The Aggregate Implications of Innovative Investment in the Garcia-Macia, Hsieh, and Klenow Model (preliminary and incomplete)

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Abstract

In this paper, we extend the model firm dynamics of Garcia-Macia, Klenow, and Hsieh (2016) (GHK) to include a description of the costs of innovative investments as in the model of Klette and Kortum (2004). In this model, aggregate productivity (TFP) grows as a result of innovative investment by incumbent and entering firms in improving continuing products and acquiring new products to the firm. This model serves as a useful benchmark because it nests both Quality-Ladders based Neo-Shumpeterian models and Expanding Varieties models commonly used in the literature and, at the same time, it provides a rich model of firm dynamics as described in GHK. We show how data on firm dynamics and firm value can be used to infer the elasticities of aggregate productivity growth with respect to changes in incumbent firms' investments in improving their incumbent products, incumbent firms' investments in acquiring products new to the firm, and entering firms' investments in acquiring new products. As discussed in Atkeson and Burstein (2015), these elasticities are a crucial input in evaluating the extent to which it is possible to alter the medium term growth path of the macroeconomy through policies aimed at stimulating innovative investments by firms. We use these methods to provide quantitative estimates of these elasticities of aggregate TFP growth with respect to changes in each of the three categories of innovative investment in the model as well as of the rate of social depreciation of innovation expenditures. We demonstrate that these quantitative implications of the model are highly sensitive to one's estimate of the baseline research intensity of the economy and to one's estimate of the baseline market value of intangible capital within firms.

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1 Introduction:

Garcia-Macia, Klenow, and Hsieh (2016) (henceforth GHK) present a tractable model of firm dynamics that captures many features of the data on firm dynamics. Firms in this model are distinguished by the technologies they posses for producing intermediate products. In this dimension, the model is a straightforward extension of Klette and Kortum (2004). This model allows for aggregate productivity growth to arise through innovation by incumbent firms to improve their own products, innovation by incumbent firms to obtain products new to the firm, and innovation by entering firms to obtain new products. Products that are new to a firm may be new to society or "stolen" from other firms. The goal of their paper is to use data on firm dynamics to estimate how much of the observed growth in aggregate productivity comes from these different types of innovation by firms.

In this paper, we extend the GHK model of firm dynamics to include a description of the costs of innovative investments that are left un-modelled in their paper. We also extend this model to allow for two simple forms of social depreciation of innovation expenditures: we allow incumbent firms to lose products due to exogenous exit of products and we allow for the productivity with which incumbent firms can produce products to deteriorate over time in the absence of innovative investments by that firm.

Our extended version of the GHK model then conveniently nests both the canonical Expanding Varieties models analyzed in Luttmer (2007), Luttmer (2011) and Atkeson and Burstein (2010) and the canonical Quality-Ladders based Neo-Shumpeterian models analyzed in Klette and Kortum (2004) and the many models based on that framework. As a result, it incorporates the increasing returns due to increased variety as well as the intertemporal knowledge spillovers from one firm's success in innovation to the social payoffs to another firm's innovative investment.

We then use this extended version of the GHK model to consider the question of how an economist who has access to rich data on firm dynamics and firm value might identify the social returns to *increased* innovative investment by firms. We measure the social returns to innovative investment by firms in terms of the increased growth of aggregate productivity that would result from an increase in innovative investment.

We are able to measure these social returns to innovative investment by firms due to several assumptions in our extended GHK model that make the model implied relationships between firms' innovative investments and the dynamics of firm size and value particularly tractable.

Under these assumptions, the current productive capacity of all firms corresponds to a measure $M_t(z)$, where $M_t(z)$ is the measure of intermediate products that can be produced with physical capital and labor at productivity $\exp(z)$. The corresponding measure of products in this economy is $M_t = \sum_z M_t(z)$. Under the assumption that markups are constant across products, aggregate TFP in the economy is given as a geometrically weighted aggregate of product level productivities

$$Z_t = \left[\sum_z \exp((\rho - 1)z) M_t(z) \right]^{1/(\rho-1)}$$

In general, to measure the social returns to firms' innovative investments, one must construct a model of how those investments translate into dynamics for the measure of productivities with which firms produce the products consumers desire $M_t(z)$. These dynamics must be inferred from data on the size of products, which corresponds to the model implied measure of size

$$s = \frac{\exp((\rho - 1)z)}{Z_t^{\rho-1}}$$

reflecting either the share of output of a product in total output, the share of production labor devoted to a product in total production labor, the share of physical capital devoted to a product in total physical capital, or the share of variable profits earned on a product in total variable profits. This inference is complicated by the fact that the growth in the number of products of a given size may be genuinely new products for society or simply a reallocation due to business stealing.

To make progress on this inference, we use two key assumptions. First, we assume that the equilibrium dynamics of z at the product level obeys a strong form of Gibrat's Law: both the exit rate of products and the increment to z for continuing products are independent of product size. This assumption is quite close to the assumptions made in GHK to derive a simple transition law for aggregate TFP as a function of the arrival rate of different types of innovations to z at the product level. (We do drop the assumption in GHK of endogenous exit of small products due to fixed operating costs.) Second, we assume that equilibrium innovative investment at the product level is proportional to product size. This second assumption is quite close to those used in Klette and Kortum (2004) and Atkeson and Burstein (2010) to aggregate innovative investment. In our model, these assumptions lead us to the following simple form of the aggregate relationship between firms' innovative investments and TFP growth

$$\log Z_{t+1} - \log Z_t = G(x_{ict}, x_{imt}, x_{et})$$

where x_{ict} is the aggregate investment incumbent firms undertake at t to lower the marginal cost with which they produce their current products, x_{imt} is the aggregate investment incumbent firms undertake at t in acquiring new products to the firm (both those new to society and those stolen from other firms), and x_{et} is the aggregate investment by entering firms at t in acquiring new products to produce at $t+1$ (both those new to society and those stolen from other firms). These innovative investments are in units of a produced input to innovation that we term the research good and are subject to a resource constraint

$$x_{ict} + x_{imt} + x_{et} = Y_{rt}$$

where Y_{rt} is the aggregate ouput of the input into innovation and P_{rt} is the price of this research good relative to consumption. We refer to $S_{rt} = P_{rt}Y_{rt}/GDP_t$ as the research intensity of the economy.

We seek to measure the social returns of firms' innovative investments in terms of the derivatives of this equilibrium function G.[1] We do so at an assumed baseline level of aggregate TFP growth $\bar{g}_Z$ and unobserved baseline levels of investment. For example, we refer to

$$G_{ic}\bar{x}_{ic}$$

as the elasticity of aggregate TFP growth with respect to a change in firms' investments in lower the marginal cost of production of their currently produced products, and likewise for the other two forms of innovative investment. The elasticity of aggregate TFP growth with respect to a proportional change in all three forms of innovative investment, is then given by

$$G_{ic}\bar{x}_{ic} + G_{im}\bar{x}_{im} + G_e\bar{x}_e$$

We also consider the elasticity of changes in innovative investment concentrated entirely on each one of the three forms of innovative investment

$$G_{ic}\bar{x}_{ic}\frac{\bar{Y}_r}{\bar{x}_{ic}}$$

[1]As discussed in Atkeson and Burstein (2015), these elasticities of aggregate TFP growth with respect to changes in real innovative investment are *impact elasticities*. That is, these elasticities are useful for measuring the elasticities of aggregate TFP growth with respect to policy-induced changes in expenditures on innovative investment on impact. As discussed in our earlier paper, the dynamics are shaped to a large extent by the degree of intertemporal knowledge spillovers. If these are large enough to ensure that the model displays fully endogenous growth, then impact elasticities are also useful in measuring the permanent change in the growth rate of aggregate TFP that arises from a policy-induced permanent change in the innovation intensity of the economy.

and likewise for the other two forms of investment.

Following Atkeson and Burstein (2015), we first show that if one imposes the assumption that there is no social depreciation of innovation expenditures, as is done in Klette and Kortum (2004) and GHK, then the quantitative implications of the model for the elasticities of aggregate productivity growth with respect to changes in innovative investments in economies with low baseline levels of TFP growth are tightly restricted by that low baseline level of TFP growth regardless of the fit of the model to the data on firm dynamics and value. Specifically, the function G is concave in a proportional increase in its three arguments so

$$G_{ic}\bar{x}_{ic} + G_{im}\bar{x}_{im} + G_e\bar{x}_e \leq \bar{g}_Z - G(0,0,0)$$

since $\bar{g}_Z$ corresponds to the growth of aggregate TFP at the baseline level of investments. We refer to the model's counterfactual implications for the growth rate of aggregate TFP if there were no investments by firms in innovation ($G(0,0,0)$) as the social depreciation of innovation expenditures. Under the assumption of no social depreciation, $G(0,0,0) = 0$, and hence the social return we wish to measure is bounded by the baseline growth rate of TFP $\bar{g}_Z$.

We then take up the question of what quantitative social returns to firms' innovative investments the extended GHK model with the two forms of social depreciation can admit once it has been fit to data on firm dynamics and firm value. We assume that the economist conducting the measurement has access to data on the size and number of products in entering and incumbent firms and data on the value of firms in excess of their stock of physical capital. We show how to identify the parameters of the function G and the levels of investment of each type from these data up to two unidentified parameters: the fraction of products that are new to entering and incumbent firms that are new to society as opposed to stolen from other firms, and the fraction of firm value that is due to the variable profits earned on existing products versus due to the added innovative capacity of the firm associated with the acquisition of these products. We are able to use the model to impose upper and lower bounds on these two unobserved parameters. Our main results are presentations of the model-implied relationships between these data on firm dynamics and firm value, the two set-identified parameters, and the elasticities of aggregate TFP growth with respect to changes in these three categories of innovative investment by firms.

We next use our theoretical results to deliver quantitative estimates of the social returns to firms' innovative investments implied by our extended GHK model and the data in the Business Dynamics Statistics database as well as calibrations of total innovative expenditure from Nakamura (2009) and firm value from McGrattan and Prescott (Forthcoming). We find elasticities of aggregate productivity growth with respect to individual categories of investment ranging from 0 to 0.44 and with respect to a proportional increase in all three categories of innovative investment ranging from 0.013 to 0.039, with the alternative estimates driven primarily by different calibrated values of the baseline market value of intangible capital within firms. We find corresponding rates of social depreciation of innovation expenditures ranging from 0.023 to 0.037 per year.

2 Model and Equilibrium Properties:

In this section, we review the key results we derive from the dynamic equations of the model. Details of how these dynamic equations are derived from a fully specified version of the model are given in subsequent sections.

2.1 Aggregate Output and Total Factor Productivity:

At each date t, aggregate output of the final good used for consumption and investment in physical capital is given by

$$Y_t = Z_t K_t^{\alpha} L_{pt}^{1-\alpha}$$

where K_t is the aggregate stock of physical capital, L_{pt} is the aggregate quantity of labor used in production of this final good, and Z_t is total factor productivity.

As is standard in this literature, we assume that this final good Y_t is produced from a continuum of intermediate products through a CES aggregator with elasticity ρ. These intermediate goods are each provided by monopolist producers who engage in monopolistic price competition. These intermediate products are produced according to production technologies

$$y_t(z) = \exp(z) k_t(z)^{\alpha} l_{pt}(z)^{1-\alpha}$$

where z indexes the position of the marginal cost curve for the producer of the intermediate good and $k_t(z)$ and $l_{pt}(z)$ denote the quantities of physical capital and labor used in production of the intermediate good with index z at date t. To simplify our notation, we assume that the support of z is a countable grid with $z_n = n\Delta$ for the integers n. Assuming constant markups $\mu > 1$ of prices to marginal costs across products, we get the result that in equilibrium total factor productivity is a CES aggregate of the indices z across the available intermediate products of the form

$$Z_t = \left[\sum_n \exp((\rho - 1) z_n) M_t(z_n) \right]^{1/(\rho-1)}$$

where $M_t(z_n)$ is the measure of intermediate products with index z_n at date t. Note that the measure of products available in the economy at date t is given by

$$M_t = \sum_n M_t(z_n)$$

2.1.1 Product Size:

In equilibrium we have that

$$s_t(z) = \frac{\exp((\rho - 1)z)}{Z_t^{\rho-1}} = \frac{y_t(z)}{Y_t} = \frac{k_t(z)}{K_t} = \frac{l_{pt}(z)}{L_{pt}}.$$

Hence, we refer to $s_t(z)$ as the *size* of a product. In data, this can be measured in terms of value added or profits or physical capital or production labor. This measure is also additive, so we can use it to refer to the *size* of categories of products as we do below.

2.1.2 Factor Shares:

In equilibrium with constant markups μ across products, fraction α/μ of output Y_t is paid to physical capital, fraction $(1 - \alpha)/\mu$ to production labor, and fraction $(\mu - 1)/\mu$ to the firms that produce the intermediate products are variable profits.

2.2 Contributions to TFP by firm and product categories:

As in Klette and Kortum (2004), firms in this economy produce a number of products j, where j is a natural number. Entering firms enter with a single product, so $j = 1$. Firms exit when the number of products that they produce drops to zero.

We say that a firm is an *incumbent firm* at t if it also produced products at $t-1$. Otherwise, firms at t are *entering firms.*

We say that a product is an *existing product* at t if it was also produced at $t-1$. Otherwise, products at t are *new products.*

New products are new to society. Not all products that a new to a firm at t are new to society. Some products that are new to a firm at t are incumbent products that were produced by some other firm at $t-1$. We refer to existing products that are produced by a different firm at t than at $t-1$ as *stolen products*. We refer to incumbent products at t that are produced by the same firm at t as at $t-1$ as *continuing products*. Note that, by definition, continuing products are produced by incumbent firms.

With this terminology, we decompose aggregate productivity at t into five components:

$$Z_{ict} = \left[\sum_n \exp((\rho-1)z_n) M_{ict}(z_n) \right]^{1/(\rho-1)}$$

where $M_{ict}(z_n)$ is the measure of continuing products with index z_n produced by incumbent firms at date t,

$$Z_{int} = \left[\sum_n \exp((\rho-1)z_n) M_{int}(z_n) \right]^{1/(\rho-1)}$$

where $M_{int}(z_n)$ is the measure of new products with index z_n produced by incumbent firms at date t,

$$Z_{ist} = \left[\sum_n \exp((\rho-1)z_n) M_{ist}(z_n) \right]^{1/(\rho-1)}$$

where $M_{ist}(z_n)$ is the measure of stolen products with index z_n produced by incumbent firms at date t,

$$Z_{ent} = \left[\sum_n \exp((\rho-1)z_n) M_{ent}(z_n) \right]^{1/(\rho-1)}$$

where $M_{ent}(z_n)$ is the measure of new products with index z_n produced by entering firms at date t, and

$$Z_{est} = \left[\sum_n \exp((\rho-1)z_n) M_{est}(z_n) \right]^{1/(\rho-1)}$$

where $M_{est}(z_n)$ is the measure of stolen products with index z_n produced by entering firms at date t.

With these definitions, we have the following decomposition of aggregate productivity

$$Z_t^{\rho-1} = Z_{ict}^{\rho-1} + Z_{int}^{\rho-1} + Z_{ist}^{\rho-1} + Z_{ent}^{\rho-1} + Z_{est}^{\rho-1}$$

As in GHK, this decomposition can be done by product categories existing and new

$$Z_t^{\rho-1} = \left(Z_{ict}^{\rho-1} + Z_{ist}^{\rho-1} + Z_{est}^{\rho-1} \right) + \left(Z_{int}^{\rho-1} + Z_{ent}^{\rho-1} \right)$$

and firm categories, incumbent and entering

$$Z_t^{\rho-1} = \left(Z_{ict}^{\rho-1} + Z_{ist}^{\rho-1} + Z_{int}^{\rho-1} \right) + \left(Z_{est}^{\rho-1} + Z_{ent}^{\rho-1} \right)$$

In using data on firm dynamics to discipline the model, we use the results that these decompositions of aggregate productivity correspond to decomposition of product categories categories by category size

$$1 = \left(\frac{Z_{ict}^{\rho-1}}{Z_t^{\rho-1}} + \frac{Z_{ist}^{\rho-1}}{Z_t^{\rho-1}} + \frac{Z_{est}^{\rho-1}}{Z_t^{\rho-1}}\right) + \left(\frac{Z_{int}^{\rho-1}}{Z_t^{\rho-1}} + \frac{Z_{ent}^{\rho-1}}{Z_t^{\rho-1}}\right)$$

and likewise for firm categories, incumbent and entering

$$1 = \left(\frac{Z_{ict}^{\rho-1}}{Z_t^{\rho-1}} + \frac{Z_{ist}^{\rho-1}}{Z_t^{\rho-1}} + \frac{Z_{int}^{\rho-1}}{Z_t^{\rho-1}}\right) + \left(\frac{Z_{est}^{\rho-1}}{Z_t^{\rho-1}} + \frac{Z_{ent}^{\rho-1}}{Z_t^{\rho-1}}\right)$$

We also use notation for a corresponding decomposition of the measure of products into five categories

$$M_{ict} = \sum_n M_{ict}(z_n)$$

and similarly for the categories is, in, es, and en.

Let

$$S_{ict} = \frac{Z_{ict}^{\rho-1}}{Z_t^{\rho-1}}$$

denote the aggregate size of continuing products at incumbent firms, and likewise let S_{int}, S_{ist}, S_{ent} and S_{est} denote the sizes of the other four product categories. Let

$$S_{imt} = S_{int} + S_{ist}$$

denote the size of those products that are new to incumbent firms at t and $S_{et} = S_{ent} + S_{est}$ the size of those products that are new to entering firms at t.

$$F_{ict} = \frac{M_{ict}}{M_t}$$

denote the fraction of products that are continuing products at incumbent firms, and likewise let F_{int}, F_{ist}, F_{ent} and F_{est} denote the sizes of the other four product categories. Let $F_{imt} = F_{int} + F_{ist}$ denote the fraction of products that are new to incumbent firms at t and $F_{et} = F_{ent} + F_{est}$ the fraction of products that are new to entering firms at t. Note that the *average size* of products in a category is given by the ratio S/F.

2.3 Technology for Innovative Investment:

We assume that firms make three types of investment in innovation as in GHK: incumbent firms invest to improve their continuing products, incumbent firms invest to acquire new products to the firm either through the creation of a new product or acquisition of a stolen product, and entering firms invest to acquire new products to the entering firm either through the creation of a new product or acquisition of a stolen product. Innovative investment is undertaken using a second final good, which we term the *research good*, as an input. The aggregate amount of this research good produced at time t is

$$Y_{rt} = A_{rt} Z_t^{\phi-1} L_{rt} \tag{1}$$

where $L_{rt} = L_t - L_{pt}$ is the quantity of labor devoted to production of the research good, A_{rt} is the level of exogenous scientific progress, and the term $Z_t^{\phi-1}$ for $\phi \leq 1$ reflects intertemporal knowledge spillovers in the production of the research good as in the model of Jones (2002) equation 5.

We denote the aggregate quantity of the research good that incumbent firms invest at t in improving z for continuing products at $t+1$ by x_{ict}. We denote the aggregate quantity of the research good that incumbent firms invest at t in acquiring products new to that firm at $t+1$ by x_{imt}. We denote the aggregate quantity of the research good that entering firms invest at t in acquiring products new to that firm at $t+1$ by x_{et}. The resource constraint for the research good is

$$x_{ict} + x_{imt} + x_{et} = Y_{rt} \tag{2}$$

As in Klette and Kortum (2004), we assume that the innovative investment undertaken by a firm is simply the sum of the investments by that firm at the product level. As in Atkeson and Burstein (2010), the equilibrium investment by incumbent firms in improving each of their continuing products is directly proportional to the size of the product. As in Klette and Kortum (2004), the equilibrium investment by incumbent firms in acquiring products new to the firm is directly proportional to the number of products produced by the firm. Finally, we assume that the amount that each entering firm invests is a parameter at t that we normalize to $1/M_t$. The technologies for innovative investment that we assume build in two key spillovers. First, for incumbent firms investing in improving a continuing product, the cost of such investment falls with aggregate TFP. Second, for incumbent and entering firms investing to acquire products new to the firm, the productivity index z of the acquired product rises with the average value of $\exp((\rho-1)z)$ across existing products and the cost of the investment falls with the measure of existing products M_t.

Furthermore, as in Klette and Kortum (2004) and Atkeson and Burstein (2010) and many other papers in the literature, we assume that these investments result in equilibrium dynamics for the scale of existing products consistent with a strong form of Gibrat's Law characterized by the following four properties of equilibrum. First, we assume that a fraction δ_0 of existing products at t cease to be produced (for exogenous reasons) at $t+1$, with this exogenous exit probability δ_0 independent of the index z of the product. Second, the probability that an existing product is stolen from an incumbent firms is independent of the index z of the product. Third, the increment to the index z for existing products that are stolen by incumbent and entering firms between t and $t+1$ is $z_{t+1} - z_t = \Delta$ independent of the index z of the product that is stolen. Fourth, the distribution of the increments to the index z between t and $t+1$ for those products that are continuing products produced by incumbent firms at $t+1$ is independent of the index z of those products at t.

We now describe the dynamics of the measures and contribution to aggregate productivity of each product category, based on our assumptions on the technologies for innovative investment at the firm level. Further details are given in the Appendix (To be written).

2.3.1 Measures by product category

The measure of products produced at $t+1$ by entering firms is given as a function of entering firms' innovative investment at t by

$$M_{est+1} + M_{ent+1} = x_{et} M_t.$$

This expression follows from the assumption that if an entering firm spends $1/M_t$ units of the research good, it acquires 1 new product. Thus if there are M_e entering firms at time t they spend M_e/M_t units of the research good and produce M_e new products at $t+1$. Further details are provided in the Appendix (to be be written).

We assume that fraction δ of these products produced by entering firms at $t+1$ are stolen from incumbent firms, with the remaining fraction $1-\delta$ being new to society. Thus, we assume that

$$M_{est+1} = \delta x_{et} M_t$$

We assume that the measure of products new to incumbent firms at $t+1$ is given as a function of these firms' investments x_{imt} at t. Specifically, this measure is

$$M_{ist+1} + M_{int+1} = h(x_{imt}) M_t$$

where $h(\cdot)$ is a strictly increasing and concave function with $h(0) = 0$ and $h(x) < 1$ for all x. This expression follows from the assumption that if each incumbent firm spends x/M_t units of the research good per product it produces, it acquires $h(x)$ new products per product – if $x = x_{imt}$ is the same for each product then x_{imt} is spent on aggregate and $h(x_{imt})M_t$ new products are created at $t+1$.

We again assume that fraction δ of these products are stolen products, so

$$M_{ist+1} = \delta h(x_{imt})M_t$$

The measure of continuing products produced by incumbent firms at $t+1$ are those products at t that do not cease to exist at $t+1$ plus those products at t that are not stolen at $t+1$. Thus

$$M_{ict+1} = (1-\delta_0)M_t - M_{ist+1} - M_{est+1} = (1-\delta_{Mt})M_t$$

where

$$\delta_{Mt} \equiv \delta_0 + \delta\left(h(x_{imt}) + x_{et}\right) \tag{3}$$

2.3.2 Dynamics of the measure of products

These assumptions imply dynamics for the total measure of products

$$\frac{M_{t+1}}{M_t} = (1-\delta_{Mt}) + h(x_{imt}) + x_{et} \tag{4}$$

2.3.3 Investment and growth of Z by product category

In this subsection, we make assumptions about the relationship between investment and the growth of Z by product category. These assumptions deliver our model's predictions for the relationship between investment and the growth of size by product category as well as for the relationship between investment and the growth rate of aggregate TFP.

We assume that, due to spillovers as in Luttmer (2007), new products in entering firms at $t+1$ have average $\exp((\rho-1)z)$ indexed by $\bar{\eta}_{en}Z_t^{\rho-1}/M_t$ so that their total contribution to aggregate productivity is

$$Z_{ent+1}^{\rho-1} = \bar{\eta}_{en}Z_t^{\rho-1}(1-\delta)x_{et}.$$

where we used the assumption that

$$\frac{M_{ent+1}}{M_t} = (1-\delta)x_{et}$$

New products obtained by incumbent firms have average $\exp((\rho-1)z)$ indexed by $\bar{\eta}_{in}Z_t^{\rho-1}/M_t$ so that

$$Z_{int+1}^{\rho-1} = \bar{\eta}_{in}Z_t^{\rho-1}(1-\delta)h(x_{imt}).$$

where we use the assumption that

$$\frac{M_{int+1}}{M_t} = (1-\delta)h(x_{imt})$$

Products that are stolen each have increment of size Δ to their index z from t to $t+1$. Thus, they have average $\exp((\rho-1)z)$ indexed by $\exp((\rho-1)\Delta)Z_t^{\rho-1}/M_t$ so

$$Z_{est+1}^{\rho-1} = \exp((\rho-1)\Delta)Z_t^{\rho-1}\delta x_{et},$$

and

$$Z_{ist+1}^{\rho-1} = \exp((\rho-1)\Delta)Z_t^{\rho-1}\delta h(x_{imt}),$$

Those products that are continuing products in incumbent firms at $t+1$ experience, on average, increment $\bar{\zeta}(x_{ict})$ to their $\exp((\rho-1)z)$ from t to $t+1$. Thus

$$Z_{ict+1}^{\rho-1} = (1-\delta_{Mt})\,\bar{\zeta}(x_{ict})Z_t^{\rho-1}$$

This expression follows from the assumptions that if an incumbent firm with a continuing product with productivity z spends x of the research good on improving that product, it draws a new productivity, conditional on survival, that is k steps from z with probability $\zeta\left(k, x\frac{Z_t^{\rho-1}}{exp((\rho-1)z)}\right)$. In equilibrium, each incumbent firm chooses the same investment per unit size $x\left(z\right) = x_{ict}\frac{exp((\rho-1)z)}{Z_t^{\rho-1}}$, so for all products the probability of drawing a new productivity that is k steps from the current level is $\zeta\left(k, x_{ict}\right)$. We define the expectation over k as $\bar{\zeta}\left(x_{ict}\right)$. Adding over all incumbent products, the aggregate resources spent in this activity are x_{ict}. We assume that $\bar{\zeta}(x_{ic})$ is a strictly increasing and concave function, and that $0 < \bar{\zeta}(0) < 1$.

2.3.4 Dynamics of TFP

These assumptions imply dynamics for aggregate productivity

$$\frac{Z_{t+1}^{\rho-1}}{Z_t^{\rho-1}} = (1-\delta_0-\delta(h(x_{imt})+x_{et}))\,\bar{\zeta}(x_{ict}) + \bar{\eta}_i h(x_{mt}) + \bar{\eta}_e x_{et} \tag{5}$$

where

$$\bar{\eta}_i = \delta\exp((\rho-1)\Delta) + (1-\delta)\bar{\eta}_{in}$$

and

$$\bar{\eta}_e = \delta\exp((\rho-1)\Delta) + (1-\delta)\bar{\eta}_{en}$$

2.3.5 Parameters and the length of the time period

The parameters δ_0, δ, $\bar{\eta}_{in}$,and $\bar{\eta}_{en}$ all represent ratios of level variables and hence are set independent of the length of the time period.

The variables x_{ic}, x_{im} and x_e represent flows, and hence are proportional to the length of the time period. Likewise, since $h(x_{im})$ reflects an arrival rate of new products, this rate should also be proportional to the length of the time period. The step size Δ for stolen products is independent of the length of the time period.

Since $\log(\bar{\zeta}(x))$ corresponds to the growth rate of z for continuing products in incumbent firms, we do require that $\log(\bar{\zeta}(x))$ converges to zero as the time interval between periods t and $t+1$ shrinks to zero.

Thus, without loss of generality, we can assume that $\exp((\rho-1)\Delta) \geq \bar{\zeta}(x)$. This corresponds to the requirement that a product that is stolen from incumbent firms is produced with a higher z at $t+1$ in its new firm than it would have had as a continuing product in the firm that previously produced it. As we discuss below, this assumption also thus corresponds to the assumption that stolen products have larger average size than continuing products in incumbent firms.

2.4 Nested Models

This model nest five commonly used models in the literature: three types of Expanding Varieties models and two types of Neo-Schumpeterian models.

If $\delta = 0$, then there is no business stealing and hence all new products acquired by incumbent and entering firms are new products for society, expanding the measure of products M_t. This is the assumption typically made in an Expanding Varieties model. Luttmer (2007) is an example of an expanding varieties model in which there is only innovative investment in entry. (Note that we do not consider the endogenous exit of products due to fixed operating costs featured in that paper and in GHK). Atkeson and Burstein (2010) is an example of an expanding varieties model in which there is innovative investment in entry and by incumbent firms in continuing products. Luttmer (2011) is an example of an expanding varieties model in which there is innovative investment in entry and in the acquisition of new products by incumbent firms.

Neo-Schumpeterian models based on the Quality-Ladders framework typically assume $\delta = 1$ and $\delta_0 = 0$. The simplest versions of these models do not accommodate growth in the measure of varieties M_t. Grossman and Helpman (1991) and Aghion and Howitt (1992) are examples of Neo-Schumpeterian models in which there is only innovative investment in entry. Klette and Kortum (2004) is an example of a Neo-Schumpeterian model in which there is innovative investment in entry and by incumbent firms in acquiring new products.

3 Disciplining the Model with Data on Firm Dynamics

We consider an economist who has data on the growth rate of the measure of products M_{t+1}/M_t as well as data on the fraction of products that are continuing products in incumbent firms F_{ict+1}, the fraction of products that are new to incumbent firms measured as the sum of those that are new to society and stolen F_{imt+1}, and the fraction of products that are produced in entering firms measured as the sum of those that are new to society and stolen F_{et+1}.

We also assume that this economist has data on the growth rate of aggregate TFP Z_{t+1}/Z_t (at least for some baseline value) and data on the aggregate size of continuing products in incumbent firms S_{ict+1}, the aggregate size of products that are new to incumbent firms measured as the sum of those that are new products and those that are stolen S_{imt+1}, and the aggregate size of products that are new to entering firms measured as the sum of those that are new products an those that are stolen S_{emt+1}.

We use the notation

$$g_{Zt+1} = \log(Z_{t+1}) - \log(Z_t)$$

to denote the growth rate of aggregate TFP and $g_{Mt+1} = \log(M_{t+1}) - \log(M_t)$ to denote the growth rate of the measure of products.

3.1 Implications of data on the dynamics of the measure of products

Data on the dynamics of the measure of products allow one to identify the values of the following on a balanced growth path (BGP)

$$(1 - \bar{\delta}_M) = \bar{F}_{ic} \exp(\bar{g}_M)$$

$$h(\bar{x}_{im}) = \bar{F}_{im} \exp(\bar{g}_M)$$

$$\bar{x}_e = \bar{F}_e \exp(\bar{g}_M)$$

3.2 Implications of data on size and aggregate TFP growth

Data on size and aggregate TFP growth, together with the data on the measures of products discussed above, allow one to identify the following model parameters.

The parameters $\bar{\eta}_e$ and $\bar{\eta}_i$ are identified from data on the average size of products that are new to entering and incumbent firms on a balanced growth path (these BGP average sizes are denoted with a bar):

$$\bar{\eta}_e = \frac{\bar{S}_e}{\bar{F}_e}\frac{\exp((\rho-1)\bar{g}_Z)}{\exp(\bar{g}_M)}$$

$$\bar{\eta}_i = \frac{\bar{S}_{im}}{\bar{F}_{im}}\frac{\exp((\rho-1)\bar{g}_Z)}{\exp(\bar{g}_M)}$$

The value of $\bar{\zeta}(x_{ict})$ is identified from data on the average size of continuing products in incumbent firms

$$\bar{\zeta}(\bar{x}_{ic}) = \frac{\bar{S}_{ic}}{\bar{F}_{ic}}\frac{\exp((\rho-1)\bar{g}_Z)}{\exp(\bar{g}_M)}$$

3.3 Parameters that are not identified

The parameters δ_0 and δ governing the share of products new to incumbent and entering firms that are stolen from other incumbent firms together with the parameters $\bar{\eta}_{in}$ and $\bar{\eta}_{en}$ governing the average value of $\exp((\rho-1)z)/Z_t^{\rho-1}$ for that fraction $(1-\delta)$ of products new to incumbent and entering firms that are new to society are not pinned down by the data on firm dynamics we have discussed. Instead, we have only identified δ_{Mt} as defined in equation (3) and $\bar{\eta}_i$ and $\bar{\eta}_e$ defined in equation (5).

Our model does impose bounds on these parameters δ_0 and δ. Both of these parameters must be non-negative. We must have $\delta_0 \leq \bar{\delta}_M$ as observed in the data $(\bar{F}_{ic}\exp(\bar{g}_M))$. We must have δ below the minimum of four upper bounds. The first of these is $\delta \leq 1$. The second of these corresponds to the value of δ implied by equation (3) with $\delta_0 = 0$ (since δ_0 cannot be negative) and the data on the exit rate of incumbent products and the fraction of new products in incumbent and entering firms. Specifically

$$\delta \leq \frac{\bar{\delta}_M}{h(\bar{x}_{im}) + \bar{x}_e} = \frac{1-\bar{F}_{ic}\exp(\bar{g}_M)}{(1-\bar{F}_{ic})\exp(\bar{g}_M)}$$

The third and fourth bound corresponds to the requirement that stolen products in incumbent firms have higher z (on average) than the products that they replace, i.e. $\exp((\rho-1)\Delta) \geq \bar{\zeta}(\bar{x}_{ic})$. Note that our model's implications for the average size of stolen products in incumbent and new firms is

$$\frac{\bar{S}_{is}}{\bar{F}_{is}} = \frac{\bar{S}_{es}}{\bar{F}_{es}} = \frac{\exp((\rho-1)\Delta)}{\exp(\rho-1)\bar{g}_Z)}\exp(\bar{g}_M)$$

and that the average size of new products in incumbent firms is the solution to

$$\frac{\bar{S}_{im}}{\bar{F}_{im}} = \delta\frac{\bar{S}_{is}}{\bar{F}_{is}} + (1-\delta)\frac{\bar{S}_{in}}{\bar{F}_{in}}$$

The requirement that stolen products be larger than those that they replace, i.e.

$$\frac{\bar{S}_{is}}{\bar{F}_{is}} \geq \frac{\bar{S}_{ic}}{\bar{F}_{ic}}$$

(or, similarly, $\exp((\rho - 1)\Delta) \geq \bar{\zeta}(\bar{x}_{ic})$) together with the requirement that new products have non-negative average size ($\bar{\eta}_{in} > 0$) implies that we must have

$$\delta \leq \frac{\bar{S}_{im}}{\bar{F}_{im}} \Big/ \frac{\bar{S}_{ic}}{\bar{F}_{ic}}$$

Similar arguments for entering firms give the fourth bound

$$\delta \leq \frac{\bar{S}_{e}}{\bar{F}_{e}} \Big/ \frac{\bar{S}_{ic}}{\bar{F}_{ic}}$$

The minimum of these bounds binds when new products in incumbent or entering firms are smaller than continuing products on average in the data.

We have imposed directly by assumption that $h(0) = 0$ and that the measure of products in entering firms at $t + 1$ is equal to zero when $x_{et} = 0$.

The parameter $\bar{\zeta}(0)$ corresponding to the counterfactual growth rate of $\exp((\rho - 1)z)$ for continuing products in the absence of investment in improving these products is also not identified.

4 Elasticities of TFP growth with respect to innovative investment

From equation (5), we can write the growth rate of TFP as a function of innovative investments as

$$\log(Z_{t+1}) - \log(Z_t) = G(x_{ict}, x_{imt}, x_{et})$$

where

$$G(x_{ic}, x_{im}, x_e) = \frac{1}{\rho - 1} \log\left((1 - \delta_0 - \delta(h(x_{im}) + x_e))\, \bar{\zeta}(x_{ic}) + \bar{\eta}_i h(x_m) + \bar{\eta}_e x_e\right)$$

We now consider the elasticities of TFP growth with respect to the three types of innovative investment. To do so, we evaluate derivatives of G at a point $(\bar{x}_{ic}, \bar{x}_{im}, \bar{x}_e)$ and $\bar{Y}_r$ that satisfies equation (2). Define $\bar{g}_Z$ to be the growth rate of TFP at those levels of investment. Define

$$\hat{x} = \log(x) - \log(\bar{x})$$

Then, to a first order approximation we have

$$\hat{g}_Z \equiv G_{ic}\bar{x}_{ic}\hat{x}_{ic} + G_{im}\bar{x}_{im}\hat{x}_{im} + G_e\bar{x}_e\hat{x}_e$$

where

$$\hat{g}_Z \approx \log(Z') - \log(Z) - \bar{g}_Z$$

and

$$G_{ic} = \frac{1}{\rho - 1}\frac{(1 - \delta_0 - \delta(h(\bar{x}_{im}) + \bar{x}_e))\,\bar{\zeta}'(\bar{x}_{ic})}{\exp((\rho - 1)\bar{g}_Z)} \tag{6}$$

$$G_{im} = \frac{1}{\rho - 1}\frac{\left(-\delta\bar{\zeta}(\bar{x}_{ic}) + \bar{\eta}_i\right) h'(\bar{x}_m)}{\exp((\rho - 1)\bar{g}_Z)} \tag{7}$$

and

$$G_e = \frac{1}{\rho - 1}\frac{\left(-\delta\bar{\zeta}(\bar{x}_{ic}) + \bar{\eta}_e\right)}{\exp((\rho - 1)\bar{g}_Z)} \tag{8}$$

4.1 Identifying or bounding elasticities

In what follows, we look to use data on firm dynamics and the value of firms to identify precisely or bound the model-implied impact elasticity $\hat{g}_Z/\hat{Y}_r$ computed from derivatives (6), (7), and (8), baseline investment levels $\bar{Y}_r, \bar{x}_{ic}, \bar{x}_{im}$ and $\bar{x}_e$, and alternative perturbations to investment $\hat{x}_{ic}$, $\hat{x}_{im}$, $\hat{x}_e$.

We consider two types of perturbations to investment. The first type is a proportional change in all categories of investment

$$\hat{x}_{ic} = \hat{x}_{im} = \hat{x}_e = \hat{Y}_r$$

The second type of perturbation is concentrated on a single form of investment

$$\hat{x}_{ic} = \frac{\bar{Y}_r}{\bar{x}_{ic}}\hat{Y}_r$$

$$\hat{x}_{im} = \frac{\bar{Y}_r}{\bar{x}_{im}}\hat{Y}_r$$

or

$$\hat{x}_e = \frac{\bar{Y}_r}{\bar{x}_e}\hat{Y}_r$$

We start our analysis with an application of the results of Atkeson and Burstein (2015) bounding the impact elasticity $\hat{g}_Z/\hat{Y}_r$ with respect to a proportional change in all three categories of investment by the difference between the baseline growth rate of TFP and the model implied counterfactual growth rate of TFP in the absence of any innovative investment, $\bar{g}_Z - G(0,0,0)$. In the original GHK model, it is assumed that $G(0,0,0) = 0$, so this proposition imposes a tight bound for advanced economies with low baseline $\bar{g}_Z$. We then show that in our specification of the GHK model, $G(0,0,0)$ can be less than zero, relaxing this bound.

Once we allow for the possibility that $G(0,0,0) < 0$, the model admits for a large value of the impact elasticity $\hat{g}_Z/\hat{Y}_r$ even in an advanced economy. As we have discussed above, data on firm dynamics allows us to identify some terms in the derivatives (6), (7), and (8), but we are not able to identify the parameter δ governing the extent of business stealing, the derivatives of the innovation technologies $\bar{\zeta}'(\bar{x}_{ic})$ and $h'(\bar{x}_m)$, and the baseline investment levels $\bar{Y}_r, \bar{x}_{ic}, \bar{x}_{im}$ and $\bar{x}_e$ needed to compute these elasticities.

In section 5 we discuss how to use data on the value of intangible capital in firms together with the condition that firms choose investment to maximize that private value to identify the derivatives of the innovation technologies $\bar{\zeta}'(\bar{x}_{ic})$ and $h'(\bar{x}_m)$ and the breakdown of expenditure on innovation into the three categories of innovative investment. In section 6 we then report on the model's quantitative implications for these elasticities for various assumed values of the parameter δ governing the extent of business stealing.

4.2 Elasticity with respect to a proportional change in all innovative investment

Following Atkeson and Burstein (2015), we are able to bound the elasticity of TFP growth with respect to a proportional change in all innovative investments as follows.

Proposition 1. *If $\hat{x}_{ic} = \hat{x}_{im} = \hat{x}_e = \hat{Y}_r$, then the elasticity of TFP growth with respect to innovative investment is bounded by the difference between the baseline growth rate of TFP and the growth rate of TFP when all investment is zero, i.e.*

$$\hat{g}_Z \leq (\bar{g}_Z - G(0,0,0))\,\hat{Y}_r$$

Proof. The proof follows from the concavity of the function $H(a)$ (see appendix) defined as

$$H(a) \equiv G(a\bar{x}_{ic}, a\bar{x}_{im}, a\bar{x}_e)$$

Specifically, if $\hat{x}_{ic} = \hat{x}_{im} = \hat{x}_e = \hat{Y}_r$, then

$$\hat{g}_Z = H'(1)\hat{Y}_r$$

The result follows from the fact that for concave functions $H'(1)1 \leq H(1) - H(0)$. □

4.3 Social Depreciation of Innovation in the GHK model

Note that the bound on the elasticity $\hat{g}_Z/\hat{Y}_r$ established in proposition 1 is independent of parameters outside of those that determine the model's implications for $\bar{g}_Z$ and $G(0,0,0)$. We refer to the growth rate of TFP that would arise if all innovative investment were set to zero, $G(0,0,0)$, as the rate of *social depreciation of innovation*.

In both Klette and Kortum (2004) and GHK, it is assumed that the rate of social depreciation of innovation $G(0,0,0) = 0$. As a result, the elasticity $\hat{g}_Z/\hat{Y}_r$ is bounded by $\bar{g}_Z$.

In our implementation of the GHK model, we do not make this assumption that there is no social depreciation of innovation. Because we allow for exogenous exit of existing varieties, denoted by δ_0, and for deterioration of the index z of continuing varieties in incumbent firms, denoted by $\bar{\zeta}(0) \leq 1$, we have social depreciation of innovation given by

$$G(0,0,0) = \frac{1}{\rho - 1} \log\left((1-\delta_0)\bar{\zeta}(0)\right) \leq 0 \tag{9}$$

(recall that since $h(x_{im})$ denotes a rate at which incumbent firms acquire new products, we impose that $h(0) = 0$). Thus, our version of the GHK model potentially allows for a higher elasticity $\hat{g}_Z/\hat{Y}_r > \bar{g}_Z$ if we allow for social depreciation by calibrating $\delta_0 > 0$ and/or $\bar{\zeta}(0) < 1$.

4.4 Bounding elasticities with respect to components of investment and social depreciation

In the experiment in which we consider proportional changes in investment, we have that the elasticity of TFP growth with respect to total innovative expenditure is equal to the sum of three individual elasticities:

$$\hat{g}_Z = \left[G_{ic}\bar{x}_{ic} + G_{im}\bar{x}_{im} + G_e\bar{x}_e\right]\hat{Y}_r$$

Using the same logic based on concavity, in this model, we can bound these elasticities of TFP growth with respect to each component of innovative investment as follows.

Corollary 2. *The individual elasticities of TFP growth with respect to the components of investment are bounded by*

$$G_{ic}\bar{x}_{ic} \leq \bar{g}_Z - G(0, \bar{x}_{im}, \bar{x}_e)$$

$$G_{im}\bar{x}_{im} \leq \bar{g}_Z - G(\bar{x}_{ic}, 0, \bar{x}_e)$$

$$G_e\bar{x}_e \leq \bar{g}_Z - G(\bar{x}_{ic}, \bar{x}_{im}, 0)$$

Proof. These bounds follow as a corollary to proposition 1 from the concavity of the functions

$$H_{ic}(a) = G(a\bar{x}_{ic}, \bar{x}_{im}, \bar{x}_e)$$

$$H_{im}(a) = G(\bar{x}_{ic}, a\bar{x}_{im}, \bar{x}_e)$$

and

$$H_e(a) = G(\bar{x}_{ic}, \bar{x}_{im}, a\bar{x}_e)$$

and the observation that the desired elasticities are given by $H'_{ic}(1)$, $H'_{im}(1)$, and $H'_e(1)$ respectively. □

The bounds on the elasticities of TFP growth with respect to changes in the components of innovative investment presented in Proposition 2 offer a procedure for bounding the change in TFP growth $\hat{g}_Z$ that would arise from a change in total innovative investment $\hat{Y}_r$ focused entirely on one of the categories of innovative investment. Specifically, consider changes in investment of the form $(\hat{x}_{ic}, 0, 0)$, $(0, \hat{x}_{im}, 0)$, and $(0, 0, \hat{x}_e)$. Then we have

$$\hat{x}_{ic} = \frac{\bar{Y}_r}{\bar{x}_{ic}} \hat{Y}_r$$

and similarly for $\hat{x}_{im}$ and $\hat{x}_e$. This observation gives us that the change in TFP growth $\hat{g}_Z$ that would arise from a change in total innovative investment $\hat{Y}_r$ focused entirely on an individual category of innovative investment $(\hat{x}_{ic}, 0, 0)$ is bounded by

$$\hat{g}_Z \leq (\bar{g}_Z - G(0, \bar{x}_{im}, \bar{x}_e)) \frac{\bar{Y}_r}{\bar{x}_{ic}} \hat{Y}_r$$

and similarly for $\hat{x}_{im}$ and $\hat{x}_e$.

We look to calculate these bounds and the corresponding exact elasticities as functions of observables below.

4.5 Exact Elasticities

Given the functional form of the transition law for TFP in equation (5), we can obtain formulas for these individual elasticities that show the relationship between the exact elasticities and these bounds. Specifically, these elasticities are given by

$$G_{ic}\bar{x}_{ic} = \frac{1}{\rho - 1} \frac{\exp((\rho - 1)\bar{g}_Z) - \exp((\rho - 1)G(0, \bar{x}_{im}, \bar{x}_e))}{\exp((\rho - 1)\bar{g}_Z)} \frac{\bar{\zeta}'(\bar{x}_{ic})\bar{x}_{ic}}{\bar{\zeta}(\bar{x}_{ic}) - \bar{\zeta}(0)} \tag{10}$$

$$G_{im}\bar{x}_{im} = \frac{1}{\rho - 1} \frac{\exp((\rho - 1)\bar{g}_Z) - \exp((\rho - 1)G(\bar{x}_{ic}, 0, \bar{x}_e))}{\exp((\rho - 1)\bar{g}_Z)} \frac{h'(\bar{x}_{im})\bar{x}_{im}}{h(\bar{x}_{im})} \tag{11}$$

$$G_e\bar{x}_e = \frac{1}{\rho-1}\frac{\exp((\rho-1)\bar{g}_Z) - \exp((\rho-1)G(\bar{x}_{ic}, \bar{x}_{im}, 0))}{\exp((\rho-1)\bar{g}_Z)} \tag{12}$$

Thus, our assumptions that $\bar{\zeta}(x)$ and $h(x)$ are concave and that $h(0) = 0$ give us tighter bounds on these elasticities than those in proposition 2. That is, for example, since

$$\frac{1}{\rho-1}\frac{\exp((\rho-1)\bar{g}_Z) - \exp((\rho-1)G(0, \bar{x}_{im}, \bar{x}_e))}{\exp((\rho-1)\bar{g}_Z)} < \bar{g}_Z - G(0, \bar{x}_{im}, \bar{x}_e)$$

for $\rho > 1$, these elasticities lie strictly below the bounds above unless we set $\rho = 1$. As $\rho \to \infty$, these elasticities converge to zero.

4.6 Exact Elasticities and Counterfactual Size of Products

To calculate these exact individual elasticities, we need to identify the concavity of the functions $\bar{\zeta}(x_{ic})$ and $h(x_{im})$. We can bound these individual elasticities, however, by expressing the terms

$$\frac{1}{\rho-1}\frac{\exp((\rho-1)\bar{g}_Z) - \exp((\rho-1)G(0, \bar{x}_{im}, \bar{x}_e))}{\exp((\rho-1)\bar{g}_Z)}$$

etc. in terms of baseline and counterfactual sizes of product categories. We do so in the next proposition.

Define

$$\tilde{S}_{ic}(0) \equiv \frac{(1-\bar{\delta}_M)\bar{\zeta}(0)}{\exp((\rho-1)\bar{g}_Z)} = \frac{\bar{F}_{ic}\exp(\bar{g}_M)}{\exp((\rho-1)\bar{g}_Z)}\bar{\zeta}(0)$$

where the second equality follows from the identification of parameters in data on firm dynamics as described above.

Proposition 3. *We have the following bounds on individual elasticities.*

$$G_{ic}\bar{x}_{ic} \leq \frac{1}{\rho-1}\left(\bar{S}_{ic} - \tilde{S}_{ic}(0)\right) \tag{13}$$

$$G_{im}\bar{x}_{im} \leq \frac{1}{\rho-1}\left(\bar{S}_{im} - \delta\frac{\bar{S}_{ic}}{\bar{F}_{ic}}\bar{F}_{im}\right) \tag{14}$$

$$G_e\bar{x}_e = \frac{1}{\rho-1}\left(\bar{S}_e - \delta\frac{\bar{S}_{ic}}{\bar{F}_{ic}}\bar{F}_e\right) \tag{15}$$

Proof. These bounds follow immediately from equations (10), (11), and (12), the concavity of $h(x)$ and $\bar{\zeta}(x)$, the assumption that $h(0) = 0$, and the definition of $\tilde{S}_{ic}(0)$. Note that equation (15) is an equality because we have a linear entry technology.

□

Note that if we impose the restriction as in GHK that incumbent firms do not suffer regress of z on continuing products even if they do not invest in improving these products (i.e. $\bar{\zeta}(0) = 1$), we then have from (13) the bound that

$$G_{ic}\bar{x}_{ic} \leq \frac{1}{\rho - 1}\left(\bar{S}_{ic} - \frac{\bar{F}_{ic}\exp(\bar{g}_M)}{\exp((\rho-1)\bar{g}_Z)}\right) \tag{16}$$

This bound can be evaluated given the data we assume is available to the economist. Alternatively, as we argue in Corollary 6, if the economist is able to estimate the elasticity $G_{ic}\bar{x}_{ic}$ using other data, one can then use (13) to put an upper bound on the value of $\bar{\zeta}(0)$. We do so in our calibration section below.

The result in equation (15) shows that the only source of uncertainty in identifying the elasticity of TFP growth with respect to entry given the data we have assumed the economist to have is the parameter δ. Likewise, the result in equation (14) allows us to bound the elasticity of TFP growth with respect to investment by incumbents in acquiring new products given knowledge of δ.

If we set δ at its minimum value of 0, so that there is no business stealing as in an Expanding Varieties model, we have

$$G_e\bar{x}_e = \frac{1}{\rho - 1}\bar{S}_e \tag{17}$$

This means that a model with $\delta = 0$ that is calibrated to a baseline growth TFP growth rate of $\bar{g}_Z = .01$, $\rho = 4$ and aggregate size of entering firms of $\bar{S}_e > .03$ implies an elasticity at least as large as 0.01 as long as the other two derivatives are non-negative. Given the bound in proposition 1, a reasonably calibrated Expanding Varieties model must imply that there is some social depreciation of innovation expenditures

Note that the corresponding elasticity of aggregate TFP to a change in innovative investment devoted entirely to investment in entry when $\delta = 0$ is then given by

$$G_e\bar{x}_e\frac{\bar{Y}_r}{\bar{x}_e} = \frac{1}{\rho - 1}\bar{S}_e\frac{\bar{Y}_r}{\bar{x}_e} \tag{18}$$

where $\bar{x}_e/\bar{Y}_r$ is the share of innovative investment undertaken by entering firms. We use data on the value of firms' intangible capital to estimate this share of innovative investment by entering firms below.

In the next section, we turn to using data on the value of firms to identify the levels of investment $\bar{x}_{ic}/\bar{Y}_r$, $\bar{x}_{im}/\bar{Y}_r$, and $\bar{x}_e/\bar{Y}_r$. We then use equilibrium conditions to identify $\bar{\zeta}'(\bar{x}_{ic})$ and $h'(\bar{x}_{im})$, which allows us to fully identify the desired elasticities up to uncertainty over δ.

5 Firm Values and the Equilibrium Levels of Innovative Investment

We now consider the features of the model that can be identified if the economist requires that the model also match data on the allocation of factor payments observed in GDP and data on the value of the intangible capital in firms relative to GDP. We assume throughout that there are no taxes or subsidies in the observed equilibrium. A full calibration exercise would have to take into account the impact of taxes and subsidies on measures of firm value. We intend this section to demonstrate the method for using data on firm value to discipline the model-implied impact elasticities of aggregate TFP growth with respect to changes in innovative investment).

Here we consider the older notion of GDP as equal to the sum of expenditures on consumption and investment in physical capital. This notion of GDP corresponds to

$$C_t + K_{t+1} - (1-\delta_K)K_t = Y_t = Z_t K_t^{\alpha} L_{pt}^{1-\alpha}$$

Standard arguments imply that fraction α/μ of GDP is paid to physical capital as either explicit or imputed rentals, fraction $(1-\alpha)/\mu$ is paid as compensation to production labor L_{pt}. The remaining fraction $(\mu-1)/\mu$ of output is available to owners of firms in the form of variable profits. A portion of these variable profits are spent on firms' innovative investments. The cost of these innovative investment is paid to research labor. Note that to match the model implications for compensation of research labor to data from the National Income and Product Accounts, one must impute compensation of research labor in the form of *sweat equity* as in McGrattan and Prescott (2010).

We assume that the compensation of research labor $W_t L_{rt}$ is equal to the total value of research output $P_{rt}Y_{rt}$ where P_{rt} represents the price of the research good in terms of consumption. Thus, the total compensation of labor is given by $W_t L_{pt} + W_t L_{rt}$. The aggregate flow of variable profits paid to firm owners in excess of aggregate costs of innovative investments is then

$$\left[\frac{\mu-1}{\mu} - S_{rt}\right] Y_t \tag{19}$$

where

$$S_{rt} = \frac{P_{rt}Y_{rt}}{Y_t}$$

denotes the research intensity of the economy.

We now turn to the equations that we use to match data on firm value on a balanced growth path.

5.1 The value of a continuing product for an incumbent firm

We guess that a product of size s at t adds value $V_t s$ to the firm that produces this product at date t. We also guess that in equilibrium, investment by incumbents in improving their own products is directly proportional to product size $s = \exp((\rho-1)z)/Z_t^{\rho-1}$, or $x_{zt}(s) = x_{zt}s$ for all s.

We then have the following recursion for this value of a product in terms of its variable profits to an incumbent

$$V_t s = \frac{\mu-1}{\mu} Y_t s - P_{rt} x_{ict} s + \frac{1}{1+R_t} V_{t+1}(1-\delta_{Mt})\bar{\zeta}(x_{ict}) \frac{Z_t^{\rho-1}}{Z_{t+1}^{\rho-1}} s$$

where R_t denotes the consumption interest rate between t and $t+1$. Using the notation

$$v_t \equiv \frac{V_t}{Y_t}$$

$\exp(g_{Yt+1}) = \frac{Y_{t+1}}{Y_t}$ we have

$$v_t = \frac{\mu-1}{\mu} - \frac{P_{rt} x_{ict}}{Y_t} + \frac{\exp(g_{Yt+1})}{1+R_t} v_{t+1} S_{ict+1} \tag{20}$$

and in a BGP

$$\frac{P_r \bar{x}_{ic}}{Y} = \frac{\mu-1}{\mu} - \left[1 - \frac{\exp(\bar{g}_Y)}{1+\bar{R}} \bar{S}_{ic}\right] \bar{v} \tag{21}$$

When an incumbent firm chooses investment x_{ic} optimally, its choice satisfies the first order condition

$$P_{rt} = \frac{1}{1+R_t} V_{t+1}(1-\delta_{Mt})\bar{\zeta}'(x_{ict}) \frac{Z_t^{\rho-1}}{Z_{t+1}^{\rho-1}}$$

which, in a BGP is equivalent to

$$\frac{P_r \bar{x}_{ic}}{Y} = \frac{\exp(\bar{g}_Y)}{1+\bar{R}} \bar{v} \frac{(1-\bar{\delta}_M)\bar{\zeta}'(\bar{x}_{ic})\bar{x}_{ic}}{\exp((\rho-1)\bar{g}_Z)} \tag{22}$$

5.2 The value of a new product in facilitating further acquisition of products

We guess that each incumbent firm invests a constant amount of the research good per product that it owns at t in acquiring new products for that incumbent firm. Thus, if x_{imt} is the total investment of this type by incumbents, then this investment per products is $x_{imt}(s) = x_{imt}/M_t$ independent of the size s of the product.

We then have the following recursion for the value of a product to an incumbent in terms of the added research capacity it gives to that incumbent

$$U_t = -\frac{P_{rt} x_{imt}}{M_t} + \frac{1}{1+R_t} V_{t+1} h(x_{imt}) \bar{\eta}_i \frac{1}{M_t} \frac{Z_t^{\rho-1}}{Z_{t+1}^{\rho-1}} +$$

$$\frac{1}{1+R_t} \left(h(x_{imt}) + (1-\delta_{Mt}) \right) U_{t+1}$$

Using the notation

$$u_t \equiv \frac{U_t M_t}{Y_t}$$

and multiplying by M_t gives

$$u_t = -\frac{P_{rt} x_{imt}}{Y_t} + \frac{\exp(g_{Yt+1})}{1+R_t} \left[v_{t+1} S_{imt+1} + u_{t+1} F_{it+1} \right]$$

where

$$F_{it+1} \equiv F_{ict+1} + F_{imt+1}$$

is the fraction of products that produced by incumbent firms at $t+1$. These include products that are new to the firm and products that are continuing.

In a BGP, we then have

$$\bar{u} = -\frac{P_r \bar{x}_{im}}{Y} + \frac{\exp(\bar{g}_Y)}{1+\bar{R}} \left[\bar{v} \bar{S}_{im} + \bar{u} \bar{F}_i \right]$$

or

$$\frac{P_r \bar{x}_{im}}{Y} = \frac{\exp(\bar{g}_Y)}{1+\bar{R}} \bar{v} \bar{S}_{im} - \left[1 - \frac{\exp(g_Y)}{1+\bar{R}} \bar{F}_i \right] \bar{u} \tag{23}$$

When an incumbent firm chooses investment x_{im} optimally, its choice satisfies the first order condition

$$P_{rt} = \frac{1}{1+R_t} \left[V_{t+1} \bar{\eta}_i \frac{Z_t^{\rho-1}}{Z_{t+1}^{\rho-1}} + U_{t+1} M_t \right] h'(x_{imt}) \tag{24}$$

In a BGP, this is equivalent to

$$\frac{P_{rt} \bar{x}_{im}}{Y} = \frac{\exp(\bar{g}_Y)}{1+\bar{R}} \left[\bar{v}(\bar{S}_{in} + \bar{S}_{is}) + \bar{u} \frac{M_t h(x_{im})}{M_{t+1}} \right] \frac{h'(\bar{x}_{im}) \bar{x}_{im}}{h(\bar{x}_{im})}$$

or

$$\frac{P_{rt} \bar{x}_{im}}{Y} = \frac{\exp(\bar{g}_Y)}{1+\bar{R}} \left[\bar{v} \bar{S}_{im} + \bar{u} \bar{F}_{im} \right] \frac{h'(\bar{x}_{im}) \bar{x}_{im}}{h(\bar{x}_{im})} \tag{25}$$

5.3 Value of a new firm

Finally, we have free entry condition

$$\frac{P_{rt}}{M_t} = \frac{1}{1+R_t}\left[V_{t+1}\frac{\bar{\eta}_e}{M_t}\frac{Z_t^{\rho-1}}{Z_{t+1}^{\rho-1}} + U_{t+1}\right] \tag{26}$$

Multiplying by $x_{et}M_t$ and dividing by Y_t gives

$$\frac{P_{rt}x_{et}}{Y_t} = \frac{\exp(g_{Yt+1})}{1+R_t}\left[v_{t+1}S_{et+1} + u_{t+1}F_{et+1}\right]$$

In a BGP, this equation becomes

$$\frac{P_r\bar{x}_e}{Y} = \frac{\exp(\bar{g}_Y)}{1+\bar{R}}\left[\bar{v}\bar{S}_e + \bar{u}\bar{F}_e\right] \tag{27}$$

5.4 Aggregate Value of Firms

Summing together equations (21), (23), and (27) gives the standard formula

$$[\bar{v}+\bar{u}] = \left[\frac{\mu-1}{\mu} - S_r\right]\left[1 - \frac{\exp(\bar{g}_Y)}{1+\bar{R}}\right]^{-1} \tag{28}$$

where $\bar{v}+\bar{u}$ is total firm value relative to GDP and, from equation (19), $\frac{\mu-1}{\mu} - S_r$ is the fraction of GDP that is paid as profits in excess of innovative investments to owners of firms. This formula values the aggregate stock of intangible capital of firms in the economy as a perpetuity that grows at the same rate as consumption in the economy.

Under the standard decentralization in which firms manage the stock of physical capital, we then have that the total value of all firms in the economy relative to GDP is given by

$$\frac{K}{Y} + v + u$$

equal to the sum of the values of the tangible and intangible capital within firms.

5.5 Implications of data on firm value for the level of each category of investment

We now consider the model's implications for the division of firm value into components v and u and the division of innovative investment into components

$$S_r = \frac{P_r x_{ic}}{Y} + \frac{P_r x_{im}}{Y} + \frac{P_r x_e}{Y}$$

To do so, assume that one has data on the division of firm intangible value into components $\bar{v}/(\bar{v}+\bar{u})$ and $\bar{u}/(\bar{v}+\bar{u})$. This split of total firm value may be difficult to identify in data. We are able to bound this fraction $\bar{v}/(\bar{v}+\bar{u})$ above and below by the requirement that both $\bar{x}_{ic}$ and $\bar{x}_{im}$ are non-negative.

Proposition 4. *In equilibrium, on a balanced growth path, the fraction of intangible firm value $\bar{v}/(\bar{v}+\bar{u})$ must lie in the interval*

$$\frac{\left[1-\frac{\exp(g_Y)}{1+\bar{R}}\bar{F}_i\right]}{\left[1-\frac{\exp(g_Y)}{1+\bar{R}}(\bar{F}_i-S_{im})\right]} \leq \frac{\bar{v}}{\bar{v}+\bar{u}} \leq \frac{\frac{\mu-1}{\mu}}{\left[\frac{\mu-1}{\mu}-S_r\right]} \frac{\left[1-\frac{\exp(g_Y)}{1+\bar{R}}\right]}{\left[1-\frac{\exp(\bar{g}_Y)}{1+\bar{R}}\bar{S}_{ic}\right]} \tag{29}$$

Proof. If one has such data on the division of firm value, then equation (21) determines $P_r x_{ic}/Y$. To ensure that the implied value of $P_r x_{ic}/Y \geq 0$, we must have

$$0 \leq \frac{\mu-1}{\mu} - \left[1-\frac{\exp(g_Y)}{1+\bar{R}}\bar{S}_{ic}\right]\frac{\bar{v}}{\bar{v}+\bar{u}}(\bar{v}+\bar{u})$$

$$\left[1-\frac{\exp(g_Y)}{1+\bar{R}}\bar{S}_{ic}\right]\frac{\bar{v}}{\bar{v}+\bar{u}}(\bar{v}+\bar{u}) \leq \frac{\mu-1}{\mu}$$

$$\frac{\bar{v}}{\bar{v}+\bar{u}} \leq \frac{1}{\left[1-\frac{\exp(g_Y)}{1+\bar{R}}\bar{S}_{ic}\right]}\frac{\frac{\mu-1}{\mu}}{\bar{v}+\bar{u}}$$

and thus the second inequality above follows from equation (28).

Likewise, given a division of firm value, then equation (23) determines $P_r x_{im}/Y$. To ensure that $P_r x_{im}/Y \geq 0$, we must have

$$0 \leq \frac{\exp(g_Y)}{1+\bar{R}}\bar{S}_{im}\frac{\bar{v}}{\bar{v}+\bar{u}} - \left[1-\frac{\exp(g_Y)}{1+\bar{R}}\bar{F}_i\right]\left(1-\frac{\bar{v}}{\bar{v}+\bar{u}}\right)$$

$$\left[1-\frac{\exp(g_Y)}{1+\bar{R}}\bar{F}_i\right] \leq \left[1-\frac{\exp(g_Y)}{1+\bar{R}}(\bar{F}_i-\bar{S}_{im})\right]\frac{\bar{v}}{\bar{v}+\bar{u}}$$

which implies the first inequality above.

□

This proposition implies that any calibration of total intangible value of firms relative to GDP, $\bar{v}+\bar{u}$, together with a division of that value that lies in the bounds set in proposition 4, we have $P_r\bar{x}_{ic}/\bar{Y}_r$ given from equation (21) and $P_r\bar{x}_{im}/\bar{Y}_r$ given from equation (23). The value of $P_r\bar{x}_e/\bar{Y}_r$ then follows from equation (27) and is immediately non-negative.

5.6 Implications of optimality of investment for the elasticity of TFP growth with respect to changes in investment in improving own products

From equation (6), we see that we must identify the derivative $\bar{\zeta}'(\bar{x}_{ic})$ to identify the elasticities that we desire. We do so using our model's implications for steady-state firm value and the assumption that investment in continuing products is chosen optimally in steady-state given in equations (21) and (22). Specifically, equation (22) identifies the desired derivative up to knowledge of $\bar{v}$ and the scale of this category of innovative investment from the condition that the private return to innovative investment in continuing products is equal to the interest rate. We then use our calibration of the scale of firms' intangible value $\bar{v} + \bar{u}$ and its division between $\bar{v}$ and $\bar{u}$ to determine $\bar{v}$ and the valuation equation (21) to determine the scale of this category of innovative investment.

These arguments give the following proposition.

Proposition 5. *Data on firm value in steady-state together with the condition that innovative investment in continuing products be privately optimal gives*

$$G_{ic}\bar{x}_{ic} = \frac{1}{\rho - 1} \frac{\frac{P_r \bar{x}_{ic}}{Y}}{\frac{\exp(\bar{g}_Y)}{1+\bar{R}} \bar{v}} \tag{30}$$

Thus, the elasticity of aggregate TFP growth with respect to a change in aggregate innovative investment directed entirely at investment by incumbent firms in improving their own products is given by

$$G_{ic}\bar{x}_{ic} \frac{\bar{Y}_r}{\bar{x}_{ic}} = \frac{1}{\rho - 1} \left[\frac{S_r}{\frac{\exp(\bar{g}_Y)}{1+\bar{R}}} \right] \frac{1}{\bar{v}} \tag{31}$$

Proof. Equation (30) follows directly from equations (6) and (22). We have

$$\frac{\bar{Y}_r}{\bar{x}_{ic}} = S_r \frac{Y}{P_r \bar{x}_{ic}}$$

With equation (30), we then have the result (31). □

Note that this procedure allows us to identify this elasticity $G_{ic}\bar{x}_{ic}$ without knowledge of the extent of business stealing. Note as well that we have the following corollary that allows us to use firm value to place an upper bound on the counterfactual value $\bar{\zeta}(0)$, i.e., the model-implied deterioration of productivity for continuing products that would occur in the absence of innovative investment by incumbent firms in those products.

Corollary 6. *The counterfactual growth rate of continuing products in incumbent firms is bounded above by*

$$\tilde{S}_{ic}(0) \leq \bar{S}_{ic} - \frac{\frac{P_r x_{ic}}{Y}}{\frac{\exp(\bar{g}_Y)}{1+\bar{R}}\bar{v}}$$

or equivalently,

$$\bar{\zeta}(0) \leq \frac{\exp((\rho-1)\bar{g}_Z)}{\bar{F}_{ic}\exp(\bar{g}_M)} \left(\bar{S}_{ic} - \frac{\frac{P_r x_{ic}}{Y}}{\frac{\exp(\bar{g}_Y)}{1+\bar{R}}\bar{v}} \right)$$

Proof. This bound follows directly from (13) and equation (30) □

5.7 Elasticities with respect to incumbent and entrant investment in new products

In the event that $\bar{\eta}_i = \bar{\eta}_e$, the first order condition for x_{im} in equation (24) and the free entry condition (26) imply that $h'(\bar{x}_{imt}) = 1$ in equilibrium for all dates t in which both investments are undertaken. This implies that $G_{im} = G_e$ at all dates t in which both investments are undertaken along the equilibrium path. In this case, we have the result that the elasticity of TFP with respect to a change in $\hat{Y}_r$ devoted entirely to either $\hat{x}_{im}$ or $\hat{x}_e$ is the same. That is

$$G_e \bar{x}_e \frac{\bar{Y}_r}{\bar{x}_e} = G_{im}\bar{x}_{im}\frac{\bar{Y}_r}{\bar{x}_{im}}$$

More generally, the first order condition and free entry condition above imply that

$$\left[\frac{V_{t+1}}{Y_{t+1}}\bar{\eta}_i \frac{Z_t^{\rho-1}}{Z_{t+1}^{\rho-1}} + \frac{U_{t+1}}{Y_{t+1}}M_t\right] h'(x_{imt}) = \left[\frac{V_{t+1}}{Y_{t+1}}\bar{\eta}_e \frac{Z_t^{\rho-1}}{Z_{t+1}^{\rho-1}} + \frac{U_{t+1}}{Y_{t+1}}M_t\right]$$

so

$$h'(x_{imt}) = \frac{v_{t+1}S_{et+1}/F_{et+1} + u_{t+1}}{v_{t+1}S_{imt+1}/F_{imt+1} + u_{t+1}}$$

We now compute the elasticities $G_{im}\bar{x}_{im}\frac{\bar{Y}_r}{\bar{x}_{im}}$ and $G_e\bar{x}_e\frac{\bar{Y}_r}{\bar{x}_e}$ in the more general case.

Proposition 7. *Data on firm value and the condition that incumbents' investments in acquiring new products be privately optimal imply that*

$$G_{im}\bar{x}_{im} = \frac{1}{\rho-1}\frac{\frac{P_r x_{im}}{Y}}{\frac{\exp(\bar{g}_Y)}{1+\bar{R}}\bar{v}} \frac{\bar{S}_{im}/\bar{F}_{im} - \delta\bar{S}_{ic}/\bar{F}_{ic}}{\bar{S}_{im}/\bar{F}_{im} + \bar{u}/\bar{v}} \tag{32}$$

The elasticity of aggregate TFP growth with respect to a change in aggregate innovative investment directed entirely at acquisition of new products by incumbent firms is given by

$$G_{im}\bar{x}_{im}\frac{\bar{Y}_r}{\bar{x}_{im}} = \frac{1}{\rho-1}\left[\frac{S_r}{\frac{\exp(\bar{g}_Y)}{1+\bar{R}}}\right]\frac{1}{\bar{v}}\left[\frac{\bar{S}_{im}/\bar{F}_{im} - \delta\bar{S}_{ic}/\bar{F}_{ic}}{\bar{S}_{im}/\bar{F}_{im} + \bar{u}/\bar{v}}\right] \tag{33}$$

Proof. From equation (11) and proposition 3, we have

$$G_{im}\bar{x}_{im} = \frac{1}{\rho - 1}\left(\bar{S}_{im}/\bar{F}_{im} - \delta\bar{S}_{ic}/F_{ic}\right)\frac{h'(\bar{x}_{im})\bar{x}_{im}}{h(\bar{x}_{im})}$$

We obtain an expression for

$$\frac{h'(\bar{x}_{im})\bar{x}_{im}}{h(\bar{x}_{im})}$$

from the first order condition for privately optimal x_{ic} in equation (25). This proves (32). We have

$$\frac{\bar{Y}_r}{\bar{x}_{im}} = S_r\frac{Y}{P_r\bar{x}_{im}}$$

With equation (32), we then have the result (33). □

We also have the following proposition regarding the elasticity of TFP growth with respect to a change in aggregate innovative investment concentrated entirely on entry.

Proposition 8. *The elasticity of aggregate TFP growth with respect to a change in aggregate innovative investment directed entirely at entering firms is given by*

$$G_e\bar{x}_e\frac{\bar{Y}_r}{\bar{x}_e} = \frac{1}{\rho - 1}\left[\frac{S_r}{\frac{\exp(\bar{g}_Y)}{1+\bar{R}}}\right]\frac{1}{\bar{v}}\left[\frac{\bar{S}_e/\bar{F}_e - \delta\bar{S}_{ic}/\bar{F}_{ic}}{\bar{S}_e/\bar{F}_e + \bar{u}/\bar{v}}\right] \tag{34}$$

Proof. We have $G_e\bar{x}_e$ given in equation (15). To compute the desired elasticity, we multiply this expression by $S_r/(P_r\bar{x}_e/Y)$. We have $P_r\bar{x}_e/Y$ given by the free entry condition (27). □

Corollary 9. *These elasticities can be ordered as follows*

$$G_{im}\bar{x}_{im}\frac{\bar{Y}_r}{\bar{x}_{im}} \leq G_{ic}\bar{x}_{ic}\frac{\bar{Y}_r}{\bar{x}_{ic}}$$

$$G_e\bar{x}_e\frac{\bar{Y}_r}{\bar{x}_e} \leq G_{ic}\bar{x}_{ic}\frac{\bar{Y}_r}{\bar{x}_{ic}}$$

In the event that the average size of new products in incumbent and entering firms is the same, as in $\bar{S}_{im}/\bar{F}_{im} = \bar{S}_e/\bar{F}_e$, then

$$G_{im}\bar{x}_{im}\frac{\bar{Y}_r}{\bar{x}_{im}} = G_e\bar{x}_e\frac{\bar{Y}_r}{\bar{x}_e}$$

Proof. The first inequality follows from the observation that

$$\left[\frac{\bar{S}_{im}/\bar{F}_{im} - \delta\bar{S}_{ic}/\bar{F}_{ic}}{\bar{S}_{im}/\bar{F}_{im} + \bar{u}/\bar{v}}\right] \leq 1$$

The second from the observation that

$$\left[\frac{\bar{S}_e/\bar{F}_e - \delta\bar{S}_{ic}/\bar{F}_{ic}}{\bar{S}_e/\bar{F}_e + \bar{u}/\bar{v}}\right] \leq 1$$

The final equality follows directly from the assumption on the equality of the average size of new products in incumbent and entering firms. □

5.8 Dynamics

To be completed.

6 Quantitative analysis

We now turn to a calibration of our model. To reiterate, we make several key assumptions regarding what the economist observes.

We assume that the economist observes the elasticity of demand for intermediate products indexed by ρ, the gap between the interest rate and the growth rate of the economy indexed by $\exp(\bar{g}_Y)/(1+\bar{R})$, the growth rate of the labor force $\bar{g}_L$.

We assume that the economist observes the baseline growth rate of aggregate TFP due to innovative investment by firms indexed by $\bar{g}_Z$ and the baseline growth rate of the number of products $\bar{g}_M$. We also assume that the economist observes product level data on the fraction of products that are continuing products in incumbent firms, $\bar{F}_{ic}$, the fraction of products that are new to incumbent firms, $\bar{F}_{im}$, the fraction of products that are new to entering firms $\bar{F}_e$, and the corresponding sizes of these three product categories $\bar{S}_{ic}$, $\bar{S}_{im}$, and $\bar{S}_e$. As discussed in section 3, these data are critical for identifying δ_M, $h(\bar{x}_{im})$, $\bar{x}_e$, $\bar{\zeta}(\bar{x}_{ic})$, $\bar{\eta}_i$, and $\bar{\eta}_e$.

We assume that the economist observes the overall innovative intensity of the economy S_r, and either the share of profits to intangible capital in GDP indexed by $(\mu-1)/\mu - S_r$ or, equivalently, the value of intangible capital in firms relative to GDP indexed by $v+u$. We also assume that the economist observes the division of that intangible value between $\bar{v}$ and $\bar{u}$. As discussed in section 5, these data are critical for identifying the division of innovative investment $\bar{S}_r$ into the three categories of investment $P_r\bar{x}_{ic}/\bar{Y}_r$, $P_r\bar{x}_{im}/\bar{Y}_r$, and $P_r\bar{x}_e/\bar{Y}_r$ and for identifying the derivatives $\bar{\zeta}'(\bar{x}_{ic})\bar{x}_{ic}/\bar{\zeta}(\bar{x}_{ic})$ and $h'(\bar{x}_{im})\bar{x}_{im}/h(\bar{x}_{im})$.

To implement a calibration of the model, we use data from the Longitudinal Business Database covering the time period 1980-2014. This data set reports on the number and employment of business establishments in the United States. These establishments are matched to the firms that own them. We make the identifying assumption that an establishment in the LBD data corresponds to an intermediate good in the model. We assume that the fraction of new products in entering firms in the model corresponds to the fraction of establishments in the data that are new and are owned by new firms. The fraction of new products in incumbent firms in the model corresponds to the fraction of establishments that are new and are owned by firms that are not new. The fraction of products that are continuing products in incumbent firms in the model corresponds to the fraction of establishments are are not new. We make the same mapping between model and data for product (establishment) size measured as the share of employment in establishments in each of the three categories above. We use averages of the data from 1980-2014 in our calibration.

We are not able to identify two key parameters in our model: the split indexed by δ of new products to firms between those that are new to society and those that are stolen from other firms and the division indexed by $\bar{v}/(\bar{v}+\bar{u})$ of firms' intangible value between components $\bar{v}$ and $\bar{u}$. The fraction δ is bounded below by zero and above as describe in section 3.3. We consider specifications of our model with δ set to its minimum and maximum values. The fraction $\bar{v}/(\bar{v}+\bar{u})$ is bounded above and below as described in proposition 4. In our baseline specification of the model, these bounds impose tight restrictions on this fraction. Hence we consider only and intermediate value of this fraction.

We are also uncertain about the level of the value of firms' intangible capital relative to GDP as indexed by $\bar{v}+\bar{u}$. Given the importance of this quantity in determining the model-implied elasticities of aggregate TFP growth with respect to changes in innovative investment by firms, we consider two alternative calibrations of this quantity in our model. To implement the alternative specification of the model with this lower level of firm intangible value, we leave the innovation intensity of the economy S_r unchanged and reduce the share of variable profits in GDP $(\mu-1)/\mu$.

6.1 Calibration

We set the parameter governing the elasticity of demand for intermediate goods at $\rho = 4$. We set the total employment growth rate at $\bar{g}_L = 0.0136$ and the growth rate of the number of products $\bar{g}_M = 0.0123$ (based on the Longitudinal Business Database described above). We set the baseline growth rate of aggregate productivity due to innovative investment by firms of $\bar{g}_Z = 0.0125$ and the baseline growth of output per capita at $\bar{g}_Y = 0.02$.

We follow McGrattan and Prescott (Forthcoming) in setting the ratio between the growth rate of output and the gross interest rate at $\frac{\exp(g_Y)}{1+R} = 0.98$, and we target $\bar{v}+\bar{u} = 1.7$ based on their estimates of the stock of business intangible capital relative to GNP in the U.S. From equation (28), we obtain the fraction of profits to intangible capital in GDP as $\frac{\mu-1}{\mu} - S_r = 0.034$. We calibrate $S_r = 0.077$ based on Nakamura (2009). This calibration implies a gross markup of $\mu = 1.125$.

We measure S_e, S_{im}, F_e, and F_{im} (and hence $S_{ic} = 1 - S_e - S_{im}$ and $F_{ic} = 1 - F_e - F_{im}$) using the Longitudinal Business Database as described above. Averaging between 1980 and 2014 for all sectors, we obtain $\bar{F}_e = 0.085$, $\bar{F}_{in} = 0.022$, $\bar{F}_{ic} = 0.893$, $\bar{S}_e = 0.030$, $\bar{S}_{in} = 0.026$ and $\bar{S}_{ic} = 0.945$. The average growth rate in the number of establishments is $\bar{g}_M = 0.012$, which we also target in our calibration, while the average growth rate in total employment is given by the target described above.

We consider two values of δ — its minimum value of $\delta = 0$ and its maximum value of $\delta = 1/3$. The first specification corresponds to a model with no business stealing. the second specification corresponds to a model with the maximum amount of business stealing consistent with the observation that new establishments in entering firms have smaller average size than do continuing establishments. Hence, in this second specification, the marginal contribution of entry to aggregate productivity growth is zero.

In our baseline calibration, the fraction $\bar{v}/(\bar{v} + \bar{u})$ is bounded between .80 and .875. We consider .85 as a baseline value for this fraction.

We consider two values for $\bar{v} + \bar{u}$. The first is 1.7 from McGrattan and Prescott (2016) as described above. The second is 0.7. This value used to illustrate the sensitivity of our model's implications to this calibrated quantity.

6.2 Alternative Specifications of business stealing and firm value

We now present results for four specifications of our model. Specification 1 is our specification with no business stealing and a high value of intangible capital ($\delta = 0$ and $\bar{v} + \bar{u} = 1.7$). Specification 2 is our specification with the maximum amount of business stealing and a high value of intangible capital ($\delta = 1/3$ and $\bar{v} + \bar{u} = 1.7$). Specification 3 is our specification with no business stealing and a moderate value of intangible capital ($\delta = 0$ and $\bar{v} + \bar{u} = 0.7$). Specification 4 is our specification with the maximum amount of business stealing and a moderate value of intangible capital ($\delta = 1/3$ and $\bar{v} + \bar{u} = 0.7$). In Specifications 1 and 2, $\frac{\mu-1}{\mu} - S_r = 0.034$. In specifications 3 and 4, μ is reduced so that $\frac{\mu-1}{\mu} - S_r = 0.014$.

In all four specifications of our model, the research intensity of the economy is fixed at $\bar{S}_r = 0.077$. In specifications 1 and 2, the implied composition of these innovative investments implied by equations (21), (23), and (27) is titled heavily toward expenditure on entry with $P_r\bar{x}_{ic}/Y = 0.0033$, $P_r\bar{x}_{im}/Y = 0.0010$, and $P_r\bar{x}_e/Y = 0.063$.

In specifications 3 and 4, these innovative investments implied by equations (21), (23), and (27) are more tilted toward investment in continuing products with $P_r\bar{x}_{ic}/Y = 0.047$, $P_r\bar{x}_{im}/Y = 0.0041$, and $P_r\bar{x}_e/Y = 0.026$.

6.3 Implied elasticities

We present the model implied impact elasticities of aggregate productivity growth with respect to each category of investment in Table 1. We report on the elasticity of aggregate TFP with respect to a change in innovative investment concentrated on entry in equation (34), the elasticity of aggregate TFP with respect to a change in innovative investment concentrated on new product acquisition by incumbent firms in equation (33), and the elasticity of aggregate TFP with respect to a change in innovative investment concentrated on continuing products in incumbent firms in equation (31). We also report on the sum of the elasticities (30), (32), and (15) corresponding to the elasticity of aggregate TFP with respect to a proportional increase in all three categories of innovative investment.

Consider first the results for specification 1 — the specification with no business stealing and a high valuation of firms' intangible capital. Here we see a moderate elasticity of aggregate TFP with respect to a proportional change in all three categories of innovative investment of 0.013. Note that this elasticity is of a similar order of magnitude as our baseline growth rate of aggregate productivity $\bar{g}_Z = 0.0125$ even though, in this specification of the model, we are allowing for social depreciation of innovation expenditures since all exit of products is assumed to be due to exogenous exit. In contrast, in specification 2 — the specification with the maximum amount of business stealing — the elasticity of aggregate TFP with respect to a proportional change in all three categories of innovative investment is much smaller at 0.002. This is due to the fact that at the maximum value of δ, the elasticity of aggregate TFP with respect to entry is equal to zero. This second specification of the model implies that proportional changes in innovative investment by firms would have only a minimal impact of aggregate TFP growth.

In both specifications 1 and 2, the elasticity of aggregate TFP growth with respect to a change in innovative investment by incumbent firms on continuing products is moderately high at 0.018. Hence, a targeted policy that did stimulate innovative investment by incumbent firms on continuing products could have a moderate impact on aggregate TFP growth, particularly in the case of specification 2 if that policy could simultaneously discourage investment in entry.

In specification 2, the bulk of innovative investment is directed at entry but this investment no marginal contribution to aggregate TFP growth.

In specifications 3 and 4, we see the quantitative importance of the value of firms' intangible capital as measured by $\bar{v} + \bar{u}$. In these two specifications, we reduce the assumed value of $\bar{v} + \bar{u}$ from 1.7 to 0.7. At the same time, we leave the division of value $\bar{v}/(\bar{v} + \bar{u})$ unchanged. Hence, from equations (31), (33), and (34), we have that all of the elasticities of aggregate TFP growth with respect to a change in each category of innovative investment is multiplied by 1.7/0.7 relative to the corresponding elasticity in specifications 1 and 2. The impact of this change in assumptions on the elasticity of aggregate TFP growth with respect to a proportional change in all three categories of innovative investment is even larger than this ratio (given the larger share of innovative investment by continuing products in this specification) — rising to 0.039 and 0.028 for specifications 3 and 4 respectively.

These results demonstrate the importance of refining our estimates of the value of intangible capital in firms for assessing the potential social gains from using policy to stimulate further innovative investment.

6.4 Implications for social depreciation of innovation expenditures

Now consider the implications of our calibrated model for the extent of social depreciation of innovation expenditures as defined in equation (9). As indicated in that equation, the rate of social depreciation of innovation expenditures is determined by parameters ρ, δ_0, and $\bar{\zeta}(0)$.

We have fixed $\rho = 4$ in all specifications of our model.

Given a choice of δ in our four specifications of the model, the data imply a value of δ_0 from equation (3). In specifications 1 and 3, in which we assume that $\delta = 0$ so that there is no business stealing, we have $\delta_0 = 0.0966$. In specifications 2 and 4 in which we have the maximum amount of business stealing consistent with the bounds on δ discussed in section 3.3, we have $\delta_0 = 0.067$. Note then that in all specifications of our model, we must assume that there is some exogenous exit of products.

We are not able to identify the parameter $\bar{\zeta}(0)$. We are able to put an upper bound on this quantity as described in corollary 6 (as well as the assumed upper bound of 1). In specifications 1 and 2 of our model, this bound in corollary 6 is higher than one, so we set $\bar{\zeta}(0) = 1$ as in GHK. In specifications 3 and 4 of our model, we set $\bar{\zeta}(0)$ equal to its upper bound of 0.992.

These parameters give the following model-implied rates of social depreciation of innovation

Table 1: Elasticities of aggregate TFP growth

Specification	x_e only	x_{im} only	x_{ic} only	Proportional change
1. $\delta = 0$, $\bar{v} + \bar{u} = 1.7$	0.012	0.016	0.018	0.013
2. $\delta = 0.33$, $\bar{v} + \bar{u}$=1.7	0.000	0.011	0.018	0.002
3. $\delta = 0$, $\bar{v} + \bar{u} = 0.7$	0.029	0.038	0.044	0.039
4. $\delta = 0.33$, $\bar{v} + \bar{u} = 0.7$	0.000	0.027	0.044	0.028

expenditures. In specification 1, $G(0,0,0) = -0.034$, in specification 2, -0.023, in specification 3, -0.037, and in specification 4, -0.026. Hence, in each specification of our model, the data call for the model to allow for moderate social depreciation of innovation expenditures.

Appendix: Concavity of H function

Define

$$H(a) = (1 - \delta_0 - \delta(h(a\bar{x}_{im}) + a\bar{x}_e))\,\zeta(a\bar{x}_{ic}) + \bar{\eta}_i h(a\bar{x}_{im}) + \bar{\eta}_e a\bar{x}_e$$

We now prove that $H''(1) < 0$. We have:

$$H'(a) = \zeta'(a\bar{x}_{ic})(1 - \delta_0 - \delta(h(a\bar{x}_{im}) + a\bar{x}_e))\,\bar{x}_{ic} - \delta(\bar{x}_{im}h'(a\bar{x}_{im}) + \bar{x}_e)\,\zeta(a\bar{x}_{ic}) + \bar{\eta}_i\bar{x}_{im}h'(a\bar{x}_{im}) + \bar{\eta}_e\bar{x}_e$$

and

$$H''(a) = \zeta''(a\bar{x}_{ic})(1 - \delta_0 - \delta(h(a\bar{x}_{im}) + a\bar{x}_e))\,\bar{x}_{ic}^2 - \zeta'(a\bar{x}_{ic})\,\delta(h'(a\bar{x}_{im})\,\bar{x}_{im} + \bar{x}_e)\,\bar{x}_{ic}$$

$$-\delta(\bar{x}_{im}h''(a\bar{x}_{im})\,\bar{x}_{im})\,\zeta(a\bar{x}_{ic}) - \delta(\bar{x}_{im}h'(a\bar{x}_{im}) + \bar{x}_e)\,\zeta'(a\bar{x}_{ic})\,\bar{x}_{ic} + \bar{\eta}_i\bar{x}_{im}h''(a\bar{x}_{im})\,\bar{x}_{im}$$

Evaluated at $a = 1$,

$$H''(1) = \zeta''(\bar{x}_{ic})(1 - \delta_0 - \delta(h(\bar{x}_{im}) + \bar{x}_e))\,\bar{x}_{ic}^2 - \zeta'(\bar{x}_{ic})\,\delta(h'(\bar{x}_{im})\,\bar{x}_{im} + \bar{x}_e)\,\bar{x}_{ic}$$

$$-\delta(\bar{x}_{im}h'(\bar{x}_{im}) + \bar{x}_e)\,\zeta'(\bar{x}_{ic})\,\bar{x}_{ic} + (\bar{\eta}_i - \delta\zeta(\bar{x}_{ic}))\,h''(\bar{x}_{im})\,\bar{x}_{im}^2$$

Given our assumptions that $h'(\bar{x}_{im}) > 0$, $\zeta'(\bar{x}_{ic}) > 0$, $\zeta''(\bar{x}_{ic}) < 0$, $h''(\bar{x}_{im}) < 0$, and $\bar{\eta}_i > \delta\zeta(\bar{x}_{ic})$, all terms are negative so $H''(1) < 0$.

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