on models with two yields factors (which were also used in Joslin et al., 2013a).

observed yields.

In our macro-finance models, the macroeconomic variables are spanned by the yield curve.[12] An alternative is to use models with unspanned macro risks as in Joslin et al. (2014), where the current short rate and yield curve depends only on the yield factors. Here we maintain the assumption that macroeconomic conditions directly affect the current short-term interest rate and yield curve, so that they are informative for inferring policy expectations under the risk-neutral measure. This specification is consistent with the expressed view of the Federal Open Market Committee (FOMC) that the short rate will be based on the unemployment and inflation rates. For further discussion of this issue and a defense of spanned macro-finance DTSMs see Bauer and Rudebusch (2015b).

2.4 Data, measurement error, and estimation

Our data consist of monthly observations of interest rates and macroeconomic variables from January 1985 to December 2014. For the short end of the yield curve, we use three-month and six-month T-bill rates. The remaining rates are smoothed zero-coupon Treasury yields with maturities of one, two, three, five, seven, and ten years from Gürkaynak et al. (2007). We measure economic activity by the unemployment gap, using the estimate of the natural rate of unemployment from the Congressional Budget Office. Inflation is measured by the year-over-year percent change in the consumer price index (CPI) for all items excluding food and energy, i.e., by core CPI inflation. We include the inflation and gap measures because these are closely linked to the target federal funds rate, the policy instrument of the Federal Reserve (Rudebusch, 2006, 2009).

Denote the vector of $J = 8$ model-implied yields by Y_t. For the affine models, we have $Y_t = A + BX_t$, with J-vector A and $J \times N$-matrix B containing the usual affine loadings. The observed bond yields used for estimation and inference are $\hat{Y}_t = Y_t + e_t$, where e_t is a vector of iid normal measurement error. We include measurement error on yields because an N-dimensional factor model cannot perfectly price $J > N$ yields. In line with the large literature on macro-finance DTSMs, we do not include measurement errors on macro variables.[13]

Estimation of the affine models is standard, both for yields-only and macro-finance models. In the estimation we assume that the yield factors are observed, as in Joslin et al. (2011) and Joslin et al. (2013b), so that μ and ϕ can be obtained using least squares and the remaining

[12] In the shadow-rate models, the macro factors are spanned by the (unobservable) shadow yields.

[13] We do not allow for measurement errors on the macro factors, because in that case "the likelihood function largely gives up on fitting the observed macro factors in favor of more accurate pricing of bonds" (Joslin et al., 2013b). Note that our affine macro-finance model corresponds to the TS^f specification in Joslin et al. (2013b), with the difference that we use $\mathcal{L} = 2$ yield factors instead of just one.

parameters are found by maximizing the likelihood function for given VAR parameters. This is particularly advantageous for macro-finance models, which have many parameters. Our estimation method delivers fast and reliable maximum likelihood estimates.[14] Instead of estimating the shadow-rate models, we take a different approach in this paper. We estimate parameters *only for the affine models* $YA(3)$ and $MA(2)$, using the pre-ZLB sample ending in December 2007. Over this period, affine and shadow-rate models are essentially indistinguishable, because yields are sufficiently far from the ZLB. Then, we use the same pre-ZLB estimated parameters in the affine models and the corresponding shadow-rate models $YZ(3)$ and $MZ(2)$, and apply the models to the full sample period until December 2014.[15]

Hence, we use shadow-rate and affine models with the same parameters estimated from the pre-ZLB sample to answer questions regarding the full sample. One important reason for this approach is that estimation of shadow-rate models incurs high computational costs, as it requires both numerical bond pricing and nonlinear filtering. This is particularly problematic for our macro-finance models due to their many parameters. In contrast, estimation of affine models is extremely fast and much more reliable. While one may be concerned about using parameters in the shadow-rate models that are not the maximum likelihood estimates, we show in Section 3 that in spite of this, shadow-rate models in fact perform very well in our data along several dimensions, and much better than the affine models. Another advantage to holding the parameters the same for each pair of affine and shadow-rate models is that the effects of the ZLB constraint when comparing each pair can be clearly seen. In addition, our approach guards against look-ahead bias from using full-sample estimates for analyzing the ZLB period. More generally, we view our use of only pre-ZLB data for estimation as a defensible compromise.

3 Model evaluation

From a theoretical perspective, shadow-rate models have a fundamental advantage over affine models in that they impose the nonnegativity of nominal interest rates. But how relevant is this in practice? In this section, we first evaluate affine and shadow-rate models during a period of near-zero interest rates. Then, we discuss and measure how the ZLB constraint affects current short rates and the distribution of future short rates.

[14] Denote by W the $\mathcal{L} \times J$ matrix with the principal component loadings. The assumption that X_t is observable, i.e., that the $\mathcal{L}$ linear combination of yields in W are priced exactly by the model, implies $X_t = W\hat{Y}_t = WY_t$ and $We_t = 0$ so that there are effectively only $J - \mathcal{L}$ independent measurement errors.

[15] We use the Kalman filter for the affine models and the Extended Kalman filter for the shadow-rate models. For details, see Bauer and Rudebusch (2015a).

3.1 Cross-sectional fit

We first assess the cross-sectional fit of model-implied yields to observed yields for affine and shadow-rate models. Table 1 shows the root mean-squared fitting errors (RMSEs) across models for the whole cross section of yields and for each yield maturity separately. The top panel reports RMSEs for the whole sample, while the bottom panel reports the fit for the ZLB subsample, here and in the following taken as the period from December 2008 to December 2014.[16] Overall, shadow-rate models fit yields better than their affine counterparts. The bottom panel of Table 1 shows that improvements in RMSEs are very substantial during the ZLB subsample. During this period, shadow-rate models have additional flexibility in fitting the cross section of yields, which behaves in an unusual way due to the pronounced nonlinearity at zero. The macro-finance models exhibit slightly worse yield fit than the yields-only models. While these models have four risk factors, more than the yields-only models, only two of these are yield factors—compared to the three yield factors in our yields-only models—hence they are more constrained in fitting the cross section of yields. For our purposes here, however, the cross-sectional fit of model $MZ(2)$ is sufficient.

3.2 Violations of the ZLB by affine models

To understand the relevance of the ZLB for term structure modeling in recent U.S. data, it is important to measure the extent to which affine models violate this constraint. One form of violation of the ZLB occurs when model-implied paths of future short rates drop below zero at some horizons. This can happen for either forward rates (i.e., expected future short rates under $\mathbb{Q}$) or for (real-world, $\mathbb{P}$-measure) short-rate expectations.[17] Table 2 shows the number of months that forward rates or expected future short rates drop below zero in each affine model. Also shown is the average length of horizon that the paths stay in negative territory. Both affine models imply short-rate paths that frequently and severely violate the ZLB constraint, and this holds for both forward curves and short-rate expectations.

Even when the expectation for the future short rate is positive, the model-implied probability distribution for the future short rate, which is Gaussian, may put nonnegligible mass on negative outcomes. Figure 1 plots the time series of conditional probabilities of negative future short rates at horizons of 6, 12, and 24 months in the future, for the period from 2000

[16] On December 16, 2008, the FOMC lowered the target for the federal funds rate to a range from 0 to 25 basis points, hence we choose December 2008 as the first month of the ZLB subsample.

[17] Throughout this paper, for simplicity we refer to $\mathbb{Q}$-measure expectations of future short rates as forward rates, although these of course differ from the actual forward rates by a convexity term. These short-rate expectations, without convexity, are available in closed form even in shadow-rate models.

to 2014. The top panel show these probabilities for model $YA(3)$, and the bottom panel for model $MA(2)$. Note that even during the extended period of monetary easing after the 2001 recession, the probability of negative future short rates was nonnegligible. For the more recent period of near-zero short rates from 2008 to 2014, both affine models imply that these probabilities are very high. The macro-finance model implies larger probabilities over this period than the yields-only models. The reason is that the high unemployment and subdued inflation toward the end of the sample imply paths of expected future short-term rates which are very low, reflecting expectations of easier future monetary policy. This leads to even higher probabilities of negative future short rates than for model $YA(3)$.

3.3 Forecasting at the ZLB

Affine models produce frequent and severe ZLB violations in the recent U.S. data. Does this matter for forecasting interest rates? While affine models may imply negative forecasts of future interest rates, a pragmatic solution is to simply set these forecasts to zero, and "fixing" them in this way may lead to sufficiently accurate forecasts. A second question is whether incorporating macroeconomic information improves interest-rate forecasts near the ZLB, because of the limited information content of yields that are constrained.

To address these questions, we investigate the out-of-sample forecast accuracy of affine and shadow-rate models during the ZLB period, focusing on the three-month T-bill rate as the forecast target. For each month from December 2008 to December 2012, we calculate model-based forecasts of this short rate for horizons up to 24 months. We use a fixed-window forecast scheme, i.e., we do not re-estimate the models but instead use our baseline parameter estimates, obtained over the estimation sample from January 1985 through December 2007.

For a given forecast date and horizon, we obtain model-based forecasts by first calculating conditional expectations of the risk factors, $E_t(X_{t+h})$, and then plugging these into the relevant yield formulas, where for the affine models we replace negative forecasts by zeros. These forecasts are not the conditional expectations of future yields, since we plug in the conditional expectations of the risk factors into non-linear functions. However, while these forecasts are not optimal under mean-squared-error loss, they are optimal under absolute error loss, because they correspond to the *median* of the forecast distribution of future yields.[18] We use the median instead of the mean because the target distribution is highly asymmetric due to

[18] The reason is that the median goes through nonlinear functions. Note in particular that these forecasts correspond to the target that is approximated by the following Monte Carlo simulation: First, simulate draws from the (Gaussian) distribution of the risk factors X_{t+h}, given information at time t. Second, calculate the model-implied three-month rate for each of these draws, replacing negative yields by zero for the affine model. Third, obtain the point forecast as the median of this model-implied forecast distribution of the short rate.

the ZLB, and the median is less affected by this asymmetry. The median is optimal under an absolute-error forecast loss function (see also Section 5).

Table 3 shows in the top panel the mean absolute forecast errors in basis points for selected forecast horizons across models. The bottom panel shows relative forecast accuracy (the ratio of mean absolute errors) for four pairs of models, with asterisks indicating the significance level of the test for equal accuracy of Diebold and Mariano (1995) and West (1996). Our main result is that the shadow-rate models predict the short rate more accurately than the affine models. The differences in forecast accuracy are very substantial, with the shadow-rate models in several cases producing forecasts that are twice as accurate as those from the affine models. In most cases, the null for equal forecast accuracy is rejected. Overall, shadow-rate models are at least as accurate and typically much more accurate than affine models when forecasting interest rates near the ZLB.

This evidence, together with the results above, demonstrates the importance of accounting for the ZLB constraint when performing inference about the yield curve during a period of near-zero short-term interest rates. While a sufficiently flexible affine model might be able to satisfactorily fit the yield curve, any type of economic inference is prone to be misleading. The ZLB has the effect that implied short-rate paths, forecasts, and term premia (which are implied by short-rate forecasts), produced by conventional DTSMs are likely to be seriously distorted and cannot be trusted.

The results in Table 3 also show the benefit of incorporating macroeconomic information for forecasting at the ZLB. With only one exception, forecasts from macro-finance models outperform those from yields-only models, and the improvements in forecast accuracy are sizable. For example, at horizons longer than six months, the forecasts from the macro-finance shadow-rate model $MZ(2)$ have average errors that are almost an order of magnitude smaller than those of the yields-only shadow-rate model $YZ(3)$. These dramatic differences in forecast accuracy illustrate the importance of accounting for macroeconomic information at the ZLB. In contrast, during normal times—away from the ZLB—the yield curve itself likely contains most or all of the information necessary to predict the future course of interest rates (Duffee, 2013; Bauer and Hamilton, 2015). But when the yield curve is constrained by the ZLB, yields cannot fully incorporate all relevant information and cannot reflect information in other important state variables. Hence it is particularly important to incorporate macroeconomic variables when making inference about monetary policy expectations near the ZLB. For these reasons, the macro-finance model $MZ(2)$ is our preferred model for the remainder of this paper.

4 The asymmetric distribution of future short rates

We now consider the model-implied distribution of future short rates. The ZLB leads to an asymmetry in this distribution which reveals how strongly the ZLB is binding, i.e., how relevant it is for the yield curve at a certain point in time. Figure 2 illustrates this asymmetry by showing the probability densities for the distributions of the future short rate and the future shadow (short) rate, as implied by model $MZ(2)$ on December 31, 2012, for a horizon of $h = 48$ months. The densities shown are for the risk-neutral ($\mathbb{Q}$-measure) distribution, and the same arguments apply to the real-world ($\mathbb{P}$-measure) distribution. For the future shadow rate, the density is Gaussian and centered around the conditional mean $E(s_{t+h}|X_t)$. The future short rate has a mixed discrete-continuous distribution: it has a point mass at zero (indicated in the graph with a vertical line) and for positive values the density equals that of the shadow rate. Therefore, its conditional mean is higher than that of the shadow rate, $E(r_{t+h}|X_t) > E(s_{t+h}|X_t)$. For what follows, it will be useful to define the mode of the short-rate distribution uniquely as $\max[0, E(s_{t+h}|X_t)]$ (as in Kim and Singleton, 2012). The distribution of the future short rate is right-skewed, the mean being higher than the mode.

The probability of a zero future short rate corresponds to the probability of a non-positive future shadow rate. During normal times, this probability is negligibly small, so that the mean and the mode of the short rate distribution approximately coincide. The more relevant the ZLB becomes, the larger the asymmetry of the distribution of future short rates, and the larger the difference between mean and mode—the "ZLB wedge." This wedge depends on the distance of yields from zero and the second moments of yield curve distribution, and measures how much the ZLB constrains the yield curve. It captures the cost of the optionality in equation (5), i.e., the value of the option of holding physical currency, which restrains nominal interest rates as they approach zero.

The *modal path* corresponds to the mode of the future short rate distribution across horizons, i.e., the most-likely path of future short rates. It is identical to expectations of future shadow rates when these are positive, and equal to zero when these are non-positive. The modal path contrasts with the *mean path*, i.e., expectations of future short rates. Figure 3 displays mean and modal paths in December 2012 and in December 2013 under both the $\mathbb{Q}$- and $\mathbb{P}$-measure. For the earlier date, the ZLB wedge between the mean and modal paths is very large and it persists out to fairly long horizons. By the end of 2013, however, there is a much smaller difference between these paths, and it becomes negligible for horizons longer than about two years. Evidently, the ZLB constraint had a greater effect constraining the yield curve in December 2012 than in December 2013.[19]

The paths under the risk-neutral measure $\mathbb{Q}$ are estimated using information in the cross section of interest rates, and the mean path under $\mathbb{Q}$ essentially corresponds to fitted forward rates. In contrast, the paths under the real-world probability measure $\mathbb{P}$ also take into account the macroeconomic information, in addition to the current shape of the yield curve. In December 2012, policy expectations under $\mathbb{P}$ and $\mathbb{Q}$ were quite similar. However, in December 2013, the $\mathbb{Q}$-measure paths were notably flatter, implying a later liftoff from the ZLB and a more gradual increase of short rates thereafter. This difference reflects a sluggish economic recovery with low underlying inflation and persistent economic slack, which in the macro-finance model results in an expectation of a very gradual easing of monetary policy.[20]

The ZLB wedge between long-term fitted and shadow interest rate measures how tightly the ZLB constrains the entire term structure of interest rates, because it equals the cumulative difference between the mean and modal paths (under $\mathbb{Q}$). Figure 4 shows the evolution over time of the fitted and shadow ten-year yields (top panel) and of the ZLB wedge between them (bottom panel). Over the period from 2009 to 2012, the difference between observed and shadow yields has increased substantially, indicating that the ZLB has increasingly constrained interest rates (see also Christensen and Rudebusch, forthcoming). This finding is consistent with Swanson and Williams (2014), who measure the tightness of the ZLB using the sensitivity of different interest rates to macroeconomic news, and document that this sensitivity has decreased for most yields over this period.[21] Conversely, over 2013 and 2014 the ZLB constraint evidently has become less restrictive, due to improving macroeconomic conditions and a resulting higher level of the ten-year yield.

5 Forecasting monetary policy liftoff

A key consideration about monetary policy expectations at the ZLB is the timing of the future policy liftoff. How can we use the information in the yield curve and macroeconomic information to forecast liftoff, based on our estimated shadow-rate model?

[19] This figure also demonstrates the limited amount of information in shadow short rates, which are similar on both dates despite very different economic situations and yield curves.

[20] Note, however, that inference about the VAR parameters μ and ϕ and about the real-world distribution of future short rates is difficult (Bauer et al., 2012; Duffee and Stanton, 2012), so the paths under $\mathbb{P}$ are subject to a substantial amount of uncertainty. In contrast, the parameters of the risk-neutral ($\mathbb{Q}$) distribution are estimated very precisely.

[21] Increases in the tightness of the ZLB often coincided with key Fed announcements of easier monetary policy, such as the switch to more explicit forward guidance by the FOMC in fall 2011, which pushed long-term interest rates closer to their lower bound, as evident also in the top panel of 4.

A natural starting point for estimating monetary policy expectations at the ZLB is an examination of the distribution of future policy liftoff. In a shadow-rate model, policy liftoff corresponds to the initial time the shadow short rate rises above a given threshold. We set this threshold at 25 basis points, which is consistent with 0 to 25 basis points range the Federal Reserve has kept during the ZLB period. Then we obtain the liftoff distribution using Monte Carlo simulation. Figure 5 shows the smoothed Kernel density of the liftoff distribution on December 31, 2012, based on simulations from our preferred model $MZ(2)$ under the risk-neutral distribution.[22] The figure also reports the mean, median, mode, and interquartile range and alternative liftoff estimates based on the mean and modal paths that we will discuss below. The liftoff distribution is strongly skewed to the right—even very distant horizons for policy liftoff are not uncommon.

The optimal forecast of policy liftoff based on this distribution depends on the forecaster's loss function. We argue that the median, which is optimal under absolute-error loss, is the appropriate forecast in this context, due to the strong asymmetry of the distribution. The mean is too strongly affected by the long-horizon right tail of the distribution, and would lead to unappealingly distant liftoff forecasts.[23] Hence, to estimate future policy liftoff, our focus will be on the median as the most natural choice for forecasting.

A crucial question is how the liftoff distribution relates to estimates of liftoff based on the mean and modal paths. It is a common practice to base estimates of liftoff on the mean path (under $\mathbb{Q}$) obtained from forward rates or money market futures rates. The time until liftoff is given by the horizon when this path rises above a certain threshold (e.g., 25 basis points).[24] At first glance, this approach may seem to be an attractive model-free approach. However, by using the mean path, it ignores the asymmetry of the distribution of the future short-term interest rate induced by the ZLB. Because of this asymmetry, the mean path does not reflect the most likely value of the policy rate at a future point in time, as noted in Section 4. Instead, we advocate using the modal path for calculating liftoff estimates, and some professional forecasters estimate liftoff in exactly this way: They first construct their

[22]Starting from the current term structure at t, we simulate 10,000 sample paths for the shadow rate using the risk-neutral dynamics of the risk factors in equation (2). For each simulation, the date of liftoff is determined by the time that the shadow rate hits the threshold. Due to the erratic nature of the sample paths, we also require that the shadow rate stays above the threshold for 12 months before we designate a policy liftoff, which leads to better-behaved liftoff distribution.

[23]Ichiue and Ueno (2012) used the mode of the liftoff distribution, but this is unappealing for the problem at hand because the skewness is completely ignored.

[24]For example, Ueno et al. (2006) take the horizon where Euroyen futures rates exceed a given threshold as an estimate of future policy liftoff by the Bank of Japan, and there are many similar examples in U.S. financial market commentary, including "Fed Likely to Push Back on Market Expectations of Rate Increase," from the June 13, 2013 issue of the Wall Street Journal.

most likely path for the future policy rate and treat the first increase in this path above a 25 basis point threshold as an estimate of policy liftoff.[25] Furthermore, liftoff estimates using the modal path essentially coincide with the median of the liftoff distribution. For example, Figure 5 shows that in December 2012, these two estimates were almost identical (33 vs. 34 months). This is true more generally, and the intuition for this concordance is the following: When the modal path crosses the liftoff threshold, the (Gaussian) shadow short rate is equally likely to be above or below the modal short-rate path. Hence, for this horizon, there is an equal probability for liftoff to occur earlier or later, meaning that it is the median of the liftoff distribution.[26]

Forecasting liftoff using the modal path is justified by the fact that this leads to an approximately optimal forecast, whereas forecasting liftoff using the mean path gives misleading results. This is illustrated by 3, where horizontal lines at 25 basis points indicate the threshold for liftoff. As the modal path is always below the mean path, the liftoff implied by the modal path is always later than that implied by the mean path. In December 2012, the difference is particularly pronounced, due to the strong asymmetry of the future short rate distribution. For the paths under the $\mathbb{Q}$-measure, the modal-path liftoff forecast at that point was 34 months, while the estimate based on the forward curve was 22 months. The following section will show the evolution of model-based liftoff forecasts over time.

6 Policy expectations over the recent ZLB period

The recent ZLB period started in December 2008 and at the time of this writing (November 2015) is still ongoing. In this section we report and discuss our model-based estimates of monetary policy expectations over this period, considering both the forecasts of liftoff and the expected subsequent pace of policy tightening.

The top panel of Figure 6 shows model-based estimates for liftoff based on the mean and modal paths, as well as the median and the interquartile range (IQR) of the forecast distribu-

[25]The responses in the Primary Dealer Survey are consistent with the view that respondents base their liftoff estimate on the modal path. Other examples of analysis in line with this approach include "Reading the Tea Leaves of Rate Expectations," Goldman Sachs US Economic Analyst from 7/3/2013.

[26]To be more precise, denote by h^* the horizon where the modal path crosses the threshold. It is equally likely for the shadow rate to be above or below the threshold at $t + h^*$. Since all paths that are above the threshold at this horizon have lifted off already, the probability mass for the event of liftoff between t and $t+h^*$ will be *at least* 0.5. Since most paths that are below the threshold have not lifted off yet, the probability of liftoff after $t + h^*$ will be below but close to 0.5. A small discrepancy between these probabilities and 0.5 arises because in some cases the shadow rate path might rise above the threshold and then fall again below it before $t + h^*$, but the chance of this happening will generally be small. Hence, the median of the liftoff distribution will always be close to h^*.

tion for the liftoff horizon. While we focus on the estimates under the $\mathbb{Q}$-measure, the same arguments apply to estimates under the $\mathbb{P}$-measure. The median of the liftoff distribution is extremely close to the liftoff based on the modal path over the entire sample, demonstrating that the modal path delivers an approximately optimal forecast of liftoff under a mean-absolute-error loss function. In contrast, the liftoff forecasts based on the mean path—corresponding to the common practice of estimating liftoff based on forward rates—imply liftoff that is much earlier, often by more than a year. Differences between liftoff estimates from the mean and modal path are particularly pronounced during 2011 and 2012, which was when the asymmetry due to the ZLB was strongest. Our interval forecasts for future liftoff—the IQR of the target distribution—illustrate the substantial uncertainty around the $\mathbb{Q}$-measure point forecasts for future policy liftoff, in particular in late in 2011 and in 2012.[27]

We now put the shadow-rate model forecasts of the time until liftoff in perspective by comparing them to alternative estimates. The bottom panel of Figure 6 shows liftoff forecasts based on the model-based modal paths under $\mathbb{P}$ and $\mathbb{Q}$ and two alternative calculations of future liftoff dates by the private sector. The first is the median of modal forecasts for the time of policy liftoff from the Survey of Primary Dealers (SPD) of the Federal Reserve Bank of New York, which is publicly available going back to January 2011.[28] The second alternative source of liftoff estimates is from Macroeconomic Advisers (MA), based on their published "policy call" and on their modal projections of the future path of the federal funds rate, i.e., the most likely scenario for Fed policy in their view. The bottom panel of Figure 6 also displays the horizons corresponding to the FOMC's calendar-based forward guidance—the "mid-2013," "late-2014," and "mid-2015" language first used in September 2011, January 2012, and September 2012, respectively. The FOMC had indicated that it expected the period of near-zero policy rates to last at least as long as these horizons, so that reasonable liftoff estimates would have to be at least as large as the horizons corresponding to these calendar dates.

Liftoff forecasts based on the modal path under $\mathbb{Q}$ reflect the views about policy liftoff that were priced into the yield curve at each month in our sample. These estimates are generally close to those from the SPD and MA (with the exception of 2009, where the MA estimates imply substantially later liftoff). Both the outside estimates and the liftoff forecast

[27]The model does not have stochastic volatility; however, as noted by Christensen and Rudebusch (forthcoming) shadow-rate models can capture some of the time variation in second moments about future monetary policy at the ZLB.

[28]See http://www.newyorkfed.org/markets/primarydealer_survey_questions.html for the questions and answers for each survey. In the survey, the respondents are asked to provide the "estimate for [the] most likely quarter and year of [the] first target rate increase." We use the middle month of the quarter to translate these responses into monthly horizons.

based on the $\mathbb{Q}$-modal path are also generally consistent with the FOMC's forward guidance. In particular, they responded to the introduction of calendar-based forward guidance by the FOMC on August 9, 2011, when the Committee noted in its statement that it expected a near-zero policy rate until at least mid-2013. This led to a substantial increase in the expected liftoff horizon (see also Swanson and Williams, 2014).

Turning to the liftoff forecasts under the real-world measure $\mathbb{P}$, we find that over the first half of the ZLB period, from 2009 to about mid-2012, these imply later liftoff than the estimates under $\mathbb{Q}$. The reason is that the former include the information in macroeconomic variables, whereas the latter are based solely on the information in the yield curve. The discrepancy between the two is due to the substantially depressed macroeconomic situation during the Great Recession and in the early years of the recovery. Our macro-finance model takes into account the elevated unemployment gap and low inflation during this period, which informs the $\mathbb{P}$-measure forecasts for short rates and liftoff. The $\mathbb{P}$-measure estimates of liftoff are also generally later than the outside estimates (the only exception being the MA forecasts in 2009), and later than the FOMC's forward guidance. It is noteworthy that the simple macro-finance shadow-rate model gave substantially longer and (with the benefit of hindsight) more reasonable estimates of liftoff during the early years of the ZLB period than most professional forecasters anticipated at the time.

The liftoff forecast can summarize the stance of monetary policy at the ZLB. Notably, it is highly correlated with the ZLB wedge in the ten-year yield—the correlation is 0.99 when using the $\mathbb{Q}$-measure modal path forecast liftoff. Intuitively, variation in the length of the expected period of near-zero policy rates is the main reason for variation in yields that are constrained by the ZLB.[29] For example, based on the liftoff estimates in Figure 6, the stance of policy became increasingly accommodative from 2009 to 2012. Federal Reserve forward guidance announcements—including the changes in the statement language about the appropriate path of the funds rate in 2008 and 2009, the explicit calendar-based forward guidance in 2011 and 2012, and the outcome-based forward guidance announcement in December 2012—typically had a noticeable impact and lengthened the estimated liftoff horizon. Accordingly, the expected liftoff horizon appears to be a fairly comprehensive univariate summary of the stance of monetary policy at the zero lower bound.

Another key dimension of monetary policy expectations at the ZLB is the expected pace

[29] In contrast, the connection between the ZLB wedge and the shadow short rate is much weaker, with a correlation coefficient of -0.47. The shadow short rate contains a more limited amount of information, lacks robustness and is hard to interpret. We also note that in contrast to estimated shadow short rates, our modal paths, forecasts for liftoff, and expected pace of tightening estimated from model $MZ(2)$ are very robust to different choices of the numerical lower bound, r_{min}.

of policy tightening after liftoff. For a *given* expected liftoff horizon, a faster expected pace implies higher interest rates, and vice versa. Hence, the anticipated pace of tightening is a crucial determinant of interest rates at medium and long maturities. To estimate the pace of tightening, a first issue is what metric to use. One possibility would be the time from liftoff until the policy rate reaches a certain higher threshold such as one or two percent. Instead, we use the cumulative anticipated increase in the policy rate over the two years after liftoff—a statistic that can easily be compared to previous monetary policy normalizations. A second issue is how to use shadow-rate models to estimate the anticipated increase in the policy rate after liftoff. We use the modal path for this purpose because again the modal path correctly accounts for the asymmetry of the short-rate distribution near zero. The mean path, in contrast, is always flatter than the modal path and would imply a slower pace of tightening.

Hence, we measure the expected pace of policy tightening as the increase in the modal path over the two-year horizon after first crossing the 25 basis point liftoff threshold. Figure 7 shows this measure for the modal path under both the $\mathbb{Q}$ and $\mathbb{P}$ measures. The first thing to note is that the pace under $\mathbb{Q}$ has been very volatile, varying considerably from below 1.5 to over 3 percentage points. This is due to the fact that this measure mostly reflects information in the cross section of interest rates, and any shift in the steepness of the yield curve translates into changes in the estimated pace. In contrast, the pace under $\mathbb{P}$ is much more stable, which reflects steadier model-based forecasts of future interest rates including macroeconomic variables. This measure, which is our preferred measure of the pace of tightening, declines over 2013 and 2014, and at the end of our sample, in December 2014, indicates an anticipated increase in the policy path of slightly below 1.5 percentage points over two years.

Figure 7 also shows the anticipated pace of policy tightening that is implied by the SPD modal policy paths. Both our model-based estimates imply a slower pace than is apparently expected by the survey respondents. While our model-based estimates imply an increase of about a 1.5 to 2 percentage points in the policy rate over the two years after liftoff, the Primary Dealers have generally anticipated fairly steadily an increase in the range of 2 to 2.5 percentage points. It is noteworthy that our preferred model-based estimates have generally implied both a later liftoff and a slower pace of tightening than outside estimates.

How do these estimates compare to historical episodes of policy tightening? During the tightening cycle from February 1994 to February 1995, the Fed increased the policy rate from 3 to 6 percentage points, which corresponds to a pace of 6 percentage points over two years. From June 1999 to May 2000 the increase was from 4.75 to 6.5 percent, at a pace of about 3.5 percentage points, and from June 2004 to June 2006 the policy rate was raised from 1 percent to 5.25 percent, an increase at a pace of 4.25 percentage points. Clearly, in all three previous policy tightening cycles, the pace of tightening was substantially faster than it is expected for the period after liftoff from the ZLB. This discrepancy may be explained by the unusual situation of the U.S. economy at the ZLB. Some Fed policymakers also expected a slow pace of policy tightening after liftoff (see, for example Dudley, 2014). In a speech in March 2015, Fed Chair Yellen (see Yellen, 2015) indicated that the pace of tightening in past monetary policy cycles may be a "highly misleading guide" to the course of monetary policy in 2015 and beyond because of perceived macroeconomic uncertainties, headwinds to the domestic and global economic outlook, and a possibly slower rate of long-run growth.

7 Conclusion

Using U.S. data, we estimate Gaussian affine and shadow-rate DTSMs with a variety of risk factors and elucidate some important issues about U.S. monetary policy at the zero bound. We estimate mean and modal paths for future short rates, taking into account the asymmetric probability distribution of future short rates at a range of projection horizons, and assess the associated dates for monetary policy liftoff from the ZLB. We argue that forecasts of policy liftoff using the term structure should be based on the modal path of future short rates, which is a near-optimal forecast and performs well empirically. We find that the increasing model-implied expectations of liftoff from 2009 to 2012 are very closely matched by private-sector and survey forecasts. Furthermore, the expected duration of the ZLB period can provide a useful measure of the stance of monetary policy and the tightness of the ZLB. Finally, we document the benefits of including macroeconomic information in shadow-rate models, which improves inference at the ZLB about future monetary policy.

An admitted shortcoming of our ZLB term structure model is the assumption of stationarity across pre-ZLB and ZLB periods, which, however, is made by all ZLB models that we are aware of. Given the unique character of a situation with near-zero policy rates, a useful direction for future research would be to allow for different macro-finance dynamics depending on whether the ZLB is binding or not. Further promising extensions of our modeling framework include imposing restrictions on the risk pricing to gain parsimony (Joslin et al., 2014; Bauer, 2015), pinning down $\mathbb{P}$-measure expectations more accurately using bias correction (Bauer et al., 2012) or survey-based interest rate forecasts (Kim and Orphanides, 2012), or using Bayesian inference for DTSM estimation (Chib and Ergashev, 2009; Bauer, 2015) to more accurately capture model and estimation uncertainty around shadow rates and estimates of monetary policy expectations at the ZLB.

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Table 1: Cross-sectional fit

Model	Total	3m	6m	1y	2y	3y	5y	7y	10y
Full sample									
YA(3)	6.7	7.0	8.5	10.0	2.7	5.8	6.0	2.8	7.2
MA(2)	11.8	16.6	9.7	14.3	12.5	10.2	8.2	7.2	13.2
YZ(3)	6.5	6.8	8.5	9.5	3.0	5.6	6.2	3.1	6.5
MZ(2)	10.4	13.5	10.4	12.4	10.5	9.3	6.9	5.7	11.8
ZLB subsample									
YA(3)	7.6	6.7	3.7	10.8	2.4	7.8	9.5	2.9	10.8
MA(2)	17.6	23.6	7.2	23.2	19.2	14.1	15.7	12.3	19.2
YZ(3)	6.8	5.5	4.5	9.2	3.4	6.8	9.7	4.3	8.2
MZ(2)	12.1	9.8	11.2	17.1	12.1	10.6	12.2	7.5	14.2

Notes: Root-mean-squared fitting errors of model-implied yields in basis points. Full sample: January 1985 to December 2014. ZLB subsample: December 2008 to December 2014.

Table 2: Violations of the ZLB

	Forward rates		Short-rate expectations	
Model	frequency	avg. length	frequency	avg. length
YA(3)	23	4.1	7	4.3
MA(2)	46	9.8	62	18.5

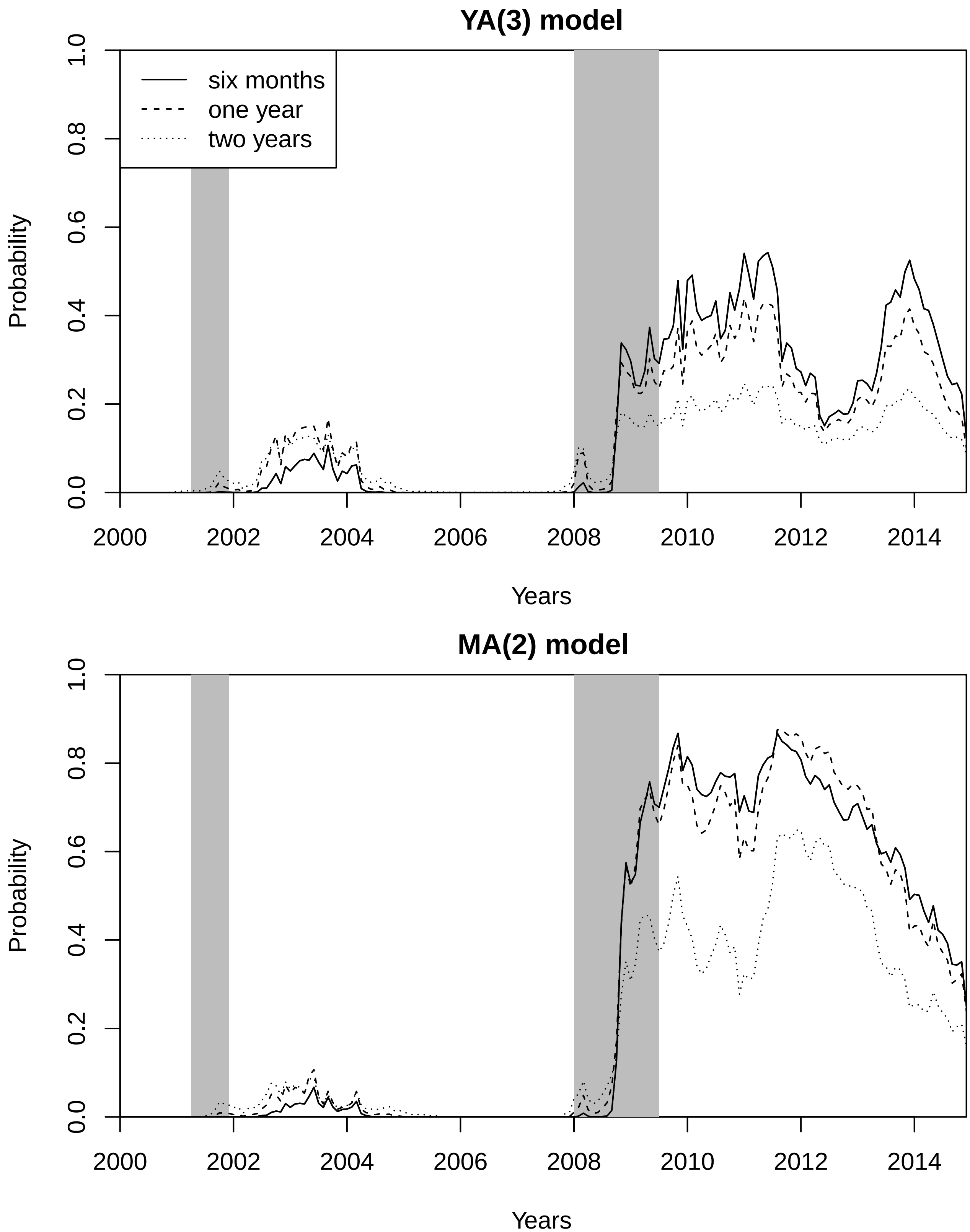
Notes: Number of months, between December 2008 and December 2014, in which some forward rates (column two) or short-rate expectations (column four) drop below zero, and the average length (in months) of horizon over which the forward curve/short-rate path stays negative.

Table 3: Out-of-sample forecast accuracy at the ZLB

Model(s)	6m	12m	18m	24m
Mean absolute forecast errors				
YA(3)	18.4	48.2	85.6	122.1
MA(2)	9.5	8.4	7.4	22.9
YZ(3)	8.0	20.5	49.7	88.2
MZ(2)	9.2	8.1	6.2	12.2
Relative mean absolute forecast errors				
YZ(3)/YA(3)	0.44^{*}	0.43^{***}	0.58^{***}	0.72^{***}
MZ(2)/MA(2)	0.97	0.96	0.84^{**}	0.54^{***}
MA(2)/YA(3)	0.52	0.17^{***}	0.09^{***}	0.19^{***}
MZ(2)/YZ(3)	1.15	0.39^{**}	0.13^{***}	0.14^{***}

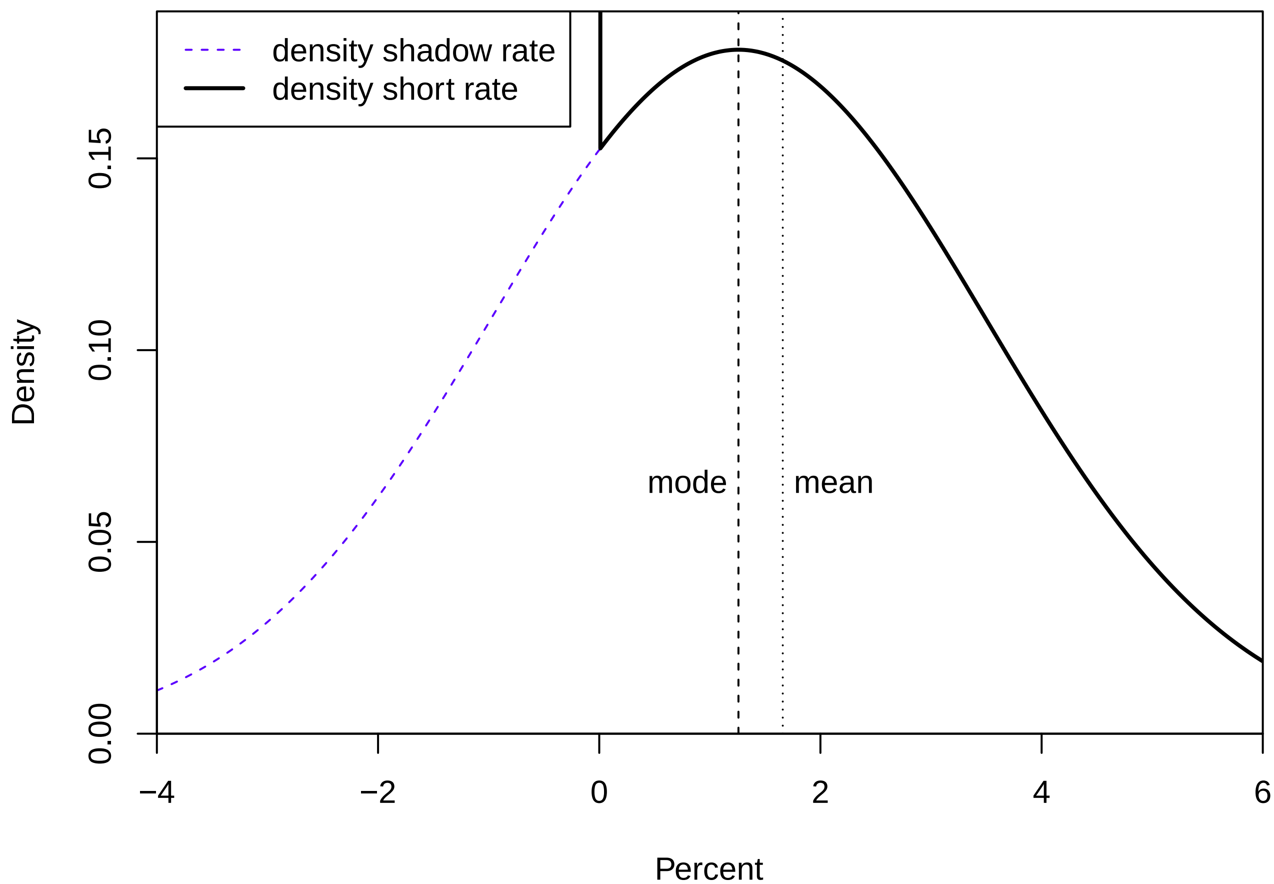
Notes: Top panel shows the mean absolute forecast errors (in basis points) for out-of-sample forecasts of the three-month T-bill rate at various forecast horizons. Bottom panel shows the relative mean absolute forecast errors for different model pairs. *, **, and *** indicate significance of the test for equal forecast accuracy at the 10%, 5%, and 1% level, respectively. Forecast period: December 2008 to December 2012.

Figure 1: Affine model probabilities of negative future short rates



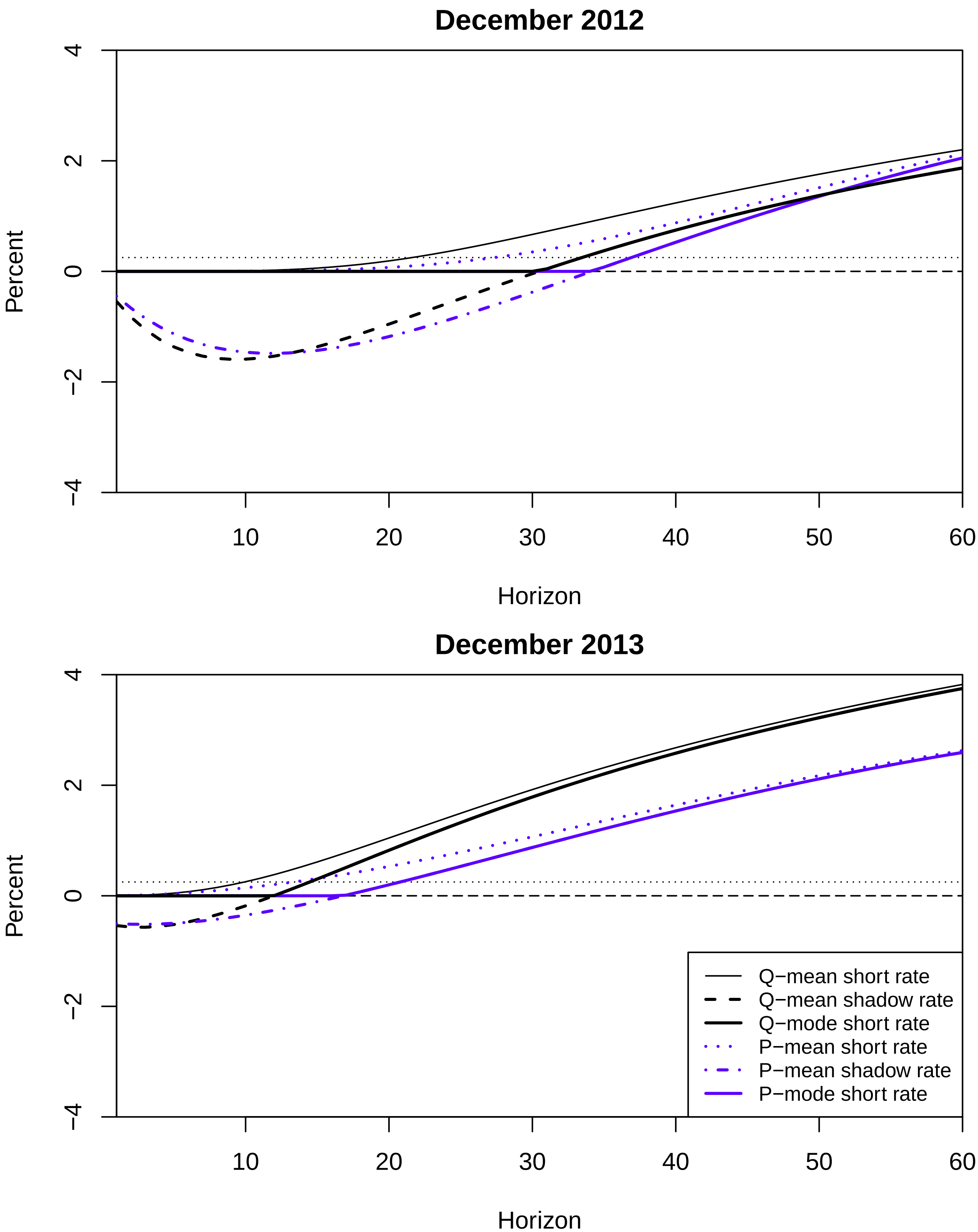
Notes: Model-implied real-world ($\mathbb{P}$) probabilities of negative future short-term interest rates at horizons of six months, one year, and two years. Shaded areas correspond to NBER recessions. Sample period: January 2000 to December 2014.

Figure 2: Distribution of future shadow rate and short rate



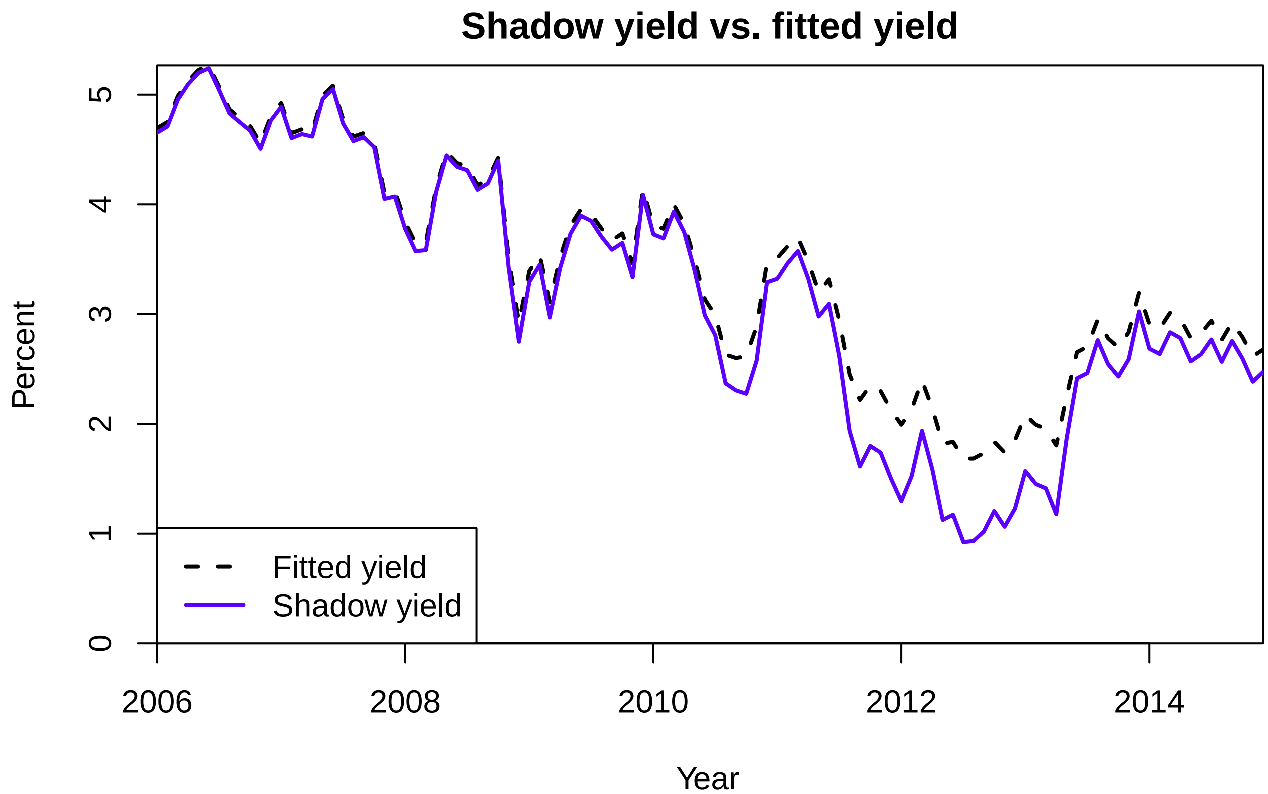
Notes: Densities of future shadow rate and of future short rate, at horizon of 48 months, on December 31, 2012. Vertical lines show the mode and mean of the distribution of the future short rate. Model: $MZ(2)$.

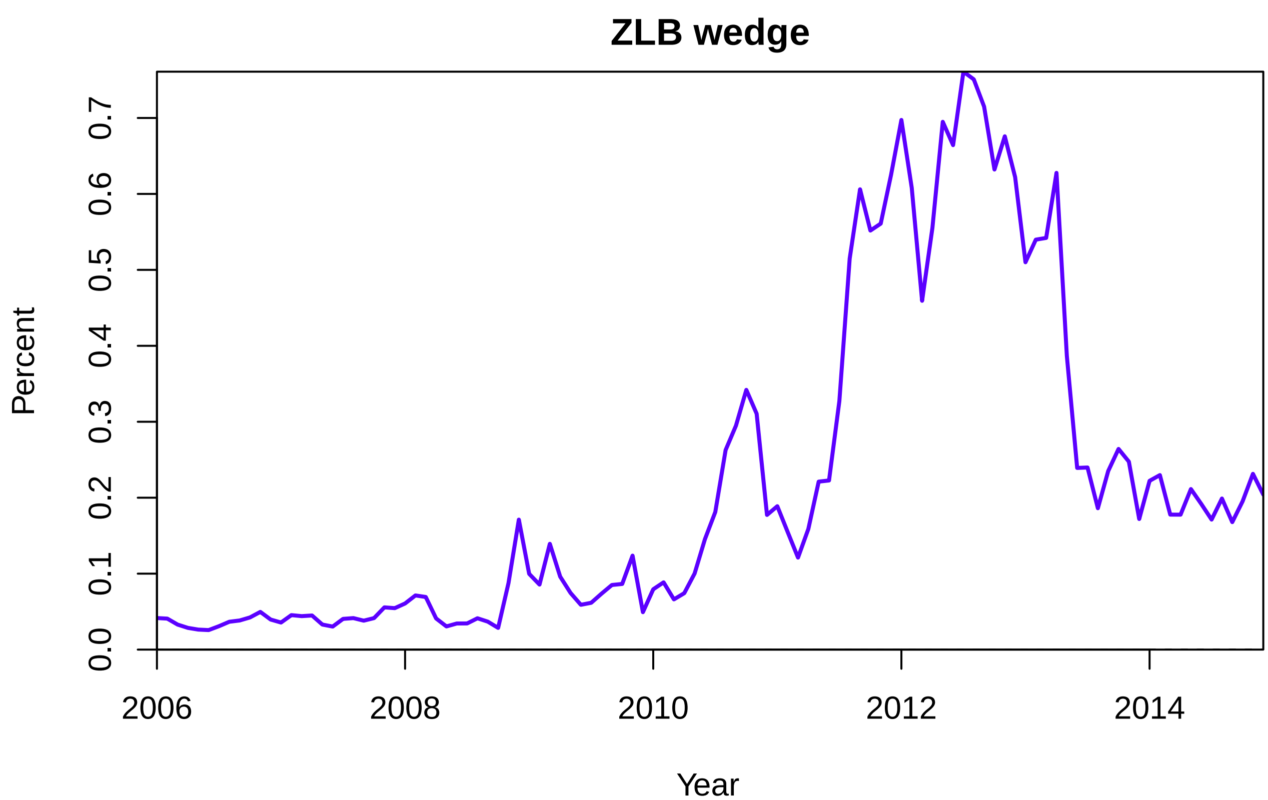
Figure 3: Policy paths



Notes: Mean and modal paths of future short rates, under real-world ($\mathbb{P}$) and risk-neutral ($\mathbb{Q}$) probability measure. The mean path under $\mathbb{Q}$ approximately corresponds to fitted forward rates. Model: $MZ(2)$.

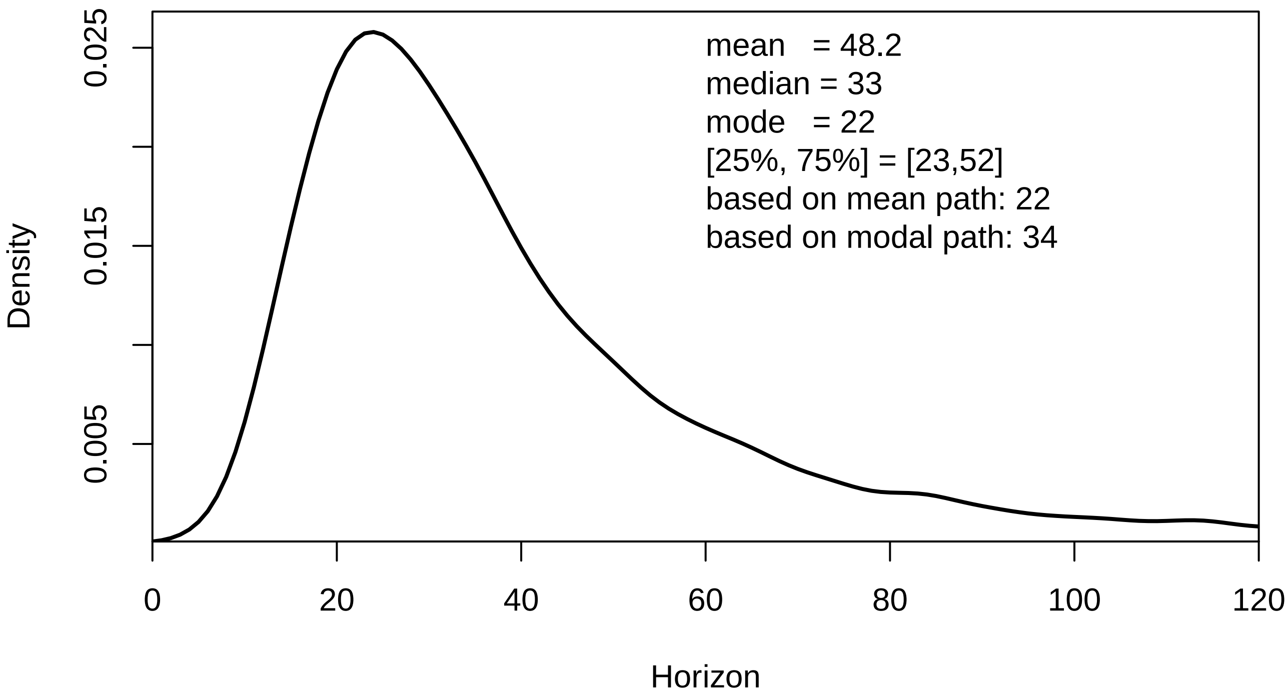
Figure 4: Ten-year yield and ZLB wedge





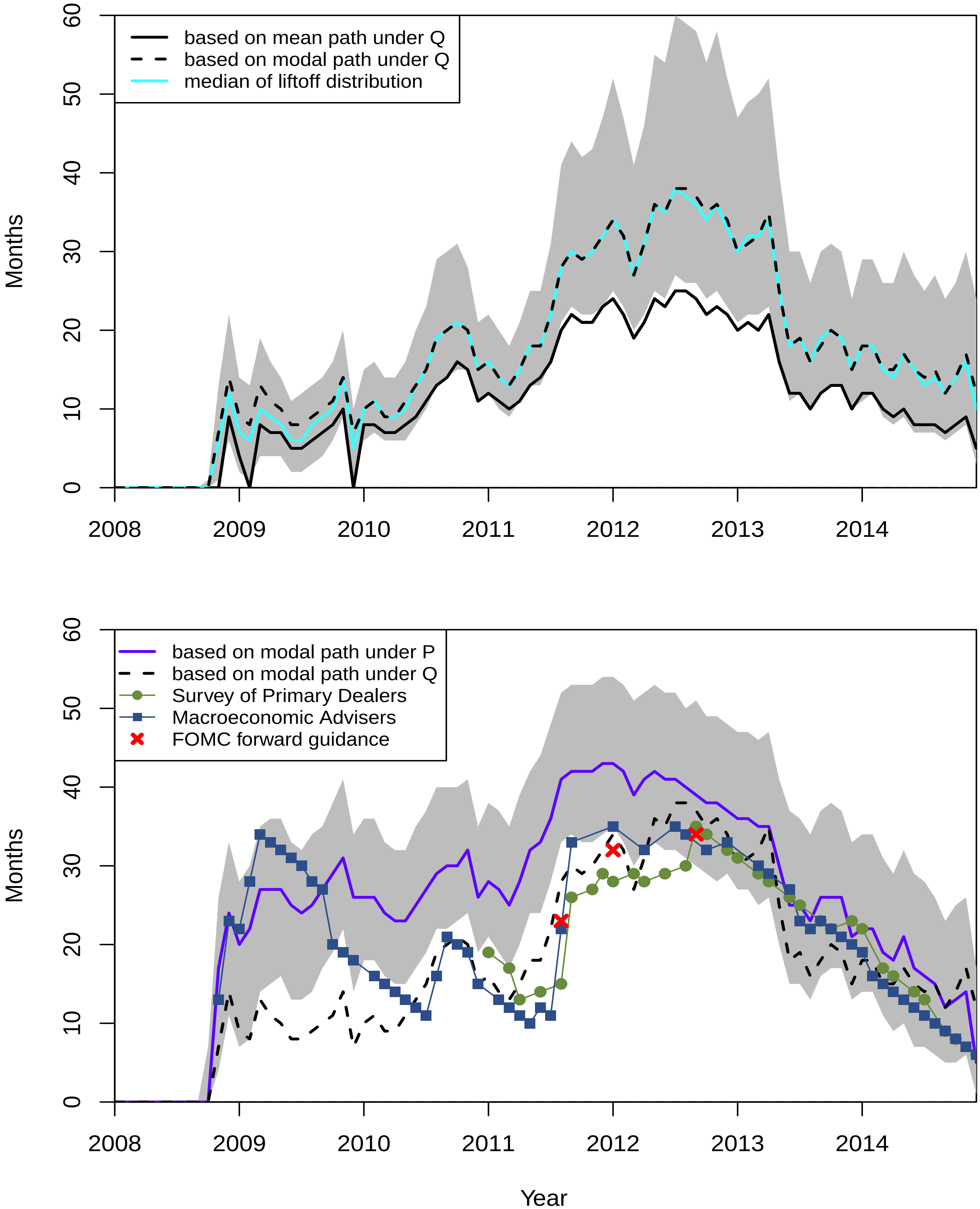
Notes: The top panel shows the fitted ten-year yield and the corresponding shadow yield. The bottom panel shows the difference between these two yields. Model: $MZ(2)$

Figure 5: Distribution of liftoff horizon



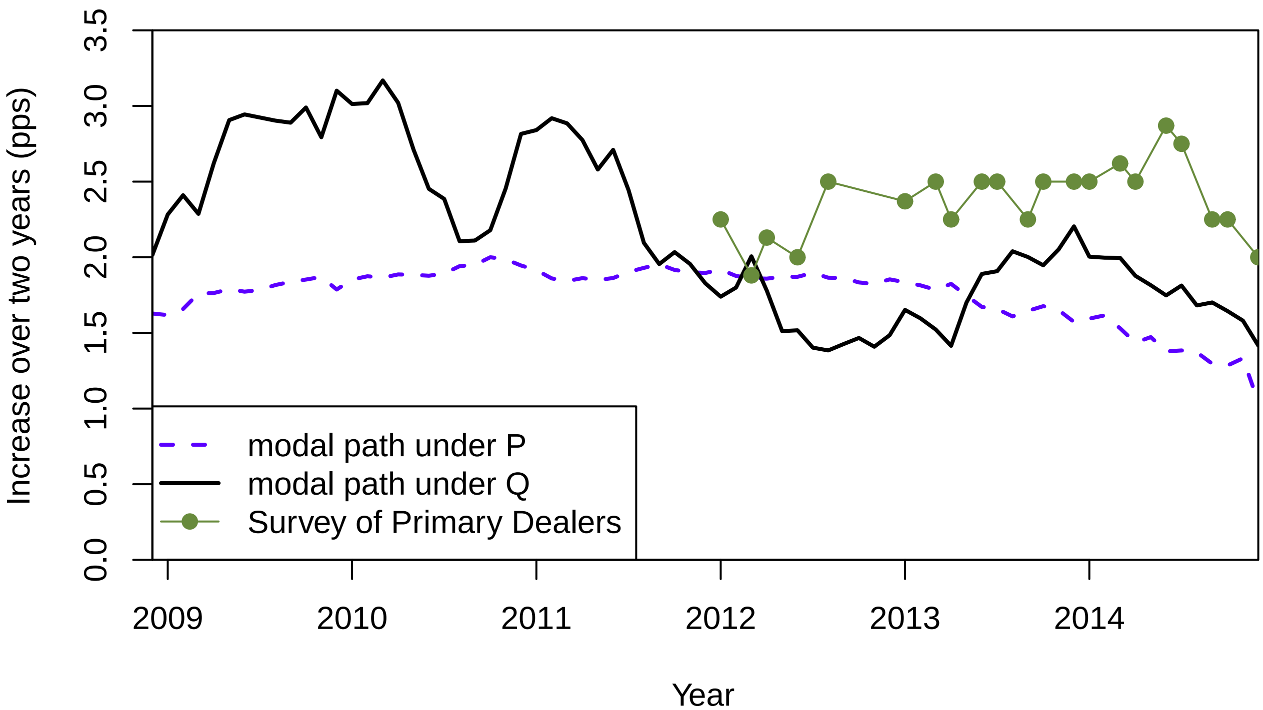
Notes: Density (under the risk-neutral probability measure) for the distribution of the liftoff horizon (in months), on December 31, 2012. Model: $MZ(2)$.

Figure 6: Liftoff



Notes: The top panel compares liftoff estimates from $MZ(2)$ based on the modal and mean paths, and the median of the liftoff distribution (all under $\mathbb{Q}$). The bottom panel compares modal-path estimates under $\mathbb{Q}$ and $\mathbb{P}$ to liftoff estimates from the Survey of Primary Dealers (median response) and from Macroeconomic Advisers, and to the FOMC's calendar-based forward guidance. Shaded areas are interquartile ranges of the liftoff distribution. Period: January 2008 to December 2014.

Figure 7: Pace of tightening



Notes: Model-based estimates of the pace of tightening after policy liftoff, measured as the increase in the modal path under the $\mathbb{Q}$- and the $\mathbb{P}$-measure during the subsequent two years, in percentage points. Also shown is the pace of tightening implied by the policy path reported in the Survey of Primary Dealers (median response). Model: $MZ(2)$. Period: December 2008 to December 2014.