Risk and Return in Segmented Markets with Expertise*

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Abstract

This paper studies the equilibrium pricing of complex securities in segmented markets by risk-averse expert investors who are subject to asset-specific risk. In our model, the investment technology of investors with more expertise is subject to less asset-specific risk. Expert demand lowers equilibrium required returns, reduces participation, and leads to endogenously segmented markets. Amongst participants, portfolio decisions and realized returns determine the joint distribution of financial expertise and financial wealth. This distribution, along with participation, then determine market-level risk bearing capacity. We show that more complex assets deliver higher equilibrium returns to expert participants. We characterize the stationary distribution as a function of the parameters that describe the complexity of the asset class in a dynamic model of industry equilibrium. We show that the stationary wealth distribution displays fatter tails in markets in which complex assets display a steeper asset-specific risk vs. expertise relation.

Key Words: segmented markets, slow moving capital, risky arbitrage, hedge funds, industry equilibrium, firm size distribution, financial expertise, intellectual capital, intermediary asset pricing.

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1 Introduction

Complex investment strategies, such as those employed by hedge funds and other sophisticated investors, require expertise that is specific to an asset class and appear to generate persistent excess returns despite free entry. Our paper aims to understand the role of expertise in determining equilibrium asset prices of complex securities that are held by experts. To do so, we develop an industry equilibrium model of risk-averse investors with heterogeneous expertise who invest in an endogenously segmented market in which the risky asset earns positive excess returns in equilibrium. We characterize how the returns to complex assets are determined by the joint distribution of expertise and financial wealth. In equilibrium, the joint distribution is in turn determined by the deep parameters which describe preferences, endowments, and technologies in our model economies, and which proxy for asset complexity.

Our model economy is populated by a continuum of agents who choose to be either non-experts who can invest only in the risk free asset or experts who can invest in both the risk free and risky assets. Investors who choose to be experts make an initial investment in expertise, which represents the investor's personnel, data, hedging and risk management technologies, back office operations and trade clearing processes, relationships with dealers, and relationships with clients.

The acquisition and management of complex assets require a joint investment in the asset and in a hedging technology which requires financial expertise. All expert investors in the market earn a common equilibrium return that clears the market. However, their returns are subject to asset-specific shocks. Expertise improves investors' hedging technology and shrinks the asset-specific volatility of the returns to the risky asset, implying that more expert investors face a higher Sharpe ratio.[1] Thus, expertise may be interpreted as the ability to hedge risks either by developing a superior model or gathering superior information.

In our stationary model, the risk is asset-specific and idiosyncratic. This is, of course, a useful assumption technically. However, we argue that it is also realistic, as argued in Merton [1987]. There is a growing literature that documents the importance of idiosyncratic risk in asset pricing, which we review below. In particular, there is a wealth of evidence that documents downward sloping demand curves in stock markets (e.g., index reconstitutions), bond markets (e.g., Treasury auctions), and other asset markets. Pontiff [2006] investigates the role of idiosyncratic risk faced by arbitrageurs in a review of the literature and argues that "The

[1] See Sharpe [1966].

literature demonstrates that idiosyncratic risk is the single largest cost faced by arbitrageurs".

Idiosyncratic risk is likely to be particularly important in markets for complex assets. Complex assets expose their owners to idiosyncratic risk through several channels. First, their constituents tend to be significantly heterogeneous, so that no two investors hold exactly the same asset. Second, the risk management of complex assets typically requires a hedging strategy that will be subject to the individual technological constraints of the investor. Third, firms which manage complex assets may be exposed to key person risk due to the importance of specialized traders, risk managers, and marketers. Finally, complex assets may introduce or amplify idiosyncratic risk on the liability side of the balance sheet, through the fact that they are difficult for outside investors to understand, but tend to be funded with external finance.[2]

In our model, funds cannot be reallocated across individual risk-averse investors. Clearly, since the risk in our economy is idiosyncratic, pooling this risk would eliminate the risk premium that experts require to hold it. Complex assets tend to be held in managed accounts. For incentive reasons, these managers cannot hedge their own exposure to their particular portfolio. In fact, Panageas and Westerfield [2009] and Drechsler [2014] provide important results for the portfolio choice of hedge fund managers who earn fees based on assets under management and portfolio performance. In particular, they show that these managers behave like constant relative risk aversion investors. This motivates why we endow expert investors in our model with CRRA preferences.

We present two closely related models to highlight the economic mechanisms driving our results. First, we discuss simple a static model, which we solve fully in closed form, taking the joint distribution of wealth and expertise as given. In this model, the risk could be asset-specific or common. We provide results for the effects of changes in this distribution, and the other model parameters, on the market clearing excess return to the complex asset, individual Sharpe ratios, and the equilibrium weighted average market Sharpe ratio. We emphasize the heterogeneity in Sharpe ratios, and the difference between individual Sharpe ratios and the market-wide risk return trade off. For example, we show that if fundamental volatility increases, there is a cutoff level of expertise below which individual Sharpe ratios decrease, and above which they increase. This result is interesting in the context of understanding changes in participation following shocks to complex asset markets, and in understanding participation patterns across asset classes.

[2]Broadly interpreted, these risks may come either from the asset side, or from the liability or fund flow side through investments in a stable investor base. We abstract from the microfoundations of risks from the liability side of funds' balance sheets, and model risk on the asset side.

Next, we present a dynamic model in continuous time. In this model, expertise varies in the cross-section but it is fixed for each agent over time. We also solve this model, including the joint stationary wealth and expertise distribution, in closed form, up to the equilibrium fixed point for expected returns.[3] In this model, the deep preference and technology parameters determine the joint distribution of wealth and expertise.

The equilibrium stationary wealth distribution is Pareto conditional on each expertise level. The decay parameter depends on investors' portfolio choice and exposure to the risky complex asset. In particular, because investors with higher expertise choose a higher exposure to the risky asset, both the drift and the volatility of their wealth will be greater, leading to a fatter tailed distribution at higher expertise levels. We use our results for how expertise-level wealth distributions are determined, along with results describing expertise-level and aggregate demand for the risky asset, to show how changes in the deep parameters of the model lead to changes in equilibrium returns which are consistent with more complex assets having higher expected returns. In our model, more complex assets pose more model risk, have higher costs of maintaining expertise, and require expertise which is more scarce. We also show that more complex assets which are characterized by a steeper expertise-risk relation lead to fatter tails in the wealth distribution, especially for high levels of expertise. Finally, we develop results for the risk-return tradeoff at the individual and market level by studying individual Sharpe ratios and the equilibrium weighted average market Sharpe ratio.

Our paper contributes results related to the industrial organization of complex asset markets and to the equilibrium pricing therein. Our IO results are as follows: First, we show how segmentation arises due to heterogenous expertise levels. Second, the wealth distribution of participants yields the fund size distribution. Finally, the joint distribution of wealth and expertise determines the market's efficiency. Equilibrium risk bearing capacity given an aggregate amount of wealth and expertise is determined by how much wealth is in the hands of high expertise investors. In terms of asset pricing results, we provide an explicit characterization of how equilibrium required returns are determined given asset complexity parameters. We also show how the equilibrium weighted average Sharpe Ratio is determined, and how it differs from investor specific Sharpe Ratios.

The paper proceeds as follows. In Section 2 we review the related literature. We present and analyze our static model in Section 3. Section 4 contains the construction and analysis of

[3] We use a numerical solution for market clearing. However, the solution is straightforward given our analytical solution for policy functions and distribution over individual states.

our dynamic model, and finally Section 5 concludes. Most proofs appear in the Appendix. In separate work (Eisfeldt et al. [2015]), we study a discrete time dynamic model with stochastic expertise, which we use to study the impact of unanticipated aggregate shocks and to develop quantitative results. In particular, using intuition developed in this paper, we show that expertise can act as an excess capacity-like barrier to entry, leading to interesting dynamics for market excess returns and volatility following shocks to investor wealth and to fundamental asset volatility.

2 Literature

Our paper contributes to a large and growing literature on segmented markets and asset pricing. Relative to the existing literature, we provide a model with endogenous entry, a continuous distribution of heterogeneous expertise, and a rich distribution of expert wealth that is determined in stationary equilibrium. Thus, we have segmented markets, but allow for a participation choice. Our market has limited risk bearing capacity, determined in part by expert wealth, but in addition to the amount of wealth, the efficiency of the wealth distribution also matters for asset pricing.

We group the existing literature into three main categories, namely financial constraints and limits to arbitrage, intermediary asset pricing, and segmented market models with alternative microfoundations to agency theory. Although our model is not one of arbitrage per se, our study shares the goal of understanding the returns to complex assets and strategies. Our model also shares the features of segmented markets and trading frictions with the limits to arbitrage literature. Gromb and Vayanos [2010b] provide a recent survey of the theoretical literature on limits to arbitrage, starting with the early work by Brennan and Schwartz [1990] and Shleifer and Vishny [1997].[4] Shleifer and Vishny [1997] emphasize that arbitrage is conducted by a fraction of investors with specialized knowledge, but similar to He and Krishnamurthy [2012], they focus on the effects of the agency frictions between arbitrageurs and their capital providers. Although we do not explicitly model risks to the liability side of investors' balance sheets, one can interpret the shocks agents in our model face to include idiosyncratic redemptions.[5]

[4]See also Aiyagari and Gertler [1999], Froot and O'Connell [1999], Basak and Croitoru [2000], Xiong [2001], Gromb and Vayanos [2002], Yuan [2005], Gabaix, Krishnamurthy, and Vigneron [2007], Mitchell, Pedersen, and Pulvino [2007], Acharya, Shin, and Yorulmazer [2009], Kondor [2009], Duffie [2010], Gromb and Vayanos [2010a], Hombert and Thesmar [2011], Edmond and Weill [2012], Mitchell and Pulvino [2012], Pasquariello [2013], and Kondor and Vayanos [2014].

[5]For other models of risks stemming from redemptions and fund outflows and the resulting asset pricing

Recently, the broader asset pricing impact of financially constrained intermediaries has been studied in the literature on intermediary asset pricing following He and Krishnamurthy [2012] and He and Krishnamurthy [2013].[6] This literature applies results from the literature on asset pricing with heterogenous agents, following Dumas [1989], to segmented markets with financial constraints.[7] In doing so, the intermediary asset pricing literature both connects to empirical applications, and to the asset price dynamics which are the focus of the limits to arbitrage literature. Finally, several papers develop alternative microfoundations to agency theory for segmented markets. Allen and Gale [2005] provide an overview of their theory of asset pricing based on "cash-in-the-market". Plantin [2009] develops a model of learning by holding. Duffie and Strulovici [2012] develop a theory of capital mobility and asset pricing using search foundations. Glode, Green, and Lowery [2012] study asset price dynamics in a model of financial expertise as an arms race in the presence of adverse selection. Kurlat [2013] studies an economy with adverse selection in which buyers vary in their ability to evaluate the quality of assets on the market, and, like us, emphasizes the distribution of expertise on the equilibrium price of the asset. Garleanu, Panageas, and Yu [2014] derive market segmentation endogenously from differences in participation costs. Kacperczyk, Nosal, and Stevens [2014] construct a model of consumer wealth inequality from differences in investor sophistication.

Our model is an example of an "industry equilibrium" model in the spirit of Hopenhayn [1992a] and Hopenhayn [1992b]. These models study the important effects of firm dynamics, entry and exit in the heterogeneous agent framework developed in Bewley [1986]. This literature focuses in large part on explaining firm growth, and moments describing the firm size distribution. Recent progress in the firm dynamics literature using continuous time techniques to solve for policy functions and stationary distributions include Miao [2005], Luttmer [2007], Gourio and Roys [2014], Moll [Forthcoming], and Achdou, Han, Lasry, Lions, and Moll [2014]. We draw on results in these papers as well as discrete time models of firm dynamics, as in recent work by Clementi and Palazzo [2014], which emphasizes the role of selection in explaining the observed relationships between firm age, size, and productivity. We also draw on work in the city size literature in Gabaix [1999] and the literature on the consumer wealth distribution with idiosyncratic risk and fiscal policy in Benhabib et al. [2014].

implications, see Berk and Green [2004], and Liu and Mello [2011].

[6] See also, for example, Adrian and Boyarchenko [2013]. For empirical applications, see for example, Adrian, Etula, and Muir [Forthcoming] and Muir [2014].

[7] For closely related work on asset pricing with heterogeneous risk aversion and segmented markets, see also Basak and Cuoco [1998], Kogan and Uppal [2001], Chien, Cole, and Lustig [2011], and Chien, Cole, and Lustig [forthcoming].

We use the the hedge fund industry, and in particular the asset backed fixed income (ABFI) segment, for some motivating empirical moments describing size and performance. As such, we draw from the literature on hedge funds performance and compensation.[8] In particular, we motivate our use of ABFI funds as our main example of a complex strategy using the evidence in Duarte, Longstaff, and Yu [2006]. They provide evidence that MBS strategies are relatively complex and earn higher returns even in comparison to other sophisticated fixed income arbitrage strategies. Several papers provide evidence for the importance of idiosyncratic risk in the hedge fund returns, following the idea in Merton [1987] that idiosyncratic risk will be priced when there are costs associated with learning about or hedging a specific asset.[9] Relatedly, Fung and Hsieh [1997] find that hedge fund returns have low and sometimes negative correlation with asset class returns. Our model features investors with constant relative risk aversion (CRRA) preferences. While we do this for tractability and parsimony to retain our focus on the effects of the joint wealth and expertise distribution, Panageas and Westerfield [2009] show that hedge fund compensation contracts with long horizons lead to portfolio choice which aligns perfectly with that of a CRRA investor. Drechsler [2014] extends these results to include variation in managers' outside options and shows the CRRA result holds as long as such reservation values are neither too high nor too low. These results extend the analysis of the impact of high-water marks in Goetzmann et al. [2003].

The majority of the assets under management in the ABFI sector are mortgage backed securities (MBS). Cash flow risk in MBS securities typically comes primarily from prepayment risk, since the largest part of the market consists of agency securities. Gabaix, Krishnamurthy, and Vigneron [2007] provide convincing evidence that returns are driven in large part by limits to arbitrage.[10] Recent work by Boyarchenko, Fuster, and Lucca [2014] extends these ideas and provides evidence that prepayment model risk explains the "smile" in MBS option adjusted spreads (OAS) and confirms that time series variation in returns is closely related to the MBS supply relative to the capital of MBS investors.[11] That idiosyncratic risk is priced in MBS

[8] Fung, Hsieh, Naik, and Ramadorai [2008] is a well known paper describing performance. Jagannathan, Malakhov, and Novikov [2010] carefully correct for selection bias and smoothed returns in a study of hedge fund performance persistence. Carlson and Steinman [2007] consider the relationship between hedge fund survival and market conditions. In a related spirit to our work, Getmansky [2012] empirically studies the effects of size and competition on hedge fund returns.

[9] See Titman and Tiu [2011] and Lee and Kim [2014]. Jurek and Stafford [Forthcoming] emphasize that scarce and specialized knowledge may drive both hedge fund returns and put pricing.

[10] Importantly, they show that although prepayment risk is partly common within a class of MBS securities, the risk in MBS investing is negatively correlated with the aggregate risks born by a representative consumer.

[11] See also Dunn and McConnell [1981a], Dunn and McConnell [1981b], Schwartz and Torous [1992], Stanton [1995], Boudoukh, Richardson, Stanton, and Whitelaw [1997], Longstaff [2005], Downing, Stanton, and Wallace

is supported by prior empirical studies. It is also consistent with the fact that different investors have different pricing and hedging models, and invest in different parts of the mortgage space. Some funds may benefit from early prepayment, while other funds may are suffer from early prepayment. Different mortgage assets have different direct interest rate exposure, and investors hedge their interest rate exposures to different extents. Finally, variation in lending standards can have opposite effects on prepayment due to default and voluntary prepayment. We implement the "model risk" inherent in funds' prepayment models via the variation in idiosyncratic risk faced by investors with varying amounts of expertise.[12]

3 Static Model

We present a static model to build intuition about the interaction between the size and expertise distribution of investors and equilibrium returns.

Setup Investors have constant relative risk aversion preferences over date 1 consumption, with coefficient of relative risk aversion γ. At date 0, they are endowed with financial wealth W and expertise X. There is a riskless asset with gross return R_f, and a risky asset with gross returns R, which are distributed log normally. We use lower case letters to denote logs.

We assume that the log return on the risky asset for any given investor, which we denote by r, is distributed according to $r \sim N(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X})$, given the distribution of W and X. We denote the variance of log returns on the fundamental asset, before expertise is applied, by σ_v^2, and call this fundamental variance, and its square root fundamental volatility. The effective variance and volatility of an investor's return on the risky asset then decreases as expertise X increases, while the innovation v itself is independent from W and X. We provide an example microfoundation for a closely related return process in the context of our dynamic model in the Appendix.

Investing in the complex asset implies a joint investment in a common market clearing return, as well as a specific risk from hedging or asset specificities. We assume the specified functional form for log return volatility for simplicity, as it allows for straightforward calculations of all expectations, and minimal parameters. It is straightforward to show that our main

[2005] for models of MBS pricing, and Agarwal, Driscoll, and Laibson [2013] for a recent model of consumer prepayment behavior solved in closed form.

[12] MBS have been described as the Swiss Army knife of asset classes, providing any risk exposure one desires. Likewise, the market for mortgage backed derivatives has been described as analogous to the heterogeneous detritus left over from butchering a pig after the desirable parts have satisfied the demands of long-only investors.

conclusions for the static model are robust to a family of functions $\frac{\sigma_v^2}{k_0+k_1X+k_2X^2+...}$, with all coefficients $k_0, k_1, k_2....$ being non-negative. In levels, expected returns $\boldsymbol{\mu}$ are the same for all investors, regardless of their individual expertise.

Solving the Portfolio Choice Problem Using the approximation described in Campbell and Viceira [2002a], and the associated appendix Campbell and Viceira [2002b], which relates log individual-asset returns to log portfolio returns over short time intervals, the investor's optimization problem becomes:

$$\max_{\theta}\left\{\theta\left(\boldsymbol{\mu}-r_f\right)-\frac{\gamma}{2}\theta^2\frac{\sigma_v^2}{X}\right\} \tag{1}$$

where r_f represents the log return on the riskless asset. In this section, for emphasis, we use bold notation to denote equilibrium returns.[13] The solution for the optimal fraction of wealth allocated to the risky asset is:

$$\theta^* = \frac{(\boldsymbol{\mu}-r_f)}{\gamma\sigma_v^2}X. \tag{2}$$

Thus, portfolio choice in a lognormal model with power utility resembles that of a mean variance investor. The allocation to the risky asset is increasing in the equilibrium average excess return, decreasing in risk aversion, and decreasing in the fundamental shock variance. Moreover, the fraction of wealth that an investor allocates to the risky asset strictly increases with expertise. The relationship is linear under our functional form assumptions.[14]

Equilibrium We now describe how the equilibrium excess return depends on the parameters for preferences, technology, and the joint distribution of wealth and expertise. We focus on comparative statics over the equilibrium average excess return, market level Sharpe ratio, and individual Sharpe ratios. We normalize the mass of investors to one, define the value of the supply of the risky asset to be S, determine the market clearing log expected return $\boldsymbol{\mu}$, and then back out the equilibrium expected level return and therefore $\boldsymbol{\alpha}$. We assume that W and X are jointly log-normally distributed. We denote the joint pdf of the log variables $f(w,x)$, with means and variances μ_w, μ_x, σ_w^2, and σ_x^2 respectively, and covariance $\rho_{w,x}\sigma_w\sigma_x$. Thus, an

[13] Because an individual investor's return volatility depends on their expertise, for the approximation to be good given our specification for log return volatility, we have to impose a technical restriction that the majority of distribution of expertise X is bounded away from zero. This assumption is unnecessary if one adopts the general functional form for volatility discussed in footnote 3.

[14] Without restrictions on the distribution of X, θ can be larger than one, implying borrowing at the risk free rate.

economy ψ is described by $\psi \equiv \{r_f, \gamma, I, \sigma_v, \mu_w, \sigma_w, \mu_x, \sigma_x, \rho_{w,x}\}$. The equilibrium log expected return $\boldsymbol{\mu}$ solves the market clearing condition:

$$\text{Supply} \equiv S = \text{Demand} = \int\int \exp(w)\theta^*\left(\exp(x)\right) f(w,x)\, dw\, dx = \frac{\boldsymbol{\mu} - r_f}{\gamma\sigma_v^2}\mathcal{X} \tag{3}$$

where $\theta^*\left(\exp(x)\right)$ is the portfolio choice given in Equation (21) and $\mathcal{X}$ is the wealth and population weighted average of expertise:

$$\int\int \exp(w+x) f(w,x)\, dw\, dx = \exp\left(\frac{1}{2}\left(\sigma_w^2 + \sigma_x^2 + 2\rho_{w,x}\sigma_w\sigma_x + 2\mu_w + 2\mu_x\right)\right) \tag{4}$$

utilizing the result for the expectation of log normally distributed variables.

Rearranging, we have:

$$\boldsymbol{\mu} - r_f = \left(\frac{\sigma_v^2}{\mathcal{X}}\right)\gamma S. \tag{5}$$

The equilibrium log expected excess return is increasing in the amount of risk relative to the risk bearing capacity of investors. We decompose the inputs into two components. The first term is the effective risk in the market, namely the fundamental risk σ_v^2, scaled down by the wealth and population weighted average of expertise. The second term is the risk aversion scaled supply of the risky asset which must be cleared. The higher is investors' coefficient of relative risk aversion, and the larger is the supply of the asset, the higher is the required return. Conversely, the wealth and population weighted average of expertise, $\mathcal{X}$, scales $\boldsymbol{\mu}$ down due to the positive impact of expertise on investors' allocation to the risky asset.

Using the equilibrium log expected return $\boldsymbol{\mu}$, we can rewrite agents' optimal portfolio allocations to the risky asset as:

$$\theta^* = \frac{X}{\mathcal{X}}S. \tag{6}$$

This expression captures the fact that, in equilibrium, the optimal portfolio allocations to the risky asset by an agent with expertise X turns out to be a fraction of total supply equal to their expertise relative to the wealth and population weighted average of expertise.

The equilibrium mean of the level of the gross risky return over the level of the gross risk free rate, $\boldsymbol{\alpha}$, is a monotonic transformation of $\boldsymbol{\mu}$. In particular, we show in the Appendix that the equilibrium α is then given by:

$$\boldsymbol{\alpha} = \exp\left(\boldsymbol{\mu}\right) - R_f \tag{7}$$

which gives a one to one mapping from μ to α conditional on parameters. Note also that writing θ^* (Equation 21) as a function of either $\boldsymbol{\mu}$ or $\boldsymbol{\alpha}$ will always yield identical equilibrium outcomes.

Lemma 3.1 *Using Equation (7) describing the equilibrium market clearing $\boldsymbol{\alpha}$, the following comparative statics can be directly calculated:*

1. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_v^2} = \exp(\boldsymbol{\mu}) \frac{\gamma S}{\mathcal{X}} > 0$. *$\boldsymbol{\alpha}$ increases with fundamental risk.*

2. $\frac{\partial \boldsymbol{\alpha}}{\partial \gamma} = \exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} S > 0$. *$\boldsymbol{\alpha}$ increases with the coefficient of relative risk aversion.*

3. $\frac{\partial \boldsymbol{\alpha}}{\partial S} = \exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma > 0$. *$\boldsymbol{\alpha}$ increases with the risky asset supply investors must absorb.*

4. $\frac{\partial \boldsymbol{\alpha}}{\partial \mu_w} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S < 0$. *$\boldsymbol{\alpha}$ decreases as aggregate wealth increases.*

5. $\frac{\partial \boldsymbol{\alpha}}{\partial \mu_x} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S < 0$. *$\boldsymbol{\alpha}$ decreases as aggregate expertise increases.*

6. $\frac{\partial \boldsymbol{\alpha}}{\partial \rho_{w,x}} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S \sigma_w \sigma_x < 0$. *As $\rho_{w,x}$ increases, there is a more efficient allocation of expertise and $\boldsymbol{\alpha}$ decreases.*

7. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_w} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S (\sigma_w + \rho_{w,x} \sigma_x)$
   - > 0 *if* $\rho_{w,x} < -\frac{\sigma_w}{\sigma_x}$, *i.e. if wealth and expertise are strongly negatively correlated.*
   - < 0 *if* $\rho_{w,x} > -\frac{\sigma_w}{\sigma_x}$, *i.e. if wealth and expertise are positively or only weakly negatively correlated.*

8. $\frac{\partial \boldsymbol{\alpha}}{\partial \sigma_x} = -\exp(\boldsymbol{\mu}) \frac{\sigma_v^2}{\mathcal{X}} \gamma S (\sigma_x + \rho_{w,x} \sigma_w)$
   - > 0 *if* $\rho_{w,x} < -\frac{\sigma_x}{\sigma_w}$, *i.e. if wealth and expertise are strongly negatively correlated.*
   - < 0 *if* $\rho_{w,x} > -\frac{\sigma_x}{\sigma_w}$, *i.e. if wealth and expertise are positively or only weakly negatively correlated.*

Proof. By direct calculation. ∎

All comparative statics are intuitive. An increase in the correlation of wealth and expertise will reduce $\boldsymbol{\alpha}$, as investors with more expertise account for a larger share of the wealth distribution. The effect of an increase in $\rho_{w,x}$ on the market clearing $\boldsymbol{\alpha}$ will be larger the larger is amount of fundamental risk, σ_v^2, the larger is the coefficient of relative risk aversion, γ, the larger is the supply of the risky asset, S, the smaller is the mean of log wealth, μ_w, and the smaller is the mean of log expertise, μ_x.

We also derive results for the equilibrium market-level and investor-specific Sharpe ratios. The market level Sharpe ratio requires a definition appropriate for our environment. Here, we define the equilibrium market level Sharpe ratio to be the equal-weighted cross-sectional average of excess returns divided by the equal-weighted cross-sectional standard deviation. Thus this market Sharpe ratio can, for example, be interpreted as the expected Sharpe ratio for an investor "behind the veil" drawing from the distribution of possible levels of expertise, before the investment stage of the model. This Sharpe ratio would be relevant, for example, in a model with entry in which an investor must decide whether to enter before drawing an expertise level from the given distribution. We then refer to what is technically the equilibrium equally weighted market Sharpe ratio as the "Sharpe ratio" for exposition purpose:

$$SR = \frac{1 - R_f \ \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}\right)\right] - 1}} = \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\sum_{k=1}^{\infty} \frac{1}{k!}\sigma_v^{2k} \exp(-k\mu_x + \frac{1}{2}k^2\sigma_x^2)}} \approx \frac{1 - R_f \exp\left(-\boldsymbol{\mu}\right)}{\sigma_v \exp(-\frac{1}{2}\mu_x + \frac{1}{4}\sigma_x^2)}, \quad (8)$$

where $\mathbb{E}$ denotes the cross-sectional expectation. This ratio aggregates all investor decisions and measures the *market level* risk return tradeoff.[15] The market-level Sharpe ratio increases as the average log expertise μ_x in this economy increases, but it decreases as the cross-sectional standard deviation of log expertise σ_x^2 increases.

Lemma 3.2 *Using Equation (8) describing the equilibrium market clearing equally weighted Sharpe ratio, the following comparative statics can be directly calculated:*

1. *Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.*
   Then, Sign $\left(\frac{\partial(SR)}{\partial\eta}\right)$ = Sign $\left(\frac{\partial\boldsymbol{\alpha}}{\partial\eta}\right)$.
2. *The signs for comparative statics with respect to parameters $\hat{\eta} \in \{\sigma_v^2, \mu_x, \sigma_x\}$, are indeterminate.*

Proof. By direct calculation, see Appendix. ■

[15] See Appendix for derivation. We also compute and analyze the market value weighted Sharpe ratio in the Appendix.

Expected returns rise proportionally relative to the volatility of the risky asset return in our static model, so that the Sharpe ratio improves with any parameter change that increases $\boldsymbol{\alpha}$. Thus, we confirm that, at the market level, parameter changes which lead to an increase in the equilibrium expected excess return in fact lead to better investment opportunities given the market risk in equilibrium.

In our model, each investor confronts a different risk-return trade-off. Since the volatility of log returns depends on individual investors' expertise, an observed increase in the market Sharpe ratio does not necessarily imply a higher Sharpe ratio for every investor in the market. Moreover, even if the Sharpe ratio improves for each agent individually, the magnitude of the improvement an individual investor faces will not, in general, coincide with the market improvement. To see this, consider the investor-specific Sharpe ratio. For an investor with wealth W and expertise X, we show in the Appendix that this investor's Sharpe ratio is given by:

$$SR\left(X\right) = \frac{1 - R_f \, \exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right) - 1}}. \tag{9}$$

Equation (9) clearly shows that the model can deliver considerable cross-sectional dispersion in investor-specific Sharpe ratios. Investors with very low effective risk, $\frac{\sigma_v^2}{X}$, face significantly higher Sharpe ratios than their counterparts with low expertise. We can determine the signs of the following comparative statics:

Lemma 3.3 *Using Equation (9) describing the investor-specific Sharpe ratio, and Equation (21) describing the portfolio allocation θ^*, the following comparative statics can be directly calculated. Let η denote any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$.*[16]

1. $\frac{\partial SR(X)}{\partial X} > 0$. *Higher expertise generates lower effective risk, and a correspondingly higher individual Sharpe ratio.*

2. $Sign\left(\frac{\partial SR(X)}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$. *All investor-specific Sharpe ratios co-move with the equilibrium excess return and the market level equilibrium Sharpe ratio.*

3. $Sign\left(\frac{\partial \boldsymbol{Var}(SR(X))}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right)$. *Whenever a parameter change in-*

[16] Derivatives with respect to μ_x and σ_x follow the same formulas as those that support parts 2 to 5 of lemma 3.3. However, the changes are not comparable to the market Sharpe ratio, as we can't determine the signs in lemma 3.2, part 2. Derivatives with respect to σ_v^2 cannot be signed generally.

creases the market level equilibrium Sharpe ratio, it leads to a larger cross-sectional dispersion in the investor-specific Sharpe ratio.

4. $Sign\left(\frac{\partial^2 SR(X)}{\partial\eta\partial X}\right) = Sign\left(\frac{\partial SR(X)}{\partial\eta}\right) = Sign\left(\frac{\partial\boldsymbol{\alpha}}{\partial\eta}\right) = Sign\left(\frac{\partial(SR)}{\partial\eta}\right)$.
Whenever a parameter change increases the investor-specific Sharpe ratio, it leads to a larger increase for high expertise investors relative to low expertise investors.

5. $Sign\left(\frac{\partial^2 \theta^*(X)}{\partial\eta\partial X}\right) = Sign\left(\frac{\partial SR(X)}{\partial\eta}\right) = Sign\left(\frac{\partial\boldsymbol{\alpha}}{\partial\eta}\right) = Sign\left(\frac{\partial(SR)}{\partial\eta}\right)$.[17]
Whenever a parameter change increases the investor-specific portfolio allocation, it leads to a larger increase for high expertise investors relative to low expertise investors.

6. $\exists \bar{X} > \underline{X} > 0$ *such that* $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial\sigma_v^2} > 0,$ *and* $\forall X < \underline{X}, \frac{\partial SR(X)}{\partial\sigma_v^2} < 0$. *An increase in the fundamental risk generates a higher Sharpe ratio for high expertise investors and a lower Sharpe ratio for low expertise investors.*

Proof. By direct calculation. See Appendix. ■

Lemma (3.3) has rich implications. First, we emphasize the co-movement between cross sectional variation in investor-specific Sharpe ratios and the level of the market Sharpe ratio. Any increase in the market-level Sharpe ratio will also increase the cross-sectional dispersion in Sharpe ratios. Furthermore, because an increase in the market level Sharpe ratio improves investment opportunities for high expertise investors by more than for low expertise investors, such an increase accordingly increases their allocation to the risky asset θ^* by more. Thus, an improvement in the market-level risk return tradeoff in large part reflects the improved risk-return trade-off faced by high expertise investors, and not by their low expertise counterparts.

Any parameter change which increases the market level Sharpe ratio increases the investor specific Sharpe ratio for high expertise by more, and increases the influence of high expertise investors' Sharpe ratios on the market level risk return tradeoff. In our model, measured improvements in the aggregate Sharpe ratio are a misleading indicator of improvements in individual investors' risk-return tradeoff, and can indeed more accurately reflect changes in the Sharpe ratio of higher expertise investors. The converse is also true.

Furthermore, part 6 of Lemma (3.3) states that changes to fundamental risk can lead to changes in individual Sharpe ratios that vary in sign. For example, if σ_v^2 increases, all investors face the same increase in the equilibrium excess return, but investors with high expertise face

[17]Except for γ, where $\frac{\partial^2\theta^*(X)}{\partial\gamma\partial X} = 0$.

a considerably smaller increase in risk. Thus, more complex assets with higher σ_v^2 can have higher market level Sharpe ratios but lower demand from non-experts. We also emphasize that because an increase in fundamental risk improves the investor-specific Sharpe ratio for some investors but not others, in a dynamic model a shock to fundamental risk can lead potentially lead to variation in investors' participation decisions. In other words, an increase in risk which improves the market level equally weighted Sharpe ratio may still lead low expertise investors to exit, or not to enter.

4 Dynamic Model

4.1 Preferences, Endowments, & Technologies

We study a model with a continuum of investors of measure one, with CRRA utility functions over consumption:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Investment Technology Investors are endowed with a level of expertise which varies in the cross section, but is fixed for each agent over time. Each individual investor is born with a fixed expertise level, x, drawn from a distribution with pdf $\lambda(x)$, and cdf $\Lambda(x)$.

Investors can choose to be experts, and have access to the complex risky asset, or non-experts, who can only invest in the risk free asset. Each investor's complex risky asset delivers a stochastic return which follows a geometric Brownian motion:

$$\frac{dP(t,s)}{P(t,s)} = \left[r_f + \alpha(s)\right] ds + \sigma(x)\, dB(t,s) \tag{10}$$

where $\alpha(s)$ is the common excess return on the risky asset and $B(t,s)$ is a standard Brownian motion which is investor specific and i.i.d. in the cross section. The volatility of the risky technology $\sigma(x)$ decreases in the investor's level of expertise x, i.e. $\frac{\partial \sigma(x)}{\partial x} < 0$. We require that $\lim_{x\to\infty} \sigma(x) = \underline{\sigma} > 0$. The lower bound on volatility, $\underline{\sigma}$, represents complex asset risk that cannot be eliminated even by the agents with the greatest expertise, and it guarantees that the growth rate of wealth is finite.

In order to invest in the risky asset, an investor must also jointly invest in a technology with a zero mean return and an idiosyncratic shock. This technology represents each investor's specific hedging and financing technologies, as well as the unique features of their particular asset.

According to its general definition, α cannot be generated by bearing systematic risk. However, capturing α is risky because it requires a model and execution, and each investor's model and execution technology is unique. For example, hedging portfolios tend to vary substantially across different investors in the same asset class.[18]

To be an expert, an investor must pay the entry cost F_{nx} to set up their specific technology for investing in the complex risky asset. Experts must also pay a maintenance cost, F_{xx} to maintain continued access to the risky technology. We consider the simplest case in which both the entry and maintenance costs are proportional to wealth:

$$\begin{aligned} F_{nx} &= f_{nx} w, \\ F_{xx} &= f_{xx} w, \end{aligned}$$

which yields value functions which are homogeneous in wealth.

Optimization We first describe the Bellman equations for non-experts and experts respectively, and characterize their value functions, as well as the associated optimal policy functions. With the value functions of experts and non-experts in hand, we then characterize the entry decision.

We begin with non-experts, who can only invest in the risk free asset. Let $w(t,s)$ denote the wealth of investors at time s with initial wealth W_t at time t. The riskless asset delivers a fixed return of r_f. All investors choose consumption, and an optimal stopping, or entry time according to the Bellman Equation:

$$\begin{aligned} V^n(w(t,s),x) &= \max_{c^n(t,s),\tau} E\left[\int_t^\tau e^{-\rho(s-t)} u(c^n(t,s))\, ds + e^{-\rho(\tau-t)} V^x(w(t,s) - F_{nx}, x)\right] && (11) \\ \text{s.t. } dw(t,s) &= (r_f w(t,s) - c^n(t,s))\, ds && (12) \end{aligned}$$

where ρ is their subjective discount factor, $c(t,s)$ is consumption at time s, F_{nx} is the entry cost, and τ is the optimal entry date.

Under the assumptions of linear entry and maintenance costs, and expertise which is fixed over time, the optimal entry date in a stationary equilibrium will be either immediately or never.

[18] In MBS, there is no agreed upon method to hedge mortgage duration risk, though most all active investors do so. Some hedge according to empirical durations, using various estimation periods and rebalancing periods. Others hedge according to the sensitivity of MBS prices yield curve shifts using their own (widely varying) proprietary model of MBS prepayments and prices.

Thus, assuming an initial stationary equilibrium, investors who choose an infinite stopping time are then non-experts, and investors who choose a stopping time $\tau = t$ are experts.[19]

Experts allocate their wealth between current consumption, a risky asset, and a riskless asset. They also choose an optimal stopping time to exit the market.

$$
\begin{aligned}
V^x\left(w\left(t,s\right),x\right) &= \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds + e^{-\rho(T-t)} V^n\left(w\left(t,s\right),x\right)\right] && (13)\\
\text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f + \theta\left(x,t,s\right)\alpha\left(t,s\right)\right) - c^x\left(t,s\right) - F_{xx}\right] ds && (14)\\
&\quad + w\left(t,s\right)\theta\left(x,t,s\right)\sigma\left(x\right) dB\left(t,s\right),
\end{aligned}
$$

where $\alpha\left(s\right)$ is the equilibrium excess return on the risky asset, $\theta\left(x,t,s\right)$ is the portfolio allocation to the risky asset by investors with expertise level x at time s, $c\left(t,s\right)$ is consumption, F_{xx} is the maintenance cost, and T is the optimal exit date.

We include exit for completeness. However, exit will not occur in this homogeneous model with fixed expertise.

Proposition 4.1 ***Value and Policy Functions:*** *The value functions are given by*

$$
\begin{aligned}
V^x\left(w\left(t,s\right),x\right) &= y^x\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} && (15)\\
V^n\left(w\left(t,s\right),x\right) &= y^n\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} && (16)
\end{aligned}
$$

where $y^x\left(x\right)$ and $y^n\left(x\right)$ are given by:

$$
\begin{aligned}
y^x\left(x\right) &= \left[\frac{\left(\gamma-1\right)\left(r_f - f_{xx}\right)+\rho}{\gamma} + \frac{\left(\gamma-1\right)\alpha^2}{2\gamma^2\sigma^2\left(x\right)}\right]^{-\gamma} \text{ and} && (17)\\
y^n\left(x\right) &= \left[\frac{\left(\gamma-1\right)r_f+\rho}{\gamma}\right]^{-\gamma}. && (18)
\end{aligned}
$$

The optimal policy functions $c^x\left(x,t,s\right), c^n\left(t,s\right)$, and $\theta\left(x\right)$ are given by:

$$
\begin{aligned}
c^x\left(x,t,s\right) &= \left[y^x\left(x\right)\right]^{-\frac{1}{\gamma}} w\left(t,s\right), && (19)\\
c^n\left(t,s\right) &= \left[y^n\left(x\right)\right]^{-\frac{1}{\gamma}} w\left(t,s\right) \text{ and} && (20)\\
\theta\left(x,t,s\right) &= \frac{\alpha(t,s)}{\gamma\sigma^2\left(x\right)}. && (21)
\end{aligned}
$$

[19] Outside of the stationary equilibrium, because α is not constant, both entry and exit are possible.

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw(t,s)}{w(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s) \tag{22}$$

Finally, investors choose to be experts if the excess return earned per unit of wealth exceeds the maintenance cost per unit of wealth:

$$\frac{\alpha^2(t,s)}{2\sigma^2(x)\,\gamma} \geq f_{xx}. \tag{23}$$

We prove this Proposition by guess and verify in the Appendix. Note that the law of motion for wealth is a sort of weighted average of the return to the risky and riskless assets, as determined by portfolio choice, net of consumption. The drift and volatility of investors' wealth are increasing in the allocation to the risky asset. This mechanism has important implications for the wealth distribution in the stationary equilibrium of our model.

4.2 The Distribution(s) of Expert Wealth

The total amount of wealth allocated to the complex risky asset, as well as the distribution of expert wealth across expertise levels, are key aggregate state variables for the the first and second moments of the equilibrium returns to the complex risky asset. Once the entry decision has been made, given that we do not clear the market for the riskless asset, the wealth of non-experts is irrelevant for the returns to the complex risky asset. We solve for the cross-sectional distribution of expert wealth in a stationary equilibrium of our model. Given that expertise is fixed over time for each investor, constructing the wealth distribution at each expertise level is sufficient to obtain the cross-sectional joint distribution of wealth and expertise.

First, we note that in order to construct a stationary equilibrium given that experts' wealth on average grows over time, it is convenient to study the ratio of individual wealth to the mean wealth of agents with highest expertise, $\mathbb{E}\left[w|_{\bar{x}}(t,s)\right]$.

$$z(t,s) = \frac{w(t,s)}{\mathbb{E}\left[w|_{\bar{x}}(t,s)\right]}.$$

Next, note that the law of motion for the mean wealth of agents with a given level of expertise x is given by

$$d\mathbb{E}w|_x(t,s) \equiv [g(x)]\,dt.$$

where $g(x)$ will be determined in equilibrium. Define the average growth rate amongst agents with the "highest" level of expertise as $g(\bar{x}) \equiv \sup_x g(x)$. Then, the ratio $z(t,s)$ follows a geometric Brownian motion given by

$$\frac{dz(t,s)}{z(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2(t,s)}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s), \tag{24}$$

where $\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t)}{2\gamma^2\sigma^2(x)} - g(\bar{x}) < 0$ represents the negative drift, or growth rate.

Let the cross-sectional p.d.f. of expert investors' wealth and expertise at time t be denoted by $\phi^x(z,x,t)$. Without additional assumptions, the relative wealth of lower expertise agents will shrink to zero. Two methods are commonly used to generate a stationary distribution. The first, for example used in Benhabib et al. [2014], is to employ a life cycle model, or Poisson elimination of agents. The second, employed by Gabaix [1999], is to introduce a reflecting barrier at a minimum wealth share, $z_{\min}$.[20] We adopt the assumption of a minimum wealth share because it leads to a more elegant expression for the wealth distribution. Moreover, for asset pricing, only the higher ends of the wealth distribution are quantitatively relevant, so this elegance comes at a low cost. We will show that the stationary distribution of wealth at each expertise level will be a Pareto distribution.[21] Note that the reflecting barrier at $z_{\min}$ implies that the growth rate of any individual agent, even those with the highest level of expertise, will grow more slowly than the mean wealth of the highest expertise agents.

Since the reflecting boundary mainly affects low wealth investors, decisions near the boundary matter little for equilibrium pricing. However, we adopt an interpretation of exit and entry at $z_{\min}$ which ensures that policies are not distorted there. Then, since both time and state variables are continuous in our model, if policies are not distorted at $z_{\min}$, then they will not be distorted elsewhere. The strategy we employ is to ensure that the value at $z_{\min}$ from adopting the optimal policy functions under non-reflecting wealth share dynamics is equal to the value of adopting those policies given that with some probability the investor will be punished by being forced to exit, and with some probability the investor will be rewarded by being able to

[20] Gabaix [1999] constructs a model of the city size distribution, and thus his share variable represents relative population shares. See also the Appendix of that paper for a related method of constructing a stationary distribution using a Kesten [1973] process, which introduces a random shock with a positive mean to normalized city size.

[21] Adopting the assumption of Poisson death with a fixed initial wealth, for example, would instead lead to a double Pareto distribution, with a cutoff at the initial value of wealth. For example, see Benhabib et al. [2014] for the wealth distribution under the alternative assumption of Poisson elimination in a closely related model. This is also the assumption we adopt in our quantitative study in Eisfeldt et al. [2015]. On the other hand, initializing agents according to the stationary distribution involves solving a challenging fixed point problem.

infuse funds themselves, or by receiving new external funds. In the case of exit, we assume the investor is replaced by a new entrant with wealth share $z_{\min}$ and the same level of expertise x as the exiting agent.[22]

We derive the Kolmogorov forward equations describing the evolution of the wealth distribution, taking $\alpha(t)$ as given, as follows:[23]

$$\begin{aligned}\partial_t \phi^x (z,x,t) = &- \partial_z \left(\left(z(r_f + \theta (x,t) \alpha (t,s)) - [y^x (x)]^{-\frac{1}{\gamma}} - f_{xx} - g(\bar{x}) \right) \phi^x (z,x,t) \right) \\ &+ \frac{1}{2} \partial_{zz} \left([z\theta (x,t) \sigma(x)]^2 \phi^x (z,x,t) \right) \\ &= -\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1) \alpha^2 (t,s)}{2\gamma^2 \sigma^2 (x)} - g(\bar{x}) \right) \phi^x (z,x,t) \right] \\ &+ \frac{1}{2} \partial_{zz} \left[\left(z \frac{\alpha (t,s)}{\gamma \sigma (x)} \right)^2 \phi^x (z,x,t) \right].\end{aligned} \tag{25}$$

We then study the stationary distribution of wealth shares, in which $\partial_t \phi^x (z,x,t) = 0$. We take as given, for now, that $\alpha(t,s)$ will be constant, as in the stationary equilibrium we define in the following section. This will be true given a stationary distribution over investors' individual state variables. A stationary distribution of wealth shares $\phi^x (z,x)$ satisfies the following equation:

$$\begin{aligned}0 \;\; = \;\; &-\partial_z \left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1) \alpha^2}{2\gamma^2 \sigma^2 (x)} - g(\bar{x}) \right) \phi^x (z,x) \right] \\ &+ \frac{1}{2} \partial_{zz} \left[\left(z \frac{\alpha}{\gamma \sigma (x)} \right)^2 \phi^x (z,x) \right].\end{aligned} \tag{26}$$

We use guess and verify to show that the stationary distribution of wealth shares at each level of expertise is given by a Pareto distribution with an expertise specific tail parameter. This tail parameter, which we denote by β, is determined by the drift and volatility of the expertise specific law of motion for wealth shares. Intuitively, the larger the drift and volatility of the expertise specific wealth process, the fatter the tail of the wealth distribution at that level of expertise will be.

Proposition 4.2 *The stationary distribution of wealth shares $\phi^x (z,x)$ has the following form:*

$$\phi(z,x) \propto C(x) z^{-\beta(x)-1},$$

[22] We discuss the interpretation we adopt in detail in the Appendix.

[23] See Dixit and Pindyck [1994] for a heuristic derivation, or Karlin and Taylor [1981] for more detail.

where

$$
\begin{aligned}
\beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\
C_1 &= 2\gamma \left(f_{xx} + \rho - r_f + \gamma g(\bar{x}) \right), \\
C(x) &= \frac{1}{\int z^{-\beta} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}.
\end{aligned}
$$

See the Appendix for the Proof, where we also show that, in the stationary distribution, $\beta > 1$, which ensures a finite integral, and confirms that the distribution satisfies stationarity. The following Corollary, which we also prove in the Appendix, gives the tail parameters for the highest expertise agents, as well as all other investors.

Corollary 4.1 *For the highest expertise agents, we have*

$$
\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}} = C_1 \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma
$$

where $\bar{z}$ is mean of normalized wealth of experts with highest expertise,

$$
\bar{z} = \int_{z_{\min}}^{\infty} z\phi(z, \bar{x}) dz = z_{\min} \left[1 + \frac{1}{\beta(\bar{x}) - 1} \right]
$$

and

$$
g(\bar{x}) = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})} \frac{1}{1 - z_{\min}/\bar{z}}
$$

For all other expertise levels, we have

$$
\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}} \right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma > 1. \tag{27}
$$

The parameter β controls the tail of each expertise specific wealth distribution. The smaller is β, the more slowly the distribution decays, and the fatter is the upper tail. Clearly, β is an increasing function of risk aversion, γ, and an increasing function of expertise level volatility, $\sigma(x)$. The dependence of the tail parameter on expertise is given by $\frac{\sigma^2(x)}{\sigma^2(\bar{x})}$. Since expertise-specific effective volatility $\sigma(x)$ is decreasing in x, the wealth distribution of experts with a higher level of fixed expertise has a fatter tail. Investors with higher expertise allocate more wealth to the risky asset, which increases the mean and volatility of their wealth growth rate. Both a higher drift, and a wider distribution of shocks, lead to a fatter upper tail for wealth. Moreover, equation (27) clearly shows that if the relation between expertise and effective volatility is steeper, indicating a more complex asset, then the difference in the size of the right tails of the wealth distribution across expertise levels increases. We can also measure the degree of wealth inequality within each expertise level as $\frac{1}{\beta(x)}$. High expertise levels exhibit greater size "inequality", and again, if the relation between expertise and effective volatility is steeper, indicating a more complex asset, then the difference in size inequality within expertise levels increases.

It is intuitive that investing more in the risky asset leads to a fatter tailed wealth distribution. However, perhaps surprisingly, as Lemma 4.1 illustrates, not every parameter which increases difference in the fraction of wealth allocated to the risky asset leads to an increase in the degree of fat tails of the expertise specific wealth distributions. We show in Lemma 4.1 that, while differences in portfolio choice driven by differences in effective volatilities lead to greater differences in decay parameters, this is not true for variation in portfolio choice driven by higher excess returns or lower risk aversion. See the Appendix for the proof.

Lemma 4.1 ***Relation Between*** $\theta(x)$ ***and*** $\beta(x)$

Consider two level of expertise, $x_{\min}$ *and* $x_{\max}$*, we have*

$$\theta(x_{\max}) - \theta(x_{\min}) = \frac{\alpha}{\gamma}\frac{\sigma^2(x_{\min}) - \sigma^2(x_{\max})}{\sigma^2(x_{\max})\,\sigma^2(x_{\min})},$$

and

$$\beta(x_{\max}) - \beta(x_{\min}) = 2\gamma^2\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(x_{\max})\,\sigma^2(x_{\min})}{\alpha^3}\left[\theta(x_{\min}) - \theta(x_{\max})\right].$$

If a larger difference in portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller. If it is due to an increase in the difference in effective volatilities, then the difference in β's is larger.

4.3 Aggregation and Stationary Equilibrium

In this section, we define a stationary equilibrium, and state the condition which determines the market clearing α in a stationary equilibrium. Slightly abusing notation by suppressing dependence on the distribution of wealth and expertise, we define aggregate investment in the complex risky asset to be I, given each sum of expertise level investment $I(x)\ \forall x$, where:

$$I = \int \lambda(x)\, I(x)\, dx. \tag{28}$$

We first define a stationary equilibrium. In order to ensure that the supply of the complex risky asset does not become negligible as investor wealth grows, we assume that the supply grows proportionally to the mean wealth of the highest expertise investors. That is, we assume:

$$S(t) = Sg(\bar{x})\, t.$$

For convenience, we assume that the support of expertise is bounded above by $\bar{x}$, although most of our results only require that $\sigma(x)$ satisfies $\lim_{x\to\infty} \sigma(x) = \underline{\sigma} > 0$.

Definition 4.1 *A stationary equilibrium consists of a market clearing α, policy functions for all investors, and a stationary distribution over investor types $i \in \{x, n\}$, expertise levels x, and wealth shares z, $\phi(i, z, x, t)$, such that given an initial wealth distribution, an expertise distribution $\lambda(x)$, and parameters $\{\gamma, \rho, S, r_f, f_{nx}, f_{xx}, \sigma_v\}$ the economy satisfies:*

1. *Investor optimality: Investors choose participation in the complex risky asset market according to Equation (23), as well as optimal consumption and portfolio choices $\{c^n(t), c(x,t), \theta(x,t)\}_{t=0}^{\infty}$ according to Equations (19)-(21), such that their utilities are maximized.*

2. *Market clearing: The equilibrium market clearing α is determined by equating supply and demand:*
$$S(t) = \int \lambda(x,t)\, \theta(x,t)\, (W(x,t) - c(x,t))\, dx.$$
In a stationary equilibrium, we have:
$$I \equiv \int \lambda(x)\, I(x)\, dx = S, \tag{29}$$
Define $Z(x)$ to be the total expertise level wealth share,
$$Z(x) = z_{\min}\left(1 + \frac{1}{\beta(x) - 1}\right).$$
Then, define $I(x)$ to be the detrended total expertise level investment in the complex risky

asset, namely,

$$
\begin{aligned}
I(x) &= \frac{\alpha}{\gamma\sigma^2(x)}\left(1-\left[y^x(x)\right]^{-\frac{1}{\gamma}}\right)Z(x), \\
&= \frac{\alpha}{\gamma\sigma^2(x)}\left[\frac{-(\gamma-1)(r_f-f_{xx})+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]Z(x). \qquad (30)
\end{aligned}
$$

3. *The distribution over all individual state variables is stationary, i.e.* $\partial_t\phi(i,z,x,t)=0$.

4.4 Asset Pricing Results

With policy functions, stationary distributions, and the equilibrium definition in hand, we turn to our asset pricing results. We begin by studying comparative statics over the equilibrium market clearing α. Next, we analyze individual Sharpe ratios. Again, we emphasize heterogeneity across investors with different levels of expertise in changes in the risk return tradeoff as fundamental volatility changes. Finally, we study market level Sharpe ratios, with a focus on the effects of the intensive and extensive margins of participation by investors with heterogeneous expertise. We focus mainly on the effect of changes in fundamental volatility in our analytical results.

Investor Demand, Aggregate Demand, and Equilibrium α We first describe the comparative statics for demand conditional on investors' expertise levels in Lemma 4.2.

Lemma 4.2 *Using equation 30 for investor demand conditional on expertise, x, we have following comparative statics. As long as the consumption share of the highest expertise agent is not too high, for example assuming the sufficient condition*

$$y^x(\bar{x})<\frac{1}{2},$$

we have that the following will hold $\forall x$:

1. $\frac{\partial I(x)}{\partial d}>0$, *where* $d=\frac{\alpha^2}{\sigma^2(x)}$
2. $\frac{\partial I(x)}{\partial \alpha}>0$
3. $\frac{\partial I(x)}{\partial \sigma_v}<0$

4. $\frac{\partial I(x)}{\partial \gamma} < 0$

If we have that $y^x(\bar{x}) < \frac{1}{1+\beta(\beta-1)}$*, then we also have:*

5. $\frac{\partial I(x)}{\partial f_{xx}} < 0$

Demand for the risky asset at each level of expertise is increasing in the squared investor specific Sharpe ratio, and it is increasing in α. Demand is decreasing in fundamental volatility, risk aversion, and, as long as the consumption share of the highest expertise agent is not too high, investor specific demand is decreasing in the maintenance cost. The reason that a restriction on consumption of the highest expertise investors is sufficient is because such a condition rules out the case in which wealth effects from improved investment opportunities are too strong for the investors with the best investment opportunities.

With expertise level total demands in hand, we can construct comparative statics for aggregate demand. We cannot express the equilibrium excess return in closed form. However, the following Proposition shows that the equilibrium excess return, α, and aggregate demand, I, form a bijection. This uniqueness result in turn ensures that α can be numerically solved for as the unique fixed point to equation (29).

Proposition 4.3 *Aggregate market demand for the complex risky asset is an increasing function of the excess return,* α*, and* α *and* I *form a bijection. Mathematically,*

$$\frac{\partial I}{\partial \alpha} > 0,$$

as long as

$$y^x(\bar{x}) < \frac{1}{3},$$

that is, if the consumption of even the highest expertise investors is less than a third of their total wealth.

The condition in Proposition 4.3 is stronger than what is needed. The Appendix gives some weaker conditions, along with a proof.

Proposition 4.4 provides comparative statics over the aggregate demand for the complex risky asset, I. Using the result in Proposition 4.3, these comparative statics also hold for α. The proof, as well as weaker conditions for the results (but with longer expressions), appear in the Appendix.

Proposition 4.4 *Using the market clearing condition, we have following comparative statics hold: As long as:*

$$y^x(\bar{x}) < \frac{1}{2},$$

we have that:

1. $\frac{\partial I}{\partial \sigma_v} < 0$, *thus* α *is an increasing function of fundamental risk*

2. $\frac{\partial I}{\partial \gamma} < 0$, *thus* α *is an increasing function of risk aversion*

If we have that $y^x(\bar{x}) < \frac{1}{1+\beta(\beta-1)}$, *then we also have:*

3. $\frac{\partial I}{\partial f_{xx}} < 0$, *thus* α *is an increasing function of the maintenance cost.*

Thus, demand for the risky asset is decreasing in fundamental volatility, risk aversion, and the maintenance cost. As a result, α is increasing in fundamental volatility, risk aversion, and the maintenance cost. We argue that an increase in these parameters proxies for greater asset complexity, and thus that our model predicts that α will be higher in more complex asset markets.

We now turn to the effect of the efficiency of the joint distribution of wealth and expertise on equilibrium pricing. In particular, we demonstrate that the equilibrium required excess return on the complex risky asset is decreasing in the amount of wealth commanded by agents with higher levels of expertise. The wealth distribution at each expertise level is a Pareto distribution with an expertise specific tail parameter. By shifting the distribution of expertise rightward, leading to a new distribution with a relatively larger fraction of higher expertise investors, relatively more wealth will reside with agents with higher expertise. Thus, with any rightward shift, the joint distribution of wealth and expertise becomes more efficient. Moreover, because the wealth distribution at higher expertise levels exhibits fatter right tails, there is an additional direct effect on overall wealth from a rightward shift in the distribution of expertise. Accordingly, Proposition 4.5 shows that if the density of experts shifts to the right, then demand for the complex risky asset will increase, and the required equilibrium excess return will decrease. The equilibrium excess return is decreasing in the amount of wealth which resides in the hands of agents with higher expertise. Note that this result can also be interpreted to state that in asset markets in which higher levels of expertise are more widespread, or less rare, equilibrium required returns will be lower. We argue that the scarcity of relevant expertise is increasing with asset complexity, again implying a higher α in more complex markets. The proof appears in the Appendix.

Proposition 4.5 *If* $\frac{\partial \sigma(x)}{\partial x} < 0$, *and* Λ_1 *exhibits first-order stochastic dominance over* Λ_2, $I(\Lambda_1) \geq I(\Lambda_2)$. *As a result* $\alpha(\Lambda_1) < \alpha(\Lambda_2)$.

Investor Specific and Market Level Sharpe ratios With the analysis of equilibrium excess returns in hand, we now turn to the equilibrium risk-return tradeoff at the investor and market level as described by the investor-specific, and market level Sharpe ratios. As in the static model, at the market level we define both the equally weighted and value weighted Sharpe ratios, but focus on the equally weighted Sharpe ratio in our analysis. In addition to offering cleaner comparative statics because it does not depend on investor portfolio choices and market shares, the equally weighted Sharpe ratio represents the expected Sharpe ratio that an investor who could pay a cost to draw from the expertise distribution above the entry cutoff would earn. In that sense, it is the "expected Sharpe ratio". Note that the Sharpe ratio for non-experts is not defined.

Given the excess return on the risky asset, we define the investor-specific Sharpe Ratio as:

$$SR(x) = \frac{\alpha}{\sigma(x)}.$$

We provide results for how investor-specific Sharpe ratios change as fundamental volatility changes under the three possible cases for the elasticity of investor specific risk with respect to fundamental volatility in Proposition 4.6. The sign of this elasticity is a key determinant of our Sharpe ratio results.

Proposition 4.6 *The comparative statics for the investor-specific Sharpe ratios with respect to fundamental volatility depend on which of the three possible cases for the elasticity of investor-specific risk with respect to fundamental volatility applies:*

1. *Case 1: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ *is a constant, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} = 0,$$

we must have that $SR(x)$ *is either an increasing or a decreasing function of fundamental risk for all expertise levels.*

2. *Case 2: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ *is a decreasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0,$$

then there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} < 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} > 0$.

3. *Case 3: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ *is an increasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} > 0,$$

there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} > 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_v} < 0$*. Further,* x^* *exists if for any small* $\varepsilon < 10^{-6}$

$$(0, \varepsilon) \subseteq \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} | \textit{ for all } x \right\} \subseteq [0, \infty).$$

Proposition 4.6 demonstrates that the effect of an increase in fundamental volatility on individual Sharpe ratios varies in the cross section, except in Case 1. The intuition is that the change in investors' Sharpe ratios depends on the percentage change in α relative to the percentage change in effective volatility. The change in α is aggregate, the same for all investors. So, the changes in individual Sharpe ratios with respect to changes in fundamental volatility depend on the percentage changes in effective volatility relative to the percentage change in fundamental volatility. If this elasticity is the same for all investors, then the percentage change in α relative to the percentage change in effective volatility is the same for all investors. If this elasticity is declining in expertise, so that higher expertise investors can weather an increase in fundamental volatility better, then Sharpe ratios increase above a cutoff level of expertise and decrease below. This case is interesting if more complex assets have higher fundamental volatility because it can explain reduced participation despite relatively higher excess returns. Note that the static model uses a functional form that satisfies Case 2, as shown in Lemma 3.3, part 6. Finally, if the elasticity of effective volatility with respect to fundamental volatility is increasing in expertise, then Sharpe ratios increase below a cutoff level of expertise and decrease above as fundamental volatility increases. This case is interesting if one interprets the increase in fundamental volatility as coming from a change in the asset which hurts incumbent higher expertise agents worse than potential new entrants. Moreover, comparative statics over fundamental volatility have large pricing effects in this case, since higher expertise agents tend hold a large share of the asset.

We now turn to the market level Sharpe ratio. We define the equally weighted market equilibrium Sharpe ratio as:

$$SR^{ew} = E\left[\frac{\alpha}{\sigma\left(x\right)}|\frac{\alpha^2}{\sigma^2\left(x\right)} \geq 2\gamma f_{xx}\right].$$

We show in the Appendix that the value weighted market equilibrium Sharpe ratio is given by:

$$\begin{aligned} SR^{vw} &= E\left[\frac{\theta\left(z-c\right)}{I}\frac{\alpha}{\sigma\left(x\right)}|\frac{\alpha^2}{\sigma^2\left(x\right)} \geq 2\gamma f_{xx}\right] \\ &= \frac{\alpha}{\gamma I}E\left[\frac{1-\left[y^x\left(x\right)\right]^{-\frac{1}{\gamma}}}{\sigma^3\left(x\right)}Z\left(x\right)|\frac{\alpha^2}{\sigma^2\left(x\right)} \geq 2\gamma f_{xx}\right]. \end{aligned}$$

We focus on comparative statics for the equally weighted market equilibrium Sharpe ratio for simplicity. We emphasize two inputs into the market level risk return tradeoff. First, incumbents' individual Sharpe ratios change. Second, as equilibrium α changes, participation also changes. We begin by describing results for bounds on the elasticity of α with respect to changes in fundamental volatility, and the implications of these bounds for participation. First, we show that the percentage change in α has to be large enough to at least satisfy the investors whose risk-return tradeoff deteriorates the least as fundamental volatility increases.

Lemma 4.3 *In the equilibrium, we have*

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\upsilon/\sigma_\upsilon} > l_{\inf}^{\sigma_\upsilon},$$

where $l_{\inf}^{\sigma_\upsilon}$ *is the lowest elasticity of the effective volatility with respect to fundamental volatility*

$$l_{\inf}^{\sigma_\upsilon} \equiv \inf\left\{\frac{\partial\log\sigma\left(x\right)}{\partial\log\sigma_\upsilon}|\frac{\alpha^2}{2\sigma^2\left(x\right)\gamma} \geq f_{xx}\right\}.$$

Next, we discuss comparative statics for the equally weighed market equilibrium Sharpe ratio. Our results depend both on which case from Proposition 4.6 applies, and on whether participation increases or decreases. We begin by showing that the equally weighted market equilibrium Sharpe ratio is increasing with fundamental volatility in Case 1 of Proposition 4.6, in general. In Case 2, a sufficient condition for the market Sharpe ratio to increase with fundamental volatility is a constraint on the difference between the highest and lowest elasticities of effective volatility with respect to fundamental volatility. In Proposition 4.8 we show that the same condition implies that participation increases as fundamental volatility increases in Case 2 of Proposition 4.6. This condition is not necessary, however, if participation decreases. It is also possible that participation declines and the market Sharpe Ratio increases as fundamental volatility increases. We discuss decreased participation in Proposition 4.9. And, we show that in our numerical example for Case 2, participation declines, and Sharpe ratios increase.

Proposition 4.7 *The equally weighted market Sharpe Ratio is increasing with fundamental risk in general equilibrium, i.e.,*

$$\frac{\partial SR^{ew}}{\partial \sigma_v} > 0,$$

if:

1. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} = 0$, *(Proposition 4.6 Case 1) or*

2. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$ *(Proposition 4.6 Case 2) and* $l_{\sup}^{\sigma_v} < \frac{2\beta(\underline{x})}{\beta(\underline{x})+1} l_{\inf}^{\sigma_v}$, *where*

$$l_{\sup}^{\sigma_v} = \sup \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_v} | \frac{\alpha^2}{2\sigma^2(x)\gamma} \geq f_{xx} \right\}$$

We now show conditions under which participation increases, i.e. under which the cutoff level of expertise for participation $\underline{x}$ declines, as fundamental volatility increases.

Proposition 4.8 *Define the entry cutoff* $\underline{x}$,

$$\underline{x} = \sigma^{-1} \left(\frac{\alpha}{\sqrt{2\gamma f_{xx}}} \right),$$

where $\sigma^{-1}(\cdot)$ *is the inverse function of* $\sigma(x)$. *We have that participation increases with fundamental volatility,*

$$\frac{\partial \underline{x}}{\partial \sigma_v} < 0$$

if the following conditions hold

1. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} \geq 0$, *(Proposition 4.6 Cases 1 or 3) or*

2. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, *(Proposition 4.6 Case 2) and* $l^{\sigma_v}_{\sup} < \frac{2\beta(\overline{x})}{\beta(\overline{x})+1} l^{\sigma_v}_{\inf}$ *where*

$$l^{\sigma_v}_{\sup} = \sup \left\{ \frac{\partial \log \sigma\left(x\right)}{\partial \log \sigma_v} | \frac{\alpha^2}{2\sigma^2\left(x\right)\gamma} \geq f_{xx} \right\}.$$

Proposition 4.8 shows that participation increases in Cases 1 and 3 as fundamental volatility increases. The reason is that demand for the complex asset by incumbent experts declines, and new wealth must be brought into the market to clear the fixed supply. However, in Case 2, it is possible that because higher expertise agents' risk-return tradeoff deteriorates by less as fundamental volatility increases, that participation declines. This can be seen in the condition for increased participation in Case 2, which requires a very small difference between the highest and lowest elasticities, since $\beta \approx 1$, and we confirm this formally in Proposition 4.9.

Corollary 4.2 *If participation increases in Case 1 or Case 2, it must be that the Sharpe ratio for all participating investors improves. Further, if all individual Sharpe ratios improve, the equally weighed market Sharpe ratio also improves.*

The proof for Corollary 4.2 follows directly from the condition defining $\underline{x}$, along with the definition of these two cases for the elasticity of effective volatility with respect to fundamental volatility.

We show that participation can decline in Case 2 of Proposition 4.6. The equally weighted market Sharpe ratio can then increase or decrease, because of selection effects. If participation declines, the market includes only higher expertise investors. Intuitively, we conjecture that the results for the market Sharpe ratio depend on the distribution of expertise x, and the elasticity of effective volatility with respect to fundamental volatility. If effective expertise is sufficiently scarce, then the Sharpe ratio should increase because the percentage change in α will need to compensate lower expertise investors in order to clear the market for the complex risky asset. We provide a numerical illustration below.

Proposition 4.9 *Define the entry cutoff* $\underline{x}$ *as in Proposition 4.8. We have*

$$\frac{\partial \underline{x}}{\partial \sigma_v} > 0$$

if the following conditions hold: $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, *(Case 2) and* $l^{\sigma_v}_{\sup} > \frac{\beta(\underline{x})+1}{2} E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} | x \geq \underline{x}\right]$, *where*

$$l_{\sup}^{\sigma_v} = \sup\left\{\frac{\partial \log \sigma(x)}{\partial \log \sigma_v} | \frac{\alpha^2}{2\sigma^2(x)\gamma} \geq f_{xx}\right\}.$$

Note that the conditions in Proposition 4.8 and Proposition 4.9 are not overlapping, because

$$\frac{\beta(\underline{x})+1}{2} > \frac{\beta(\underline{x})+1}{2} \geq \frac{2\beta(\overline{x})}{\beta(\overline{x})+1}.$$

4.5 Numerical Examples

This section presents complementary numerical results and comparative statics for the three cases for the elasticity of effective volatility with respect to fundamental volatility from Proposition 4.6. The model generates closed form policy functions and wealth distributions conditional on expertise levels. To provide intuition for the effects of equilibrium pricing, we provide the comparative statics in both partial equilibrium and general equilibrium. In partial equilibrium, the excess return is given exogenously, and held fixed, while aggregate demand (and hence implicitly supply) varies. In general equilibrium, the excess return is computed endogenously given a fixed supply of the risky asset. Because α and I form a bijection (Proposition 4.3 provides conditions for which they are one to one and onto), for any given supply of the complex risky asset, we can solve for the market equilibrium α in the following steps:

1. Choose an upper and a lower bound for α, namely α_1 and α_2, $(\alpha_1 > \alpha_2)$.
2. Let $\alpha = \frac{\alpha_1+\alpha_2}{2}$, and compute the total demand for the risky asset

$$\int \lambda(x) I(x) dx$$

3. If $S - \int \lambda(x) I(x) dx < -10^{-4}$, let $\alpha_1 = \alpha$ and back to step 1; if $S - \int \lambda(x) I(x) dx > 10^{-4}$, let $\alpha_2 = \alpha$ and back to step 1; otherwise, STOP.

We provide results under specific parametric assumptions. The following are examples of the three cases for the elasticity of investor specific risk with respect to fundamental volatility in Proposition 4.6.

Case 1: $\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}$ is a constant, $\sigma(x) = \left(a + x^{-b}\right)\sigma_v^2$

Case 2: $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, $\sigma(x) = a + x^{-b}\sigma_v^2$ [24]

[24] x^{-b} can be replaced by any function $f(x)$ as long as $\frac{\partial f(x)}{\partial x} < 0$.

Case 3: $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} > 0$, $\sigma(x) = a\sigma_v^2 + x^{-b}$.

Our baseline parameters are summarized in Table 1. The time interval is one quarter. The risk-free rate is 1%. The discount factor is 1%. The maintenance cost is also 1%. The coefficient of relative risk-aversion is 5. The log-normal distribution of expertise has a mean of 0 and volatility of 5. The minimum wealth share is set to 0.4. The fundamental standard deviation of the risky asset return is 20%. We set $a = 0.28$ and $b = 2$ in Cases 1 and 3. This implies that the highest expertise investors, with effective variance $a\sigma_v^2$, can eliminate 47% of fundamental risk, and face an effective standard deviation of 10.5%. We then choose $a = 0.0112$ and $b = 2$ in Case 2, so that we have the same effective volatility for the highest expertise agents as in Cases 1 and 3. The model generates a stationary equilibrium in all cases.

Case 1 We begin with the constant elasticity case, Case 1. The entry cutoff is $x = 1.45$, which implies that the total measure of experts is 47%. The average wealth of experts is 0.33. In aggregate, experts invest 88.2% of their total wealth in the complex risky asset.

Figures 1 - 5 show the model comparative statics in both general equilibrium and partial equilibrium. All blue lines represent model results in partial equilibrium with a fixed excess return and a perfectly elastic supply of the risky asset. All red lines represent model results in general equilibrium with a fixed supply of the risky asset and the market clearing equilibrium value for the excess return. Figure 1 plots model statistics with different values for the maintenance cost. A higher maintenance cost represents a higher cost of being an expert. There are fewer experts in both partial equilibrium and general equilibrium at higher maintenance costs. Demand for the risky asset is smaller in partial equilibrium as a result, resulting in a higher excess return in general equilibrium to clear the market. Also, the market equally weighted Sharpe ratio increases as the maintenance cost becomes larger. This is both because α must increase to clear the market with lower participation, and because selection lowers effective volatilities. The higher maintenance cost represents an entry barrier for lower expertise investors, and we argue that this cost proxies for asset complexity. More complex assets require larger investments in expertise. Experts earn a higher excess return, and the overall market represents a better risk-return tradeoff, for more complex strategies in which the higher investment costs deter lower-expertise entrants.

Figure 2 displays the model statistics as a function of risk aversion. In partial equilibrium, if investors are more risk averse, they invest less in the risk asset. Also, there are fewer experts in the market. Market risk is lower because of the selection effect of only "better" experts operating in the market. These results change somewhat in general equilibrium, since α increases in general equilibrium. The positive effects on entry because of the increase in α dominate the negative effects from increased γ. Thus, there are more experts in the market, and the worse selection of experts implies higher market-level effective risk. However, the market Sharpe ratio increases in both general equilibrium and partial equilibrium as a result of increased risk aversion because the effect of the higher α dominates.

Figure 3 plots the model statistics with different fundamental risk levels. The comparative statics for fundamental risk share some similar patterns with the effects of risk aversion. The results differ, however, for the market-level risk and Sharpe ratio in partial equilibrium. With increasing fundamental risk, there are fewer experts in partial equilibrium and more experts in general equilibrium, parallel to the results for an increase in γ. However, the market risk is higher in both partial and general equilibrium because the increased value of fundamental risk dominates the selection effects on entry. However, in general equilibrium the effect of higher risk on α still dominates and the Sharpe ratio improves.

Figure 4 and Figure 5 show the model results with different value of b. The expertise parameter b has two effects in our model. First, b represents the difference in investor-specific risk between high and low expertise investors. A higher b means a larger difference. Second, b controls the entry cutoff for experts. Because a higher b increases the effect of expertise at all expertise levels, a higher b results in a lower entry cutoff. Thus, this parameter has somewhat opposing effects. To see this, Figure 4 considers both effects, while Figure 5 eliminates the second effect. In Figure 4, with a higher value of b, we have a lower entry cutoff in partial equilibrium. The effects of lowered risk because of higher b dominate the negative selection effects on market risk. The standard deviation of market risk decreases as the value of b increases in both partial and general equilibrium. The decreased risk and increased fraction of experts results in a lower market excess return. α decreases faster than market risk in general equilibrium. Thus we see a higher Sharpe ratio in general equilibrium. In Figure 5, we counterbalance the selection effects of b on entry with by appropriately scaling the value of a to ensure that the entry cutoff does not change with varies value of b in partial equilibrium. In this way, we can focus on the comparative statics from variation in the complexity of the asset strategy, as proxied for by b. With a higher value of b, a has to be higher to keep the entry cutoff constant. Figure 5 displays several different implications for comparative statics over b. First, the standard deviation of overall market risk is an increasing function of b. More complex strategies, in which there is a bigger difference in the risk faced by high expertise investors vs. low expertise investors, display more market risk. Second, the market demand for the risky asset is a decreasing function of b because of this higher market risk. And, the general equilibrium excess return is an increasing function of b to compensate the higher risk. Third, the increased variance dominates the increased market return, so that the market Sharpe Ratio is a decreasing function of b in both partial and general equilibrium.

Case 2 Figures 5 to 8 show comparative statics for the model under Case 2 for the maintenance cost, risk aversion parameter, and fundamental volatility respectively. Figure 5 shows that as the maintenance cost increases, participation declines in both partial and general equilibrium, and the equilibrium α must increase to compensate this decline in demand. Selection implies that market-level risk declines, and the Sharpe ratio improves from both the numerator and denominator effects. Figure 7 shows the effects of changes in risk aversion. Figure 8 studies the effects of changes in fundamental volatility. Importantly, these comparative statics show that, in this Case 2 example, participation declines (unlike in Case 1) despite the fact that the equilibrium equally weighted market level Sharpe Ratio increases.

Case 3 Figures 5 to 8 show comparative statics for the model under Case 3. Note that as conjectured, Figure 8 shows that in Case 3 changes in effective volatility have large effects on equilibrium α since such changes heavily impact high expertise investors with large market shares.

5 Conclusion

We study the equilibrium returns to complex risky assets in segmented markets with expertise. We show that required returns increase with asset complexity, as proxied for by higher fundamental volatility, higher costs of maintaining expertise, and by exertise being scarce in the population. We emphasize heterogeneity in the risk-return tradeoff faced by investors with different levels of expertise. Accordingly, we show that, under certain conditions, improvements in market level Sharpe ratios can be accompanied by lower market participation. Finally, we describe the implications of our model for the industrial organization of markets for complex risky assets. We show that the stationary wealth distribution displays fatter tails in markets in which assets are more complex, meaning that they display a steeper asset-specific risk vs. expertise relation.

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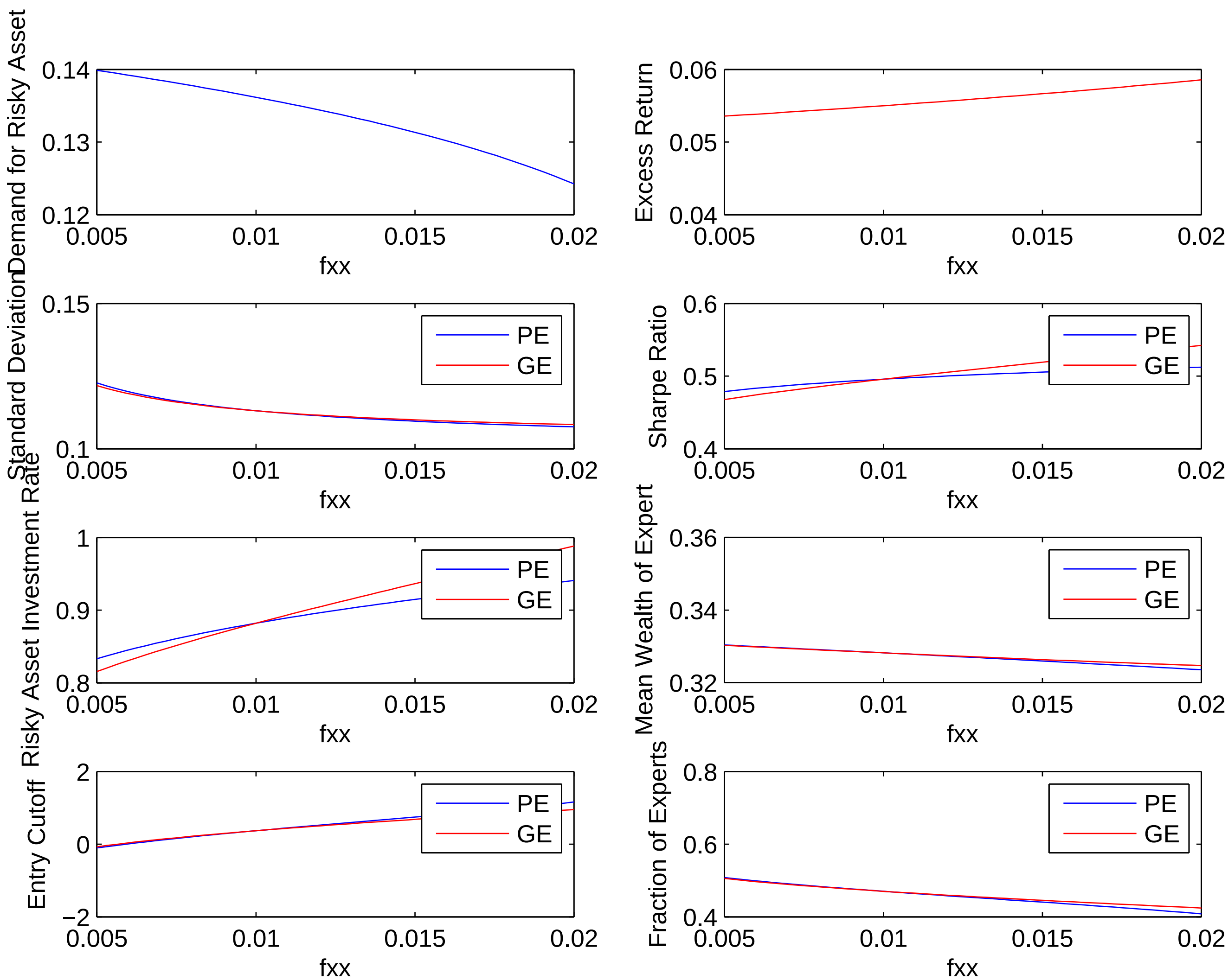

Figure 1: Case 1 Model comparative statics: maintenance cost

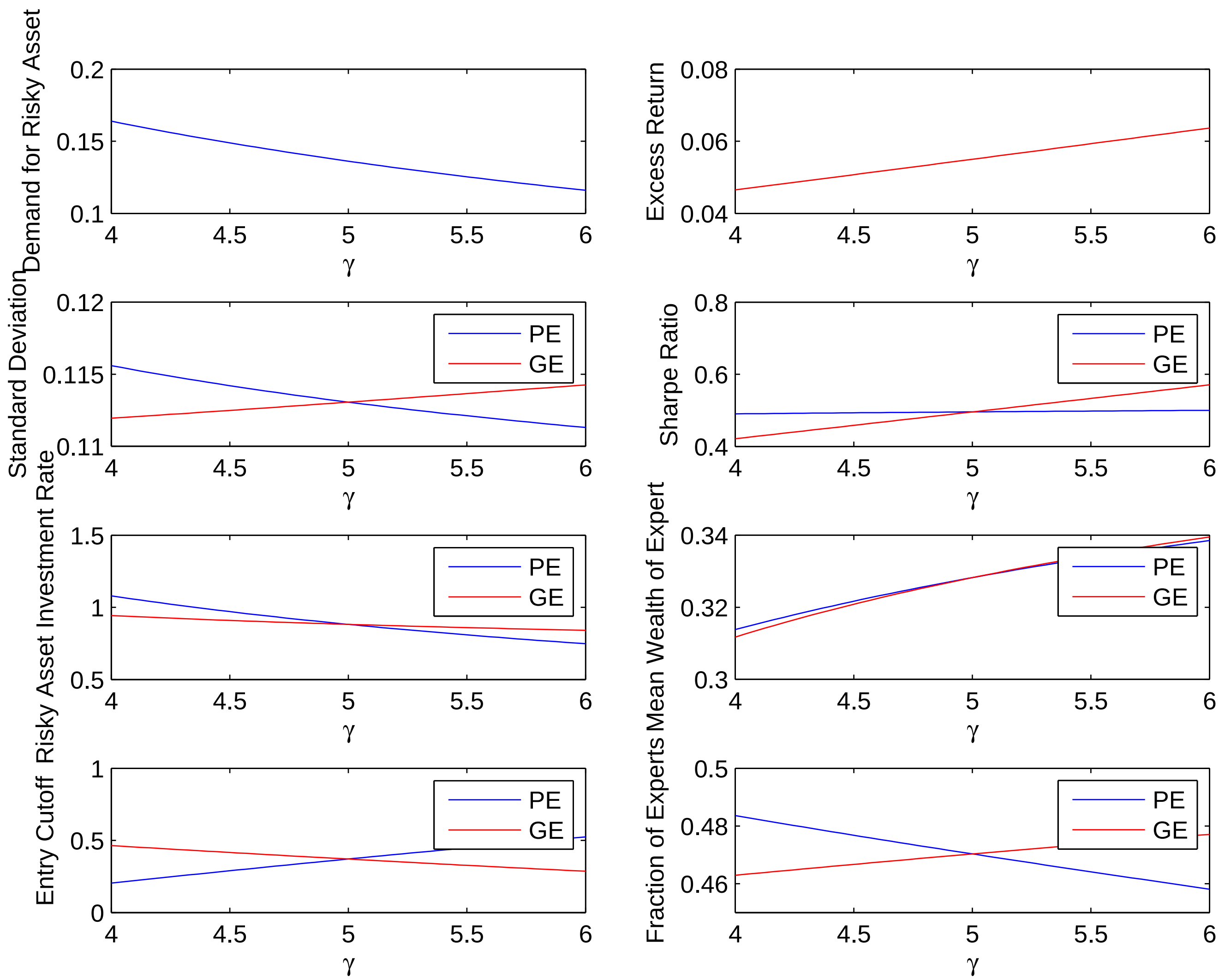

Figure 2: Case 1 Model comparative statics: risk aversion

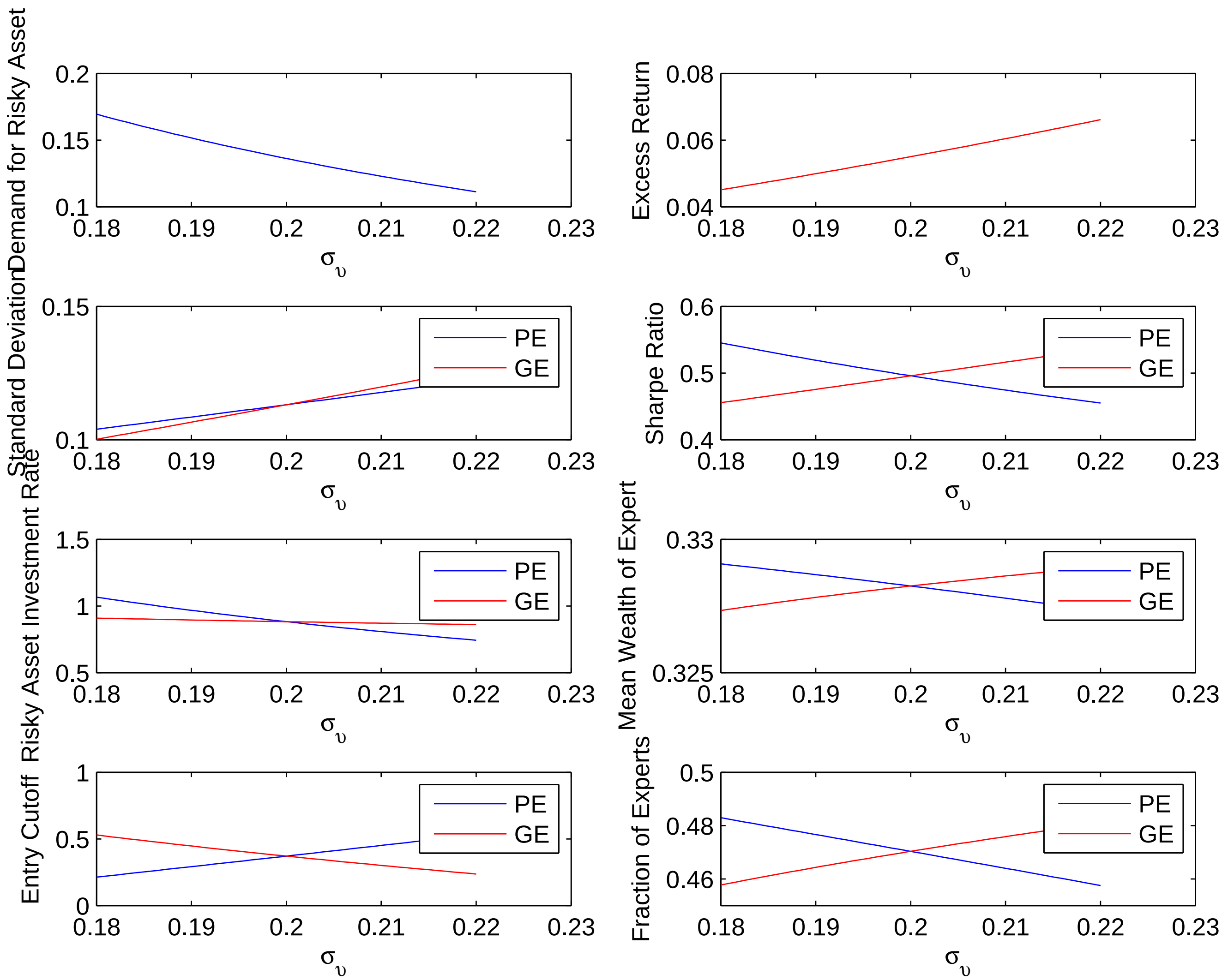

Figure 3: Case 1 Model comparative statistics: fundamental risk

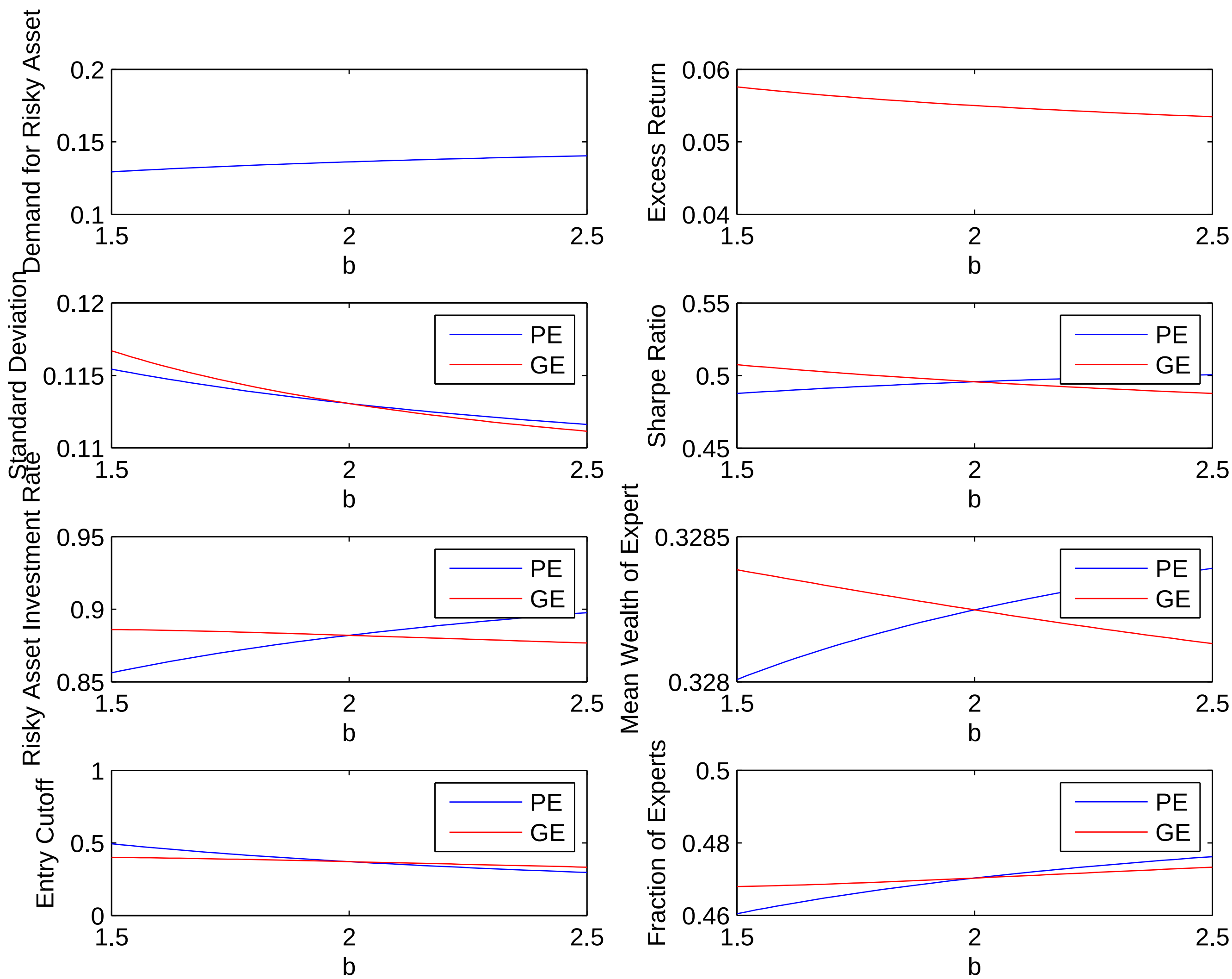

Figure 4: Case 1 Model comparative statics: b with fixed a

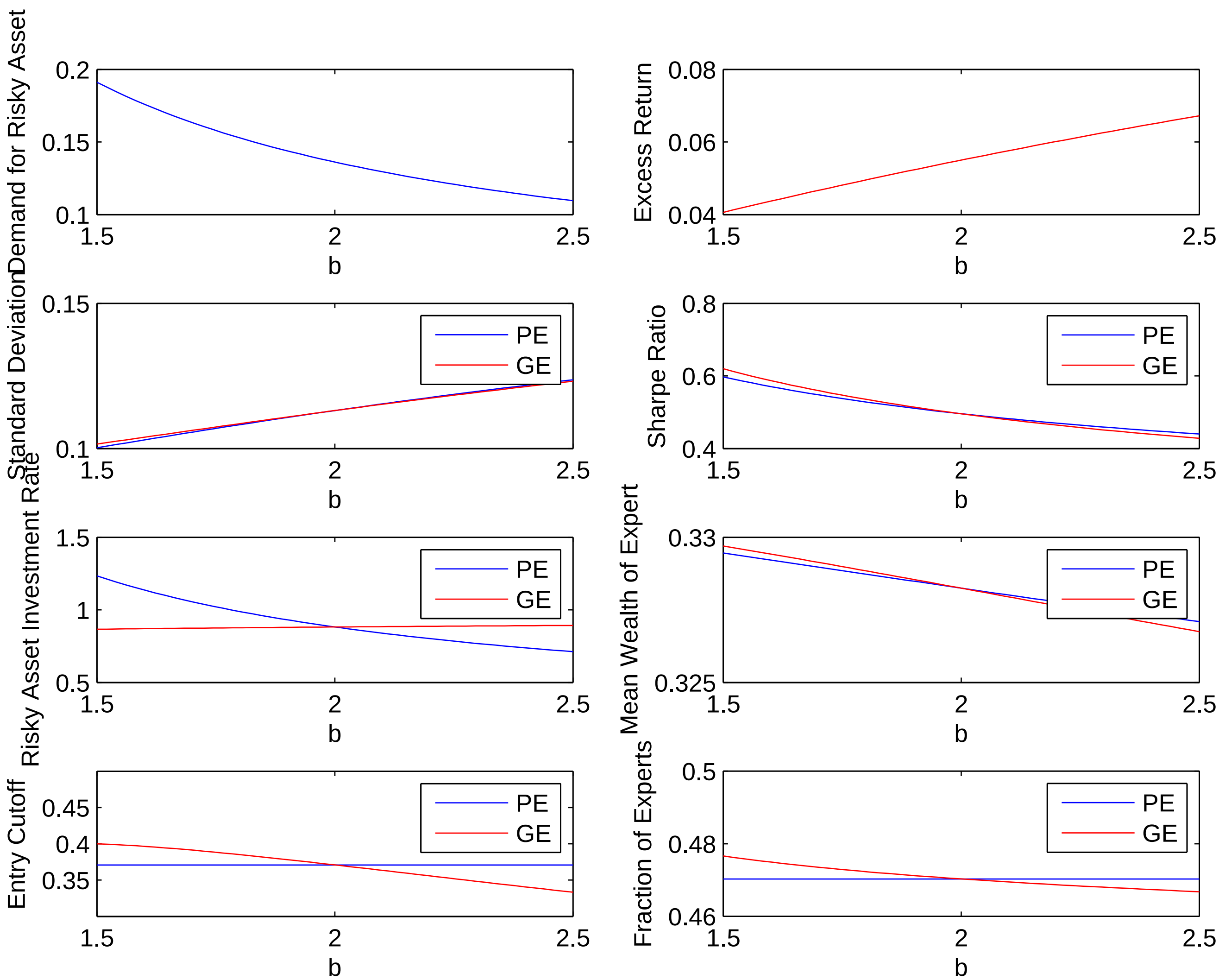

Figure 5: Case 1 Model comparative statics: b with flexible a

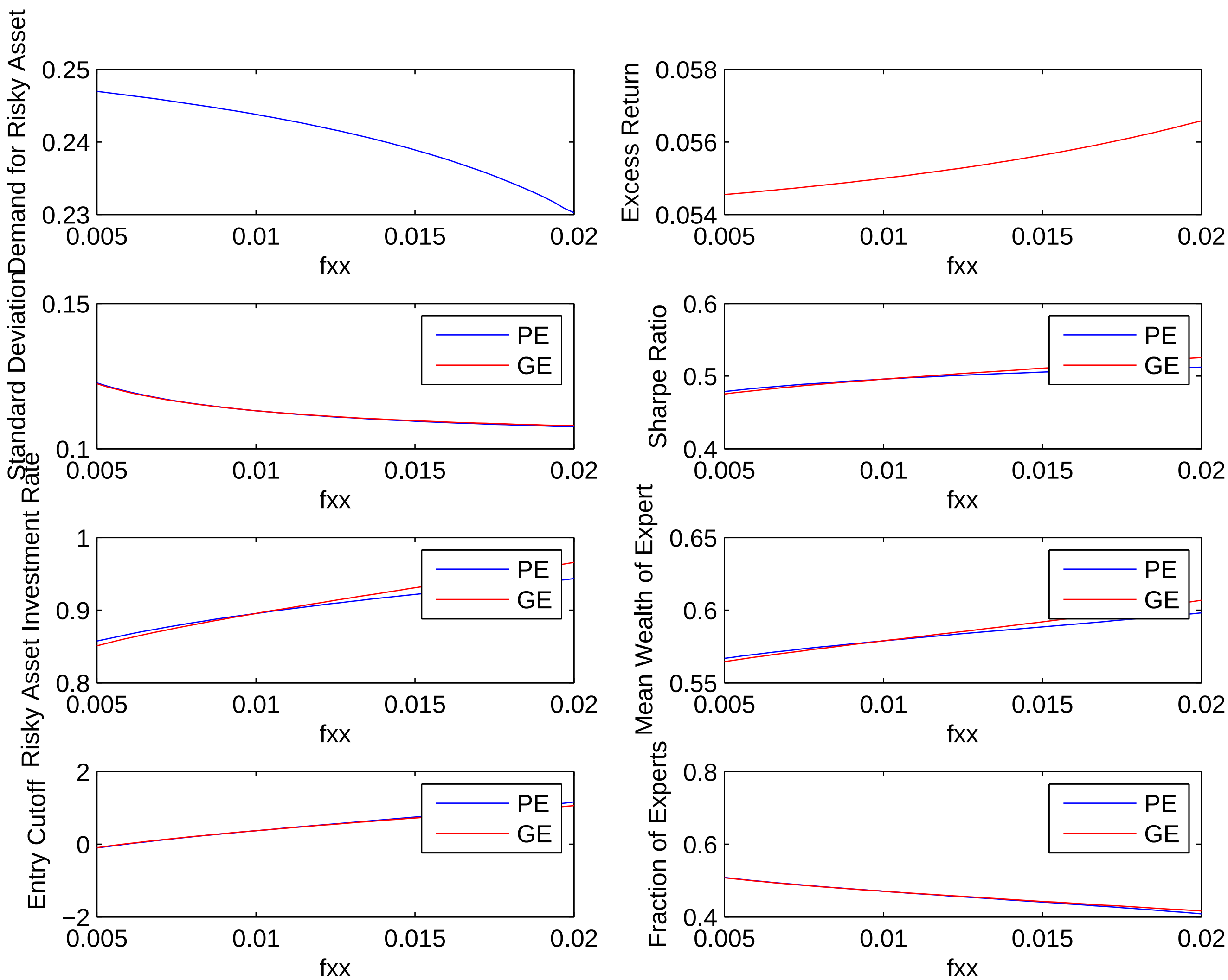

Figure 6: Case 2 Model comparative statics: maintenance cost

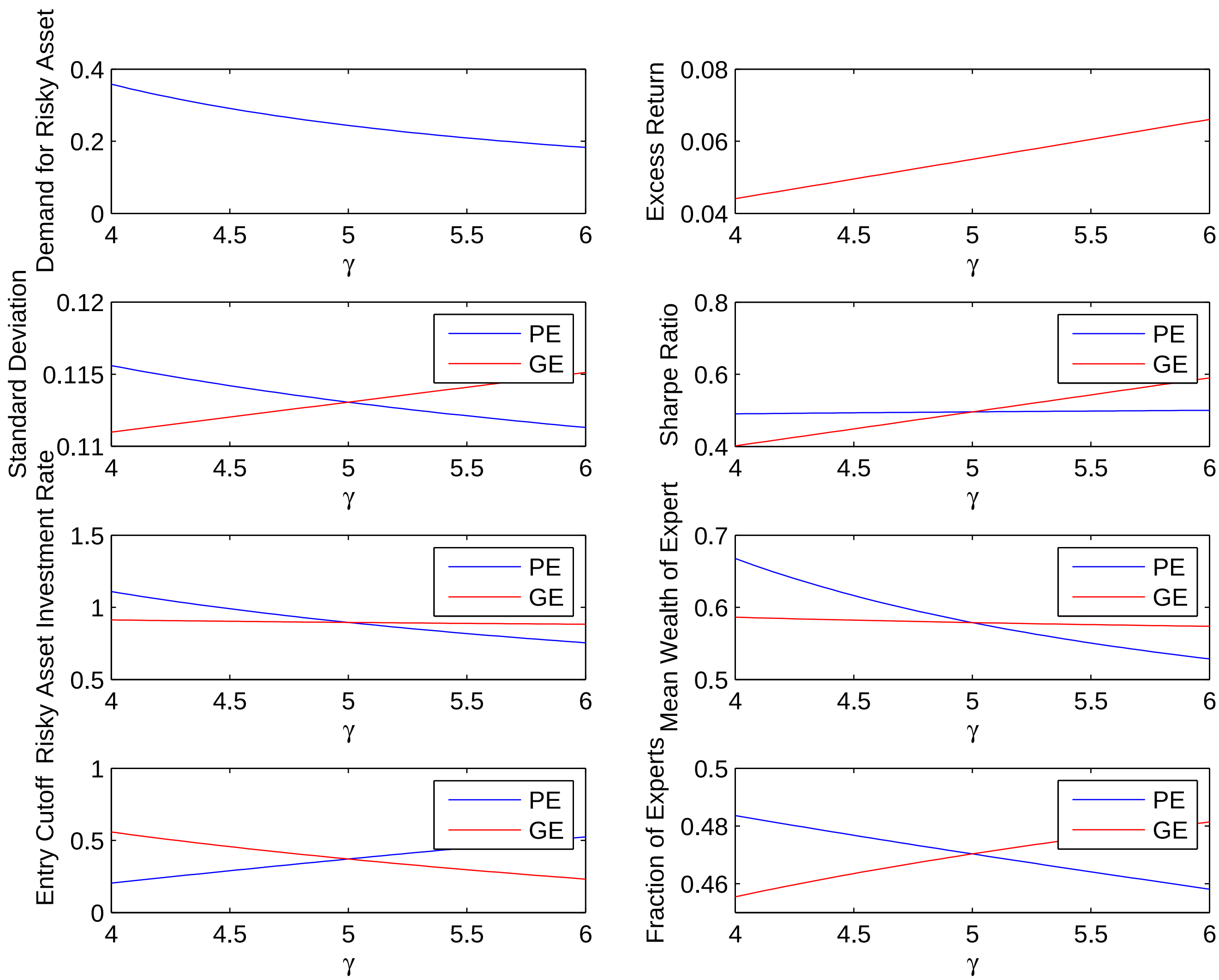

Figure 7: Case 2 Model comparative statics: risk aversion

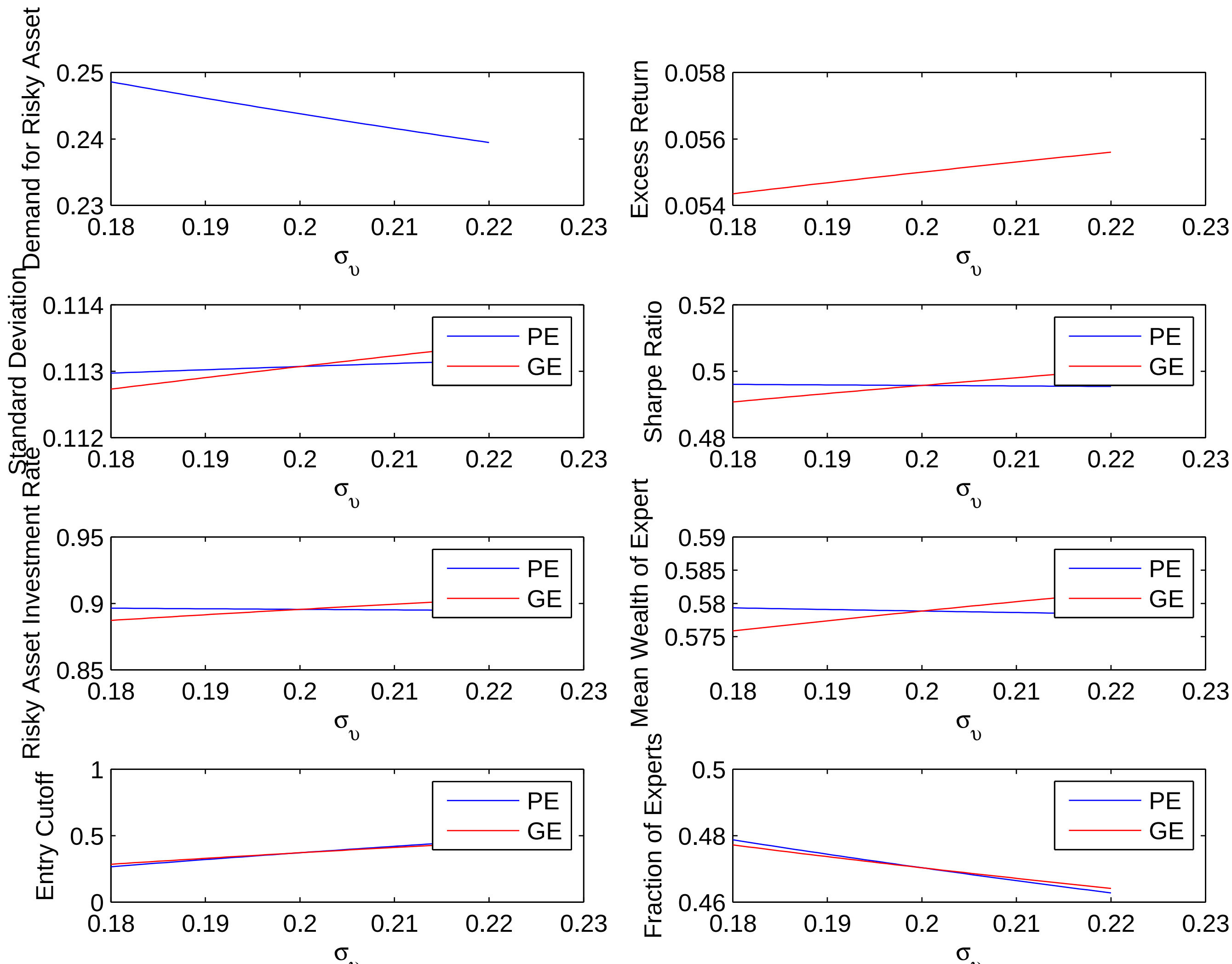

Figure 8: Case 2 Model comparative statics: fundamental risk

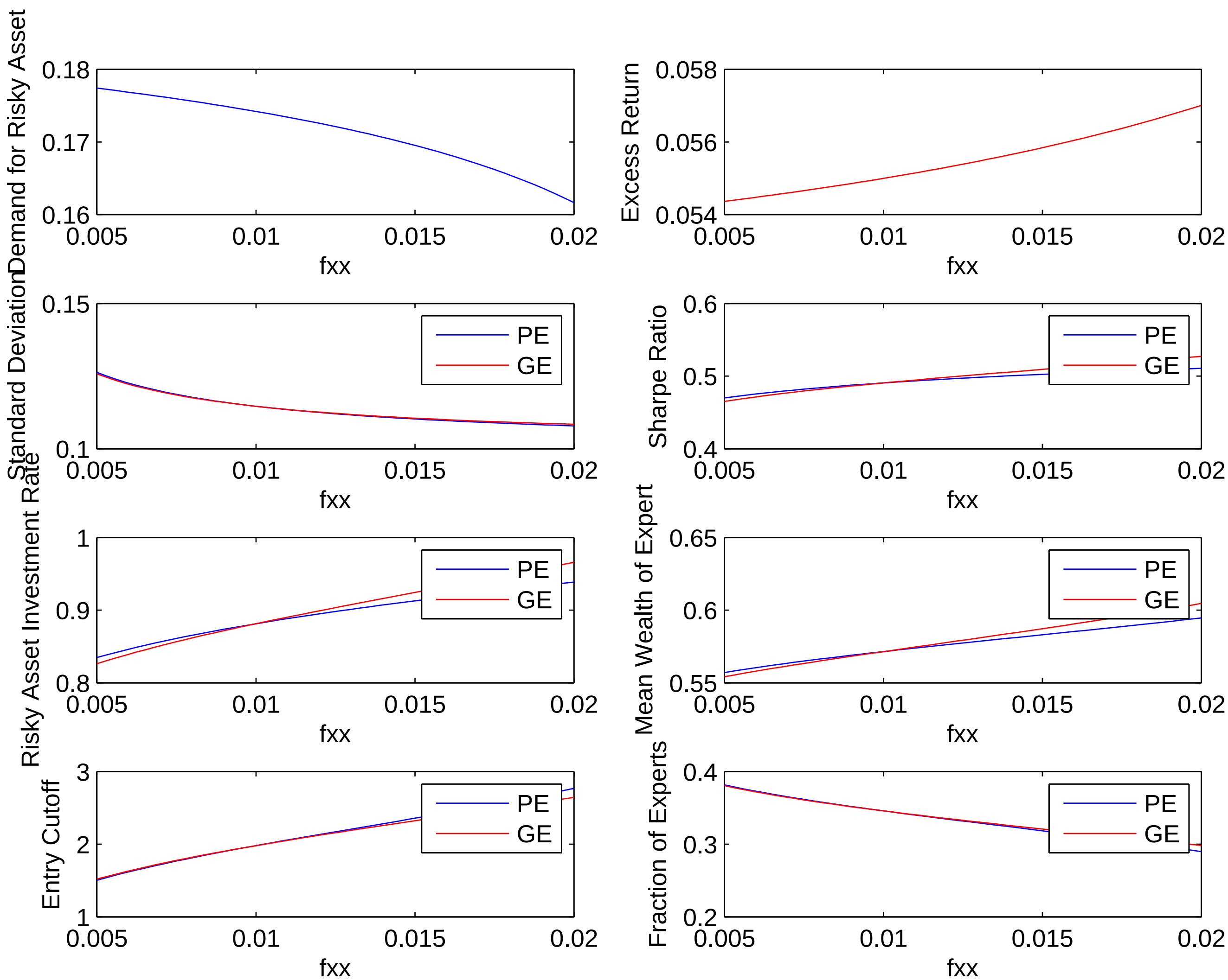

Figure 9: Case 3 Model comparative statics: maintenance cost

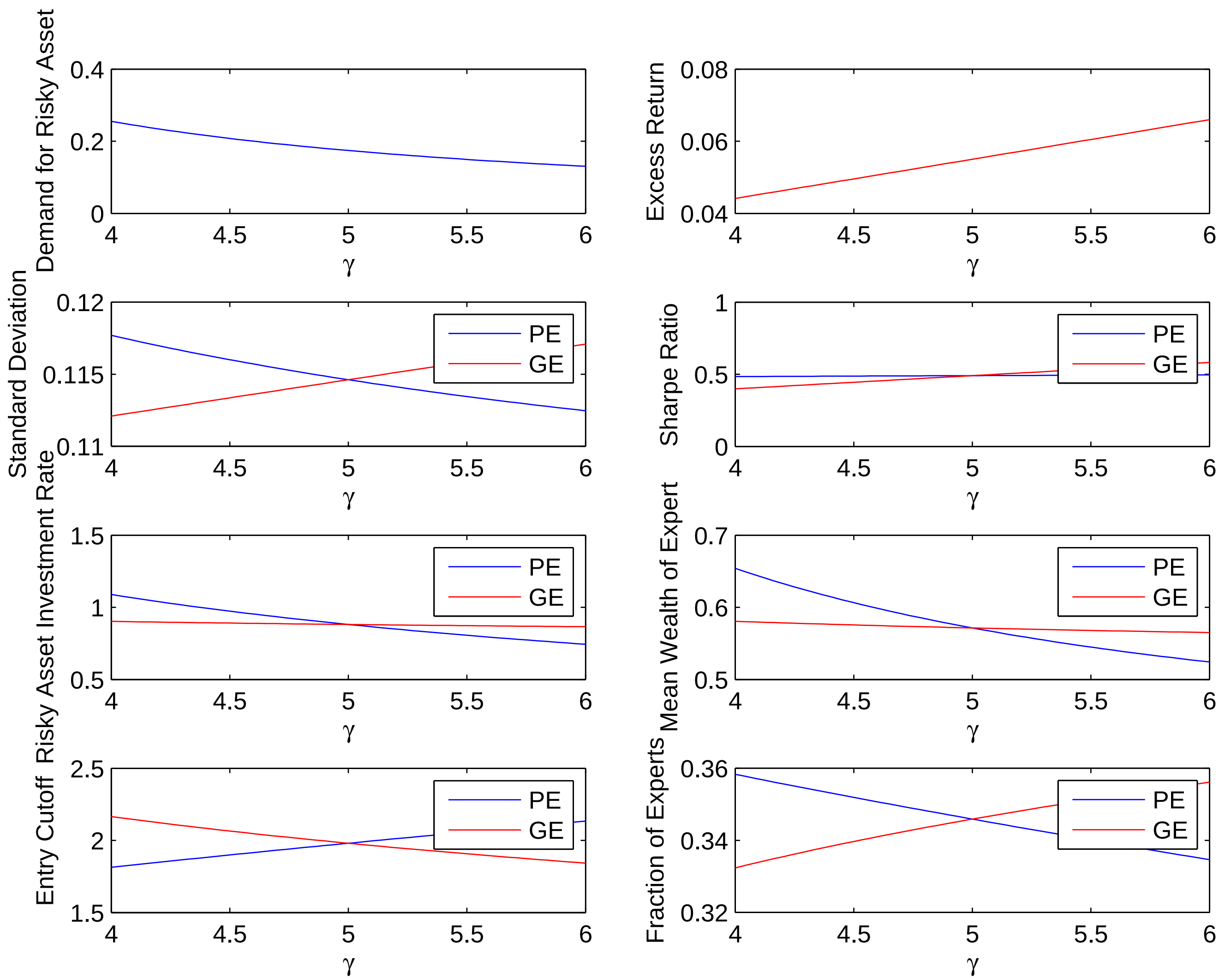

Figure 10: Case 3 Model comparative statics: risk aversion

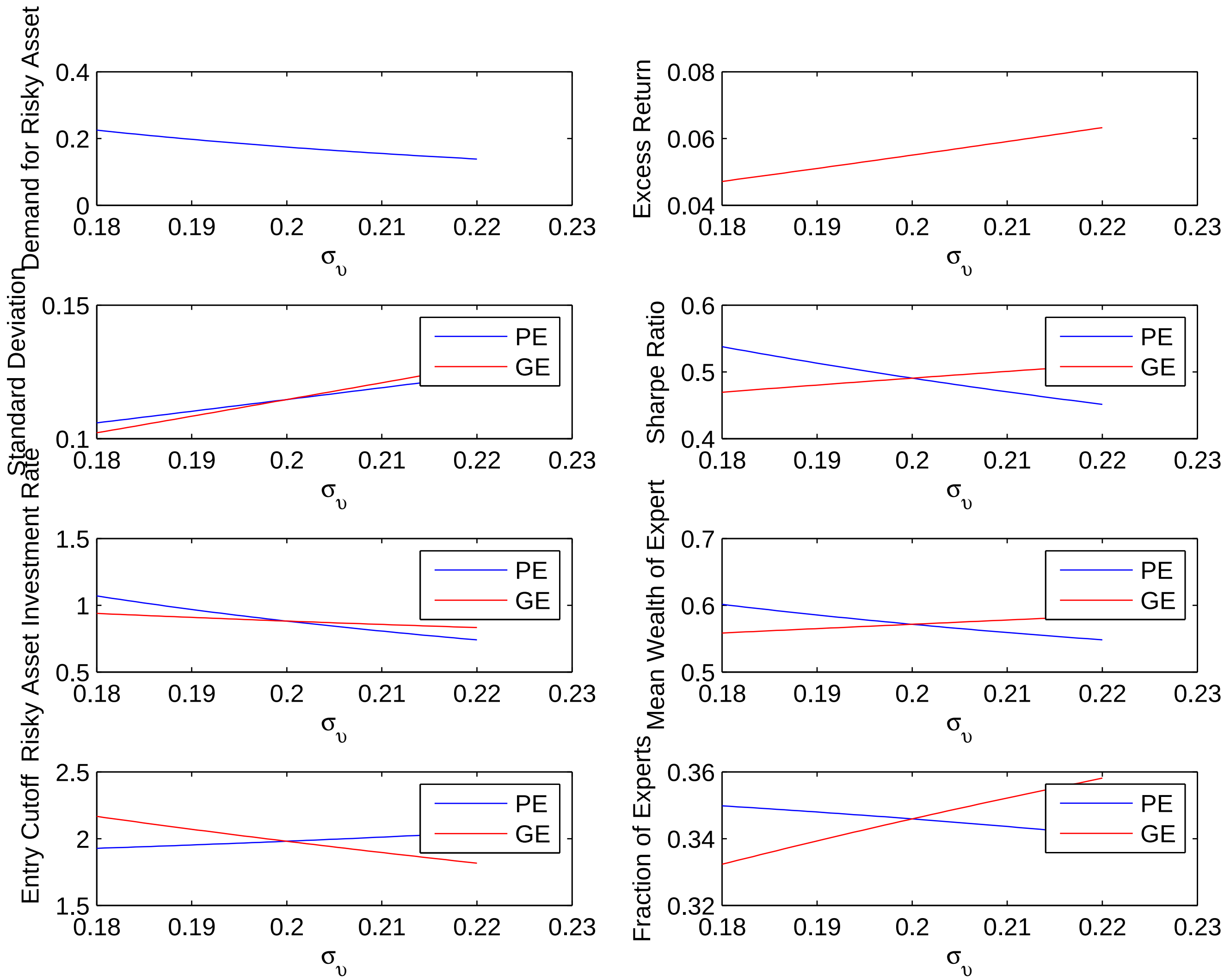

Figure 11: Case 3 Model comparative statics: fundamental risk

Table 1: Parameter Values: Numerical Example for Market Clearing in the Dynamic Model:

Parameter	Symbol	Value	Target
Discount factor	ρ	0.01	Annual interest rate
Risk-free rate	r_f	0.01	Annual interest rate
Risk aversion	γ	5	Data/mean portfolio choice
Entry cost	fnx	0.03	
Maintenance cost	fxx	0.01	
Risky asset supply	S	0.14	$\alpha = 5.5\%$
Volatility of risky asset return	σ_v	20%	
Mean of expertise process	μ_x	0	
Volatility of expertise process	σ_x	5	
Constant in σ_x^2	a	0.28	
Slope of σ_x^2	b	2	
Minimum wealth share	z_{min}	0.4	

Appendix A: Static Model

This section contains proofs and additional results for the static model.

Optimal Portfolio Choice

This section describes how to solve the optimal portfolio allocation problem in the static model. We also use upper case letters for level variables, and lower case letters for log variables. Under the assumptions in the main text, the optimization problem for an investor with wealth W and expertise X, can be written as:

$$v(W,X) = \max_{\theta} \mathbb{E}\left[\frac{(WR_p)^{1-\gamma}}{1-\gamma}\right]$$

subject to

$$\begin{aligned} R_p &= \theta R + (1-\theta) R_f, \\ r|(W,X) &\sim N\left(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right). \end{aligned}$$

Campbell and Viceira [2002a] and Campbell and Viceira [2002b] show that the log portfolio return r_p over a short time horizon with bounded variance, can be approximated by:

$$r_p \approx r_f + \theta(r - r_f) + \frac{1}{2}\theta(1-\theta)\frac{\sigma_v^2}{X}.$$

As a result,

$$r_p|(W,X) \sim N\left(r_f + \theta(\boldsymbol{\mu} - r_f) - \frac{1}{2}\theta^2\frac{\sigma_v^2}{X}, \theta^2\frac{\sigma_v^2}{X}\right).$$

Then the value function equals:

$$v(W,X) = \max_{\theta} \frac{W^{1-\gamma}}{1-\gamma} \exp\left((1-\gamma) r_f + (1-\gamma)\theta(\boldsymbol{\mu} - r_f) - \frac{1}{2}\gamma(1-\gamma)\theta^2\frac{\sigma_v^2}{X}\right).$$

Hence, The investor's optimization problem becomes:

$$\max_{\theta}\left\{\theta(\boldsymbol{\mu} - r_f) - \frac{\gamma}{2}\theta^2\frac{\sigma_v^2}{X}\right\}.$$

Equilibrium Market Excess Return

This section describes how to derive the equilibrium market excess return, $\boldsymbol{\alpha}$, from the log expected return, $\boldsymbol{\mu}$, given all parameters. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_v^2}{X},\frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right)=\exp\left(\boldsymbol{\mu}\right).$$

In addition,

$$\mathbb{E}\left[R\right]=\mathbb{E}\left[\mathbb{E}\left(R|W,X\right)\right].$$

Hence,

$$\mathbb{E}\left[R\right]=\exp\left(\boldsymbol{\mu}\right).$$

Finally,

$$\alpha=\exp\left(\boldsymbol{\mu}\right)-R_f.$$

Equilibrium Equally Weighted Market Sharpe ratio

This section describes how to derive the equilibrium equally weighted market Sharpe ratio, SR, from the log expected return, $\boldsymbol{\mu}$, given all parameters. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_{\upsilon}^{2}}{X},\frac{\sigma_{\upsilon}^{2}}{X}\right),$$

Then

$$\boldsymbol{Var}\left(R|W,X\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)-1\right).$$

In addition, we have proven that

$$\mathbb{E}\left(R|W,X\right)=\mathbb{E}\left[R\right]=\exp\left(\boldsymbol{\mu}\right).$$

Therefore, the equally weighted variance of the risky asset, is given by:

$$\boldsymbol{Var}\left[R\right]=\mathbb{E}\left(\boldsymbol{Var}\left(R|W,X\right)\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]-1\right).$$

Hence, the equally weighted market Sharpe ratio, can be written as:

$$SR=\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]-1}},$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_{\upsilon}^{2k}\mathbb{E}[\exp(-kx)]$, using a Taylor expansion of $\exp(\sigma_{\upsilon}^{2}X^{-1})=1+\sigma_{\upsilon}^{2}X^{-1}+\frac{1}{2!}\sigma_{\upsilon}^{4}X^{-2}+\frac{1}{3!}\sigma_{\upsilon}^{6}X^{-3}+\ldots$, which is equivalent to:

$$\mathbb{E}\left[\exp\left(\frac{\sigma_{\upsilon}^{2}}{X}\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_{\upsilon}^{2k}[\exp(-k\mu_{x}+\frac{1}{2}k^{2}\sigma_{x}^{2})],$$

where we have used the moment-generating function of the normal distribution. Hence, the equally weighted market Sharpe ratio, can be written as:

$$SR=\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\sum_{k=0}^{\infty}\frac{1}{k!}\sigma_{\upsilon}^{2k}\exp(-k\mu_{x}+\frac{1}{2}k^{2}\sigma_{x}^{2})-1}},$$

Provided that σ_{υ}^{4} is small enough, this SR is approximately equal to the following expression:

$$SR\approx\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sigma_{\upsilon}\exp(-\frac{1}{2}\mu_{x}+\frac{1}{4}\sigma_{x}^{2})}.$$

Equilibrium Investor-Specific Sharpe ratio

This section describes how to derive the equilibrium investor-specific Sharpe ratio, $SR\left(X\right)$, from log expected return, $\boldsymbol{\mu}$, given all parameters. For an investor with wealth W and expertise X, Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_{v}^{2}}{X},\frac{\sigma_{v}^{2}}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right)=\exp\left(\boldsymbol{\mu}\right),$$

And

$$\boldsymbol{Var}\left(R|W,X\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_{v}^{2}}{X}\right)-1\right).$$

Hence, the investor-specific Sharpe ratio is given by:

$$SR\left(X\right)=\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_{v}^{2}}{X}\right)-1}}$$

$$E\left[SR\left(X\right)\right]=E\left[\frac{1-R_{f}\exp\left(-\boldsymbol{\mu}\right)}{\sqrt{\exp\left(\frac{\sigma_{v}^{2}}{X}\right)-1}}\right]$$

Proof of Lemma 3.2 and 3.3

This section describes how to prove lemma 3.2 and 3.3. From equations (21), (7), (8) and (9), we can derive that, if η denotes any parameter $\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,x}\}$:

1. $\frac{\partial \boldsymbol{\alpha}}{\partial \eta} = \exp(\boldsymbol{\mu}) \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
2. $\frac{\partial (SR)}{\partial \eta} = \frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\mathbb{E}\left(\exp\left(\frac{\sigma_\upsilon^2}{X}\right)\right) - 1}} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
3. $\frac{\partial SR(X)}{\partial \eta} = \frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right) - 1}} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
4. $\frac{\partial \boldsymbol{Var}(SR(X))}{\partial \eta} = 2\left(1 - R_f \exp(-\boldsymbol{\mu})\right) R_f \exp(-\boldsymbol{\mu}) \boldsymbol{Var}\left(\frac{1}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right) - 1}}\right) \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
5. $\frac{\partial^2 SR(X)}{\partial \eta \partial X} = \frac{\partial\left(\frac{R_f \exp(-\boldsymbol{\mu})}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right) - 1}}\right)}{\partial\left(\frac{\sigma_\upsilon^2}{X}\right)} \frac{\partial\left(\frac{\sigma_\upsilon^2}{X}\right)}{\partial X} \frac{\partial \boldsymbol{\mu}}{\partial \eta}$;
6. $\frac{\partial^2 \theta^*(X)}{\partial \eta \partial X} = \frac{1}{\gamma \sigma_\upsilon^2} \frac{\partial \boldsymbol{\mu}}{\partial \eta}, \forall \eta \neq \gamma$, and $\frac{\partial \theta^*(X)}{\partial \gamma} = 0$
7. $\frac{\partial SR(X)}{\partial \sigma_\upsilon^2} = \frac{R_f \exp(-\boldsymbol{\mu}) \frac{\boldsymbol{\mu} - r_f}{\sigma_\upsilon^2} - \frac{1}{2}\left(1 - R_f \exp(-\boldsymbol{\mu})\right) \frac{\exp\left(\frac{\sigma_\upsilon^2}{X}\right) \frac{1}{X}}{\exp\left(\frac{\sigma_\upsilon^2}{X}\right) - 1}}{\sqrt{\exp\left(\frac{\sigma_\upsilon^2}{X}\right) - 1}}$.

$$\text{Hence,} \quad Sign\left(\frac{\partial \boldsymbol{\mu}}{\partial \eta}\right) = Sign\left(\frac{\partial \boldsymbol{\alpha}}{\partial \eta}\right) = Sign\left(\frac{\partial (SR)}{\partial \eta}\right) = Sign\left(\frac{\partial SR(X)}{\partial \eta}\right)$$
$$= Sign\left(\frac{\partial \boldsymbol{Var}(SR(X))}{\partial \eta}\right) = Sign\left(\frac{\partial^2 SR(X)}{\partial \eta \partial X}\right) = Sign\left(\frac{\partial^2 \theta^*(X)}{\partial \eta \partial X}\right)$$

In addition, we have:

1. Because $\frac{\exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right)-1} > \frac{1}{X}$, $\forall X$,

   then $\frac{\partial SR(X)}{\partial \sigma_v^2} < \frac{R_f \exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2} - \frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)\frac{1}{X}}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right)-1}} < 0$, $\forall X < \underline{X}$,

   where $\underline{X} = \frac{\frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)}{R_f\exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2}} > 0$;

2. $0 = \frac{1}{\sigma_v^2}\left(1+\frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} < \frac{1}{\sigma_v^2}\exp\left(\frac{\sigma_v^2}{X}\right) - \frac{1}{X} - \frac{1}{\sigma_v^2} = \left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\left(\frac{1}{X}+\frac{1}{\sigma_v^2}\right) - \exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}$,

   then $\frac{\exp\left(\frac{\sigma_v^2}{X}\right)\frac{1}{X}}{\exp\left(\frac{\sigma_v^2}{X}\right)-1} < \frac{1}{X}+\frac{1}{\sigma_v^2}$,

   and $\frac{\partial SR(X)}{\partial \sigma_v^2} > \frac{R_f \exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2} - \frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)\left(\frac{1}{X}+\frac{1}{\sigma_v^2}\right)}{\sqrt{\exp\left(\frac{\sigma_v^2}{X}\right)-1}}$, $\forall X$.

   Hence, if $\bar{X} = \frac{1}{\frac{R_f\exp(-\boldsymbol{\mu})\frac{\boldsymbol{\mu}-r_f}{\sigma_v^2}}{\frac{1}{2}\left(1-R_f\exp(-\boldsymbol{\mu})\right)} - \frac{1}{\sigma_v^2}} > 0$, then $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} > 0$.

   $\bar{X} > 0$ if and only if $F\left(\boldsymbol{\mu}-r_f\right) \equiv \exp\left(-\left(\boldsymbol{\mu}-r_f\right)\right)\left(\boldsymbol{\mu}-r_f+\frac{1}{2}\right) - \frac{1}{2} > 0$.
   We can prove $F\left(\boldsymbol{\mu}-r_f\right) > 0$ if and only if $0 < \boldsymbol{\mu}-r_f < 1.2564$, but $\boldsymbol{\mu}-r_f = 1.2564$ corresponds to an $\boldsymbol{\alpha}$ around 250%. Then we can conclude $\bar{X} > 0$ for all reasonable parameters.

3. we can prove by direct computation that $\bar{X} > \underline{X}$ whenever $\bar{X} > 0$.

In sum, for all reasonable parameters, $\exists \bar{X} > \underline{X} > 0$ such that $\forall X > \bar{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} > 0$, and $\forall X < \underline{X}, \frac{\partial SR(X)}{\partial \sigma_v^2} < 0$. The general functional form for effective risk yields similar results.

Equilibrium Value-Weighted Market Sharpe ratio

This section shows that our main conclusions still hold with respect to the value-weighted equilibrium market Sharpe ratio. Because

$$r|\left(W,X\right)\sim N\left(\boldsymbol{\mu}-\frac{1}{2}\frac{\sigma_v^2}{X},\frac{\sigma_v^2}{X}\right),$$

Then

$$\mathbb{E}\left(R|W,X\right)=\exp\left(\boldsymbol{\mu}\right).$$

Then the value-weighted market expected return also equals to:

$$\exp\left(\boldsymbol{\mu}\right),$$

In addition,

$$\boldsymbol{Var}\left(R|W,X\right)=\exp\left(2\boldsymbol{\mu}\right)\left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right).$$

Therefore, the value-weighted variance of the risky asset, is given by:

$$\int\int\boldsymbol{Var}\left(R|W,X\right)\frac{\exp(w)\theta^*\left(\exp(x)\right)f(w,x)}{\int\int\exp(w)\theta^*\left(\exp(x)\right)f(w,x)\,dw\,de}\,dw\,de,$$

Which equals to

$$\exp\left(2\boldsymbol{\mu}\right)\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\exp(w+x)\right]}{\mathcal{X}}$$

Hence, the value-weighted market Sharpe ratio, can be written as:

$$\frac{1-R_f e^{-\mu}}{\sqrt{\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\exp(w+x)\right]}{\mathcal{X}}}}.$$

where $\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}+w+x\right)\right]$, using a Taylor expansion of $\exp(\sigma_v^2X^{-1}+w+x)=1+\sigma_v^2X^{-1}+w+x+\frac{1}{2!}(\sigma_v^2X^{-1}+w+x)^2+\frac{1}{3!}(\sigma_v^2X^{-1}+w+x)^3+\ldots$, which is equivalent to:

$$\mathbb{E}\left[\exp\left(\frac{\sigma_v^2}{X}+w+x\right)\right]=\sum_{k=0}^{\infty}\frac{1}{k!}(\sigma_v^2X^{-1}+w+x)^k)],$$

This will be approximately equal to

$$\begin{aligned}E[\exp(\sigma_v^2X^{-1}+w+x)] \quad &\approx 1+E[\sigma_v^2X^{-1}]+E[w]+E[x],\\ &+\frac{1}{2}E[\sigma_v^4X^{-2}+w^2+x^2+2wx\sigma_v^4X^{-2}+2w^2x\sigma_v^2X^{-1}+2wx^2\sigma_v^2X^{-1}]\end{aligned}$$

The moment-generating function is given by:

$$M(t_1, t_2) = E[\exp(t_1 w)\exp(t_2 x)] = \exp\left(t_1\mu_x + t_2\mu_w + (1/2)(t_1\sigma_x^2 + t_2\sigma_w^2 + 2t_1t_2wx\rho_{w,x}\sigma_x\sigma_w)\right)$$

$\partial M(t_1, t_2)$

Then, if η denotes any parameter $\eta \in \{\gamma, S\}$,

$$\frac{\delta\,(SR)}{\delta\eta} = \frac{R_f e^{-\mu}}{\sqrt{\frac{\mathbb{E}\left[\left(\exp\left(\frac{\sigma_v^2}{X}\right)-1\right)\exp(w+x)\right]}{\mathcal{X}}}}\frac{\delta\mu}{\delta\eta},$$

However, unlike in the case of the equally weighted market equilibrium Sharpe ratio, for the value weighted Sharpe ratio, the derivatives needed to sign the comparative statics in lemma 3.2 and 3.3 for $\eta \in \{\mu_w, \sigma_w, \rho_{w,x}\}$ are indeterminate.

Wealth effect of Expertise

This section shows that while savings rates can theoretically be slightly decreasing in expertise, due to the wealth effect from higher expertise and the associated larger present value of investment opportunities, this effect tends to be dominated by the portfolio choice effect.

The static model with a consumption savings decision can be written as:

$$v(W,X) = \max_{(I,\theta)} \frac{(W-I)^{1-\gamma}}{1-\gamma} + \beta I^{1-\gamma}\mathbb{E}\left[\frac{R_p^{1-\gamma}}{1-\gamma}\right]$$

subject to:

$$\begin{aligned} R_p &= \theta R + (1-\theta) R_f, \\ r|\left(W,X\right) &\sim N\left(\boldsymbol{\mu} - \frac{1}{2}\frac{\sigma_v^2}{X}, \frac{\sigma_v^2}{X}\right). \end{aligned}$$

Clearly, the portfolio choice problem is independent from the consumption savings decision, and the solution to the portfolio choice problem coincides with that of the static model without the consumption saving decision. For any choice of investment I, the optimal portfolio allocation always solves the same problem, maximizing the expected utility derived from the chosen investment level, given the return process for the riskless and risky assets. Therefore, we can plug the optimal portfolio choice back into the value function, and then derive the optimal investment. Finally we get:

$$I^* = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}$$

where

$$E\left[R_p^{1-\gamma}\right] = \exp\left(\left(1-\gamma\right) r_f + \frac{1}{2}\frac{\left(1-\gamma\right)}{\gamma}\frac{\left(\boldsymbol{\mu} - r_f\right)^2}{\frac{\sigma_v^2}{X}}\right).$$

Then, we can show that:

$$\frac{\partial I^*}{\partial X} = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{\left(1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right)^2}\frac{1}{2}\frac{\left(\gamma-1\right)}{\gamma^2}\frac{\left(\boldsymbol{\mu}-r_f\right)^2}{\left(\frac{\sigma_v^2}{X}\right)^2}\frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X}.$$

Observe that the saving rate decreases with the expertise if and only if $\gamma > 1$.

However, for the investment in the risky asset, $I^*\theta^*$, we have:

$$I^*\theta^* = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}\frac{(\boldsymbol{\mu}-r_f)}{\gamma\frac{\sigma_v^2}{X}}.$$

Then,

$$\frac{\partial\left(I^*\theta^*\right)}{\partial X} = W\frac{\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}\frac{(\boldsymbol{\mu}-r_f)}{\gamma\left(\frac{\sigma_v^2}{X}\right)^2}\left(\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1\right)\frac{\partial\left(\frac{\sigma_v^2}{X}\right)}{\partial X}$$

There are two cases, depending on the coefficient of relative risk aversion:

1. If $\gamma < 1$, the saving rate does not fall with the expertise, neither does the investment in the risky asset.
   We have $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<0$.
   Therefore, $\frac{\partial(I^*\theta^*)}{\partial X}>0, \forall X$.
2. If $\gamma > 1$, the saving rate falls with the expertise, while the investment in the risky asset doesn't, as long as the expertise level is not too high.
   We have $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}-1$, $\forall X$.
   Then $\frac{1}{2}\frac{(\gamma-1)}{\gamma^2}\frac{(\boldsymbol{\mu}-r_f)^2}{\frac{\sigma_v^2}{X}}\frac{1}{1+\left(\beta E\left[R_p^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}}-1<0$, $\forall X<\bar{X}$, where $\bar{X}=\frac{2\gamma^2}{(\gamma-1)}\frac{\sigma_v^2}{(\boldsymbol{\mu}-r_f)^2}$.
   Therefore, $\frac{\partial(I^*\theta^*)}{\partial X}>0, \forall X<\bar{X}$. The signs for comparative statics for $\forall X>\bar{X}$ are indeterminate.

In sum, investment in the risky asset increases with expertise, as long as the expertise level is not too high. The general functional form for effective risk yields similar results.

Appendix B: Dynamic Model

Proof. Proposition 4.1. We prove this Proposition by guess and verify. First, we write the HJB equations of our model

$$
\begin{aligned}
\max_{c^x(t,s),\theta(x,t,s)} 0 &= u\left(c^x\left(t,s\right)\right) + V_w^x\left[w\left(t,s\right)\left(r_f+\theta\left(x,t,s\right)\alpha\left(t,s\right)\right) - c^x\left(t,s\right) - f_{xx}w\left(t,s\right)\right] \\
& \frac{\theta^2\left(x\right)\sigma^2\left(x\right)w\left(t,s\right)^2}{2}V_{ww}^x - \rho V^x \\
\max_{c^n(t,s)} 0 &= u^n\left(c\left(t,s\right)\right) + V_w^n\left(r_f w\left(t,s\right) - c^n\left(t,s\right)\right) - \rho V^n
\end{aligned}
$$

The first order conditions are

$$
\begin{aligned}
u'\left(c\left(t,s\right)\right) &= V_w^x, \\
u'\left(c\left(t,s\right)\right) &= V_w^n, \\
V_w^x\alpha\left(t,s\right) + \theta\left(x,t,s\right)\sigma^2\left(x\right)w\left(t,s\right)V_{ww}^x &= 0.
\end{aligned}
$$

Next, we guess that

$$
\begin{aligned}
V^x\left(w\left(t,s\right),x\right) &= y^x\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma}, \\
V^n\left(w\left(t,s\right),x\right) &= y^n\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
c^x &= \left[y^x\left(x,t,s\right)\right]^{-\frac{1}{\gamma}}w\left(t,s\right), \\
c^n &= \left[y^n\left(x,t,s\right)\right]^{-\frac{1}{\gamma}}w\left(t,s\right),
\end{aligned}
$$

and portfolio choice is given by

$$
\theta\left(x,t,s\right) = \frac{\alpha\left(t,s\right)}{\gamma\sigma^2\left(x\right)}.
$$

Plugging these choices into the HJB equations, we get

$$
\begin{aligned}
0 &= \left[y^x\left(x,t,s\right)\right]^{-\frac{1-\gamma}{\gamma}} + y^x\left(x,t,s\right)\left(r_f + \frac{\alpha^2\left(t,s\right)}{\gamma\sigma^2\left(x\right)} - \left[y^x\left(x,t,s\right)\right]^{-\frac{1}{\gamma}} - f_{xx}\right)\left(1-\gamma\right) \\
&\quad -\frac{\alpha^2\left(t,s\right)}{2\gamma\sigma^2\left(x\right)}y^x\left(x,t,s\right)\left(1-\gamma\right) - \rho y^x\left(x,t,s\right) \\
&= \gamma\left[y^x\left(x,t,s\right)\right]^{-\frac{1-\gamma}{\gamma}} + y^x\left(x,t,s\right)\left(r_f + \frac{\alpha^2\left(t,s\right)}{2\gamma\sigma^2\left(x\right)} - f_{xx}\right)\left(1-\gamma\right) - \rho y^x\left(x,t,s\right), \\
0 &= \gamma\left[y^n\left(x,t,s\right)\right]^{-\frac{1-\gamma}{\gamma}} + y^n\left(x,t,s\right)\left(1-\gamma\right)r_f - \rho y^n\left(x,t,s\right).
\end{aligned}
$$

Rearranging the equations, we solve for $y^x(x,t,s)$ and $y^n(x,t,s)$,

$$\begin{aligned} y^x(x,t,s) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right]^{-\gamma}, \\ y^n(x,t,s) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma}\right]^{-\gamma}. \end{aligned}$$

Given all policy functions, we get the experts' wealth growth rates:

$$\frac{dw(t,s)}{w(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s)$$

Finally, given homogeneity of the value functions in wealth, the participation cutoff is constructed by direct comparison between $y^x(x,t,s)$ and $y^n(x,t,s)$. ■

Proof of equivalence of policy functions under the reflecting barrier $\mathbf{z_{min}}$

Interpretation of $z_{\min}$: We assume that one of two things can happen to an investor at $z_{\min}$. With probability q, the investor is eliminated from the market, and replaced with a new agent with wealth share $z_{\min}$ and the same expertise as the exiting agent. Note that elimination in isolation would cause the incumbent agent to be conservative, to avoid $z_{\min}$. With probability $1-q$, the agent is rewarded by being able to infuse funds themselves, or by receiving new external funds, and the wealth share reflects. Note that this reward in isolation would cause the agent to risk shift, to take advantage of limited liability at $z_{\min}$. We require that $E[V^x(z,x)_{true}] = qE[V^x(z,x)_{die}] + (1-q)E[V^x(z,x)_{reflect}]$, conditional on the optimal policies under the true wealth share dynamics. Since the value under the true, non-reflecting, dynamics lies between the punishment value of dying and the reward value of reflection, we conjecture that there exists some probability, conditional on parameters, that this is the case. For simplicity, we assume that $V^x(z,x)_{die} = 0$. It seems quite realistic that investors face uncertainty about what will happen to them as their assets fall below a threshold level. Will they be liquidated, or rescued? Note that our proof offers a technical contribution, since in Gabaix [1999] cities do not choose size, unlike the case for our investors, who choose their savings and portfolio allocations.

We show that the optimal policies in the model with reflecting barrier z_{min} are equivalent to those in the original model under our assumptions of a zero value at death, which is traded off with the positive value of reflection. Our proof assumes an optimal exit date. This is without loss of generality in a stationary equilibrium with no entry or exit. Model 1:

$$\begin{aligned}
V^x\left(w\left(t,s\right),x\right) &= \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}u\left(c^x\left(t,s\right)\right)ds + e^{-\rho(T-t)}V^n\left(w\left(t,s\right),x\right)\right] \\
\text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f+\theta\left(x,t,s\right)\alpha\left(t,s\right)\right)-c^x\left(t,s\right)-F_{xx}\right]ds \\
&\quad +w\left(t,s\right)\theta\left(x,t,s\right)\sigma\left(x\right)dB\left(t,s\right),
\end{aligned}$$

Model 2:

$$\begin{aligned}
V^y\left(w\left(t,s\right),y\right) &= \max_{c^y(t,s),T,\theta(y,t,s)} \max\left\{V^x\left(w\left(t,s\right),y\right),\mathbb{E}\left[\begin{array}{c} \int_t^{s'} e^{-\rho(s-t)}u\left(c^y\left(t,s\right)\right)ds \\ +\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},y\right)\end{array}\right]\right\} \\
\text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f+\theta\left(y,t,s\right)\alpha\left(t,s\right)\right)-c^y\left(t,s\right)-F_{yy}\right]ds \\
&\quad +w\left(t,s\right)\theta\left(y,t,s\right)\sigma\left(y\right)dB\left(t,s\right)
\end{aligned}$$

Assume $F_{xx}=F_{yy}$. They are both linear in wealth. By definition, we have

$$V^y\left(w\left(t,s\right),x\right)=\left(1-q\right)V^y\left(w_{\min},x\right), \text{ for } w\left(t,s\right)\leq w_{\min}.$$

Define

$$q\left(w\left(t,s\right),w_{\min}\right)=1-\left[\frac{w\left(t,s\right)}{w_{\min}}\right]^{1-\gamma}, \text{ for } w\left(t,s\right)\leq w_{\min}.$$

Therefore, we have

$$V^x\left(w\left(t,s\right),x\right)=\left(1-q\right)V^x\left(w_{\min},x\right), \text{ for } w\left(t,s\right)\leq w_{\min}.$$

It suffices to show that

$$V^y\left(w\left(t,s\right),x\right)=V^x\left(w\left(t,s\right),x\right), \text{ for all } x \text{ and } w\left(t,s\right),$$

when agent's wealth hits $w_{\min}$ before he/she exits the market. That is

$$\begin{aligned}
V^y\left(w\left(t,s\right),y\right) &= \max_{c^y(t,s),\theta(y,t,s)} \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u\left(c^y\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},y\right)\right] \\
\text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f+\theta\left(y,t,s\right)\alpha\left(t,s\right)\right)-c^y\left(t,s\right)-F_{yy}\right]ds \\
&\quad +w\left(t,s\right)\theta\left(y,t,s\right)\sigma\left(y\right)dB\left(t,s\right)
\end{aligned}$$

First,

$$\begin{aligned}
V^y\left(w_{\min},x\right) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u\left(c^y\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},x\right)\right] \\
&\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u\left(c^x\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},x\right)\right],
\end{aligned}$$

that is,

$$\begin{aligned} & E\int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds \\ \leq\; & E\int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds \\ =\; & \frac{1}{1-\mathbb{E}\left[(1-q)\,e^{-\rho(s'-t)}\right]} V^y\left(w_{\min},x\right). \end{aligned}$$

Second,

$$\begin{aligned} V^x\left(w_{\min},x\right) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds + e^{-\rho(s'-t)} V^x\left(w\left(t,s'\right),x\right)\right] \\ &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds + (1-q)\,e^{-\rho(s'-t)} V^x\left(w_{\min},x\right)\right] \\ &\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds + (1-q)\,e^{-\rho(s'-t)} V^x\left(w_{\min},x\right)\right], \end{aligned}$$

that is,

$$\begin{aligned} & E\int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds \\ \leq\; & E\int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds \\ =\; & \frac{1}{1-\mathbb{E}\left[(1-q)\,e^{-\rho(s'-t)}\right]} V^x\left(w_{\min},x\right). \end{aligned}$$

Therefore, we must have

$$E\int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds = E\int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds,$$

and

$$V^y\left(w_{\min},x\right) = V^x\left(w_{\min},x\right).$$

Next,

$$\begin{aligned}
V^{y}\left(w\left(t,s\right),x\right) &= \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{y}\left(t,s\right)\right)ds + \left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},x\right)\right] \\
&\geq \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{x}\left(t,s\right)\right)ds + \left(1-q\right)e^{-\rho(s'-t)}V^{x}\left(w_{\min},x\right)\right] \\
&= \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{x}\left(t,s\right)\right)ds + e^{-\rho(s'-t)}V^{x}\left(w\left(t,s'\right),x\right)\right] \\
&= V^{x}\left(w\left(t,s\right),x\right), \text{ for all } w\left(t,s\right)
\end{aligned}$$

with equality iff $c^{x}(t,s) = c^{y}(t,s)$ and $\theta^{x}(x,t,s) = \theta^{y}(x,t,s)$.

Lastly,

$$\begin{aligned}
V^{x}\left(w\left(t,s\right),x\right) &= \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{x}\left(t,s\right)\right)ds + e^{-\rho(s'-t)}V^{x}\left(w\left(t,s'\right),x\right)\right] \\
&\geq \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{y}\left(t,s\right)\right)ds + \left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},x\right)\right] \\
&= V^{y}\left(w\left(t,s\right),x\right), \text{ for all } w\left(t,s\right)
\end{aligned}$$

with equality iff $c^{x}(t,s) = c^{y}(t,s)$ and $\theta^{x}(x,t,s) = \theta^{y}(x,t,s)$.

Therefore,

$$\begin{aligned}
V^{y}\left(w\left(t,s\right),x\right) &= V^{x}\left(w\left(t,s\right),x\right), \text{ for all } x \text{ and } w\left(t,s\right). \\
c^{x}\left(t,s\right) &= c^{y}\left(t,s\right), \\
\theta^{x}\left(x,t,s\right) &= \theta^{y}\left(x,t,s\right).
\end{aligned}$$

Proof. Proposition 4.2 We prove this Proposition by guess and verify. We guess that:

$$\phi(z,x) = Cz^{-\beta-1},$$

Then, we have

$$\begin{aligned}
0 = & -\partial_z\left(z^{-\beta}\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right)\right) \\
& + \frac{1}{2}\partial_{zz}\left(z^{1-\beta}\frac{\alpha^2}{\gamma^2\sigma^2(x)}\right) \\
& = \beta\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right) \\
& - \frac{1}{2}\beta(1-\beta)\left[\frac{\alpha}{\gamma\sigma(x)}\right]^2 \\
& = \beta\left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta)}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right]
\end{aligned}$$

Thus

$$\begin{aligned}
\beta &= C_1\frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\
C_1 &= 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right), \\
C &= \frac{1}{\int z^{-\beta-1}dz} = \frac{C_1\frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1\frac{\sigma^2(x)}{\alpha^2}+\gamma}}.
\end{aligned}$$

Note there are two roots of equation

$$0 = \beta\left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2(\gamma+\beta)}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right].$$

We only take the root that is larger than 1 to ensure the mean wealth has a finite mean. ■

Proof. Corollary 4.1. For the highest expertise agents, we have

$$\bar{z} = \int_{z_{\min}}^{\infty} z\phi(z,\bar{x})dz = \int_{z_{\min}}^{\infty} Cz^{-\beta(\bar{x})}dz = z_{\min}\left[1 + \frac{1}{\beta(\bar{x}) - 1}\right].$$

This gives us another expression of $\beta(\bar{x})$,

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}}.$$

Also, we know

$$\beta\left(\bar{x}\right)=2\gamma\left(f_{xx}+\rho-r_{f}+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^{2}\left(\bar{x}\right)}{\alpha^{2}}-\gamma$$

Therefore, we have

$$2\gamma\left(f_{xx}+\rho-r_{f}+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^{2}\left(\bar{x}\right)}{\alpha^{2}}-\gamma=\frac{1}{1-z_{\min}/\bar{z}},$$

Rearrange the above equation, we get

$$g\left(\bar{x}\right)=\frac{r_{f}-f_{xx}-\rho}{\gamma}+\frac{\alpha^{2}}{2\gamma\sigma^{2}\left(\bar{x}\right)}+\frac{\alpha^{2}}{2\gamma^{2}\sigma^{2}\left(\bar{x}\right)}\frac{1}{1-z_{\min}/\bar{z}}.$$

Plug $g\left(\bar{x}\right)$ into $\beta\left(x\right)$, we derive

$$\beta\left(x\right)=\left(\gamma+\frac{z_{\min}/\bar{z}}{1-z_{\min}/\bar{z}}\right)\frac{\sigma^{2}\left(x\right)}{\sigma^{2}\left(\bar{x}\right)}-\gamma.$$

■

Proof. Lemma 4.1

$$\begin{aligned}\text{Recall that:}\theta\left(x\right) &= \frac{\alpha}{\gamma\sigma^{2}\left(x\right)}\\ \beta\left(x\right) &= 2\gamma\left(f_{xx}+r-r_{f}+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^{2}\left(e\right)}{\alpha^{2}}-\gamma\end{aligned}$$

Consider two levels of expertise, $x_{\min}$ and $x_{\max}$, we have

$$\begin{aligned}\theta\left(x_{\max}\right)-\theta\left(x_{\min}\right) &= \frac{\alpha}{\gamma}\left[\frac{1}{\sigma^{2}\left(x_{\max}\right)}-\frac{1}{\sigma^{2}\left(x_{\min}\right)}\right]\\ &= \frac{\alpha}{\gamma}\frac{\sigma^{2}\left(x_{\min}\right)-\sigma^{2}\left(x_{\max}\right)}{\sigma^{2}\left(x_{\max}\right)\sigma^{2}\left(x_{\min}\right)},\end{aligned}$$

and

$$\begin{aligned}\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right) &= 2\gamma\left(f_{xx}+r-r_{f}+\gamma g\left(\bar{x}\right)\right)\frac{1}{\alpha^{2}}\left[\sigma^{2}\left(x_{\max}\right)-\sigma^{2}\left(x_{\min}\right)\right]\\ &= 2\gamma^{2}\left(f_{xx}+r-r_{f}+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^{2}\left(x_{\max}\right)\sigma^{2}\left(x_{\min}\right)}{\alpha^{3}}\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right].\end{aligned}$$

If a larger dispersion of portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller, since:

$$
\begin{aligned}
\frac{\partial\left[\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right)\right]}{\partial\alpha} &< 0, \text{ and } \frac{\partial\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]}{\partial\alpha}>0 \\
\frac{\partial\left[\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right)\right]}{\partial\gamma} &> 0, \text{ and } \frac{\partial\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]}{\partial\gamma}<0
\end{aligned}
$$

Consider the case where $\sigma^2\left(e_{\max}\right)\sigma^2\left(e_{\min}\right)$ is a constant, then

$$
\frac{\partial\left[\beta\left(e_{\max}\right)-\beta\left(e_{\min}\right)\right]}{\partial\left[\theta\left(e_{\min}\right)-\theta\left(e_{\max}\right)\right]}=2\gamma^2\left(f_{xx}+r-r_f+\gamma g\left(\bar{e}\right)\right)\frac{\sigma^2\left(e_{\max}\right)\sigma^2\left(e_{\min}\right)}{\alpha^3}.
$$

A larger dispersion in portfolio choice, resulting from a larger difference between effective volatility, implies a larger dispersion of tail distribution. The condition on the product of the effective variances is not necessary, however, as can be seen by simple algebra. ■

Proof. Proof of Lemma 4.2 Direct calculation. We use 1 to denote a positive sign.

We have

$$
\begin{aligned}
& sign\left(\frac{\partial I\left(x\right)}{\partial d}\right) \\
=\ & -sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^2\left(x\right)}\right) \\
=\ & -sign\left[\begin{array}{c}\frac{\alpha}{\gamma\sigma^4(x)}\left[-\left(1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}\right)+\frac{(\gamma-1)\alpha^2}{\gamma^2\sigma^2(x)}\right]Z\left(x\right) \\ +\frac{\alpha}{\gamma\sigma^2(x)}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{-1}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2}\frac{C_1}{\alpha^2}\end{array}\right] \\
\geq\ & -sign\left[-\left(1-\frac{\left(\gamma-1\right)\left(r_f-f_{xx}\right)+\rho}{\gamma}\right)+\frac{\left(\gamma-1\right)\alpha^2}{\gamma^2\sigma^2\left(x\right)}\right] \\
=\ & 1
\end{aligned}
$$

Second, for each level of expertise, we have

$$\begin{aligned}
sign\left(\frac{\partial I(x)}{\alpha}\right) &= sign\left[\begin{array}{c} \frac{1}{\gamma\sigma^2(x)}\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}Z(x)-\frac{1}{\gamma\sigma^2(x)}\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}Z(x) \\ +\frac{\alpha}{\gamma\sigma^2(x)}\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{2C_1\frac{\sigma^2(x)}{\alpha^3}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2} \end{array}\right] \\
&= sign\left[\begin{array}{c} 1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)} \\ +\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{z_{\min}}{Z(x)}\frac{2C_1\frac{\sigma^2(x)}{\alpha^2}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2} \end{array}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})}\right] \\
&= 1,
\end{aligned}$$

Third, for each level of expertise, we have

$$\begin{aligned}
& sign\left(\frac{\partial I(x)}{\partial\sigma_\upsilon}\right) \\
=\ & sign\left(\frac{\partial I(x)}{\partial\sigma^2(x)}\frac{\partial\sigma^2(x)}{\partial\sigma_\upsilon}\right) \\
=\ & sign\left(\frac{\partial I(x)}{\partial\sigma^2(x)}\right)sign\left(\frac{\partial\sigma^2(x)}{\partial\sigma_\upsilon}\right) \\
=\ & -1.
\end{aligned}$$

Fourth, for each level of expertise:

$$\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial\gamma}\right) &= sign\left[\begin{array}{c} -\frac{1}{\gamma^2}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]Z(x) \\ +\frac{1}{\gamma}\left[-\frac{r_f-f_{xx}}{\gamma^2}+\frac{\rho}{\gamma^2}-\frac{-\gamma+2}{2\gamma^3}\frac{\alpha^2}{\sigma^2(x)}\right]Z(x) \\ -\frac{1}{\gamma}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]z_{\min}\frac{1}{[\beta-1]^2}\frac{\beta}{\gamma} \end{array}\right] \\
&= sign\left[\begin{array}{c} -\left[1-\frac{(\gamma-2)\left(r_f-f_{xx}\right)}{\gamma}-\frac{2\rho}{\gamma}+\frac{-2\gamma+3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right] \\ -\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]\frac{1}{\beta-1} \end{array}\right] \\
&\leq sign\left[-\left(1-\frac{(\gamma-2)(r_f-f_{xx})+2\rho}{\gamma}-\frac{2\gamma-3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right)\right] \\
&\leq sign\left[-1+\frac{2(\gamma-1)(r_f-f_{xx})+2\rho}{\gamma}+\frac{2\gamma-2}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right] \\
&= -1
\end{aligned}$$

Lastly, for each level of expertise:

$$\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= sign\left[\begin{array}{c} \frac{\alpha}{\gamma\sigma^2(x)}\frac{(\gamma-1)}{\gamma}Z(x) \\ -\frac{\alpha}{\gamma\sigma^2(x)}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] z_{\min}\frac{2\gamma}{[\beta-1]^2}\frac{\sigma^2(x)}{\alpha^2} \end{array}\right] \\
&= sign\left[\begin{array}{c} \frac{(\gamma-1)}{2\gamma}\frac{\alpha^2}{\sigma^2(x)} \\ -\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{1}{\beta(\beta-1)} \end{array}\right] \\
&= -sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\left(1+\beta(\beta-1)\right)\right] \\
&= -1 \text{ if } y^x(\bar{x}) < \frac{1}{1+\beta(\beta-1)}
\end{aligned}$$

■

Weaker conditions for Proposition 4.3 Some weaker conditions are:

$$\begin{aligned}
\underbrace{1-\frac{(\gamma-1)\rho_f+\rho}{\gamma}}_{\text{investment of non-expert}}+\underbrace{\frac{(\gamma-1)f_{xx}}{\gamma}}_{\text{cost of being expert}}-\underbrace{\frac{3(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})}}_{\text{benefit of being expert}} &> 0, \\
or\ y^x(\bar{x}) &< 1-\frac{(\gamma-1)}{\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})} \\
or\ y^x(\bar{x}) &< \frac{1}{3}\left[1+\frac{2(\gamma-1)(r_f-f_{xx})+2\rho}{\gamma}\right] \\
or\ \frac{\alpha^2}{\sigma^2(\bar{x})} &< \frac{2\gamma^2}{3(\gamma-1)}\left[1-\frac{(\gamma-1)(\rho_f-f_{xx})+\rho}{\gamma}\right]
\end{aligned}$$

Proof. Proof of Proposition 4.3 For each level of expertise, we have

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\alpha}\right) &= sign\left[\begin{array}{c} \frac{1}{\gamma\sigma^2(x)}\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}Z(x)-\frac{1}{\gamma\sigma^2(x)}\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}Z(x) \\ +\frac{\alpha}{\gamma\sigma^2(x)}\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{2C_1\frac{\sigma^2(x)}{\alpha^3}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2}\end{array}\right] \\
&= sign\left[\begin{array}{c} 1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)} \\ +\left[\frac{-(\gamma-1)\left(r_f-f_{xx}\right)+\gamma-\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{z_{\min}}{Z(x)}\frac{2C_1\frac{\sigma^2(x)}{\alpha^2}}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2}\end{array}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] \\
&\geq sign\left[1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{3(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})}\right] \\
&= 1,\ \text{for all } x \text{ such that } \frac{\alpha^2}{2\sigma^2(x)\gamma}\geq f_{xx}
\end{aligned}
$$

And when α is higher, more experts enter. Thus

$$\frac{\partial I}{\partial\alpha}>0.$$

■

Weaker conditions for Proposition 4.4 Some weaker conditions are:

$$
\begin{aligned}
1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{\gamma^2\sigma^2(x)} &> 0, \\
or\ y^x(\bar{x}) &< 1-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(\bar{x})} \\
or\ y^x(\bar{x}) &< \frac{1}{2}\left[1+\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}\right] \\
or\ \frac{\alpha^2}{\sigma^2(\bar{x})} &< \frac{\gamma^2}{\gamma-1}\left[1-\frac{(\gamma-1)(r-f_{xx})+\rho}{\gamma}\right]
\end{aligned}
$$

Proof. Proof of Proposition 4.4 Direct calculation. We use 1 to denote a positive sign.

$$\begin{aligned}
& sign\left(\frac{\partial I(x)}{\partial \sigma_v}\right) \\
= \;& sign\left(\frac{\partial I(x)}{\partial \sigma^2(x)}\frac{\partial \sigma^2(x)}{\partial \sigma_v}\right) \\
= \;& sign\left(\frac{\partial I(x)}{\partial \sigma^2(x)}\right) sign\left(\frac{\partial \sigma^2(x)}{\partial \sigma_v}\right).
\end{aligned}$$

We also have

$$\begin{aligned}
& -sign\left(\frac{\partial I(x)}{\partial d}\right) \\
= \;& sign\left(\frac{\partial I(x)}{\partial \sigma^2(x)}\right) \\
= \;& sign\left[\begin{array}{c} \frac{\alpha}{\gamma\sigma^4(x)}\left[-\left(1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}\right)+\frac{(\gamma-1)\alpha^2}{\gamma^2\sigma^2(x)}\right] Z(x) \\ +\frac{\alpha}{\gamma\sigma^2(x)}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right] z_{\min}\frac{-1}{\left[C_1\frac{\sigma^2(x)}{\alpha^2}-\gamma-1\right]^2}\frac{C_1}{\alpha^2} \end{array}\right] \\
\leq \;& sign\left[-\left(1-\frac{(\gamma-1)(r_f-f_{xx})+\rho}{\gamma}\right)+\frac{(\gamma-1)\alpha^2}{\gamma^2\sigma^2(x)}\right] \\
= \;& -1
\end{aligned}$$

Thus for each level of expertise, when fundamental risk is higher, the demand for the complex risky asset is smaller. And when σ_v is higher, fewer experts enter the complex risky asset market. Thus

$$\frac{\partial I}{\partial \sigma_v}<0.$$

Next, for each level of expertise:

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial\gamma}\right) &= sign\left[\begin{array}{c}
-\frac{1}{\gamma^2}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]Z(x)\\
+\frac{1}{\gamma}\left[-\frac{r_f-f_{xx}}{\gamma^2}+\frac{\rho}{\gamma^2}-\frac{-\gamma+2}{2\gamma^3}\frac{\alpha^2}{\sigma^2(x)}\right]Z(x)\\
-\frac{1}{\gamma}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]z_{\min}\frac{1}{[\beta-1]^2}\frac{\beta}{\gamma}
\end{array}\right]\\
&= sign\left[\begin{array}{c}
-\left[1-\frac{(\gamma-2)\left(r_f-f_{xx}\right)}{\gamma}-\frac{2\rho}{\gamma}+\frac{-2\gamma+3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]\\
-\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]\frac{1}{\beta-1}
\end{array}\right]\\
&\leq sign\left[-\left(1-\frac{(\gamma-2)\left(r_f-f_{xx}\right)+2\rho}{\gamma}-\frac{2\gamma-3}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right)\right]\\
&\leq sign\left[-1+\frac{2(\gamma-1)\left(r_f-f_{xx}\right)+2\rho}{\gamma}+\frac{2\gamma-2}{2\gamma^2}\frac{\alpha^2}{\sigma^2(x)}\right]\\
&= -1
\end{aligned}
$$

Lastly, for each level of expertise:

$$
\begin{aligned}
sign\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= sign\left[\begin{array}{c}
\frac{\alpha}{\gamma\sigma^2(x)}\frac{(\gamma-1)}{\gamma}Z(x)\\
-\frac{\alpha}{\gamma\sigma^2(x)}\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]z_{\min}\frac{2\gamma}{[\beta-1]^2}\frac{\sigma^2(x)}{\alpha^2}
\end{array}\right]\\
&= sign\left[\begin{array}{c}
\frac{(\gamma-1)}{2\gamma}\frac{\alpha^2}{\sigma^2(x)}\\
-\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\right]\frac{1}{\beta(\beta-1)}
\end{array}\right]\\
&= -sign\left[1-\frac{(\gamma-1)\left(r_f-f_{xx}\right)+\rho}{\gamma}-\frac{(\gamma-1)\alpha^2}{2\gamma^2\sigma^2(x)}\left(1+\beta(\beta-1)\right)\right]\\
&= -1 \text{ if } y^x(\bar{x}) < \frac{1}{1+\beta(\beta-1)}
\end{aligned}
$$

Therefore:

$$\frac{\partial I}{\partial\gamma} < 0 \text{ and } \frac{\partial I}{\partial f_{xx}} < 0$$

■

Proof. Proof of Proposition 4.5 We have

$$sign\left(\frac{\partial I(x)}{\partial x}\right) = sign\left(\frac{\partial I(x)}{\partial d}\frac{\partial d}{\partial x}\right) = sign\left(-\frac{\partial I(x)}{\partial d}\frac{\partial\sigma(x)}{\partial x}\right) = 1$$

And

$$\begin{aligned}
I(\Lambda_1) - I(\Lambda_2) &= \int [\lambda_1(x) - \lambda_2(x)]\, I(x)\, dx \\
&= -I(x)\left[\Lambda_1(\underline{x}) - \Lambda_2(\underline{x})\right] - \int \frac{\partial I(x)}{\partial x}\left[\Lambda_1(x) - \Lambda_2(x)\right] dx \\
&> 0
\end{aligned}$$

■

Proof. Proof of Proposition 4.6. Given

$$\frac{\partial SR(x)}{\partial \sigma_\upsilon} = \frac{\frac{\partial \alpha}{\partial \sigma_\upsilon}\sigma(x) - \alpha \frac{\partial \sigma(x)}{\partial \sigma_\upsilon}}{\sigma^2(x)}$$

we have

$$\frac{\partial SR(x)}{\partial \sigma_\upsilon} > 0 \text{ iff } \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon} < \frac{\partial \log \alpha}{\partial \log \sigma_\upsilon}.$$

If $\frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}$ is a constant, we must have either $\frac{\partial \log \alpha}{\partial \log \sigma_\upsilon} > \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}$ for all x or $\frac{\partial \log \alpha}{\partial \log \sigma_\upsilon} < \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}$ for all x.

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}}{\partial x} < 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_\upsilon} = \frac{\partial \log \alpha}{\partial \log \sigma_\upsilon},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\upsilon} < 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\upsilon} > 0$.

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}}{\partial x} > 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_\upsilon} = \frac{\partial \log \alpha}{\partial \log \sigma_\upsilon},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\upsilon} > 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\upsilon} < 0$. ■

Value Weighted Equilibrium Sharpe ratio The market value weighted Sharpe ratio can be written as

$$
\begin{aligned}
SR^{vw} &= E\left[\frac{\theta(z-c)}{I}\frac{\alpha}{\sigma(x)}|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\
&= E\left[\frac{\theta z\left(1-[y^x(x)]^{-\frac{1}{\gamma}}\right)}{I}\frac{\alpha}{\sigma(x)}|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\
&= E\left[\frac{\alpha}{\gamma\sigma^2(x)}\frac{1}{I}\frac{\alpha}{\sigma(x)}\epsilon^x\left(Z(x)|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right)\right] \\
&= \frac{\alpha}{\gamma I}E\left[\frac{1-[y^x(x)]^{-\frac{1}{\gamma}}}{\sigma^3(x)}Z(x)|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right]
\end{aligned}
$$

Proof. Proof of Lemma 4.3 Proof by contradiction. Suppose σ_v is increased by 1%, but the equilibrium α is increased by less than $l_{\inf}^{\sigma_v}$, that is

$$
\frac{\partial\alpha/\alpha}{\partial\sigma_v/\sigma_v} \leq l_{\inf}^{\sigma_v}
$$

We have

1. Less participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial\alpha/\alpha}{\partial\sigma_v/\sigma_v} < l_{\inf}^{\sigma_v}$, $\underline{x}$ is higher.
2. Less investment in the complex risky asset:

$$
\frac{\partial \log I(x)}{\partial\sigma_v} < 0, \text{ for all } x.
$$

Therefore, in the new equilibrium, the total demand for risky asset is less than the total supply. Contradiction. It must be that

$$
\frac{\partial\alpha/\alpha}{\partial\sigma_v/\sigma_v} > \inf\left\{\frac{\partial\log\sigma(x)}{\partial\log\sigma_v}|\frac{\alpha^2}{2\sigma^2(x)\gamma} \geq f_{xx}\right\}.
$$

■

Proof. Proof of Proposition 4.7 The first part follows directly from the proof of Proposition 4.3 since all elasticities are constant in x in Case 1. For the second part, first we prove that

$\frac{\partial SR^{ew}}{\partial \sigma_\upsilon} > 0$ if $\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} > l^{\sigma_\upsilon}_{\text{sup}}$. We have

$$\begin{aligned}
& \frac{\partial SR^{ew}}{\partial \sigma_\upsilon} \\
= \; & E\left[\frac{1}{\sigma(x)}\frac{\partial \alpha}{\partial \sigma_\upsilon} - \frac{\alpha}{\sigma^2(x)}\frac{\partial \sigma(x)}{\partial \sigma_\upsilon}|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] - \frac{\alpha}{\sigma(x)}|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}}\frac{\partial \underline{x}}{\partial \sigma_\upsilon} \\
= \; & \frac{\alpha}{\sigma_\upsilon}E\left[\frac{1}{\sigma(x)}\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon/\sigma_\upsilon}\right)|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] - \frac{\alpha}{\sigma(x)}|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}}\frac{\partial \underline{x}}{\partial \sigma_\upsilon} \\
> \; & \frac{\alpha}{\sigma_\upsilon}E\left[\frac{1}{\sigma(x)}\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon/\sigma_\upsilon}\right)|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\
> \; & \frac{\alpha}{\sigma_\upsilon}E\left[\frac{1}{\sigma(x)}\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon/\sigma_\upsilon}\right)|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\
> \; & 0.
\end{aligned}$$

Next, we show that if $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}}{\partial x} < 0$, and $l^{\sigma_\upsilon}_{\text{sup}} < \frac{2\beta(\overline{x})}{\beta(\overline{x})+1} l^{\sigma_\upsilon}_{\text{inf}}$, we have

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} > l^{\sigma_\upsilon}_{\text{sup}}.$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\upsilon}}{\partial x} < 0$, assume $\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} \leq l^{\sigma_\upsilon}_{\text{sup}} < \frac{2\beta(\underline{x})}{\beta(\underline{x})+1} l^{\sigma_\upsilon}_{\text{inf}}$, We have

- Less participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} \leq l^{\sigma_\upsilon}_{\text{sup}}$, $\underline{x}$ is higher.
- Less investment in the complex risky asset:

$$\begin{aligned}
& \frac{\partial \log I(x)}{\partial \sigma_\upsilon} \\
= \; & -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon} \\
& +\frac{1}{\sigma_\upsilon}\left[1 + \frac{2(\beta(x)+\gamma)}{(\beta(x)-1)^2} - \frac{1}{1-y^x(x)}\frac{(\gamma-1)}{2\gamma^2}\frac{2\alpha^2}{\sigma^2(x)}\right]\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon/\sigma_\upsilon} + \frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon}\right]
\end{aligned}$$

1) For x such that $\frac{\partial \alpha/\alpha}{\partial \sigma_\upsilon/\sigma_\upsilon} < \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon/\sigma_\upsilon}$,

$$\frac{\partial \log I(x)}{\partial \sigma_\upsilon} < 0.$$

2) For x such that $\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\upsilon/\sigma_\upsilon} < \frac{\partial\alpha/\alpha}{\partial\sigma_\upsilon/\sigma_\upsilon} < l_{\sup}^{\sigma_\upsilon} < \frac{2\beta(\underline{x})}{\beta(\underline{x})+1} l_{\inf}^{\sigma_\upsilon}$,

$$\begin{aligned}
&\frac{\partial \log I(x)}{\partial \sigma_\upsilon} \\
< \;& -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon} + \frac{1}{\sigma_\upsilon}\left[1 + \frac{2}{\beta(x)-1} - \frac{2y^x(x)}{1-y^x(x)}\right]\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial\sigma_\upsilon/\sigma_\upsilon} + \frac{\partial\alpha/\alpha}{\partial\sigma_\upsilon/\sigma_\upsilon}\right] \\
< \;& -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon} + \frac{1}{\sigma_\upsilon}\frac{\beta(\overline{x})+1}{\beta(\overline{x})-1}\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial\sigma_\upsilon/\sigma_\upsilon} + \frac{\partial\alpha/\alpha}{\partial\sigma_\upsilon/\sigma_\upsilon}\right] \\
< \;& \frac{1}{\sigma_\upsilon}\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon} + \frac{\beta(\overline{x})+1}{\beta(\overline{x})-1}\left(l_{\sup}^{\sigma_\upsilon} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\upsilon}\right)\right]
\end{aligned}$$

Next

$$\begin{aligned}
\frac{\partial I}{\partial \sigma_\upsilon} &\leq \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial \sigma_\upsilon} d\Lambda(x) \\
&\leq I(\overline{x}) \int_{\underline{x}}^{\infty} \frac{\partial \log I(x)}{\partial \sigma_\upsilon} d\Lambda(x) \\
&< \frac{I(\overline{x})}{\sigma_\upsilon} \int_{\underline{x}}^{\infty} \left\{-l^{\sigma_\upsilon}(x)\frac{2\beta(\overline{x})}{\beta(\overline{x})-1} + \frac{\beta(\overline{x})+1}{\beta(\overline{x})-1} l_{\sup}^{\sigma_\upsilon}\right\} d\Lambda(x) \\
&= \frac{I(\underline{x})(1-\Lambda(\underline{x}))}{\sigma_\upsilon}\frac{\beta(\overline{x})+1}{\beta(\overline{x})-1}\left\{-\frac{2\beta(\overline{x})}{\beta(\overline{x})+1} E\left[l^{\sigma_\upsilon} | x \geq \underline{x}\right] + l_{\sup}^{\sigma_\upsilon}\right\} \\
&< 0.
\end{aligned}$$

Therefore, in the new equilibrium, the total demand for the complex risky asset is less than the total supply. Contradiction. Therefore, it must be that

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\upsilon/\sigma_\upsilon} > l_{\sup}^{\sigma_\upsilon} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\upsilon/\sigma_\upsilon}.$$

■

Proof. Proof of Proposition 4.8 First,

$$\frac{\partial \underline{x}}{\partial \sigma_\upsilon} < 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\upsilon} > 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\upsilon} = 2\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\upsilon/\sigma_\upsilon} - \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\upsilon/\sigma_\upsilon}\right)$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\upsilon} > 0 \text{ iff } \frac{\partial\alpha/\alpha}{\partial\sigma_\upsilon/\sigma_\upsilon} > \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\upsilon/\sigma_\upsilon}$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} \geq 0$, from Proposition 4.3 we have

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\text{inf}}^{\sigma_v} = \frac{\partial \sigma \left(\underline{x}\right) / \sigma \left(\underline{x}\right)}{\partial \sigma_v / \sigma_v}.$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_v}}{\partial x} < 0$, assume $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \leq l_{\text{sup}}^{\sigma_v} < \frac{2\beta(\underline{x})}{\beta(\underline{x})+1} l_{\text{inf}}^{\sigma_v}$, We have

- Less participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v}$, $\underline{x}$ is higher.
- Less investment in the complex risky asset:

$$\begin{aligned} & \frac{\partial \log I\left(x\right)}{\partial \sigma_v} \\ = \quad & -\frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v} \\ & + \frac{1}{\sigma_v} \left[1 + \frac{2\left(\beta\left(x\right) + \gamma\right)}{\left(\beta\left(x\right) - 1\right)^2} - \frac{1}{1 - y^x\left(x\right)} \frac{\left(\gamma - 1\right)}{2\gamma^2} \frac{2\alpha^2}{\sigma^2\left(x\right)} \right] \left[-\frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \end{aligned}$$

1) For x such that $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < \frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v}$,

$$\frac{\partial \log I\left(x\right)}{\partial \sigma_v} < 0.$$

2) For x such that $\frac{\partial \sigma(x) / \sigma(x)}{\partial \sigma_v / \sigma_v} < \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\text{sup}}^{\sigma_v} < \frac{2\beta(\underline{x})}{\beta(\underline{x})+1} l_{\text{inf}}^{\sigma_v}$,

$$\begin{aligned} & \frac{\partial \log I\left(x\right)}{\partial \sigma_v} \\ < \quad & -\frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v} + \frac{1}{\sigma_v} \left[1 + \frac{2}{\beta\left(x\right) - 1} - \frac{2y^x\left(x\right)}{1 - y^x\left(x\right)} \right] \left[-\frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \\ < \quad & -\frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v} + \frac{1}{\sigma_v} \frac{\beta\left(\overline{x}\right) + 1}{\beta\left(\overline{x}\right) - 1} \left[-\frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v / \sigma_v} + \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} \right] \\ < \quad & \frac{1}{\sigma_v} \left[-\frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v} + \frac{\beta\left(\overline{x}\right) + 1}{\beta\left(\overline{x}\right) - 1} \left(l_{\text{sup}}^{\sigma_v} - \frac{\partial \sigma\left(x\right) / \sigma\left(x\right)}{\partial \sigma_v} \right) \right] \end{aligned}$$

Next

$$\begin{aligned}
\frac{\partial I}{\partial \sigma_v} &\leq \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial \sigma_v} d\Lambda(x) \\
&\leq I(\overline{x}) \int_{\underline{x}}^{\infty} \frac{\partial \log I(x)}{\partial \sigma_v} dG(x) \\
&< \frac{I(\overline{x})}{\sigma_v} \int_{\underline{x}}^{\infty} \left\{ -l^{\sigma_v}(x) \frac{2\beta(\overline{x})}{\beta(\overline{x})-1} + \frac{\beta(\overline{x})+1}{\beta(\overline{x})-1} l_{\sup}^{\sigma_v} \right\} d\Lambda(x) \\
&= \frac{I(\underline{x})(1-\Lambda(\underline{x}))}{\sigma_v} \frac{\beta(\overline{x})+1}{\beta(\overline{x})-1} \left\{ -\frac{2\beta(\overline{x})}{\beta(\overline{x})+1} E\left[l^{\sigma_v} | x \geq \underline{x}\right] + l_{\sup}^{\sigma_v} \right\} \\
&< 0.
\end{aligned}$$

Therefore, in the new equilibrium, the total demand for the complex risky asset is less than the total supply. Contradiction. Therefore, it must be that

$$\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\sup}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}.$$

■

Proof. Proof of Proposition 4.9 First,

$$\frac{\partial \underline{x}}{\partial \sigma_v} > 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} < 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} = 2\left(\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} - \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v} \right)$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_v} < 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} < l_{\sup}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}$$

If $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > l_{\sup}^{\sigma_v} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_v / \sigma_v}$, we have

- More participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial \alpha / \alpha}{\partial \sigma_v / \sigma_v} > e_{\sup}^{\sigma_v}$, $\underline{x}$ is lower.

- More investment in the complex risky asset:

$$
\begin{aligned}
& \frac{\partial \log I(x)}{\partial \sigma_v} \\
= \; & -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v} \\
& +\frac{1}{\sigma_v}\left[1+\frac{2(\beta(x)+\gamma)}{(\beta(x)-1)^2}-\frac{1}{1-y^x(x)}\frac{(\gamma-1)}{2\gamma^2}\frac{2\alpha^2}{\sigma^2(x)}\right]\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v}+\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v}\right] \\
> \; & \frac{1}{\sigma_v}\left\{-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v}+\left[1+\frac{2}{\beta(x)-1}-\frac{2y^x(x)}{1-y^x(x)}\right]\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v}+\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v}\right]\right\} \\
> \; & \frac{1}{\sigma_v}\left\{-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v}+\frac{2}{\beta(x)-1}\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v}+\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v}\right]\right\} \\
> \; & \frac{1}{\sigma_v}\left\{-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v}+\frac{2}{\beta(\underline{x})-1}\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_v/\sigma_v}+\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v}\right]\right\}
\end{aligned}
$$

Next

$$
\begin{aligned}
\frac{\partial I}{\partial \sigma_v} & \geq \int_{\underline{x}}^{\infty}\frac{\partial I(x)}{\partial \sigma_v}d\Lambda(x) \\
& \geq I(\underline{x})\int_{\underline{x}}^{\infty}\frac{\partial \log I(x)}{\partial \sigma_v}dG(x) \\
& > \frac{I(\underline{x})}{\sigma_v}\int_{\underline{x}}^{\infty}\left\{-l^{\sigma_v}(x)\frac{\beta(\underline{x})+1}{\beta(\underline{x})-1}+\frac{2}{\beta(\underline{x})-1}l^{\sigma_v}_{\text{sup}}\right\}d\Lambda(x) \\
& = \frac{I(\underline{x})}{\sigma_v}\frac{2(1-\Lambda(\underline{x}))}{\beta(\underline{x})-1}\left\{-\frac{\beta(\underline{x})+1}{2}E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_v}|x\geq \underline{x}\right]+l^{\sigma_v}_{\text{sup}}\right\} \\
& > 0.
\end{aligned}
$$

Therefore, in the new equilibrium, the total demand for risky asset is more than total supply. Contradiction. Therefore, it must be that

$$\frac{\partial \alpha/\alpha}{\partial \sigma_v/\sigma_v} < l^{\sigma_v}_{\text{sup}} = \frac{\partial \sigma(\underline{x})/\sigma(\underline{x})}{\partial \sigma_v/\sigma_v}.$$

■