Third-Party Sale of Information

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Abstract

We characterise the optimal selling mechanism by a third party adviser who has sole access to information on the continuous value of the good for the buyer in a seller-buyer relationship. The adviser sells binary advice (buy or not) contingent on the seller's price and, with positive probability, recommends a buy decision when the price exceeds the value. We extend the analysis to the case that the buyer has additional private information on the value of the good. If the buyer has private information the adviser may improve the welfare of the trading parties but, if not, the adviser is detrimental to them.

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1 Introduction

In many markets there exist third-party information providers who, for a fee, give information or advice to buyers about goods which they are considering purchasing. For example, experts provide assessments of fine arts or antiques. Or the buyer may be a potential purchaser of a firm and the third-party information provider a consultant who has knowledge about the value of the firm, including its fit to the potential investor. In this paper we study a situation in which there is a single seller of an indivisible good, a single potential buyer who may or may not have some private information about their valuation of the good, and an adviser who has some information about the buyer's valuation which neither the buyer nor the seller possesses. The adviser has a monopoly on this information, which we model as a realization w of a continuously distributed random variable, and is able to commit to conveying it truthfully to the buyer. The adviser sets a fee for information and designs the form which information disclosure will take, assuming the buyer agrees to pay the fee. For example, he could fully disclose the value of w, or he could adopt a binary structure (report whether w is above or below a given threshold), or he could adopt a structure with two thresholds (report whether w is 'high', 'medium' or 'low'). Alternatively he could report a noisy signal of w. The adviser is able to make the form of disclosure contingent on the price of the good and the seller sets the price in the knowledge of the adviser's policy. We are interested, among other aspects, in the form which information disclosure takes, the interaction between the strategic interests, and hence price formation, of the seller and the adviser, and the effects of the adviser on the welfare of the buyer and seller.

The adviser, in designing his fee and information policy, takes into account its effect on the seller's price. On the one hand, he would like to design the policy so as to reduce this price because that would increase the buyer's surplus and, in principle, the adviser might be able to extract this extra surplus through his information fee. On the other hand, he faces a number of constraining factors. In the first place, if the seller's price is low the buyer may not be willing to buy information and will simply buy the good without the information. Secondly, the seller has the option of setting a low price precisely in order to induce the buyer to buy outright (as opposed to buying information first and then buying the good only with a lower probability). The adviser has to allow the seller enough surplus that she does not want to bypass the adviser in this way.

We first study the case in which the buyer has no private information. The seller's equilibrium price is at least as high as the buyer's expected value. If the seller's cost of production is zero, or close to zero, the price is equal to the expected value, $E(w)$. The intuition for this is that if the price were higher than $E(w)$ the value of information would increase if the price were lowered (because the alternative for the buyer is not to buy the good at all) and if the price were lower the value of information would increase if the price were increased (because the alternative is to buy the good without information). The adviser therefore designs his fee and information policy in such a way as to induce the seller to set the price equal to $E(w)$.

In equilibrium, the buyer buys information but the adviser does not provide full information to him. Instead, he sells him binary information, i.e., a threshold structure. Moreover, the equilibrium threshold is below the equilibrium price - in effect, the adviser advises the buyer to buy the good when its value is below its price. The reason for this is that it gives a relatively high probability of sale, thereby increasing the seller's payoff and reducing her incentive to reduce her price and bypass the adviser.

We also show, for the uniform distribution, that the presence of the adviser is detrimental - he does not improve the buyer's expected payoff, and reduces that of the seller (as well as reducing total surplus).

Secondly, we study a model in which the buyer has private information and the adviser can sell information in the form of a collection of thresholds (for each threshold z the buyer can tell whether w is above or below z). In this case it may no longer be optimal for the adviser to have a binary information policy. If the distribution of the buyer's information is convex and the highest type of buyer does not buy information then the information structure has either one or two thresholds. If, on the other hand, the distribution is concave and the lowest type is excluded then there is full information disclosure over the range of values relevant to those types who buy information.

For the uniform distribution case we show that the adviser is beneficial to the trading parties if his information is 'small', i.e., if it is relatively less important than the buyer's private information. He strictly improves the payoff of some buyer types without reducing that of any buyer type or of the seller. This therefore contrasts with the case in which his information is very 'large' - the buyer has no private information - since, as mentioned above, the adviser is detrimental in that case.

There is a large literature which examines the incentives of sellers to provide information to buyers in markets and auctions. Among others, Lewis and Sappington (1994), Ottaviani and Prat (2001), Johnson and Myatt (2006), Bergemann and Pesendorfer (2007) and Eso and Szentes (2007a) study situations in which a seller chooses both a selling mechanism and a rule for disclosing information to buyers. Our paper focusses instead on a setting in which the seller and information provider are distinct agents, and on the strategic interaction between them.

There is also a literature on certification intermediaries. For example, in Lizzeri (1999) a monopoly intermediary who is informed about a seller's quality sets a fee and commits to an information disclosure policy. The seller then decides whether to pay the intermediary or sell direct. In the unique equilibrium all sellers pay the intermediary who reveals no information beyond the fact that the seller has paid to be certified. A seller who does not pay the intermediary is believed to be the worst type. The difference between this (and other papers in this literature, such as Albano and Lizzeri (2001) and Biglaiser (1993)) and our paper is that our third party sells information (in our case, to the buyer) which is not known to the seller.

Eso and Szentes (2007b), like us, study an adviser who sells information rather than a good. Their focus, however, is different from ours. An adviser gives information about the value of a project to a decision-maker. The decision-maker has private information about this value, but the adviser can give him an additional signal. The main result is that the adviser can get exactly the same surplus if she does not observe the signal which she gives as if she observes it perfectly. In the optimal mechanism there is a menu of pairs of prices - a payment for the signal and a further payment conditional on whether the project was undertaken.

The literature on Bayesian persuasion (e.g., Kamenica and Gentzkow (2011), Rayo and Segal (2010)) is also concerned with design of information disclosure policies. Our model is different in that a third party supplies a signal and that both information and a product are sold. Prices are crucial strategic variables. In the language of Kamenica and Gentzkow, we combine two ways in which an agent can be induced to do something, by pricing and by changing beliefs.

Other papers which study the sale of information, but not by a third party in the context of a trading relationship, are Bergemann, Bonatti and Smolin (2016), Hörner and Skrzypacz (2015). Cabrales and Gottardi (2014) study a model in which buyers can acquire information at a cost and sell it to other buyers if it is not relevant to them. The focus of the paper is on the efficiency of information acquisition.

Roesler and Szentes (2016) characterize the information structure which is optimal for the buyer, assuming the seller knows the structure but does not observe the signal which the buyer receives. This structure is efficient and generates a unit-elastic demand curve for the seller. This would be the best structure for our adviser to provide to the buyer if he could extract the surplus. However, he cannot do so because, since the signal structure is efficient, the buyer would be unwilling, once the seller has named her price, to pay a positive price for the signal. Our paper characterizes the structure of information which obtains if it has to be bought from a monopoly provider.

2 Model

A seller (S) has an indivisible object which she can sell to a single potential buyer (B). The value of the object to B, denoted by w, is distributed according to a CDF F with support $[0, 1]$. Neither S nor B knows the value of w, but there is a third party, A (for 'adviser'), who does know w and can sell information about w to B.

All parties are risk-neutral and have quasi-linear utility for money. Thus, if trade between the buyer and seller takes place at price p and the buyer pays an amount f to the adviser then the seller's payoff is $p - c$ where $c < E(w)$ is the cost of transaction/production, the buyer's payoff is $w - p - f$ and the adviser's is f. We make the following assumption about the distribution F.

Assumption 1 F has a strictly positive density.

The adviser A is able to decide what kind of information to offer to the buyer - for example, he could reveal the true value of w, or he could reveal whether or not it is above a certain value, or he could provide a stochastic signal which is imperfectly informative about the value of w. We are interested, among other things, in examining the equilibrium structure of information provided by A.

Let an *Information Structure* be a function $\Sigma : [0, 1] \to R$, where R is the set of real-valued random variables, which satisfies the measurability condition that, for any $p \in \Re_+$, $\{(\tilde{w}, \sigma(\tilde{w}))|E(w|\sigma(\tilde{w})) \geq p\}$ is a Borel-measurable set, where $\sigma(\tilde{w})$ is the realization of $\Sigma(\tilde{w})$. For $w \in [0, 1]$, $\Sigma(w)$ is the signal, possibly stochastic, which B receives from A if he buys information and the true state is w.

We denote the set of Information Structures by Φ.

A *Price-Contingent Information Scheme* is a function $M : \Re_+ \to \Phi$. For $p \in \Re_+$, $M(p)$ is the information structure which B obtains if he buys information and S has charged price p.

We model the interaction between the three parties by means of the following game, Γ.

1. A announces a fee $f \in \Re_+$ and a price-contingent information scheme M.
2. S observes A's announcement and announces a price p.
3. B observes (f, M) and p and chooses either (a) to buy information for fee f and then, given the realization of $M(p)$, either to buy the good or not; or (b) to buy the good outright at price p; or (c) not to buy information or the good.

We assume that if B is indifferent between buying the good and not doing so, he buys the good. Therefore, having bought information, B then buys the good if and only if $E[w|m] \geq p$, where m is the realization of $M(p)$. Similarly, if B is indifferent between buying information and not doing so, he buys information. In the sequel we take it for granted that B's equilibrium strategy satisfies this tie-breaking rule. Since the buyer's value is bounded above by 1, we assume, without loss of generality, that S's price is in the interval $[c, 1]$.

Suppose that A has announced (f, M) and S has charged p. If B does not buy information, then, if B buys the good, B's payoff is $E(w) - p$ and S's profit is $p - c$; if B does not buy the good B's payoff is zero and so is S's profit. If B buys information, B's expected payoff is

$$pr[E(w|m) \geq p][E(w|E(w|m) \geq p) - p] - f.$$

B chooses his optimal action and so his maximized payoff is

$$max[E(w) - p, pr[E(w|m) \geq p][E(w|E(w|m) \geq p) - p] - f, 0].$$

Given B's behaviour as described above, the game following A's announcement is equivalent to a two-stage two-player game of complete information and the solution concept we use is subgame-perfect equilibrium (that respects B's tie-breaking rule above).

One family of information structures which will be significant is the family of deterministic threshold structures. For any $\hat{w} \in [0, 1]$, the *$\hat{w}$-Threshold Structure* is given by $\Sigma(w) = I_{\hat{w}}(w)$, where

$$I_{\hat{w}} = 1 \quad \text{with} \quad \text{probability} \quad 1 \quad \text{if} \quad w \geq \hat{w}$$

and

$$I_{\hat{w}} = 0 \quad \text{with} \quad \text{probability} \quad 1 \quad \text{if} \quad w < \hat{w}.$$

In other words, this is the deterministic structure according to which A tells B whether or not w is at least $\hat{w}$.

3 Equilibrium Pricing and Information Disclosure

One possible course of action for A is to offer full information to B, or, equivalently from B's point of view, to offer the scheme $M(p) = I_p$. For any price p charged by S, this tells B whether or not his value is above p. From an *ex post* standpoint this is optimal for the buyer. If A offers this he could then extract this maximal rent via the upfront fee. However, this ignores the fact that A, by a judicious choice of information scheme, could oblige S to charge a lower price, thereby in principle increasing B's extractible rent. The question of which scheme is optimal for B in this *ex ante* sense has been answered in a recent paper by Roesler and Szentes (2016) for the case of $c = 0$. They show that if the information structure is optimal for the buyer then the buyer buys the good after any signal, i.e. the outcome is efficient. The structure has the property that it minimizes the seller's optimum price subject to this efficiency constraint. In the least-informative optimal structure, the buyer's posterior expectation of w is distributed in a unit-elastic fashion over an interior interval $[p^*, \bar{p}]$ with an atom at $\bar{p}$. As a result, S is indifferent between all prices $p \in [p^*, \bar{p}]$, and selects price p^*.

Unfortunately for A, he cannot extract this *ex ante* maximal rent from B. If A sells the information at a strictly positive price it cannot be that the information is buyer-optimal, because, once the seller has set her price, the buyer has no incentive to pay for the information if he knows that he will buy the good regardless of the signal he receives. It must be, therefore, that the information structure designed by A is inefficient.

We say that a fee f is *achievable given information scheme M* if there exists a subgame-perfect continuation equilibrium following A's announcement of (f, M) in which B buys information. f is *achievable* if there exists some M such that it is achievable given M.

A fee f is *strongly achievable given information scheme* M if, following A's announcement of (f, M), B buys information in any subgame-perfect continuation equilibrium. f is *strongly achievable* if there exists some M such that it is strongly achievable given M.

f^* is *optimal* if it is the maximum achievable f.

First, Lemma 1 below shows that we can assume without loss of generality that A's information structure takes a threshold form. A simply recommends whether or not B should buy the good; more specifically, he recommends that B buys if and only if w is above a threshold, where the threshold may depend on the price. In effect, A is in the position of a mechanism designer, albeit one who aims to maximize the fee paid by B for information. By a logic similar to that of Myerson (1982), he can recommend a price to S and recommend a price-contingent buy decision to B. Moreover, to maximize the extractible surplus, he will recommend B to buy when w is high, hence the threshold rule.

Lemma 1 Suppose that f is achievable; that is, there exists a price-contingent information scheme M and an equilibrium μ following (f, M) in which B buys information. Then there exists a price-contingent information scheme $\hat{M}$ such that, for each p, $\hat{M}(p)$ is a threshold structure and such that there exists a continuation equilibrium following $(f, \hat{M})$ in which B buys information. In this continuation equilibrium, A and S have the same payoff as in μ while B's payoff is weakly higher than in μ.

A can always ensure that she gets a strictly positive payoff, as stated in the next lemma. This follows from the fact that if she sets a low enough strictly positive fee and then, for any price p, tells B whether w is above p (the *ex post* optimal information structure for B) then B will buy information in any equilibrium.

Lemma 2 There exists a strictly positive strongly achievable fee f.

Suppose that $f_1 > 0$ is achievable given M_1 and, in a continuation equilibrium in which B buys information, let p_1 be the price which S charges on the equilibrium path and let θ_1 be the equilibrium probability of sale of the good. Then S's equilibrium payoff is $\pi_1 = (p_1 - c)\theta_1$.

One simple observation is that, since the maximum available ex ante surplus is $E_c(w) := [E(w|w > c) - c][1 - F(c)] < E(w)$ and since A's equilibrium payoff is strictly positive, S's equilibrium payoff π_1 is less than $E_c(w)$. Another is that, for any p such that $\pi_1 + c < p < E(w)$, B must buy information. To see this, note that since $p < E(w)$ B must either buy information or buy outright. If he buys outright then S can profitably deviate to price p and get payoff $p - c > \pi_1$. Hence we have

Lemma 3 In any continuation equilibrium in which $f_1 > 0$, S sets price p_1, B buys information and S's payoff is $(p_1 - c)\theta_1$, (i) $(p_1 - c)\theta_1 < E_c(w)$; (ii) if S sets price $p \in ((p_1 - c)\theta_1 + c, E(w))$, B buys information.

The following provides a characterization of the achievable fees.

Lemma 4 $f_1 > 0$ is achievable if and only if there exists $(p_1, \theta_1) \in [c, 1] \times [0, 1]$ such that

$$f_1 \leq U_1(p_1, \theta_1) \equiv \int_0^{(p_1-c)\theta_1+c} ((p_1-c)\theta_1+c-w)dF(w) \quad \text{if} \quad (p_1-c)\theta_1+c \leq E(w), \tag{1}$$

$$f_1 \leq U_2(p_1, \theta_1) \equiv \int_0^{w_1} (p_1 - w)dF(w) \tag{2}$$

and

$$f_1 \leq U_3(p_1, \theta_1) \equiv \int_{w_1}^1 (w - p_1)dF(w), \tag{3}$$

where $1 - F(w_1) = \theta_1$.

Note that if the price is $\tilde{p}$ and the information threshold (the value of w above which B would buy having bought information) is $\tilde{w}$ then the net value of buying information rather than buying outright is

$$\int_0^{\tilde{w}} (\tilde{p} - w) dF(w)$$

and the net value of buying information rather than not buying at all is

$$\int_{\tilde{w}}^1 (w - \tilde{p}) dF(w).$$

Therefore (1) corresponds to the requirement that B prefers to buy information than buy outright when the price is $(p_1 - c)\theta_1 + c$ and the information structure is $I_{(p_1-c)\theta_1+c}$. (2) corresponds to the same requirement when the price is p_1 and the information structure is I_{w_1}. (3) is the condition that B prefers to buy information than not to buy at all in the latter case. Here p_1 is the equilibrium price, θ_1 is the equilibrium sale probability, and w_1 is the threshold used after price p_1. (2) and (3) must be true because B buys information on the equilibrium path. (1) must be true because, by Lemma 3, it must be weakly optimal for B to buy information when the price is $(p_1 - c)\theta_1 + c \leq E(w)$ and the information structure is *ex post* optimal for B.

In equilibrium A sets the maximum achievable fee f^*, i.e. chooses $(f_1, p_1, \theta_1) \in \Re \times [c, 1] \times [0, 1]$ to maximize f_1 subject to (1) – (3). As we show in the proof of Proposition 1 below, the optimal (p^*, θ^*) satisfies

$$(p^* - c)\theta^* + c < E(w) \leq p^*$$

and constraints (1) and (3) bind at the optimum. Given p^*, the optimal threshold w^* must maximize U_3 subject to $U_3 = U_1$, i.e. it must be the maximizer of the problem $(P1(p))$ for $p = p^*$, where $(P1(p))$ is:

$$max_{\tilde{w} \in [0,1]} \int_{\tilde{w}}^1 (w - p) dF(w)$$

subject to

$$\int_{\tilde{w}}^{1}(w-p)dF(w)=\int_{0}^{[1-F(\tilde{w})][p-c]+c}([1-F(\tilde{w})][p-c]+c-w)dF(w). \qquad (*)$$

The following Lemma shows how the optimal threshold varies with p.

Lemma 5 There exists $\hat{p}\in(E(w),1)$ such that $(P1(p))$ has a solution $\tilde{w}_p$ if and only if $p\leq\hat{p}$. The solution is unique. $\tilde{w}_p$ increases in $p\in[E(w),\hat{p}]$ from $\tilde{w}_{E(w)}\in(0,E(w))$ to $\tilde{w}_{\hat{p}}>\hat{p}>E(w)$.

Let p^c be defined by $\tilde{w}_{p^c}=c$ and denote by $f^*(p)$ the maximized value of $(P1(p))$ for $p\in[E(w),\hat{p}]$. The following result characterizes the equilibrium of our game.

Proposition 1 An equilibrium of Γ exists. In any equilibrium $[(f,M),p^*]$,

(i) If $c\leq\tilde{w}_{E(w)}$ then $p^*=E(w)$;

(ii) If $c>\tilde{w}_{E(w)}$ then $p^*=p^c$;

(iii) $f=f^*(p^*)=f^*$, $M(p^*)=I_{\tilde{w}_{p^*}}$ and B buys information on the equilibrium path;

(iv) B's equilibrium payoff is zero, A's is $f^*(p^*)$ and S's is $[1-F(\tilde{w}_{p^*})][p^*-c]$.

As noted above, the adviser's information structure takes a threshold form: after the seller has set the equilibrium price p^*, the adviser tells the buyer whether or not w is above $\tilde{w}_{p^*}$ (and the buyer buys the good only if it is).

When c is zero (and also when it is positive but close to zero) the seller's price is $E(w)$. The underlying reason for this is as follows. If $p>E(w)$ then for B the alternative to buying information is not to buy at all; in this case reducing p increases the value of information to B. If $p<E(w)$ then the alternative to buying information is to buy outright; in this case increasing p makes information more attractive. Therefore A designs the fee and information scheme in such a way that S sets price $E(w)$.

Given price $E(w)$, the expected value of information to the buyer, namely

$$\int_{\tilde{w}_{E(w)}}^{1} (w - E(w))dF(w),$$

is all extracted by the adviser via the upfront fee f^*. The fact that $\tilde{w}_{E(w)} < E(w)$ means that the buyer, with positive probability, gets a negative ex post payoff - the information structure is not ex post optimal for him. Therefore, if A were to increase the threshold above $\tilde{w}_{E(w)}$ A could extract more surplus. However this would violate the constraint in $(P1(p^*))$, which is that S should not be able to deviate to a low price (just above $[E(w)][1 - F(\tilde{w}_{E(w)})]$) and sell outright to the buyer.

When c is higher, the above solution is inefficient in the sense that the threshold is below the cost of production. In that case A does better by increasing the threshold, reducing the probability of sale, and simultaneously raising the price above $E(w)$ to compensate the seller.

There are other schemes which achieve f^*. However, the information structure which follows the equilibrium price p^* must be behaviourally equivalent to I_{w^*}; that is, B, having bought information, buys the good if and only if $w \geq w^* = \tilde{w}(p^*)$. Suppose this were not so - there is a scheme M' such that, after A offers (M', f^*), S sets price p^* and the sale probability is θ^*, but the information structure is not I_{w^*}. Then it would be possible for A to offer the same scheme except for the case of price p^*, in which case he offers a threshold structure with a threshold slightly lower than w^*; this would mean a higher payoff, following p^*, both for S and for B - for S, because the sale probability is higher, for B because I_{w^*} is the unique optimal information structure with sale probability θ^*. A could then charge a fee slightly higher than f^* and it would still be the case that S would set price p^* and B would buy information, contradicting the optimality of f^*.

Any $f < f^*$ is strongly achievable given scheme M^* as just defined. This is because, with fee f and scheme M^*, B strictly prefers to buy information for any $p \in [(p^* - c)\theta^* + c, E(w)] \cup \{p^*\}$ since, with the higher fee f^*, B would be indifferent between buying information and buying outright at price $(p^* - c)\theta^* + c$ and strictly prefer the former for any higher price. Therefore, in any continuation equilibrium, S must set price p^* and B must buy information.

Example: $\quad$ *F uniformly distributed on* $[0,1]$.

In the case of the uniform distribution,

$$\int_{\tilde{w}}^{1}(w-p)dF(w)=\frac{1-\tilde{w}^2}{2}-p(1-\tilde{w})$$

and

$$\int_{0}^{[1-\tilde{w}][p-c]+c}([1-\tilde{w}][p-c]+c-w)dw=\frac{\big([1-\tilde{w}][p-c]+c\big)^2}{2}.$$

When $c=0$, $p^*=E(w)=1/2$; equating the above two expressions, we get $\tilde{w}=1/5$ and f^*, which is their common value, is $2/25$.

To solve for general c:

$$\tilde{w}_{E(w)}=\frac{3-2\sqrt{1-2c}-2c}{5-4c+4c^2}\begin{cases}>c & if \quad c<c^*\approx 0.258056\\ <c & if \quad c^*<c<0.5.\end{cases}$$

For $c\in(c^*,0.5)$, as p increases from $E(w)=0.5$, $\tilde{w}$ increases to reach c at $p^c=\frac{-1+c-c^2+c^3+\sqrt{(1-c)^2(2+c^2)}}{(1-c)^2}$ and $(1-c)(p^c-c)+c<E(w)=0.5$. Therefore, the equilibrium price is 0.5 for $c<c^*$ but p^c for $c\in(c^*,0.5)$ and consequently, the optimal fee is $\tilde{w}(1-\tilde{w})/2$ for $c<c^*$ while it is $\int_c^1(w-p^c)dF(w)=(1-c)[(1+c)/2-p^c]$ for $c\in(c^*,0.5)$.

It is instructive to compare the solution for $c=0$ with the case in which A fully discloses w. In that case, for uniform F, B's expected payoff from buying information when the price is p is

$$(1-p)[\frac{1+p}{2}-p]-f=\frac{(1-p)^2}{2}-f$$

and his payoff from buying outright is $1/2-p$. Therefore B buys outright if $p<\sqrt{2f}$, buys information if $p\in[\sqrt{2f},1-\sqrt{2f}]$ (if this interval is non-empty) and otherwise does not buy. This implies that, if information is bought, it must be that $f \leq 1/8$, so that the interior interval is non-empty. Given such an f, S chooses between setting a low price, $\sqrt{2f}$, and selling outright, and setting price $1/2$ and selling with probability $1/2$ after B buys information (S's optimal price in the interval $[\sqrt{2f}, 1-\sqrt{2f}]$, assuming B is informed, is $1/2$). A's optimal strategy is to set the highest f such that S is willing to do the latter, i.e., such that $\sqrt{2f} \leq 1/4$. This gives $f = 1/32$.

In the full information scheme B's payoff is strictly positive ($3/32$). The reason that A cannot extract this surplus by increasing the fee is that she is constrained by the threat that S will bypass her by charging a low price. In A's optimal scheme it is not the case, as might be expected, that A's scheme is designed to force down S's price so that surplus is created which A can extract from B through the fee. In fact the seller's price ($1/2$) is the same in the optimal scheme as in the full information scheme. Instead, A relaxes the constraint that S should not want to bypass him by withholding information from B and thereby increasing the probability of sale (from $1/2$ to $4/5$). This creates an increase in total surplus, all of which, and more, is given to S, reducing her incentive to bypass A. A is then able to increase the fee, reducing B's payoff to zero. S's payoff is $2/5$, compared to $1/4$ in the full information case.

We can also compare A's optimal scheme to two benchmarks in which there is no adviser available: one in which B is uninformed and one in which he is fully informed (he has free access to w before he meets S).

No adviser; B uninformed. In this case, S sets price $1/2$, the probability of sale is 1, S's payoff is $1/2$ and B's payoff is zero. The outcome is therefore efficient. The presence of the adviser reduces efficiency and reduces S's payoff from $1/2$ to $2/5$, but leaves B's payoff unchanged. The adviser has negative value for the two original parties.

No adviser; B fully informed. S sets price $1/2$ and sells with probability $1/2$. S's payoff is $1/4$ and B's is $1/8$. So, having information provided at a cost by A improves efficiency and benefits the seller but the buyer is strictly worse off.

3 The Case of a Privately Informed Buyer

In this section we examine a model in which the buyer has some private information about his valuation for the good. We assume that his valuation takes an additive form; that is, it equals $u + w$, where the buyer is informed about u and the adviser is informed about w. One interpretation is that w represents information about the objective quality of the good (its technical capacities, for example) while u represents the buyer's idiosyncratic subjective preference.

We assume that w is distributed according to a cumulative distribution function F with support $[0, k]$ and u is distributed, independently of w, according to a cumulative distribution function G with support $[0, 1]$. G and F both have a strictly positive density.

We also assume that the information structures which A may offer for sale are those which take the form of a set of thresholds and that B, having bought information, can select any one of these thresholds and buy the good if and only if w is above the threshold. That is, there is a set of thresholds $\{z_1, z_2, ...z_j, ..\} \subseteq [0, k]$, with $z_j < z_{j+1}$ for all j, and A reports which interval $(z_j, z_{j+1}]$ w falls in. The number of thresholds may be finite or infinite; in the latter case we assume that there exist one or more intervals of $[0, k]$ for which A reports the exact value of w. We refer to such a structure as an *interval information structure.* We model the interaction between the three players as follows.

(a) A announces, before learning the value of w, a collection of price-contingent fees and interval information structures $\{f(p), M(p)\}_{p \in \Re_+}$.

(b) S announces price $p \in \Re_+$ after observing A's announcement.

(c) B observes the announcements of A and S and chooses either to buy outright, not buy at all, or buy information.

(d) If B chooses to buy information, he pays $f(p)$ to A, A learns $\tilde{w}$, the realization of w, and reports to B which interval of $M(p)$ $\tilde{w}$ belongs to. B then decides whether or not to buy the good for price p.

The payoff functions of the three players are as in the preceding section, except that $u + w$ replaces w in the (type-u) buyer's payoff function and we assume in this section that the seller's cost c is zero. The game following an announcement by A is now one of incomplete information; accordingly we use Perfect Bayesian Equilibrium as our solution concept.

Consider an arbitrary equilibrium σ in which A sells information with positive probability for a strictly positive price. In this equilibrium, let the seller's price be p and let $f(p) = f$. Suppose that $M(p)$ has a finite number of thresholds. Let the infimum and supremum of the types which buy information be, respectively, u_l and u_h. Let the lowest threshold which is used on the equilibrium path be z_1 and let the highest be z_n (these are not necessarily the two extreme points of the offered structure - if necessary, discard more extreme thresholds and re-label). It is straightforward to show that the probability of sale must be weakly increasing in u, so, without loss of generality, types strictly below u_l buy with probability zero, types strictly above u_h buy outright and types in $[u_l, u_h]$ all buy information. Moreover, types in $[u_l, u_h]$ are partitioned into intervals $[u_l, u_{n-1,n}, u_{n-2,n-1}, ..., u_{2,3}, u_{1,2}, u_h]$ such that, for $k \in 1, ..., n$, types in $(u_{k,k+1}, u_{k-1,k}]$ use threshold z_k, types in $(u_l, u_{n-1,n}]$ use threshold z_n and types in $(u_{1,2}, u_h]$ use threshold z_1. That is, higher types use a lower threshold.

If $u_l > 0$ then type u_l is indifferent between buying information and not buying at all; therefore, since if he buys information he then uses threshold z_n,

$$-f + [1 - F(z_n)][u_l + E(w|w \geq z_n) - p] = 0,$$

so

$$u_l = p + \frac{f}{1 - F(z_n)} - E(w|w \geq z_n).$$

Similarly, u_h (if $u_h < 1$) is indifferent between buying information, subsequently using threshold z_1, and buying outright, hence

$$-f + [1 - F(z_1)][u_h + E(w|w \geq z_1) - p]] = u_h + E(w) - p,$$

so

$$u_h = p - \frac{f}{F(z_1)} - E(w|w \leq z_1).$$

Type $u_{k,k+1}$ is indifferent between using thresholds z_k and z_{k+1}, which implies that

$$u_{k,k+1} = p - E(w|w \in [z_k, z_{k+1}]).$$

If the density of G is monotonic then we can derive the following proposition about the structure of information offered by A in equilibrium.

Proposition 2 (i) Suppose that $G'(u)$ is increasing in u. Then there exists an equilibrium in which the equilibrium path information structure has at most two thresholds. If $G'(u)$ is strictly increasing then, in any equilibrium in which a positive measure of types buy outright, the equilibrium path information structure must have at most two thresholds.

(ii) Suppose that $G'(u)$ is decreasing in u. Then there exists an equilibrium in which the equilibrium path information structure gives full information. If $G'(u)$ is strictly decreasing then, in any equilibrium in which a positive measure of types buy neither information nor the good, the equilibrium path information structure must have a continuum of thresholds.

The idea of the proof is as follows. Suppose that there exists an interval information structure $\tilde{M}$ such that if $\tilde{M}$ replaces $M(p)$, everything else, including $f(p)$ and all off-equilibrium path information structures, being the same, then (i) given p, the set of types buying information is unchanged, i.e. u_h and u_l are unchanged; and (ii) the seller's payoff is strictly higher than in the equilibrium. Suppose first that $G'(u)$ is strictly increasing in u, i.e., G is a strictly convex function, and that $u_h < 1$. For small $\epsilon > 0$, let A offer $(f(p), \tilde{M})$ when the seller's price is $p + \epsilon$ and, for any other price, offer the same fee and information structure as in the equilibrium σ. The increase in seller's price implies that all the buyer thresholds $u_l, u_{n,n-1}, ...u_{1,2}, u_h$ move ϵ to the right. A's equilibrium expected payoff is $f[G(u_h) - G(u_l)]$ and his payoff after this deviation, assuming the seller responds by setting price $p+\epsilon$, is $f[G(u_h+\epsilon)-G(u_l+\epsilon)]$

which, because G is strictly convex is strictly greater than $f[G(u_h) - G(u_l)]$. This would imply that A has a profitable deviation. Moreover, if G is strictly convex then, given price p, the seller's expected sales and hence payoff can indeed be increased without altering u_h or u_l, by removing one of the interior thresholds z_k. Iterating this argument, we conclude that when G' is strictly increasing any equilibrium in which some types buy outright must involve at most two thresholds, namely the ones used by types u_h and u_l. A similar argument shows that if G' is strictly decreasing and some types do not buy at all, it must be that there is a continuum of thresholds, i.e., A provides full information, at least in some interval $[z_1, z_n]$.

Example: The Uniform case

In the case in which F and G are both uniform distributions it is possible to solve explicitly for the equilibrium.

Recall that

$$u_l = p + \frac{f}{1 - F(z_n)} - E(w|w \geq z_n)$$

and

$$u_h = p - \frac{f}{F(z_1)} - E(w|w \leq z_1)$$

are the lower and upper bounds respectively of the set of types who buy information, assuming that they both lie in $[0, 1]$. Suppose that, for any p, $(\bar{f}, \bar{z}_1, \bar{z}_n)$ is the unconstrained maximum of $f[u_h(p, f, z_1) - u_l(p, f, z_n)]$. Suppose also that $\bar{p}$ maximizes S's expected payoff if, for any p, the information fee is $\bar{f}$ and the information structure is given by the two thresholds $\bar{z}_1$ and $\bar{z}_2$, and that $0 \leq u_l(\bar{p}, \bar{f}, \bar{z}_n) < u_h(\bar{p}, \bar{f}, \bar{z}_1) \leq 1$. Then it is optimal for A to offer this fee and two-threshold structure.

For uniform G, $f[u_h(p, f, z_1) - u_l(p, f, z_n)]$

$$= f[\frac{k + z_n}{2} - \frac{z_1}{2} - \frac{kf}{z_1} - \frac{kf}{k - z_n}]$$

and maximizing this expression gives $z_1 = k/3, z_n = 2k/3$ and $f = k/18$. With this structure, we have

$$u_l = p - 2k/3$$

$$u_h = p - k/3$$

and

$$u_{1,n} = p - k/2.$$

If $k < 2/3$ then S optimally sets $p = (1/2) + (k/4)$ so $u_l = (1/2) - (5k/12) > 0$, $u_h = (1/2) - (k/12)$ and $u_{1,n} = (1/2) - (k/4)$.

Compared with the case in which there is no adviser, efficiency is improved and the *ex ante* welfare of buyers is also improved. Suppose there were no adviser. Then the valuation of type u is $u + (k/2)$. The optimum price for the seller is $(1/2) + (k/4)$ and the types who buy are types u such that $u \geq (1/2) - (k/4)$. When there is an adviser, the seller's price is the same and types in the interval $[(1/2) - (5k/12), (1/2) - (k/12)]$ buy on average with probability 0.5, so expected demand is also the same. However, efficiency is increased because types in the top half of this interval buy only if $w \geq (k/3)$ and types in the bottom half, who buy only if $w \geq (2k/3)$, would not buy if there were no adviser. All types in the interior of this interval are strictly better off as a result. The adviser takes some, but not all, of the added social surplus as rent.

In the previous section, for the uniform example, we saw that introducing an adviser reduces efficiency and makes the original trading parties strictly worse off. That model, with $u = 0$, represents the limiting case in which the information of the adviser is large, i.e., very important to determining the value of the relationship. The case considered here, with k relatively small, represents the opposite case, in which A's information is useful but not large in the above sense. We see therefore that an adviser may be beneficial if his information is small but detrimental if his information is large.

Appendix

Proof of Lemma 1 We partition the set of prices into three, according to B's response in the continuation equilibrium μ following (f, M). P_1 is the set of prices after which B buys information, P_2 is the set of prices after which B buys outright, and P_3 is the set after which B does not buy at all. For $p \in P_1$, let $\theta(p)$ be the probability, conditional on buying information, that B buys the good when the price is p and the equilibrium is μ. Let $w(p)$ satisfy $1-F(w(p)) = \theta(p)$. Then $\hat{M}$ is defined as follows.

If $p \in P_1$

$$\hat{M}(p) = I_{w(p)}$$

and if $p \in P_2 \cup P_3$

$$\hat{M}(p) = I_0.$$

For $p \in P_1$ we can partition the set of signals as $b(p) \cup n(p)$ such that, when the information structure is $M(p)$, B, having bought information, buys the good after signals in $b(p)$ and not after signals in $n(p)$. Then the probability of $b(p)$ is $\theta(p)$. If instead of observing the signal B were only told whether it was in $b(p)$ or $n(p)$ then it would be optimal to buy if the former and not buy if the latter. A fortiori, if B is told only that w is above $w(p)$, an event which has the same probability as $b(p)$, then it is optimal for him to buy the good since his expectation of w conditional on $w \in [w(p), 1]$ is weakly greater than his expectation conditional on $b(p)$. Similarly, if he is told that w is below $w(p)$ then it is optimal for him not to buy the good. Furthermore, after $p \in P_1$, B's expected payoff conditional on buying information is weakly higher given $\hat{M}(p)$ than it is given $M(p)$.

It follows that, given $(f, \hat{M})$ it is optimal for B to buy information if $p \in P_1$. It is optimal for B to buy outright if $p \in P_2$ and not to buy if $p \in P_3$; this is because $\hat{M}(p)$ gives no information in these cases. Therefore there is an equilibrium in which B's behaviour is the same after any p as it is in μ and thus in which S charges the same price, say $\tilde{p}$, as in μ. B buys information in this equilibrium. A's payoff is f and S's is $(\tilde{p} - c)\theta(\tilde{p})$, as in μ. QED.

Lemma A1 Suppose, for some $p \in [0, 1)$ and some $f > 0$, B weakly prefers to buy information than to buy outright when the price is p, the fee is f and the information structure is the p-threshold structure I_p. Then, for any $\tilde{p} \in [0, 1]$ such that $\tilde{p} > p$, B strictly prefers to buy information than to buy outright when the fee is f, the price is $\tilde{p}$ and the information structure is $I_{\tilde{p}}$.

Proof With information structure I_p, price p and fee f, B weakly prefers to buy information than to buy outright if

$$-f + (1 - F(p))[E(w|w \geq p) - p] \geq E(w) - p$$

i.e., if

$$f \leq p[\int_0^p (1 - \frac{w}{p})dF(w)] \tag{4}$$

and strictly prefers to do so if this inequality is strict. The RHS of (4) is strictly increasing in p. QED

Proof of Lemma 2 Let

$$\bar{\pi} = max_p(p - c)[1 - F(p)]$$

and let

$$\bar{p} = argmax_p(p - c)[1 - F(p)].$$

Then $\bar{p} - c > \bar{\pi} > 0$. Take $\underline{p} - c \in (0, \bar{\pi})$. Suppose A announces M such that, for each price p, $M(p) = I_p$. There exists $\underline{f} > 0$ sufficiently small that (i) (4) is satisfied for ($f = \underline{f}$,$p = \underline{p}$), so that, given any price $p \geq \underline{p}$, B strictly prefers to buy information than to buy outright, and (ii) given price $\bar{p}$, B prefers to buy information than not to buy at all. Suppose A announces this fee $\underline{f}$. Then S must set price $\bar{p}$ since this will give her a payoff of $\bar{\pi}$, which is the highest payoff she can get conditional on B buying information, and any price at which B will buy outright is at most $\underline{p} < \bar{\pi}$. Therefore B must buy information in any continuation equilibrium. QED.

Let

$$\hat{\pi} = max_{p \leq E(w)}(p - c)[1 - F(p)],$$

the largest *argmax* of which is denoted by $\hat{p}$. Then $\hat{p} \in (\hat{\pi} + c, E(w)]$.

Define information structure $M^I(p)$ by $M^I(p) = I_p$ if $p \leq E(w)$ and $M^I(p) = I_0$

if $p > E(w)$.

Proof of Lemma 4 ('Only if'). Suppose that f_1 is achievable. Let the corresponding mechanism be M_1 and let the price and sale probability in the equilibrium continuation be p_1 and θ_1 respectively. By Lemma 3(i) $(p_1 - c)\theta_1 < E_c(w)$. Define $\tilde{M}_1$ by $\tilde{M}_1(p_1) = I_{w_1}$ and $\tilde{M}_1(p) = M^I(p)$ for $p \neq p_1$. Given $(f_1, \tilde{M}_1)$, B would prefer to buy information than to buy outright when the price is $(p_1 - c)\theta_1 + c \leq E(w)$ since, by Lemma 3, he does so given (f_1, M_1). This implies (1). Similarly, he also prefers to buy information when the price is p_1, which implies (2) and (3).

('If'). Define mechanism $\tilde{M}_1$ as above, for the given p_1 and w_1. Suppose A announces $(f_1, \tilde{M}_1)$. In the continuation, let S's strategy be to set p_1 if $\hat{\pi} < (p_1 - c)\theta_1$ and otherwise set $\hat{p}$. Define $\underline{p}$ by

$$[1 - F(\underline{p})][E(w|w \geq \underline{p}) - \underline{p}] - f_1 = E(w) - \underline{p}.$$

Since (2) implies that B prefers to buy information at p_1 if information structure is I_{p_1}, Lemma A1 implies $\underline{p} \in (0, p_1)$.

First, consider the case that $(p_1 - c)\theta_1 + c \leq E(w)$. Since B prefers to buy outright if $p = 0$, Lemma A1 and (1) imply that $\underline{p} \in (0, (p_1 - c)\theta_1 + c]$. Let B's strategy be: (i) for prices $p < \underline{p}$, buy outright; (ii) for prices $p \in [\underline{p}, E(w)]$ and for price p_1, buy information (and then buy the good if and only if w is above the threshold); (iii) for all other prices, buy neither information nor the good.

Consider first B's strategy. For $p > E(w)$ such that $p \neq p_1$ the mechanism gives no information, hence it is optimal not to buy at all. (1), (2) and (3) imply that it is optimal for B to buy information at $(p_1 - c)\theta_1 + c$ and at p_1. Together with Lemma A1 and the definition of $\underline{p}$, this shows that B's strategy is optimal. If S sets a price such that B buys outright then S gets at most $\underline{p} - c \leq (p_1 - c)\theta_1$. If she sets price p_1 B buys information and S's payoff is $(p_1 - c)\theta_1$. This is optimal for S unless $\hat{\pi} > (p_1 - c)\theta_1$, in which case $\hat{p}$ is optimal (and, again, B buys information because S's payoff from a price inducing outright purchase is bounded above by $(p_1 - c)\theta_1$ as asserted above). Therefore S's stated strategy is optimal. This therefore defines an equilibrium which achieves f_1.

Next, consider the case that $(p_1 - c)\theta_1 + c > E(w)$. Let B's strategy be: (i) for prices $p < \min\{\underline{p}, E(w)\}$, buy outright; (ii) for prices $p \in [\underline{p}, E(w)]$ and for price p_1, buy information (and then buy the good if and only if w is above the threshold); (iii) for all other prices, buy neither information nor the good.

Consider first B's strategy. For $p > E(w)$ such that $p \neq p_1$ the mechanism gives no information, hence it is optimal not to buy at all. (2) and (3) imply that it is optimal for B to buy information at p_1. Together with Lemma A1 and the definition of $\underline{p}$, this shows that B's strategy is optimal. If S sets a price such that B buys outright then S gets at most $\min\{\underline{p}, E(w)\} - c < (p_1 - c)\theta_1$. If she sets price p_1 B buys information and S's payoff is $(p_1 - c)\theta_1$. This is optimal for S unless $\hat{\pi} > (p_1 - c)\theta_1$, in which case $\hat{p}$ is optimal (and, again, B buys information). Therefore S's stated strategy is optimal. This therefore defines an equilibrium which achieves f_1. QED

Proof of Lemma 5 Note that the derivative of the LHS of the constraint (*) with respect to $\tilde{w}$ is $-(\tilde{w} - p)F'(\tilde{w})$ which is positive for $\tilde{w} < p$ and negative for $\tilde{w} > p$, so the LHS of the constraint increases in $\tilde{w}$ from $E(w) - p \leq 0$ at $\tilde{w} = 0$, peaks at $\tilde{w} = p$, then decreases down to 0 at $\tilde{w} = 1$. Observe that the RHS of (*) has its derivative wrt to $\tilde{w}$ as $-F'(\tilde{w})[p - c]F([1 - F(\tilde{w})][p - c] + c) < 0$.

For $p = E(w)$, therefore, it suffices to show that the LHS exceeds the RHS when $\tilde{w} = E(w)$, which is the case because

$$\int_{E(w)}^{1} (w - E(w))dF(w) = \int_{0}^{E(w)} (E(w) - w)dF(w)$$

and $[1 - F(\tilde{w})][E(w) - c] + c < E(w)$.

Note that $-(\tilde{w} - p)$ is 0 when $\tilde{w} = p$, then decreases in $\tilde{w} \in (p, 1)$, whereas $-[p - c]F([1 - F(\tilde{w})][p - c] + c)$ increases in $\tilde{w} \in (0, 1)$. This implies that the derivatives of the LHS and RHS of (*) may coincide at most once for $\tilde{w} \in (p, 1)$. As p increases from $E(w)$ the LHS decreases while the RHS increases for all $\tilde{w}$ and exceeds the LHS at $\tilde{w} = 1$, this in turn implies that the unique solution $\tilde{w}_p$ increases until p reaches a threshold $\hat{p} < 1$ at which the graphs of the LHS and RHS are tangent at

$\tilde{w} = \tilde{w}_{\hat{p}}$, and after which the solution does not exist because the RHS exceeds the LHS for all $\tilde{w}$. Note further that $\tilde{w}_{\hat{p}} > \hat{p}$ because the slope is negative at the tangency point. QED

Proof of Proposition 1 Let f^e be the maximized value of the problem $(P2)$:

$$max_{(p_1,\theta_1)} min\{U_1(p_1,\theta_1), U_2(p_1,\theta_1), U_3(p_1,\theta_1)\}$$

subject to $(p_1,\theta_1) \in D \equiv \{(p,\theta) \in [c,1] \times [0,1] \,|\, (p-c)\theta + c \leq E(w)\}$.

It will follow that f^e is optimal, i.e. $f^e = f^*$. This implies that f^* is obtained by (p_1,θ_1) such that $(p_1 - c)\theta_1 + c \leq E(w)$, which we assume here for expositional ease and prove below. The closure of D is compact and U_1, U_2 and U_3 are continuous functions so there exists an optimum for the problem $(P2)$.

Claim 1 f^e is the maximized value of the problem $(P3)$:

$$max_{(p_1,\theta_1)} U_3(p_1,\theta_1)$$

subject to $(p_1,\theta_1) \in \{(p,\theta) \in [c,1] \times [0,1] \,|\, (p-c)\theta + c < E(w) \leq p\}$ and $U_1(p_1,\theta_1) = U_3(p_1,\theta_1)$.

Proof

Consider a solution (p^e,θ^e) to $(P2)$, with maximized value f^e. Let w^e be defined by $1 - F(w^e) = \theta^e$. Note that $f^e > 0$ because $(p_1,\theta_1) = (E(w),\epsilon) \in D$ and $U_3(p_1,\theta_1) = U_2(p_1,\theta_1) > 0$ for small $\epsilon > 0$ as well as $U_1(p_1,\theta_1) > 0$. Hence, $\theta^e < 1$ because $\theta^e = 1$ would imply $w^e = 0$ and thus, $U_2(p^e,\theta^e) = 0$.

If $p^e < E(w)$ then (2) implies that (3) is satisfied and slack. Increasing p increases both $U_2(p,\theta)$ and $U_1(p,\theta)$ (the latter because $(p^e - c)\theta^e + c < p^e < E(w)$ holds). Therefore it is feasible to increase f. Contradiction. Therefore $p^e \geq E(w)$.

Suppose $(p^e - c)\theta^e + c = E(w)$, so that $p^e > E(w)$. Then

$$U_1(p^e,\theta^e) = \int_0^{E(w)} (E(w) - w)dF(w) = F(E(w))[E(w) - E(w|w < E(w))]$$

while

$$U_3(p^e,\theta^e) < \int_{w^e}^{1} (w - E(w))dF(w)$$

$$\leq \int_{E(w)}^{1} (w - E(w))dF(w) = (1 - F(E(w)))[E(w|w > E(w)) - E(w)]$$

so that

$$U_1(p^e,\theta^e) - U_3(p^e,\theta^e) > F(E(w))[E(w) - E(w|w < E(w))]$$

$$-(1 - F(E(w)))[E(w|w > E(w)) - E(w)] = -F(E(w))E(w|w < E(w))$$

$$-(1 - F(E(w)))E(w|w > E(w)) + E(w) = 0,$$

thus (1) cannot bind if $(p^e - c)\theta^e + c = E(w)$.

If only (1) binds, then $(p^e - c)\theta^e + c < E(w)$ and it is feasible to increase $(p^e - c)\theta^e$, thereby increasing $U_1(p,\theta)$, hence increasing f. Contradiction. Therefore if (1) binds the other relevant constraint(s) must also bind.

Suppose $p^e > E(w)$. Then, (3) implies (2). Suppose that (3) binds and (1) is slack. Then reducing p increases $U_3(p,\theta)$, so f can feasibly be increased. Contradiction. Hence (3) and (1) must both bind.

Suppose $p^e = E(w)$. Then, (2) and (3) are the same condition. Suppose (2) and (3) bind and (1) is slack. In this case $U_3 = U_2 = \int_0^{w^e}(E(w) - w)dF(w)$. Maximizing this with respect to w^e gives $w^e = E(w)$. Hence we conclude that $w^e = E(w)$, otherwise it would be possible to increase f by varying θ (hence w^e). However, if (2) binds and (1) is slack, we have $f^e = U_2(p^e,\theta^e) < U_1(p^e,\theta^e)$, i.e.,

$$\int_0^{E(w)} [E(w) - w]dF(w) < \int_0^{(E(w)-c)\theta^e+c} [(E(w) - c)\theta^e + c - w]dF(w),$$

which is false since $(E(w) - c)\theta^e + c \leq E(w)$. This shows that $f^e = U_1(p^e,\theta^e) = U_2(p^e,\theta^e) = U_3(p^e,\theta^e)$ if $p^e = E(w)$.

Summarizing, (1) and (3) must both bind at (p^e,θ^e) and $p^e \geq E(w)$. Together with the fact that $U_2(p,\theta) \geq U_3(p,\theta)$ if $p \geq E(w)$, this establishes that the solution to $(P2)$ also solves $(P3)$. This proves the Claim.

Let (p^*, θ^*) solve $(P2)$ with maximized value f^* and let $1 - F(w^*) = \theta^*$. Denote by M^I the information scheme defined by $M^I(p) = I_p$ if $p \leq E(w)$ and $M^I(p) = I_0$ if $p > E(w)$. Denote by M^* the scheme defined by $M^*(p^*) = I_{w^*}$ and, for $p \neq p^*$, $M^*(p) = M^I(p)$, where $1 - F(\tilde{w}(p^*)) = \theta^*$. That is, for price p^*, the threshold is $\tilde{w}(p^*)$ and for any other price p, B learns nothing if p is above $E(w)$ and learns whether w is at least p otherwise. Then M^* achieves the optimal fee.

Claim 2 The price-contingent information scheme M^* achieves the optimal fee f^* with price p^* and sale probability θ^*. B's equilibrium strategy, given (f^*, M^*), is to buy outright for $p \in [0, (p^* - c)\theta^* + c)$, buy information for $p \in [(p^* - c)\theta^* + c, E(w)] \cup \{p^*\}$ and otherwise not to buy at all.

Proof of Claim 2 The mechanism $\tilde{M}_1$ described in the 'If' part of the proof of Lemma 4 is the same as M^*, for $p_1 = p^*$ and $w_1 = w^*$. Therefore, taking $f_1 = f^*$, the argument in the proof of Lemma 4 shows that after A offers (M^*, f^*), there is an equilibrium continuation in which S sets price p^* and B's strategy is as described in the Lemma, as long as $\underline{p} = (p^* - c)\theta^* + c$ and $\hat{\pi} \leq (p^* - c)\theta^*$. $\underline{p}$ is uniquely defined by

$$[1 - F(\underline{p})][E(w|w \geq \underline{p}) - \underline{p}] - f^* = E(w) - \underline{p}.$$

However, this equation is true for $\underline{p} = (p^* - c)\theta^* + c$ since (1) is binding for (f^*, p^*, θ^*). Therefore $\underline{p} = (p^* - c)\theta^* + c$. Suppose, to establish a contradiction, that $\hat{\pi} > (p^* - c)\theta^*$. Then, as argued in the proof of Lemma 4, given (M^*, f^*) there is an equilibrium which achieves f^* in which S sets price $\hat{p}$ and the sale probability is $1 - F(\hat{p})$. Hence $(p_1 = \hat{p}, \theta_1 = \hat{\theta} = 1 - F(\hat{p}))$ solves $(P2)$ because $(\hat{p} - c)\hat{\theta} + c < (E(w) - c) + c = E(w)$. By Claim 1, (1) is binding, i.e. $U_1(\hat{p}, 1 - F(\hat{p})) = f^*$. Since (p^*, θ^*) also solves $(P2)$, $U_1(p^*, \theta^*) = f^*$ so

$$\int_0^{(p^* - c)\theta^* + c} ((p^* - c)\theta^* + c - w)dF(w) = \int_0^{\hat{\pi} + c} (\hat{\pi} + c - w)dF(w).$$

This implies that $\hat{\pi} = (p^* - c)\theta^*$, a contradiction.

It remains to show that the value f^* obtained from the solution (p^*, θ^*) to $(P2)$ as explained above, is the maximum achievable f_1 characterized in Lemma 4. Recall from the proof of Claim 1 above that $p^* \geq E(w)$, (1) and (3) both bind at (p^*, θ^*), and $(p^* - c)\theta^* + c < E(w)$.

To reach a contradiction, suppose there is a higher $f^{**} > f^*$ achievable from equilibrium price p^{**} and θ^{**}. Then, $(p^{**} - c)\theta^{**} + c \geq E(w)$ must hold because otherwise p^{**} and θ^{**} should solve $(P2)$, contradicting f^* being the value of $(P2)$. The inequality also implies $p^{**} > E(w)$ because $\theta^{**} < 1$, which in turn implies that $f^{**} \leq U_3(p^{**}, \theta^{**}) < U_2(p^{**}, \theta^{**})$. Define $p_t = (1-t)p^* + tp^{**}$ for $t \in (0,1)$, and let θ_t be a continuous function with $\theta_0 = \theta^*$ and $\theta_1 = \theta^{**}$ such that $U_3(p_t, \theta_t)$ is a continuously and strictly increasing function of t from f^* at $t = 0$ to $U_3(p^{**}, \theta^{**}) \geq f^{**}$ at $t = 1$.[3] Then, since $U_1(p_t, \theta_t)$ is continuous function of t such that$U_1(p_0, \theta_0) = U_1(p^*, \theta^*) < U_1(p_1, \theta_1) = U_1(p^{**}, \theta^{**})$, there must be some $\tau \in (0,1)$ such that $(p^* - c)\theta^* + c < (p_\tau - c)\theta_\tau + c < \min\{E_c(w) + c, E(w)\}$ and thus, $U_1(p^*, \theta^*) < U_1(p_\tau, \theta_\tau)$. As $U_2(p_\tau, \theta_\tau) > U_3(p_\tau, \theta_\tau)$ because $p_t > E(w)$ for all $t \in (0,1)$, we would have $\min\{U_1(p_\tau, \theta_\tau), U_2(p_\tau, \theta_\tau), U_3(p_\tau, \theta_\tau)\} > f^*$, contradicting f^* solving $(P2)$. This establishes that f^* is optimal, i.e, the maximum achievable fee. This proves Claim 2.

Finally, we need to establish the values of p^* specified in Proposition 1. Consider the solution $\tilde{w}$, as a function of p, that solves $(P1(p))$ and thus, satisfies the constraint $(*)$:

$$\int_{\tilde{w}}^{1} (w - p)dF(w) = \int_{0}^{[1-F(\tilde{w})][p-c]+c} ([1 - F(\tilde{w})][p - c] + c - w)dF(w).$$

Differentiating both sides wrt to p,

$$\tilde{w}'(p-\tilde{w})F'(\tilde{w}) - (1-F(\tilde{w})) = \big[1 - F(\tilde{w}) - \tilde{w}'F'(\tilde{w})(p-c)\big]F([1-F(\tilde{w})][p-c]+c) \quad (5)$$

where $\tilde{w}' = \tilde{w}'(p)$ and $\tilde{w} = \tilde{w}(p)$ for short. Recall from Lemma 5 that $\tilde{w}' > 0$. (i) If $c < \tilde{w}$, so that $p - c > p - \tilde{w}$, the value of (5) is negative because otherwise, the RHS

[3] This is always possible to construct.

being positive, i.e, $1 - F(\tilde{w}) \geq \tilde{w}' F'(\tilde{w})(p - c)$, would imply the LHS being negative, a contradiction. If $c < \tilde{w}(E(w))$, therefore, as p increases from $E(w)$, the maximized value of $(P1)$ decreases. Consequently, the optimal fee is obtained at $p^* = E(w)$ and $\theta^* = 1 - F(\tilde{w}(E(w)))$.

(ii) If $c > \tilde{w}$, so that $p-c < p-\tilde{w}$, the value of (5) is positive by an argument analogous to above. If $c > \tilde{w}(E(w))$, therefore, as p increases from $E(w)$, the maximized value of $(P1(p))$ increases until $\tilde{w}(p)$ hits c, say at $p^c > E(w)$, after which it decreases by (i) above. Moreover, $[1-F(\tilde{w}(p^c))][p^c-c]+c < E(w)$ must hold because otherwise the solution to $(P2)$ would be at $p^* \in (E(w), p^c]$ such that $[1-F(\tilde{w}(p^*))][p^*-c]+c = E(w)$, which contradicts the earlier finding (in the proof of Claim 1) that (1) binds at the solution to $(P2)$ and thus $[1 - F(\tilde{w}(p^*))][p^* - c] + c < E(w)$. Therefore, the optimal fee is obtained at $p^* = p^c$ and $\theta^* = 1 - F(\tilde{w}(p^c)))$. QED

Proof of Proposition 2 Take an equilibrium σ in which A's payoff is strictly positive. (There must exist an equilibrium in which a positive measure of types buy information for a strictly positive price since, for any $p \in (0, k)$, there exists a single threshold for which some types are willing to pay a sufficiently low positive fee.)

Suppose first that the equilibrium path information structure $M(p)$ consists of a finite number of thresholds $\{z_1, z_2, ..., z_n\}$. Then types in $[u_l, u_{n-1,n}]$ buy with probability $1 - F(z_n)$, types in $[u_{k,k+1}, u_{k-1,k}]$ buy with probability $1 - F(z_k)$, types in $[u_{12}, u_h]$ buy with probability $1 - F(z_1)$ and types above u_h buy with probability 1.

Therefore S's expected payoff is $p\Psi(p)$, where expected equilibrium sales $\Psi(p)$ are given by

$$\Psi(p) = 1 - G(u_h) + [G(u_h) - G(u_{12})][1 - F(z_1)] + [G(u_{12}) - G(u_{23})][1 - F(z_2)]$$

$$+... + [G(u_{k-2,k-1}) - G(u_{k-1,k})][1 - F(z_{k-1})] + [G(u_{k-1,k}) - G(u_{k,k+1})][1 - F(z_k)]$$

$$+[G(u_{k,k+1}) - G(u_{k+1,k+2})][1 - F(z_{k+1})] + ... + [G(u_{n-2,n-1}) - G(u_{n-1,n})][1 - F(z_{n-1})]$$

$$+[G(u_{n-1,n}) - G(u_l)][1 - F(z_n)].$$

Now define a new partition threshold structure $\tilde{M}$ which is the same as $M(p)$ except that one interior threshold, z_k, has been removed. Suppose that A offers $\tilde{M}$ instead of $M(p)$ for price p, the rest of his strategy remaining unchanged. If S sets price p her expected payoff will be $p\tilde{\Psi}(p)$, where

$$\tilde{\Psi}(p) = 1 - G(u_h) + [G(u_h) - G(u_{12})][1 - F(z_1)] + [G(u_{12}) - G(u_{23})][1 - F(z_2)]$$

$$+ ... + [G(u_{k-2,k-1}) - G(\tilde{u}_{k-1,k+1})][1 - F(z_{k-1})] + [G(\tilde{u}_{k-1,k+1}) - G(u_{k+1,k+2})][1 - F(z_{k+1})]$$

$$+ ... + [G(u_{n-2,n-1}) - G(u_{n-1,n})][1 - F(z_{n-1})] + [G(u_{n-1,n}) - G(u_l)][1 - F(z_n)]$$

and $\tilde{u}_{k-1,k+1} = p - E(w|z_{k-1} \leq w \leq z_{k+1})$ is the type which is indifferent between using thresholds z_{k-1} and z_{k+1}.

Therefore

$$\tilde{\Psi}(p) - \Psi(p) = [G(u_{k-2,k-1}) - G(\tilde{u}_{k-1,k+1})][1 - F(z_{k-1})] + [G(\tilde{u}_{k-1,k+1}) - G(u_{k+1,k+2})][1 - F(z_{k+1})]$$

$$-[G(u_{k-2,k-1}) - G(u_{k-1,k})][1 - F(z_{k-1})] - [G(u_{k-1,k}) - G(u_{k,k+1})][1 - F(z_k)]$$

$$-[G(u_{k,k+1}) - G(u_{k+1,k+2})][1 - F(z_{k+1})]$$

$$= G(u_{k-1,k})[F(z_k) - F(z_{k-1})] + G(u_{k,k+1})[F(z_{k+1}) - F(z_k)] - G(\tilde{u}_{z_{k-1},z_{k+1}})[F(z_{k+1}) - F(z_{k-1})].$$

S's payoff increases, compared with the equilibrium, if and only if

$$G(u_{k-1,k})[F(z_k) - F(z_{k-1})] + G(u_{k,k+1})[F(z_{k+1}) - F(z_k)] \geq G(\tilde{u}_{k-1,k+1})[F(z_{k+1}) - F(z_{k-1})]. \tag{6}$$

Let

$$\alpha = \frac{F(z_k) - F(z_{k-1})}{F(z_{k+1}) - F(z_{k-1})}.$$

Then (6) is

$$\alpha G(p - E(w|z_{k-1} \leq w \leq z_k)) + (1-\alpha) G(p - E(w|z_k \leq w \leq z_{k+1})) \geq G(p - E(w|z_{k-1} \leq w \leq z_{k+1})).$$

But

$$p - E(w|z_{k-1} \leq w \leq z_{k+1}) = \alpha(p - E(w|z_{k-1} \leq w \leq z_k) + (1-\alpha)(p - E(w|z_k \leq w \leq z_{k+1}).$$

Therefore, if G is convex, S's payoff is weakly increased compared with the equilibrium. It follows that it is optimal for S to set price p in the continuation. A's payoff is unchanged since it depends only on p, f, u_h and u_l, all of which are unchanged. Iterating this argument, we conclude that there must be an equilibrium in which A offers just two thresholds at most (the highest and lowest offered in the original equilibrium). If G is concave, we can argue, conversely, that A can add a threshold in between any two existing thresholds (e.g. their average) and S's payoff will weakly increase. Therefore there is an equilibrium in which A offers full information (a continuum of thresholds $[0, k]$).

Suppose that G is strictly convex and, in a given equilibrium, more than two thresholds are used and $u_h < 1$. Then deleting an interior threshold strictly increases S's payoff and leaves A's unchanged. Again, denote this altered structure by $\tilde{M}$. For small $\epsilon > 0$, let A offer $(f(p), \tilde{M})$ when the seller's price is $p+\epsilon$ and, for any other price, offer the same fee and information structure as in the equilibrium σ. The increase in seller's price implies that all the buyer thresholds $u_l, u_{n,n-1}, ...u_{1,2}, u_h$ move ϵ to the right. A's equilibrium expected payoff is $f[G(u_h) - G(u_l)]$ and his payoff after this deviation, assuming the seller responds by setting price $p+\epsilon$, is $f[G(u_h+\epsilon) - G(u_l+\epsilon)]$ which, because F is strictly convex, is strictly greater than $f[G(u_h) - G(u_l)]$. This would imply that A has a profitable deviation. Since, for price p, $\tilde{M}$ gives S strictly more than her equilibrium payoff, the same is true for price $p + \epsilon$ if ϵ is small enough. Since a structure with infinitely many thresholds can be approximated arbitrarily closely by one with finitely many thresholds, the conclusion (i) follows.

In the concave case, we can make a similar argument, involving a small reduction in S's price. This proves the Proposition.

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