Love and Chance: Equilibrium and Identification in a Large NTU Matching Markets With Stochastic Choice

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Matchings: a new class of discrete choice models

- Review of standard discrete choice models
- Two items: A and B with characteristics X_A and X_B respectively.
- If A is chosen, then utility maximization implies that $U(X_A) + \epsilon_A > U(X_B) + \epsilon_B$.
- MLE or other inference methods based on this restriction.

Problem: Endogenously-Determined Utility and Choice Set

- Why the standard discrete choice model fails?
- Example: Tom marries Tina instead of Jennifer
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  - Tom prefers Tina.
  - Jennifer does not like Tom.
  - Other 10 men all want to marry Jennifer
- Externality
  - The utility is determined by the actions of other players. (similar to normal form game; e.g. Ciliberto and Tamer, 2009)

Problem: Endogenously-Determined Utility and Choice Set

- Availability-Constrained Choice Set
  - Every man can only choose his spouse from the set of women who are willing to marry him, and vice versa.
  - Exclusiveness: if someone is married, he/she is not available and hence cannot be included in the choice set of other agents.

Structural Models based on Individual-level Matchings

- The classical matching theory predicts stable matchings at the individual level (Roth and Sotomayor, 90).
- Simulation-based approaches: Agarwal (2014), Boyd et al. (2013), and Logan, Hoff and Newton (2008).
  - Identification?
  - Computational issues similar to estimating exponential random graph models.

Aggregate Matchings

- From policy makers' point of view, it is much easier to analyze the policy implications by investigating the aggregate statistics derived from the individual matchings; e.g., sorting patterns of education attainment in marriage market, family background and school characteristics...etc.
- Pioneered by Dagsvik (2000), Choo and Siow (2006), and Echenique, Lee, Shum and Yenmez (2013).
- Possible to derive equilibrium aggregate statistics, without solving individual-level matchings
- Easy to analyze identification and simple estimators.

Large Matching Markets as a Two-Sided Demand System

- Assume agents have preference over partners' observed (discrete) characteristics.
- If man i marries woman j: $\alpha_{ij} = \alpha_{x_i y_j} + \epsilon_{i y_j}$
- If woman j marries man i: $\gamma_{ij} = \gamma_{x_i y_j} + \eta_{x_i j}$
- Discrete choice demand system (a la BLP...)

Large Matching Markets as a Two-Sided Demand System

- Suppose there are n_x type-x men and m_y type-y women
- Define $P^m_{y|x} = Prob\{y = argmax_{y \in \mathcal{Y}_0}(\alpha_{xy} + \epsilon_{iy})\}$
- Define $P^w_{x|y} = Prob\{x = argmax_{x \in \mathcal{X}_0}(\gamma_{xy} + \eta_{xj})\}$
- $n_x \cdot P^m_{y|x} \neq m_y \cdot P^w_{x|y}$ $(70 \neq 50)$, given arbitrary utility parameters $(\alpha_{xy}, \gamma_{xy})$.
- Need a transfer technology (price mechanism) to clear the market

Choo and Siow's (JPE 2006) Solution

- Transferable Utility Matchings: type-x men has to pay τ_{xy} in order to marry type-y women.
- $P^m_{y|x} = Prob\{y = argmax_{y \in \mathcal{Y}_0}(\alpha_{xy} - \tau_{xy} + \epsilon_{iy})\}$;
  $P^w_{x|y} = Prob\{x = argmax_{x \in \mathcal{X}_0}(\gamma_{xy} + \tau_{xy} + \eta_{xj})\}$
- If there are more type-x men who demand type-y women, one can increase τ_{xy}, the price of type-xy marriage, until $n_x \cdot P^m_{y|x} = m_y \cdot P^w_{x|y}$ $(60 = 60)$.
- Decompose aggregate matchings into two discrete choice problems, subject to market clearing condition.

This Paper: the case of NTU Matchings

- No actual price mechanism exists, but one can define a shadow price system to support the equilibrium matching.
  - Easy to define "aggregate" stable matchings
  - Efficient algorithm for large type space

- There is no way to shift agents' preference, but at least $\mu_{xy} = \min\{n_x \cdot P^m_{y|x}, m_y \cdot P^w_{x|y}\}(50 = \min\{70, 50\})$ matchings can be created at this moment. This 50 couples get their best allocations, and hence are stable.

- It is as if men are charged $\tau_{xy} > 0$ and women receive zero transfer such that $n_x \cdot Prob\{y = argmax_{y \in \mathcal{Y}_0}(\alpha_{xy} - \tau_{xy} + \epsilon_{iy})\} = m_y \cdot Prob\{x = argmax_{x \in \mathcal{X}_0}(\gamma_{xy} + \eta_{xj})\}$.

- Either men or women should pay the shadow price (but not both), depending on who are on the long side.

Contributions

- Characterize the set of aggregate stable matchings as a nonlinear complementarity problem. It has a shadow price interpretation.
- Provide two aggregate versions of the Deferred-Acceptance algorithm.
- Derive a Leontief marriage matching function.
- Partial and point identification results.
- Implementing likelihood-based inference. Can accommodate continuous characteristics.

Definition of Aggregate Matchings

- Men and women can be grouped according to their observed discrete characteristics, denoted by $\mathcal{X}$ and $\mathcal{Y}$.
- n_x men of type $x \in \mathcal{X}$ and m_y women of type $y \in \mathcal{Y}$
- An aggregate matching, is a contingency table $(\mu_{xy})_{x\in\mathcal{X},y\in\mathcal{Y}}$ counting the number of matches between x-type men and y-type women.
- The number of x-type men who remain single is denoted by $\mu_{x0} = n_x - \sum_{y\in\mathcal{Y}} \mu_{xy}$ and the number of y-type women who remain single is denoted by $\mu_{0y} = m_y - \sum_{x\in\mathcal{X}} \mu_{xy}$.

Assumptions on the Unobserved Heterogeneity

- If man i marries woman j: $\alpha_{ij} = \alpha_{x_i yj} + \epsilon_{iy_j}$
- If woman j marries man i: $\gamma_{ij} = \gamma_{x_i y_j} + \eta_{x_i j}$
- a) For any man i such that $x_i = x$, ϵ_{iy} is a $|\mathcal{Y}_0|$-dimensional random vector drawn from a zero-mean distribution $\mathbf{P}_x$;
- b) For any woman j such that $y_j = y$, η_{xj} is a $|\mathcal{X}_0|$-dimensional random vector drawn from a zero-mean distribution $\mathbf{Q}_y$;
- c) $\mathbf{P}_x$ and $\mathbf{Q}_y$ have full support and are absolutely continuous with respect to the Lebesgue measure.

Deriving Aggregate Stability Condition form Individual-level Stability Condition

- Let u_i and v_j be the equilibrium payoffs of agents under the matching $\tilde{\mu}$.

  No-Blocking-Pair implies that a pair of agents should not be able to both increase their welfare by deviating from their current marriage, which is expressed as

  $$\max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) \geq 0$$

  $$\tilde{\mu}_{ij} > 0 \text{ implies } \max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) = 0.$$

Deriving Aggregate Stability Condition form Individual-level Stability Condition

- One can derive the aggregate version of no-blocking pair condition

$$\max\left(U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}\right) \geq 0$$

- $\mu_{xy} > 0$ implies $\max\left(U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}\right) = 0$
- By the full support assumption on $\mathbf{P}_x$ and $\mathbf{Q}_y$, $\mu_{xy} > 0$ for all (x, y)
- (U_{xy}, V_{xy}) are the endogenously determined systemic utility.

Tools from Galichon and Salanie 2014 (General TU Matchings)

- Consider the discrete choice problem based on the endogenous (equilibrium) systemic utility: $U_{xy} + \epsilon_{iy}$ and $V_{xy} + \eta_{xj}$. (McFadden 76, Harsanyi 73, Machina 85 and Fudenberg et al 2014.)
- The corresponding total indirect surplus of men is

$$G(U) = \sum_{x \in \mathcal{X}} n_x \mathbb{E}_{\mathbf{P}_x} \left[\max_y \left(U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right) \right],$$

- (Similar to the inclusive value in the logit regression)

Tools from Galichon and Salanie 2014 (General TU Matchings)

- Its Legendre-Fenchel transformation is given by

$$G^* (\mu) = \sup_U \left\{ \sum_{xy} \mu_{xy} U_{xy} - G(U) \right\}$$

- By the Fenchel conjugation relations:

$$G(U) = \sup_\mu \left\{ \sum_{xy} \mu_{xy} U_{xy} - G^* (\mu) \right\}.$$

Tools from Galichon and Salanie 2014 (General TU Matchings)

- We define similarly

$$H(V) = \sum_{y \in \mathcal{Y}} m_y \mathbb{E}_{\mathbf{Q}_y} \left[\max_x \left(V_{xy} + \eta_{xj}, \eta_{0j} \right) \right], \text{ and}$$

$$H^*(\mu) = \sup_V \left\{ \sum_{xy} \mu_{xy} V_{xy} - H(V) \right\}.$$

Notion of Semi-Matchings

- Results from convex analysis:
- $\mu = \nabla G(U)$(Williams-Daly-Zachary theorem) if and only if $U = \nabla G^*(\mu)$ (similar to CCP inversion)
- Given arbitrary $(n_x, m_y, \mathbf{P}_x, \mathbf{Q}_y, U, V)$, one can compute the *semi-matching* of men via $\mu^m = \nabla G(U)$, and the semi-matching of women via $\mu^w = \nabla H(V)$. The (aggregate) market clearing condition requires demand equal to supply: $\mu^w = \mu^m = \mu$.
- Note: the notion of semi-matching has nothing to do with TU/NTU matchings per se.

Large NTU Matching Markets: Equilibrium Characterization

- Given $(n_x, m_y, \mathbf{P}_x, \mathbf{Q}_y, \alpha, \gamma)$, $(\mu_{xy}, U_{xy}, V_{xy})$ constitutes an aggregate stable matching if
- $\sum_y \mu_{xy} \leq n_x$, $\sum_x \mu_{xy} \leq m_y$ (Feasibility),
- $\mu = \nabla G(U) = \nabla H(V)$ (Market Clearing).
- $\max(U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}) = 0$ (No-Blocking Pair),
- The difference between endogenous and exogenous systemic utility $U_{xy} - \alpha_{xy}(V_{xy} - \gamma_{xy})$is interpreted as *shadow price*

Large NTU Matching Markets: Equilibrium Characterization

- Equivalent to find μ that solves the following nonlinear complimentarity problem

$$\max\left(\nabla G^{*}\left(\mu\right)-\alpha,\nabla H^{*}\left(\mu\right)-\gamma\right)=0. \tag{1}$$

- We propose two variants of Deferred-Acceptance algorithms to solve this problem, and prove existence.

Large NTU Matching Markets: Implied Marriage Matching Function

- In the logit case (ϵ_{iy}, η_{xj} are i.i.d Gumbel distributed), the master equation becomes $\max\left(\log(\mu_{xy}/\mu_{x0}) - \alpha_{xy}, \log(\mu_{xy}/\mu_{0y}) - \gamma_{xy}\right) = 0$
- That is,

$$\begin{aligned}
\mu_{xy} &= \min\left(\mu_{x0}\exp(\alpha_{xy}), \mu_{0y}\exp(\gamma_{xy})\right), \\
\mu_{x0} + \textstyle\sum_{y\in\mathcal{Y}} \mu_{xy} &= n_x, \\
\mu_{0y} + \textstyle\sum_{x\in\mathcal{X}} \mu_{xy} &= m_y.
\end{aligned}$$

- Aggregate marriage matching function implied by NTU matchings is Leontief.

Large NTU Matching Markets: Implied Marriage Matching Function

- Most of the MMF in literature are Cobb-Douglas (see Mourifie and Siow 2014 for a survey). Logit assumption is the key
- Choo-Siow: $\mu_{xy} = \mu_{x0}^{1/2} \mu_{0y}^{1/2} \exp(\alpha_{xy} + \gamma_{xy})$
- Dagsvik-Menzel: $\mu_{xy} = \mu_{x0} \mu_{0y} \exp(\alpha_{xy} + \gamma_{xy})$

Implied Marriage Matching Function and Identification

- Even without the logit assumption...
- Theorem: Given (μ_{xy}, n_x, m_y) satisfying the feasibility condition , and $(\mathbf{P}_x, \mathbf{Q}_y)$ satisfying the assumption 1, we have
- (i) The contour level I_μ of μ (set of (α, γ) that leads to μ) takes the Leontief form.

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- (ii) The kink point of the Leontief contour level is given by $(\hat{\alpha}, \hat{\gamma}) = (\nabla G^* (\mu), \nabla H^* (\mu))$. Namely, $I_\mu = \left\{ (\alpha_{xy}, \gamma_{xy})_{xy} : \min (\alpha_{xy} - \hat{\alpha}_{xy}, \gamma_{xy} - \hat{\gamma}_{xy}) = 0 \right\}$.

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- (iii) If $(\alpha, \gamma) = (\hat{\alpha}, \hat{\gamma})$, everyone is assigned to his/her most preferred type under μ.

Identification

- Identified set is fully characterized by $\left(\nabla G^{*}(\mu), \nabla H^{*}(\mu)\right)$ thanks to the Leontief form.
- Can be directly computed via any "CCP inversion" routine; e.g., Chiong, Galichon and Shum (2014). No need to run our deferred-acceptance algorithms to find it.
- $(\alpha_{xy}, \gamma_{xy})$ can be separately point identified if there exists two markets.

Implied Social Surplus

- In TU matchings, Galichon and Salanie (2014) show that

$$(\alpha - \tau^{TU}, \gamma + \tau^{TU}) = (\nabla G^* (\mu), \nabla H^* (\mu)),$$

- As a result, the social surplus $\Phi^{TU} = \alpha + \gamma$ (a la Choo and Siow (2006)) is identified by $\nabla G^* (\mu) + \nabla H^* (\mu)$
- $= (\log(\mu_{xy}/\mu_{x0}) + \log(\mu_{xy}/\mu_{0y}))$ in the logit case (Choo-Siow).

Implied Social Surplus

- In NTU matchings, we show that the kink point $(\hat{\alpha}, \hat{\gamma})$ is identified by

$$(\hat{\alpha}, \hat{\gamma}) = (\nabla G^* (\mu), \nabla H^* (\mu)),$$

- As a result, $\nabla G^* (\mu) + \nabla H^* (\mu)$ identifies the minimum possible social surplus.
- $\Phi^{NTU} \geq \Phi^{TU}$. (Intuitively, this is because part of the social surplus cannot be fully revealed in the observed matching μ due to market friction.)
- $\Phi^{NTU}_{GH} > \Phi^{NTU}_{DM}$ in the logit case.

DASMC: Deferred-Acceptance with Squential Market Clearing

- Step 1. Each man proposes to his most preferred type of women, if that type of women are still available. If unavailable, he continues to propose to the second preferred type, and so on.
- Step 2. Based on men's proposals, calculate the type-x men's aggregate demand for the type-y women D^m_{xy}.
- Step 3. Each woman proposes to her most preferred type of men, if that type of men are still available. If unavailable, she continues to propose to the second preferred type, and so on.
- Step 4. Based on women's proposals, calculate the type-y women's aggregate demand for the type-x men D^w_{xy}.

DASMC: Deferred-Acceptance with Squential Market Clearing

- Step 5. The number type-xy matches created in this round is given by $\mu_{xy}^{t} = \min(D_{xy}^{m}, D_{xy}^{w})$. $D_{xy}^{m} - \mu_{xy}^{t}$ ($D_{xy}^{w} - \mu_{xy}^{t}$) is the number of unmatched type-x men (type-y women) who shall repeat step 1-4.

- Step 6. The algorithm stops at iteration T when there is no more residual demand: $D_{xy}^{m} - \mu_{xy}^{T} = 0$ and $D_{xy}^{w} - \mu_{xy}^{T} = 0$ for all $(x, y) \in \mathcal{X}_0 \times \mathcal{Y}_0$. $\mu_{xy} = \sum_{t=1}^{T} \mu_{xy}^{T}$.

DASMC: An Numerical Example

- $(\alpha_{xy}, \gamma_{xy}) = 0; x, y \in \{1, 2\}$. α_{x0} and γ_{0y} are normalized to $-\infty$ so no one will remain single.
- Implies random preference list $R^m_{12|1} = R^m_{12|2} = R^w_{12|1} = R^w_{12|2} = 0.5$.
- marginal distribution of types $(n_1, n_2) = (m_1, m_2) = (70, 30)$
- want to determine the joint distribution of types μ

type of men/women	1	2	row sum
1	μ_{11}	μ_{12}	70
2	μ_{21}	μ_{22}	30
column sum	70	30	

Step 1. Create the Initial Matching Market

- Men propose first
- bimatrix representation (men's demand in the left and women's demand in the right).

men/ women	type 1	type 2	men's type dist.
type 1	(35,35)	(35,15)	70
type 2	(15,35)	(15,15)	30
women's type dist.	70	30	

Step 2. Create Matchings and Compute Residual Demands

- matrix of the number of couples created, C1

men/ women	type 1	type 2	men's type dist.
type 1	35	15	70
type 2	15	15	30
women's type dist.	70	30	

- bimatrix of residual demands.

men/ women	type 1	type 2	men's type dist.
type 1	(0,0)	(20,0)	
type 2	(0,20)	(0,0)	
women's type dist.			

Matching Market in the 2nd Round

- bimatrix of demand/supply

men/ women	type 1	type 2	men's type dist.
type 1	(20,20)	(0,0)	
type 2	(0,0)	(0,0)	
women's type dist.			

- matrix of the number of couples created, C2

men/ women	type 1	type 2	men's type dist.
type 1	20	0	70
type 2	0	0	30
women's type dist.	70	30	

Final Output

- bimatrix of the residual demands

men/ women	type 1	type 2	men's type dist.
type 1	(0,0)	(0,0)	70
type 2	(0,0)	(0,0)	30
women's type dist.	70	30	

- sum over the number of matches created in each round: $C1 + C2 =$

men/ women	type 1	type 2	men's type dist.
type 1	55	15	70
type 2	15	15	30
women's type dist.	70	30	

DASMC: Comments

- Only 2 iterations! Regardless of the number of players
- At each iteration, everyone is assigned to his/her best available partner, and hence the resulting matching is stable.
- The contingency table implied by the independent copula

men/ women	type 1	type 2	men's type dist.
type 1	49	21	70
type 2	21	9	30
women's type dist.	70	30	

- Since $55 > 49$, married couples' types are positively correlated. But the underlying utility functions have nothing to do with their types. Observed sorting pattern is completely driven by marginal distributions(Becker and Murphy (2000); Graham (2011).)

DASMC: Pros and Cons

- In general, only need small number of iterations.
- Intuitive.
- Need to evaluate the probability of all possible preference lists
  - High memory cost and hence only works for small problems.
  - Besides logit case, need high dimensional numerical integration.
- Hard to analyze its property.

DARUM: Deferred-Acceptance with Random Uitlity Models

- The intuition of DASMC: assign agents to their most preferred type of marriage, subject to availability at each iteration. Implementation is based on CCP and hence are memory-intensive.
- Equivalently, one can maximize men/women's social surplus subject to availability: theory of availability constrained discrete choice
- DARUM: Implementation is based on iterative utility maximization, and it works for large problems. Algorithm is based on fixed point iteration, and hence need more iterations in general.
  - Fortunately, DARUM is parallelizable! (DASMC can't)

DARUM

- We need to keep track of μ^A_{xy}, the number of offers of men x to women y which have not been rejected. Initially, one sets $\mu^{A,0}_{xy} = n_x m_y$
- At step k, men propose women among those who have not yet rejected them:

$$\mu^{P,k} = \arg\max_{\mu} \{ \sum_{xy} \mu_{xy} \alpha_{xy} - G^* (\mu) : \mu_{xy} \leq \mu^{A,k-1}_{xy} \}$$

DARUM

- In turn, women retain the offer they prefer among the ones they have received if any:

$$\mu^{D,k} = \arg\max_{\mu} \{ \sum_{xy} \mu_{xy} \gamma_{xy} - H^* (\mu) : \mu_{xy} \leq \mu^{P,k}_{xy} \}$$

- the count of offers that have not been rejected gets updated: a number $\mu^{P,k}_{xy} - \mu^{D,k}_{xy}$ of offers from men x to women y have been rejected, so

$$\mu^{A,k}_{xy} = \mu^{A,k-1}_{xy} - \left(\mu^{P,k}_{xy} - \mu^{D,k}_{xy} \right).$$

- The algorithms iterates until no offer gets rejected, that is $\mu^{P,k}_{xy} = \mu^{D,k}_{xy}$.

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Likelihood Inference

- Parametric model for utility: $\alpha_{xy} = x_i' A y_j$ or $\beta |x_i - y_j|$
- Likelihood (multinomial choice)is given by

$$\log L(\lambda) = 2 \sum_i \sum_j \tilde{\mu}_{ij} \log \mu_{ij} + \sum_i \tilde{\mu}_{i0} \log \mu_{i0} + \sum_j \tilde{\mu}_{0j} \log \mu_{0j}$$

Equilibrium constraints under logit:

$$\begin{aligned} \mu_{i0} + \sum_j \min\left(\mu_{i0} e^{\alpha_{x_i y_j}}, \mu_{0j} e^{\gamma_{x_i y_j}}\right) &= 1 \ \forall i \\ \mu_{0j} + \sum_i \min\left(\mu_{i0} e^{\alpha_{x_i y_j}}, \mu_{0j} e^{\gamma_{x_i y_j}}\right) &= 1 \ \forall j \\ \min\left(\mu_{i0} e^{\alpha_{x_i y_j}}, \mu_{0j} e^{\gamma_{x_i y_j}}\right) &= \mu_{ij} \ \forall i, j \end{aligned}$$