Cross-Border Equity Holdings and External Accounts*

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Abstract

Recent evidence on the importance of cross-border equity flows calls for a rethinking of the standard theory of external adjustment. We introduce equity holdings and portfolio choice into an otherwise conventional open-economy dynamic equilibrium model. Our model is simple and it admits an exact closed-form solution regardless of whether financial markets are complete or incomplete. In this simple framework, we are able to establish interconnections between the real side of the economy, represented by the trade balance, the current account, and consumption allocations and the financial side, such as portfolio holdings, stock prices and valuation changes. Methodologically, we contribute to the existing DSGE literature on international portfolio choice that so far has either relied on approximation methods or used simplified asset market structures.

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1. Introduction

Since David Hume's seminal work in 1752, "Of the Balance of Trade", economists have been confronted with the question of external account sustainability. The issue, in principle, is a simple one: are a country's sources of income today and in the future capable of financing its consumption plan? In practice, however, the answer has been far more elusive. In recent years, the swelling of cross-border equity holdings and capital gains on these holdings has complicated the matter even further. The recent empirical literature has argued that these capital gains, currently misreported in the national accounts because of the lack of marking to market, could account for most of the required adjustment in developed nations during the last decade. Although more work lies ahead, the empirical literature has advanced significantly. The theoretical counterpart, on the other hand, has encountered several obstacles. First, most open economy macroeconomics models abstract from equity markets and portfolio choice, which makes them inadequate for studying capital gains. Second, the few models that do take into consideration cross-country holdings typically require that financial markets are complete. In his Ohlin lecture, Maurice Obstfeld remarks that "Portfolio choice under incomplete markets is largely *terra incognita*". A handful of recently-developed models that make progress in this direction rely on new approximation methods that are still in the process of being developed.

In this paper, we offer a simple framework in which we incorporate portfolio choice and asset pricing in an otherwise standard open economy dynamic equilibrium model. We are able to describe asset prices, exchange rates, portfolio holdings, capital gains, and external accounts exactly, in closed form, regardless of whether financial markets are complete or incomplete. Our model allows us to understand the interconnections between portfolio decisions and the measures of external imbalances in the international economics literature. For example, in our model, a country that holds a riskier external portfolio vis-à-vis the rest of the world would enjoy a risk premium, which is used towards financing its trade deficit. A country like this can potentially run a persistent trade deficit and a current account deficit, effectively receiving wealth transfers from the rest of the world. Of course, during financial crises, such a country suffers more severe losses on its portfolio relative to the rest of the world. As Gourinchas and Rey (2007b) and Gourinchas, Govillot, and Rey (2010) document, this description fits the US, whose external portfolio is indeed riskier than that of the rest of the world, with its assets dominated by risky equity-type assets and the liabilities by safe government bonds. The US has been running a current account deficit for the last three decades, which has long puzzled many commentators. Our framework allows us to rationalize this fact and to also shed some light on conditions under which a particular risk profile is optimal for a country, how a country's external accounts respond to a shock in financial markets, and whether market incompleteness matters.

In our model, each country is endowed with a Lucas tree, producing a country-specific good. Each tree's output is driven by its own supply shock, and stocks are claims to the Lucas trees. There is also a bond that provides opportunities for riskless borrowing and lending. Each country is populated by a representative consumer/investor who has log-linear preferences over the two goods. Uncertainty in the economy is due to output shocks in each country and to the consumers' demand shocks. Each demand shock reflects a shift in preferences towards a particular good, as in Dornbusch, Fischer, and Samuelson (1977). These demand shocks are a convenient source of heterogeneity in cross-country appetites for risky assets as they give rise to hedging portfolios that are different across countries. There are more shocks than there are assets, and so markets are incomplete. In this model, we define the capital-gains adjusted current account, given by a change in the net foreign asset position of a country. Since the countries' hold long-lived assets (stocks), this notion of the current account differs from the conventional current account (as reported in the national accounts) in that it incorporates capital gains on Net Foreign Assets (NFA) positions (the so-called valuation effects).

Our model allows us to revisit several important relationships and mechanisms that underpin the traditional intertemporal approach to the current account. First, in contrast to the traditional view, it is entirely possible in our model for a debtor country to run long persistent current account deficits, including running a current account deficit forever. This is possible because we allow for cross-border equity holdings, and (positive) equity holdings of a country that is a net borrower in the international debt market can finance the current account deficits. Second, we find that there is a one-to-one mapping between the trade balance, a traditional measure of external imbalances based on real variables, and the size of the NFA position, a measure of external imbalances based on financial holdings. Any shock to an NFA position of a country must then be accompanied by an offsetting adjustment of the trade balance; otherwise the consumption profile becomes unaffordable. This is an equilibrium relationship that holds under both complete and incomplete markets. A similar result has been obtained in traditional bonds-only models of international economics; it turns out to be robust to the introduction of cross-border equity trade.

Third, we establish that the response of NFA to real shocks depends on the (endogenous) correlation of the cross-country stock returns and on the optimal portfolios of the countries. Our analysis demonstrates that the direction of the adjustment of the current account following a shock depends on whether a country is a net creditor or net debtor. For example, applied to the recent financial crisis, our model predicts that the drop in output in the US and UK reduces these countries' current account deficits, while the exact same drop in output in China and Japan reduces their surpluses. While the current accounts improve for the first group of countries and deteriorate for the second, however, the stock prices of all countries fall in tandem. Our model highlights that when the correlations of stock market returns are endogenous and portfolio holdings are determined in equilibrium, the question of how external accounts change after a particular shock cannot be answered without understanding the dynamics of asset returns and the compositions of the international portfolios.

1.1. Related Literature

Our analysis is motivated by the recent empirical literature that has documented the importance of valuation changes (Gourinchas and Rey (2007a), Gourinchas and Rey (2007b), Lane and Milesi-Ferretti (2001), Lane and Milesi-Ferretti (2007), Tille (2003), and Tille (2008)). This literature emphasizes that capital gains on countries' NFA positions are large, volatile, serially uncorrelated, and effectively dominate the movements in NFA (Kollmann (2006) and Lane and Shambaugh (2010)). Moreover, these capital gains are negatively correlated with the trade balance, and so capital gains stabilize countries' NFA positions by offsetting trade balance movements (Gourinchas and Rey (2007b), Devereux and Sutherland (2010)). Our model has the time series implications that are consistent with this empirical evidence. However, exploring these implications is beyond the scope of this paper. We undertake this exercise in a companion paper, Pavlova and Rigobon (2010), whose main goal is to calibrate a simpler version of the model developed here and to apply it to study several questions within the global imbalances literature. In contrast, the goal of this paper is to make progress on the "Obstfeld's agenda" (incomplete markets) by establishing a methodology for solving international portfolio models under incomplete markets. This methodology does not cover just the case of incomplete (missing) markets. Another natural application of our methodology is to the case of restricted participation or other types of portfolio constraints.

Our work belongs to the strand of literature that introduces valuation changes in models of international macroeconomics. This literature relies primarily on approximation methods. Ghironi, Lee, and Rebucci (2006) and Kollmann (2006) compute portfolios and NFA changes using standard first-order approximations around a deterministic steady state. The second-generation methodologies that grow out of Samuelson (1970) and Judd and Guu (2001) were developed by Devereux and Sutherland (2011), Engel and Matsumoto (2009), Evans and Hnatkovska (2007), and Tille and van Wincoop (2010), who employ higher-order approximations to analyze countries' portfolios and external accounts. To this day, little is known about the behavior of these economies away from the deterministic steady state, where the underlying volatilities are not small.[1] By contrast, our model admits an exact analytical solution, even under incomplete markets, and as such can serve as a natural benchmark to assess the accuracy of these popular approximation methods.

In terms of the modeling framework, our paper is closely related to Pavlova and Rigobon (2007) and Pavlova and Rigobon (2008), both of which are analytically tractable international macro-finance models. The main difference from these papers is that neither of them examines external adjustment. In fact, in those models the current account equals to zero. We extend their framework to produce meaningful implications for the current account. On the technical side, this is an extension of their framework to incomplete markets, which is non-trivial and thus constitutes a methodological contribution of this paper. The fact that in the real world markets are incomplete hardly requires any motivation. If markets were complete and all risks could be perfectly hedged, there would be no role for a policy response and the subject of external imbalances would be purely academic. Our model is also related to Cole and Obstfeld (1991), Helpman and Razin (1978), and Zapatero (1995). All these are tractable multi-asset multi-good models like ours, but in contrast to our work, in each of these papers markets are complete or effectively complete. Our paper complements Kraay and Ventura (2000) and Kraay and Ventura (2003), who assume a flat supply of capital, eliminating valuation effects and focusing on investment. We employ an endowment economy, and hence investment is not modeled, and focus entirely on valuation effects.[2] Finally, two recent papers that embark on an agenda very close to ours are Gourinchas, Govillot, and Rey (2010) and Maggiori (2012). Like us, they consider two-country endowment economies and, like

[1]More generally, the accuracy and performance and of numerical algorithms for solving multi-country dynamic stochastic general equilibrium (DSGE) models remain the subject of an ongoing debate. The complexity of the problem makes some of the algorithms quite inaccurate, especially away from the steady state. An excellent reference for the current state of this debate is a recent special issue of the Journal of Economic Dynamics and Control on numerical methods for multi-country DSGE models (see Den Haan, Judd, and Juillard (2010)).

[2]Also related, but, unlike ours, cast in the context of a production economy, is the elegant analysis of Devereux and Saito (2006).

us, link persistent current account deficits to asymmetric risk sharing between the two countries. Asymmetries between countries in these papers stem from differences in financial development. The papers have appeared after the first draft of our paper was circulated, and they lend strong support to our framework and our intuitions.

In terms of the methodology, our paper is closely related to He and Pearson (1991), Cuoco and He (1994), and Basak and Cuoco (1998). At a partial equilibrium level, He and Pearson derive a solution to a consumption-portfolio problem under incomplete markets.[3] Our contribution is to embed their technique in a general equilibrium setting and establish the effects of incomplete markets on asset prices. Cuoco and He develop a method for solving for equilibrium under incomplete markets via a "planner" with stochastic weights. Basak and Cuoco were the first to apply this method to study financial markets with frictions (restricted participation, in their case). None of these two papers, however, offers a model with multiple risky assets and incomplete markets that can be analyzed analytically.

The rest of the paper is organized as follows. Section 2 describes the model and characterizes its equilibrium. Section 3 derives a number of implications of our model for the current account and its dynamics. Section 4 presents several special cases of our economy in which the characterization of the countries' portfolios simplify significantly. Section 5 highlights the external adjustment mechanism in our model, Section 6 discusses several caveats and desirable extensions, and Section 7 concludes.

2. The Model

2.1. The Economic Setting

We consider a continuous-time pure-exchange world economy with a finite horizon, $[0, T]$ along the lines of Pavlova and Rigobon (2007).[4] The main advantage of using continuous time is tractability—an analogous discrete-time model does not admit a closed-form solution. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined a standard four-dimensional Brownian motion $\vec{w}(t) = (w(t), w^*(t), w^\gamma, w^\theta)^\top$, $t \in [0, T]$. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by $\vec{w}$. All stated (in)equalities

[3] See Schroder and Skiadas (2003) for a related analysis under a more general class of preferences.

[4] The assumption of finite horizon is for simplicity. We rely on existence results and regularity conditions developed in asset pricing for finite horizon models. T can be arbitrarily large but finite.

involving random variables hold P-almost surely. In what follows, given our focus, we assume all processes introduced to be well-defined, without explicitly stating regularity conditions ensuring this.

There are two countries in the world economy: Home and Foreign. The Home country represents a large industrialized country, while Foreign stands for the rest of the world. Each country is endowed with a Lucas tree producing a strictly positive amount of a country-specific perishable good, Y and Y^*, respectively:

$$dY(t) \;=\; \mu_Y(t)\,Y(t)\,dt + \sigma_Y(t)\,Y(t)\,dw(t) \qquad \text{(Home)}, \tag{1}$$

$$dY^*(t) \;=\; \mu_{Y^*}(t)\,Y^*(t)\,dt + \sigma_{Y^*}(t)\,Y^*(t)\,dw^*(t) \qquad \text{(Foreign)}, \tag{2}$$

where μ_Y, μ_Y^*, $\sigma_Y > 0$, and $\sigma_Y^* > 0$ are arbitrary adapted processes. The claims to the trees, Home and Foreign stocks S and S^*, respectively, are available for trade by all investors and are in fixed supply of one share each. The prices of the Home and Foreign goods are denoted by p and p^*, respectively. We fix the world numeraire basket to contain $a \in (0,1)$ units of the Home good and (1-a) units of the Foreign good, and normalize the price of this basket to be equal to unity. The terms of trade, q, are defined as the price of the Home good relative to that of the Foreign good: $q \equiv p/p^*$.

In addition to the stocks S and S^*, there is also the "world" bond with the price B available for investment, which is a money market account locally riskless in units of the numeraire. The bond is in zero net supply. Since there are four independent Brownian motions driving the economy and only three investment opportunities in place, financial markets are incomplete. To fix notation, the posited dynamics of the investment opportunity set of the agents is given by

$$dB(t) \;=\; B(t)r(t)dt, \tag{3}$$

$$dS(t) + p(t)Y(t)dt \;=\; S(t)[\mu_S(t)dt + \sigma_S(t)d\vec{w}(t)], \tag{4}$$

$$dS^*(t) + p^*(t)Y^*(t)dt \;=\; S^*(t)[\mu_{S^*}(t)dt + \sigma_{S^*}(t)d\vec{w}(t)], \tag{5}$$

where the interest rate r, the stocks expected returns μ_S and μ_{S^*} and their volatilities σ_S and σ_{S^*} are to be determined in equilibrium. The *volatility matrix* of the stock returns is then defined as $\sigma \equiv \begin{bmatrix} \sigma_S \\ \sigma_{S^*} \end{bmatrix}$.

The initial shareholdings of a representative consumer-investor of each country consist of no shares of the bond and a total supply of the stock market of his country. Thus, the initial wealth of the Home resident is $W_H(0) = S(0)$ and that of the Foreign resident is $W_F(0) = S^*(0)$. Each consumer i, $i \in \{H, F\}$, chooses nonnegative consumption of each good $(C_i(t), C_i^*(t))$ and a portfolio of the available securities $x_i(t) \equiv (x_i^S(t), x_i^{S^*}(t))$, where x_i^j denotes the fraction of wealth of consumer i invested in asset j. The dynamic budget constraint of each consumer has the standard form

$$\begin{aligned} dW_i(t) &= \Big[W_i(t)\, r(t) + x_i^S(t) W_i(t)(\mu_S(t) - r(t)) + x_i^{S^*}(t) W_i(t)(\mu_{S^*}(t) - r(t))\Big] dt \\ &\quad + \Big[x_i^S(t) W_i(t)\sigma_S(t) + x_i^{S^*}(t) W_i(t)\sigma_{S^*}(t)\Big] d\vec{w}(t) - \Big[p(t)C_i(t) + p^*(t)C_i^*(t)\Big] dt\,, \end{aligned} \tag{6}$$

where $W_i(T) \geq 0$, $i \in \{H, F\}$. Preferences of consumer i, are represented by a time-additive log-linear utility function defined over consumption of both goods:[5]

$$E\left[\int_0^T e^{-\rho t} u_i(C_i(t), C_i^*(t))\, dt\right], \qquad \rho > 0, \quad i \in \{H, F\}, \tag{7}$$

where

$$\begin{aligned} u_H(C_H(t), C_H^*(t)) &= \gamma_H(t)\Big[\theta_H(t)\log C_H(t) + (1 - \theta_H(t))\log C_H^*(t)\Big], \\ u_F(C_F(t), C_F^*(t)) &= \gamma_F\Big[(1 - \theta_F)\log C_F(t) + \theta_F \log C_F^*(t)\Big]. \end{aligned}$$

In our specification of the countries' utilities, we allow for the possibility of preference shifts towards the home or the foreign good (or "demand shocks"), modeled along the lines of Dornbusch, Fischer, and Samuelson (1977). There are several reasons why we consider demand shocks. First, in the absence of the demand shocks, free trade in goods makes stock prices perfectly correlated and financial markets irrelevant (Helpman and Razin (1978), Cole and Obstfeld (1991), Zapatero (1995)). Second, empirical evidence indicates that demand shocks are important drivers of the real-world dynamics. For example, Stockman and Tesar (1995) argue that preference shocks are roughly 85% of the size of supply shocks. Third, these demand shocks represent a convenient modeling device that allows to analyze international differences in appetite for risky assets in a very tractable way. For example, our model is more tractable than the one in which the two countries differ in their risk tolerance, but it can deliver similar implications. As argued in Pavlova and

[5] Our choice of logarithmic preferences is dictated by tractability; otherwise, we do not obtain closed-form solutions for stock prices and portfolios. In Section 6 we discuss how one may try to relax this assumption. It is well-known that these preferences give rise to small equity risk premia. While it is not our goal in this paper to match the asset-pricing moments, an extension of our model that does so is clearly desirable. Some promising recent work along this dimension has been done by Stathopoulos (2008) who extends our framework and analyzes a two-country two-good model in which agents have logarithmic preferences with external habits (see Section 6 for more discussion).

Rigobon (2007), the demand shocks can have a broad range of interpretations such as differences of opinion, catching up with the Joneses, or pure demand shifts.

The stochastic term γ_H, as we demonstrate below, gives rise to intertemporal hedging motives. The stochastic term θ_H represents an intratemporal demand shift towards a particular good. As we are going to show below, it does not give rise to any additional hedging portfolios, which is why we find it useful to separate the component of the demand shock that operates across time (γ_H) from the component that operates across goods (θ_H). Formally, we assume that γ_H and θ_H follow

$$d\gamma_H(t) = \sigma_{\gamma_H}(t)d\vec{w}(t), \quad d\theta_H(t) = \mu_{\theta_H}(t)dt + \sigma_{\theta_H}(t)d\vec{w}(t).$$

We additionally require that γ_H and $\gamma_H \times \theta_H$ are strictly positive adapted stochastic processes and martingales. In the analysis that follows, we consider primarily two types of demand shocks: (i) demand shocks that are completely independent of the supply (output) shocks w and w^* and (ii) demand shocks that are allowed to be correlated with the supply shocks. We assume that there are no demand shocks at Foreign. This asymmetry does not limit the ability of our model to produce a wide range of risk-sharing patterns: there is enough heterogeneity in this simpler version, and the extended version would not yield many additional insights. Additionally, this specification is better for analytical tractability. If we were to introduce demand shocks at Foreign, we can solve the model without relying on approximations only in some special cases.

2.2. Countries' Portfolio Choice

The first step in our solution procedure is to derive the countries' optimal portfolios at a partial equilibrium level. To do so, we are going to employ techniques developed in the portfolio choice literature. However, relative to that literature, there are two non-standard ingredients in the optimization problem that the countries are facing: multiple consumption goods and incomplete markets. We address them in turn.

For concreteness, we focus our exposition on the Home consumer. The portfolio of the Foreign consumer is derived analogously. Following the early literature in finance (Breeden (1979), Adler and Dumas (1983)), we decompose the problem of maximizing his utility (7) subject to the budget constraint (6) into two parts. First, at each t, we derive the consumer's demands for the Home and the Foreign goods, keeping the overall consumption expenditure fixed. Second, we derive his optimal consumption expenditure process and the optimal portfolio.

The first step is the standard static consumer problem under certainty:

$$\max_{\{C_H(t),\, C_H^*(t)\}} \gamma_H(t)\left[\theta_H(t)\log C_H(t) + (1-\theta_H(t))\log C_H^*(t)\right]$$

$$\text{s.t.} \qquad p(t)C_H(t) + p^*(t)C_H^*(t) \le \mathbb{C}_H(t),$$

where $\mathbb{C}_H(t)$ denotes overall consumption expenditure at time t. Solving this problem, we obtain the following demands for the individual goods as fractions of the overall expenditure:

$$\overline{C}_H(t) = \theta_H(t)\frac{\mathbb{C}_H(t)}{p(t)}, \quad \overline{C}_H^*(t) = (1-\theta_H(t))\frac{\mathbb{C}_H(t)}{p^*(t)}. \tag{8}$$

The indirect utility function defined as $U_H\left(\mathbb{C}_H(t);\, p(t), p^*(t)\right) \equiv u_H\left(\overline{C}_H(t),\, \overline{C}_H^*(t)\right)$ is then given by

$$U_H\left(\mathbb{C}_H(t);\, p(t), p^*(t)\right) = \gamma_H(t)\log\mathbb{C}_H(t) + F(\gamma_H(t),\, p(t),\, p^*(t)),$$

where $F(\cdot)$ is a function the form of which does not affect our analysis. This function F depends only on the variables that are exogenous from the viewpoint of the consumer and therefore, because of the separability of the indirect utility, it drops out of his portfolio choice.

The second step is to reformulate the portfolio choice problem of the consumer in terms of his indirect utility:

$$\max_{x_H^S,\, x_H^{S^*},\, \mathbb{C}_H} E\left[\int_0^T e^{-\rho t}\gamma_H(t)\log\mathbb{C}_H(t)\,dt\right] \tag{9}$$

$$\text{s.t.} \qquad dW_H(t) = \Big[W_H(t)\,r(t) + x_H^S(t)W_H(t)(\mu_S(t) - r(t)) + x_H^{S^*}(t)W_H(t)(\mu_{S^*}(t) - r(t))\Big]dt$$

$$+\Big[x_H^S(t)W_H(t)\sigma_S(t) + x_H^{S^*}(t)W_H(t)\sigma_{S^*}(t)\Big]d\vec{w}(t) - \mathbb{C}_H(t)dt\,. \tag{10}$$

The optimization problem is thus formally equivalent to a familiar single-good consumption-investment problem, with consumption expenditure $\mathbb{C}_H$ replacing consumption. Consumption of individual goods can then be recovered from (8). It is important to note that the prices of the individual goods, p and p^*, and hence the terms of trade have dropped out of the optimization problem. This implies that fluctuations in the terms of trade do not pose a risk that the consumer desires to hedge. In contrast, he does desire to hedge his demand shocks, but since θ_H has dropped out of his optimization problem, he only wishes to hedge the part of the demand shocks that governs his intertemporal choice—the state variable γ_H. The intuition for why θ_H drops out of the optimization is simple. The country wants to consume more of the Home good in the high-θ_H states, and so it desires higher wealth in those states. But these states are precisely the ones in which it desires fewer imports ($1-\theta_H$ is low). The two effects exactly balance each other.

The next issue we need to address is market incompleteness. A technique for solving such problems in a single-good framework via martingale methods has been developed in a seminal contribution of He and Pearson (1991). He and Pearson show that, just like for the case of complete markets, one can replace the dynamic optimization problem (9)–(10) by the following static variational problem:

$$\max_{\mathbb{C}_H} E\left[\int_0^T e^{-\rho t}\gamma_H(t)\log\mathbb{C}_H(t)\,dt\right] \tag{11}$$

$$\text{s.t.} \qquad E\left[\int_0^T \xi_\nu(t)\mathbb{C}_H(t)dt\right] \leq W_H(0), \tag{12}$$

where ξ_ν denotes an appropriate state price density—i.e., an Arrow-Debreu state price per unit of probability P. The difficulty arises from the fact that in incomplete markets, there is an infinite number of such state price densities consistent with no arbitrage and hence potentially an infinite number of static budget constraints (12). However, this set of budget constraints is known to possess some special structure. Let m denote the market price of risk process

$$m(t) \equiv \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}(\mu(t)-r(t)\mathbf{1}), \tag{13}$$

where $\mu \equiv (\mu_S, \mu_{S^*})^\top$ and $\mathbf{1}$ is a two-dimensional vector of ones. Then the set of state price densities can be represented as (He and Pearson, Proposition 1):

$$d\xi_\nu(t) = -r(t)\xi_\nu(t)dt - (m(t)+\nu(t))^\top\xi_\nu(t)d\vec{w}(t), \tag{14}$$

with $\nu(t)\in\mathbb{R}^4$ satisfying $\sigma(t)\nu(t)=0$, $\forall t\in[0,T]$ and $\int_0^T ||\nu(t)||^2 dt < \infty$.[6] It is easy to see that if the volatility matrix σ is a nondegenerate square matrix, the condition $\sigma(t)\nu(t)=0$ can be satisfied only for $\nu(t)=\mathbf{0}$, where $\mathbf{0}$ is a four-dimensional vector of zeros. This is precisely the case when markets are complete: the state price density is unique and $\nu(t)=\mathbf{0}$ at all t. If, however, the volatility matrix has fewer rows than there are Brownian motions, many possible $\nu(t)$'s can satisfy the restriction $\sigma(t)\nu(t)=0$. This is the case when markets are intrinsically incomplete.

He and Pearson go on to prove that there exists a unique individual-specific ν, which we denote by ν_H, that minimizes the maximum expected utility in (11). We derive the expression for it in the proposition below. The only relevant budget constraint in (12) is then the one corresponding to ν_H.

[6]The notation $||z||^2$ stands for the dot product $z\cdot z$.

Establishing the portfolio that solves the optimization problem (11)–(12) is then straightforward. We report this portfolio, as well as the portfolio of Foreign, in the following proposition.

Proposition 1. *(i) The fractions of wealth x_H and x_F invested in the risky stocks by the Home and the Foreign country, respectively, are given by*

$$x_H(t) = \underbrace{(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, m(t)}_{\textit{mean-variance portfolio}} + \underbrace{(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, \frac{\sigma_{\gamma_H}(t)^\top}{\gamma_H(t)}}_{\textit{hedging portfolio}} \tag{15}$$

$$x_F(t) = (\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, m(t). \tag{16}$$

The fractions of wealth invested in the bond by Home and Foreign are given by $1 - \mathbf{1}^\top x_H(t)$ and $1 - \mathbf{1}^\top x_F(t)$, respectively.

(ii) The processes ν_H and ν_F, entering the specification of the personalized state price densities of Home and Foreign, respectively, are given by

$$\nu_H(t)^\top = -\frac{\sigma_{\gamma_H}(t)}{\gamma_H(t)}(I_4 - \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)) \quad \textit{and} \quad \nu_F(t) = \mathbf{0}, \tag{17}$$

where I_4 is a 4-dimensional identity matrix.

Consider first the portfolio of the Home consumer. It consists of two parts: the mean-variance efficient portfolio and the hedging portfolio. This decomposition is standard in the portfolio choice literature. The optimal mean-variance portfolio was first derived by Markowitz (1952) in a one-period setting and later generalized by Merton (1971) to a continuous-time stochastic environment. Furthermore, Merton shows that in addition to the mean-variance portfolio an investor optimally selects a hedging portfolio whose role is to offset fluctuations in the state variables in his optimization problem. As is well-known, investors with logarithmic preferences do not wish to hedge against changes in their investment opportunity set (stock and bond price dynamics)—in that sense they behave myopically. However, they do wish to hedge against fluctuations in the state variables entering their preferences, namely the preference shifts. When markets are complete (or effectively complete), the gains made by the hedging portfolio are perfectly positively correlated with the fluctuations in state variable Home desires to hedge: γ_H. (This is the state variable entering Home's objective function (9)). When markets are incomplete, not every payoff can be replicated and so it is typically not possible to construct a portfolio whose gains are perfectly correlated with a state variable. In that case, the Home investor chooses the portfolio most highly correlated with γ_H.

In contrast, the Foreign investor demands no hedging portfolio. This is because the term γ_F entering his objective function is non-stochastic. Consequently, the inability to hedge perfectly under incomplete markets does not hurt the Foreign investor: in contrast to that of the Home investor, his personalized ν_F remains the same as it would be under complete markets.

As we elaborate later, (heterogeneous) hedging demands is the key vehicle for generating trade in financial markets in equilibrium. For example, in the absence of preference shifts, agents have no hedging demands and hence they have no reason to trade assets.

2.3. Characterization of Equilibrium

The techniques used so far were developed for partial equilibrium analyses. We now embed these techniques in our general equilibrium framework. An equilibrium in our economy is defined in a standard way: it is a collection of goods and asset prices (p, p^*, S, S^*, B) and consumption-investment policies ($C_i(t)$, $C_i^*(t)$, $x_i^S(t)$, $x_i^{S^*}(t)$), $i \in \{H, F\}$ such that (i) each consumer-investor maximizes his utility (7) subject to the budget constraint (6) and (ii) goods, stock, and bond markets clear.

In the economy with incomplete markets the equilibrium allocation would not be Pareto optimal. Hence, the usual construction of a representative agent's (planner's) utility as a weighted sum, with constant weights, of individual utility functions is not possible. Instead, we are going to employ a fictitious representative agent with stochastic weights (introduced in an important contribution by Cuoco and He (1994)), with these stochastic weights reflecting the effects of market incompleteness.[7] This fictitious representative agent maximizes his utility subject to the resource constraints:

$$\max_{\{C_H, C_H^*, C_F, C_F^*\}} \quad E\left[\int_0^T e^{-\rho t}\left(u_H(C_H(t),\, C_H^*(t)) + \lambda(t)\, u_F(C_F(t),\, C_F^*(t))\right)dt\right]$$

$$\text{s. t.} \quad C_H(t) + C_F(t) = Y(t)\,,$$

$$C_H^*(t) + C_F^*(t) = Y^*(t)\,,$$

where we have normalized the weight on the Home consumer to be equal to one and assigned the weight λ to the Foreign consumer. λ is simply the ratio of the marginal utilities of either good of the two countries. When markets are complete, this ratio is constant and the allocation is Pareto

[7] Alternatively, we could have solved for equilibrium directly from the system of equilibrium equations. We prefer the method we are presenting because of the clarity of the ensuing intuitions. The construction of a representative agent with stochastic weights has been employed extensively in dynamic asset pricing models with financial market frictions. See, for example, Basak and Croitoru (2000), Basak and Cuoco (1998), and Detemple and Serrat (2003). A related approach is the extra-state-variable methodology of Kehoe and Perri (2002) and Marcet and Marimon (1999). For the original solution method utilizing weights in the representative agent, see Negishi (1960).

optimal. When markets are incomplete, the marginal utilities are no longer proportional, and so λ becomes stochastic. It enters as an additional (endogenous) state variable, which reflects the relative importance the "planner" assigns to the Foreign country's utility.

Solving the representative agent's optimization problem, we obtain the sharing rules

$$C_H(t) = \frac{\gamma_H(t)\theta_H(t)}{\gamma_H(t)\theta_H(t) + \lambda(t)\gamma_F(1-\theta_F)}\, Y(t), \quad C_H^*(t) = \frac{\gamma_H(t)(1-\theta_H(t))}{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\gamma_F\theta_F}\, Y^*(t), \tag{18}$$

$$C_F(t) = \frac{\lambda(t)\gamma_F(1-\theta_F)}{\gamma_H(t)\theta_H(t) + \lambda(t)\gamma_F(1-\theta_F)}\, Y(t)\,, \quad C_F^*(t) = \frac{\lambda(t)\gamma_F\theta_F}{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\gamma_F\theta_F}\, Y^*(t)\,. \tag{19}$$

We can now derive the terms of trade that prevail in a competitive equilibrium. They are identified with the ratio of either country's marginal utilities of the Home and Foreign goods:

$$q(t) = \frac{\gamma_H(t)\theta_H(t) + \lambda(t)\gamma_F(1-\theta_F)}{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\gamma_F\theta_F}\, \frac{Y^*(t)}{Y(t)}\,. \tag{20}$$

We next use the no-arbitrage valuation principle to obtain stock prices and equilibrium wealth of the countries.

Lemma 1. *Equilibrium stock prices in our economy are given by*

$$S(t) = \frac{1-e^{-\rho(T-t)}}{\rho}\, \frac{q(t)}{aq(t)+1-a}\, Y(t)\,, \tag{21}$$

$$S^*(t) = \frac{1-e^{-\rho(T-t)}}{\rho}\, \frac{1}{aq(t)+1-a}\, Y^*(t) \tag{22}$$

and the wealth of the countries by

$$W_H(t) = \frac{\gamma_H(t)}{\gamma_H(t)\theta_H(t) + \lambda(t)\gamma_F(1-\theta_F)} S(t)\,,\ W_F(t) = \frac{\lambda(t)\gamma_F}{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\gamma_F\theta_F} S^*(t)\,. \tag{23}$$

Lemma 1 yields a simple interpretation of the weight λ. One can see that

$$\lambda(t) = \frac{W_F(t)}{W_H(t)} \frac{\gamma_H(t)}{\gamma_F}. \tag{24}$$

That is, incomplete markets enrich the dynamics of the economy with an additional state variable λ, which is related to the wealth distribution, but not given exactly by the wealth distribution unless $\gamma_H(t)$ is constant. We have already discussed the role of $\gamma_H(t)$ earlier in our analysis: it was the state variable giving rise to the hedging portfolio held by Home.

Lemma 1 allows us to characterize the dynamics of stock returns and the market price of risk in equilibrium, which are tedious but straightforward to compute. Equation (24) lets us pin down the weight λ. We relegate the details of the necessary calculations to the Appendix, and report the resulting dynamics of λ below.

Proposition 2. *(i) In an equilibrium, the weight of the Foreign country in the fictitious representative agent follows*

$$d\lambda(t) = -\lambda(t)\,\nu_H(t)\,d\vec{w}(t), \quad \textit{with}\;\; \lambda(0) = \gamma_H(0)(1-\theta_H(0))/(\gamma_F(1-\theta_F)).$$

(ii) When such equilibrium exists, the volatility matrix σ, the market price of risk m, and hence ν_H can be computed as functions of exogenous state variables. They are reported in the Appendix.

Note that our characterizations the terms of trade, consumption, and stock prices presented in this section all involve the exogenous state variables of the model and one endogenous quantity: the weight λ. With this weight λ now characterized in Proposition 2, we can then pin down these equilibrium quantities and their dynamics. Moreover, the countries' portfolios held in equilibrium are also fully determined now, with the volatility matrix of stock returns and the market price of risk characterized fully in terms of exogenous state variables (see the Appendix). Admittedly, the equilibrium characterizations of the portfolios are not particularly transparent. To develop intuition, in Section 4 we consider several special cases in which the expressions for the portfolios are simple. The analysis of these special cases relies in part on the result of the following lemma.

Lemma 2. *The countries hold no bond in their portfolios if and only if the value of the hedging portfolio demanded by Home is equal to zero.*

Proof. Suppose that bondholdings of the countries are zero. This is equivalent to saying that the fraction of wealth each country invests in the stocks is equal to one:

$$\begin{aligned}
\mathbf{1}^\top x_H(t) &= \mathbf{1}^\top \left[(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\,m(t) + (\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\,\frac{\sigma_{\gamma_H}(t)}{\gamma_H(t)}\right] = 1 \\
\mathbf{1}^\top x_F(t) &= \mathbf{1}^\top (\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\,m(t) = 1,
\end{aligned}$$

where we have substituted the formulas for the portfolios derived in Proposition 1. This can happen only if $\mathbf{1}^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\,\frac{\sigma_{\gamma_H}(t)}{\gamma_H(t)} = 0$—i.e., the fraction of wealth invested in the hedging portfolio is zero.

Conversely, if the value of the hedging portfolio is zero, then $\mathbf{1}^\top x_H(t) = \mathbf{1}^\top x_F(t) = \mathbf{1}^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\,m(t)$. Bond market clearing then implies that $\mathbf{1}^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)m(t) = 1$. ■

3. External Accounts

From the theoretical and practical point of view, the measure of external sustainability of countries has evolved through time. The original measure reflected simply the trade balance of goods, then it got revised to reflect the trade balance of goods and services, then switched to the (conventionally-defined) current account, and now it is changing again so as to better capture changes in net foreign asset positions. In policy circles, it is not uncommon to disregard all previous measures of external imbalances once a new measure comes to the fore.[8]

In this section, we define the trade balance, the conventional current account, and the capital-gains adjusted current account in the context of our model. The first one refers to the balance of imported and exported goods, the second one adds dividend and interest payments to the trade balance, and the last one adds the balance of capital gains to the conventional current account. Our goal is to study within our model the relationships between the traditional measures of sustainability, such as the trade balance and the current account, and the measure that has received most of the attention lately, the NFA position. We show that the trade balance and the capital-gains adjusted current account are closely intertwined, and link the conventional current account to the value of the countries' hedging portfolios.

3.1. The Trade Balance and the Conventional Current Account

In our model, the *trade balance*—defined as exports minus imports—is simply

$$TB_H(t) = (p(t)(Y(t) - C_H(t)) - p^*(t)C_H^*(t))dt.$$

The conventional measure of the *current account* differs from the trade balance in that it also includes net dividend and interest payments (but not capital gains). For expositional simplicity, let us concentrate on the Home country. The conventionally-defined current account in our model is given by

$$CA_H(t) = TB_H(t) + \Bigg[\underbrace{s_H^{S^*}(t)p^*(t)Y^*(t) - s_F^S(t)p(t)Y(t)}_{\text{Net Dividend Payments}} + \underbrace{s_H^B(t)B(t)r(t)}_{\text{Net Interest Payments}}\Bigg]dt, \qquad (25)$$

where s_i^j denotes the number of shares of asset j held by country i. The second and the third terms in (25) are dividend receipts from foreign assets minus dividend payments to Foreign, and the last

[8] As the former US Treasury Secretary Paul O'Neill once memorably remarked, the (conventional) current account had become a "meaningless concept."

term is the interest paid on current bondholdings. Recall that each of the above quantities in our model is defined as a rate (e.g., the export rate, the dividend rate, etc.) and hence need to be scaled by a time increment. This is the reason behind the term "dt" appearing in (25).

An often cited shortcoming of pure-exchange models with log-linear preferences is that they are unable to generate nontrivial current accounts. Having a current account equal to zero at all times would clearly hinder any quantitative analysis of current account deficits that we intend to undertake in this paper. It is therefore worth highlighting the situations under which the current account is zero in our model.

Lemma 3. *The current account of the Home country can be represented as follows:*

$$CA_H(t) = s_H^B(t)B(t)\left(r(t) - \frac{\rho}{1-e^{-\rho(T-t)}}\right)dt. \tag{26}$$

Proof. Note that

$$\begin{aligned} s_H^B(t)B(t) &= W_H(t) - s_H^S(t)S(t) - s_H^{S^*}(t)S^*(t) \\ &= \frac{\gamma_H(t)}{\gamma_H(t)\theta_H(t) + \lambda(t)\,\gamma_F(1-\theta_F)}S(t) - (1-s_F^S(t))S(t) - s_H^{S^*}(t)S^*(t) \\ &= \frac{1-e^{-\rho(T-t)}}{\rho}\left[p(t)Y(t)\frac{\gamma_H(t)(1-\theta_H(t)) - \lambda(t)\,\gamma_F(1-\theta_F)}{\gamma_H(t)\theta_H(t) + \lambda(t)\,\gamma_F(1-\theta_F)} - s_H^{S^*}(t)p^*(t)Y^*(t) + s_F^S(t)p(t)Y(t)\right], \end{aligned}$$

where the second equality follows from Lemma 1 and stock market clearing ($s_H^S(t) = 1 - s_F^S(t)$), and the last one, again, from Lemma 1. On the other hand, by substituting (18) into (25) and simplifying, one can show that

$$\begin{aligned} CA_H(t) &= \left[-p(t)Y(t)\frac{\gamma_H(t)(1-\theta_H(t)) - \lambda(t)\,\gamma_F(1-\theta_F)}{\gamma_H(t)\theta_H(t) + \lambda(t)\,\gamma_F(1-\theta_F)} + s_H^{S^*}(t)p^*(t)Y^*(t) - s_F^S(t)p(t)Y(t) + s_H^B(t)B(t)r(t)\right]dt \\ &= s_H^B(t)B(t)\left(-\frac{\rho}{1-e^{-\rho(T-t)}} + r(t)\right)dt. \end{aligned}$$

This proves the statement in (26). ■

This lemma reveals that the first sufficient condition for the current account to be equal to zero is that the Home country (and hence the Foreign) holds no bonds. While it is indeed a common implication of models with log-linear preferences to have zero net bond holdings in equilibrium, nothing in our model prevents the bond holdings from being different from zero. That is, preference shifts may potentially introduce enough heterogeneity among the countries so that they are willing to trade in all available financial assets for risk-sharing purposes. The second condition under which the current account is zero is when the interest rate $r(t)$ is equal to $\rho/(1-e^{-\rho(T-t)})$. The latter quantity is deterministic, while the interest rate is a stochastic process. Hence, it is true only on the measure zero set of parameter values.

3.2. The Capital-Gains Adjusted Current Account

We now define the current account as the change in the net foreign asset position of a country. Unlike the conventional current account CA, this measure fully accounts for the capital gains on the NFA positions. We therefore label it "$CGCA$," which stands for the "*capital-gains adjusted current account.*"

$$CGCA_H(t) \equiv d\,NFA_H(t) = d\Big[s_H^{S^*}(t)S^*(t) - s_F^S(t)S(t) + s_H^B(t)B(t)\Big], \tag{27}$$

where the first two terms in the square brackets are Home's investment in the Foreign stock minus Foreign' investment in the Home stock, and the last term is Home's balance on the bond account.

Note that, by market clearing, $s_F^S(t) = 1 - s_H^S(t)$ and that, by definition, Home's financial wealth equals its portfolio value, $W_H(t) = s_H^S(t)S(t) + s_H^{S^*}(t)S^*(t) + s_H^B(t)B(t)$. Hence, we can rewrite (27) as

$$CGCA_H(t) = dW_H(t) - dS(t). \tag{28}$$

3.3. Congruence between NFA and Trade Balance

To conclude this section, we derive the expression for NFA in our model and draw a connection between the NFA position and the trade balance. Note that

$$\begin{aligned} NFA_H(t) &= W_H(t) - S(t) \\ &= \frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\,(p(s)\,C(s) + p^*(s)\,C^*(s))ds\right] - \frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\,p(s)\,Y(s)\,ds\right]. \end{aligned}$$

Hence, by definition of a trade balance,

$$NFA_H(t) = -\frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\underbrace{(p(s)(Y(s) - C_H(s)) - p^*(s)C_H^*(s))ds}_{=TB_H(s)}\right]. \tag{29}$$

Equation (29) is nothing else but the familiar statement that the NFA position is given by the present value of the future trade deficits. The traditional intertemporal approach to external adjustment—that ignores changes in the state price density (or the stochastic discount factor) ξ_{ν_H}—says that, for example, for a country with a negative NFA position, adjustment must come through future trade surpluses. Recent literature challenges this conclusion and draws attention to the "valuation channel" of the external adjustment that operates precisely through changes in the stochastic discount factor (Gourinchas and Rey (2007b)). It is argued that such changes are large and volatile, and hence the valuation channel should have a substantial contribution to the NFA dynamics. Surprisingly, it turns out that in our model, after the endogenous responses of asset prices and hence the stochastic discount factor to underlying shocks are taken into account, the NFA adjustment takes place instantaneously and entirely through the trade balance. In that sense, the traditional and the new views are not at all inconsistent.[9]

Lemma 4 (Congruence between NFA and trade balance). *The relationship between the net foreign assets and the trade balance is given by*

$$TB_H(t) = -\frac{\rho}{1 - e^{-\rho(T-t)}} \, NFA_H(t)dt. \tag{30}$$

The net foreign asset position of Home is

$$NFA_H(t) = \frac{\gamma_H(t)(1 - \theta_H(t)) - \lambda(t)\,\gamma_F(1 - \theta_F)}{\gamma_H(t)\theta_H(t) + \lambda(t)\,\gamma_F(1 - \theta_F)}\, S(t)\,. \tag{31}$$

We note that this result is not due to the fact that we only consider permanent shocks. For example, the supply shocks in our model are general stochastic processes and in particular, they can be mean-reverting. In the related literature, a similar result has been obtained as a first-order approximation (Tille and van Wincoop (2010)). Here, the formula is exact and approximation-free. The perfect negative correlation between the current trade balance and the NFA position certainly relies on our assumption that the agents have log-linear preferences that rule out intertemporal hedging motives. It is important to evaluate the robustness of this result under alternative preferences that give rise to intertemporal hedging.

Our congruence relationship may remind the reader of the textbook formula from the intertemporal approach to the current account (Obstfeld and Rogoff (1996), Chapter 2). That formula is derived in the context of an economy with no stocks, under many simplifying assumptions. The economy we are considering is significantly more complicated—featuring endogenous portfolio recompositions and trade in equities under incomplete markets—and yet the relationship between the

[9] It is important to note that the "connection" is between the NFA and the trade balance of goods and services, and not between NFA and the conventional current account.

NFA position and the trade balance comes out similar to that in the classical external adjustment theory.

4. Equilibrium Portfolios

As we have been stressing, in our model there is a close relationship between the countries' external accounts and their equilibrium portfolio compositions. These interconnections take place through different mechanisms—related to the terms of trade, asset prices, hedging motives, or portfolio choices. In this section, we present special cases of our model that highlight such mechanisms. To establish a benchmark, we start by replicating some results of the earlier literature—in particular, that under certain conditions financial markets could be irrelevant for the allocation of consumption. In this special case of our model, there is perfect risk sharing and the stock market returns are perfectly correlated around the world. Risk diversification occurs through terms of trade fluctuations. Second, we study the case in which the stock market returns are not perfectly correlated, but the optimal portfolios again fully diversify international risk. In this case, the channel of diversification comes from the countries' holdings, and not (entirely) from the terms of trade. The case we study is one in which shocks to any of the countries have no implications for the conventional current account, portfolio holdings are constant through time and therefore, capital flows are zero. The third example is one in which markets are intrinsically incomplete and hence risk sharing is imperfect. We have set up this example in such a way that the value of the hedging portfolio is zero, which in turn implies that the bond holdings and the conventional current account are always equal to zero. However, the capital-gains adjusted current account does vary in response the underlying shocks because asset prices and optimal portfolios are not constant. On a separate note, it is important to stress that even though in this example markets are incomplete, the (conventional) current account is always zero. Finally, we study a case in which the value of the hedging portfolio is different from zero. We specialize our setup so that markets are complete, but all external accounts are nonzero. We analyze the relationship between the current account and portfolio compositions in this environment. In summary, the objective of this section is to cumulatively develop the intuitions behind the interconnections that our model exhibits, which we do via examples that capture several different aspects of the workings of the model.

4.1. Example 1: The Irrelevance Result

The first example we study is one in which the financial markets' structure is irrelevant, there are no net portfolio flows, and therefore, capital gains on financial assets play no role in the international adjustment process. This is the case considered in Helpman and Razin (1978), Cole and Obstfeld (1991), and Zapatero (1995). The three examples that follow relax some of the assumptions we make here in order to clarify the role that capital gains play.

In our model, we obtain our irrelevance result by specializing the Home consumer's preferences so that γ_H and θ_H are constant. As is well-known, under this specification the returns on the two stocks are perfectly correlated in equilibrium. Hence, portfolio allocations into these stocks are indeterminate. Only positions in the composite stock market, $S(t) + S^*(t)$, can be uniquely determined. Using an argument analogous to that we employed in Section 2.2, we can derive the investors' optimal fractions of wealth invested in the composite stock market. One can easily see that the portfolios demanded by the two agents are going to be identical: in the absence of preference shifts, they both demand the mean-variance but no hedging portfolios.

It is also well-known that financial markets are effectively complete in this special case. This is equivalent to saying that $\nu_H(t) = \nu_F(t) = \mathbf{0}$ at all times and hence, from Proposition 2, the weight λ is constant. Thus, Pareto optimality obtains despite market incompleteness—or, in other words, markets are effectively complete. Investors are not adversely affected by market incompleteness because they do not make use of financial markets to construct portfolios hedging against fluctuations in any state variables: there is no state variable either agent desires to hold a hedge against. The intuition for why the financial markets are not needed in this case comes from the fact that movements in the terms of trade exactly offset output shocks and hence the values of the dividends on the Home and the Foreign stock markets, $p(t)Y(t)$ and $p^*(t)Y^*(t)$, respectively, are always the same (up to a multiplicative constant). Fluctuations in the terms of trade therefore fully offset the supply shocks; i.e., with no demand uncertainty the capital gains on the two stocks are always perfectly correlated. This feature of our model is due to the way we specified preferences (log-linear) and endowments (shares of trees), and represents a simple benchmark for comparison.

Finally, unlike holdings of individual stocks, the bond holdings of the countries are uniquely determined: there are equal to zero at all times.[10] This is because, the two countries demand the

[10] This result is not new. See Cass and Pavlova (2004).

same portfolio and in particular, wish to invest the same fraction of wealth in the bond. For the bond market to clear, this fraction has to be zero. Consequently, $CA(t) = 0$ at all times. Moreover, the capital-gains adjusted current account $CGCA$ is also zero. Each country is holding the same portfolio and stock markets are perfectly correlated. Therefore, the NFA positions (and changes in the NFA positions) are zero at all times.

4.2. Example 2: No Hedging and No Capital Flows

The purpose of this example is to highlight the importance of international asset cross-holdings and portfolio rebalancing. We consider a version of our model in which portfolio holdings are unique (as opposed to indeterminate as in the previous case), but portfolio holdings are fixed (hence there is never a rebalancing after any of the shocks). We show that in this case the current account is zero at all times. Any movements in the capital-gains adjusted current account are due to fluctuations in asset prices, and not due to portfolio rebalancing.

To highlight these dynamics, consider the special case of the model in which γ_H remains constant ($\sigma_{\gamma_H}(t) = \mathbf{0}$), and θ_H is stochastic. In the presence of preference shocks—even one type of shock, as we specify here—the two stocks are no longer perfectly correlated, the volatility matrix is invertible, and hence the expressions in Propositions 1 and 2 readily apply. It turns out that the equity portfolios of the countries, expressed as numbers of shares, take a particularly simple form

$$s_H(t) = \left(\frac{\gamma_H}{\gamma_H + \lambda(t)\,\gamma_F}, \frac{\gamma_H}{\gamma_H + \lambda(t)\,\gamma_F} \right) \quad \text{and} \quad s_F(t) = \left(\frac{\lambda(t)\gamma_F}{\gamma_H + \lambda(t)\,\gamma_F}, \frac{\lambda(t)\gamma_F}{\gamma_H + \lambda(t)\,\gamma_F} \right), \qquad (32)$$

where $s_i \equiv (s_i^S, s_i^{S^*})$ are obtained from x_i using Lemma 1.

No hedging portfolio is held by either country and $\nu_H(t) = \mathbf{0}$. These results is immediate from Proposition 1. Intuitively, the only state variable that the Home investor desires to hedge against in our economy is γ_H, and that variable is non-stochastic. It is of no surprise then that it turns out that $\nu_H(t) = \mathbf{0}$: the investor is able to achieve the same efficiency of hedging as under complete markets. Consequently (Proposition 2), again, the weight λ is constant and markets are effectively complete. Since λ is constant, the portfolios reported above are also constant, and hence there are no capital flows between countries at any time. In contrast to the no preference shifts case, however, one can see that effective market completeness does not lead to the indeterminacy of equilibrium portfolios.

This example is also useful in drawing a distinction between portfolio home bias and consumption home bias. In our model, the parameters controlling the degree of home bias are θ_H and θ_F. In other words, the home bias is coming because the demand shock is affecting home demand for home goods (this is equivalent to explicitly modeling shocks to the non-tradable demand which has been already highlighted in the literature). It is important to highlight that the home bias in consumption has *nothing* to do with the home bias in portfolios in this case. As one can see from (32), the countries always hold an identical number of shares of the Home and Foreign stock.

Furthermore, just like in the previous special case, the countries invest nothing in the bond. To see this, recall that the hedging portfolio is zero at all times. The result then follows from Lemma 2. Intuitively, there is not enough heterogeneity across countries to justify borrowing and lending. With γ_H being constant, the Home agent has no desire to hedge his demand shocks. Consequently, the two countries demand the same portfolio (in fractions of wealth; see Proposition 1) and in particular, wish to invest the same fraction of wealth in the bond. For the bond market to clear, this fraction has to be zero.

Finally, note that

$$CA_i(t) = 0, \qquad i \in \{H, F\}.$$

The (conventional) current account is zero because none of the countries invests in the bond (Lemma 3). Except for time $t = 0$, the trade balance and the capital-gains adjusted current account are generally non-zero, however. Although there is no portfolio rebalancing (capital flows) taking place, the capital-gains adjusted current account does fluctuate. This is because the prices of stocks move in response to the underlying shocks. These stock price fluctuations result in capital gains that get transmitted to the capital-gains adjusted current account. We examine the effects of the shocks on the capital-gains adjusted current account in the next section.

4.3. Example 3: Incomplete Markets

We now consider a more general case of our model in which the current account is still identically equal to zero but the hedging portfolio is now different from zero and there are capital flows across countries. The external adjustment process in this example is therefore driven entirely by the valuation effects and has nothing to do with the traditional channels.

We consider a special case of the model in which the demand shocks γ_H and θ_H load only on the Brownian motions w^γ and w^θ, and not on w and w^*—i.e., the demand shocks are completely independent of the supply shocks. Formally, we assume that $\sigma_{\gamma_H}(t) = (0,\, 0,\, \sigma_{\gamma_1}(t),\, \sigma_{\gamma_2}(t))$ and

$\sigma_{\theta_H}(t) = (0,\, 0,\, \sigma_{\theta_1}(t),\, \sigma_{\theta_2}(t))$, with $\sigma_{\gamma_H} > 0$ and $\sigma_{\theta_H} > 0$. We derive the stockholdings of Home and Foreign, respectively, to be as follows:

$$s_H = \frac{1}{G}\left(\lambda\gamma_F\theta_F\sigma_{\gamma_H}\sigma_{\theta_H}^\top + \gamma_H^2||\sigma_{\theta_H}||^2, \quad -\lambda\gamma_F(1-\theta_F)\sigma_{\gamma_H}\sigma_{\theta_H}^\top + \gamma_H^2||\sigma_{\theta_H}||^2\right), \tag{33}$$

$$s_F = 1 - s_H\,, \tag{34}$$

where $G \equiv \gamma_H^2||\sigma_\theta||^2(1+\lambda\gamma_F) - \lambda\gamma_F(1-\theta_H-\theta_F)\sigma_{\gamma_H}\sigma_{\theta_H}^\top$. In the expressions above and for the remainder of this section, we suppress the argument t. It already becomes clear at this point that depending on parameter values, our model can produce large gross portfolios.

The hedging portfolio, h, held by Home, in numbers of shares, is

$$h = \frac{1}{G}\left((\lambda\gamma_F\theta_F + \gamma_H(1-\theta_H))\sigma_{\gamma_H}\sigma_{\theta_H}^\top,\ (\lambda\gamma_F(1-\theta_F) + \gamma_H\theta_H)\sigma_{\gamma_H}\sigma_{\theta_H}^\top\right). \tag{35}$$

Consider the gain on the hedging portfolio, which turns out to be:

$$dh = [\ldots]\,dt + \frac{W_H\sigma_{\gamma_H}\sigma_{\theta_H}^\top}{\gamma_H||\sigma_{\theta_H}||^2}(0,\, 0,\, \sigma_{\theta_1},\, \sigma_{\theta_2})\,d\vec{w}, \tag{36}$$

and compare it to the fluctuations in the state variable, γ_H, that Home desires to hedge against:

$$d\gamma_H = (0,\ 0,\ \sigma_{\gamma_1},\ \sigma_{\gamma_2})\,d\vec{w}.$$

(Recall that the discount factor γ_H is the state variable entering Home's objective function (9).) Comparing the above two equations, one can see that the hedge is generally not perfect. The extent to which a country is able to hedge its γ_H shocks depends on the correlation between γ_H and θ_H (or on their covariance $\sigma_{\gamma_H}\sigma_{\theta_H}^\top$). One extreme is when they are perfectly correlated, in which case the hedge is perfect and hence markets become effectively complete. In that case, the dynamics of h matches the dynamics of γ_H (the vectors $(0,\, 0,\, \sigma_{\theta_1},\, \sigma_{\theta_2})$ and $(0,\ 0,\ \sigma_{\gamma_1},\ \sigma_{\gamma_2})$ are collinear). The other extreme is when the correlation is zero (and, of course, $\sigma_{\gamma_H}\sigma_{\theta_H}^\top = 0$), in which case the hedging portfolio is constant and agents are unable to hedge any of the γ_H-risk.

Note that the holdings of the two stocks in the hedging portfolio have the opposite sign, and this sign depends on the sign of G. This implies that our model can produce a home bias or a reverse home bias in portfolios. Note that in this case the presence of home bias in consumption ($\theta_H > 1/2$ and $\theta_F > 1/2$) is sufficient to generate home bias in portfolios.

The inability to hedge perfectly is indicative of the fact that market incompleteness matters. Indeed, in equilibrium,

$$\nu_H = \left(0, \quad 0, \quad \frac{\sigma_{\theta_2}(\sigma_{\theta_1}\sigma_{\gamma_2} - \sigma_{\theta_2}\sigma_{\gamma_1})}{\gamma_H||\sigma_\theta||^2}, \quad -\frac{\sigma_{\theta_1}(\sigma_{\theta_1}\sigma_{\gamma_2} - \sigma_{\theta_2}\sigma_{\gamma_1})}{\gamma_H||\sigma_\theta||^2}\right), \tag{37}$$

and hence the weight λ follows a stochastic process. The zeros in the first two positions of ν_H are not accidental. Since the preference shifts that the Home country faces are uncorrelated with the output shocks, it demands a hedge correlated with the Brownian motions w^γ and w^θ but not with w and w^*. Constructing such a hedging portfolio is possible: one can easily show that any zero-cost portfolio of the two stocks is going to be uncorrelated with the output shocks. The hedging portfolio h must then have a value of zero, and one can easily verify from (35) and Lemma 1 that this is indeed the case. As a corollary, none of the countries holds any bond (Lemma 2) and hence their (conventional) current accounts are zero (Lemma 3):

$$CA_i(t) = 0, \qquad i \in \{H, F\}.$$

In this example, however, the capital-gains adjusted current account is different from zero. We examine it closely in the next section.

Remark 1 (A limiting case of "small risks"). There are two types of risks that we highlight in this example: the demand shocks due to fluctuations in γ_H and those due to the fluctuations in θ_H. The limit of $\sigma\gamma_H \to 0$ is discussed in detail in the previous subsection. The main conclusion that becomes apparent from comparing the case of stochastic and deterministic γ_H is that if uncertainty about γ_H disappears, so does market incompleteness.

Let us now examine the case of uncertainty about θ_H becoming small. We first establish whether markets remain incomplete. To do so, we take the limit in (37) as $\sigma_{\theta_1} \to 0$ and $\sigma_{\theta_2} \to 0$. One can easily see that this limit does not exist. For example, if we first let $\sigma_{\theta_1} \to 0$ and then $\sigma_{\theta_2} \to 0$, the limit of ν_H is $(0,\, 0,\, -\sigma_{\gamma_1}/\gamma_H,\, 0)$, and if we do the reverse, it is $(0,\, 0,\, 0,\, -\sigma_{\gamma_2}/\gamma_H)$. Hence, the limit does not exist. This is alarming because many of the existing methods for solving portfolio models in international macro-finance rely on approximations around a deterministic case. The limit of the countries' portfolios, however, is well-defined. Home's holdings of the two stocks converge to

$$s_H = \left(\frac{\theta_F}{\theta_H + \theta_F - 1},\quad -\frac{1-\theta_F}{\theta_F + \theta_H - 1} \right).$$

If we complete the markets in our model by adding two more bonds—such as, for example, the local-good denominated bonds—we are going to get the same portfolios as the ones occurring in this limit.

4.4. Example 4: Complete Markets and the Current Account

In the examples we have considered so far, we have obtained rich implications for asset prices, portfolio holdings and therefore on the NFA positions (which we discuss further in the next section). All three examples, however, imply that the conventional current account is zero at all times. The main reason is that in none of the examples agents hold any bond positions. In this section, we consider a special case of our model in which bondholdings are nontrivial. For simplicity, we set up the example in such a way that markets end up being complete. This highlights that market incompleteness is neither necessary nor sufficient to generate movements in the (conventional) current account.

To induce the countries to trade in bonds, we now allow for the correlation between the demand and output shocks. We reduce the number of Brownian motions driving the economy from four to two. In particular, we shut down Brownian motions w^γ and w^θ and require that all processes are adapted to the filtration generated by the output shocks w and w^*. Under this modification, all four-dimensional vectors in our analysis in Section 2 and the Appendix become two-dimensional. This implies further that the volatility matrix of stock returns σ is a 2×2 square matrix. If this matrix is nondegenerate—which is always the case in the presence of stochastic preference shifts—financial markets are complete. Equilibrium allocation is then Pareto optimal and the weight λ is constant. It is important to emphasize that a constant λ does not imply constant wealth distribution; as one can see from equation (24), the relative wealth fluctuates together with γ_H.

In the interest of space, we do not report the countries' portfolios in this case. It suffices to say that now portfolios depend on all of the parameters of the model except for the drifts of outputs. As to be expected, the gain on the hedging portfolio in this case is perfectly correlated with the fluctuations in γ_H:

$$d\,h = [\ldots]\,dt + \frac{W_H}{\gamma_H}(\sigma_{\gamma_1},\, \sigma_{\gamma_2})\,d\vec{w},$$

which simply reflects the fact that when markets are complete, agents can perfectly hedge against fluctuations in pertinent state variables. In contrast to all the special cases we have considered so far, however, the value of the hedging portfolio is not equal to zero. Lemma 2 then implies that now the countries engage in borrowing and lending. Let us consider an example in which we set $\sigma_{\gamma_H} = (\sigma_{\gamma_1},\, 0)$ and $\sigma_{\theta_H} = (0,\, \sigma_{\theta_2})$. In this example, the value of the bondholdings of the Home country becomes

$$-\frac{\frac{\left(1-e^{-\rho(T-t)}\right)}{\rho} Y\,Y^* \lambda \gamma_H \gamma_F \sigma_{\gamma_1} \sigma_{\theta_2}}{(1-a)Y\lambda\gamma_F(\gamma_H(1-\theta_H)+\lambda\gamma_F\theta_F)(1-\theta_H-\theta_F)\sigma_{Y^*}\sigma_{\gamma_1} + aY^*\gamma_H(\gamma_H+\lambda\gamma_F)(\gamma_H\theta_H+\lambda\gamma_F(1-\theta_F))\sigma_Y\sigma_{\theta_2}},$$

and under some restrictions on parameters, we can get this quantity to be always below zero. This implies that Home borrows from Foreign to finance its hedging portfolio, whose value is always greater than zero in this case. This example demonstrates that in our model it is possible to have a negative bond position and hence possibly a current account deficit (Lemma 3) forever. This does not in any way contradict sustainability of a country's external position: if an equilibrium exists, the budget constraints of both countries are always satisfied, and so a negative position in the bond account is offset by positive positions in the stocks. Having a negative debt position in equilibrium implies that Home is holding a leveraged portfolio. That portfolio is riskier than that of the Foreign country. The Home country is compensated for taking a greater share of world risk by collecting risk premia on its risky asset holdings. These risk premia can be used to finance current account deficits.[11] In a recent paper Gourinchas and Rey (2007a) argue that the US, whose external liabilities are mainly Treasury bills and bonds (with relatively low expected returns) while its external assets are riskier instruments (with relatively high expected returns), collects additional income from positive risk premia on its NFA positions. Our model allows for a calibration that captures this possibility.

According to Lemma 3, the countries' current accounts are always nonzero. This is the first time we encounter a nonzero current account in this section. As the case we are considering here demonstrates, enough heterogeneity in hedging demands that is sufficient to give rise to trade in bonds for risk sharing purposes guarantees that the current account deviates from zero.

5. Capital Gains and External Adjustment

We now explore the dynamics of NFA in our model, and, in particular, how NFA respond to demand and supply shocks. Let us focus on the NFA of, say, Home. To understand the intuition behind the role the capital gains in the dynamics of NFA, note first that the budget constraint of Home (6) can be equivalently represented as

$$\begin{aligned} dW_H(t) &= \Big[s_H^B(t)\, B(t)\, r(t) + s_H^S(t) S(t)\mu_S(t) + s_H^{S^*}(t) S^*(t)\mu_{S^*}(t)\Big] dt \\ &\quad + \Big[s_H^S(t) S(t)\sigma_S(t) + s_H^{S^*}(t) S^*(t)\sigma_{S^*}(t)\Big] d\vec{w}(t) + TB_H(t) - p(t)Y(t)dt\,. \end{aligned}$$

We can then substitute this expression into (28) and use (4) and the stock market clearing

$$\begin{aligned} CGCA_H(t) &= TB_H(t) + \Big[s_H^{S^*}(t) S^*(t)\mu_{S^*}(t) - s_F^S(t) S(t)\mu_S(t) + s_H^B(t) B(t) r(t)\Big] dt \\ &\quad + \Big[s_H^{S^*}(t) S^*(t)\sigma_{S^*}(t) - s_F^S(t) S(t)\sigma_S(t)\Big] d\vec{w}(t), \end{aligned} \tag{38}$$

Define an *unexpected* change in a variable Z at time t as the diffusion term, $\sigma_Z(t)d\vec{w}$, in the dynamics $dZ(t) = \mu_Z(t)dt + \sigma_Z(t)d\vec{w}(t)$. (The trend term, $\mu_Z(t)dt$, is the *expected* change.) The un-

[11] We thank Mick Devereux for suggesting this intuition to us.

expected changes dominate the short-term fluctuations in the NFA positions because the Brownian-motion term dwarfs the trend term. We therefore focus on the unexpected capital gains. They represent the movement in the NFA in response to a shock, on impact. Making use of (4)–(5) and (25), we can rewrite (38) as

$$CGCA_H(t) \quad = \quad CA_H(t) + s_H^{S^*}(t)dS^*(t) - s_F^S(t)dS(t) \tag{39}$$

From (39), we can see that, on impact, only valuation changes of the two stocks contribute to the unexpected capital gains on the NFA. There are no unexpected capital gains on the bond, since its return over the next instant is riskless.[12] The conventional current account is also riskless over the next instant.

We find that in our model the signs of the responses of the NFA to the underlying innovations depend on whether the country is a net debtor ($NFA < 0$) or a net creditor ($NFA > 0$). This generalizes an important result derived by Kraay and Ventura (2000). Recall from Lemma 4 that

$$NFA_H = \frac{\gamma_H(1-\theta_H) - \lambda\,\gamma_F(1-\theta_F)}{\gamma_H\theta_H + \lambda\,\gamma_F(1-\theta_F)}\,S\,.$$

We thus need the following condition:

Condition NC: Home is Net Creditor. $\gamma_H(1-\theta_H) - \lambda\,\gamma_F(1-\theta_F) > 0$.

To keep our discussion focused, we now specialize our economy to the setting in Example 3. In order to better highlight the effects of incomplete markets, we postulate further that the demand shock γ_H loads only on the Brownian motion w^γ and the demand shock θ_H only on the Brownian motion w^θ. That is, the demand shocks are not only independent of the output shocks but also independent of each other. Formally, we assume that $\sigma_{\gamma_H}(t) = (0, 0, \sigma_{\gamma_1}(t), 0)$ and $\sigma_{\theta_H}(t) = (0, 0, 0, \sigma_{\theta_2}(t))$, with $\sigma_{\gamma_1} > 0$ and $\sigma_{\theta_2} > 0$. A similar analysis can be performed in the context of Example 2, but we focus on Example 3 for more generality. Our goal is to sign (or characterize) the direction of the valuation effects. Table 1 presents the unexpected gains/losses on the terms of trade, the stocks and the ensuing gains/losses on the NFA position of the Home country.

Table 1 reveals that on impact, both stocks yield unexpected capital gains in response to a positive output shock in either country (dw or dw^*).[13] This is because a positive output shock in say, Home, raises the dividend on the Home tree. In response, the Home stock goes up. At the

[12] An important and realistic extension would be to introduce long-maturity bonds. Capital gains on these bonds will be part of the (unexpected) capital gains on NFA. We leave this extension for future research.

[13] To establish this result, we have explicitly computed the diffusion terms $\sigma_S S$ and $\sigma_{S^*} S^*$ in the equilibrium processes for the stock prices (4)–(5) and signed them. See the proof of Proposition 2 in the Appendix.

	dw	dw^*	dw^γ	dw^θ
Unexpected change in q	$-$	$+$	0	$+$
Unexpected change in S	$+$	$+$	0	$+$
Unexpected change in S^*	$+$	$+$	0	$-$
Unexpected change in NFA_H	$+^{NC}$	$+^{NC}$	0	$-$

Table 1: The valuation effects: Unexpected gains on the terms of trade, stocks and Home's the net foreign assets in response to the underlying shocks. The superscript ^{NC} indicates that a necessary and sufficient condition under which a sign obtains is Condition NC.

same time, it causes a deterioration of Home's terms of trade because the Home good becomes less scarce. This in turn improves Foreign's terms of trade and hence raises the value of the output of the Foreign tree. Hence, the Foreign stock rises. An analogous argument applies for the output shock at Foreign. Output shocks, then, always cause the two stock markets to comove (as in Cole and Obstfeld (1991)). The reaction of stock prices to the demand shocks has a distinctly different pattern: stock prices always move in opposite directions. As Home shifts its preference towards the Home good (in response to a positive realization of dw^θ), there is an excess demand for the Home good in the world. This pushes the price of the Home good up, or equivalently, causes an appreciation of the terms of trade, q. This raises the value of the Home output relative to Foreign. Consequently, the price of the Home stock increases, while that of the Foreign stock falls. An analogous argument applies for the demand shift towards the Foreign good (a negative realization of dw^θ).

The most subtle of the impact responses presented in Table 1 is that to a shock to the discount factor γ_H. At the outset, one would expect that such a shock should have an impact on the terms of trade and stock prices because Home desires to consume more at times when γ_H is high. For example, if it prefers one good over the other (e.g., if it has consumption home bias), then, following a shock to γ_H, the price of that good should go up, in relative terms. This intuition would have been correct under complete markets, and the fact that it fails here highlights the effects of market incompleteness in our model. Indeed, the Home agent would have liked to consume more in high-γ_H states, but we have set up our example in such a way that he is unable to purchase a state-contingent claim that allows him to do so. Recall from our discussion of equation (36) that the Home agent's ability to hedge γ_H shocks is governed by the correlation between θ_H and γ_H. When the correlation is perfect, the agent is able to hedge perfectly, and markets are complete. The opposite case—the case of zero correlation that we highlight in this example—features an extreme degree of market incompleteness, in which no hedging of γ_H is possible. Because of this, neither consumption nor prices respond to w^γ shocks.

To determine the unexpected capital gains on the NFA following a shock, we use equation (39). All we need to know is the capital gains on Home and Foreign stock and the countries' portfolio holdings. In the case we are considering here, the countries' portfolios take an especially simple form. Equations (33)–(34) reduce to

$$s_H = \left(\frac{\gamma_H}{\gamma_H + \lambda\gamma_F}, \frac{\gamma_H}{\gamma_H + \lambda\gamma_F}\right), \quad s_F = \left(\frac{\lambda\gamma_F}{\gamma_H + \lambda\gamma_F}, \frac{\lambda\gamma_F}{\gamma_H + \lambda\gamma_F}\right).$$

Importantly for our discussion here, both countries have positive holdings of each stock.

We are now ready to examine the adjustment of the NFA following a shock. The mechanism is the simplest for the demand shocks. Following a discount-factor shock (a movement in w^γ), obviously, there is no response of the NFA because the stock prices remain the same. Now consider a preference shift towards the Home good (a positive dw^θ). Home stock rises and foreign residents make gains on their holdings at Home, while Foreign stock falls and home residents suffer capital losses on their position abroad. Hence, the NFA of Home falls. A preference shift towards the Foreign good (a negative dw^θ) obviously causes the opposite reaction. Home stock falls and Foreign stock rises. Hence, Home gains on its investment in the Foreign stock while Foreign loses on its investment in the Home stock, and therefore the NFA of Home increases.

The responses to the output shocks are somewhat more difficult to analyze. This is because output shocks induce comovement of stocks across countries. A positive output shock makes both the Home and the Foreign stock go up. So both countries make a gain on their positions abroad simultaneously. The NFA movement then must reflect the relative magnitude of the gains to Home vs. Foreign investors, which in turn depends on their respective stock positions abroad. Let us consider two extremes. Suppose first that Home holds most of the supply of the risky stocks. Because none of the countries holds the bond, Home is holding most of the world's assets, which makes it a net creditor to the world. In response to a positive output shock, Home enjoys significant capital gains, both at home and abroad. The capital gains of Foreign are significantly smaller (in both markets) because Foreign's stock positions are much smaller. Hence, the net capital gain of Home a is larger and therefore Home's NFA increase. The other extreme is when Foreign holds most of the supply of the risky stocks, and hence is a net creditor. Then, the capital gains on Foreign's stock positions (both at home and abroad) by far exceed those of Home. Hence, Foreign's NFA increase, which implies that Home's NFA position deteriorates. The knife-edge case turns out to be the one in which Home turns from being a net creditor to a net debtor to the world, which is reflected in the signs reported in Table 1. The responses to an output shock at Foreign follow the same pattern.

It is important to emphasize that in our model the response of a country's NFA to an output shock is positive if and only if the country is a net creditor (i.e., under Condition NC). So only for countries like China or Japan, who are net creditors to the world, do the NFA positions improve following good news about productivity. But for a country like the US or the UK, whose NFA are large and negative, the NFA position should deteriorate in response to good news about productivity. The opposite, of course, is true for a drop in output (a recession) in either country.

6. Discussion

Some of the recent literature has drawn attention to the relevance of the quality of international assets for the discussion of global imbalances (see Caballero, Farhi, and Gourinchas (2008), Dooley, Folkerts-Landau, and Garber (2004), and Blanchard, Giavazzi, and Sa (2005) for the link to global imbalances, and Kouri (1982) for an earlier discussion). Because in our model the entire output of each country is capitalizable and there are no restrictions on capital flows, financial assets do not vary in their quality. But we believe that differences in asset quality is an important feature of international capital markets, and therefore it would be interesting to extend our framework to include this element into the analysis.

Extending the framework beyond log-linear preferences may also prove fruitful. This would introduce some of the intertemporal hedging motives that have been shut down in our model. Moving away from the log-linear specification, however, has the drawback that the model loses its tractability. For instance, for the case of CES preferences, it is not possible to obtain analytical characterizations for portfolios and asset prices. There are three ways in which one can tackle such a model. First, one can attempt to solve the model numerically. To our knowledge, this has been done only for the complete-markets case (Gourinchas and Rey (2006))—an extension to the incomplete-markets case is a daunting task. Second, one can follow, for example, Devereux and Sutherland (2010) and Tille and van Wincoop (2010) and approximate around a deterministic steady state to derive equilibrium characterizations for small volatilities. Finally, one can recognize that log-linear preferences are a special case of CES preferences and build on our model to find an approximate solution for the CES case. To do so, one can perturb the equilibrium in our economy by expanding around the unitary elasticity of substitution, for which the solutions are analytical.[14] The advantage of this approach is that the approximation is done around a stochastic equilibrium as opposed to a deterministic steady state.

To be able to fit the model to the data and to generate a realistic equity premium, we need to extend our model to include investment. In our framework, there is no difference between savings and the external accounts, and the introduction of investment breaks this tight link. In this paper we have decided to concentrate on the valuation effect while holding capital fixed. In the data, of course, movements in investment play an important role. Since our model does not produce realistic equity premia and some other asset-pricing moments, expected capital gains (anticipated valuation changes) are smaller than those documented by Gourinchas and Rey (2007b). To improve the performance of our model along this dimension, one could integrate an external habit into the preferences' specification. Single-good models with external habits are known to match key asset pricing moments very well (Campbell and Cochrane (1999)). A recent work by Stathopoulos (2008) shows that external habits are also successful in matching standard asset pricing moments in a two-country two-good model with logarithmic preferences (and external habits), with the tractability offered by a logarithmic utility still being preserved.

Some theoretical properties linking financial assets and external accounts derived in our model are easily testable using new datasets that account for valuation effects (Gourinchas and Rey (2007b), Lane and Milesi-Ferretti (2007)). Tests of these relationships, as well as a calibration of our model are beyond the scope of this paper. A calibration exercise is undertaken in Pavlova and Rigobon (2010), who find that our model is consistent with the data along several dimensions. For example, capital gains on countries' NFA positions are large, volatile, serially uncorrelated, and effectively dominate the movements in NFA (Kollmann (2006) and Lane and Shambaugh (2010)). Furthermore, these capital gains are negatively correlated with the trade balance, and so capital gains stabilize countries' NFA positions by offsetting trade balance movements (Gourinchas and Rey (2007b), Devereux and Sutherland (2010)).

[14] This idea is closely related to the works of Judd (1998) and Kogan and Uppal (2003) who develop applications of perturbation methods to solving problems in economics and finance.

7. Conclusion

In his Harms Lecture at the Kiel Institute, Obstfeld (2004) stresses that "recent changes in the functioning of international capital markets require a new view of external adjustment" and moreover, that any notion of "external balance adjustment cannot be defined without reference to the structure of national portfolios." In this paper, we take a step in that direction. We develop an open economy model with endogenous portfolio decisions, in which we investigate the interaction between capital markets and the external adjustment process.

From the methodological point of view, our contribution is to construct a framework that is rich enough to include multiple risky assets, incomplete markets, and supply- and demand-side uncertainty, while at the same time simple enough to allow for closed-form characterizations of asset prices, net foreign asset positions, and equity portfolios. It is within this framework that we are able to establish the interconnections between the real side of the economy represented by the trade balance, current account, and consumption allocations and the financial side such as portfolio holdings, stock prices, and valuation changes.

Appendix

Proof of Proposition 1. In this proof, we closely follow He and Pearson (1991). Their analysis is presented in the context of a single-good economy, but this does not present a difficulty for us because (in the main text) we have reduced our problem to a representation that is equivalent to a familiar single-good one. In particular, the first-order conditions for the consumer problem (11)–(12) have the familiar form

$$e^{-\rho t}\frac{\gamma_i(t)}{\overline{\mathbb{C}}_i(t)} = y_i \xi_{\nu_i}(t)\,, \qquad i \in \{H,\, F\}, \tag{A.1}$$

where the Lagrange multiplier y_i is such that the budget constraint evaluated at the optimal consumption expenditure, $\overline{\mathbb{C}_i}$, is satisfied with equality:

$$E\left[\int_0^T \xi_{\nu_i}(t)\overline{\mathbb{C}}_i(t)dt\right] = W_i(0)\,, \qquad i \in \{H,\, F\}.$$

It follows that, by no-arbitrage, the time-t wealth of a consumer is given by

$$W_i(t) \quad = \quad E_t\left[\int_t^T \frac{1}{\xi_{\nu_i}(t)}\xi_{\nu_i}(s)\overline{\mathbb{C}}_i(s)ds\right]\,, \qquad i \in \{H,\, F\},$$

and hence, making use of (A.1) and the assumption that γ_i are martingales, we have

$$W_i(t) = \frac{\gamma_i(t)}{y_i \xi_{\nu_i}(t)}\,\frac{e^{-\rho t} - e^{-\rho T}}{\rho}, \qquad i \in \{H,\, F\}. \tag{A.2}$$

Of course, for the case of the Foreign country, the argument γ_F is constant over time.

To find optimal portfolios, we apply Itô's lemma to (A.2) and match the diffusion term with that in the dynamic budget constraint (10). This operation yields

$$x_i^\top(t)\sigma(t) = \frac{\sigma_{\gamma_i}(t)}{\gamma_i(t)} + (m(t) + \nu_i(t))^\top, \tag{A.3}$$

where we have used equation (14). Recall that in incomplete markets the matrix σ is not a square matrix, and hence the above system of equations contains 4 equations (dimensionality of the vector of Brownian motions) in 2 unknowns (the number of stocks). It has a solution if and only if its right-hand side lies in Span(σ). This entails a restriction

$$(I_4 - \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t))\frac{\sigma_{\gamma_i}(t)^\top}{\gamma_i(t)} + \nu_i(t) = 0, \tag{A.4}$$

where we have applied the projection operator $I_4 - \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)$ to both sides of (A.3). Equation (17) then follows immediately. Note that, for the case of Foreign, equation (A.4) simplifies to yield $\nu_F(t) = \mathbf{0}$ because σ_{γ_F} is equal to zero.

The optimal portfolios are obtained from (A.3) via simple algebraic manipulations that, in particular, make use of the property that $\sigma(t)\nu_i(t) = 0$. ■

Proof of Lemma 1. We use the construct of the representative agent to value stocks in the economy. The representative agent's utility evaluated at the aggregate output is given by

$$u(Y(t),\, Y^*(t); \lambda(t)) = \max_{\substack{C_H(t)+C_F(t)=Y(t),\\ C_H^*(t)+C_F^*(t)=Y^*(t)}} u_H(C_H(t),\, C_H^*(t)) + \lambda(t) u_F(C_F(t),\, C_F^*(t)).$$

It follows from this definition that the marginal utilities of the representative agent and the individual agents, evaluated at the optimum, are related as

$$\nabla u(Y(t),\, Y^*(t); \lambda(t)) = \nabla u_H(\overline{C}_H(t),\, \overline{C}_H^*(t)) = \lambda(t) \nabla u_F(\overline{C}_F(t),\, \overline{C}_F^*(t)),$$

where the symbol ∇ is used to denote the gradient. From the first-order conditions of the Home consumer,

$$\nabla u_H(\overline{C}_H(t),\, \overline{C}_H^*(t)) = \Big(y_H p(t) \xi_{\nu_H}(t),\; y_H p^*(t) \xi_{\nu_H}(t) \Big).$$

To derive this we used the fact that $\nabla u_H(\overline{C}_H(t),\, \overline{C}_H^*(t)) = (\gamma_H(t)\theta_H(t)/\overline{C}_H(t),\; \gamma_H(t)(1-\theta_H(t))/\overline{C}_H^*(t))$ combined with (8) and (A.1). Substituting the sharing rules of the representative agent (18), we can then derive the personalized state price density of the Home consumer and hence that of the representative agent:

$$\xi_{\nu_H}(t) = e^{-\rho t} \frac{p(0)}{p(t)} \frac{\overline{C}_H(0)}{\overline{C}_H(t)} = e^{-\rho t} \frac{p(0) Y(0)}{p(t) Y(t)} \frac{\gamma_H(t)\theta_H(t) + \lambda(t)\gamma_F(1-\theta_F)}{\gamma_H(0)\theta_H(0) + \lambda(0)\,\gamma_F(1-\theta_F)}. \tag{A.5}$$

This state price density can be used to price the risky stocks by no-arbitrage:[15]

$$S(t) = \frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\, p(s)\, Y(s)\, ds\right], \quad S^*(t) = \frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\, p^*(s)\, Y^*(s)\, ds\right].$$

Hence, the price of the Home stock is

$$\begin{aligned} S(t) &= \frac{e^{\rho t} p(t)\, Y(t)}{\gamma_H(t)\theta_H(t) + \lambda(t)\gamma_F(1-\theta_F)} E_t\left[\int_t^T e^{-\rho s}(\gamma_H(s)\theta_H(s) + \lambda(s)\gamma_F(1-\theta_F)) ds\right] \\ &= \frac{1-e^{-\rho(T-t)}}{\rho} p(t) Y(t) + \frac{e^{\rho t}\gamma_F(1-\theta_F)\, p(t)\, Y(t)}{\gamma_H(t)\theta_H(t) + \lambda(t)\gamma_F(1-\theta_F)} E_t\left[\int_t^T e^{-\rho s}(\lambda(s)-\lambda(t)) ds\right], \end{aligned} \tag{A.6}$$

where we used the fact that $\gamma_H\theta_H$ is a martingale (i.e., $E_t[\gamma_H(s)\theta_H(s)] = \gamma_H(t)\theta_H(t)$). Analogously, using the fact that $\gamma_H(1-\theta_H)$ is a martingale, we find the price of the Foreign stock to be

$$\begin{aligned} S^*(t) &= \frac{e^{\rho t} p^*(t)\, Y^*(t)}{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\gamma_F\theta_F} E_t\left[\int_t^T e^{-\rho s}(\gamma_H(s)(1-\theta_H(s)) + \lambda(s)\gamma_F\theta_F) ds\right] \\ &= \frac{1-e^{-\rho(T-t)}}{\rho} p^*(t)\, Y^*(t) + \frac{e^{\rho t}\gamma_F\theta_F\, p^*(t)\, Y^*(t)}{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\gamma_F\theta_F} E_t\left[\int_t^T e^{-\rho s}(\lambda(s)-\lambda(t)) ds\right]. \end{aligned} \tag{A.7}$$

There are two ways to proceed in evaluating the above conditional expectations. The first is to assume that λ is a martingale (and hence $E_t[\lambda(s) - \lambda(t)] = 0$) and then verify that it is indeed the case in equilibrium. From Proposition 2, however, we can only conclude that λ is a local martingale.

[15] All stocks are in positive net supply. The arguments of Loewenstein and Willard (2007) rule out bubbles in the stocks' valuation.

In all special cases that we consider in Section 4, it is easy to verify that λ is also a true martingale under some additional mild regularity conditions imposed on the preference shifts.[16] However, for the general case it is not immediate how to show it.

An alternative approach is to use the following, less direct, argument based on market clearing. In particular, from (A.1)–(A.2), we have

$$W_H(t) + W_F(t) = (\overline{\mathbb{C}}_H(t) + \overline{\mathbb{C}}_F(t))\frac{1-e^{-\rho(T-t)}}{\rho} = p(t)\,Y(t)\,\frac{1-e^{-\rho(T-t)}}{\rho} + p^*(t)\,Y^*(t)\,\frac{1-e^{-\rho(T-t)}}{\rho},$$

where in the last equality we used the fact that the total consumption expenditure at time t equals $p(t)Y(t) + p^*(t)Y^*(t)$. On the other hand, from stock market clearing, we have

$$W_H(t) + W_F(t) = S(t) + S^*(t).$$

Combining the resulting restriction that

$$S(t) + S^*(t) = p(t)\,Y(t)\,\frac{1-e^{-\rho(T-t)}}{\rho} + p^*(t)\,Y^*(t)\,\frac{1-e^{-\rho(T-t)}}{\rho}$$

with (A.6)–(A.7), we conclude that

$$S(t) = p(t)\,Y(t)\,\frac{1-e^{-\rho(T-t)}}{\rho} \quad \text{and} \quad S^*(t) = p^*(t)\,Y^*(t)\,\frac{1-e^{-\rho(T-t)}}{\rho}.$$

This establishes (21)–(22).

To derive (23), we combine (A.1)–(A.2) with (18)–(19) and use the representation of the stock prices S and S^* derived in this lemma. ■

Proof of Lemma 4. Equation (31) follows from $NFA_H(t) = W_H(t) - S(t)$ and Lemma 1. To derive (30), we use the definition of the trade balance, $TB_H(t) = (p(t)(Y(t) - C_H(t)) - p^*(t)Y^*(t))dt$, substitute the equilibrium expressions for consumption and the terms of trade, (18) and (20), and simplify. ■

Before we proceed to the rest of the proofs, we need to define several auxiliary vectors to be used throughout the remainder of this appendix. Let

$$i_1 \equiv (1,\,0,\,0,\,0), \quad i_2 \equiv (0,\,1,\,0,\,0), \quad \text{and} \tag{A.8}$$

$$\begin{aligned} A(t) \equiv\; & \frac{\gamma_H(t)\sigma_{\gamma_H}(t) + \theta_H(t)\sigma_{\theta_H}(t) - \gamma_F(1-\theta_F)\,\lambda(t)\,\nu_H(t)^\top}{\gamma_H(t)\theta_H(t) + \lambda(t)\,\gamma_F(1-\theta_F)} \\ & - \frac{(1-\theta_H(t))\sigma_{\gamma_H}(t) - \gamma_H\sigma_{\theta_H}(t) - \gamma_F\theta_F\,\lambda(t)\nu_H(t)^\top}{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\,\gamma_F\theta_F} - \sigma_Y(t)\,i_1 + \sigma_{Y^*}(t)\,i_2. \end{aligned} \tag{A.9}$$

[16] The only special case that requires these additional assumptions is that presented in Section 4.3. In particular, one needs to bound the preference shifts in such a way that the expression in (37) satisfies $E\left[e^{\frac{1}{2}\int_0^T \nu_H(t)^\top \nu_H(t)dt}\right] < \infty$. This condition is known as the Novikov condition.

Proof of Proposition 2. By substituting (A.2) into (24) we derive

$$\lambda(t) = \frac{y_H\, \xi_{\nu_H}(t)}{y_F\, \xi_{\nu_F}(t)}.$$

Applying Itô's lemma and using the representation of the countries' state price densities from (14), we have

$$d\lambda(t) = -\lambda(t) m(t)^\top \nu_H(t) dt - \lambda(t) \nu_H(t) d\vec{w}(t), \tag{A.10}$$

where we have substituted the finding that $\nu_F(t) = \mathbf{0}$ established in Proposition 1. To show that the drift term in (A.10) is equal to zero, we use the definition of m from (13) and the restriction that $\sigma(t)\nu_H(t) = 0$.

To determine $\lambda(0)$, note from Lemma 1 that the initial financial wealth of, say, the Home country is given by (23) evaluated at $t = 0$. On the other hand, $W_H(0) = S(0)$ because the initial portfolio of Home consists of one share of the Home stock. This allows us to pin down $\lambda(0)$. It is easy to show that $\lambda(0) = \gamma_H(0)(1 - \theta_H(0))/\gamma_F(1 - \theta_F)$.

We now report the volatility matrix of stock returns.

$$\sigma(t) = \begin{bmatrix} \frac{1-a}{a\,q(t)+1-a}\, A(t) + \sigma_Y(t)\, i_1 \\ -\frac{a\,q(t)}{a\,q(t)+1-a}\, A(t) + \sigma_{Y^*}(t)\, i_2 \end{bmatrix},$$

where $A(t)$, i_1, and i_2 are defined in (A.8)–(A.9). This volatility matrix is obtained by applying Itô's lemma to the closed-form expressions for the stock prices (21)–(22).

The market price of risk process m can be derived from the dynamics of ξ_H in (14). Using the identity $\xi_H(t) = ap(t)\xi_H(t) + (1-a)p^*(t)\xi_H(t)$ and equations (20) and (A.5), we derive

$$\begin{aligned} \xi_H(t) \;=\; & a e^{-\rho t} p(0) \frac{\gamma_H(t)\theta_H(t) + \lambda(t)\,\gamma_F(1-\theta_F)}{\gamma_H(0)\theta_H(0) + \lambda(0)\,\gamma_F(1-\theta_F)} \left(\frac{Y(0)}{Y(t)}\right) \\ & + (1-a) e^{-\rho t} p(0)\, \frac{\gamma_H(t)(1-\theta_H(t)) + \lambda(t)\,\gamma_F\theta_F}{\gamma_H(0)\theta_H(0) + \lambda(0)\,\gamma_F(1-\theta_F)} \left(\frac{Y(0)}{Y^*(t)}\right). \end{aligned} \tag{A.11}$$

Applying Itô's lemma and identifying the diffusion term with that in the representation of ξ_H in (14), we obtain

$$\begin{aligned} m(t) \;=\; & -\frac{p(0)e^{-\rho t}}{\xi_H(t)} \frac{Y(0)}{\gamma_H(0)\theta_H(0) + \lambda(0)\,\gamma_F(1-\theta_F)} \Big[a\, \frac{\gamma_H(t)\sigma_{\gamma_H}(t) + \theta_H(t)\sigma_{\theta_H}(t) - \gamma_F(1-\theta_F)\lambda(t)\,\nu_H(t)}{Y(t)} \\ & + (1-a)\, \frac{(1-\theta_H(t))\sigma_{\gamma_H}(t) - \gamma_H\sigma_{\theta_H}(t) - \gamma_F\theta_F\lambda(t)\,\nu_H(t)}{Y^*(t)} \\ & - a\, \frac{\gamma_H(t)\theta_H(t) + \gamma_F(1-\theta_F)\lambda(t)}{Y(t)} \sigma_Y(t) i_1 - (1-a)\, \frac{\gamma_H(t)(1-\theta_H(t)) + \gamma_F\theta_F\lambda(t)}{Y^*(t)} \sigma_{Y^*}(t) i_2 \Big] - \nu_H(t). \end{aligned}$$

This completes the proof of the proposition. ■

Derivation of the remaining equilibrium quantities. We first report the interest rate r and the stocks' expected returns μ_S and μ_{S^*} and then explain how we derived these expressions.

$$\begin{aligned} r(t) = \rho &+ \frac{aq(t)}{aq(t)+1-a}\left(\mu_Y(t)-\sigma_Y(t)^2\right) + \frac{1-a}{aq(t)+1-a}\left(\mu_{Y^*}(t)-\sigma_{Y^*}(t)^2\right) \\ &+ \frac{aq(t)}{aq(t)+1-a}\,\frac{\sigma_Y(t)i_1(\gamma_H(t)\sigma_{\theta_H}(t)^\top + \theta_H(t)\sigma_{\gamma_H}(t)^\top - \lambda(t)\,\nu_H(t)\,\gamma_F(1-\theta_F))}{\gamma_H(t)\theta_H(t)+\lambda\gamma_F(1-\theta_F)(t)} \\ &+ \frac{1-a}{aq(t)+1-a}\,\frac{\sigma_{Y^*}(t)i_2(-\gamma_H(t)\sigma_{\theta_H}(t)^\top + (1-\theta_H(t))\sigma_{\gamma_H}(t)^\top - \lambda(t)\,\nu_H(t)\,\gamma_F\theta_F)}{\gamma_H(t)(1-\theta_H(t))+\lambda\gamma_F\theta_F}, \end{aligned} \tag{A.12}$$

$$\mu_S(t) = \rho + \mu_Y(t) + \frac{1-a}{aq(t)+1-a}\left(\mu_q(t) - \frac{aq(t)}{aq(t)+1-a}||A(t)||^2 + \sigma_Y(t)A(t)i_1^\top\right), \tag{A.13}$$

$$\mu_{S^*}(t) = \rho + \mu_{Y^*}(t) + \frac{aq(t)}{aq(t)+1-a}\left(-\mu_q(t) + \frac{aq(t)}{aq(t)+1-a}||A(t)||^2 - \sigma_{Y^*}(t)A(t)i_2^\top\right), \tag{A.14}$$

where μ_q is the expected improvement in the terms of trade, given by

$$\begin{aligned} \mu_q(t) \;=\; & \mu_{Y^*}(t) - \mu_Y(t) + \frac{1}{2}||A(t)||^2 - \frac{1}{2}\frac{||\gamma_H(t)\sigma_{\theta_H}(t)^\top + \theta_H(t)\sigma_{\gamma_H}(t)^\top - \lambda(t)\,\nu_H(t)\,\gamma_F(1-\theta_F)||^2}{(\gamma_H(t)\theta_H(t)+\lambda\gamma_F(1-\theta_F)(t))^2} \\ &+\frac{1}{2}\frac{||-\gamma_H(t)\sigma_{\theta_H}(t)^\top + (1-\theta_H(t))\sigma_{\gamma_H}(t)^\top - \lambda(t)\,\nu_H(t)\,\gamma_F\theta_F||^2}{(\gamma_H(t)(1-\theta_H(t))+\lambda\,\gamma_F\theta_F(t))^2} + \frac{1}{2}\sigma_Y(t)^2 - \frac{1}{2}\sigma_{Y^*}(t)^2, \end{aligned}$$

and where $A(t)$, i_1, and i_2 are defined in (A.8)–(A.9).

The interest rate r in (A.12) is equal to the drift term from the Itô expansion of the equilibrium state price density reported in (A.11). The formulas in (A.13)–(A.14) are obtained by applying Itô's lemma to the closed-form expressions for the stock prices (21)–(22) and the terms of trade (20), and then using the definitions of μ_S and μ_{S^*} from (4)–(5). Throughout these derivations, we have used our assumption that γ_H and $\theta_H\gamma_H$ are martingales. ■

Derivations for Section 4. All derivations for the special cases examined in Section 4 are tedious but straightforward. Perhaps the easiest way to obtain the formulas and signs reported in that section is to use *Mathematica* to simplify the expressions derived above and manipulate them in *Mathematica* to verify the desired properties. Our programs are available upon request.

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