A dynamic model of optimal creditor dispersion

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Abstract

Borrowing from multiple creditors exposes firms to liquidation risks due to coordination problems among creditors, but it also improves the firms' repayment incentives, thereby increasing pledgeability. Based on this trade-off, I develop a dynamic debt rollover model to analyze the evolution of creditor dispersion. Consistent with empirical findings, firms optimally increase the number of creditors when they perform badly, while in the cross-section, high growth firms can support more dispersed debt. Policies that promote ex-post efficient coordination lower firms' ex-ante pledgeability and therefore exacerbate rollover risk. Finally, frequent debt rollover diminishes the additional pledgeability from having multiple creditors.

1 Introduction

Many firms borrow from multiple creditors. Having more creditors risks coordination problems, making it more difficult for the firms to restructure their debt. At face value, this seems to suggest that during bad times, firms should consolidate their existing creditors, so that they can renegotiate the distressed debt more easily and avoid bankruptcy. Surprisingly, empirical evidence suggests the opposite. Companies increase the number of lenders when they are in trouble.[1] An obvious explanation is that their existing lenders refused to throw more good money after bad, leaving the company no choice but to borrow from new (and therefore more) creditors. However, this is only part of the story at best. The question remains: why are the new creditors willing to step in and "throw good money", while the incombent ones are busy "cashing out" and "rushing for the exit"?

From the policy perspective, many mechanisms exist to avoid the devastating consequences of coordination failures among creditors. For instance, the "automatic stay" clause in the bankruptcy stage prevents the creditors from panically grabbing debtor's assets. Moreover, distressed debt investors often buy dispersed debt and profit from negotiating efficiently with the company in private outside formal bankruptcy. Do such mechanisms always help prolong the firms' life as intended? How does the commitment to have an ex-post efficient negotiation affect the frms' ex-ante choice of creditor dispersion, which in turn affects the likelihood and the outcome of the negotiation?

To shed light on these questions, I develop a parsimonious dynamic model, in which a firm with insufficient internal resources must finance a long-term project by repeatedly rolling over short-term debt. A dispersed creditor structure creates coordination problems, which in bad times can result in inefficient liquidation. In good times, however, the same coordination problems enhance pledgeability by making it harder for the firm to opportunistically hold up its creditors. The firm optimally readjusts the number of creditors in every period by

[1] Farinha and Santos (2002) show that firms are more likely to abandon a single creditor structure and borrow from multiple ones when their performance measures worsen.

trading off the risk of rollover failure against the benefit of better commitment.

In contrast to the existing literature, which has mostly relied on static models to examine creditor structure,[2] the dynamic debt rollover framework is arguably closer to the reality. First of all, 47% of the Compustat firms during fiscal years 2012 and 2013 have insufficient operating cash flow to repay their maturing debt and thereby have to rely on debt rollover.[3] Perhaps more importantly, debt rollover itself is fundamentally a dynamic concept: the ability to roll over debt today depends on whether the firm's new creditors anticipate that they will be able to, in turn, roll over their debt in the future, which in turn depends on whether creditors anticipate that rollover will be possible even further in the future. Despite the model's parsimony, it generates a rich set of predictions, especially in the time series.

First, my model delivers predictions on how many creditors a firm has as well as when it decides to seek more creditors or to consolidate the existing ones. Consistent with empirical evidence, I show that firms increase the number of creditors when their performance deteriorates. This is because in the time series, when firms perform poorly, the required leverage endogenously increases. Firms must increase the number of creditors to make future repayment more credible and thereby support a higher debt capacity. In the cross-section, on the contrary, I show that firms with higher growth prospects can have more creditors, which is also consistent with empirical findings.[4] Intuitively, if a firm has a higher growth rate, then it has more upside to pledge and is less vulnerable to the downside rollover risk. Hence, it can support more dispersed creditors. The seeming contradiction between the cross-sectional and the time series predictions highlights the necessity of a dynamic model.

Second, I challenge the received wisdom that having multiple creditors and the resulting coordination problems are responsible for firms' difficulties in rolling over their debts. Indeed,

[2]For example, Berglöf and von Thadden (1994), Bolton and Scharfstein (1996), and Diamond (2004).

[3]I use the Compustat variables *total debt in current liability* (*DLC* — the total amount of short-term notes and the current portion of long-term debt that is due in one year) and *EBITDA* as the proxies for maturing debt and operating cash flow. For 47% of the firms, EBITDA is smaller than total debt in current liability.

[4]Farinha and Santos (2002) find that firms that refinance from more creditors tend to have better growth perspectives, as measured by sales growth.

many failed companies often suffer from coordination failure, such as lawsuits and disputes among their creditors. People are tempted to claim that the dispersed creditor structure prevented private debt restructuring that could have led to a more efficient resolution.[5] Implicit in such views is the idea that the firm would have had an easier time if it had had fewer creditors. But this counterfactual ignores the fact that borrowing from more creditors is an endogenous choice made by the firm in the past. Without such a choice, the firm could have failed even earlier. To make a more meaningful comparison, I contrast the expected liquidation probability and the firm value in my model with the ones in a counterfactual model in which the firm can borrow from only one creditor. I show that firms would have an even higher chance of liquidation and lower value, if they were forced to borrow from just one lender. An interesting implication is that policies, such as the automatic stay clause, that eliminate coordination failure among dispersed creditors may reduce a firm's ability to raise money ex-ante and result in lower welfare due to a more difficult debt rollover early on.

Finally, the dynamic model can predict how renegotiation frequency affects pledgeability. I find that in the limit when the firm can instantaneously renegotiate its debt, the additional borrowing capacity from more creditors disappears.[6] This result is perhaps surprising because dispersed short-term debt has been viewed by many existing studies as a mechanism to alleviate the commitment problem caused by renegotiation. In a dynamic world, my analysis suggests that the power of this mechanism can fade away when the commitment problem is severe. Intuitively, the ultimate source of the additional pledgeability comes from the growth of firm value between two negotiation dates. Having more creditors enables the firm to borrow more against the incremental value before the next negotiation, but not against the full value of the project. With extremely frequent negotiation, the interim growth vanishes, as does the additional debt capacity.

[5] For instance, many distressed firms turn to shadow banking system as the last borrowing source. Oliver Keene, the CEO of a failed jewelry-store chain, Brodkey Brothers Inc, describes shadow lending as the funding that "[y]ou can't get out". Similarly, Gary Rabin, the CEO of Advanced Cell Technology Inc, describes it "like a financing that never goes away". (See Dugan 2013)

[6] Because potential renegotiation occurs whenever the debt matures in the model, the finding can also be interpreted as one for debt maturity or rollover frequency.

2 Related Literature

Having multiple creditors can cause coordination problems. Perhaps the most famous example is bank (creditor) run. Diamond and Dybvig (1983) show that in a static setting, socially inefficient bank run equilibria generally exist. Goldstein and Pauzner (2005) further characterize the probability of a bank run under a global game framework. He and Xiong (2012a) study the dynamic evolution of a panic-based run on staggered corporate debt.

If borrowing from multiple lenders is subject to costly runs, then why do firms continue doing so? Berglöf and von Thadden (1994) claim that having multiple lenders specialize in lending at different maturities is a superior structure. The short-term creditors can impose externalities on the long-term creditors at the renegotiation stage, thereby increasing the ex-post repayment incentives and in turn the ex-ante efficiency. Following this line of thinking, Diamond (2004) demonstrates that when enforcing a debt contract is difficult, a single lender with a large stake in the firm has limited or no incentive to discipline the firm, since such actions also hurt the lender himself. The firm, knowing that disciplinary actions are not credible, will misbehave ex-ante. In the case of multiple creditors, the one who takes the action can claim against the whole firm, thereby hurting other creditors. The improved incentive for lenders to be active ex-post forces the borrowers to behave and thus increases the amount of money that can be raised ex-ante. These papers share the key insight that potential coordination failure with multiple creditors disciplines the firm and can potentially improve the ex-ante outcome. However, they vary the number of creditors exogenously and therefore are silent on when firms endogenously change the creditor structure.

Bolton and Scharfstein (1996) is, to my knowledge, the first paper that endogenizes the optimal number of creditors. They study the optimal choice between one and two creditors with the trade-off between liquidation risk and commitment power. The firms in their model can strategically default and renegotiate the debt even when they have the money to repay. The creditor(s), upon (either strategic or fundamental-based) default by the firm, can inefficiently sell the project to an outside investor. Under a multilateral bargaining setup, the benefit of having two creditors is to increase their collective bargaining power against the firm following a strategic default. In this case, the creditors can extract higher repayments from the firm. However, it lowers the expected payoff following a bad state, where this stronger collective bargaining power makes it less likely for the creditors to get an outside investor. A common feature of these papers is that they are all static, i.e. a one-time choice of the number of creditors upfront. My model shares the idea that having multiple lenders is a costly mechanism to induce correct behavior from the borrowers, but instead I bring the trade-off to a dynamic world. This dynamic feature is particularly important since firms usually do not have sufficient operating cash flow to pay back the maturing debt and must rely on repeatedly rollover.

Several other papers also explicitly investigate creditor structure from various perspectives. Detragiache, Garella, and Guiso (2000) present a completely different trade-off. If banks can fail, then having multiple banking relationships is beneficial because financing is more robust and will not fail unless all banks do. However, when all banks actually do fail, having more relationship banks is a stronger negative signal and therefore increases firms' refinancing costs. Brunnermeier and Oehmke (2013) study another aspect of debt contracts, namely the choices of debt maturities by dispersed creditors, and explain why an excessively short maturity structure may prevail in equilibrium, despite the increased rollover risks. Petersen and Rajan (1995) propose a model that illustrates how lenders' market power affects the quality of the financed firms and the cost of credit. They take the lenders' market power as an exogenous parameter. Among other differences, my paper endogenizes the variation of bargaining power by explicitly modeling the game between the firm and its creditors. Furthermore, the empirical studies in Petersen and Rajan (1994,1995) suggest that having more creditors is associated with a higher cost of credit in equilibrium, which is consistent with my model's prediction.

The effects of debt rollover and renegotiation on credit risk and debt prices have been studied from an asset-pricing perspective. Mella-Barral and Perraudin (1997) and Mella-

Barral (1999) study the asset-pricing implications when the firm can renegotiate and service the troubled debt, rather than just defaulting directly as in Leland (1998). He and Xiong (2012b) investigate how creditors with different maturities strategically interact with each other when they decide whether or not to roll over the maturing debt. Similar to the work of Diamond (2004), the creditors' decisions not to roll over pose externalities on other incumbent creditors with claims not yet matured. Hege and Mella-Barral (2005) look at an economy in which a firm can exchange liquidation rights for coupon concessions on debt and study how that feature affects the credit risk premia as the number of creditors changes. With the creditor structure exogenously fixed, these papers focus on pricing the debt claims given the possibility of renegotiation or rollover frictions. My paper, in contrast, focuses on the optimal choice of creditor dispersion.

3 The Model

3.1 The Project and Financing

Time t is discrete and the discount rate is normalized to 1. A risk-neutral penniless entrepreneur starts a firm at $t = 0$ with a project.[7] The project requires an upfront investment I_0 and only generates a final liquidating dividend at a random project maturity τ_π. At each date, the project matures with probability π. The final dividend depends on a stochastic firm-specific state $\theta_t \in \{G(ood), B(ad)\}$ and a fundamental process $Y_t = Y_0 \Pi_{1 \leq s \leq t} z_s$, where z_s are i.i.d. positive random variables with continuous density $g(z)$.[8] Assume $g(z)$ has a compact support $[\underline{z}, \overline{z}]$. Denote the mean $E(z_s) = \mu > 1$ and assume $\underline{z} < 1$. The random processes τ_π, θ_t and Y_t are independent. The final dividend is Y_t only if the state is good at

[7]Henceforth, I do not distinguish between the entrepreneur and the firm and use the two terms interchangeably.

[8]In the case where the entrepreneur and the outside investors share the *same* stochastic discount rate, one can simply redefine z_s to be the random shock multiplied by the discount rate. The model essentially stays the same. The analysis is more complicated, if the players discount the future dividend differently. This is left for future research.

maturity ($\theta_{\tau_\pi} = G$); otherwise, it is 0. The Markov process θ_t has a transition probability $p^\theta = Prob(\theta_{t+1} = \theta | \theta_t = \theta)$ (where $\theta = G, B$). It can be interpreted, for example, as the demand shock for the firm's output or the firm-specific productivity shock. To ensure that the project has a finite value, I impose the following parameter assumption:

$$(1-\pi)\mu < 1. \tag{1}$$

Given the initial state θ_1 and the fundamental Y_0, the expected final dividend is

$$E(\mathbf{1}_{\theta_{\tau_\pi}=G} Y_{\tau_\pi} | \theta_1, Y_0). \tag{2}$$

Lemma 1 *If the project is carried through to its random maturity τ_π, then its expected value defined in (2) conditional on the current state $\theta = G, B$ and fundamental Y is given by*

$$\begin{aligned} E(\mathbf{1}_{\theta_{\tau_\pi}=G} Y_{\tau_\pi} | \theta = G, Y) &= \tfrac{\pi[1-(1-\pi)\mu p^B]\mu}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]} Y, \\ E(\mathbf{1}_{\theta_{\tau_\pi}=G} Y_{\tau_\pi} | \theta = B, Y) &= \tfrac{\pi(1-p^B)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]} Y. \end{aligned} \tag{3}$$

At any time t, the project can be liquidated prematurely for λY_t. The liquidation value is assumed to be independent of θ because it is possible to sell the project to other firms that are not subject to this firm-specific shock. I assume that liquidation is always inefficient even in the bad state, namely $E(\mathbf{1}_{\theta_{\tau_\pi}=G} Y_{\tau_\pi} | B, Y) > \lambda Y$. By (3), this is equivalent to

$$\lambda < \frac{\pi(1-p^B)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]}. \tag{4}$$

Under this assumption, the values specified by (2) are indeed first best, which can be realized if the entrepreneur has enough cash to make the up-front investment I_0. Denote by $V_{FB}^{\theta\star}(Y)$ the first best values in (3).

In order to highlight the rollover frictions, I assume that the firm can only issue *one-period debt* to *short-lived* creditors. Since the project does not generate any interim cash flow, the firm must repeatedly issue new debt to finance the payment to the maturing creditors. The remainder of this section details a dynamic rollover game between the entrepreneur and the creditors.

3.2 Dynamic Debt Rollover Game

Figure 1 outlines the key ingredients of the dynamic debt rollover game. The firm enters period t with N_t incumbent creditors and a total promised face value F_t. The current state θ_t and the previous fundamental Y_{t-1} are also publicly known. At the beginning of the period, a new shock z_t (or equivalently Y_t) is realized, and then the project matures with probability π. If it matures, the game ends with a final dividend $Y_t \mathbf{1}_{\theta_t=G}$. Otherwise, a new state θ_{t+1} is realized, and the game moves on to the *repayment stage*. At this stage, the entrepreneur has three options: (a) voluntarily liquidating the project and ending the game, (b) making the promised repayment F_t, or (c) initiating a repayment negotiation. In case (b), the amount that the firm must refinance is $X_t = F_t$. In case (c), the firm meets each creditor sequentially in a *random* order and makes a take-it-or-leave-it offer S_i to the ith negotiating creditor. The offer history is public information. If any creditor rejects the offer, the negotiation fails and the firm is liquidated. Otherwise, the repayment negotiation is successful and the firm needs to refinance $X_t = \sum_1^{N_t} S_i$.

Following the repayment stage, the game enters the *refinancing stage*, in which the firm must raise the *exact* repayment X_t by offering new debt with an aggregate face value F_{t+1} to N_{t+1} new identical creditors. The new creditors simultaneously accept or reject the new debt offerings. Any rejection results in a liquidation. Otherwise, if all N_{t+1} new creditors accept, the firm survives period t and the next period begins with the new state variables $(\theta_{t+1}, Y_t, N_{t+1}, F_{t+1})$.

3.3 Terminal Payoffs and Markov Strategies

The entrepreneur is long-lived and the creditors live for only one period. If the firm survives period t, then the ith incumbent creditor receives $-\frac{X_{t-1}}{N_t} + S_i$ in the case of having a negotiation or $-\frac{X_{t-1}}{N_t} + \frac{F_t}{N_t}$ otherwise, where $-\frac{X_{t-1}}{N_t}$ captures the capital provided by the incumbent creditor during the previous period.

Alternatively, the game can end at date t if one of the following four events occurs: (i) the project matures, (ii) the entrepreneur voluntarily liquidates the project, (iii) the negotiating creditor forces liquidation, or (iv) the refinancing offer is rejected. In case (i), each of the N_t incumbent creditors equally gets $-\frac{X_{t-1}}{N_t} + \frac{1}{N_t}\min(\mathbf{1}_{\theta_t=G}Y_t, F_t)$ and the entrepreneur gets the residual $\max(0, \mathbf{1}_{\theta_t=G}Y_t - F_t)$. In case (ii) and (iv), each incumbent creditor receives $-\frac{X_{t-1}}{N_t} + \frac{1}{N_t}\min(\lambda Y_t, F_t)$ and the entrepreneur gets $\max(0, \lambda Y_t - F_t)$. Finally, in case (iii), I assume that the negotiating creditor who forces liquidation has priority over the liquidating cash flow, and gets $-\frac{X_{t-1}}{N_t} + \min(\frac{F_t}{N_t}, \lambda Y_t)$. Every remaining creditor receives $-\frac{X_{t-1}}{N_t} + \min(\frac{F_t}{N_t}, \frac{1}{N_t-1}\max(0, \lambda Y_t - \frac{F_t}{N_t}))$, and $\max(0, \lambda Y_t - F_t)$ goes to the entrepreneur.

A pure Markov strategy profile includes the following items. The firm has a negotiation strategy for the ith creditor $S_i^{\theta_{t+1}}(\sum_{j<i} S_j, F_t, Y_t, N_t) \in \mathbb{R}_+$ as a function of the total negotiated repayment in this period untill now $\sum_{j<i} S_j$, the originally promised face value F_t, the current fundamental Y_t, the realized next period state θ_{t+1}, and the number of incumbent creditors N_t. With a slight abuse of notation, $S_0^{\theta_{t+1}}(F_t, Y_t, N_t) \in \{L, F\}$ denotes a voluntary liquidation or a full repayment of F_t. The firm also has a set of financing strategies that specify the new number of creditors $N_+^{\theta_{t+1}}(X_t, F_t, Y_t) \in \mathbb{N}$ and the total face value $F_+^{\theta_{t+1}}(X_t, F_t, Y_t) \in \mathbb{R}_+$, as functions of the required financing amount X_t, the fundamental Y_t, and the state θ_{t+1}. In addition, in each period, the ith incumbent creditor has an acceptance strategy after receiving an offer S_i: $s_i^{\theta_{t+1}}(\sum_{j<i} S_j, S_i, F_t, Y_t, N_t) \in \{A, R\}$. Finally, given any refinancing offer (F_+, N_+), each new creditor $i \leq N_+$ has an acceptance strategy: $r_i^{\theta_{t+1}}(X_t, F_+, Y_t, N_+) \in \{A, R\}$.

Remark 1: Rather than taking an optimal contract design approach, I instead assume that only standard debt contracts are possible. I make this assumption because, unlike the other papers, I take the cross-externality among investors as given and focus on how creditor dispersion evolves dynamically. In addition, I do not allow the firm to privately save. However, as will be discussed in section 9, I do not expect the possibility of savings to change the firm's choice of creditor dispersion.

Remark 2: Assigning cash flow priority to the rejecting creditor when the repayment negotiation fails may appear to be a counterfactual assumption. Indeed, in the event of an actual default, ex-ante similar creditors are often treated equally ex-post by the bankruptcy court. The notion of "liquidation" in this model should be more broadly understood as, for example, costs of not being able to roll over financing. Such costs may include using valuable cash reserves, cutting back profitable investment, partial firesale of the assets, and so on. One scenario where the negotiating creditor enjoys the de facto priority is that a company has a staggered debt structure. The maturing creditor can potentially refuse a renegotiation offer. The firm may then have to incur the "liquidation costs", such as partial firesale, to honor the debt in full. The ex-post random negotiation order is an inconsequential technical simplification. A fuller analysis of the model under staggered debt structure is postponed to subsection 9.1.

The economic intuition behind the priority assumption is that the rejecting creditor can force liquidation and secure his own stake at the cost of other creditors. This creates a coordination problem among creditors which is at the heart of the model. The intuition of the model remains valid as long as the coordination problem exists and is more severe with a larger pool of creditors. In fact, these conditions are widely documented empirically.[9] From a theoretical perspective, Berglöf and von Thadden (1994), Bolton and Scharfstein (1996), and Diamond (2004) show coordination failure may naturally stem from optimal financing contracts.

[9]Using a natural experiment, Hertzberg, Liberti and Paravisini (2011) show that creditors reduce lending when they anticipate other incumbent creditors to learn negative information about the firm. Gilson John and Lang (1990) and Brunner and Krahnen (2008) both show that creditor dispersion adversely affects the probability of a successful workout for distressed firms.

3.4 Equilibrium Definition

Given any pure strategy profile, define $D^{\theta_t}_{N_t}(F_t, Y_{t-1})$ to be the total value of debt claims at the *beginning* of each period. Here I keep the time indices to make the evolution of the state variables transparent.

$$
\begin{aligned}
D^{\theta_t}_{N_t}(F_t, Y_{t-1}) \quad &= \quad E\{\pi \min(F_t, Y_{t-1}z_t)\mathbf{1}_{\theta_t=G} + (1-\pi) \\
&\quad\quad [\mathbf{1}_{RO}X_t + (1-\mathbf{1}_{RO})\min(F_t, \lambda Y_{t-1}z_t)]\},
\end{aligned} \tag{5}
$$

where $\mathbf{1}_{RO} \equiv (\Pi_{i\leq N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\leq N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})$ is the indicator function of a successful rollover, and the expectation is taken over the random variables z_t and θ_{t+1}. In the future, when the time indices are omitted, I use θ' to denote the next period state θ_{t+1}. The first term captures the payout to the debt holders upon project maturity, which happens with probability π. If the project does not mature, then the total (possibly negotiated) repayment X_t is honored if every player chooses not to liquidate. Otherwise, the liquidation payoff is distributed. Let τ_L be the stopping time when any player chooses liquidation. Define $\tau_S = \min(\tau_\pi, \tau_L)$ to be the time when the game ends. The total value of the firm at the beginning of each period can be naturally defined as

$$
V^{\theta_t}_{N_t}(F_t, Y_{t-1}) = E(\mathbf{1}_{\tau_S=\tau_\pi}\mathbf{1}_{\theta_{\tau_\pi}=G}Y_{\tau_\pi} + \mathbf{1}_{\tau_S=\tau_L}\lambda Y_{\tau_L}). \tag{6}
$$

In this paper, I focus on Markov perfect equilibria, meaning that the strategy profiles $\{(S_i^{\theta*}, N_+^{\theta*}, F_+^{\theta*}), s_i^{\theta*}, r_i^{\theta*}\}$ depend only on the above-mentioned payoff-relevant state variables and are subgame perfect.

1. Given other equilibrium strategies, the firm's negotiation offer $S_i^{\theta*}$ maximizes the net equity value:

$$
\mathbf{1}_{RO}(V^{\theta}_{N_+^{\theta*}}(F_+^{\theta*}, Y) - X) + (1-\mathbf{1}_{RO})(\lambda Y - F), \tag{7}
$$

where $S_i^{\theta*}$ implicitly feeds into the rollover indicator $\mathbf{1}_{RO}$, the total negotiated repayment X

and therefore next period debt structure $(N_+^{\theta*}, F_+^{\theta*})$ as well.

2. Given other equilibrium strategies, the rollover strategy $(N_+^{\theta*}, F_+^{\theta*})$ maximizes equity value (7).

3. The ith negotiating creditor weakly prefers to accept offer S_i^θ, (namely, $s_i^\theta = A$) if and only if the payoff from unilateral rejection is dominated:

$$\min(\frac{F}{N}, \lambda Y) \leq \begin{cases} \min(\frac{F}{N}, \frac{1}{N-1}\max(0, \lambda Y - \frac{F}{N})), & \text{if negotiation fails later} \\ \frac{1}{N}\min(F, Y), & \text{if refinancing fails later} \\ S_i^\theta & \text{if } \mathbf{1}_{RO} = 1 \end{cases}$$

4. The potential creditors at the refinancing stage weakly prefer to accept the new debt issuance (F_+, N_+) if and only if they can expect to at least break even:

$$-X + D_{N_+}^\theta(F_+, Y) \geq 0.$$

4 Equilibrium Characterization

Define the debt capacity from N creditors as follows:

$$DC_{N_t}^{\theta_t}(Y_{t-1}) \equiv \max_{F_t} D_{N_t}^{\theta_t}(F_t, Y_{t-1}) \tag{8}$$

and the total debt capacity as

$$DC^{\theta_t}(Y_{t-1}) \equiv \max_{N_t} DC_{N_t}^{\theta_t}(Y_{t-1}). \tag{9}$$

Finally, define $\overline{F}_{N_t}^{\theta_t}$ to be the face value that maximizes the value of debt, given the number of creditors N_t, fundamental Y_{t-1}, and state θ_t. If several values of F deliver this maximum, set $\overline{F}^{\theta_t}_{N_t}$ to be the smallest one:

$$\overline{F}^{\theta_t}_{N_t}(Y_{t-1}) \equiv \min[\arg\max_{F_t} D^{\theta}_{N}(F_t, Y_{t-1})]. \tag{10}$$

As will soon be transparent, the $V^{\theta}_{N}(F, Y)$ is decreasing in F, so the entrepreneur has no incentive to offer a new face value $F_+ > \overline{F}^{\theta}_{N_+}$. The first proposition characterizes the equilibrium strategies and the value functions.

Proposition 1 *Consider the following strategies:*

1. *The entrepreneur always offers the negotiating creditor his reservation value from unilateral liquidation:*
$$S^{\theta\star}_i = \min(\frac{F}{N}, \lambda Y). \tag{11}$$

2. *The entrepreneur's financing strategy $N^{\theta\star}_+(X, F, Y)$ and $F^{\theta\star}_+(X, F, Y)$ solves*
$$\begin{aligned} &\max\nolimits_{N_+} V^{\theta}_{N_+}(F_+, Y) \\ s.t. \quad &F_+ \text{ is the smallest solution to } \; D^{\theta}_{N_+}(F_+, Y) = X. \end{aligned} \tag{12}$$

   If there is no combination of (N_+, F_+) such that (12) holds, then the firm chooses $N^{\theta\star}_+ = 1$ and $F^{\theta\star}_+ = 0$.[10]

3. *The ith creditor accepts the offer S_i (i.e., $s^{\theta\star}_i(\sum_{j<i} S_j, S_i, F, Y, N) = A$) if and only if $S_i \geq \min(\frac{F}{N}, \lambda Y)$ and*
$$\sum_{j<i} S_j + S_i + \sum_{j>i} S^{\theta\star}_i \leq DC^{\theta}(Y). \tag{13}$$

4. *The potential new creditors accept the financing offers $r^{\theta\star}_i(X, F_+, Y, N_+) = A$ if and only if $D^{\theta}_{N_+}(F_+, Y) \geq X$.*

Under the proposed strategies, for any state $\theta = G, B$ and any number of creditors N,

[10] In fact, in this case, the financing strategy can be arbitrary because it will be rejected by the potential new creditors.

1. *the value of debt* $D_N^\theta(F,Y)$ *is continuous and homogeneous of degree one (HD1) in* (F,Y)*;*

2. *the minimum face value that achieves the* N *creditor debt capacity is linear in* Y*, i.e.,* $\overline{F}_N^\theta(Y) = \overline{f}_N^\theta Y$ *for some constant* $\overline{f}_N^\theta$*;*

3. *the debt capacity from* N *creditors is linear in* Y*, i.e.,* $DC_N^\theta(Y) = \kappa_N^\theta Y$ *for some constant* κ_N^θ*.*

Define

$$\kappa^\theta \equiv \max_N \kappa_N^\theta. \tag{14}$$

If $\min(\kappa^G, \kappa^B) > \lambda$*, then the proposed strategies indeed constitute a subgame perfect equilibrium. In addition, the firm's value function* $V_N^\theta(F,Y)$ *satisfies*

1. $V_N^\theta(F,Y) \geq \kappa_N^\theta Y$*, for any* $F \leq \overline{F}_N^\theta(Y)$*;*

2. $V_N^\theta(F,Y)$ *is continuous, HD1 in* (F,Y)*, weakly decreasing in* F*, and increasing in* Y*.*

The proof in the appendix takes a guess and verify approach. The key to this construction lies in finding a consistent (κ^G, κ^B) that gives linear debt capacities: $DC^\theta = \kappa^\theta Y$. Given this conjecture, the equilibrium strategies then imply that rollover is possible in state θ (i.e. $\mathbf{1}_{RO} = 1$), only when the total repayment offered is feasible:

$$\min(F, N\lambda Y) \leq DC^\theta(Y) = \kappa^\theta Y. \tag{15}$$

The debt value (5) becomes

$$\begin{aligned} D_N^\theta(F,Y) \;&=\; E\{\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi) \\ &\qquad [\mathbf{1}_{\min(F,N\lambda Yz)\leq\kappa^{\theta'}Yz}\min(F, N\lambda Yz) + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}\min(F,\lambda Yz)]\}, \end{aligned} \tag{16}$$

where the expectation is taken over z and θ'. It is easy to see that this is HD1 in (F, Y). Linearity of debt capacities is then just a simple corollary:

$$\begin{aligned}\kappa_N^\theta &= \max_f E\{\pi \min(f, z)\mathbf{1}_{\theta=G} + (1-\pi) \\ &\quad [\mathbf{1}_{\min(F,N\lambda z)\leq\kappa^{\theta'}z} \min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^{\theta'}z} \min(f, \lambda z)]\}\end{aligned} \tag{17}$$

Clearly (17) depends on the initial conjecture of (κ^G, κ^B), and it has to arrive at the same debt capacity in equilibrium by equating (14). Any guess of (κ^G, κ^B) that solves this fixed point problem is consistent and can be supported in an equilibrium.

One can observe from (17) that a low debt capacity tomorrow reduces the pledgeable amount, resulting in a lower debt capacity today. This self-fulfilling feature could result in multiple equilibria, but the results in this paper do not depend on which κ^θ I choose.

Despite the potential multiplicity, the existence of any equilibrium is not obvious at all. This is because the right-hand side in (17) as a function of κ^θ is not continuous. For example, when $\kappa^\theta \geq \lambda N$, it is always possible to roll over. However, as soon as κ^θ decreases to just below λN, there is a nontrivial chance that the firm will be liquidated, which lowers the ex-ante borrowing capacity discontinuously. Fortunately, despite the discontinuity of (17), the right-hand side is still order preserving and Tarski's fixed point theorem guarantees a solution.

With a consistent conjecture of κ^θ held fixed, the firm's value function (6) can be expressed recursively:

$$\begin{aligned}V_N^\theta(F, Y) &= \max_{N_+, F_+} E\{\pi Y z \mathbf{1}_{\theta=G} + (1-\pi) \\ &\quad [\mathbf{1}_{\min(F,N\lambda Yz)\leq\kappa^{\theta'}Yz} V_{N_+}^{\theta'}(F_+, Yz) + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz} \lambda Y z]\}\end{aligned} \tag{18}$$

where F_+ is the minimum solution to

$$D_{N_+}^{\theta'}(F_+, Y) = \min(F, N\lambda Y z). \tag{19}$$

Establishing continuity in V_N^θ is challenging, since a small change in (F, Y) can result in a discontinuous change in the minimum solution F_+. Therefore, the constraint correspondence $(F, Y) \mapsto \{(N_+, F_+)|\text{ s.t. (19) holds}\}$ is discontinuous and the standard theorem of maximum does not apply. Even so, one can show that the value function in equilibrium is indeed continuous. After proving the properties of the value functions V_N^θ, it is relatively straightforward to verify that the constructed strategy profile is indeed subgame perfect.

Despite the complicated construction and verification, the equilibrium is quite intuitive. The entrepreneur has all the bargaining power, so he just needs to *credibly* offer each creditor his liquidation payoff $\min(\frac{F}{N}, \lambda Y)$ as in (11). On the other hand, for an incumbent creditor to accept an offer S_i, it must be weakly higher than the liquidation payoff. In addition, condition (13) implies that the offer must be credible in the sense that following the proposed strategies, the total repayment can be financed.

The cost and benefit of having more creditors are immediately transparent from (15) and (16). With a higher N, the left-hand side of (15) weakly increases, causing a weakly higher chance of liquidation. On the other hand, having more creditors lowers the stake of an individual creditor relative to the whole firm and therefore effectively grants creditors higher bargaining power. The total actual repayment conditional on successful rollover in (16) weakly increases, as does the pledgeability.

4.1 Bounds on the Number of Creditors

The next lemma orders the debt capacities in the two states $\theta = G, B$.

Lemma 2 *The debt capacity is strictly higher in the good state, i.e.,* $\kappa^G > \kappa^B$.

Proof. Suppose otherwise if $\kappa^G \leq \kappa^B$, then

$$\begin{aligned} & \mathbf{1}_{\min(F,N\lambda z)\leq\kappa^G z}\min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^G z}\min(f, \lambda z) \\ \leq\; & \mathbf{1}_{\min(F,N\lambda z)\leq\kappa^B z}\min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^B z}\min(f, \lambda z). \end{aligned}$$

So (17) implies:

$$\begin{aligned}
\kappa^B &\leq \max_{N,f}(1-\pi)E[\mathbf{1}_{\min(F,N\lambda z)\leq\kappa^B z}\min(f,N\lambda z)+\mathbf{1}_{\min(F,N\lambda z)>\kappa^B z}\min(f,\lambda z)] \\
&\leq (1-\pi)E\kappa^B z=(1-\pi)\mu\kappa^B \\
&< \kappa^B.
\end{aligned}$$

Contradiction! So it must be $\kappa^G > \kappa^B$. ∎

This result is somewhat surprising because I do not pose any assumption on the transition probability p^θ. One might think that when p^θ is small, a bad state is very likely to transit into a good state in the next period and vice versa, rendering the debt capacity even higher in the bad state, because the future looks brighter. This intuition does not hold due to the infinitely recursive nature of debt rollover. Even though the state may be good in the immediate next period, this good state is itself short-lived. In fact, the above proof shows that in order to achieve the debt capacity in a bad state, the firm must rely on a higher borrowing capacity from a good state in the next period.

Given this lemma,[11] I can now define *creditor capacity* as

$$\overline{N}\equiv[\frac{\max(\kappa^G,\kappa^B)}{\lambda}]+1=[\frac{\kappa^G}{\lambda}]+1, \tag{20}$$

beyond which the equilibrium outcome no longer depends on the number of creditors. In fact, when $N\geq\overline{N}$, the rollover condition (15) becomes $F\leq\kappa^\theta Y$, and the firm is liquidated whenever the maturing debt is not fully repaid. In this case, both the commitment power and the rollover risk are at maximum. Using the simplified rollover condition, it is easy to check that the value of debt in (16) and the firm's problem (18) and (19) are independent of N. Without loss of generality, we can limit the firm's choice of the number of creditors to weakly below $\overline{N}$. The finite bound is also important for establishing equilibrium existence.

[11]The bracket denotes the floor function: $[a]$= the largest natural number weakly smaller than a.

Finally, I define the *safe number of creditors*:

$$\underline{N} \equiv [\frac{\kappa^B}{\lambda}]. \tag{21}$$

When $N \leq \underline{N}$, rollover is always possible since condition (15) always holds. In this case, having more creditors enhances pledgeability without an immediate risk of liquidation. However, as will be transparent shortly, it does not imply that the firm always prefers to have at least $\underline{N}$ creditors.

5 Key Trade-offs and Empirical Predictions

Only in this section, I study the comparative statics of exogenous changes in creditor dispersion. To do so, I change the incumbent number of creditors as if it is a parameter and keep the equilibrium continuation strategies. In other words, I study the outcome of a one shot deviation of the number of creditors in equilibrium. This exercise highlights the trade off between pledgeability and the liquidation risk that the firm faces when choosing creditor dispersion. Many empirical predictions can be carried through in equilibrium, whereas others may be reversed by the firm's selection effect. The empirical implications will be further discussed in section 6.2.

5.1 Pledgeability

5.1.1 Value of Debt, Debt Capacity, and Interest Rate

Notice from (16) that having multiple creditors has two offsetting effects on the value of debt. On the one hand, the entrepreneur's pledgeability improves with more creditors, as the actual repayment $\min(F, N\lambda Yz)$ increases. It in turn raises the value of debt for any given face value. On the other hand, having more creditors reduces the ex-post financial flexibility and leads to more liquidation, because the rollover condition $\min(F, N\lambda Yz) \leq \kappa^{\theta'} Yz$ is more stringent. The second effect lowers the debt value. The next result highlights the first channel.

Proposition 2 *Suppose that one of the following three conditions holds: (a)*$\underline{N} \geq N_2 > N_1$*, (b)*$\overline{N} > N_2 > N_1 > \underline{N}$*, or (c)*$N_2 > N_1 = 1$*. Then,*

1. *for any face value* F *and fundamental* Y*, the value of debt* $D^{\theta}_{N_2}(F, Y) \geq D^{\theta}_{N_1}(F, Y)$*,*
2. *as an immediate consequence of 1, the debt capacity is higher with more creditors, i.e.,* $\kappa^{\theta}_{N_2} \geq \kappa^{\theta}_{N_1}$*,*
3. *also as an immediate consequence of 1, the required interest rate is lower with more creditors: for any* θ *and* $X \leq DC^{\theta}_{N_1}(Y)$*, let* F^{θ}_{k} *(*$k = N_1, N_2$*) be the minimum solution to* $X = D^{\theta}_{k}(F^{\theta}_{k}, Y)$*. Then the solutions exist and* $F^{\theta}_{N_2} \leq F^{\theta}_{N_1}$*.*

The conditions (a) and (b) in proposition 2 control for rollover risks as N changes. In case (a), rollover is always possible, while in case (b), the firm is liquidated only in the bad state when the maturing debt cannot be paid in full. Therefore, in both cases, only the pledgeability effect remains. Case (c) is different. The value of debt is lowest with a single creditor because the actual repayment is just the liquidation payoff $\min(F, \lambda Y)$ regardless of whether or not rollover is possible.[12] It is easy to see from (16) that multiple creditors can at least secure a repayment of the liquidation value. Thus, having multiple creditors always weakly improves pledgeability. The single creditor debt capacity is attained when $F \geq \lambda Y \overline{z}$ and

$$\kappa^{\theta}_{1} = [\pi \mathbf{1}_{\theta = G} + (1 - \pi)\lambda]\mu. \tag{22}$$

As stated in proposition 2, the enhanced pledgeability lowers the required interest rate proxied by $\frac{F^{\theta}_{N}}{X}$. Note that one should not expect the negative correlation between the number of creditors and the interest rates to hold *in equilibrium*. I will postpone this discussion until section 6.2. As a preview, in equilibrium, firms choose more creditors when they do badly.

[12] The actual repayment is conditional on the project not maturing.

Creditors demand higher interest rates because the debt is more risky. Therefore, having more creditors is associated with poorer performance, which in turn causes higher interest rates.

5.1.2 Probability of Renegotiation and Default

I call it *renegotiation* whenever the firm *successfully* rolls over the maturing debt with an actual repayment strictly less than the promised face value. This occurs when $N\lambda Yz < \min(F, \kappa^\theta Yz)$ and the firm continues by repaying each creditor the liquidation value λYz. Similarly, I call it *default* whenever the creditors do not receive the full repayment F. Mathematically, default means $F > \min(\kappa^\theta, N\lambda)Yz$ when the project does not mature;[13] or $F > Yz\mathbf{1}_{\theta=G}$ when the project matures with insufficient final dividend to repay the creditors in full. Notice that a firm can renegotiate or default multiple times over its life cycle. To avoid any confounding effect, denote by τ_R and τ_D the *first* time that the firm renegotiates and defaults, and let

$$R_N^{\theta,T}(F,Y) = Prob(\tau_R \leq T \text{ and } \tau_R \leq \tau_\pi) \tag{23}$$

$$DFT_N^{\theta,T}(F,Y) = Prob(\tau_D \leq T \text{ and } \tau_D \leq \tau_\pi) \tag{24}$$

be the probability that firm does so at least once during the next $T \leq \infty$ periods before or when the project matures. Rewrite the probabilities recursively:

$$R_N^{\theta,T}(F,Y) = (1-\pi)E[R_{N_+^{\theta'\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{F\leq\min(\kappa^{\theta'},N\lambda)Yz} + \mathbf{1}_{N\lambda Yz<\min(F,\kappa^{\theta'}Yz)}], \tag{25}$$

[13] Default contains renegotiation as a special case and it also includes liquidation.

and

$$
\begin{aligned}
DFT_N^{\theta,T}(F,Y) &= \pi[Prob(Yz < F)\mathbf{1}_{\theta=G} + \mathbf{1}_{\theta=B}] + (1-\pi) \\
&\quad E[DFT_{N_+^{\theta'\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{F\le\min(\kappa^{\theta'},N\lambda)Yz} + \mathbf{1}_{F>\min(\kappa^{\theta'},N\lambda)Yz}].
\end{aligned} \tag{26}
$$

The meaning of formulation (25) is straightforward. With probability $1-\pi$, the firm enters the repayment stage. Renegotiation occurs if $N\lambda Yz < \min(F, \kappa^\theta Yz)$; otherwise, if rollover is possible with a full repayment F, the continuation probability of renegotiation in the next $T-1$ periods is calculated by using the equilibrium refinancing strategies for the next period number of creditors $N_+^{\theta'\star}$ and face value $F_+^{\theta'\star}$. The expression (26) can be similarly interpreted.

If the project continues without a renegotiation or default, the creditors are paid F in full regardless of the number of creditors N. Therefore, the continuation probabilities $R_{N_+^{\theta\star}}^{\theta,T-1}(F_+^{\theta\star}, Yz)$ in (25) and $DFT_{N_+^{\theta\star}}^{\theta,T-1}(F_+^{\theta\star}, Yz)$ in (26) are independent of N as well. On the other hand, as N increases, the region in which the firm makes the full repayment widens, since $F \le \min(\kappa^\theta, N\lambda)Yz$ is more likely to hold. Finally, the probability of a renegotiation or a default in the immediate next period is lower. The next proposition summarizes the discussion.

Proposition 3 *The probabilities of renegotiation and default are lower with more creditors, i.e., $R_{N_2}^{\theta,T}(F,Y) \le R_{N_1}^{\theta,T}(F,Y)$ and $DFT_{N_2}^{\theta,T}(F,Y) \le DFT_{N_1}^{\theta,T}(F,Y)$, for any $N_2 > N_1$, θ, F, and Y.*

Proposition 3 is another way to demonstrate the pledgeability channel. Having more creditors provides a better repayment incentive and therefore reduces the probability that the firm willingly (renegotiation) or unwillingly (default) cuts debt repayment.

5.2 Rollover Risk

5.2.1 Probability of Liquidation

Recall that τ_L and τ_π are the random times of liquidation and project maturity. Define

$$L_{N_t}^{\theta_t,T}(F_t, Y_{t-1}) = Prob(\tau_L \leq t + T \text{ and } \tau_L < \tau_\pi), \tag{27}$$

at the beginning of period period t, to be the expected probability of liquidation in the next $T \leq \infty$ periods before the project matures. Since liquidation occurs if and only if (15) is violated, the liquidation probability L must satisfy the recursive formulation:

$$L_N^{\theta,T}(F,Y) \;=\; (1-\pi)E[L_{N_+^{\theta'\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{\min(F,N\lambda Yz)\leq\kappa^{\theta'}Yz} + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}] \tag{28}$$

With probability $1-\pi$, the firm enters the repayment stage. A failed negotiation results in an immediate liquidation; otherwise, the continuation probability of liquidation in the next $T-1$ periods is calculated by using the equilibrium refinancing strategies $(N_+^{\theta\star}, F_+^{\theta\star})$.

A direct consequence of having more creditors is that the immediate liquidation probability

$$L_N^{\theta,1}(F,Y) \;=\; (1-\pi)P(\min(F, N\lambda Yz) > \kappa^{\theta'}Yz)$$

increases because the rollover condition (15) is less likely to hold with a bigger N. I state this simple result as a lemma.

Lemma 3 *The one-period-ahead liquidation probability increases with the number of creditors, i.e.,* $L_{N_2}^{\theta,1}(F,Y) \geq L_{N_1}^{\theta,1}(F,Y)$ *for all* $N_2 > N_1$*,* θ*,* F*, and* Y*.*

Lemma 3 highlights the cost of having more creditors arising from a higher chance of an immediate liquidation. It is also helpful to compare lemma 3 with a seemingly contradictory result proposition 3. Fundamentally unlike liquidation, renegotiation as I defined in subsection 5.1.2 pose no direct welfare loss, since it does not lead to an inefficient termination of the project. Instead, it (negatively) reflects the entrepreneur's commitment power. With more creditors, the entrepreneur commits to make more repayment at the cost of a more likely ex-post liquidation.

One can interpret renegotiation and default as financial distress and liquidation as a *costly* outcome (for example, formal bankruptcy proceedings). Under this interpretation, the results in this subsection state that with more creditors, the firm ex-ante is less likely to end up in distress. Once it is in distress, however, the creditors are less likely to strike a deal. This prediction is confirmed by Gilson, John, and Lang (1990), who find that financially distressed firms with more creditors are less likely to turn around and emerge from a private debt restructuring.

5.2.2 Firm Value

Exogenous increase in the number of creditors reduces the total firm value. An immediate consequence of having more creditors is a greater liquidation risk in the next period. The long-run effect is the higher actual repayment which permanently increases the future liquidation probability. Both effects lower the firm value. The second channel is also why firms in general are *not* indifferent when $N < \underline{N}$, as previously mentioned at the end of section 4.

Proposition 4 *Given other state variables, the firm value is lower with more creditors:* $V^{\theta}_{N_1}(F, Y) \geq V^{\theta}_{N_2}(F, Y)$ *for any* θ, F, Y, *and* $N_1 < N_2$.

In summary, the discussions in this section show that having exogenously more creditors improves pledgeability by lowering the probability of renegotiation and default, resulting in a higher value of debt. On the other hand, it also poses higher rollover risks and reduces firm value. The next section studies the evolution of creditor dispersion in equilibrium.

6 Creditor Dynamics

The dynamics associated with the number of creditors is determined by the firm's refinancing problem (12). As in proposition 2, the benefit to borrow from more creditors N_+ is a potentially lower refinancing cost F_+. The cost, as in lemma 3, is a higher immediate liquidation threat in the next period. The firm optimally chooses N_+ by trading off the benefit against the cost. Unfortunately, for a dynamic discrete choice problem like this one, an analytical solution is not generally available. However, all the numerical examples that I have calculated unanimously show that the cost of having more creditors *always* outweighs the benefit. The firm chooses more creditors only when refinancing from fewer ones is infeasible.

In this section, I first provide a sufficient condition under which the firm must increase the number of creditors and utilize the higher borrowing capacity. This is when the firm performs poorly. Then, with a representative numerical example, I show that some empirical predictions with exogenous variations in creditor dispersion can change completely when such variations are endogenously determined in equilibrium. Finally, I study the cross-sectional creditor dispersion by varying the distribution of fundamental shock z_t. In contrast to the time-series prediction, better companies with superior fundamental can support more creditors.

6.1 When Do Firms Choose More Creditors?

The next result shows that firms with poor historical performance must borrow from more creditors to refinance the required repayment. Here, I keep the time subscripts to avoid any confusion.

Proposition 5 *Suppose that the realized fundamental is low $F_t \geq N_t \lambda Y_t$, the state is bad $\theta_{t+1} = B$, and rollover is possible $N_t \lambda \leq \kappa^B$. Then the continuation number of creditors must strictly increase, $N_+^{B\star}(N_t \lambda Y_t, F_t, Y_t) > N_t$.*

Proof. Providing the proof here is worthwhile. Equilibrium strategy (11) combined with the assumption $N_t\lambda Y_t \leq F_t$ suggests that the firm offers liquidation value to each creditor λY_t. By assumption, $N_t\lambda Y_t \leq \kappa^B Y_t$, so the repayment is feasible. Because the realized repayment to each creditor in the next period $t+1$ is at most $\min\{\frac{F_{t+1}}{N_{t+1}}, \lambda Y_{t+1}\} \leq \lambda Y_{t+1}$, the debt capacity $\kappa^B_{N_{t+1}} Y_t$ from (16) in the bad state with N_{t+1} creditors is bounded by

$$(1-\pi)N_{t+1}E(\lambda Y_{t+1}) = (1-\pi)\mu(N_{t+1}\lambda Y_t).$$

Since $(1-\pi)\mu < 1$ by assumption (1), the above bound is in turn dominated by $N_{t+1}\lambda Y_t$. The firm chooses a continuation number of creditors N_+ at least to finance the required repayment $N_t\lambda Y_t$. Therefore,

$$N_t\lambda Y_t \leq \kappa^B_{N_+} Y_t < N_+\lambda Y_t.$$

So it must be $N_+ > N_t$. ■

The economics behind this result is straightforward. As both the fundamental and the firm-specific state are poor, the required leverage endogenously increases. In order to support the higher leverage, the firm must refinance from more creditors and make itself more credible. This result has also been empirically documented by Farinha and Santos (2002), who show that firms are more likely to abandon a single creditor structure when the performance measures are worse.[14]

6.2 A Numerical Example and Interest Rate in Equilibrium

The numerical example is based on the following parameter choices: the per period fundamental shock $z \sim uniform(0.63, 1.83)$, the transition probabilities of the firm-specific state $(p^G, p^B) = (0.8, 0.3)$, the project maturing probability per period $\pi = 0.2$, the liquidation coefficient $\lambda = 1$, and the required up-front investment $I_0 = 1$. Even though $\lambda = 1$, liqui-

[14]The performance measures include liquidity, cash flow, leverage and so on.

dation is still inefficient since the future growth opportunities are lost. The key equilibrium variables, debt capacities, are calculated to be $(\kappa_1^G, \kappa_2^G, \kappa_1^B, \kappa_2^B) = (1.23, 1.273, 0.984, 1.022)$ and $\kappa^\theta = \kappa_2^\theta$. Under this parameterization, the creditor capacity $\overline{N} = 2$, and therefore the relevant choice for the new creditors N_+ is between 1 and 2. The numerical example is not designed to match any data, but its qualitative features are robust to parameter and distribution choices.

Figure 2 plots the total firm value normalized by the fundamental (i.e. $\frac{V_N^\theta(F,Y)}{Y}$) against the normalized value of the debt (i.e. $\frac{D_N^\theta(F,Y)}{Y}$) or equivalently the amount that has to be borrowed $\frac{X}{Y}$ in problem (12). The solid and the dashed lines are the firm value with a single creditor when the state $\theta = G$ and $\theta = B$ respectively. Similarly, the dotted and dash-dotted lines depict the firm value with two creditors conditional on the state being good and bad. The thick solid segments on the lower curves can be supported only by two creditors.

A quick observation is that when the value of debt is low, the values of firms with one and two creditors coincide. This is because the firm is essentially committed to repay the full face value regardless of the state variable realizations and the number of lenders.[15] As the value of debt increases, the two lines diverge and, when both are feasible, the single creditor case always delivers a higher firm value. This pattern suggests that the cost of inefficient liquidation is greater than the benefit of interest reduction (lower continuation face value F_+). However, since the curves end on the x-axis at κ_N^θ,[16] the lower curves for two creditor firms indeed extend farther than their single creditor counterparts. This means that when the firm needs to borrow beyond its single creditor debt capacities, it has to seek two creditors.

Figure 3 is a typical sample path of the firm. Areas are shaded when the state is bad. The solid and dashed lines denote the exogenous fundamental process Y_{t-1} and the endogenous face value process F_t respectively. I use bold segments when the firm chooses two creditors.

[15] When $\frac{F}{Y}$ is sufficiently small such that $\frac{F}{Y\underline{z}} \leq 1$, the firm will repay F in the next period in order to continue.

[16] Notice $\frac{D_N^\theta(F,Y)}{Y} \leq \kappa_N^\theta$ by definition.

The values plotted at each period t are the state variables entering this period: number of creditors N_t, the promised face value F_t, state θ_t, and fundamental process Y_{t-1}. Finally, the dotted bars plot the interest rates $\frac{F_t}{D_{N_t}^{\theta_t}(F_t, Y_{t-1})}$ during each period.

The firm starts by borrowing the required investment $I_0 = 1$ from one creditor in a good state with an interest rate of 9% (a level of 1.09 in the plot). During period 1, the fundamental drops to 0.7. With a single creditor, the firm negotiates the actual payment down to the liquidation value 0.7 and issues new debt with a face value of 0.76 and an interest rate of 9% to finance the repayment. During period 2, the fundamental keeps deteriorating to 0.49 and the state θ_3 switches to bad. The firm again negotiates the actual payment down to 0.49. With a bad state, the firm must refinance this payment from two creditors, because the debt capacity with a single creditor is insufficient to cover the liquidation value. The interest rate soars to 63%. The firm enters period 3 with face value $F_3 = 0.8$. During period 3, even though the state θ is still bad, the fundamental dramatically improves and the firm is able to make the promised repayment 0.8 and roll over the debt with a single creditor. The required interest rate reduces to 49%. What happens during period 4 is very similar to period 1. The state θ returns to good and the firm pays out and refinances the liquidation value by borrowing from one creditor at an interest rate of 9%. At period 5, the fundamental continues to improve to 1.23, and the firm can even issue risk free debt to finance the 0.77 debt obligation. This is possible since even if the project matures with the lowest shock realization $z = \underline{z} = 0.63$, the full value of the debt can still be repaid.[17] Period 6 and 7 are similar to period 2 and 3: the state switches to bad, the financing costs for the firm increases and two creditors are eventually required. Finally, during period 8, the state θ returns to good and the realized fundamental improves to 1.14. Even so, the borrowing capacity is only $1.14 \times 1.27 = 1.45$, which is not high enough to cover the promised amount of 1.55 to the two creditors. The firm is then liquidated.

The first noticeable feature in figure 3 is that the firm switches to two creditors only in

[17] To be specific, $1.23 \times 0.63 > 0.77$.

the bad state $\theta = B$ when the fundamental deteriorates and consolidates back to a single creditor structure when its performance improves. In the model, the firm is never liquidated with a single incumbent creditor. Therefore, the firm only uses the costly borrowing capacity from two creditors as a last line of defense.

Second, the interest rates are higher in general with more creditors. Why does this not contradict with proposition 2, which states that having more creditors reduces interest rates? Even though an exogenous increase in the number of creditors may increase pledgeability and lower the interest rate, once the number of creditors is endogenized in equilibrium, the selection effect dominates. The firm only chooses to have more creditors when they are in worse shape, which in turn causes higher interest rates. Empirically, Petersen and Rajan (1994) find that companies with more banking relationships also have higher cost of credit.

6.3 Creditor Dispersion in the Cross-section

When the per period fundamental shock z_t on average improves, the future of the firm becomes more promising. It has several effects. A direct effect is that the firm has a higher liquidation value on average in the next period, which increases the bargaining position of the creditors. Second, the firm in the next period is more likely to have the resources to make the promised repayment. Finally, bad fundamental shocks are less often, reducing the likelihood of inefficient rollover failures. All effects improve the value of debt as well as the debt capacity, and therefore more creditors can be supported.

Proposition 6 *Suppose g_i ($i = a, b$) are two density functions for z, and g_a first-order stochastically dominates g_b. Then for any equilibrium under g_b, there exists an equilibrium under g_a such that $\kappa^{\theta,a} \geq \kappa^{\theta,b}$, where $\kappa^{\theta,i}$ are the corresponding debt capacity coefficients.*[18] *In addition, the creditor capacity and the safe number of creditors are both higher under g_a, i.e., $\overline{N^a} \geq \overline{N^b}$ and $\underline{N^a} \geq \underline{N^b}$.*

[18] The opposite direction holds too. That is, for any equilibrium under g_a, there exists an equilibrium under g_b such that $\kappa^{\theta,a} \geq \kappa^{\theta,b}$.

Since first-order stochastic dominance implies that the average growth rate is higher, a direct implication is that firms with higher growth rates can be associated with more creditors. This is consistent with the empirical evidence documented by Farinha and Santos (2002), who find that firms with better growth perspectives, as measured by sales growth, tend to have more creditors.

Notice that the cross-sectional result is qualitatively opposite to the time series prediction in proposition 5, which states that firms increase the number of creditors after they perform poorly. The startling contrast highlights the necessity to draw time series conclusions from a dynamic model.

7 The Value of Coordination Failure and Policy Implications

Quite often, firms in distress or even default are more valuable as going concerns than they are being liquidated piecemeal. However, coordination failure among creditors may prevent the more efficient outcome. With this friction in mind, many policies are designed to reduce or eliminate coordination problems. For example, the *automatic stay* clause, which halts creditors' actions to claim a debtor's assets, and *Chapter 11 reorganization*, which promotes a constructive renegotiation with all creditors collectively, both fall into this category. Broadly speaking, collateral arrangement and the existence of distress debt investors also improve ex-post coordination among creditors.[19] If the policies indeed eliminate all ex-post coordination failure and commit multiple creditors to negotiate the dispersed debt as one, then I show that such policies may potentially cause ex-ante higher chances of liquidation and lower firm values.

Committing to an ex-post efficient negotiation is equivalent to a counterfactual model

[19] Many distressed debt investors acquire dispersed debt and attempt an efficient negotiation with the troubled company.

in which the firm can borrow *only* from one creditor. With one creditor, the firm always renegotiates the repayment down to the liquidation value if the project does not mature. Therefore, the debt capacities are (κ_1^N, κ_1^B) given by (22) in this counterfactual case.

In the bad state, the debt capacity is less than the liquidation value, i.e. $\kappa_1^B = (1-\pi)\mu\lambda < \lambda$. When the realized fundamental Y is sufficiently weak ($F > \lambda Y$), repayment negotiation fails because the firm cannot credibly pledge the liquidation payoff $\min(F, \lambda Y) = \lambda Y > \kappa_1^B Y$. Therefore, the single-creditor counterfactual case has effectively no room for negotiation in a bad state. On the contrary, in the true model if the firm is allowed to have multiple creditors, it can pledge at least λY (in fact, $\kappa^B Y$), so a single creditor never liquidates. Therefore, the expected probability of liquidation $L_1^{\theta,T}(F,Y)$ is lower for the true model compared with the counterfactual one.

Using the same example as in section 6, figure 4 plots the expected probability of liquidation $L_1^{\theta,\infty}(F,Y)$ against the expected value of the debt conditional on the current state $\theta = G$ (top panel) and $\theta = B$ (bottom panel). The solid (dashed) line is the liquidation probability with a single creditor (two creditors) in the full model. The dotted line is for the counterfactual model in which the number of creditors is exogenously fixed at one.

As figure 4 illustrates, having two creditors generally means a higher liquidation probability than having a single creditor in the true model because of the following two adverse effects. The short-term effect is a higher probability of an immediate liquidation in the next period, captured by lemma 3. The long-term effect is that more creditors can secure a bigger repayment, which requires a larger continuation face value, which in turn causes a higher liquidation probability in the future. Even so, consistent with previous discussion, the option to have two creditors is still beneficial. It uniformly reduces the liquidation probability with a single creditor in the true model compared with the counterfactual.

Firm values tell a similar story. Although establishing strict inequalities in a dynamic programming framework requires some work, the economics behind it is intuitive. Without the costly mechanism to support a higher leverage with more creditors, the firm fails even sooner, lowering its value.

Proposition 7 *Let $V^{\theta}_{CF}(F, Y)$ be the firm value in the counterfactual world. Then for any $F > 0$, $V^{\theta}_{CF}(F, Y) < V^{\theta}_{1}(F, Y)$, and for any $N > 1$, there exists a nonempty set $\mathbb{F}$ (may depend on N) such that $V^{\theta}_{CF}(F, Y) < V^{\theta}_{N}(F, Y)$ for all $F \in \mathbb{F}$.*

In this economy, since the creditors always break even, the total value of the firm is a welfare criterion. Proposition 7 states that eliminating the possibility of a coordination failure is socially inefficient. More interestingly, the second half of proposition 7 suggests that forcing a single creditor structure may be even more inefficient than suboptimally having multiple creditors. This double counterfactual intuition is also confirmed by the liquidation probability in the previous example. In figure 4, for a substantial range of fundamental values, the liquidation probability with *two creditors* in the true model is strictly lower compared with the *single* creditor counterfactual.

The findings raise caution regarding ex-post efficient procedures that eliminate coordination failures. If the threat of coordination problem makes the firm more credible, such policies may have a side effect: reducing the firm's pledgeability and thereby limiting its borrowing capacity early on. This channel can lead to more likely liquidation, lower firm value and lower welfare ex-ante.

8 Renegotiation Frequency

Several existing studies interpret dispersed short-term debt as a mechanism to alleviate debtors' commitment problem (e.g. Berglöf and von Thadden 1994) and to promote creditor activism (e.g. Diamond 2004). However, if the firm can renegotiate when the debt matures, then shortening debt maturity has an unintended consequence of more frequent renegotiation, which exacerbates the commitment problem. How does renegotiation frequency affect the equilibrium outcome? With a dynamic model we can conveniently answer this question. I find, perhaps surprisingly, that when renegotiation becomes very frequent, the borrowing capacity converges to liquidation value and the benefit of dispersed short-term debt vanishes.

Instead of shrinking the debt maturity directly, I keep one-period debt and extend the expected project duration while holding the quality of the project constant. This transformation effectively compresses the debt maturity under the original calendar time and makes renegotiation more frequent. I denote the variables with hats as the ones after the timescale change. Let

$$\hat{\pi} = \frac{\pi}{T}, \tag{29}$$

and the expected project duration becomes $E(\tau_{\hat{\pi}}) = \frac{T}{\pi} = TE(\tau_\pi)$, T times longer than in the original model. Equivalently, renegotiation is T times more frequent.

To highlight the economic intuition, I simplify the model such that the shock $z_t = \mu$ is a constant and the transition matrix is symmetric $p^G = p^B = p$. Choose the growth rate $\hat{\mu}$ and the switching probability $\hat{p}$ for the new timescale to match the first best firm values as defined in (3):

$$\hat{V}_{FB}^{\theta\star}(Y|\hat{\mu}, \hat{\pi}, \hat{p}) = V_{FB}^{\theta\star}(Y|\mu, \pi, p) \tag{30}$$

for both $\theta = G, B$, so that the quality of the project is unaffected. The next lemma confirms that the proposed modifications are natural in the following sense. First, the parameters of the game after the timescale change are well defined. Second, when the period length is very small, the (probabilities of) changes in the state variables are also very small.

Lemma 4 *The new set of parameters after the timescale change* $\hat{\pi} = \frac{\pi}{T} \in (0,1)$, $\hat{\mu} = \frac{T\mu}{T\mu-\mu+1} > 1$, *and* $\hat{p} = \frac{T-1+p(1-\pi)}{T-\pi} \in (0,1)$ *are well defined. In addition, as the effective debt maturity goes to* 0, *i.e., when* $T \to \infty$, *the new parameters satisfy* $\hat{\pi} = \frac{\pi}{T} \to 0$, $\hat{\mu} \to 1$, $\hat{p} \to 0$, *and* $(1-\hat{\pi})\hat{\mu} < 1$.

From (22), an immediate implication of lemma 4 is that the single creditor debt capacity converges to the liquidation value $\hat{\kappa}_1^\theta \to \lambda$. The next result shows that with more frequent negotiation, in the limit, the pledgeable amount in the bad state $\theta = B$ approaches the liquidation value as well. Recall from the example in section 6 and proposition 5 that the debt capacity in the bad state is crucial for when the firm increases the number of creditors. Therefore, the benefit of having multiple creditors becomes negligible as the firm renegotiates more frequently.

Proposition 8 *When $T \to \infty$, the debt capacities $\hat{\kappa}^B \to \lambda$.*

To understand this result, recall that the firm can pledge the liquidation value with a single creditor. Although more creditors can indeed force more repayment by proposition 2, the ultimate source of this extra repayment is from the *fundamental growth between two negotiation dates*. As the renegotiation becomes very frequent, the time horizon between the two negotiation dates vanishes, together with the growth and the incremental pledgeability with more creditors.

Since renegotiation is closely related to the debt maturity in the model and a troubled firm typically negotiates the repayment at maturity in practice, the renegotiation frequency can be interpreted as the debt maturity. With very short maturity,[20] having multiple creditors provides no extra pledgeability.

9 Possible Extensions

9.1 Staggered Debt

In this section, I explicitly consider the staggered debt structure. Everything stays the same except for the (re)financing stage. After the entrepreneur decides the number of new creditors, he specifies the order in which the new debt claims sequentially mature in the next period. The creditors also know their position in the maturity sequence, which also controls the bilateral renegotiation order. Denote n to be the creditor whose debt matures in the nth place. Clearly, same as before, the ones who renegotiate earlier are in better positions.

[20] For example, an overnight repo agreement.

Hence, the value of debt also depends on n:

$$\begin{aligned} D^{\theta}_{n,N}(F,Y) &= E\{\pi \tfrac{\min(F,Yz)}{N}\mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F,N\lambda Yz)\le\kappa^{\theta'}Yz}\min(\tfrac{F}{N},\lambda Yz) \\ &\quad +\mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}\min(\tfrac{F}{N},\max(\lambda Yz-\tfrac{n-1}{N}F,0))]\}. \end{aligned}$$

The first two terms are the same as in (16). The last term captures the fact that if rollover fails, the previous $n-1$ creditors will reject the firm's offers and claim $\min(\frac{n-1}{N}F,\lambda Yz)$ against the firm's liquidation proceeds. The nth creditor can claim the remaining cash from liquidation up to the full value of his claim. Define the total value of debt naturally as

$$D^{\theta}_{N}(F,Y) = \sum_{n=1}^{N} D^{\theta}_{n,N}(F,Y).$$

Then it is easy to see that $D^{\theta}_{N}(F,Y)$ defined above coincides with (16). Therefore, the total value of debt and the firm's problem are essentially unchanged.

9.2 Trading of the Debt Claims in the Secondary Market

Because debt claims may be more valuable if held by some different number of investors, so the creditors may have incentives to trade with others. Suppose after the firm issues new debt claims in the refinancing stage, the new creditors can trade these claims free of transaction costs. Then naturally trades will occur until the number of creditors eventually maximizes the debt value. In the numerical example from section 6.2, for instance, even if the firm issues debt to a single creditor, it is in the creditor's best interest to sell half of the claim to a second investor. As a matter of fact, it can be shown that the number of creditors is often suboptimally high, which hurts the firm value by Proposition 4.

This result is surprising because it suggests that better liquidity in the secondary debt market undermines the firm's ability to control its creditor dispersion and therefore could potentially be bad for the firm. However, I must highlight that I am not claiming a secondary corporate debt market is necessarily bad and should be banned all together. In fact, He and Xiong (2012b) have studied the consequence of a liquidity crunch in the secondary market. What is indeed worth noting here is that when the coordination problem among the creditors is a major concern, trades among them may render the pool of debt holders inefficiently large and thereby exacerbate rollover risks.

It is also interesting to contrast the result with Dewatripont's comment that the possibility of trading leads to ex-post efficient consolidation of the claims.[21] To get Dewatripont's effect in this model, we need to allow for trades after the uncertainty is realized (*before* the renegotiation). When rollover is going to fail in the current model, the creditors have incentive to consolidate, similar to the discussion in section 7. Here the timing is different. Trading *after* the new debts are issued in the refinancing stage leads to sub-optimally large creditor pool, as dispersed ownership may make the debt more valuable.

9.3 Uneven Concentration

So far, I have assumed that the firm must evenly distribute the face value of the debt when refinancing from new creditors. However, this assumption is not crucial for the intuition to work. The key economic force here is that having more creditors means that each one of them can pose greater externalities on the others, causing coordination problems, which, on the other hand, improves their collective bargaining position against the entrepreneur. Allowing creditors to have different shares of the loan does not eliminate these channels.

Of course, the exact amount of externalities they create certainly depends on the specific distribution of creditor size. For example, suppose there are two creditors. One is large and the other is small. Renegotiating with the smaller creditor will be more difficult, while forcing concession from the larger one will be easier. In the limit, if the large lender holds almost the entire outstanding debt, then the outcome approaches the single creditor case. The same economic forces can also potentially endogenize the optimal debt concentration,

[21] The comment is made at the Nobel foundation conference on corporate finance (Stockholm, August 1995). See page 410 in Tirole (2006) for detailed discussions.

an aspect that can be investigated by future research.

9.4 Private Savings by the Entrepreneur

Suppose the entrepreneur can save; that is, instead of raising just enough money to roll over maturing debts, the firm can now borrow more and keep internal cash. The relevant question regarding the equilibrium creditor dispersion becomes whether the firm wishes to borrow from more creditors and save for the future. A rigorous analysis of this problem is beyond the scope of this paper, but intuitively I conjecture that the firm has no incentive to do so.

First, having more creditors increases the firm's probability of liquidation. Moreover, internal cash in the current model is unlikely to serve as a "cushion" that provides "the last source of repayment", as one might imagine. Recall that the project has no cash flow. Therefore, additional cash can only be raised by promising an even higher repayment (weakly positive interest rate). Because the internal cash can always be seized so, when the debt becomes due in the next period, this additional repayment may hardly be renegotiated down even when the liquidation value of the project is very low. Thus, the private cash savings will be insufficient to meet the associated additional repayment, let alone to be the source of funds for the original level of debt. To summarize, having internal cash may not benefit the firm. It gives each creditor a stronger bargaining position as the liquidation value of the firm (including both project and cash) increases, which in turn exacerbates the coordination problem among the lenders.

9.5 Entrepreneur's Liquidation Incentive

The endogenous parameter assumption $\kappa^{\theta} \geq \lambda$ in proposition 1 rules out the entrepreneur's incentive to voluntarily liquidate the project. Without this assumption, the entrepreneur may wish to liquidate in equilibrium. For example, in a bad state, if the entrepreneur definitely foresees a liquidation tomorrow, he is better off voluntarily liquidating today, since the liquidation payoff λY_t is higher than the continuation value $(1-\pi)\lambda E_t(Y_t z_{t+1}) = (1-\pi)\mu\lambda Y_t$. An interesting study would investigate how creditor dispersion interacts with the entrepreneur's liquidation decision. This topic is left for future research.

10 Conclusion

I build a dynamic model in which the firm must repeatedly roll over debt and can renegotiate repayment. Having more creditors brings the disadvantage of coordination problems, which in bad times make it harder for a firm to restructure its debt to avoid liquidation. In good times, however, these same coordination problems enhance pledgeability by making it harder for a firm to opportunistically hold up its creditors. In the model, the firm actively chooses the number of creditors over time by optimally trading off pledgeability with the liquidation probability.

Analysis of the model shows that firms increase the number of creditors when they perform badly. Doing so increases the liquidation probability and lowers the firm value. Allowing for coordination failure in equilibrium is valuable and policies that commit the creditors to ex-post efficient coordination reduce the firm value and may raise the liquidation probability. If the firm can renegotiate the debt very frequently, the enhanced pledgeability from multiple creditors diminishes.

The model's implications highlight the potential for selection bias in empirical studies that investigate the effect of creditor dispersion. For example, an exogenous increase in the number of creditors lowers the required interest rate due to the firm's better repayment incentives. In equilibrium, however, this relationship is reversed because firms choose more creditors when they are in trouble, which in turn leads to higher interest rate.

Finally, having outstanding debt may provide the entrepreneur with the incentive to inefficiently continue the project, for example, risk shifting and gambling for survival. The received wisdom is that a higher level of debt exacerbates the problem and increases the inefficiency associated with such continuation bias. In this paper, I make parameter assumptions such that continuing the project is always efficient.[22] Therefore, there is no debt-equity conflict in continuing the project inefficiently. Instead, if abandoning the project is optimal in certain states, then having outstanding debt generates non-monotonic outcomes in my model, in contrast with the aforementioned intuition. When leverage is low, the entrepreneur implements the first best liquidation strategy. When leverage is high, the efficient liquidation can still be implemented. In this case, even though the entrepreneur is willing to gamble for survival, the creditors refuse to rollover and force an efficient termination. In addition, an intermediate case may exist, in which the debt level is high enough to distort the entrepreneur's liquidation incentive, but not too high to spur the creditors into action. Intuitively, having more creditors in this intermediate case may facilitate restoring the efficient liquidation strategy and correct the entrepreneur's continuation bias. A more rigorous analysis is required to further investigate this problem and I look forward to future research that can shed light on this issue.

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Appendix

Lemma A-1 *(Multi-dimensional Blackwell's Sufficient Condition) Let* $X \subseteq \mathbb{R}^K$ *and* $B^L(X)$ *be the space of bounded vector-valued functions:* $v = (v_1, v_2, ..., v_L) : X \to \mathbb{R}^L$*, where* $L < \infty$*. Equipe* $B^L(X)$ *with the sup norm over coordinates, i.e.* $||v|| = \max_{i \leq L}\{\sup_x v_i(x)\}$*. Suppose* $v, w \in B^L(X)$*, and define* $v \geq w$ *if and only if* $v_i \geq w_i$ *for all* $i \leq L$*. If the operator* $T : B^L(X) \to B^L(X)$ *satisfies that*

1. *(monotonicity) if* $v \geq w$*, then* $T(v) \geq T(w)$*, and*

2. *(discounting) there exists a constant* β *such that for any constant* a*,* $T(v+a) \leq T(v) + \beta a$*,*

then T *is a contraction mapping with coefficient* β*, namely* $||Tv - Tw|| \leq \beta||v - w||$ *for any* $v, w \in B^L(X)$*.*

Proof. Since $w \leq v + ||w - v||$, so monotonicity of T implies $T(w) \leq T(v + ||w - v||)$. The latter expession is in turn bounded by $T(v) + \beta||w - v||$ by discounting. Therefore,

$$T(w) - T(v) \leq \beta||w - v||.$$

Similarly, one can derive the opposite side $T(v) - T(w) \leq \beta||w - v||$. By the definition of the norm on $B^L(X)$, $||T(w)-T(v)|| \leq \beta||w-v||$. T is therefore a contraction mapping with coefficient β. ■

Proof of Lemma 1: In order to be consistent with the notations in the main text following the lemma, denote the values given in (2) by $V_{FB}^{\theta\star}(Y)$. They can be recursively formulated as following:

$$\begin{aligned} V_{FB}^{G\star}(Y) &= E\{\pi Y z + (1-\pi)[p^G V_{FB}^{G\star}(Yz) + (1-p^G)V_{FB}^{B\star}(Yz)]\} \\ V_{FB}^{B\star}(Y) &= (1-\pi)E[p^B V_{FB}^{B\star}(Yz) + (1-p^B)V_{FB}^{G\star}(Yz)]. \end{aligned} \tag{A-31}$$

The first part πYz captures the final dividend, which is materialized only in the good state $\theta = G$. This case occurs with probability π. The second part captures the continuation payoff taking into account a potential switch in the state θ. Normalizing by Y and letting $v_{FB}^{\theta}(Y) = \frac{V_{FB}^{\theta\star}(Y)}{Y}$, (A-31) becomes

$$\begin{aligned} v_{FB}^{G}(Y) &= E\{\pi z + (1-\pi)[p^G v_{FB}^{G}(Yz)z + (1-p^G)v_{FB}^{B}(Yz)z]\} \\ v_{FB}^{B}(Y) &= (1-\pi)E[p^B v_{FB}^{B}(Yz)z + (1-p^B)v_{FB}^{G}(Yz)z]. \end{aligned} \tag{A-32}$$

For any bounded continuous functions on $\mathbb{R}_+$: $v_{FB}^{\theta} \in B^1(\mathbb{R}_+)$, $(\theta = G, B)$, it is easy to check that the right hand side of (A-32) induces a natural operator $T : C_B^2(\mathbb{R}_+) \to C_B^2(\mathbb{R}_+)$ as following:

$$T(v_{FB}^G, v_{FB}^B) = \begin{cases} E\{\pi z + (1-\pi)[p^G v_{FB}^{G}(Yz)z + (1-p^G)v_{FB}^{B}(Yz)z]\} \\ (1-\pi)E[p^B v_{FB}^{B}(Yz)z + (1-p^B)v_{FB}^{G}(Yz)z]. \end{cases}$$

Clearly T satisfies the monotonicity condition in lemma A-1. To verify the discounting condition, notice

$$T(v_{FB} + a) = T(v_{FB}) + (1-\pi)E(az) = T(v_{FB}) + (1-\pi)\mu a.$$

By assumption (1) and lamma A-1, T is a contraction. Therefore, Banach fixed point theorem states that T has a unique fixed point, which implies (A-32) and thereby (A-31) have a unique solution. Finally, to find this solution, observe that (A-32) has a constant solution $(v_{FB}^{G\star}, v_{FB}^{B\star})$ that satisfies:

$$\begin{aligned} v_{FB}^{G} &= \pi\mu + (1-\pi)\mu[p^G v_{FB}^G + (1-p^G)v_{FB}^B]\} \\ v_{FB}^{B} &= (1-\pi)\mu[p^B v_{FB}^B + (1-p^B)v_{FB}^G]. \end{aligned}$$

Solving the above system for (v_{FB}^G, v_{FB}^B) gives (3).

Proof of Proposition 1: The proof contains three parts to verify the proposed equilibrium. First, given the conjectured properties stated in the proposition, I show that the conjectured strategy profile indeed constitutes a subgame perfect equilibrium. Part II (III) proves that the conjectured properties for the value of debt (firm) indeed hold in this equilibrium. In the following proof, the time indices and the arguements in the strategies are sometimes omitted when there is no confusion.

Part I: Given the stated properties of D_N^θ and V_N^θ, I check that the proposed strategy profile is subgame perfect. If the firm survives the period t stage game, then following the equilibrium strategies, the expected payoff to the entrepreneur is

$$V_{N_+^\star}^{\theta_{t+1}}(F_+^\star, Y_t) - \min(F_t, N\lambda Y_t) = \max_{N_+} V_{N_+}^{\theta_{t+1}}(F_+(X^*), Y_t) - X^*,$$

where $X^* = \min(F_t, N\lambda Y_t)$ and $F_+(X)$ is the smallest solution to $D_{N_+}^\theta(F_+, Y) = X$. By the definition of κ^θ, the conjectured property that $V_{N_+}^\theta(F_+, Y) \geq \kappa_{N_+}^\theta Y$, and the endogenous assumption $\kappa^\theta \geq \lambda$, the above equality implies:

$$V_{N_+^\star}^{\theta_{t+1}}(F_+^\star, Y_t) - \min(F_t, N\lambda Y_t) \geq \kappa^{\theta_{t+1}} Y - F_t \geq \lambda Y_t - F_t.$$

Therefore, the continuation payoff is weakly higher than the liquidation payoff. Thus the firm has no strict incentive to voluntarily liquidate nor to offer $S_i < \min(\frac{F}{N}, \lambda Y)$ and induce an immediate liquidation. Suppose the firm offers $S_i > \min(\frac{F}{N}, \lambda Y)$. Two possible cases can happen. If the offer is infeasible, i.e., $\sum_{j \leq i} S_j + (N-i)\min(\frac{F}{N}, \lambda Y) > DC^{\theta}(Y)$, then the creditor rejects the offer and the project is liquidated. This case is clearly dominated by the equilibrium outcome as discussed before. Alternatively if the offer is feasible. Let X be the total negotiated repayment following S_i. Clearly, it must be $X > X^*$, which implies $F_+(X) > F_+(X^*)$ for any given N_+. Because we have conjectured that $V_N^{\theta}(F, Y)$ is weakly decreasing in F, so

$$V_{N_+}^{\theta_{t+1}}(F_+(X^*), Y_t) - X^* \geq V_{N_+}^{\theta_{t+1}}(F_+(X), Y_t) - X^* > V_{N_+}^{\theta_{t+1}}(F_+(X), Y_t) - X,$$

for any N_+. Therefore, the entrepreneur is strictly worse off by offering any $S_i > \min(\frac{F}{N}, \lambda Y)$. In all, the offering strategy $S_i^{\star} = \min(\frac{F}{N}, \lambda Y)$ is optimal.

The entrepreneur's financing strategy $(N_+^{\theta\star}, F_+^{\theta\star})$ is just a repetition of the equilibrium definition. The ith incumbent creditor clearly has no incentive to accept any offer lower than the liquidation payoff. On the other hand, if the payoff is not feasible such that (13) fails, the project will be liquidated following the equilibrium strategies by other creditors. In this case, creditor i either gets $\min(\frac{F_t}{N_t}, \frac{1}{N_t - 1}\max(0, \lambda Y_t - \frac{F_t}{N_t}))$ or $\frac{1}{N_t}\min(F_t, \lambda Y_t)$, both are weakly dominated by $\min(\frac{F_t}{N_t}, \lambda Y_t)$. Finally, the optimality of the new creditors' strategies $r_i^{\theta\star}$ is trivial to verify.

Part II: Given the above strategies, I now show that there exists a consistent linear conjecture of the debt capacities, i.e. $DC^{\theta}(Y) = \kappa^{\theta} Y$ for some constants κ^{θ}. In addition, the value of debt $D_N^{\theta}(F, Y)$ is continuous and HD1 in (F, Y).

Under the conjecture $DC^{\theta}(Y) = \kappa^{\theta} Y$, the equilibrium strategies (condition (13) in particular) imply that rollover is possible if and only if (15) holds. Under this condition, the value of debt can be rewritten as (16). The value of debt $D_N^{\theta}(F, Y)$ is clearly HD1, because one can verify that

$$D_N^{\theta}(F, Y) = Y D_N^{\theta}(\frac{F}{Y}, 1) \equiv Y D_N^{\theta}(f, 1).$$

where $f \equiv \frac{F}{Y}$. The ratio $\frac{\overline{F}_N^\theta(Y)}{Y}$ being a constant independ of Y is a simple corollary of HD1. In fact, one can readily see $\overline{f}_N^\theta(Y) = \arg\max_f D_N^\theta(f,1)$. In addition, the debt capacity with N creditors is linear as given by (17). Finally, $D_N^\theta(f,1)$ is continuous in f, since it can be expressed as the sum of integrals in the form of $\int_{B(f)}^{A(f)} C(f,z)dz$, where A, B, C are continuous functions in their arguments. For example, when $N > \frac{\max_\theta \kappa^\theta}{\lambda}$,

$$\begin{aligned} D_N^\theta(f,1) \quad \equiv \quad & \pi \int_{\underline{z}}^{\overline{z}} \min(f,z)\mathbf{1}_{\theta=G} g(z)dz + (1-\pi)\sum\nolimits_{\theta'=G,B} \\ & P(\theta'|\theta)[\int_{\max(\underline{z},\frac{f}{\kappa^{\theta'}})}^{\max(\overline{z},\frac{f}{\kappa^{\theta'}})} g(z)fdz + \int_{\min(\underline{z},\frac{f}{\kappa^{\theta'}})}^{\min(\overline{z},\frac{f}{\kappa^{\theta'}})} \lambda z g(z)dz] \end{aligned} \tag{A-33}$$

which is clearly continuous in f. The remaining cases are similar. Finally, I show that there exists a consistent conjecture of $\{\kappa_N^\theta, \kappa^\theta\}_{N\in\mathbb{N}}^{\theta=G,B}$. Notice that (17) is a function of (κ^G, κ^B). Denote $\hat{\kappa}_N^\theta(\kappa^G, \kappa^B)$ to be this function and let $L^\theta(\kappa^G, \kappa^B) \equiv \max\{\hat{\kappa}_1^\theta, ..., \hat{\kappa}^\theta_{[\frac{\max_\theta k^\theta}{\lambda}]+1}\}$. So a consistent conjecture of $\{\kappa_N^\theta, \kappa^\theta\}_{N\in\mathbb{N}}^{\theta=G,B}$ is a solution to (14) which is in turn a fixed point of $L \equiv (L^G, L^B) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$. Equipe $\mathbb{R}_+^2$ with the usual partial order $\leq$ such that $x \leq y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Apparently L is order-preserving, since D_N^θ is weakly increasing in κ^θ. I shall then construct a complete lattice $\Omega \subseteq \mathbb{R}_+^2$ such that $L(\Omega) \subseteq \Omega$. By Tarski's fixed point theorem, L has a fixed point and therefore a solution to (14) exists. The remainder of the proof is to construct such an Ω.

By (1), it is possible to choose M big enough such that

$$(\pi + (1-\pi)M)\mu < M. \tag{A-34}$$

Let $\Omega \equiv [0, M] \times [0, M]$ be a complete lattice. Suppose $(\kappa^G, \kappa^B) \in \Omega$, (17) and (A-34) imply that for all $N \leq [\frac{M}{\lambda}] + 1$,

$$
\begin{aligned}
\hat{\kappa}_N^\theta &= \max_f E\{\pi \min(f, z)\mathbf{1}_{\theta=G} + (1-\pi) \\
&\quad [\mathbf{1}_{\min(f,N\lambda z)\leq \kappa^{\theta'} z} \min(f, N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)> \kappa^{\theta'} z} \min(f, \lambda z)]\} \\
&\leq \pi E(z) + (1-\pi)E[\mathbf{1}_{\min(f,N\lambda z)\leq \kappa^{\theta'} z}\kappa^{\theta'} z + \mathbf{1}_{\min(f,N\lambda z)> \kappa^{\theta'} z}\lambda z] \\
&\leq \pi\mu + (1-\pi)M\mu \\
&< M.
\end{aligned}
$$

Therefore, $L^\theta(\kappa_N^\theta) < M$, which implies that Ω is invariant under L . This completes the proof.

Part III: Finally, for any pair of $\kappa^\theta \geq \lambda$, I will show there exists a unique continuous HD1 function $V_N^\theta(F, Y)$ which is increasing in Y, weakly decreasing in F and $V_N^\theta(F, Y) \geq \kappa_N^\theta Y$ for any $F \leq \overline{F}_N^\theta$. By the discussion following proposition 1, in the conjectured equilibrium, the firm's problem can be rewritten as a dynamic programming problem (18) and (19). By the definition of $\overline{N}$ in (20) and the discussion following it, we can confine the choice of N_+^θ to $\{1, 2, ..., \overline{N}\}$ without loss of generality.

Define an auxiliary problem:

$$
v_N^\theta(f) = E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\leq \kappa^{\theta'} z} \max_{N_+} v_{N_+}^{\theta'}(f_+) z + \mathbf{1}_{\min(f,N\lambda z)> \kappa^{\theta'} z}\lambda z]\} \tag{A-35}
$$

where $f_+(\frac{f}{z}, N)$ is the minimum solution to

$$
D_{N_+}^{\theta'}(f_+, 1) = \min(\frac{f}{z}, N\lambda). \tag{A-36}
$$

By the definition of $\overline{f}_N^\theta$ in proposition 1, it must be $f_+ \leq \overline{f}_{N_+}^{\theta'}$. Denote $T_N^\theta : B^{2\overline{N}} \to B$ to be the operator on $(v_i^\theta)_{i\leq \overline{N}}^{\theta=G,B}$ induced by the right-hand side of (A-35) and let $T \equiv (T_N^\theta) : B^{2\overline{N}} \to B^{2\overline{N}}$.

First, notice that if $v \in B^{2\overline{N}}$ is bounded by some $M > 1$, then $||Tf(v)|| \leq \pi(1 + \mu) +$

$(1-\pi)M(1+\mu)$ is also bounded. So T is indeed well-defined. Then I prove that T is a contraction mapping by verifying monotonicity and discounting conditions in lemma A-1. Monotonicity is trivial. For any constant a, $T(v+a) \le T(v) + (1-\pi)(1+\mu)a$. So the discounting condition holds by (1).

Denote $C_{a,l} = \{v \ : \ v \text{ is bounded, continuous, decreasing, and } v|_{[0,a]} \ge l\} \subseteq B^1$ to be the subset of all bounded continuous decreasing functions taking values in $[l,\infty)$ when restricted to $[0,a]$. Consider $C \equiv \times_{N\le\overline{N},\theta=G,B} C_{\overline{f}^\theta_N,\kappa^\theta_N}$. Clearly C is a closed subset of $B^{2\overline{N}}$. Next I show $T(C)\subseteq C$. Suppose $v\in C$ and $f_1 \le f_2$. By the definition of f_+, we have $f_+(\frac{f_1}{z},N) \le f_+(\frac{f_2}{z},N)$. To simplify notation, let $f_{1+} \equiv f_+(\frac{f_1}{z},N)$. The following inequalities must hold:

$$
\begin{aligned}
T^\theta_N(v)(f_1) &= E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_1,N\lambda z)\le\kappa^{\theta'}z} \max\nolimits_{N_+} v^{\theta'}_{N_+}(f_{1+})z + \mathbf{1}_{\min(f_1,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_1,N\lambda z)\le\kappa^{\theta'}z} \max\nolimits_{N_+} v^{\theta'}_{N_+}(f_{2+})z + \mathbf{1}_{\min(f_1,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_2,N\lambda z)\le\kappa^{\theta'}z} \max\nolimits_{N_+} v^{\theta'}_{N_+}(f_{2+})z + \mathbf{1}_{\min(f_2,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&= T^\theta_N(v)(f_2).
\end{aligned}
$$

The last inequality is because that $v^\theta_{N_+}(f_{2+}) \ge \kappa^\theta \ge \lambda$ and $\{z|\min(f_1,N\lambda z) \le \kappa^\theta z\} \supseteq \{z|\min(f_2,N\lambda z) \le \kappa^\theta z\}$ for $\theta = G,B$. So each coordinate in $T(v)$ is also a decreasing function. In addition,

$$
\begin{aligned}
T^\theta_N(v)(\overline{f}^\theta_N) &= E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}^\theta_N,N\lambda z)\le\kappa^{\theta'}z} \max\nolimits_{N_+} v^{\theta'}_{N_+}(f_+)z + \mathbf{1}_{\min(\overline{f}^\theta_N,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}^\theta_N,N\lambda z)\le\kappa^{\theta'}z} \kappa^{\theta'} z + \mathbf{1}_{\min(\overline{f}^\theta_N,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}^\theta_N,N\lambda z)\le\kappa^{\theta'}z} \min(\overline{f}^\theta_N, N\lambda z) + \mathbf{1}_{\min(\overline{f}^\theta_N,N\lambda z)>\kappa^{\theta'}z}\min(\overline{f}^\theta_N, N\lambda z)]\} \\
&= \kappa^\theta_N
\end{aligned}
$$

The first inequality uses the fact $\max_{N_+} v^\theta_{N_+}(f_+) \ge \max_{N_+} \kappa^\theta_{N_+} = \kappa^\theta$ for both $\theta = G,B$, since $v\in C$. The second inequality holds because $\min(\overline{f}^\theta_N, N\lambda z) \le \kappa^\theta z$ over the relevant region. The last equality is by the definition of $\overline{f}^\theta_N$ and (17). Because $T^\theta_N(v)$ is a decreasing function, so $T^\theta_N(v)|_{[0,\overline{f}^\theta_N]} \ge \kappa^\theta_N$. Finally, I show that $T^\theta_N(v)$ must be a continuous function.

Consider $\frac{f_2}{f_1} = 1 + \delta$. By definition (A-35)

$$\begin{aligned}
T_N^\theta(v)(f_2) &= \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)\textstyle\sum_{\theta'=G,B} P(\theta'|\theta) \\
&\quad [\textstyle\int_{\min(f_2,N\lambda z)\le\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{2+})zg(z)dz + \int_{\min(f_2,N\lambda z)>\kappa^{\theta'}z} \lambda zg(z)dz] \\
&= \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)\textstyle\sum_{\theta'=G,B} P(\theta'|\theta) \\
&\quad [\textstyle\int_{\min(f_1,N\lambda z')\le\kappa^{\theta'}z'} \max_{N_+} v_{N_+}^{\theta'}(f_{1+})(1+\delta)^2 z' g[z'(1+\delta)]dz' \\
&\quad + \textstyle\int_{\min(f_1,N\lambda z')>\kappa^{\theta'}z'} \lambda(1+\delta)z'g[z'(1+\delta)]dz']
\end{aligned}$$

where the change of variable $z = (1+\delta)z'$. Notice that, by assumption, v_N^θ are bounded by some constant M and g is a density function, so, as $\delta \to 0$, the functions under the integrals in the above expression are dominated by $2Mz'g(2z')$. Because the random variable z has a finite mean, so $\int 2Mz'g(2z')dz' < \infty$. The dominated convergence theorem then implies that as $\delta \to 0$, the last expression converges to $T_N^\theta(v)(f_1)$. Therefore, the function $T_N^\theta(v)$ is continuous. In all, I have established that the contraction mapping T maps C into itself.

By contraction mapping theorem, the operator T has a unique fixed point $v^\star \in B^{2\overline{N}}$. Furthermore, this fixed point must belong to C. Define

$$V_N^\theta(F,Y) = v_N^{\theta\star}(\frac{F}{Y})Y. \tag{A-37}$$

which is decreasing in F. It is very easy to verify that the constructed solution satisfies the original recursive problem (18) with (19). Because $v_N^{\theta\star}(\frac{F}{Y})$ is increasing in Y, so V_N^θ as defined above is also increasing in Y. This completes the full proof of this proposition.

Proof of Proposition 2: First, if $N_1 = 1$, then by (16),

$$\begin{aligned}
D_{N_2}^\theta(F,Y) &= E\{\pi\min(F,Yz)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(F,N_2\lambda Yz)\le\kappa^{\theta'}Yz}\min(F,N_2\lambda Yz) + \mathbf{1}_{\min(F,N_2\lambda Yz)>\kappa^{\theta'}Yz}\min(F,\lambda Yz)]\} \\
&\ge E[\pi\min(F,Yz)\mathbf{1}_{\theta=N} + (1-\pi)\min(F,\lambda Yz)] \\
&= D_1^\theta(F,Y).
\end{aligned}$$

If $\underline{N} \geq N_2 > N_1$, by the definition of $\underline{N}$ in (21), then the liquidation region $\{z | \min(F, N_i \lambda Y z) > \kappa^\theta Y z\} = \emptyset$ for $\theta = G, B$. Therefore,

$$
\begin{aligned}
D^\theta_{N_2}(F, Y) &= E[\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi)\min(F, N_2\lambda Y z)] \\
&\geq E[\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi)\min(F, N_1\lambda Y z)] \\
&= D^\theta_{N_1}(F, Y).
\end{aligned}
$$

Finally, if $\overline{N} > N_2 > N_1 \geq \underline{N}$, then $\{z | \min(F, N_i\lambda Y z) > \kappa^G Y z\} = \emptyset$ and $\{z | \min(F, N_i\lambda Y z) > \kappa^B Y z\} = \{z | F > \kappa^B Y z\}$. Therefore,

$$
\begin{aligned}
D^\theta_{N_2}(F, Y) &= E\{\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi)\{P(G|\theta)\min(F, N_2\lambda Y z) \\
&+ P(B|\theta)[\mathbf{1}_{F \leq \kappa^B Y z}\min(F, N_2\lambda Y z) + \mathbf{1}_{F > \kappa^B Y z}\min(F, \lambda Y z)]\}\} \\
&\geq E\{\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi)\{P(G|\theta)\min(F, N_1\lambda Y z) \\
&+ P(B|\theta)[\mathbf{1}_{F \leq \kappa^B Y z}\min(F, N_1\lambda Y z) + \mathbf{1}_{F > \kappa^B Y z}\min(F, \lambda Y z)]\}\} \\
&= D^\theta_{N_1}(F, Y).
\end{aligned}
$$

So statement 1 holds. Higher debt capacity with N_2 in each category ($\kappa^\theta_{N_2} \geq \kappa^\theta_{N_1}$) is a direct implication of the previous statement.

Finally, to show the last statement, by definition (16), $D^\theta_N(F, Y)$ is continuous in F with $D^\theta_N(0, Y) = 0$. Intermediate value theorem guarantees the existence of the solutions $F^\theta_{N_i}$. Utilizing statement 1,

$$
D^\theta_{N_2}(F^\theta_{N_2}, Y) = S = D^\theta_{N_1}(F^\theta_{N_1}, Y) \leq D^\theta_{N_2}(F^\theta_{N_1}, Y).
$$

Again by intermediate value theorem, the minimum solution to $D^\theta_{N_2}(F^\theta_{N_2}, Y) = S$ must be within $(0, F^\theta_{N_1}]$, completing the proof of the proposition.

Proof of Proposition 4: For any continuation number of creditors N_+, define F_{+,N_i} ($i = 1, 2$) to be the minimum solution such that $D^\theta_{N_+}(F_{+,N_i}, Yz) = \min(F, N_i\lambda Y z)$. For a given N_+

$$\begin{aligned} D^{\theta}_{N_+}(F_{+,N_2}, Yz) &= \min(F, N_2\lambda Yz) \\ &\geq \min(F, N_1\lambda Yz) \\ &= D^{\theta}_{N_+}(F_{+,N_1}, Yz). \end{aligned}$$

Thus $F_{+,N_2} \geq F_{+,N_1}$, so for any given continuation number of creditors N_+, having more incumbent creditors $N_2 > N_1$ implies higher continuation face value $F_{+,N_2} \geq F_{+,N_1}$. By the recursive formulation (18) and proposition 1 we have:

$$\begin{aligned} V^{\theta}_{N_2}(F, Y) &= E\{\pi Yz\mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F,N_2\lambda Yz)\leq\kappa^{\theta'}Yz} \max\nolimits_{N_+} V^{\theta'}_{N_+}(F_{+,N_2}, Yz) + \mathbf{1}_{\min(F,N_2\lambda Yz)>\kappa^{\theta'}Yz}\lambda Yz]\} \\ &\leq E\{\pi Yz\mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F,N_1\lambda Yz)\leq\kappa^{\theta'}Yz} \max\nolimits_{N_+} V^{\theta'}_{N_+}(F_{+,N_2}, Yz) + \mathbf{1}_{\min(F,N_1\lambda Yz)>\kappa^{\theta'}Yz}\lambda Yz]\} \\ &\leq E\{\pi Yz\mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F,N_1\lambda Yz)\leq\kappa^{\theta'}Yz} \max\nolimits_{N_+} V^{\theta'}_{N_+}(F_{+,N_1}, Yz) + \mathbf{1}_{\min(F,N_1\lambda Yz)>\kappa^{\theta'}Yz}\lambda Yz]\} \\ &= V^{\theta}_{N_1}(F, Y). \end{aligned}$$

The first equality is by definition. The second inequality is because $\{z|\min(F, N_2\lambda Yz) \leq \kappa^{\theta}Yz\} \subseteq \{z|\min(F, N_1\lambda Yz) \leq \kappa^{\theta}Yz\}$ and $V^{\theta}_{N_+}(F_+, Yz) \geq \lambda Yz$ by proposition 1. The third inequality is because $F_{+,N_2} \geq F_{+,N_1}$ and the fact that $V^{\theta}_{N_+}(F_+, Yz)$ is decreasing in F_+ by proposition 1. Thus $V^{\theta}_{N_2}(F, Y) \leq V^{\theta}_{N_2}(F, Y)$.

Proof of Proposition 6: The proof shares the same spirit as the existence proof of κ^{θ} in proposition 1 part II. Define the same order-preserving function $L : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ as in the proof of proposition 1 with the expectations taken under the distribution g_a. Pick any pair of $\kappa^{\theta,b}$. I shall prove that there exists a fixed point $\kappa^{\theta,a} \in \Omega$ of L, where $\Omega = [\kappa^{G,b}, M] \times [\kappa^{B,b}, M]$ is a complete lattice and M is given by (A-34). For any N and $\kappa^{\theta,a} \in \Omega$,

$$\begin{aligned}
\hat{\kappa}_N^{\theta,a} &= \max\nolimits_f E_{g_a}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\le\kappa^{\theta',a}z}\min(f,N\lambda z)+\mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',a}z}\min(f,\lambda z)]\} \\
&\ge \max\nolimits_f E_{g_a}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\le\kappa^{\theta',b}z}\min(f,N\lambda z)+\mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',b}z}\min(f,\lambda z)]\} \\
&\ge \max\nolimits_f E_{g_b}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\le\kappa^{\theta',b}z}\min(f,N\lambda z)+\mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',b}z}\min(f,\lambda z)]\} \\
&= \kappa_N^{\theta,b}.
\end{aligned}$$

The first inequality is because $\min(f,N\lambda z)\ge\min(f,\lambda z)$ and $\{z|\min(f,N\lambda z)\le\kappa^{\theta,b}z\}\subseteq\{z|\min(f,N\lambda z)\le\kappa^{\theta,a}z\}$. The second inequality uses first order stochastic dominance and the fact that the function under the expectation is weakly increasing in z. Therefore, for any $\kappa^{\theta,a}\in\Omega$, $L^\theta(\kappa^{\theta,a})=\max_N\hat{\kappa}_N^{\theta,a}\ge\max_N\kappa_N^{\theta,b}\ge\kappa^{\theta,b}$. So $L(\Omega)\subseteq\Omega$ and Tarski's fixed point theorem completes the argument. The omitted proof for the other direction is very similar, with the auxiliary set $\Omega=[0,\kappa_N^{\theta,a}]$.

Proof of Proposition 7: First I show $V_{CF}^\theta(F,Y)<V_1^\theta(F,Y)$. Recall the function space C and the mapping T defined in proposition 1 part III. Define a new closed subset of functions in $B^{2(\overline{N}+1)}$: $C_A=\{(v_{CF}^\theta,v_N^\theta)_{N\le\overline{N}}^{\theta=G,B}|(v_N^\theta)_{N\le\overline{N}}^{\theta=G,B}\in C \text{ and } v_{CF}^\theta\le v_1^\theta\}\subset B^{2(\overline{N}+1)}$. Let $C_B=\{(v_{CF}^\theta,v_N^\theta)_{N\le\overline{N}}^{\theta=G,B}\in C_A|v_{CF}^\theta<v_1^\theta \text{ for all } f>0\}\subset C_A$. Finally let $C_\beta=\{(v_{CF}^\theta,v_N^\theta)_{N\le\overline{N}}^{\theta=G,B}\in C_A|v_{CF}^\theta(f)<v_1^\theta(f) \text{ for all } f>\beta\}$. Clearly $C_B=C_0\subset C_{\beta_2}\subset C_{\beta_1}\subset C_\infty=C_A$ for any $\beta_2<\beta_1$. Define a new mapping $TC(v_{CF}^\theta,v_N^\theta)_{N\le\overline{N}}^{\theta=G,B}=\Big(T_{CF}[(v_{CF}^\theta)^{\theta=G,B}],T[(v_N^\theta)_{N\le\overline{N}}^{\theta=G,B}]\Big)$ on $B^{2(\overline{N}+1)}$, where $T_{CF}=(T_{CF}^G,T_{CF}^B)$ is given by

$$T_{CF}^\theta(v_{CF}) \;=\; \pi\mu\mathbf{1}_{\theta=G}+(1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa_1^{\theta'}z}v_{CF}^{\theta'}(f_{+,1})z+\mathbf{1}_{\min(f,\lambda z)>\kappa_1^{\theta'}z}\lambda z]$$

where $f_{+,N}$ is an abbreviation for $f_+(\frac{f}{z},N)$, the minimum solution to $D_N^\theta(f_+,1)=\min(\frac{f}{z},\lambda N)$ as before. Similar to the proof in proposition 1 part III, it is straight forward to check that

TC defined above satisfies the monotonicity and discounting conditions stated in lemma A-1. So TC must have a unique fixed point $v^\star$ in $B^{2(\overline{N}+1)}$. Our goal is to show this $v^\star \in C_B$.

Claim: there exists a decreasing sequence of $\beta_n \to 0$ such that $\beta_0 = \infty$ and $TC(C_{\beta_n}) \subseteq C_{\beta_{n+1}}$.

Given this claim, we have

$$TC(C_A) = TC(C_\infty) \subseteq C_{\beta_1} \subseteq C_A. \tag{A-38}$$

The contraction mapping theorem states that the unique fixed point can be derived from repeated iterations starting from any point v, i.e., $v^\star = \lim_{n\to\infty} TC^{(n)}(v)$. Because the set C_A is closed, one can start the iteration from any point $v \in C_A$ and the limiting point $v^\star$ will stay in C_A by (A-38). Furthermore, for any n, one can argue $v^\star \in TC^{(n)}(C_A) \subseteq C_{\beta_n}$. As $n \to \infty$, $v^\star \in \lim_{n\to\infty} TC^{(n)}(C_A) \subseteq \lim_{n\to\infty} C_{\beta_n} = C_0 = C_B$. Therefore, $v^\star \in C_B$. Let $V^\theta_{CF}(F,Y) = v^{\theta\star}_{CF}(\frac{F}{Y})Y$. Following the same procedures in proposition 1 part III, one can check that it is indeed the firm's value function in the counterfactual case. The fact $v^\star \in C_B$ implies $V^\theta_{CF}(F,Y) < V^\theta_1(F,Y)$ for all $F > 0$, completing the first half of the statement in the proposition.

Finally, when $F < \lambda \underline{z} Y < \kappa^\theta Y$, the actual repayment in the true model must be $F = \min(F, \lambda N z Y)$ regardless of the number of incumbent creditors N. The firm always survives the next period. Therefore, it is easy to see from (18) and (19) that the firm values do not depend on N when $F < \lambda \underline{z} Y$. Combining with the result we just proved, it is immediate that $V^\theta_N(F,Y) = V^\theta_1(F,Y) > V^\theta_{CF}(F,Y)$ for all N, establishing the proposition.

Proof of the claim: Suppose $(v^\theta_{CF}, v^\theta_N)^{\theta=G,B}_{N\leq\overline{N}} \in C_A$. By the construction of the operator T^θ_{CF}, we have

$$\begin{aligned} T^\theta_{CF}(v_{CF}) &= \pi\mu \mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\leq \kappa^{\theta'}_1 z} v^{\theta'}_{CF}(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa^{\theta'}_1 z}\lambda z] \\ &\leq \pi\mu \mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\leq \kappa^{\theta'}_1 z} v^{\theta'}_1(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa^{\theta'}_1 z}\lambda z] \end{aligned} \tag{A-39}$$

Because $\kappa_1^\theta \leq \kappa^\theta$ and $\kappa_1^B = (1-\pi)\mu\lambda < \lambda < \kappa^B$, so whenever $f > (1-\pi)\mu\lambda\underline{z}$ there is a positive probability that $f > \kappa_1^B z$. In addition, because $\max_{N_+} v_{N_+}^\theta(f_{+,N_+}) \geq \max_{N_+} \kappa_{N_+}^\theta = \kappa^\theta > \lambda$, the last expression in (A-39) is strictly dominated by

$$
\begin{aligned}
\text{(A-39)} \quad &< \quad \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\leq\kappa^{\theta'}z}\max_{N_+} v_{N_+}^{\theta'}(f_{+,N_+})z + \mathbf{1}_{\min(f,\lambda z)>\kappa^{\theta'}z}\lambda z] \\
&= \quad \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\max_{N_+} v_{N_+}^{\theta'}(f_{+,N_+})z] \\
&= \quad T_1^\theta((v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B}).
\end{aligned}
\tag{A-40}
$$

Therefore, $TC((v_{CF}^\theta, v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B}) \in C_{(1-\pi)\mu\lambda\underline{z}}$ and we can pick $\beta_1 = (1-\pi)\mu\lambda\underline{z}$. Let $\beta_{n+1} = (1-\pi)\beta_n$. I shall prove that $TC(v) \in C_{(1-\pi)\beta_n}$ for all $v \in C_{\beta_n}$. Suppose $(v_{CF}^\theta, v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B} \in C_{\beta_n}$ and consider any $f \in (\beta_n(1-\pi), \beta_n]$. On one hand, from the rollover condition (A-36) and the fact that $\frac{f}{z} \leq \frac{f}{\underline{z}} \leq \frac{\beta_n}{\underline{z}} < \frac{\beta_1}{\underline{z}} = (1-\pi)\mu\lambda < \lambda$, we have

$$D_1^B(f_+^B, 1) = \min(\frac{f}{z}, \lambda) = \frac{f}{z}.$$

On the other hand, from expression (16) and the fact $f \leq \beta_n \leq \lambda\underline{z}$, we have

$$D_1^B(f_+^B, 1) = (1-\pi)f_+^B.$$

The above two equalities together imply that $f_{+,1}^B = \frac{f}{z(1-\pi)} > \frac{\beta_n}{z}$, which in turn implies that there is positive possibility that $f_{+,1}^B > \beta_n$. By the construction of the set C_{β_n}, $v_{CF}^B(f_{+,1}) < v_1^B(f_{+,1})$ holds strictly when $f_{+,1} > \beta_n$. Therefore, the inequality (A-39) holds strictly in this case. On the other hand, the weak inequality between (A-39) and (A-40) is trivial, so we again have $T_{CF}^\theta(v_{CF})(f) < T^\theta((v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B})(f)$ for all $f > (1-\pi)\beta_n$. Therefore, we have established $TC(C_{\beta_n}) \subseteq C_{\beta_{n+1}}$ for the constructed sequence of β_n that converges to zero, completing the proof of the claim and the whole proposition.

Proof of Lemma 4: By definition (29), $\lim_{T\to\infty}\hat{\pi} = \lim_{T\to\infty}\frac{\pi}{T} = 0$ is obvious. Rewrite

(30) using (3):

$$\begin{aligned}
\frac{\hat{\pi}[1-(1-\hat{\pi})\hat{\mu}\hat{p}]\hat{\mu}}{[1-(1-\hat{\pi})\hat{\mu}][1-(1-\hat{\pi})\hat{\mu}(2\hat{p}-1)]} &= \frac{\pi[1-(1-\pi)\mu p]\mu}{[1-(1-\pi)\mu][1-(1-\pi)\mu(2p-1)]}, && \text{(A-41)}\\
\frac{\hat{\pi}(1-\hat{p})(1-\hat{\pi})\hat{\mu}^2}{[1-(1-\hat{\pi})\hat{\mu}][1-(1-\hat{\pi})\hat{\mu}(2\hat{p}-1)]} &= \frac{\pi(1-p)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(2p-1)]}. && \text{(A-42)}
\end{aligned}$$

Adding the above two equations, we have

$$\frac{\hat{\pi}\hat{\mu}}{1-(1-\hat{\pi})\hat{\mu}} = \frac{\pi\mu}{1-(1-\pi)\mu}. \tag{A-43}$$

Plugging in $\hat{\pi} = \frac{\pi}{T}$ from (29), one can solve for $\hat{\mu} = \frac{T\mu}{T\mu-\mu+1} \to 1$ as $T \to \infty$. Finally, in order to calculate $\hat{p}$, divide (A-41) by (A-42) and then we have

$$\frac{1-(1-\hat{\pi})\hat{\mu}\hat{p}}{(1-\hat{p})(1-\hat{\pi})\hat{\mu}} = \frac{1-(1-\pi)\mu p}{(1-p)(1-\pi)\mu}.$$

Subtract 1 from both sides and multiply it by (A-43),

$$\frac{\hat{\pi}}{(1-\hat{p})(1-\hat{\pi})} = \frac{\pi}{(1-p)(1-\pi)}.$$

Plug in $\hat{\pi} = \frac{\pi}{T}$ and we can solve for $\hat{p} = \frac{T-1+p(1-\pi)}{T-\pi}$. Clearly, when $T \geq 1$, $\hat{\pi}, \hat{p} \in (0,1)$. In addition, $\lim_{T\to\infty} \hat{\pi} = 0$ and $\lim_{T\to\infty} \hat{p} = 1$. Finally,

$$(1-\hat{\pi})\hat{\mu} = \frac{\mu(T-\pi)}{T\mu-\mu+1} = 1 - \frac{1-\mu(1-\pi)}{T\mu-\mu+1} < 1,$$

by assumption (1). Therefore, the new parameters are well defined.

Proof of Proposition 8: First, notice $\hat{\kappa}^G$ must be bounded as $T \to \infty$. This is because

$$\hat{\kappa}_N^G Y = \max_F D_N^G(F,Y) \leq V_{FB}^{G\star}(Y).$$

So $\hat{\kappa}^G = \max_N \hat{\kappa}_N^G$ must be bounded by some upper bound M ($\frac{V_{FB}^{G\star}(Y)}{Y}$ for example) that is independent of T. Let $\hat{N}$ be the number of creditors such that $\hat{\kappa}^B_{\hat{N}}$ attains the total debt capacity $\hat{\kappa}^B$, then

$$\begin{aligned} \hat{\kappa}^B &= \max_f (1-\hat{\pi})\{\hat{p}[\mathbf{1}_{\min(f,\hat{N}\lambda\mu)\le\hat{\kappa}^B\hat{\mu}} \min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})>\hat{\kappa}^B_2\hat{\mu}} \min(f,\lambda\hat{\mu})] \\ &+ (1-\hat{p})[\mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})\le\hat{\kappa}^G_2\hat{\mu}} \min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\mu)>\hat{\kappa}^G_2\hat{\mu}} \min(f,\lambda\hat{\mu})]\}. \quad \text{(A-44)} \\ &\le \max_f (1-\hat{\pi})\{\hat{p}[\mathbf{1}_{\min(f,\hat{N}\lambda\mu)\le\hat{\kappa}^B\hat{\mu}} \min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})>\hat{\kappa}^B_2\hat{\mu}} \min(f,\lambda\hat{\mu})] \\ &+ (1-\hat{p})\min(f,\hat{N}\lambda\hat{\mu})\}. \quad \text{(A-45)} \end{aligned}$$

Let $f^\star$ be the optimal f such that (A-44) attains $\hat{\kappa}^B$. Suppose $f^\star \le \hat{\kappa}^B\hat{\mu}$. Notice that the expression in (A-45) is increasing in $f \in [0, \hat{\kappa}^B\hat{\mu}]$ and $(1-\hat{\pi})\hat{\mu} < 1$ by lemma 4, so

$$\hat{\kappa}^B \le (1-\hat{\pi})\min(\hat{\kappa}^B\hat{\mu}, \hat{N}\lambda\hat{\mu}) < \hat{\kappa}^B.$$

Contradiction! On the other hand, if $\hat{N}\lambda \le \hat{\kappa}^B$, then it is optimal to set $f^\star$ arbitrarily large in (A-45) and $\hat{\kappa}^B = (1-\hat{\pi})\hat{N}\lambda\hat{\mu} < \hat{N}\lambda$. Again a contradiction! Therefore, at $f = f^\star$, it must be $\min(f^\star, \hat{N}\lambda\hat{\mu}) > \hat{\kappa}^B\hat{\mu}$ and (A-44) becomes:

$$\begin{aligned} \hat{\kappa}^B &= (1-\hat{\pi})\{\hat{p}\min(f^\star,\lambda\hat{\mu}) \\ &+ (1-\hat{p})[\mathbf{1}_{\min(f^\star,\hat{N}\lambda\hat{\mu})\le\hat{\kappa}^G\hat{\mu}} \min(f^\star,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f^\star,\hat{N}\lambda\hat{\mu})>\hat{\kappa}^G\hat{\mu}} \min(f^\star,\lambda\hat{\mu})]\} \\ &\le (1-\hat{\pi})\{\hat{p}\min(f^\star,\lambda\hat{\mu}) + (1-p)\hat{\kappa}^G\hat{\mu}\} \\ &\le (1-\hat{\pi})\{\hat{p}\lambda\hat{\mu} + (1-\hat{p})\hat{\mu}M\}, \quad \text{(A-46)} \end{aligned}$$

where the last inequality uses the fact that $\hat{\kappa}^G \le M$, which is independent of T. As $T \to \infty$, lemma 4 states $\hat{\pi} \to 0$, $\hat{\mu} \to 1$, and $\hat{p} \to 1$, so the upper bound given by (A-46) approaches λ. Finally, because $\hat{\kappa}^B \ge \hat{\kappa}^B_1 \to \lambda$ as $T \to \infty$, so we conclude $\lim_{T\to\infty} \hat{\kappa}^B = \lambda$.

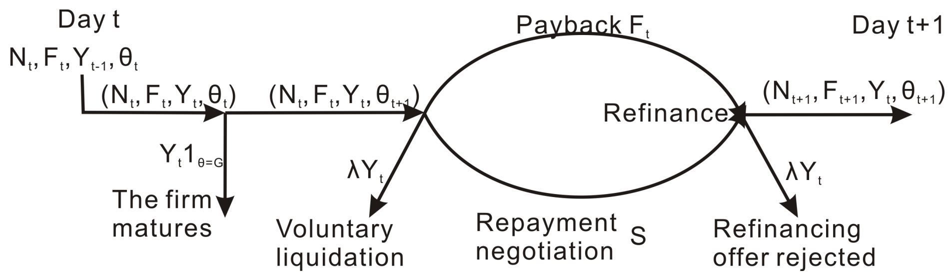

Figure 1 *The timeline and the evolution of the state variables.*

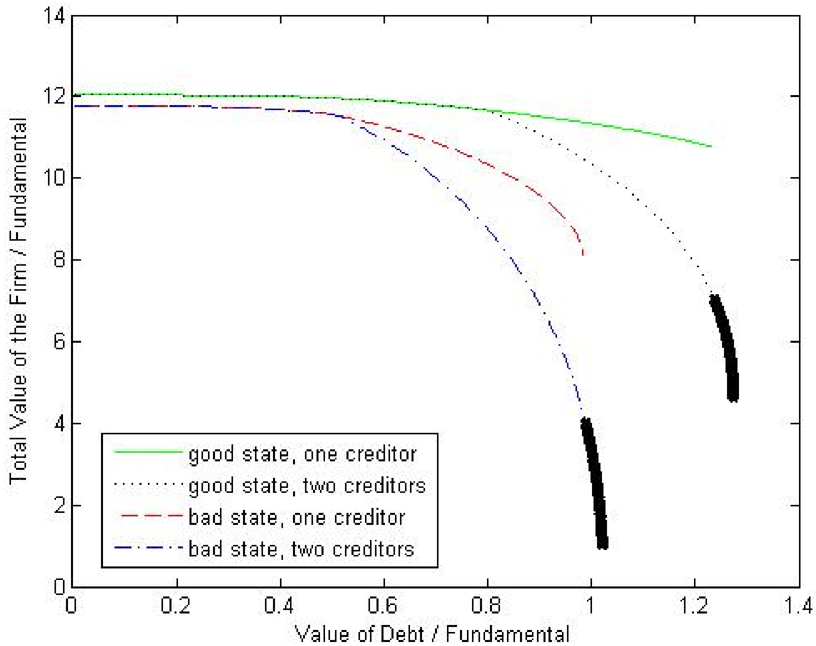

Figure 2 *The figure plots the expected total firm value against the expected value of the debt. The solid (dashed) line is the firm value with a single creditor when the fundamental* $\theta = G$ $(\theta = B)$. *The dotted (dash-dotted) line is the firm value with two creditors when the fundamental* $\theta = G$ $(\theta = B)$. *The thick solid black segments can be supported only by two creditors. Although the firm values are comparatively much lower along the thick lines, the firm cannot even reach that portion with just one creditor. When the value of debt is very low the choice between one and two creditors is irrelevant. As the fundamental worsens, the two groups of lines diverge and, when both are feasible, the single creditor structure always delivers a higher firm value.*

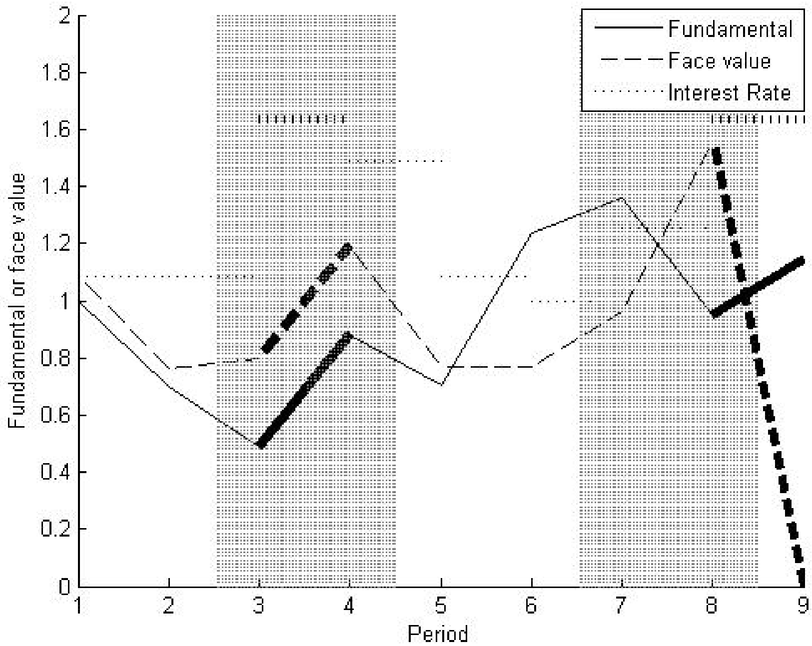

Figure 3 *The figure plots a typical sample path of the firm. Areas are shaded when the state is bad. The solid (dashed) line denotes the exogenous fundamental process* Y_{t-1} *(the face value process* F_t *determined in equilibrium). I use bold segments when the firm chooses two creditors. The values plotted at each period* t *are the state variables entering period* t*: number of creditors* N_t*, the promised face value* F_t*, state* θ_t*, and fundamental process* Y_{t-1}*. Finally, the dotted bars plot the interest rates* $\frac{F_t}{D_{N_t}^{\theta_t}(F_t, Y_{t-1})}$ *during each period.*

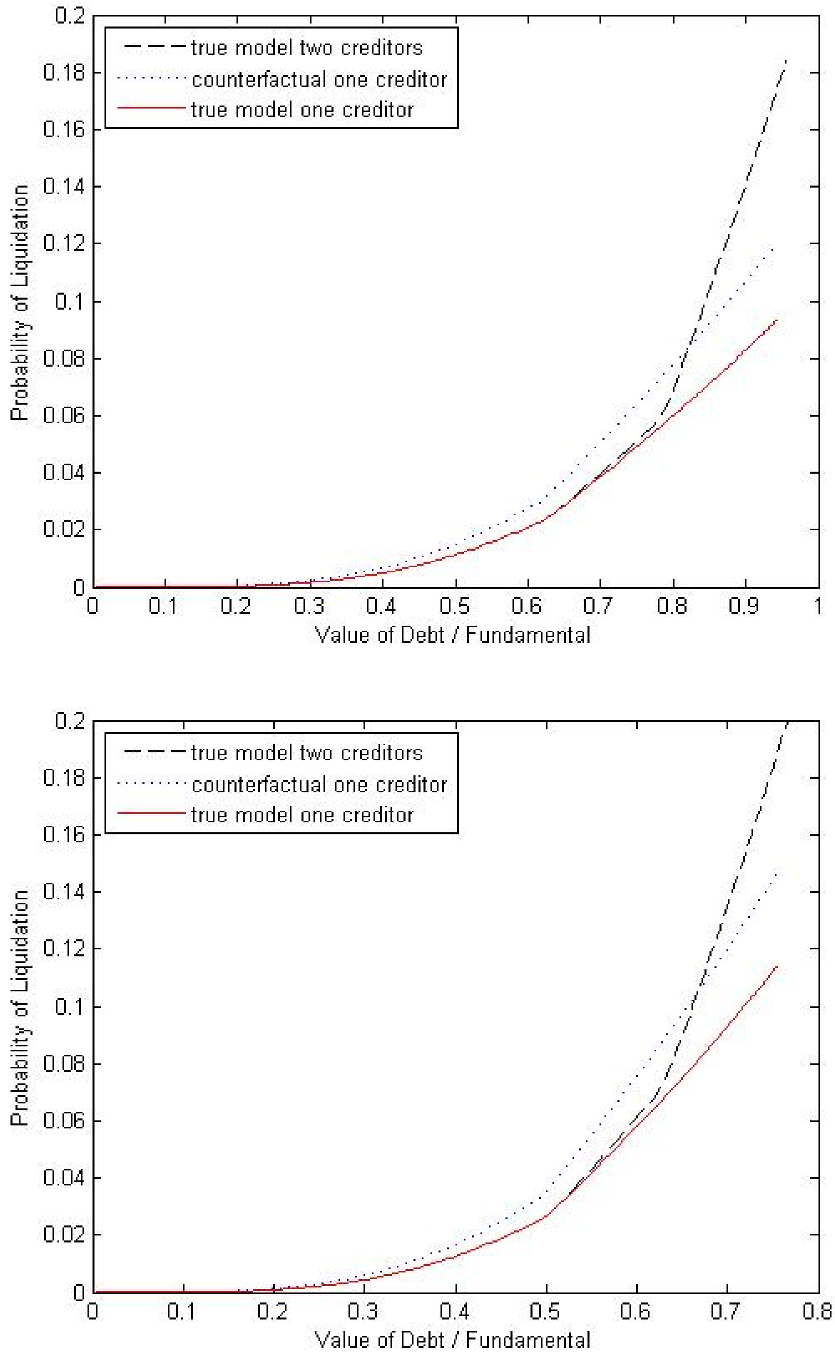

Figure 4 *The figure plots the expected probability of liquidation* $L_1^{\theta,\infty}(F,Y)$ *against the expected value of the debt conditional on the current state* $\theta = G$ *(top panel) and* $\theta = B$ *(bottom panel). The solid (dashed) line is the liquidation probability with a single creditor (two creditors) in the full model. The dotted line is for the counterfactual model in which the number*

of creditors is exogenously fixed at one. It is easy to see that having a single creditor in the true model means a lower liquidation probability compared to having two creditors as well as the counterfactual one creditor model. For a substantial range of fundamental values, the liquidation probability with two creditors *in the true model is strictly lower compared with the* single *creditor counterfactual.*