Markets with Multidimensional Private Information*

Veronica Guerrieri
University of Chicago

Robert Shimer
University of Chicago

August 26, 2015

Abstract

This paper explores price formation when sellers are privately informed about their preferences and the quality of their asset. There are many equilibria, including a semi-separating one in which each seller's price depends on a one-dimensional index of her preferences and asset quality. This multiplicity does not rely on off-the-equilibrium path beliefs and so is not amenable to standard signaling game refinements. The semi-separating equilibrium may be not Pareto efficient, even if it is not Pareto dominated by any other equilibrium. Instead, efficient allocations may require transfers across uninformed buyers, inconsistent with any equilibrium.

JEL Codes: D52, D82, D53, E44

Keywords: adverse selection, multiple equilibria, price dispersion, Pareto efficiency, market mechanisms

*Guerrieri: University of Chicago, Booth School of Business, 5807 South Woodlawn Avenue, Chicago, IL, 60637 (e-mail: vguerrie@chicagobooth.edu); Shimer: University of Chicago, Department of Economics, 1126 East 59th Street, Chicago, IL, 60637 (e-mail: robert.shimer@gmail.com). We thank Briana Chang, Piero Gottardi, Robert Hall, Guido Lorenzoni, and numerous seminar audiences for comments on earlier versions of this paper. For research support, Guerrieri is grateful to the Alfred P. Sloan Foundation and Guerrieri and Shimer are grateful to the National Science Foundation.

1 Introduction

This paper develops and explores a canonical exchange economy in which the initial owner of an asset has private information both about the quality of the asset and about her preferences. We are interested in understanding how market mechanisms work, whether such mechanisms involve price dispersion for different assets, whether markets reallocate assets towards more productive users, and whether equilibrium allocations are Pareto efficient.

The model we develop in this paper is abstract, but it may be useful to keep a particular example in mind, the market for used cars (Akerlof, 1970). The car's initial owner may have private information about some of the car's attributes, such as its reliability. She may also have private information about her own preferences for newer versus older model cars. When the owner sets a price, she may perceive a trade-off: if she asks for a higher price, it will take her longer to sell the car, but she will get more money when she succeeds in selling it. Given these perceptions, the price she sets will depend both on her preferences and on the attributes of the car. If she has only a weak desire for a newer model and the car is reliable, she will prefer to set a high sale price and risk failing to sell it. She will set a lower sale price and sell the car faster in the opposite circumstance.

Turn now to a used car buyer. When he sees a car with a high asking price conditional on its observable attributes, he should conclude that either the car is reliable or that the seller has a weak desire to sell it. If in expectation he believes that higher prices are associated with higher quality cars, he may be willing to pay a higher price. This behavior can support an equilibrium with price dispersion.

We can answer the questions in the first paragraph using the terminology from this example. We find that observationally identical cars may sell for heterogeneous prices, depending on the owner's preferences. Conversely, at any given price, heterogeneous cars are typically available. While markets generally transfer assets from sellers with a strong taste for new cars to buyers with a greater willingness to purchase a used car, some sales shift low quality cars away from their natural owners. Finally, our analysis shows that under some conditions the equilibrium allocation is Pareto efficient. If it is inefficient, there may be other Pareto superior equilibria which cross-subsidize sellers through pooling prices. Alternatively, there may be no Pareto superior equilibrium, but instead a Pareto improving allocation may require cross-subsidizing uninformed buyers. The remainder of the introduction explains these findings in more detail and places them in the context of the existing literature.

Our model economy is abstract but is designed to capture the real-world trade-offs faced by buyers and sellers. It is populated by a continuum of risk-neutral investors who live for two periods. Investors are heterogeneous in their discount factor between the periods.

At the start of the first period, investors are endowed with a perishable consumption good (potential buyers), with assets that produce dividends in the second period (potential sellers), or possibly with both goods and assets. Assets are heterogeneous in their quality, defined as the amount of the second period consumption good that they produce.

At the beginning of the first period each investor privately observes his discount factor and the quality of any assets that he owns. Next, investors may exchange the first period consumption good for assets. Investors may use their consumption goods to buy assets, sell their assets for the consumption good, engage in both activities if they are endowed both with assets and consumption goods, or simply consume their endowment. We allow investors to buy or sell at any price, forming beliefs about the probability that they will be able to trade at that price and about the composition of assets offered for sale at that price. Trade is rationed by the short side of the market at every price, with all traders on the long side of the market equally likely to trade.

We show that our model features multiple equilibria. First, we describe a *semi-separating equilibrium* of the sort we highlighted in the used car example. Sellers with higher continuation values, defined as the product of their discount factor and their asset quality, set higher prices and sell their assets with lower probability. As long as sellers with higher continuation values have higher quality assets on average, buyers rationally perceive that they will get more by paying more, and so are willing to buy at a range of different prices. In such an equilibrium, identical quality assets sell at different prices, reflecting heterogeneity in the sellers' preferences, while heterogeneous assets sell at the same price if the two sellers have the same continuation value. Then we show that, under some assumptions, there is also a *one-price* equilibrium, in which all trade takes place at a single price. Any seller with a continuation value below this price sells for sure at that price, while sellers with higher continuation values do not sell. There are also many other equilibria, for example equilibria in which all trade takes place at n different prices for arbitrary n, and *mixed* equilibria that combine some pooling mass points and some semi-separating intervals.

What drives the multiplicity of equilibria? One might conjecture that this is a consequence of the signaling aspect of our model: a seller's price is a noisy signal of her asset's quality. Signaling games often exhibit many equilibria, unless one imposes restrictions on off-the-equilibrium-path beliefs, i.e. beliefs about who would sell for a price that no one actually sets in equilibrium (Cho and Kreps, 1987; Banks and Sobel, 1987). We include such restrictions directly in our definition of equilibrium and so this is not the source of our multiplicity. Instead, multiple equilibria are a consequence of the multidimensional private information. Sellers with the same continuation value have the same preferences and so may all be indifferent about setting any price in a nontrivial interval. This implies that the single crossing condition holds only weakly in our environment. Buyers care which seller sets which price even when sellers are indifferent. This creates the scope for multiple equilibria.

We also introduce a more restrictive definition of equilibrium with *unidimensional* private information. This definition reduces the sellers' private information to a single dimension, their continuation value. We impose that (i) all sellers with the same continuation value set the same price; and (ii) buyers believe that all sellers with the same continuation value are equally likely to select any sale price not chosen in equilibrium. We prove that the semi-separating equilibrium is unique under this more restricted notion of equilibrium.

Nevertheless, we believe it is a mistake to collapse the seller's multiple dimensional private information down to a single dimension. Preferences and endowments are distinct in an Arrow-Debreu economy and so it makes sense to keep them distinct in an economy with private information as well. Distinguishing between preferences and endowments is important for understanding how markets reallocate goods across heterogeneous investors. For example, in addition to the anticipated finding that private information reduces the amount of trade, particularly for high quality goods, we find that it also leads to some pairwise inefficient trades. Some assets are sold by a more patient seller to a less patient buyer.

We explore the efficiency of the equilibrium, particularly the semi-separating equilibrium, within the set of incentive-feasible allocations. It is well known that a pooling allocation can Pareto dominate a separating allocation (Rothschild and Stiglitz, 1976). The benefit of a pooling allocation is that separation is wasteful, as it reduces the trading probabilities. The cost is that sellers cross-subsidize each other, and so pooling is less attractive to high quality sellers. If there are few low quality sellers, the benefit of pooling outweighs the cost for everyone and hence a pooling allocation Pareto dominates a separating allocation. This might suggest that the one-price equilibrium will Pareto dominate the semi-separating equilibrium under similar conditions.

It turns out it is easy to construct examples where this is not the case, even if there are arbitrarily few low quality sellers. Even though both a semi-separating and a one-price equilibrium exist, the equilibria are not Pareto comparable. Either some buyers or some sellers prefer each equilibrium to the other. We also consider mixed equilibria, where small subsets of sellers set a common price, while other sellers set different prices. We show that under a mild regularity condition, the semi-separating equilibrium is not Pareto dominated by a mixed equilibrium.

Finally, we find that efficient allocations may require cross-subsidizing uninformed buyers, which is impossible in any market equilibrium. In equilibrium, buyers may be indifferent over a range of different prices, rationally anticipating that they will obtain sufficiently high average quality at high prices so as to offset the high cost. In contrast, implementing an efficient allocation may require buyers to lose money at some prices and make money at other prices. Such cross-subsidization of buyers is not a natural feature of a market environment, but we show that it may be important for a Pareto optimal allocation.

The analysis of our model with multidimensional private information differs from our previous work in which investors' discount factors were observable (Guerrieri and Shimer, 2014) and so there was only a single dimension of private information. In that model, we found a unique fully-separating equilibrium in which higher quality assets trade at higher prices in less liquid markets. The predictions of our two models differ in a number of ways. Most importantly, we find that a continuum of equilibria exist in this environment, which allows us to compare welfare across equilibria. Moreover, we show that Pareto efficient allocations may require cross-subsidization of both buyers and sellers.

In addition, even if we were to focus on the semi-separating equilibrium in this environment, the nature of the equilibrium differs qualitatively between the two papers. With multidimensional private information there is price dispersion for assets of the same quality and heterogeneous assets sell for the same price. This is relevant for any empirical analysis of the relationship between price and quality. It is also important for information transmission through prices. In a semi-separating equilibrium, buyers can learn something about the quality of their asset from its price, but they cannot learn everything. In our prior work, there was a one-to-one mapping between asset quality and price, so equilibrium prices fully-revealed an asset's quality. This leads to the possibility that in a dynamic extension of our model, multidimensional private information leads to the gradual loss of private information in secondary markets. We also find that with multidimensional private information, some investors may be willing to buy and sell, depending on both their preferences and the quality of their asset. In contrast, with observable preferences, investors' decision to buy or sell depends only on their preferences.

Although the setup of our paper is deliberately abstract, we believe the analysis offers insight into many real-world markets, not just the market for used cars. The market for existing homes shares many of the same characteristics as the used car market. Sellers have multiple hidden motives for putting their home on the market and buyers only care about some dimensions of the sellers' private information. Our approach may be useful for understanding the joint determination of prices and sales volume in this important market. Private information may also be important in some securities markets. For example, a number of recent empirical papers have analyzed the extent to which mortgage originators have private information about the quality of mortgage pools, particularly for low-documentation loans. (see, for example, Keys, Mukherjee, Seru and Vig, 2010; Demiroglu and James, 2012; Jiang, Nelson and Vytlacil, 2014a,b; Piskorski, Seru and Witkin, 2015). Our framework may be useful for understanding price formation in this market as well. Moreover, to the extent that buyers in primary markets learn some of the mortgage originators' private information from the transaction price, information asymmetries may persist in secondary markets, a possibility that the empirical literature has thus far neglected. Finally, the nature of equilibrium may have important implications for the efficacy of particular policy interventions. Although a serious analysis of these possibilities goes beyond the scope of our paper, we comment more on these possibilities in the conclusion.

Our notion of equilibrium builds on Guerrieri, Shimer and Wright (2010), which in turn builds on prior research, most notably Wilson (1980), Gale (1996), and Ellingsen (1997). All of these papers share the idea that price dispersion can arise in the presence of adverse selection, as privately-informed sellers can use a high selling price to signal a high quality asset, if higher sale prices are associated with lower sale probabilities.

The closest related paper in this research stream is Chang (2014). The two papers address different questions. First, while Chang also assumes sellers differ both in their preferences and in the quality of their asset, she collapses the analysis to a single dimension, equivalent to our continuation value. This avoids the source of our multiple equilibria and hence most of the questions we address in this paper. Second, Chang focuses on a number of specific policy proposals, while we examine the efficiency properties of equilibrium more generally. Third, we impose a restriction on parameters, that sellers with higher continuation values have higher quality assets on average. The most novel parts of Chang are concerned with situations in which this restriction is violated.

There are numerous other small but important differences between the papers. Chang (2014) looks at an environment in which the role of an investor as a buyer or seller is determined exogenously, while we allow investors to choose whether to buy assets, sell assets, do both, or do neither. Chang assumes that all buyers value any asset more than the average seller does, which implies that in equilibrium, all assets are sold with a positive probability. In our model, investors are heterogeneous and the decision to buy and sell is endogenous. As a result, we find that some investors may choose not to attempt to sell their assets and that some assets are transferred from investors who value them more to investors who value them less. We believe these insights may be important for understanding real-world trading patterns. For example, in the market for existing homes, the decision to sell or buy is endogenous and such an endogeneity may be an important determinant of the equilibrium allocation.

There is a related line of research that studies how optimal mechanisms can allow for separation when sellers are privately informed, in the spirit of Maskin and Tirole (1992). In DeMarzo and Duffie (1999), sellers can commit to retain a portion of an asset in order to signal its quality. In a similar spirit, in Chari, Shourideh and Zetlin-Jones (2014), buyers offer sellers a menu of contracts, inducing sellers of high quality assets to sell a small amount of their holdings at a high price. Both of these papers focus on environments in which asset quality is private information but sellers' preferences are common knowledge, while we allow for multidimensional private information. More fundamentally, we show that markets can achieve the same outcome through a shortage of buyers and rationing. While there is no mathematical difference between probabilistic sales and sales of a fraction of asset holdings in our environment, the distinction may again be important for understanding real-world trading patterns. For example, sellers do not retain a fraction of their home to signal its quality.

Daley and Green (2012) obtain a separating outcome using a different approach, again in a model with homogeneous sellers who are privately informed about their asset quality. They show that delay in a dynamic model plays a similar role to sale probabilities in our static setting. In their equilibrium, a sequence of short-lived buyers offer an increasing sequence of sale prices. Sellers with a low valuation sell quickly while those with a high valuation sell later, again dissipating some of the gains from trade. We show that the same dissipation of rents can occur in a static environment through an endogenous shortage of buyers at high prices.

Still other papers have developed models of adverse selection in which all trade occurs at a single price. In some of these papers, such as Eisfeldt (2004) and Kurlat (2013), investors are not allowed to consider trading at a different price. In other papers, such as Tirole (2012) and Chiu and Koeppl (2011), the equilibrium is characterized by a pooling price for traded assets.

The paper proceeds as follows. Section 2 lays out the basic model. In Section 3 we define our notion of equilibrium with multidimensional private information and establish by construction that our model exhibits a continuum of equilibria, including the semi-separating and one-price equilibria. In Section 4 we define an equilibrium with unidimensional private information, which imposes that sellers with identical preferences behave identically and that buyers believe that they will behave identically. We also establish uniqueness of the semi-separating equilibrium under this restricted notion of equilibrium. In Section 5 we explore whether equilibria can be Pareto ranked and show that a semi-separating equilibrium may be Pareto dominated by an incentive-feasible allocation, but not by any other equilibrium. This is because, even though sellers may pool, buyers cannot cross-subsidize each other in equilibrium. Section 6 concludes with a discussion of why the notion of equilibrium may be important.

2 Model

The economy lasts for two periods, $t = 1, 2$. It is populated by a unit measure of risk-neutral investors. A typical investor $i \in [0, 1]$ has a discount factor $\beta_i \geq 0$ and is endowed with $e_i \geq 0$ units of the period 1 consumption good and $a_i \geq 0$ units of an asset that produces the period 2 consumption good as a dividend in period 2. Assets are heterogeneous in their dividend. If $a_i > 0$, let $\delta_i \geq 0$ denote the amount of the period 2 consumption good that each unit of i's asset produces.[1] Both consumption goods and assets are divisible. Consumption must be nonnegative in each period.

At the beginning of period 1, each investor privately observes his type, that is, his discount factor β_i and his endowment (e_i, a_i, δ_i). Next, there is a market in which period 1 consumption goods and assets are exchanged. We refer to an investor with $e_i > 0$ as a potential buyer and an investor with $a_i > 0$ as a potential seller and suppress the word "potential" in the remainder of the paper. We allow for the possibility, but do not require, that some investors are both buyers and sellers. We assume that an investor can only buy assets using the period 1 consumption good that he holds at the start of the period, and so must consume any period 1 consumption goods he gets from selling his asset.[2] After the market meets, investors consume any remaining period 1 consumption good, $c_1 \geq 0$. In period 2, each investor consumes the dividends generated by the assets he holds in that period, $c_2 \geq 0$. An investor with discount factor β seeks to maximize $\mathbb{E}(c_1 + \beta c_2)$, where expectations recognize that the investor may be uncertain about whether he will succeed in buying and selling assets and about the quality of the assets that he buys.

The identity of individual investors is unimportant for our analysis, only the distribution of goods and assets across investors with different preferences. Let $G_b(\beta) \equiv \int_0^1 \mathbb{I}(\beta_i \leq \beta) e_i di$ with closed support $B \subset \mathbb{R}_+$ denote the initial measure of the period 1 consumption good held by investors with discount factor smaller than β, where $\mathbb{I}$ is an indicator function, equal to 1 if its argument is true and zero otherwise. Let $G_s(\beta, \delta) \equiv \int_0^1 \mathbb{I}(\beta_i \leq \beta \cap \delta_i \leq \delta) a_i di$ with closed support $S \subset \mathbb{R}_+^2$ denote the initial measure of assets with dividend less than δ held by investors who have a discount factor less than β. It will also be useful to define a seller's continuation value per unit of asset that is not sold, $v \equiv \beta\delta$. Let $H(v) \equiv \int_0^1 \mathbb{I}(\beta_i \delta_i \leq v) a_i di$ with closed support $V \subset \mathbb{R}_+$ denote the measure of assets held by sellers with continuation value less than v. It is useful to define the lowest continuation value, $\underline{v} \equiv \min V$. For expositional convenience, we assume that G_b, G_s, and H are atomless and let g_b, g_s, and h denote the associated densities.

[1] We assume for notational convenience that an investor only holds one type of asset.

[2] Other assumptions are possible here. While they would change some of our calculations, we do not believe that changing this "consumption-good-in-advance" constraint would alter our main results.

Finally, let $\Gamma : V \to \mathbb{R}_+$ denote the expected dividend conditional on an investor's continuation value v. It is straightforward to prove that

$$\Gamma(v) \equiv \frac{\int g_s\left(\frac{v}{\delta}, \delta\right) d\delta}{\int \frac{1}{\delta} g_s\left(\frac{v}{\delta}, \delta\right) d\delta},$$

We focus our analysis on the case where the following restriction holds:[3]

Assumption 1 Γ *is continuous and increasing.*

Note that Γ depends on G_s and so this is an assumption on a primitive model object. We also believe that this is a natural assumption: knowing that a seller's continuation value $\beta\delta$ is slightly higher leads us to conclude that her asset quality δ is slightly higher. Not surprisingly, it is easy to find distribution functions that satisfy this restriction.

Two concrete examples may help to illuminate this assumption. Suppose β and δ have independent Pareto distributions, $G_s(\beta, \delta) = (1-\beta^{-\alpha_\beta})(1-\delta^{-\alpha_\delta})$ on $[1, \infty)^2$ for some positive constants α_β and α_δ. Then

$$H(v) = 1 - \frac{\alpha_\beta v^{-\alpha_\delta} - \alpha_\delta v^{-\alpha_\beta}}{\alpha_\beta - \alpha_\delta} \text{ and } \Gamma(v) = \frac{(\alpha_\beta - \alpha_\delta)(v^{\alpha_\beta - \alpha_\delta + 1} - 1)}{(\alpha_\beta - \alpha_\delta + 1)(v^{\alpha_\beta - \alpha_\delta} - 1)},$$

both continuous and increasing on $[1, \infty)$.

Alternatively, suppose $G_s(\beta, \delta) = \beta^{\alpha_\beta}\delta^{\alpha_\delta}$ on $[0, 1]^2$ for some positive constants α_β and α_δ. Then

$$H(v) = \frac{\alpha_\beta v^{\alpha_\delta} - \alpha_\delta v^{\alpha_\beta}}{\alpha_\beta - \alpha_\delta} \text{ and } \Gamma(v) = \frac{(\alpha_\delta - \alpha_\beta)\left(1 - v^{\alpha_\delta - \alpha_\beta + 1}\right)}{(\alpha_\delta - \alpha_\beta + 1)\left(1 - v^{\alpha_\delta - \alpha_\beta}\right)},$$

again both continuous and increasing on $[0, 1]$.

3 Multidimensional Private Information

This section defines and characterizes equilibrium with multidimensional private information. We describe two such equilibria and explain how to construct many more. Despite the multiplicity of equilibrium, our structure puts some restriction on outcomes, and so we conclude the section by discussing those.

[3] Much of the analysis in Chang (2014) is focused on environments in which Γ is not monotonic.

3.1 Definition of Equilibrium

We start by developing our notion of equilibrium. During the first period, a continuum of markets, each characterized by a nonnegative price, opens up. Each buyer has to decide whether to consume his endowment of the period 1 consumption good or to use it to try to buy assets and, if he tries to buy assets, he has to decide at which price, $p_b(\beta)$. Each seller has to decide whether to try to sell his assets or not and, if he sells, he has to decide at which price, $p_s(\beta, \delta)$. Each unit of asset and each unit of the period 1 consumption good can be brought to only one market, so an effort to sell (or buy) an asset at a price p is also a commitment not to sell (or buy) the asset at any other price.[4]

In making their decisions, investors must form beliefs about the trading probability and the type of assets for sale at any nonnegative price, even those not offered in equilibrium. Let $\Theta(p) \in \mathbb{R}_+ \cup \infty$ denote the market tightness associated with price p, that is, the ratio of the amount of the consumption good that buyers want to use to buy at price p, relative to the cost of the assets that sellers want to sell at price p. If $\Theta(p) < 1$, there are not enough goods to buy all the assets for sale at price p and the sellers are randomly rationed. If instead $\Theta(p) > 1$, there are more goods than needed to buy all the assets for sale at price p and the buyers are randomly rationed. Specifically, a seller who attempts to trade at price p expects to sell with probability $\min\{\Theta(p), 1\}$, or equivalently to sell a fraction $\min\{\Theta(p), 1\}$ of his assets. Similarly, a buyer who attempts to trade at price p expects to buy with probability $\min\{\Theta(p)^{-1}, 1\}$, or equivalently to use a fraction $\min\{\Theta(p)^{-1}, 1\}$ of his goods to buy assets. A seller who is rationed keeps his assets and in period 2 consumes the dividend produced by it. A buyer who is rationed consumes his period 1 consumption goods.

In addition, let $\Delta(p)$ denote buyers' belief about the average dividend among the assets offered for sale at a price p. If some assets are sold at a price p, these beliefs must be consistent with the quality of assets offered for sale. Our definition of equilibrium also rules out equilibria sustained by unreasonable beliefs about the quality of assets for sale in markets that are inactive. Our definition builds on our prior work (Guerrieri, Shimer and Wright, 2010), which in turn builds on earlier research, most notably Wilson (1980), Gale (1996), and Ellingsen (1997).

Definition 1 *An equilibrium with multidimensional private information is four functions* $p_s : S \to \mathbb{R}_+$, $p_b : B \to \mathbb{R}_+$, $\Theta : \mathbb{R}_+ \to \mathbb{R}_+ \cup \infty$, *and* $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying the following conditions:*

[4] We again assume for notational convenience that each investor must choose a single buy price and a single sell price. Allowing an investor to divide his assets or consumption good and attempt to trade at different prices would not affect the set of equilibria.

1. *Optimal Selling Decision: given* Θ*, for all* $(\beta, \delta) \in S$

$$p_s(\beta, \delta) \in \arg\max_{p \geq \beta\delta} \big(\min\{\Theta(p), 1\}(p - \beta\delta) \big);$$

2. *Optimal Buying Decision: given* Θ *and* Δ*, for all* $\beta \in B$

$$p_b(\beta) \in \arg\max_{p \geq 0} \left(\min\{\Theta(p)^{-1}, 1\} \left(\frac{\beta\Delta(p)}{p} - 1 \right) \right);$$

3. *Beliefs: For all* $p \in \mathbb{R}_+$ *with* $\Theta(p) < \infty$*,*

   (a) *if there exists a* $(\beta, \delta) \in S$ *with* $p_s(\beta, \delta) = p$*,* $\Delta(p) = \mathbb{E}(\delta | p_s(\beta', \delta') = p)$*; otherwise*

   (b) *there exists a* $(\beta_1, \delta_1), (\beta_2, \delta_2) \in S$ *with* $p \geq \max\{\beta_1\delta_1, \beta_2\delta_2\}$*,* $\delta_1 \leq \Delta(p) \leq \delta_2$*, and*

$$p = \arg\max_{p' \geq \beta_1\delta_1} \big(\min\{\Theta(p'), 1\}(p' - \beta_1\delta_1) \big) = \arg\max_{p' \geq \beta_2\delta_2} \big(\min\{\Theta(p'), 1\}(p' - \beta_2\delta_2) \big);$$

4. *Market Clearing: for all* $p \geq 0$*,* $d\mu_b(p) = \Theta(p)\, d\mu_s(p)$*, where*

$$\mu_s(p) \equiv \iint_{p_s(\beta,\delta) \leq p} g_s(\beta, \delta)\, d\delta\, d\beta \text{ and } \mu_b(p) \equiv \int_{p_b(\beta) \leq p} \frac{g_b(\beta)}{p_b(\beta)}\, d\beta$$

   are the measure of assets for sale at prices below p *and the purchasing power of goods at prices below* p*. Moreover, if there exists a* $(\beta, \delta) \in S$ *with* $p_s(\beta, \delta) = p$ *and* $\Theta(p) > 0$*, then there exists a* $\beta' \in B$ *with* $p_b(\beta') = p$*; and if there exists a* $\beta \in B$ *with* $p_b(\beta) = p$ *and* $\Theta(p) < \infty$*, then there exists a* $(\beta', \delta') \in S$ *with* $p_s(\beta', \delta') = p$*.*

The first condition requires that sellers set optimal prices given their beliefs about the difficulty of selling at each price. Each seller (β, δ) sets a price p for her asset, recognizing that she will only succeed in selling with probability $\min\{\Theta(p), 1\}$, or equivalently only sells this fraction of her assets.[5] She gets p units of the consumption good per unit of asset sold in period 1 but gives up δ units of the consumption good in period 2, which she values at $\beta\delta$. If she fails to sell, she gains nothing. We also impose the restriction that sellers never set a price below their continuation value $\beta\delta$, since such a strategy is weakly dominated.

The second condition requires that buyers set optimal prices given their beliefs about the difficulty of buying at each price and the quality of assets available at each price. Each buyer β sets a price p for buying assets, recognizing that he will only succeed in buying with probability $\min\{\Theta(p)^{-1}, 1\}$, or equivalently only buys using this fraction of his period

[5]There is no loss of generality in assuming that she attempts to sell the asset. Attempting to sell at any price $p \geq \beta\delta$ always weakly dominates not selling the asset.

1 consumption good. He gets $1/p$ units of assets per unit of the consumption good, each of which produces an expected dividend $\Delta(p)$ next period.[6] If he fails to buy, he gains nothing.

The third condition imposes restrictions on buyers' beliefs. In particular, condition 3(a) imposes that buyers' beliefs about asset quality are consistent with the observed trading patterns whenever possible. If at least one seller sets a price p, then the expected dividend must be the average among the sellers who set that price.

Condition 3(b) describes beliefs at prices that nobody sets, a refinement in the vein of the intuitive criterion (Cho and Kreps, 1987) or divinity (Banks and Sobel, 1987). We require that buyers must be able to rationalize the expected dividend as coming from some probability distribution over sellers, each of whom finds this price weakly optimal. This means that there must either be some seller $(\beta, \Delta(p))$ with $p \geq \beta\Delta(p)$ who finds it optimal to set the price p, or that there must be both a seller with a higher quality asset and a seller with a lower quality asset who find this price optimal. In the latter case, appropriate weights on those two sellers justify the expectation $\Delta(p)$.[7]

One way to think about condition 3(b) is to imagine what would happen if a single buyer set a price p that was not previously set in the market. Some sellers would respond by offering some assets at that price, driving down the buyer-seller ratio until some investors are indifferent between p and another price and no investor finds p strictly optimal. The assumption states that buyers believe that if they purchase at this price, they will not buy from some combination of the sellers who find this price weakly optimal.

The fourth condition imposes market clearing. It requires that the buyer-seller ratio $\Theta(p)$ at any price p is equal to the ratio of the measure of the purchasing power of buyers at price p to the measure of sellers selling at that price. The last piece of this condition ensures that this holds even if both measures are zero, yet a finite number of buyers or sellers sets price p. For notational convenience alone, we do not impose that the buyer-seller ratio is exactly equal to $\Theta(p)$ in this case.

3.2 Partial Characterization

Equilibrium imposes some restrictions on behavior. The first observation is that in order for some sellers to be willing to set a high price and others to be willing to set a low price, there must be a trade-off between the selling price and the selling probability. Moreover,

[6] In any equilibrium with trade, $\Theta(p) = \infty$ at sufficiently low prices p. Therefore buyers can always be sure to consume in period 1 by setting a low price and so we do not give buyers the explicit option not to buy.

[7] In our previous research (Guerrieri, Shimer and Wright, 2010; Guerrieri and Shimer, 2014), the analogous condition defined a probability distribution over seller types at each price p. None of the results in this paper would change if we used that definition. We adopt this one for its notational simplicity.

sellers with different continuation values perceive the trade-off differently. If a seller with some continuation value prefers the low price to the high price, then any seller with a lower continuation value must have the same preferences. This leads to our first proposition:

Proposition 1 *Consider an equilibrium with multidimensional private information. Take any seller who sells with a positive probability. Then any other seller with a lower continuation value sells with a weakly higher probability at a weakly lower price.*

The second observation is that buyers' behavior is determined simply by the value they place on period 2 consumption. Patient buyers buy, impatient buyers don't buy, and the marginal buyer is indifferent about buying all the assets. This implies that assets are priced using the preferences of the marginal buyer:

Proposition 2 *Consider an equilibrium with multidimensional private information. There is a marginal buyer with discount factor $\hat{\beta}$ who is indifferent about paying any price at which assets are sold. All buyers who are more patient use all their period 1 consumption good to buy assets and are indifferent about which price they pay. All buyers who are less patient do not buy assets.*

We prove these propositions in Appendix B.

Equilibrium with multidimensional private information imposes some other restrictions on behavior. For example, suppose a range of sellers with different continuation values pool at a common price. Then a seller with a slightly lower continuation value must set a discretely lower price; otherwise buyers would prefer to buy from the pool. The seller must also trade with a discretely higher probability; otherwise she would prefer setting the pooling price. Symmetrically, a seller with a slightly higher continuation value must set a discretely higher price; otherwise buyers would prefer to buy from this seller rather than the pool. And the sale probability must be discretely lower; otherwise sellers in the pool would prefer setting this price. Still, these restrictions are quite weak. We illustrate this by discussing some of the possible equilibria in the remainder of this section.

3.3 Semi-Separating Equilibrium

Suppose Assumption 1 holds and there are gains from trade with the average seller who has the lowest continuation value. Appendix A.1 makes this "gains from trade" condition precise and then characterizes an equilibrium with multidimensional private information in which sellers set different prices if and only if they have different continuation values. We call such an equilibrium semi-separating (rather than separating) to emphasize that sellers with the same continuation value have different preferences and hold different quality assets. In a

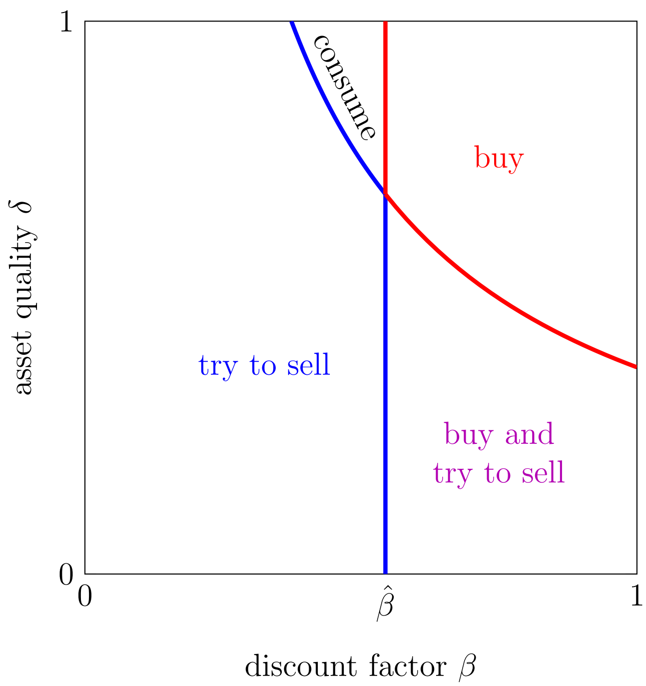

Figure 1: Behavior in a semi-separating equilibrium.

semi-separating equilibrium, heterogeneous assets sell at the same price and the same asset sells at heterogeneous prices. These outcomes are a joint consequence of the heterogeneity in sellers' preferences and the presence of multidimensional private information.

In a semi-separating equilibrium, sellers set a price that is strictly increasing in their continuation value, $p_s(\beta, \delta) = P(\beta\delta)$. They perceive a cost of setting a higher sale price, the shortage of buyers at high prices. That is, $\Theta(p)$ is decreasing. The single crossing property that drives Proposition 1 ensures that sellers with higher continuation values set higher prices, because they are less concerned with the risk of failing to sell their assets. The difference in seller continuation values across sale prices then delivers a steep enough relationship between expected asset quality and sale price, $\Delta(p)$, so as to leave buyers indifferent about the price they pay, consitent with Proposition 2.

Figure 1 illustrates investors' behavior in the semi-separating equilibrium. Investors are divided into four groups. Patient investors with high quality assets buy other assets. Impatient investors with low quality assets try to sell their asset. There are also patient investors with low quality assets who try to sell their asset and buy other assets; and somewhat impatient investors with high quality assets who neither buy nor sell asset but simply consume their endowment in each period.

3.4 One-Price Equilibrium

We can also construct an equilibrium with multidimensional private information in which all trade takes place at a single price. In this equilibrium, sellers perceive a simple choice: they can sell for sure at p^* or they cannot sell. From the perspective of a buyer, the quality of assets available at prices above p^* does not justify the higher price, and so buyers are only willing to buy at p^*. The existence of this equilibrium imposes some restrictions on the support of the seller's type distribution S; see Appendix A.2 for details.

Our construction of the one-price equilibrium ensures that some seller sets every price between p^* and the highest continuation value in the seller population. By doing so, we avoid imposing any restrictions on buyers' off-the-equilibrium-path beliefs. For this reason, a one-price equilibrium with multidimensional private information is robust to standard equilibrium refinements based on forward inducation (Cho and Kreps, 1987; Banks and Sobel, 1987).

Eisfeldt (2004) and Kurlat (2013) assume that all trade occurs at price p^*. They restrict trading opportunities so a seller has no technology for selling his asset at a price different than p^*. We allow sellers to set such prices, yet all trade occurs at p^* in a one-price equilibrium. Our approach clarifies that the existence of a one-price equilibrium is sensitive to buyers' beliefs $\Delta(p)$ at prices $p > p^*$. It might be most natural to think that all sellers with continuation value just above p^* set a price just above p^*. If that were the case, and Assumption 1 holds, buyers would anticipate being able to purchase an asset with expected quality just above $\Gamma(p^*)$ at such prices. Since the expected quality of assets for sale at p^*, $\Delta(p^*)$, is discretely less than this—it is the average quality of assets held by sellers with continuation values less than or equal to p^*—buyers would find it more profitable to pay this higher price, breaking the one-price equilibrium.

Instead, we support the one-price equilibrium through buyers' belief that sellers with a continuation value just above p^* will set a price just above p^* only if they have the lowest quality asset consistent with the continuation value. This pushes down buyers' beliefs and supports the equilibrium. Moreover, by construction these beliefs are consistent with sellers' actual behavior in equilibrium; some sellers do set a price just above p^*, justifying these beliefs.

3.5 Other Equilibria

Once one understands how to construct the one-price equilibrium, it is easy to construct many other equilibria. For example, we show in Appendix A.3 that our model admits a continuum of one-price equilibria, each characterized by a sale price p^*, a marginal buyer

$\hat{\beta}$, and a sale probability $\theta^* < 1$. At lower prices, the sale probability is higher than θ^*, eventually reaching 1 at some $\underline{p} < p^*$. The sale probability $\Theta(p)$ in this interval leaves the seller with the lowest continuation value indifferent about charging any price $p \in [\underline{p}, p^*]$ and induces sellers with higher continuation values to set the equilibrium price p^*. Buyers do not prefer buying at a higher price because they believe that they will only encounter sellers with low quality assets relative to their continuation value, as we have discussed above. They also do not prefer buying at a lower price because they again anticipate getting lower quality assets, lower than the average quality held by sellers with the lowest continuation value.

Building on this logic, we show in Appendix A.3 that our model can also admit a continuum of equilibrium with n prices for any positive n. It may also exhibit a continuum of semi-separating equilibria, again distinguished by the highest selling price (and lowest selling probability) of an investor with the lowest continuation value. The behavior that supports these equilibria is similar to that which supports the other one-price equilibria.

Finally, we show in Appendix A.4 that our model also admits equilibria that combine some pooling prices that attract a positive measure of buyers and sellers with some intervals where sellers with different continuation values set different prices. We call these "mixed equilibria." We use limiting versions of these equilibria, with very small mass points, in our normative analysis in Section 5, treating them as perturbations of the semi-separating equilibrium.

4 Unidimensional Private Information

In this section, we propose a more restrictive equilibrium definition that effectively collapses private information to one dimension. We characterize such an equilibrium and, extending the results in our previous work, we show that it is unique.

4.1 Definition of Equilibrium

An equilibrium with unidimensional private information imposes that all sellers with the same preference ordering over lotteries set the same selling price and that all buyers believe that such sellers always do so:

Definition 2 *An equilibrium with unidimensional private information is an equilibrium with multidimensional private information* $\{p_s, p_b, \Theta, \Delta\}$ *satisfying*

1. *Seller Behavior: for all* $(\beta, \delta), (\beta', \delta') \in S$ *with* $\beta\delta = \beta'\delta'$, $p_s(\beta, \delta) = p_s(\beta', \delta')$*; and*

2. Buyer Belief: for all $p \in \mathbb{R}_+$ with $\Theta(p) < \infty$, there exists a $v_1, v_2 \in V$ with $p \geq \max\{v_1, v_2\}$, $\Gamma(v_1) \leq \Delta(p) \leq \Gamma(v_2)$, and

$$p = \arg\max_{p' \geq v_1} \big(\min\{\Theta(p'), 1\}(p' - v_1) \big) = \arg\max_{p' \geq v_2} \big(\min\{\Theta(p'), 1\}(p' - v_2) \big).$$

We tighten the definition of equilibrium with multidimensional private information in two ways. First, we modify condition 1 by restricting sellers with the same continuation value to set the same selling price. Second, we modify condition 3(b) by imposing that buyers believe that sellers with the same continuation value behave in the same manner. Both modifications are important for the uniqueness result that follows. For example, the construction of the one-price equilibrium with multidimensional private information in Section 3.4 relies on sellers with the same continuation value behaving differently in equilibrium. The first condition in the definition of equilibrium with unidimensional private information precludes this possibility. However, we could also support the same allocation with sellers with the same continuation value setting the same equilibrium price, but buyers not believing that this is the case at prices that no one sets in equilibrium. The second condition precludes this possibility.

The definition of equilibrium with unidimensional private information might be appealing because it imposes that sellers with the same cardinal preferences over prices behave and are expected to behave in the same way. For example, Chang (2014) defines a seller's type to be her continuation value, rather than the separate components (β, δ). This hard-wires the restriction into her analysis. However, we believe the notion of equilibrium with multidimensional private information is useful for several reasons.

First, we prove in Proposition 3 below that there is a unique equilibrium with unidimensional private information and so multiple equilibria are intimately connected to multidimensional private information. Many of our normative results concern welfare comparisons across equilibria, and so having multiple equilibria is central to these results.

Second, defining a seller's type to be her continuation value v rather than the separate components (β, δ) obscures the economics of the model. In an Arrow-Debreu economy, individuals' preferences (β) and endowments (δ) are distinct concepts. Private information does not eliminate the distinction between preferences and endowments and there is no good reason why it should. This distinction is important for a number of issues. For example, we are interested in understanding the extent to which markets transfer assets from low value (low β) sellers to high value (high β) buyers and whether inefficient trades from high value sellers to low value buyers can occur in equilibrium. We are also interested in understanding whether high or low quality assets are more likely to trade in equilibrium. Neither of these questions is naturally asked when the seller's type is defined directly as her continuation value and the buyer's value is simply assumed to be $\Gamma(v)$.

Third, and most fundamentally, there is no good theoretical justification for imposing that all individuals with the same preferences behave the same. For example, equilibrium often requires that buyers with the same preferences pay different prices. A restriction that any two investors with the same preferences over lotteries behave the same would generally preclude the existence of equilibrium.

4.2 Unique Equilibrium

Aware of those concerns, we still think it is useful to characterize an equilibrium with unidimensional private information. In particular, we prove that such an equilibrium is unique:

Proposition 3 *Under Assumption 1, there exists a unique equilibrium with unidimensional private information. In particular,*

1. *if* $\beta\Gamma(\underline{v}) \leq \underline{v}$ *for all* $\beta \in B$*, the equilibrium features no trade at any positive price;*

2. *otherwise, the equilibrium features trade and is semi-separating: sellers with higher continuation value sell at a higher price with lower probability;*

We prove this result in Appendix B. The proposition extends uniqueness results in our earlier work (Guerrieri, Shimer and Wright, 2010; Guerrieri and Shimer, 2014), but the proof strategy is completely different. This is because those papers assumed that there were a finite number of types of sellers and single type of buyer, while here we allow for a continuum of "types" v as well as heterogeneous buyers.

When $\beta\Gamma(\underline{v}) \leq \underline{v}$ for all $\beta \in B$, there are no gains from trade with the seller who has the lowest continuation value. In this case, the unique equilibrium with unidimensional information features no trade. If there were trade of any other type of asset, the owners of the worst asset would pretend to own such an asset and break the equilibrium. Otherwise, the unique equilibrium with unidimensional private information is equivalent to the semi-separating one we described in Section 3.3. The seller with the lowest continuation value sells for sure at a low price, while sellers with higher continuation values, up to some threshold $\bar{p}$, sell with lower probabilities at higher prices. Sellers with still higher continuation values fail to sell their assets. Buyers are willing to pay heterogeneous prices because they expect to get higher quality assets when they pay a higher prices.

4.3 The Role of Beliefs

We comment briefly on the role of off-the-equilibrium path beliefs in the definition of an equilibrium. Consider a relaxed version of the definitions. Rather than part 3(b) in the definition of equilibrium with multidimensional private information and the second part of the definition of equilibrium with unidimensional private information (the analog of 3(b)), we require only that if no seller sets a price p, there must be a $v \in V$ with $\Gamma(v) = \Delta(p)$. That is, we require that buyers have some belief about the seller who offers an off-equilibrium price p, but do not require that they believe it is the seller with the strongest incentive to do so. This alternative assumption opens the door to additional equilibria. For example the one-price equilibrium in Section 3.4 would be an equilibrium with unidimensional private information, according to this relaxed definition.

How does the set of allocations consistent with this relaxed version of equilibrium with unidimensional private information compare to the set of allocations consistent with equilibrium with multidimensional private information? In general, it is easiest to support a particular allocation by giving buyers the most pessimistic "off-the-equilibrium-path" beliefs abut sellers' asset quality. In the relaxed unidimensional private information problem, this means buyers believe that $\Delta(p) = \Gamma(\underline{v})$ at any price not offered in equilibrium. In the multidimensional private information problem, buyers believe $\Delta(p)$ is equal to the lowest asset quality among the sellers who find price p to be weakly optimal.[8]

These two beliefs are not the same, and so in general the sets of allocations are different. To be concrete, consider the other semi-separating equilibria with multidimensional private information mentioned in Section 3.5. In these equilibria, there is a one-to-one mapping between a seller's continuation value and the price she sets, $p_s(\beta, \delta) = P(\beta\delta)$; however, in contrast to the usual semi-separating equilibrium, even sellers with the lowest continuation value are rationed. This is because buyers believe that if they pay less than $P(\underline{v})$, they will get an asset quality below the average of the sellers with the lowest continuation value, $\Gamma(\underline{v})$. Such beliefs are inconsistent with the relaxed version of equilibrium with unidimensional private information, where the worst possible belief is $\Delta(p) = \Gamma(\underline{v})$, so the hypothetical seller is average among those with the lowest continuation value.

Conversely, we find that if the support of the buyer type distribution is an interval B and the support of the seller type distribution is a rectangle $S = B \times D$ for some interval D, then any relaxed equilibrium with unidimensional private information is also an equilibrium with

[8] In the multidimensional private information problem, beliefs need not be off-the-equilibrium-path. Instead, sellers may actually offer all prices in equilibrium. See the again the construction of the one-price equilibrium with multidimensional private information in Appendix A.2.

multidimensional private information.[9] If for some continuation values, the upper bound on the support of the seller's discount factor is less than the highest buyer type, we can reverse this conclusion as well.

5 Efficient Allocations

This section explores the efficiency of the equilibria of our model. We address this in two ways. First, given the plethora of equilibria with multidimensional private information, we ask whether the equilibria can be Pareto ranked. In particular, we examine whether pooling some or all of the sellers can improve welfare relative to the semi-separating equilibrium. Second, we ask whether other incentive feasible allocations—that is, allocations that respect the investors' private information and the economy's resource constraint—Pareto dominate the semi-separating equilibrium. We find that this may be the case, even when the semi-separating equilibrium is not Pareto dominated by another equilibrium. This highlights some of the constraints that equilibrium places on allocations.

We focus most of our discussion in the text on a particular example and leave a more general analysis, including a formal definition of the set of incentive feasible allocations, to Appendix C. We assume β and δ have independent Pareto distributions, $G_s(\beta,\delta) = (1-\beta^{-\alpha-1})(1-\delta^{-\alpha})$ on $[1,\infty)^2$ for some positive constant α. This implies

$$H(v) = 1-(\alpha+1)v^{-\alpha}+\alpha v^{-\alpha-1} \quad \text{and} \quad \Gamma(v) = \frac{1+v}{2}.$$

In the semi-separating equilibrium, the seller's trading probability depends on the identity of the marginal buyer $\hat{\beta}$, but is independent of the parameter α:

$$\Theta(P(v)) = \begin{cases} \left(\frac{\hat{\beta}+v(\hat{\beta}-2)}{2(\hat{\beta}-1)}\right)^{\frac{\hat{\beta}}{2-\hat{\beta}}} & \hat{\beta}>2 \text{ or } \left[\hat{\beta}<2 \text{ and } v<\hat{\beta}/(2-\hat{\beta})\right] \\ 0 \qquad\qquad \text{if} & \hat{\beta}<2 \text{ and } v\geq \hat{\beta}/(2-\hat{\beta}) \\ e^{1-v} & \hat{\beta}=2. \end{cases}$$

The identity of the marginal buyer depends on the distribution of the consumption good

[9] Consider, for example, the one price equilibrium, where all sellers with $v \leq p^*$ sell for p^* and the marginal buyer has a discount factor $\hat{\beta}$. To support this allocation, it is sufficient to show that buyers believe that they will get an asset with dividend less than $\Delta(p^*)$ when they pay a higher price. In a relaxed equilibrium with unidimensional private information, buyers believe the dividend is $\Gamma(\underline{v})$; this is smaller than $\Delta(p^*)$, since Γ is increasing and $\Delta(p^*)$ is equal to the average value of $\Gamma(v)$ among all $v \leq p^*$. In an equilibrium with multidimensional private information, buyers believe the dividend is $p^*/\max B$, the worst asset held by a seller with continuation value p^* (using the seller's full support assumption). This again must be smaller than $\Delta(p^*)$, because buyers' indifference condition requires $\Delta(p^*) = p^*/\hat{\beta}$ and $\hat{\beta} \leq \max B$.

across buyers $G_b(\beta)$. Through a judicious choice of G_b, we can set this equal to any desired value. For this reason, we treat $\hat{\beta}$ parametrically in this section.

5.1 Semi-Separating versus One-Price Equilibrium

A cost of the semi-separating equilibrium is that a seller with a high continuation value must convince a buyer that she has a high quality asset by undertaking an inefficient activity, setting a price that ensures she is unlikely to sell her good. In contrast, in the one-price equilibrium, any seller who wants to sell at that price can sell for sure. We know from Rothschild and Stiglitz (1976) and many other papers that pooling can be Pareto superior to separating. The cost of pooling is that sellers with low continuation values are cross-subsidized, while the benefit is that it eliminates the wasteful signal. Since in this example, there are very few sellers with a low continuation value—formally, $H(1) = H'(1) = 0$—one might expect that the one-price equilibrium Pareto dominates the semi-separating equilibrium. It turns out that this is not the case.

This is easiest to see if the marginal buyer has a high discount factor, $\hat{\beta} \geq 2$, in the semi-separating equilibrium. In that case, all sellers sell with some probability in the semi-separating equilibrium, but not in the one-price equilibrium. Any seller whose continuation value exceeds the one price p^* prefers the semi-separating equilibrium, in which she trades, to the one-price equilibrium, in which she lives in autarky.

The same is true if the identity of the marginal buyer in the semi-separating equilibrium is lower, $\hat{\beta} < 2$, so all sellers with a continuation value below $\hat{\beta}/(2 - \hat{\beta})$ sell with positive probability. The proof is a bit more cumbersome. First, it is straightforward to prove that all buyers are better off in any equilibrium where the identity of the marginal buyer is lower. Therefore, the identity of the marginal buyer in the one-price equilibrium must be weakly lower than the identity of the marginal buyer in the semi-separating equilibrium, say $\hat{\beta}_p \leq \hat{\beta}$, if the former is to Pareto dominate the latter. Next, we can prove algebraically that if $\hat{\beta}_p \leq \hat{\beta}$, the one-price equilibrium price, p^*, is strictly less than $\hat{\beta}/(2 - \hat{\beta})$. Sellers with continuation values in the open interval $(p^*, \hat{\beta}/(2-\hat{\beta}))$ again trade in the semi-separating equilibrium but live in autarky in the one-price equilibrium, and so must prefer the former to the latter, a contradiction.

Of course, it is possible to construct other examples where the one-price equilibrium Pareto dominates the semi-separating equilibrium.[10] There may be very little trade in the semi-separating equilibrium because buyers value the asset held by sellers with a low con-

[10] Convesely, it is easy to show that the semi-separating equilibrium does not Pareto dominate the one-price equilibrium for any buyer and seller distributions G_b and G_s for which both equilibria exist. Either the seller with the lowest continuation value or buyers or both prefer the one-price equilibrium.

tinuation value only a bit more than do the sellers. A one-price equilibrium may allow for trade at a much higher price, where the buyers' valuations significantly exceed the sellers'.

5.2 Perturbing the Semi-Separating Equilibrium

Even when the one-price equilibrium does not Pareto dominate the semi-separating equilibrium, like in our example, the basic idea that pooling sellers may be welfare enhancing seems plausible. In this section, we argue that, under mild assumptions, this idea is incorrect, at least if we consider only allocations that are consistent with our notion of equilibrium with multidimensional private information.

We consider smaller pools of sellers. For example, we could look at an equilibrium in which all sellers with continuation values in some interval, say $\beta\delta \in (v - \varepsilon, v + \varepsilon)$, set a common price. Sellers with other continuation values set a price that is strictly increasing in their continuation value. More generally, in Appendix A.4, we describe a *mixed equilibrium* as a set of points $v_1 < \cdots < v_n$, and non-overlapping pools of positive radius ε_1, ..., ε_n around those points. Any two sellers set the same price if and only if they have the same continuation value or are members of the same pool.[11] Our analysis in that Appendix shows that we can construct an equilibrium with multidimensional private information of this form under fairly general conditions; however equilibrium imposes some strong restrictions on prices and trading probabilities. Once we account for these restrictions, we find that such equilibria generally cannot Pareto dominate the semi-separating equilibrium.

The first restriction is that the pooling price must be fair given the quality of the assets sold in the pool. A fair price leaves buyers indifferent between buying at the pooling price or buying at a price offered by a different seller who is not part of the pool. This implies that a seller with continuation value $v_i - \varepsilon_i$ (respectively, $v_i + \varepsilon_i$) must set a price discretely below (respectively, above) the pooling price, $P(v_i - \varepsilon_i) < P(v_i) < P(v_i - \varepsilon_i)$, reflecting the difference in asset quality.

The second restriction is that the trading probabilities must give sellers an incentive to set the postulated price. This means that the sale probability must drop discretely at both ends of the pool, $\Theta(P(v_i - \varepsilon_i)) > \Theta(P(v_i)) > \Theta(P(v_i - \varepsilon_i))$, and the size of the drop must leave sellers with continuation value $v_i - \varepsilon_i$ and $v_i + \varepsilon_i$ just indifferent about setting the pooling price.

Although it is difficult to analyze the full scope of such equilibria, we consider the limit as the radius of the pools vanishes. That is, we look at small perturbations of the semi-separating equilibrium. We find that under weak conditions, weaker than our specific Pareto

[11] That is, for all (β, δ) and (β', δ'), $p_s(\beta, \delta) = p_s(\beta', \delta')$ if and only if $\beta\delta = \beta'\delta'$ or there exists an $i \in \{1, \ldots, n\}$ with $\beta\delta \in (v_i - \varepsilon_i, v_i + \varepsilon_i)$ and $\beta'\delta' \in (v_i - \varepsilon_i, v_i + \varepsilon_i)$.

example, no such equilibrium Pareto dominates the semi-separating equilibrium:

Proposition 4 *Impose Assumption 1 and that the elasticity of* Γ *is smaller than* 2*. There is no mixed equilibrium with arbitrarily small pools* $\{\varepsilon_i\}_{i=1}^n$ *that Pareto dominates the semi-separating equilibrium.*

We prove this result in Appendix B. We first show that if the identity of the marginal buyer is different in the mixed equilibrium, the two equilibria are not Pareto comparable. We then prove that if the marginal buyer were the same and all sellers were weakly better off in the mixed equilibrium, then the mixed equilibrium would require transferring more resources from buyers to sellers in period 1 and back again in period 2. This is inconsistent with a constant identity of the marginal buyer.

The condition on the elasticity of Γ being smaller than 2 is only a sufficient condition and we view it as a weak one. For example, if all sellers have the same discount factor, the elasticity is 1. In our Pareto example, the elasticity is always strictly smaller than 1 and so there is no Pareto improving perturbation of the semi-separating equilibrium. An elasticity larger than 2 means that a seller with twice as high a continuation value has an asset that is at least four times as good on average, and so on average a much lower continuation value.

5.3 Cross-Subsidizing Buyers

If pooling sellers is not welfare enhancing, does that mean that the semi-separating equilibrium is Pareto efficient? Within the set of equilibrium allocations, the only feasible deviation from the semi-separating equilibrium involves pooling sellers, so the answer is "yes." But the underlying economic environment allows for other possibile allocations. We find that it is sometimes feasible to raise welfare by cross-subsidizing buyers who pay different prices.

We return to the example to illustrate this point. Assume now that $\alpha = 3$ and $\hat{\beta} = 2$ in the semi-separating equilibrium.[12] We construct a Pareto improvement by creating two pools of sellers, one containing all individuals with $v \in [1, 1.01]$, the other containing all individuals with $v \in [8.3, 11.3]$. Every seller who is a member of a pool trades with the raw average value of $\Theta(P(v))$ for that group of sellers in the semi-separating equilibrium. Every seller who is not a member of a pool trades with the same probability as in the semi-separating equilibrium. Finally, the sale price ensures that each seller finds it optimal to trade at the appropriate price.

The proposed allocation is not an equilibrium because buyers are not indifferent about which price they pay, inconsistent with Proposition 2. It turns out that the quality of assets

[12]The identity of the marginal buyer $\hat{\beta}$ is endogenous. Such an equilibrium exists if and only if $\int_{\beta \geq 2} g_b(\beta) d\beta = 1.280 = \int_1^\infty P(v)\Theta(P(v))h(v)dv$, where $P(v) = \hat{\beta}\Gamma(v) = 1 + v$ and $\Theta(P(v)) = e^{1-v}$.

sold by the pool of low value sellers is high relative to the price of the pool, while the quality of assets sold by the pool of high value sellers is low relative to the price of the pool. In equilibrium, buyers get to choose their buying price and so no buyer would be willing to buy from the high value pool; however, we find that the marginal buyer would prefer buying a share of both pools rather than not participating in the market at all. That is, the proposed allocation requires cross-subsidizing buyers.

There is no inherent reason why cross-subsidizing buyers is infeasible, although it is ruled out by our notion of equilibrium. For example, in another environment, a buyer might be able to decide whether he wants to buy, but he is not able to decide on the purchase price. Instead, he buys a lottery ticket and is committed to buying whatever bundle the lottery instructs him to buy. The proposed allocation yields a sufficiently low average price that the marginal buyer is willing to buy a lottery ticket.

Whether allocating buyers to prices through lotteries is feasible depends on the commitment technology available. If the buyer can refuse to buy when he does not like the outcome of the lottery, this outcome may be unattainable. Instead, a Pareto improvement from the semi-separating equilibrium may require explicit cross-subsidization of buyers who are instructed to pay different prices.

Despite this example, there are also situations in which we can conclude that the semi-separating equilibrium is Pareto efficient. Establishing this requires a significant investment in new notation and so we place the relevant results in Appendix C, where we define the set of feasible allocations. Propositions 6 and 7 in the appendix give necessary and sufficient conditions for the semi-separating equilibrium to be Pareto efficient. In our example, we find that the semi-separating equilibrium is never Pareto efficient if the Pareto parameter is big, $\alpha > 2$, and the identity of the marginal buyer is large, $\hat{\beta} \geq 2$. Otherwise, Pareto efficiency hinges on the distribution of buyer preferences.

6 Conclusion

This paper develops and analyzes an environment in which sellers have multidimensional private information and buyers only care about one dimension of the private information. Our model delivers multiple equilibria which may sometimes be Pareto-rankable. Collapsing the private information to single dimension eliminates the equilibrium multiplicity and delivers a unique semi-separating equilibrium. In such an equilibrium, sellers who are eager to sell set a low price and sell with a high probability, while less motivated sellers set a high price and sell with a low or zero probability. Buyers are willing to pay a range of prices, knowing that they will be rewarded with high quality on average when they pay a high price. In particular, the marginal buyer is indifferent about buying any of the assets offered for sale. We also find that the semi-separating equilibrium can be Pareto inefficient even if no equilibrium Pareto dominates it. This is because efficient allocations may require cross-subsidization of uninformed buyers, not just informed sellers.

The type of equilibrium matters for two reasons that we have not yet discussed. The first concerns how private information is transmitted in a dynamic setting. Suppose, for example, that only the initial owner (say a mortgage originator) of an asset (a mortgage pool) can observe its quality, but future owners know the price they paid for the asset. In a one-price equilibrium, most information about asset quality is lost in the secondary market since prices are a coarse aggregator of information. In a semi-separating equilibrium, buyers receive more nuanced information since different buyers pay different prices. Therefore, in a semi-separating equilibrium in which past transactions prices are not observed by other market participants, private information can get transmitted to secondary asset markets. It would be interesting to explore this possibility both theoretically, in a dynamic extension of this model, and empirically, investigating the secondary market for certain mortgage-backed securities, such as those backed by low-documentation loans.

Second, the response to policy interventions is likely to depend on the nature of the equilibrium. For example, a small amount of bad assets can have a big effect on a semi-separating equilibrium, in an extreme case leading to a breakdown in trade. An asset purchase program that removes the lowest-quality assets from the market can then have a big impact on asset prices and trading volumes. In contrast, a small intervention is unlikely to substantially alter a one-price equilibrium, since the equilibrium by its nature depends on all the inframarginal traders. While our model is too stylized to be calibrated, we believe these observations are likely to carry over to a quantitatively serious model of private information.

Finally, the multiplicity of equilibria suggests new possibilities and new concerns for policy intervention. On the one hand, there may be policies that move the economy from one equilibrium to another by changing traders' beliefs about the two key equilibrium objects, the market tightness Θ and the average quality of available assets Δ. To the extent that the economy is trapped in a Pareto inefficient equilibrium, this opens the door to inexpensive welfare-enhancing interventions. On the other hand, any policy intervention must affect traders' beliefs and hence risks shifting beliefs towards an undesirable equilibrium. This points towards the need for an improved understanding of equilibrium selection in an environment with multidimensional private information.

References

Akerlof, George A., "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 1970, *84* (3), 488–500.

Banks, Jeffrey S. and Joel Sobel, "Equilibrium Selection in Signaling Games," *Econometrica*, 1987, *55* (3), 647–661.

Chang, Briana, "Adverse Selection and Liquidity Distortion," 2014. University of Wisconsin Mimeo.

Chari, V.V., Ali Shourideh, and Ariel Zetlin-Jones, "Reputation and Persistence of Adverse Selection in Secondary Loan Markets," *American Economic Review*, 2014, *104* (12), 4027–4070.

Chiu, Jonathan and Thorsten V. Koeppl, "Trading Dynamics with Adverse Selection and Search: Market Freeze, Intervention and Recovery," 2011. Bank of Canada Mimeo.

Cho, In-Koo and David M. Kreps, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 1987, *102* (2), 179–221.

Daley, Brendan and Brett Green, "Waiting for News in the Dynamic Market for Lemons," *Econometrica*, 2012, *80* (4), 1433–1504.

DeMarzo, Peter and Darrell Duffie, "A Liquidity-Based Model of Security Design," *Econometrica*, 1999, *67* (1), 65–99.

Demiroglu, Cem and Christopher James, "How Important is Having Skin in the Game? Originator-Sponsor Affiliation and Losses on Mortgage-backed Securities," *Review of Financial Studies*, 2012, *25* (11), 3217–3258.

Eisfeldt, Andrea L., "Endogenous Liquidity in Asset Markets," *Journal of Finance*, 2004, *59* (1), 1–30.

Ellingsen, Tore, "Price Signals Quality: The Case of Perfectly Inelastic Demand," *International Journal of Industrial Organization*, 1997, *16* (1), 43–61.

Gale, Douglas, "Equilibria and Pareto Optima of Markets with Adverse Selection," *Economic Theory*, 1996, *7* (2), 207–235.

Guerrieri, Veronica and Robert Shimer, "Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality," *American Economic Review*, 2014, *104*, 1875–1908.

— , — , and Randall Wright, "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 2010, *78* (6), 1823–1862.

Jiang, Wei, Ashlyn Aiko Nelson, and Edward Vytlacil, "Liar's loan? Effects of origination channel and information falsification on mortgage delinquency," *Review of Economics and Statistics*, 2014, *96* (1), 1–18.

— , — , and — , "Securitization and Loan Performance: A Contrast of Ex Ante and Ex Post Relations in the Mortgage Market," *Review of Financial Studies*, 2014, *27* (2), 454–483.

Keys, Benjamin J., Tanmoy Mukherjee, Amit Seru, and Vikrant Vig, "Did Securitization Lead to Lax Screening? Evidence from Subprime Loans," *Quarterly Journal of Economics*, 2010, *125* (1), 306–362.

Kurlat, Pablo, "Lemons Markets and the Transmission of Aggregate Shocks," *American Economic Review*, 2013, *103* (4), 1463–1489.

Maskin, Eric and Jean Tirole, "The Principal-Agent Relationship with an Informed Principal, II: Common Values," *Econometrica*, 1992, *60* (1), 1–42.

Piskorski, Tomasz, Amit Seru, and James Witkin, "Asset Quality Misrepresentation by Financial Intermediaries: Evidence from RMBS Market," 2015. Forthcoming, *Journal of Finance*.

Rothschild, Michael and Joseph Stiglitz, "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," *Quarterly Journal of Economics*, 1976, *90* (4), 629–649.

Tirole, Jean, "Overcoming Adverse Selection: How Public Intervention Can Restore Market Functioning," *American Economic Review*, 2012, *102* (1), 29–59.

Wilson, Charles, "The Nature of Equilibrium in Markets with Adverse Selection," *Bell Journal of Economics*, 1980, *11* (Spring), 108–130.

Appendix

A Examples of Equilibria

For the following analysis, it is useful to introduce some additional notation. In parallel with $\underline{v} = \min V$, let $\bar{v} = \max V$, $\underline{\beta} = \min B$, and $\bar{\beta} = \max B$.

A.1 Semi-Separating Equilibrium

Impose Assumption 1 and assume there exists a $\beta \in B$ with $\beta\Gamma(\underline{v}) > \underline{v}$. The semi-separating equilibrium is characterized by a discount factor for the marginal buyer, $\hat{\beta} \in B$, which is determined in equation (1) below. For now, fix $\hat{\beta}$ and assume, as we verify below, that $\hat{\beta}\Gamma(\underline{v}) > \underline{v}$.

We next define two critical prices. The lowest price with trade is $\underline{p} \equiv \hat{\beta}\Gamma(\underline{v})$, the value that the marginal buyer places on an asset sold by the seller with the lowest continuation value. Note that, given our assumption, $\underline{p} > \underline{v}$, so a seller with the lowest continuation value strictly prefers selling his asset for $\underline{p}$ rather than retaining it. The second critical price is the highest one with trade. Let $\bar{p}$ be the smallest price satisfying $\bar{p} = \hat{\beta}\Gamma(\bar{p})$, or $\bar{p} = \infty$ if there is no such price. That is, $\hat{\beta}\Gamma(v) > v$ whenever $v < \bar{p}$.

In the semi-separating equilibrium, the equilibrium buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & p < \underline{p} \\ \exp\left(-\int_{\underline{p}}^{p} \frac{1}{p' - \Gamma^{-1}(p'/\hat{\beta})} dp'\right) \text{ if } & p \in [\underline{p}, \bar{p}] \\ 0 & p > \bar{p}. \end{cases}$$

Facing this market tightness, the first part of the definition of equilibrium implies that any seller (β, δ) with continuation value $\beta\delta < \bar{p}$ maximizes his profit by setting the sale price $p_s(\beta, \delta) = \hat{\beta}\Gamma(\beta\delta)$. A seller (β, δ) with a higher continuation value cannot sell his asset at any price satisfying $p \geq \beta\delta$ and $\Theta(p) > 0$. Although such a seller is indifferent between all sale prices at which he cannot sell his asset, i.e. with $\Theta(p) = 0$, his behavior still matters in equilibrium since it influences buyers' beliefs. We assume that such an investor sets price $p_s(\beta, \delta) = \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$.

Turn next to the buyers' belief about the quality of asset offered at each price. At prices $p < \underline{p}$, buyers are unable to find sellers, $\Theta(p) = \infty$, and so beliefs are undefined. Intermediate prices, $p \in [\underline{p}, \bar{p}]$, are offered only by investors with continuation value $v = \Gamma^{-1}(p/\hat{\beta})$. Since the average quality asset held by these sellers is $\Gamma(v) = p/\hat{\beta}$, part 3(a) of the definition of equilibrium imposes $\Delta(p) = p/\hat{\beta}$ when $p \in [\underline{p}, \bar{p}]$. Finally, at still higher prices, $\Delta(p) \leq p/\hat{\beta}$. Such beliefs are rational, since by construction an investor with continuation value $v > \bar{p}$ always sets a price at least equal to $\hat{\beta}\Gamma(\beta\delta)$.[13]

Given these beliefs, we now use part 2 of the definition of equilibrium. An investor with discount factor $\beta > \hat{\beta}$ maximizes his profit by buying at any price $p \in [\underline{p}, \bar{p}]$, weakly prefers buying at those prices rather than any higher price, and strictly prefers buying at these prices rather than a lower price where there are no sellers. An investor with discount factor $\beta < \hat{\beta}$ prefers to offer a price $p < \underline{p}$, which ensures that he fails to buy in equilibrium.

The last piece of equilibrium is the determination of the marginal discount factor. In order to ensure that the supply of assets is equal to the demand, we require

$$\int_{\beta > \hat{\beta}} g_b(\beta) d\beta = \iint_{\beta\delta < \bar{p}} p_s(\beta, \delta)\Theta(p_s(\beta, \delta))g_s(\beta, \delta)\, d\beta d\delta \tag{1}$$

The left hand side is the total supply of the period 1 consumption good brought to the market by investors with discount factors greater than $\hat{\beta}$. The right hand side is the total cost of purchasing up the assets brought to the market by investors with continuation values $\beta\delta < \bar{p}$. We prove in the proof of Proposition 3 that there is a unique solution to this equation. Finally, we allocate buyers with $\beta > \hat{\beta}$ to markets so as to equate supply and demand at each price, in accordance with part 4 of the definition of equilibrium.

A.2 One-Price Equilibrium

Assume that the support of the buyer's type distribution is an interval B and the support of the seller type distribution is a rectangle $S = B \times D$ for some interval D. Under these restrictions, we prove the existence of a one-price equilibrium characterized by two numbers, the trading price p^* and the identity of the marginal buyer $\hat{\beta} \in B$.

In a one-price equilibrium, an investor can purchase an asset at any price greater than or equal to p^* and can sell an asset at any price less than or equal to p^*:

$$\Theta(p) = \begin{cases} \infty \\ 1 & \Leftrightarrow p \lesseqgtr p^*. \\ 0 \end{cases}$$

Part 1 of the definition of equilibrium implies that, taking $\Theta(p)$ as given, an investor (β, δ) with a continuation value $\beta\delta \leq p^*$ will choose to sell for $p_s(\beta, \delta) = p^*$. Investors with a

[13] Part 3(a) of the definition of equilibrium, together with the assumption that $p_s(\beta, \delta) = \max\{\beta\delta, \hat{\beta}\Gamma(\beta\delta)\}$ imposes additional restrictions on $\Delta(p)$, but these are unimportant for our analysis.

higher continuation value, $\beta\delta > p^*$, set a higher sale price. To support the equilibrium, we choose one such price, $p_s(\beta,\delta) = \bar{\beta}\delta$ if $\beta\delta > p^*$.

Turn next to buyers' beliefs. At prices $p < p^*$, buyers cannot find any seller so beliefs are undefined. At $p = p^*$, Part 3(a) of the definition of equilibrium implies that buyers expect

$$\Delta(p^*) = \frac{\iint_{\beta\delta \le p^*} \delta g_s(\beta,\delta)\, d\delta\, d\beta}{\iint_{\beta\delta \le p^*} g_s(\beta,\delta)\, d\delta\, d\beta},$$

the average quality asset held by investors with a continuation value below p^*. At $p > p^*$, beliefs are also pinned down by condition 3(a): $\Delta(p) = \max\{p/\bar{\beta}, \min D\}$ whenever $p \in (p^*, \bar{v}]$. This is the worst quality asset held by an investor with continuation value p. Finally, we assume $\Delta(p) = \max D$ when $p > \bar{v}$, consistent with condition 3(b).

Now turn to part 2 of the definition of equilibrium. Let $\hat{\beta} = p^*/\Delta(p^*)$. Given the beliefs we just constructed, buyers with discount factor $\beta > \hat{\beta}$ find it strictly optimal to buy at price p^*, while buyers with lower discount factors find it better to offer a price $p < p^*$ at which they cannot buy.

Finally, we close the model using the market clearing condition, part 4 of the definition of equilibrium:

$$\int_{\beta > \hat{\beta}} g_b(\beta) d\beta = p^* \iint_{\beta\delta < p^*} g_s(\beta,\delta)\, d\beta d\delta. \tag{2}$$

The left hand side is the amount of the period 1 consumption good held by investors with discount factors greater than $\hat{\beta}$ and the right hand side is the cost of buying the assets held by investors with continuation value less than p^*.

A one-price equilibrium is a pair $(\hat{\beta}, p^*)$ solving $\hat{\beta}\Delta(p^*) = p^*$ and equation (2). Depending on functional forms, one or more one-price equilibrium may exist.

A.3 Other Equilibria

We illustrate the full multiplicity of equilibria through a parametric example. Assume $G_s(\beta,\delta) = \beta\delta^2$ on $[0,1]^2$, so $\underline{v} = 0$, $\bar{v} = 1$, $\Gamma(v) = \frac{1+v}{2}$, and $H(v) = v(2-v)$.

Other Semi-Separating Equilibria We start by showing there is a continuum of semi-separating equilibria. These equilibria are indexed by the identity of the seller with the highest continuation value, $\bar{p} \in [0.456, 1]$. Given $\bar{p}$, let $\underline{p} = \bar{p}/(1+\bar{p})$, $\hat{\beta} = 2\underline{p}$, and

$$\hat{\theta} = \frac{(1-\bar{p})(2+\bar{p})(3+\bar{p})}{4\bar{p}^2(6-\bar{p})}. \tag{3}$$

The restriction on the range of $\bar{p}$ ensures that $\hat{\theta} \in [0,1]$. In such an equilibrium, the buyer-seller ratio is

$$\Theta(p) = \begin{cases} \infty & \text{if } p < \hat{\theta}\underline{p} \\ \hat{\theta}\underline{p}/p & \text{if } p \in [\hat{\theta}\underline{p}, \underline{p}) \\ \hat{\theta}\left(\frac{\bar{p}-p}{\underline{p}\bar{p}}\right)^{\bar{p}} & \text{if } p \in [\underline{p}, \bar{p}] \\ 0 & \text{if } p > \bar{p}, \end{cases}$$

while the expected quality of assets offered for sale at prices above $\hat{\theta}\underline{p}$ is $\Delta(p) \leq p/\hat{\beta}$, with equality if $p \in [\underline{p}, \bar{p}]$.

To prove this is an equilibrium, we need to discuss buying and selling behavior. Start with selling. For any investor (β, δ) with continuation value with $\beta\delta \in (0, \bar{p})$, the unique optimal selling price is $p_s(\beta, \delta) = \hat{\beta}\Gamma(\beta\delta)$. For investors with the lowest continuation value, $\beta\delta = 0$, any $p_s(\beta, \delta) \in [\hat{\theta}\underline{p}, \underline{p}]$ is optimal; we assume $p_s(\beta, \delta) = \underline{p}$. For investors with higher continuation values, $\beta\delta \geq \bar{p}$, any $p_s \geq \beta\delta$ is optimal; we assume $p_s(\beta\delta) = \beta\delta$.

Given these beliefs,

$$\Delta(p) = \begin{cases} 0 & \text{if } p \in [\hat{\theta}\underline{p}, \underline{p}) \\ p/\hat{\beta} & \text{if } p \in [\underline{p}, \bar{p}] \\ (1+p)/2 & \text{if } p > \bar{p}. \end{cases}$$

Note that we are free to assign any beliefs at prices $p \in [\hat{\theta}\underline{p}, \underline{p})$, since all investors with $\beta = 0$ find such prices optimal. We choose to assign beliefs that only those investors who have $\delta = 0$ set these prices. Given these beliefs, optimal buying behavior sets any price $p_b(\beta) \in [\underline{p}, \bar{p}]$ if $\beta \geq \hat{\beta}$ and any prices $p_b(\beta) < \hat{\theta}\underline{p}$ if $\beta < \hat{\beta}$.

Finally, we can verify that equation (3) ensures that the goods market clears.

Building on this logic, we can construct a continuum of semi-separating equilibria whenever the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by investors with the lowest continuation value. If the support of (β, δ) is a rectangle, this requires that the lowest continuation value is zero, but otherwise it may hold more generally.

Other One-Price Equilibria The same logic supports a continuum of one-price equilibria with rationing at the equilibrium trading price. Equilibria are now characterized by three numbers, the equilibrium trading price p^*, the probability of trade at that price $\theta_1 \in [0,1]$, and the discount factor of the marginal buyer $\hat{\beta}$, but only two equations. First, the marginal buyer must be indifferent about buying all the assets offered for sale at p^*:

$$p^* = \hat{\beta}\frac{3 - p^*2}{3(2 - p^*)},$$

where $(3 - p^*2)/3(2 - p^*)$ is the average quality of assets held by investors with continuation value $v < p^*$. Second, the goods market must clear:

$$1 - \hat{\beta} = \theta_1 p^* 2(2 - p^*),$$

where $p^*(2-p^*)$ is the fraction of sellers at the price p^*. There is a solution to these equations with $\theta_1 \in [0, 1]$ if $p^* \in [0.426, 0.634]$, giving

$$\theta_1 = \frac{3 - 6p^* + 2p^*2}{p^*2(2 - p^*)(3 - p^*2)}$$

and

$$\hat{\beta} = \frac{3p^*(2 - p^*)}{3 - p^*2}.$$

In such an equilibrium, the buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & \text{if } p < \theta_1 p^* \\ \theta_1 p^*/p & \text{if } p \in [\theta_1 p^*, p^*] \\ 0 & \text{if } p > p^*, \end{cases}$$

while the expected quality of assets for sale relative to the price is maximized at p^*.

To construct an equilibrium of this sort, we again discuss buying and selling behavior. All investors (β, δ) with continuation value $\beta\delta < p^*$ set price p^* in equilibrium, while those with higher continuation values set price $p_s(\beta, \delta) = \delta$. This pins down buyers' beliefs at prices above p^*. At prices between $\theta_1 p^*$ and p^*, rational beliefs requires that investors anticipate meeting sellers with zero continuation value. To support the equilibrium, we assume that they anticipate meeting sellers with zero-quality assets:

$$\Delta(p) = \begin{cases} 0 & \text{if } p < p^* \\ \frac{3-p^*2}{3(2-p^*)} & \text{if } p = p^* \\ p & \text{if } p > p^* \end{cases}$$

One can verify that $\Delta(p)/p$ is maximized at p^* for all $p^* \leq 0.634$, so buyers with $\beta \geq \hat{\beta}$ in fact prefer to pay this single price: $p_b(\beta) = p^*$ if $\beta \geq \hat{\beta}$ and $p_b(\beta) = 0$ otherwise. Finally, one can verify that the goods market clears.

Again, this logic shows how to construct a continuum of one-price equilibria whenever the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by investors with the lowest continuation value.

n-Price Equilibria Our model also admits an n-dimensional set of n-price equilibria. Denote the prices by $p^1 < \cdots < p_n$; in equilibrium all trade occurs at these prices. Also let $\theta_1 > \cdots > \theta_n$ denote the buyer-seller ratios at these prices, with $\theta_1 \in (0,1]$. Let $v_1 < \cdots < v_n$ denote the n critical continuation values who are indifferent between neighboring prices (so v_i is indifferent between setting prices p_i and p_{i+1} and v_n is indifferent between setting price p_n and setting a higher price at which she cannot sell). Finally, let $\hat{\beta}$ denote the discount factor of the marginal buyer. This gives us a total of $3n+1$ variables. These must satisfy $2n+1$ equations. The first n equations come from the indifference conditions of the marginal sellers:

$$\theta_i(p_i - v_i) = \theta_{i+1}(p_{i+1} - v_i) \text{ for } i \in \{1, \ldots, n-1\}$$

and $p_n = v_n$. The next n equations come from the marginal buyer's indifference about buying at any price. With our functional forms, this gives

$$p_i = \hat{\beta} \frac{3 - v_{i-1}^2 - v_{i-1}v_i - v_i^2}{3(2 - v_{i-1} - v_i)},$$

where $v_0 = 0$. The fraction is the average value of the assets held by investors with continuation value $v \in [v_{i-1}, v_i]$. Finally, the goods market must clear:

$$1 - \hat{\beta} = \sum_{i=1}^{n} \theta_i p_i (v_i(2 - v_i) - v_{i-1}(2 - v_{i-1})),$$

where $v_i(2 - v_i) - v_{i-1}(2 - v_{i-1})$ is the measure of investors who set price p_i, those with continuation values $v \in [v_{i-1}, v_i]$.

In equilibrium, the buyer-seller ratio satisfies

$$\Theta(p) = \begin{cases} \infty & p < p_0 \\ \frac{\theta_i(p_i - v_i)}{p - v_i} \text{ if} & p \in [p_i, p_{i+1}],\ i \in \{0, \ldots, n-1\} \\ 0 & p > p_n, \end{cases}$$

where $p_0 = \theta_1 p_1$ and $\theta_0 = 1$. Given this structure, only sellers with continuation value v_i find prices $p \in (p_i, p_{i+1})$ optimal for $i \in \{0, \ldots, n-1\}$. To support the equilibrium, we assume buyers anticipate that investors (β, δ) with $\beta = 1$ and $\delta = v_i$ set these prices. Finally, investors with $\beta\delta > p_n$ and $\delta = p$ set price $p > p_n$. This pins down buyers' beliefs. The remainder of the construction of equilibrium is now standard.

In our parametric example, first suppose $\theta_1 = 1$. We find that for any value of $\theta_2 \in [0, 0.832]$, it is possible to construct an equilibrium with trade at two prices. Higher values of θ_2 are associated with lower values of p_1 (falling from 0.426 to 0.371), lower values of $p_2 = v_2$ (falling from 0.527 to 0.446), lower values of v_1 (falling from 0.426 to 0), and higher values of $\hat{\beta}$ (rising from 0.714 to 0.743). It does not seem possible to construct equilibria with $\theta_2 > 0.832$, because the system of equations would imply $v_1 < 0$. For lower values of θ_1, there is a smaller interval of θ_2 corresponding to an equilibrium, but the interval always exists.

The possibility that $\theta_1 < 1$ again hinges on the assumption that the lowest asset quality held by investors with the lowest continuation value is smaller than the average asset quality held by these investors. However, the remaining construction does not rely on this restriction and so appears to be completely general. For example, there are many n-price equilibria in the independent Pareto example that we use throughout the text.

Qualitatively an n-price equilibrium looks very similar to the semi-separating equilibrium. Investors with higher continuation values set weakly higher sale prices and sell with a weakly lower probability. Indeed, we conjecture that in the limit as n converges to infinity, the functions $\Theta(p)$ and $\Delta(p)$ in any n price equilibrium will be close to their values in some semi-separating equilibrium in the sense of the sup-norm.

A.4 Mixed Equilibria

Equilibria may also feature a mix of mass points and continuous distributions. We discuss how to construct such equilibria here; it will be useful later in our analysis, so we break it into a separate section.

Take a set of points $v_1 < \cdots < v_n$, and construct pools of positive radius $\varepsilon_1, \ldots, \varepsilon_n$ around those points with $v_1 - \varepsilon_1 > \underline{v}$ and $v_i - \varepsilon_i > v_{i-1} + \varepsilon_i$ for all $i \in 2, \ldots, n$. We look for an equilibrium where any two sellers with continuation values in the same pool set the same price. That is, for all (β, δ) and (β', δ'), $p_s(\beta, \delta) = p_s(\beta', \delta')$ if and only if there exists an $i \in \{1, \ldots, n\}$ with $\beta\delta \in (v_i - \varepsilon_i, v_i + \varepsilon_i)$ and $\beta'\delta' \in (v_i - \varepsilon_i, v_i + \varepsilon_i)$.

Within each pool, the equilibrium price reflects the quality of the pool:

$$p_i = \hat{\beta} \frac{\int_{v_i - \varepsilon_i}^{v_i + \varepsilon_i} \Gamma(v) dH(v)}{H(v_i + \varepsilon_i) - H(v_i - \varepsilon_i)} \tag{4}$$

Using the known functional forms, it is possible to simplify this and the subsequent expressions. Outside of these pools, the price is the fair one, $p_s(\beta,\delta) = \hat{\beta}\Gamma(\beta\delta)$.

We turn next to the sale probability. For sellers (β,δ) with the lowest continuation value, $\beta\delta \in [\underline{v}, v_1 - \varepsilon_1]$, the sale price is $\hat{\beta}\Gamma(\beta\delta)$ and the sale probability is as in the semi-separating equilibrium with the same value of the marginal buyer $\hat{\beta}$,

$$\Theta(\hat{\beta}\Gamma(\beta\delta)) = \exp\left(-\int_{\underline{p}}^{\hat{\beta}\Gamma(\beta\delta)} \frac{1}{p' - \Gamma^{-1}(p'/\hat{\beta})} dp'\right)$$

We then proceed recursively. Assume for some $i \in \{1,\dots,n\}$, we have already found $\Theta(\hat{\beta}\Gamma(v_i - \varepsilon_i))$, the trading probability at the bottom of the pool. A seller with this continuation value must be indifferent about charging the separating price $\hat{\beta}\Gamma(v_i - \varepsilon_i)$ or charging the pooling price p_i:

$$\Theta(\hat{\beta}\Gamma(v_i - \varepsilon_i))\big(\hat{\beta}\Gamma(v_i - \varepsilon_i) - (v_i - \varepsilon_i)\big) = \Theta(p_i)\big(p_i - (v_i - \varepsilon_i)\big). \tag{5}$$

This pins down the trading probability in the pool, $\Theta(p_i)$. Next, we turn to the seller with continuation value $v_i + \varepsilon_i$. He must be indifferent between separating and pooling as well,

$$\Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i))\big(\hat{\beta}\Gamma(v_i + \varepsilon_i) - (v_i + \varepsilon_i)\big) = \Theta(p_i)\big(p_i - (v_i + \varepsilon_i)\big), \tag{6}$$

which we solve for $\Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i))$. Finally, sellers (β,δ) with $\beta\delta \in [v_i + \varepsilon_i, v_{i+1} - \varepsilon_{i+1}]$ must find the price $\hat{\beta}\Gamma(\beta\delta)$ optimal. It is straightforward to prove that this price is locally optimal if and only if the sale probability is proportional to its level in the semi-separating equilibrium. The value of $\Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i))$ pins down the constant of proportionality:

$$\Theta(\hat{\beta}\Gamma(\beta\delta)) = \Theta(\hat{\beta}\Gamma(v_i + \varepsilon_i)) \exp\left(-\int_{\hat{\beta}\Gamma(v_i+\varepsilon_i)}^{\hat{\beta}\Gamma(\beta\delta)} \frac{1}{p' - \Gamma^{-1}(p'/\hat{\beta})} dp'\right). \tag{7}$$

This completes the recursion.

All that remains is pinning down the beliefs of a buyer at prices without trade. We make those as pessimistic as possible. In our example, this means that a buyer believes that if a price p is weakly optimal for a seller with continuation value v but no such seller sets the price, $\Delta(p) = v$ (and so the seller's discount factor is $\beta = 1$). Such beliefs always support the mixed equilibrium.

B Omitted Proofs

Proof of Proposition 1. Fix $(\beta_1, \delta_1) \in S$ and $(\beta_2, \delta_2) \in S$ with $\beta_1\delta_1 < \beta_2\delta_2$. Part 1 of the definition of equilibrium implies

$$\min\{\Theta(p_s(\beta_1,\delta_1)), 1\}(p_s(\beta_1,\delta_1) - \beta_1\delta_1) \geq \min\{\Theta(p_s(\beta_2,\delta_2)), 1\}(p_s(\beta_2,\delta_2) - \beta_1\delta_1) \tag{8}$$

$$\min\{\Theta(p_s(\beta_2,\delta_2)), 1\}(p_s(\beta_2,\delta_2) - \beta_2\delta_2) \geq \min\{\Theta(p_s(\beta_1,\delta_1)), 1\}(p_s(\beta_1,\delta_1) - \beta_2\delta_2). \tag{9}$$

Add the inequalities and simplify to get

$$\left(\min\{\Theta(p_s(\beta_1,\delta_1)), 1\} - \min\{\Theta(p_s(\beta_2,\delta_2)), 1\}\right)(\beta_2\delta_2 - \beta_1\delta_1) \geq 0$$

or $\min\{\Theta(p_s(\beta_1,\delta_1)), 1\} \geq \min\{\Theta(p_s(\beta_2,\delta_2)), 1\}$.

Now assume the seller with the higher continuation value sells with a positive probability, $\Theta(p_s(\beta_2,\delta_2)) > 0$. Divide the left hand side of inequality (9) by $\min\{\Theta(p_s(\beta_2,\delta_2)), 1\}$ and the right hand side by the larger quantity $\min\{\Theta(p_s(\beta_1,\delta_1)), 1\}$ to prove $p_s(\beta_2,\delta_2) \geq p_s(\beta_1,\delta_1)$.

Finally, suppose to find a contradiction that prove $\Theta(p_s(\beta_1,\delta_1)) < \Theta(p_s(\beta_2,\delta_2))$. If $\Theta(p_s(\beta_1,\delta_1)) < 1$, this contradicts the first step. On the other hand, if $\Theta(p_s(\beta_1,\delta_1)) \geq 1$, $\min\{\Theta(p_s(\beta_2,\delta_2)), 1\} = 1$ as well. But then inequality (8) implies $p_s(\beta_1,\delta_1) \geq p_s(\beta_2,\delta_2)$, a contradiction. ■

Proof of Proposition 2. Let $\hat{\beta}$ be the infimal value of $p/\Delta(p)$ among p with $\Theta(p) < \infty$. This means that for all $\beta > \hat{\beta}$, there exists a p with $\Theta(p) < \infty$ such that $\beta > p/\Delta(p)$, or equivalent $\beta\Delta(p)/p > 1$. Part 2 of the definition of equilibrium implies that buyers with that discount factor buy at some such price. If $\beta < \hat{\beta}$, then for any p with $\Theta(p) < \infty$, $\beta\Delta(p)/p < 1$. Buyers with this discount factor are better off not buying, i.e. setting a price such that $\Theta(p_b(\beta)) = \infty$.

Now suppose there is a seller $(\beta, \delta) \in S$ with $0 < \Theta(p_s(\beta,\delta)) < \infty$. The definition of $\hat{\beta}$ implies $\hat{\beta} \leq p_s(\beta,\delta)/\Delta(p_s(\beta,\delta))$. If the inequality were strict, part 2 of the definition of equilibrium implies there is there is no buyer who finds the price $p_s(\beta,\delta)$ optimal, contradicting part 4 of the definition of equilibrium. Therefore $\hat{\beta} = p_s(\beta,\delta)/\Delta(p_s(\beta,\delta))$. It follows immediately that all buyers with $\beta \geq \hat{\beta}$ are indifferent about buying any of the assets sold in equilibrium. ■

We turn next to the proof of Proposition 3. To prove this, we first state and prove five preliminary Lemmas.

Lemma 1 *Consider an equilibrium with multidimensional private information. Take any* $p_1 < p_2$. *If* $0 < \Theta(p_1) < \infty$, $\Theta(p_2) < \min\{1, \Theta(p_1)\}$. *If* $\Theta(p_1) = 0$, $\Theta(p_2) = 0$ *as well.*

Proof of Lemma 1. Suppose there is a $p_1 < p_2$ with $0 < \Theta(p_1) < \infty$. Part 3 of the definition of equilibrium implies that there is a (β, δ) who finds p_1 an optimal sale price. In particular, $p_1 \geq \beta\delta$ and p_1 gives weakly higher profits than p_2:

$$\min\{\Theta(p_1), 1\}(p_1 - \beta\delta) \geq \min\{\Theta(p_2), 1\}(p_2 - \beta\delta). \tag{10}$$

Now if $\Theta(p_2) \geq 1$, $\min\{\Theta(p_2), 1\}(p_2 - \beta\delta) = p_2 - \beta\delta$. Also, since $p_1 \geq \beta\delta$, $p_1 - \beta\delta \geq \min\{\Theta(p_1), 1\}(p_1 - \beta\delta)$. Combine these inequalities with (10) to get $p_1 \geq p_2$, a contradiction.

Now suppose $0 < \Theta(p_1) \leq \Theta(p_2)$. Then $0 < \min\{\Theta(p_1), 1\} \leq \min\{\Theta(p_2), 1\}$. Divide equation (10) by this inequality to get $p_1 \geq p_2$, a contradiction. This proves $\Theta(p_2) < \min\{1, \Theta(p_1)\}$ when $0 < \Theta(p_1) < \infty$.

Finally, assume $\Theta(p_1) = 0$. Since the left hand side of inequality (10) is zero, the right hand side must be as well. Given that $p_2 > p_1 \geq \beta\delta$, it must be that $\Theta(p_2) = 0$. ■

Lemma 2 *Consider an equilibrium with multidimensional private information. Let* $\mathbb{V} : \mathbb{R}_+ \rightrightarrows V$ *denote the set of sellers' continuation values* v *for which the price* $p \geq v$ *is weakly optimal:*

$$v \in \mathbb{V}(p) \Leftrightarrow p \in \arg\max_{p' \geq v} \min\{\Theta(p'), 1\}(p' - v).$$

If $\Theta(p) < \infty$, *the set* $\mathbb{V}$ *is nonempty. Take any* $p_1 < p_2$ *with* $\Theta(p_1) < \infty$, $v_1 \in \mathbb{V}(p_1)$, *and* $v_2 \in \mathbb{V}(p_2)$. *If* $\Theta(p_1) \geq 0$ *then* $v_1 \leq v_2$. *Moreover,* $\mathbb{V}(p)$ *is convex and closed for all* p *with* $\Theta(p) < \infty$.

Proof of Lemma 2. First observe that part 3(b) of the definition of equilibrium with multidimensional private information implies that the set $\mathbb{V}(p)$ is nonempty when $\Theta(p)$ is finite. Next, by the definition of $\mathbb{V}$, a seller with continuation value v_1 weakly prefers p_1 to p_2:

$$\min\{\Theta(p_1), 1\}(p_1 - v_1) \geq \min\{\Theta(p_2), 1\}(p_2 - v_1), \tag{11}$$

Similarly, a seller with continuation value v_2 either finds the price p_1 suboptimal because $p_1 < v_2$ or prefers p_2 to p_1. First suppose $p_1 < v_2$. Since $v_1 \in \mathbb{V}(p_1)$, $v_1 \leq p_1$, proving $v_1 < v_2$. Second suppose v_2 prefers p_2 to p_1 and $p_1 \geq v_2$:

$$\min\{\Theta(p_2), 1\}(p_2 - v_2) \geq \min\{\Theta(p_1), 1\}(p_1 - v_2). \tag{12}$$

If $\Theta(p_2) = 0$, $\Theta(p_1) > 0$ implies $p_1 = v_2$ so again $v_1 \leq v_2$. If instead $\Theta(p_2) > 0$, multiply inequalities (11) and (12) and simplify to get $(p_2 - p_1)(v_2 - v_1) \geq 0$, which proves that $v_2 \geq v_1$.

To prove $\mathbb{V}(p)$ is convex, take any p and $v_1 < v_2$ with $v_1, v_2 \in \mathbb{V}(p)$. Fix any $\tilde{v} \in (v_1, v_2)$ and $\tilde{p}$ such that $\tilde{v} \in \mathbb{V}(\tilde{p})$. Note that such a $\tilde{p}$ must exist; set $\tilde{p} = p_s(\beta, \delta)$ for any $\beta\delta = \tilde{v}$. If $p < \tilde{p}$, $v_2 \in \mathbb{V}(p)$ and $\tilde{v} \in \mathbb{V}(\tilde{p})$ contradicts the first part of the lemma. If $\tilde{p} < p$, $v_1 \in \mathbb{V}(p)$ and $\tilde{v} \in \mathbb{V}(\tilde{p})$ contradicts the first part of the lemma. Therefore $p_s(\beta, \delta) = p$ for all $\beta\delta \in (v_1, v_2)$.

To prove $\mathbb{V}(p)$ is closed, suppose there exists a sequence $\{v_n\} \to v$ with $v_n \in \mathbb{V}(p)$ for all n but $v \notin \mathbb{V}(p)$. Since $p \geq v_n$ for all n, $p \geq v$ as well. The definition of $\mathbb{V}$ then implies that there exists a $\tilde{p} \geq v$ with

$$\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v) - \min\{\Theta(p), 1\}(p - v) \equiv \varepsilon > 0$$

But since $\{v_n\} \to v$, there exists an N such that for all $n > N$,

$$\big(\min\{\Theta(\tilde{p}), 1\} - \min\{\Theta(p), 1\}\big)(v_n - v) < \varepsilon.$$

Using the definition of ε, this implies $\min\{\Theta(\tilde{p}), 1\}(\tilde{p} - v_n) > \min\{\Theta(p), 1\}(p - v_n)$, and in particular $\tilde{p} > v_n$, which contradicts $v_n \in \mathbb{V}(p)$. ■

Lemma 3 *Impose Assumption 1 and consider an equilibrium with unidimensional private information. Take any p with $0 < \Theta(p) < \infty$. Then $p \geq \underline{v}$ and there exists a $\mathcal{V}(p) \in [\underline{v}, \min\{p, \bar{v}\}]$ such that $\mathbb{V}(p) = \{\mathcal{V}(p)\}$ and hence $\Delta(p) = \Gamma(\mathcal{V}(p))$. Moreover, $\mathcal{V}$ and Δ are continuous.*

Proof of Lemma 3. Since $\Theta(p) < \infty$, the second part of the definition of equilibrium with unidimensional private information implies that there is a (β, δ) who finds p an optimal sale price and in particular $p \geq \beta\delta \geq \underline{v}$. If $p = 0$, these inequalities imply $\beta\delta = \underline{v} = 0$ as well, so $\mathbb{V}(0) = \{\mathcal{V}(0)\}$, where $\mathcal{V}(0) = 0$.

Otherwise, $p > 0$ and in order to find a contradiction, suppose $\mathbb{V}(p) \neq \{\beta\delta\}$. Then Lemma 2 implies it must be an interval, $\mathbb{V}(p) = [v_1, v_2]$. Lemma 2 also implies that p is the only optimal sale price for $v \in (v_1, v_2)$: $p_s(\beta, \delta) = p$ if $\beta\delta \in (v_1, v_2)$, while any investor who finds a lower (higher) sale price optimal must have a lower (higher) continuation value. Then part 3(a) of the definition of equilibrium with multidimensional private information implies $\Delta(p)$ is the average quality asset held by sellers with this continuation value:

$$\Delta(p) = \frac{\int_{v_1}^{v_2} \Gamma(v)h(v)dv}{\int_{v_1}^{v_2} h(v)dv}.$$

Monotonicity of Γ (Assumption 1) implies

$$\Gamma(v_1) < \Delta(p) < \Gamma(v_2) \leq \Delta(p')$$

for any price $p' > p$. Note that the last inequality uses the definition of unidimensional private information.

We can use this to prove that no buyer finds setting the price p optimal. If there were such a buyer, he must have $\beta\Delta(p) \geq p$ by part 2 of the definition of equilibrium with multidimensional private information. So take any $p' > p$ with $p'\Delta(p) < p\Delta(p')$; this is a nonempty interval since $\Delta(p') > \Delta(p)$ and $p > 0$. Since $\Theta(p) < \infty$, Lemma 1 implies $\Theta(p') < 1$. Then

$$\min\{\Theta(p)^{-1}, 1\}\left(\frac{\beta\Delta(p)}{p} - 1\right) \leq \frac{\beta\Delta(p)}{p} - 1 < \frac{\beta\Delta(p')}{p'} - 1 = \min\{\Theta(p')^{-1}, 1\}\left(\frac{\beta\Delta(p')}{p'} - 1\right).$$

The first inequality uses $\min\{\Theta(p)^{-1}, 1\} \leq 1$ and $\beta\Delta(p) \geq p$. The second uses $\Delta(p)/p < \Delta(p')/p'$. The equality uses $\Theta(p') < 1$. All buyers prefer p' to p.

We now have a contradiction. The measure of buyers setting price p is zero, $d\mu_b(p) = 0$, while the measure of sellers setting price p is positive, $d\mu_s(p) = \int_{v_1}^{v_2} h(v)dv$. This is inconsistent with part 4 of the definition of equilibrium with multidimensional private information, $d\mu_b(p) = \Theta(p)d\mu_s(p)$.

Now suppose $\mathcal{V}$ has a discontinuity at p. Using the arguments in Lemma 2, all $v \in \left(\liminf_{p'\to p} \mathcal{V}(p'), \limsup_{p'\to p} \mathcal{V}(p')\right)$ must find price p optimal, which contradicts the first part of this result. Finally, $\Delta(p) = \Gamma(\mathcal{V}(p))$ by the second part of the definition of equilibrium with unidimensional private information, and continuity of Δ follows from Assumption 1. ■

Lemma 4 *Impose Assumption 1 and consider an equilibrium with unidimensional private information. Take any $p_1 < p_2$ with $\Theta(p_1) < \infty$ and $\Theta(p_2) > 0$. Then $\mathcal{V}(p_1) \leq \mathcal{V}(p_2)$ with strict inequality if $\mathcal{V}(p_1) < \max V$.*

Proof of Lemma 4. Since $\Theta(p_1) < \infty$, Lemma 1 implies $\Theta(p) < 1$ for all $p > p_1$. And since $\Theta(p_2) > 0$, the same Lemma implies $\Theta(p) > 0$ for all $p < p_2$. Then Lemma 3 implies

$\mathbb{V}(p) = \{\mathcal{V}(p)\}$, a singleton, and Lemma 2 implies $\mathcal{V}(p)$ is weakly increasing on this interval, $\mathcal{V}(p_1) \leq \mathcal{V}(p_2)$.

Now to find a contradiction, suppose $v = \mathcal{V}(p_1) = \mathcal{V}(p_2) < \max V$ and let $p_3 = \max p$ such that $v = \mathcal{V}(p)$. By definition, $v \leq p_1 < p_2 \leq p_3$. Moreover, for any price $p \in (p_1, p_3)$, the fact that sellers with continuation value v find p an optimal price implies

$$\Theta(p) = \min\{\Theta(p_1), 1\} \frac{p_1 - v}{p - v}. \tag{13}$$

If $v = p_1$, $\Theta(p_2) = 0$, a contradiction. Therefore $v < p_1$ and $\Theta(p) > 0$.

Now since $v < \max V$, we can find a continuation value $\tilde{v} > v$ that satisfies the following two restrictions: (1) $\Gamma(\tilde{v}) < \Gamma(v)p_3/p_1$ and (2) $\tilde{v} < p_3$. The first restriction is feasible since Γ is continuous by Assumption 1 and $p_1 < p_3$ by assumption. The second restriction is feasible because $v < p_3$. Finally, let $\tilde{p}$ denote an optimal price for a seller with continuation value $\tilde{v}$, say $\tilde{p} = p_s(\tilde{\beta}, \tilde{\delta})$ for some $\tilde{\beta}\tilde{\delta} = \tilde{v}$. By Lemma 3, $\tilde{v} > v$ implies $\tilde{p} > p_3$.

The fact that a seller with continuation value $\tilde{v}$ sets price $\tilde{p}$ implies in particular that

$$\Theta(\tilde{p})(\tilde{p} - \tilde{v}) \geq \Theta(p_3)(p_3 - \tilde{v}).$$

Since $\Theta(p_3) > 0$ and $p_3 > \tilde{v}$, the right hand side is positive. The left hand side must therefore be as well, so in particular $\Theta(\tilde{p}) > 0$, so some buyers offer price $\tilde{p}$.

Now consider the value to a buyer of offering $p \in (p_1, p_3)$ rather than $\tilde{p}$:

$$\min\{\Theta(p)^{-1}, 1\}\left(\frac{\beta\Gamma(v)}{p} - 1\right) = \frac{\beta\Gamma(v)}{p} - 1 > \frac{\beta\Gamma(\tilde{v})}{\tilde{p}} - 1 = \min\{\Theta(\tilde{p})^{-1}, 1\}\left(\frac{\beta\Gamma(\tilde{v})}{\tilde{p}} - 1\right)$$

The first equality holds because $\Theta(p) < 1$. The inequality holds because $\Gamma(v)/p > \Gamma(\tilde{v})/p_3 > \Gamma(\tilde{v})/\tilde{p}$, first by construction of $\tilde{v}$, second by $\tilde{p} > p_3$. The second equality holds because $\Theta(\tilde{p}) < 1$ as well. But then $\tilde{p}$ is not an optimal price for any buyer, inconsistent with part 4 of the definition of equilibrium with multidimensional private information. This contradiction implies $\mathcal{V}(p_1) < \mathcal{V}(p_2)$. ■

Lemma 5 *Impose Assumption 1 and consider an equilibrium with unidimensional private information. Take any* $p_1 < p_2$ *with* $\Theta(p_1) < \infty$ *and* $\Theta(p_2) > 0$. *Then*

$$\frac{-\Theta(p_2)}{p_1 - \mathcal{V}(p_1)} \geq \frac{\Theta(p_2) - \Theta(p_1)}{p_2 - p_1} \quad \textit{and} \quad \frac{\Theta(p_2) - \min\{\Theta(p_1), 1\}}{p_2 - p_1} \geq \frac{-\min\{\Theta(p_1), 1\}}{p_2 - \mathcal{V}(p_2)}, \tag{14}$$

both strictly if $\mathcal{V}(p_1) < \max V$. *In particular, if* $0 < \Theta(p) < 1$ *and* $p > \mathcal{V}(p)$, $\Theta'(p) =$

$-\Theta(p)/(p-\mathcal{V}(p))$.

Proof of Lemma 5. Lemma 4 implies

$$\min\{\Theta(p_1),1\}(p_1-\mathcal{V}(p_1)) \geq \min\{\Theta(p_2),1\}(p_2-\mathcal{V}(p_1)),$$

and strictly so if $\mathcal{V}(p_1) < \max V$, since a seller with continuation value $\mathcal{V}(p_1)$ only finds the price p_1 optimal. Note that this implies $p_1 > \mathcal{V}(p_1)$ as well. Since $\Theta(p_1) < \infty$, Lemma 1 implies $\Theta(p_2) < 1$, while $p_1 > \mathcal{V}(p_1)$ implies $\Theta(p_1)(p_1-\mathcal{V}(p_1)) \geq \min\{\Theta(p_1),1\}(p_1-\mathcal{V}(p_1))$. Combining these inequalities yields the first inequality in condition (14).

Similarly, Lemma 4 implies

$$\min\{\Theta(p_2),1\}(p_2-\mathcal{V}(p_2)) \geq \min\{\Theta(p_1),1\}(p_1-\mathcal{V}(p_2)),$$

and strictly so if $\mathcal{V}(p_1) < \max V$, since $\mathcal{V}(p_1)$ is the only seller who finds price p_1 weakly optimal. Again note that $\Theta(p_2) < 1$ but now $\Theta(p_1) > 1$ is possible. This then leads to the second inequality in condition (14).

Now suppose $0 < \Theta(p) < 1$ and consider an arbitrary sequence of prices $\{\tilde{p}\}$ with $0 < \Theta(\tilde{p}) < 1$ and converging to p. Since $\mathcal{V}$ is continuous by Lemma 3, $\mathcal{V}(\tilde{p}) \to \mathcal{V}(p)$ as well. At every point in the sequence, condition (14) implies

$$\Theta(p)\frac{p-\mathcal{V}(p)}{\tilde{p}-\mathcal{V}(p)} \geq \Theta(\tilde{p}) \geq \Theta(p)\frac{p-\mathcal{V}(\tilde{p})}{\tilde{p}-\mathcal{V}(\tilde{p})}.$$

The two bounds converge to $\Theta(p)$, proving that $\Theta(\tilde{p}) \to \Theta(p)$. Finally, condition (14) also implies

$$\frac{-\Theta(\tilde{p})}{p-\mathcal{V}(p)} \geq \frac{\Theta(\tilde{p})-\Theta(p)}{\tilde{p}-p} \geq \frac{-\Theta(p)}{\tilde{p}-\mathcal{V}(\tilde{p})}.$$

Again both bounds converge to $-\Theta(p)/(p-\mathcal{V}(p))$, establishing the result. ■

Proof of Proposition 3. First we assume that $\beta\Gamma(\underline{v}) \leq \underline{v}$ for all $\beta \in B$ and show that we can construct an equilibrium with no trade. Set $P(v) = \max\{\bar{\beta}\Gamma(v), v\}$ for all v, where $\bar{\beta} = \max B$. Also set $\Theta(p) = 0$ for all $p \geq P(\underline{v})$ and $\Theta(p) = \infty$ otherwise. Finally, assume $\Delta(p) \leq p/\bar{\beta}$ for all p. It is easy to verify that this is an equilibrium with unidimensional private information.

Now to find a contradiction, suppose that $\beta\Gamma(\underline{v}) \leq \underline{v}$ for all $\beta \in B$ and there is an equilibrium with $\Theta(p) > 0$ for some $p > \underline{v}$. Sellers' optimality implies $P(\underline{v}) > \underline{v}$ with $\Theta(P(\underline{v})) > 0$. Lemma 4 implies only sellers with the lowest continuation value set this price, and therefore $\Delta(P(\underline{v})) = \Gamma(\underline{v})$. Stringing together these inequalities gives $\beta\Delta(P(\underline{v})) < P(\underline{v})$ for all $\beta \in B$, and so part 2 of the definition of equilibrium with multidimensional private information implies no buyer sets this price. This contradicts part 4 of the definition of equilibrium with multidimensional private information.

For the remainder of the proof, assume $\bar{\beta}\Gamma(\underline{v}) > \underline{v}$. We first rule out the possibility of an equilibrium in which $\Theta(p) = \infty$ for $p < \bar{p}$ and $\Theta(p) = 0$ for $p > \bar{p}$ for some $\bar{p} \geq 0$. If $\bar{p} > \underline{v}$ and $\Theta(\bar{p}) < 1$, there is no $p_s(\beta, \delta)$ consistent with part 1 of the definition of equilibrium with multidimensional private information for $\beta\delta \in [\underline{v}, \bar{p})$, a contradiction. If $\bar{p} > \underline{v}$ and $\Theta(\bar{p}) \geq 1$, $P(v) = \bar{p}$ for $v \in [\underline{v}, \bar{p})$, which contradicts Lemma 3. Therefore it must be the case that $\underline{v} \geq \bar{p}$ and hence $\bar{\beta}\Gamma(\underline{v}) > \bar{p}$. Part 1 of the definition of equilibrium with multidimensional private information implies $p_s(\beta\delta) \geq \beta\delta > \bar{p}$ for all $\beta\delta > \underline{v}$, hence sellers (β, δ) don't sell, i.e. the market must shut down. But the second part of the definition of equilibrium with unidimensional private information implies $\Delta(v) \geq \Gamma(\underline{v})$ for all v, which implies $\bar{\beta}\Delta(v) > \bar{p}$: a sufficiently patient buyer would make money buying at a price just above $\bar{p}$, contradicting the market clearing condition. This cannot be an equilibrium.

Now using Lemma 1, there are thresholds $\underline{p} < \bar{p}$ where $\Theta(p) = \infty$ if $p < \underline{p}$, $\Theta(p) = 0$ if $p > \bar{p}$, and $\Theta(p) \in (0, 1)$ if $p \in (\underline{p}, \bar{p})$. The differential equation for Θ in Lemma 5 then applies in this range, giving

$$\Theta(p) = \lambda \exp\left(-\int_{\underline{p}}^{p} \frac{1}{\tilde{p} - \mathcal{V}(\tilde{p})} dp\right) \tag{15}$$

for all $p \in (\underline{p}, \bar{p})$ and some constant of integration $\lambda > 0$. In addition, Lemma 1 ensures that $\lambda \leq 1$ so that $\Theta(p) < 1$ for all $p > \underline{p}$. And if $\lambda < 1$, an investor with continuation value $v = \mathcal{V}(p)$ where $\underline{p} < p < (\underline{p} - \Theta(p)p)/(1 - \Theta(p))$ would earn higher profits selling with probability 1 at price $\underline{p} - \varepsilon$ for some sufficiently small ε, rather than selling at price p, a contradiction. Therefore $\lambda = 1$.

Turn now to the buyers' problem. Buyers know that only a seller with continuation value $\mathcal{V}(p)$ sells at price p, so $\Delta(p) = \Gamma(\mathcal{V}(p))$. For buyers to be willing to purchase at all prices $p \in [\underline{p}, \bar{p}]$, it must be the case that $p/\Delta(p) = \hat{\beta}$ for some constant $\hat{\beta}$, or equivalently $\Gamma(\mathcal{V}(p)) = p/\hat{\beta}$. Substituting this into equation (15) and changing the variable of integration gives

$$\Theta(P(v)) = \exp\left(-\int_{\underline{v}}^{v} \frac{\hat{\beta}\Gamma'(\tilde{v})}{\hat{\beta}\Gamma(\tilde{v}) - \tilde{v}} d\tilde{v}\right) \tag{16}$$

for $v \in (\underline{v}, \bar{p})$. Moreover, we can extend this to $v = \underline{v}$ since buyers must be willing to purchase assets from the sellers with the lowest continuation value as well, which requires that $\Theta(\underline{p}) \leq 1$.

Given equation (16), sellers with continuation value $v < \bar{p}$ set price $P(v) = \hat{\beta}\Gamma(v)$, while all sellers with higher continuation values are indifferent about all prices $p > \bar{p}$ and in particular are willing to set prices such that $P(v) \geq \hat{\beta}\Gamma(v)$. This ensures that buyers with $\beta > \hat{\beta}$ are indifferent about buying at any price $p \in [\underline{p}, \bar{p}]$ and prefer those prices to higher prices. Buyers with lower continuation values set lower prices and do not succeed in buying. To find an equilibrium, we simply allocate the buyers to the different prices in a way that ensures the appropriate buyer-seller ratio at each price. This is feasible if the total wealth of buyers with $\beta > \hat{\beta}$ is exactly enough to purchase the assets sold by sellers with $v \in [\underline{v}, \bar{p}]$:

$$\int_{\beta \geq \hat{\beta}} g_b(\beta) d(\beta) = \int_{v \leq \bar{p}} P(v)\Theta(P(v))h(v)dv. \tag{17}$$

This is the same as equation (1). The left hand side is decreasing in $\hat{\beta}$, equal to 0 when $\hat{\beta} = \bar{\beta}$. The right hand side is strictly positive when $\hat{\beta} = \bar{\beta}$ and increasing in $\hat{\beta}$. To prove monotonicity, note first that $\bar{p}$, defined as the smallest solution to $\bar{p} \geq \hat{\beta}\Gamma(\bar{p})$ is nondecreasing in $\hat{\beta}$. So are $P(v) = \hat{\beta}\Gamma(v)$ and $\Theta(P(v))$ defined in equation (16). There is a unique $\hat{\beta}$ that solves equation (17). ■

Proof of Proposition 4. Let $\hat{\beta}$ denote the marginal buyer in the semi-separating equilibrium and $\hat{\beta}_m$ denote the marginal buyer in the mixed equilibrium. We first prove that if a mixed equilibrium Pareto dominates the semi-separating equilibrium, it must have $\hat{\beta}_m = \hat{\beta}$. If $\hat{\beta}_m > \hat{\beta}$, all buyers with $\beta > \hat{\beta}$ are worse off in the mixed equilibrium, since they live in autarky rather than buying, and so it does not Pareto dominate the semi-separating equilibrium. If $\hat{\beta}_m < \hat{\beta}$, consider a seller with the lowest continuation value $\underline{v}$. He sells for sure in both equilibria, earning a price equal to $\Gamma(\underline{v})$ times the discount factor of the marginal buyer. A reduction in the discount factor of the marginal buyer therefore makes him worse off and so again the mixed equilibrium does not Pareto dominate the semi-separating equilibrium.

The bulk of the proof then compares a semi-separating equilibrium with a given value of $\hat{\beta}$ to a mixed equilibrium with the same value of $\hat{\beta}_m = \hat{\beta}$. Here we focus on the limiting case where the pool sizes are arbitrarily small. For the construction of the mixed equilibrium when the pool sizes are not necessarily small, see Appendix A.4; we use those results here.

We start by approximating the price charged by each pool. Taking a Taylor expansion of equation (4) in a neighborhood of v_i, we obtain

$$p_i \approx \hat{\beta}\Gamma(v_i) + \frac{\hat{\beta}}{6}\left(\Gamma''(v_i) + \frac{2\Gamma'(v_i)h'(v_i)}{h(v_i)}\right)\varepsilon_i^2 + O(\varepsilon_i^3). \tag{18}$$

We next turn to the trading probabilities. For notational convenience, we let $\Theta(P(v))$ and $\omega(v)$ denote the equilibrium trading probabilities for a seller with continuation value v in the semi-separating and mixed equilibrium. The latter probabilities are defined recursively in equations (5), (6) and (7).

First, we claim that the mixed equilibrium makes all sellers better off than the the semi-separating equilibrium if and only if $\omega(v_i + \varepsilon_i) \geq \Theta(P(v_i + \varepsilon_i))$ for $i = 1, \ldots, n$. For $v_i + \varepsilon_i$, pooling alters the probability of trade but not the price, and so this seller is better off if and only if $\omega(v_i + \varepsilon_i) \geq \Theta(P(v_i + \varepsilon_i))$. This proves the "only if" part of the statement. To prove the "if" part, we turn to the other sellers. In the mixed equilibrium, all individuals with $v \in (v_i - \varepsilon_i, v_i + \varepsilon_i)$ trade with the same probability and at the same price. Moreover, $v_i + \varepsilon_i$ is indifferent about trading at that price. If trading in the pool makes $v_i + \varepsilon_i$ better off, then it makes all the members of the pool, who have lower quality assets better off. Additionally, for individuals who are separating in the mixed equilibrium, $v \in (v_i + \varepsilon_i, v_{i+1} - \varepsilon_{i+1})$, pooling raises the probability of trade by the same proportion as it raises the probability of trade for $v_i + \varepsilon_i$ and again it has no effect on their trading price. This establishes the claim.

Next we compute $\omega(v_i + \varepsilon_i)/\Theta(P(v_i + \varepsilon_i))$. Use equations (5) and (6) to get

$$\frac{\omega(v_i + \varepsilon_i)}{\omega(v_i - \varepsilon_i)} = \frac{\left(\hat{\beta}\Gamma(v_i - \varepsilon_i) - (v_i - \varepsilon_i)\right)\left(p_i - (v_i + \varepsilon_i)\right)}{\left(\hat{\beta}\Gamma(v_i + \varepsilon_i) - (v_i + \varepsilon_i)\right)\left(p_i - (v_i - \varepsilon_i)\right)}.$$

Equation (7) implies

$$\frac{\omega(v_i + \varepsilon_i)}{\omega(v_i - \varepsilon_i)} = \exp\left(-\int_{v_i - \varepsilon_i}^{v_i + \varepsilon_i} \frac{\hat{\beta}\Gamma'(\tilde{v})}{\hat{\beta}\Gamma(\tilde{v}) - \tilde{v}} d\tilde{v}\right).$$

Finally, the relative trading probabilities between the mixed and semi-separating equilibria are constant within the separating regions:

$$\frac{\omega(v_i - \varepsilon_i)}{\Theta(P(v_i - \varepsilon_i))} = \frac{\omega(v_{i-1} + \varepsilon_{i-1})}{\Theta(P(v_{i-1} + \varepsilon_{i+1}))},$$

where $v_0 + \varepsilon_0 \equiv \underline{v}$ and $\omega(\underline{v}) = \Theta(P(\underline{v})) = 1$. Combining these three equalities and solving the recursion gives

$$\frac{\omega(v_i + \varepsilon_i)}{\Theta(P(v_i + \varepsilon_i))} = \prod_{j=1}^{i} \frac{\left(\hat{\beta}\Gamma(v_j - \varepsilon_j) - (v_j - \varepsilon_j)\right)\left(p_j - (v_j + \varepsilon_j)\right)}{\left(\hat{\beta}\Gamma(v_j + \varepsilon_j) - (v_j + \varepsilon_j)\right)\left(p_j - (v_j - \varepsilon_j)\right)} \exp\left(\int_{v_j - \varepsilon_j}^{v_j + \varepsilon_j} \frac{\hat{\beta}\Gamma'(\tilde{v})}{\hat{\beta}\Gamma(\tilde{v}) - \tilde{v}} d\tilde{v}\right)$$

for $i = 1, \ldots, n$. Thus all sellers are better off in the mixed equilibrium than the semi-separating equilibrium if and only if

$$\sum_{j=1}^{i}\left(\log\left(\frac{\big(\hat\beta\Gamma(v_i-\varepsilon_i)-(v_i-\varepsilon_i)\big)\big(p_i-(v_i+\varepsilon_i)\big)}{\big(\hat\beta\Gamma(v_i+\varepsilon_i)-(v_i+\varepsilon_i)\big)\big(p_i-(v_i-\varepsilon_i)\big)}\right)+\int_{v_j-\varepsilon_j}^{v_j+\varepsilon_j}\frac{\hat\beta\Gamma'(\tilde v)}{\hat\beta\Gamma(\tilde v)-\tilde v}d\tilde v\right)\geq 0$$

for $i=1,\dots,n$. Finally, perform a Taylor expansion of this sum near $\varepsilon_j=0$, using the approximation for p_i given in equation (18). This gives

$$\log\left(\frac{\omega(v_i+\varepsilon_i)}{\Theta(P(v_i+\varepsilon_i))}\right)=\sum_{j=1}^{i}\left(\frac{2\hat\beta\Gamma'(v_j)}{3(\hat\beta\Gamma(v_j)-v_j)^2}\left(\frac{h'(v_j)}{h(v_j)}-\frac{2-\hat\beta\Gamma'(v_j)}{\hat\beta\Gamma(v_j)-v_j}\right)\varepsilon_j^3+O(\varepsilon_j^4)\right). \quad (19)$$

All sellers are better off in the mixed equilibrium if and only if this is non-negative.

Now we turn to the amount of consumption goods buyers use to purchase assets in the first period. We compute the difference in this cost in the mixed equilibrium compared to the semi-separating equilibrium:

$$\sum_{i=1}^{n}\left(p_i\omega(v_i)\big(H(v_i+\varepsilon_i)-H(v_i-\varepsilon_i)\big)+\frac{\omega(v_i+\varepsilon_i)}{\Theta(P(v_i+\varepsilon_i))}\int_{v_i+\varepsilon_i}^{v_{i+1}-\varepsilon_{i+1}}\hat\beta\Gamma(v)\omega(v)dH(v)\right)$$
$$-\int_{v_1-\varepsilon_1}^{v_{n+1}-\varepsilon_{n+1}}\hat\beta\Gamma(v)\Theta(P(v))dH(v).$$

The first term on the first line is the cost within the pooling regions and the second term is the cost in the separating regions in the mixed equilibrium. The second line is the cost in the semi-separating equilibrium.

We substitute the previous expressions for $\omega(v_i)$ and $\omega(v_i+\varepsilon_i)/\Theta(P(v_i+\varepsilon_i))$ into this expressions, then perform a Taylor expansion of this sum near $\varepsilon_j=0$, using the approximation for p_i given in equation (18). This gives that the increase in first period cost is

$$\sum_{i=1}^{n}\frac{2\hat\beta\Gamma'(v_i)}{3(\hat\beta\Gamma(v_i)-v_i)^2}\left(\left(\frac{h'(v_i)}{h(v_i)}-\frac{2-\hat\beta\Gamma'(v_i)}{\hat\beta\Gamma(v_i)-v_i}\right)\int_{v_i}^{\infty}\hat\beta\Gamma(v)\Theta(P(v))dH(v)\right.$$
$$\left.+\hat\beta\Theta(P(v_i))h(v_i)(2\Gamma(v_i)-v_i\Gamma'(v_i))\right)\varepsilon_i^3+\sum_{i=1}^{n}O(\varepsilon_i^4). \quad (20)$$

A quick comparison of this expression with equation (19) shows that the first period cost of the mixed equilbrium exceeds the first period cost of the semi-separating equilibrium with the same value of $\hat\beta$ whenever the mixed equilibrium leaves all sellers better off and the elasticity of Γ is smaller than 2. Because $G_b(\beta)$ is continuous, this is inconsistent with Part

4 of the definition of equilibrium, market clearing. ■

C Pareto Efficient Allocations

C.1 Incentive Feasible Allocations

We start by using the revelation principle to define the set of incentive compatible and feasible allocations. Each seller and buyer reports his or her private information to a mechanism, which then recommends certain trades. Without loss of generality, we focus on incentive-compatible mechanisms and we verify that the resulting trades are feasible.

We start with buyers. Each buyer reports her discount factor β to the mechanism and receives consumption $c_1^B(\beta)$ in period 1 and $c_2^B(\beta)$ in period 2. The mechanism must be incentive compatible, so a buyer prefers to report her true type β rather than misreporting it as some other $\tilde{\beta}$:

$$u^B(\beta) = c_1^B(\beta) - 1 + \beta c_2^B(\beta) \geq c_1^B(\tilde{\beta}) - 1 + \beta c_2^B(\tilde{\beta}) \tag{21}$$

for all β and $\tilde{\beta}$, where $u^B(\beta)$ is the buyer's gain from trade. In addition, the mechanism must satisfy the buyer's participation constraint, $u^B(\beta) \geq 0$ for all β.

Turning now to sellers, each seller reports his continuation value v to the mechanism, getting expected consumption $c^S(v)$ in period 1 and giving up his asset with probability $\omega(v)$.[14] Again, the mechanism must be incentive compatible, so a seller prefers to report his true type v rather than misreporting it as some other $\tilde{v}$:

$$u^S(v) = c^S(v) - v\omega(v) \geq c^S(\tilde{v}) - \omega(\tilde{v})v$$

for all v and $\tilde{v}$, where $u^S(v)$ is the seller's gain from trade. In addition, the mechanism must satisfy the seller's participation constraint, $u^S(v) \geq 0$ for all v.

Standard arguments imply that the seller's mechanism is incentive compatible if and only if $\omega(v) \in [0,1]$ is non-increasing and

$$c^S(v) = \int_v^{\bar{v}} \omega(x)dx + v\omega(v) + k$$

[14]We assume that a seller only reports his continuation value, rather than both his discount factor and his asset quality. It is an open question whether a mechanism that allows a seller to separately report his asset quality and discount factor would do better still.

for some constant k. Substituting this back into the expression for $u^S(v)$ in the previous paragraph gives

$$u^S(v) = \int_v^{\bar{v}} \omega(x)dx + k. \tag{22}$$

The seller's participation constraint imposes that $k \geq 0$, which in turn also guarantees that $c^S(v) \geq 0$ for all v.

We next turn to feasibility, i.e. the cost of this mechanism. We start with the buyers' cost. In period 1, a buyer with discount factor β consumes $c_1^B(\beta)$ units of the consumption good per unit of endowment. Allowing for free disposal, the cost is therefore

$$C_1^B \geq \int_{\underline{\beta}}^{\bar{\beta}} (c_1^B(\beta) - 1)dG_1(\beta). \tag{23}$$

In period 2, the buyers have no endowment and receive $c_2^B(\beta)$ units of the consumption good per unit of endowment. Thus the cost is simply

$$C_2^B \geq \int_{\underline{\beta}}^{\bar{\beta}} c_2^B(\beta)dG_1(\beta). \tag{24}$$

Now turn to the sellers' cost. In period 1, the sellers receive $c^S(v)$ units of the consumption good, so the cost is

$$C_1^S \geq \int_{\underline{v}}^{\bar{v}} c^S(v)h(v)dv = \int_{\underline{v}}^{\bar{v}} \omega(v)(H(v) + vh(v))dv + k, \tag{25}$$

where the equality uses incentive compatibility and integration by parts. The total cost of the mechanism in period 2 is negative, given by the amount of dividends collected from the sellers:

$$C_2^S \geq -\int_{\underline{v}}^{\bar{v}} \omega(v)\Gamma(v)h(v)dv. \tag{26}$$

The buyers' and sellers' mechanisms are feasible if total costs are zero in each period, $C_1^B + C_1^S = C_2^B + C_2^S = 0$.

It is straightforward to verify that any equilibrium allocation is incentive compatible and feasible, but the converse is not true. The most important difference lies in the trading probability $\omega(v)$. Incentive compatibility and feasibility only restricts $\omega(v)$ to lie between 0 and 1 and be non-increasing. We argued in Section 3.2 that equilibrium imposes additional restrictions on ω; for example, a pair of discontinuities must surround any constant portion of ω. It follows that some incentive compatible and feasible allocations cannot be supported in any equilibrium.

C.2 Buyer Efficiency

We say an allocation is *buyer efficient* if it is incentive-compatible, feasible, and Pareto optimal for buyers among all the incentive-compatible, feasible allocations with the same buyer cost (C_1^B, C_2^B). We prove that any buyer efficient allocation is characterized by a threshold $\hat{\beta}$. Buyers with discount factor $\beta < \hat{\beta}$ consume only in the first period, while buyers with $\beta > \hat{\beta}$ consume only in the second period. A buyer with discount factor $\hat{\beta}$ is indifferent between consuming in the two periods.

Proposition 5 *Let b and $\hat{\beta}$ solve*

$$C_1^B = \big(bG_b(\hat{\beta}) - 1\big) \text{ and } C_2^B = \frac{(G_b(\bar{\beta}) - G_b(\hat{\beta}))b}{\hat{\beta}}. \tag{27}$$

If this defines $b \geq 1$, then any buyer efficient allocation has

$$c_1^B(\beta) = \begin{cases} b & \text{if } \beta < \hat{\beta} \\ 0 & \text{if } \beta > \hat{\beta} \end{cases} \quad \text{and} \quad c_2^B(\beta) = \begin{cases} 0 & \text{if } \beta < \hat{\beta} \\ b/\hat{\beta} & \text{if } \beta > \hat{\beta} \end{cases}.$$

Otherwise there is no incentive-compatible, feasible allocation with cost (C_1^B, C_2^B).

Proof of Proposition 5. The proposed allocation is incentive compatible, has $c_1^B(\beta)$ and $c_2^B(\beta)$ nonnegative, and satisfies the feasibility constraints (23) and (24). Now consider a competitive equilibrium of an economy in which each individual with $\beta < \hat{\beta}$ has an endowment of b in period 1 and 0 in period 2, while each individual with $\beta \geq \hat{\beta}$ has an endowment of 0 in period 1 and $b/\hat{\beta}$ in period 2. It is easy to verify the equilibrium involves no trade. The first welfare theorem implies this allocation is Pareto optimal among all allocations satisfying the two feasibility constraints. It is therefore Pareto optimal among the smaller set of allocations that also satisfy the incentive constraint (21). ■

A corollary of this result is that any equilibrium of our model is buyer efficient. This is not surprising, since there is no interesting information problem on the buyer's side of the market. A buyer is privately informed about her discount factor, but a seller does not care about the buyer's discount factor when they trade.[15] This contrasts with the seller's side of the market, since a buyer cares about a seller's expected valuation v, which is private information. We turn to the seller's problem next.

[15] Comparing Propositions 2 and 5 shows that a buyer efficient allocation with $b > 1$ can never be supported in equilibrium. Such an allocation would require an initial redistribution of $b - 1$ units of the period 1 consumption good to each buyer.

C.3 Seller Efficiency

An allocation is *seller efficient* if it is incentive compatible, feasible, and Pareto optimal for sellers among all the incentive-compatible, feasible allocations with the same seller cost (C_1^S, C_2^S). This section provides necessary and sufficient conditions for a semi-separating equilibrium to be seller efficient:

Proposition 6 *If Assumption 1 holds and $\Gamma(\underline{v}) > 0$, the semi-separating equilibrium is seller efficient if and only if there exist non-negative numbers ψ_1 and ψ_2 satisfying the following conditions:*

- $\psi_1 \geq 1$,
- $J(\underline{v}) = 0$,
- $J(v)$ *nondecreasing for* $v \in [\underline{v}, \bar{p}]$,
- $J(\bar{p}) = 1$, *and*
- $\int_{\bar{p}}^{v} J(x)dx/(v - \bar{p}) \geq 1$ *for* $v > \bar{p}$,

where $J(v) \equiv \psi_1(H(v) + vh(v)) - \psi_2\Gamma(v)h(v)$.

Proof of Proposition 6. To start, assume that the semi-separating equilibrium is seller efficient. This means that there are nondecreasing integrated Pareto weights $\Lambda(v)$ with $\Lambda(\underline{v}) \geq 0$ and $\Lambda(\bar{v}) = 1$,[16] such that the allocation maximizes the Pareto-weighted sum of seller utilities,

$$\int_{\underline{v}}^{\bar{v}} u^S(v)d\Lambda(v),$$

among all incentive compatible and feasible allocations. Eliminate $u^S(v)$ using equation (22) and perform integration-by-parts to rewrite the Pareto-weighted sum of utilities as

$$\int_{\underline{v}}^{\bar{v}} \omega(v)(\Lambda(v) - \Lambda(\underline{v}))dv + k. \tag{28}$$

Any seller-efficient allocation maximizes (28) subject to $\omega(v) \in [0, 1]$ non-increasing, $k \geq 0$, and the two resource constraints (25) and (26) for some nondecreasing integrated Pareto weights $\Lambda(v)$.

[16]The integrated Pareto weight $\Lambda(v)$ is the sum of the Pareto weights on sellers with continuation value less than or equal to v, so the Pareto weight on v is $d\Lambda(v)$.

Write the Lagrangian of the Pareto-weighted maximization problem, placing nonnegative multipliers ψ_1 and ψ_2 on the two constraints (25) and (26):

$$\mathcal{L} = \int_{\underline{v}}^{\bar{v}} \omega(v)\phi(v)dv + (1-\psi_1)k + \psi_1 C_1^S + \psi_2 C_2^S \tag{29}$$

subject to $k \geq 0$, and $\omega(v) \in [0,1]$ non-increasing, where $\phi(v) \equiv \Lambda(v) - \Lambda(\underline{v}) - J(v)$ with

$$J(v) \equiv \psi_1(H(v) + vh(v)) - \psi_2 \Gamma(v)h(v).$$

The Lagrangian is linear in k, which implies that $\psi_1 \geq 1$; otherwise raising k would increase the Lagrangian without bound. In addition, integration by parts implies

$$\int_{\underline{v}}^{\bar{v}} \omega(v)\phi(v)dv = \omega(\bar{v})\Phi(\bar{v}) - \int_{\underline{v}}^{\bar{v}} \Phi(v)d\omega(v),$$

where $\Phi(v) \equiv \int_{\underline{v}}^{v} \phi(x)dx$. Therefore the Lagrangian is also linear in $d\omega(v)$, which implies that $\omega(v)$ is constant at any v that does not maximize $\Phi(v)$. In the semi-separating equilibrium, $\Theta(P(v))$ is strictly decreasing for all $v \in [\underline{v}, \bar{p}]$. Therefore if the equilibrium is Pareto efficient, all values of v in this interval must maximize $\Phi(v)$. We use this to characterize the conditions for Pareto efficiency.

Now assume there is a pair (ψ_1, ψ_2) such that the five conditions in the statement of the proposition hold. Set $\Lambda(v) = J(v)$ for $v \in [\underline{v}, \bar{p}]$ and $\Lambda(v) = 1$ for $v > \bar{p}$. The first condition ensure that $k = 0$ is optimal with these Pareto weights and Lagrange multipliers. The next three conditions ensure that $\Lambda(\underline{v}) = 0$, $\Lambda(v)$ is nondecreasing, and $\Lambda(\bar{p}) = 1$, so $d\Lambda(v)$ are valid Pareto weights. By construction $\phi(v) = \Phi(v) = 0$ for all $v \in [\underline{v}, \bar{p}]$ and $\Phi(v) = \int_{\bar{p}}^{v}(1 - J(x))dx \leq 0$ for all $v > \bar{p}$ using the final condition. Therefore any function $\omega(v)$ that is strictly decreasing on $[\underline{v}, \bar{p}]$ and 0 at higher values of v maximizes the Lagrangian. In particular, the semi-separating equilibrium is Pareto optimal.

Conversely, suppose there is no pair (ψ_1, ψ_2) satisfying these five conditions. If the first condition failed, the Lagrangian would not have a maximum and so the semi-separating equilibrium allocation would not maximize it. If any of the next three conditions failed, any nondecreasing Pareto weight $\Lambda(v)$ would have $\phi(v) = \Lambda(v) - \Lambda(\underline{v}) - J(v) \neq 0$ for some $v \in [\underline{v}, \bar{p}]$; therefore not all $v \in [\underline{v}, \bar{p}]$ would maximize $\Phi(v)$ and any solution to the Lagrangian must have $d\omega(v)$ constant at such v, inconsistent with the semi-separating equilibrium allocation. And if the fifth condition failed, $\Phi(v) > 0$ at some $v > \bar{v}$, so again any solution to the Lagrangian must have $d\omega(v) = 0$ at all $v \leq \bar{v}$, inconsistent with the semi-separating equilibrium allocation. ■

We use this proposition to prove the results in Section 5.3. As in the text, assume

$$\Gamma(v) = \frac{1+v}{2} \text{ and } H(v) = 1 - (\alpha+1)v^{-\alpha} + \alpha v^{-\alpha-1}.$$

First assume $0 < \alpha \leq 2$.[17] If $\hat{\beta} < 2$, set $\psi_1 = \psi_2 = \big(H(\bar{p}) + \bar{p}h(\bar{p}) - \Gamma(\bar{p})h(\bar{p})\big)^{-1} > 1$. If $\hat{\beta} \geq 2$, set $\psi_1 = \psi_2 = 1$. It is easy to verify that $J(v)$ is increasing with $J(1) = 0$ and $J(\bar{p}) = 1$. Proposition 6 implies that the semi-separating equilibrium is seller efficient.

If instead $\alpha > 2$ and $\hat{\beta} \geq 2$ (so $\bar{p} = \infty$), then the semi-separating equilibrium is not seller efficient. If $\psi_1 < \psi_2$, $J'(1) < J(1) = 0$, which implies $J(v)$ is negative at values of v slightly above 1, inconsistent with a seller-efficient allocation. If $\psi_1 \geq \psi_2$, $J(v)$ is decreasing at sufficiently large values of v, again inconsistent with a seller efficient allocation when $\bar{p} = \infty$.

To construct a Pareto improvement in this example, it is not enough to pool a single group of sellers. That will always either reduce some sellers' utility or raise costs in one of the periods. Instead, we must pool investors within two separate intervals.

We illustrate this with a concrete example. As in the text, assume $\alpha = 3$ and $\hat{\beta} = 2$, so that $\Theta(P(v)) = e^{1-v}$. First consider a pool with radius ε in the neighborhood of some $v > 1 + \varepsilon$, setting $\omega(v)$ equal to the average value of $\Theta(P(v))$ within this pool, $\omega(v) = e^{1-v}(e^{\varepsilon} - e^{-\varepsilon})/(2\varepsilon)$. By construction, this increases welfare relative to the semi-separating equilibrium for all $v' \in (v - \varepsilon, v + \varepsilon)$, while welfare is unchanged for other sellers; see equation (22). Thus the pool is Pareto improving if it is cost feasible.

Taking a Taylor expansion of costs in a neighborhood of $\varepsilon = 0$, we find that the first period cost of the pool in excess of the cost of the semi-separating allocation, is

$$\int_{v-\varepsilon}^{v+\varepsilon} \big(\omega(v) - \Theta(P(x))\big)(H(x) + xH'(x))dx = \frac{8e^{1-v}(3-2v)}{v^5}\varepsilon^3 + O(\varepsilon^4)$$

which is negative if $v > 3/2$. The second period cost of this pool, again in excess of the cost of the semi-separating allocation, is

$$-\int_{v-\varepsilon}^{v+\varepsilon} \big(\omega(v) - \Theta(P(x))\big)\Gamma(x)H'(x)dx = \frac{4e^{1-v}(3v^2-5)}{v^6}\varepsilon^3 + O(\varepsilon^4),$$

which is negative if $v < \sqrt{5/3}$. Since these regions do not overlap, any single pool must raise costs in one of the two periods.

But now consider two such pools, one with radius ε_1 in a neighborhood of some $v_1 < \sqrt{5/3}$

[17] With $\alpha \leq 1$, total dividends held by sellers are infinite, $\int_1^\infty \Gamma(v)H'(v)\,dv = \infty$, which might seem worrisome for constructing an equilibrium. Nevertheless, total dividends sold are bounded above by the dividends of the worst asset: $\int_1^\infty \Theta(P(v))\Gamma(v)H'(v)\,dv < 1$ for any value of $\hat{\beta}$, so the market clearing condition can hold.

and the other with radius ε_2 in a neighborhood of some $v_2 > 3/2$. Manipulating the above expressions, we find that for small values of ε_1 and ε_2, the costs are negative in both periods if

$$\left(\frac{e^{1-v_2}(2v_2-3)v_1^5}{e^{1-v_1}(3-2v_1)v_2^5}\right)^{1/3} > \frac{\varepsilon_1}{\varepsilon_2} > \left(\frac{e^{1-v_2}(3v_2^2-5)v_1^6}{e^{1-v_1}(5-3v_1^2)v_2^6}\right)^{1/3}$$

Simplifying these inequalities, we find that if $v_1 \in (1, 10/9)$ and $v_2 > \frac{5(3-2v_1)}{10-9v_1}$, the inequalities on $\varepsilon_1/\varepsilon_2$ define a non-empty open interval on the nonnegative real line. This means that in a neighborhood of such v_1 and v_2, we construct two small pools. Each pool alone would raise costs in one of the periods, but the two pools together reduce costs in both periods.

We can also compute the expected price within such a pool, the ratio of the buyer's cost in period 1 to the amount of dividends he gets in period 2. Again using a Taylor expansion, this is

$$\frac{\omega(v+\varepsilon)+(v+\varepsilon)\omega(v)}{\omega(v)\frac{\int_{v-\varepsilon}^{v+\varepsilon}\Gamma(x)H'(x)dx}{H(v+\varepsilon)-H(v-\varepsilon)}} = 2 + \frac{2(v^2+3v-5)}{3v(v^2-1)}\varepsilon^2 + O(\varepsilon^3),$$

which is bigger than 2 when v is bigger than $(\sqrt{29}-3)/2 \approx 1.19$ and smaller than 2 at lower values. In other words, buyers get a low price when they buy from the low pool, $v \in (v_1-\varepsilon_1, v_1+\varepsilon_1)$, but they pay a high price when they buy from the high pool, $v \in (v_2-\varepsilon_2, v_2+\varepsilon_2)$. Randomizing between both pools allows them to make money in expected value. The example in the text, with pools for $v \in [1, 1.01]$ and $v \in [8.3, 11.3]$, is based on this calculation but does not use limits as the radius of the pools vanish.

C.4 Local Pareto Efficiency

In the previous two sections, we asked whether, starting from a semi-separating equilibrium, it is possible to improve the welfare first of buyers and then of sellers without affecting the other group of investors, i.e. taking the costs C_1^B, C_2^B, C_1^S, and C_2^S as given. This section examines the possibility of achieving a Pareto improvement by moving costs across periods in a manner consistent with the resource constraint.

To understand the scope for this, we need to understand how buyers' and sellers' utility is affected by changes in the costs. We focus here on marginal changes in the costs, again starting from a semi-separating equilibrium. We say an allocation is *locally Pareto efficient* if it is buyer- and seller-efficient and if no small resource-feasible change in the costs generates a Pareto improvement.

Proposition 7 *Impose Assumption 1 and* $\Gamma(\underline{v}) > 0$ *and suppose that the semi-separating equilibrium is seller efficient. If there exists a pair* (ψ_1, ψ_2) *that not only satisfies all condi-*

tions in Proposition 6 but also

$$\frac{G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})}{g(\hat{\beta})} > \frac{\psi_2}{\psi_1} > \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})},$$

then the semi-separating equilibrium is locally Pareto efficient. Otherwise it is not locally efficient.

Proof of Proposition 7. Proposition 5 describes the buyer efficient allocation. Buyers' utility is

$$u^B(\beta) = \begin{cases} b - 1 & \text{if } \beta < \hat{\beta} \\ \beta b / \hat{\beta} - 1 & \text{if } \beta \geq \hat{\beta}, \end{cases}$$

where b and $\hat{\beta}$ depend on C_1^B and C_2^B through equation (27). Implicitly differentiating this expression, we get that a change in (C_1^B, C_2^B) of magnitude (dC_1^B, dC_2^B) raises the utility of buyers with $\beta < \hat{\beta}$ if and only if

$$dC_1^B + \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})} dC_2^B > 0$$

The same change raises the utility of buyers with $\beta > \hat{\beta}$ if and only if

$$\frac{g_b(\hat{\beta})}{G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})} dC_1^B + dC_2^B > 0.$$

Note that

$$\frac{G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})}{g_b(\hat{\beta})} \geq \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})},$$

as can be confirmed algebraically. This means that if buyers $\beta < \hat{\beta}$ like the perturbation (dC_1^B, dC_2^B) with $dC_2^B \geq 0$, all buyers like the perturbation. And if buyers $\beta > \hat{\beta}$ like the perturbation (dC_1^B, dC_2^B) with $dC_2^B \leq 0$, all buyers like the perturbation.

Next, a feasible change in the costs satisfies $dC_1^S = -dC_1^B$ and $dC_2^S = -dC_2^B$ and so in particular $\psi_1 dC_1^S + \psi_2 dC_2^S = -\psi_1 dC_1^B - \psi_2 dC_2^B$. Proposition 6 then implies that if $\psi_1 dC_1^B + \psi_2 dC_2^B < 0$ for any (ψ_1, ψ_2) consistent with the conditions in the Proposition, the equilibrium is not locally Pareto efficient.

Putting these results together, the equilibrium is locally Pareto efficient if there exists a (ψ_1, ψ_2) consistent with the conditions in Proposition 6 such that

1. for any $dC_2^B > 0$, $dC_1^B + \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1 - G_b(\hat{\beta}) + \hat{\beta} g_b(\hat{\beta})} dC_2^B < 0$ or $\psi_1 dC_1^B + \psi_2 dC_2^B \geq 0$, and

2. for any $dC_2^B < 0$, $\frac{g_b(\hat{\beta})}{G_b(\hat{\beta})+\hat{\beta}g_b(\hat{\beta})}dC_1^B + dC_2^B < 0$ or $\psi_1 dC_1^B + \psi_2 dC_2^B \geq 0$.

Part 1 holds if and only if $\frac{\psi_2}{\psi_1} > \frac{\hat{\beta}^2 g_b(\hat{\beta})}{1-G_b(\hat{\beta})+\hat{\beta}g_b(\hat{\beta})}$, while part 2 holds if and only if $\frac{G_b(\hat{\beta})+\hat{\beta}g_b(\hat{\beta})}{g(\hat{\beta})} > \frac{\psi_2}{\psi_1}$. ■

The Lagrange multipliers ψ_1 and ψ_2 give the marginal value of funds to the sellers in each period, thus their ratio is the marginal rate of substitution of funds across the two periods. The first ratio involving $G_b(\hat{\beta})$ is the marginal rate of substitution for active buyers, those with $\beta > \hat{\beta}$. The last ratio is the marginal rate of substitution for inactive buyers, those with $\beta < \hat{\beta}$. If the marginal rate of substitution for sellers lies in between these two marginal rates of substitution, there is no way to make all investors better off by reallocating resources across periods.

To see how to apply this Proposition, we build on our previous example with independent Pareto distributions. Assume $0 \leq \alpha_\delta \leq 1$. For any $\psi_2 > \psi_1$, $J'(1) < 0$, so there is no associated seller-efficient allocation, while any ratio $\psi_2/\psi_1 \geq 0$ gives us valid Pareto weights for the semi-separating equilibrium. Therefore the semi-separating equilibrium is locally Pareto efficient if and only if $\frac{\hat{\beta}^2 g_b(\hat{\beta})}{1-G_b(\hat{\beta})+\hat{\beta}g_b(\hat{\beta})} < 1$.[18] Since these conditions hinge on the value of $G_b(\hat{\beta})$ and $g_b(\hat{\beta})$, they may or may not hold in any particular economy.

[18] If $1 < \alpha_\delta \leq 2$, there is also a lower bound on the ratio ψ_2/ψ_1 for generating valid Pareto weights, say $\psi_2/\psi_1 \geq \bar{\psi}$, where $\bar{\psi} \in (0, 1]$. We therefore also require $\frac{G_b(\hat{\beta})+\hat{\beta}g_b(\hat{\beta})}{g(\hat{\beta})} > \bar{\psi}$ in order for the semi-separating equilibrium to be locally Pareto efficient.