# Grounded by Gravity: A Well-Behaved Trade Model with Industry-Level Economies of Scale*

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**Abstract**

This paper presents a multi-industry trade model with industry-level economies of scale that nests a Ricardian model with Marshallian externalities as well as multi-industry versions of Krugman (1980) and Melitz (2003). The behavior of the model depends on two industry-level elasticities: the trade elasticity and the scale elasticity. For the case of two countries, we show that the equilibrium is unique if and only if the product of the trade and scale elasticities is weakly lower than one in all industries. Extensive simulation analysis strongly suggests that this result extends to the case of more than two countries. If the condition for uniqueness is satisfied, then all countries gain from trade, even when the scale elasticity varies across industries. The presence of scale economies tends to lower the gains from trade except if the country specializes in industries with relatively high scale elasticities. On the other hand, scale economies amplify the gains from trade liberalization except if it leads to specialization in industries with relatively low scale elasticities. These and other results are explored at the quantitative level for different values of the scale elasticity.

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## 1. Introduction

The field of international trade has made great strides in recent years by "mapping theory to data" in the new quantitative trade models (or so-called "gravity models"). This has led to important insights into the consequences of globalization. But a fundamental issue has been missing from these models: the role of localized and industry-specific external economies of scale. These externalities played a large role in the world economy at the time of Marshall (1890, 1930), and recent anecdotal and empirical evidence suggests that they play, if anything, an even larger role in the global economy today.[1]

There is good reason for this omission. Early models yielded some discomforting results, including "a bewildering variety of [multiple] equilibria" (Krugman, 1995) so that trade patterns need not conform to comparative advantage, along with the "paradoxical implication that trade motivated by the gains from concentrating production need not benefit the participating countries" (Grossman and Rossi-Hansberg, 2010). At the heart of these "pathologies" seemed to lay the *compatibility assumption* of increasing returns and perfect competition (see Chipman, 1965), namely that firms take productivity as given even though productivity depends on total industry output. This leads to a circularity whereby the scale of an industry affects its productivity, while an industry's productivity affects its scale through the impact on the pattern of comparative advantage and specialization. In the standard analysis, this leads to multiple equilibria.[2]

Grossman and Rossi-Hansberg (2010, henceforth GRH) recently proposed a two-country Ricardian model with national industry-level external economies of scale (or *Marshallian externalities*), which attacks this compatibility assumption head-on. Instead of perfect competition, GRH assume Bertrand competition so that firms in each industry understand the implications of their decisions on industry output and productivity, ensuring that in equilibrium we have the "right" allocation of industries across countries in a similar fashion to that of the constant returns to scale framework of Dornbusch, Fischer and Samuelson (1977). While the framework successfully eliminates the "pathologies" in a world free of trade costs, Lyn and Rodríguez-Clare (2013a,b)

[1]See Krugman (2011) for a nice exposition of recent anecdotal evidence. For empirical evidence see, for instance, Caballero and Lyons (1989, 1990, 1992), Chan, Chen and Cheung (1995), Segoura (1998), and Henriksen, Steen and Ulltveit-Moe (2001).

[2]See early work exploring this by Graham (1923), Ohlin (1933), Matthews (1949), Kemp (1964), Melvin (1969), Markusen and Melvin (1981), and Ethier (1982a,b).

illustrate circumstances under which multiple equilibria arise in the presence of trade costs. Coupled with the fact that the equilibrium has mixed strategies for some levels of trade costs, the framework quickly becomes intractable, with little hope of extending it to a multi-country setting with trade frictions.

In this paper we present a Ricardian model with Marshallian externalities that admits a unique equilibrium under intuitive parameter restrictions. Unlike GRH, we leave the compatibility assumption intact and approach the problem from a different angle by relaxing the implicit assumption in the standard framework (and in GRH) that firms within each industry are producing a homogeneous good. In particular, we allow for intra-industry heterogeneity as in Eaton and Kortum (2002, henceforth EK) and find that this adds some "curvature" that helps in establishing uniqueness of equilibrium as long as the strength of Marshallian externalities is not too high. The framework yields the standard gravity-type equation and so provides a platform to assess quantitatively the importance of these externalities for the welfare effects of trade.[3]

The system of equations that characterizes the equilibrium of the Ricardian multi-industry model with Marshallian externalities turns out to be isomorphic to the equilibrium system of a more general version of the multi-industry Krugman (1980) model of product differentiation with internal economies of scale.[4] The existence and uniqueness result that we prove for our Ricardian setting can then be seamlessly applied to the multi-industry Krugman model. As far as we know, we are the first to establish uniqueness of equilibrium for this general case. Not surprisingly, the isomorphism extends also to the multi-industry Melitz (2003) model if the productivity distribution is Pareto as in Chaney (2008) and the fixed exporting costs are paid in units of labor of the destination country.

The common mathematical structure that characterizes the equilibrium in all these multi-industry gravity models is governed by two elasticities that can vary across industries: the elasticity of bilateral trade flows to bilateral trade costs, commonly referred to as the *trade elasticity*; and the elasticity of productivity with respect to industry size,

[3]Our analysis restricts to the case of Marshallian externalities, which operate inside each industry. An alternative case is the one in which some of the externalities operate across industries. Yatsynovich (2014) has recently shown conditions under which a model with such cross-industry externalities exhibits a unique equilibrium for the case with frictionless trade.

[4]Abdel-Rahman and Fujita (1990), Allen *et al.* (2015) and Redding (2016) explore similar isomorphisms for spatial equilibrium models in the economic geography literature.

which we will refer to as the *scale elasticity*. The condition for uniqueness is that (in all industries) the product of these two elasticities is not higher than one.[5] If this condition is violated, then there exist multiple equilibria, so the condition is both necessary and sufficient.

In the Ricardian model the scale elasticity is given directly by the strength of Marshallian externalities, so the condition for uniqueness is that the strength of these externalities is not higher than the inverse of the trade elasticity. In the Krugman or Melitz-Pareto models the scale elasticity is given by the inverse of the trade elasticity, hence we are always at the edge of the region of uniqueness. One can easily add flexibility to the Krugman and Melitz-Pareto models to break the tight link between the two elasticities. For example, if we allow the elasticity of substitution across varieties from different countries to differ from the elasticity of substitution across varieties from the same country (with nested CES preferences) then the product of the scale and trade elasticities can be different than one.

When formulating the equilibrium conditions in our model, we explicitly allow for corner equilibria in which industries shut down in some countries. We show that if the product of trade and scale elasticities is less than one then every country is active in all industries, while if the product is one then the equilibrium may exhibit corners. In particular, as is known in the literature, the multi-industry Krugman model can have countries completely specialized in some subset of industries as an equilibrium outcome. Remarkably, however, the existing literature lacks a proof of uniqueness of equilibria in the multi-industry Krugman model while appropriately dealing with the complementary slackness conditions relevant for this case.

In this paper we show existence and uniqueness of equilibrium in a setting with multiple industries and two countries while allowing for complete specialization and for endogenous wages. Our existence result is valid for more than two countries, while we have only been able to extend the uniqueness result to more than two countries under frictionless trade or with exogeneous wages. The theoretical difficulty lies in the fact that the excess labor demand system *does not* satisfy the gross substitutes property —

[5]Strictly speaking, uniqueness also requires that the matrix of bilateral trade freeness parameters be invertible. This is a mild condition that is needed only when the product of the trade elasticity and the scale elasticity is exactly one, and its violation leads to multiplicity in allocations but — as far as we know — not in real wages.

a sufficient condition for unique wages that is often satisfied in similar environments. We have explored alternative approaches to prove equilibrium uniqueness. One is the Index Theorem, which requires showing that the determinant of the negative of the Jacobian of the excess labor demand system is positive at any equilibrium wage vector (see Kehoe, 1980). Unfortunately, signing this determinant is extremely difficult. Another approach is the one proposed by Allen *et al.* (2014), but it turns out that their sufficient condition for uniqueness is violated whenever our multi-sector gravity model exhibits economies of scale.

We have done extensive numerical simulations to see whether there are cases in which multiplicity of equilibria emerge in economies where the product of the trade and scale elasticities is weakly lower than one in all sectors. We checked more than 1.5 million random parameterizations of our economy with three or four countries and two sectors and did not find any examples of multiplicity. We have also developed an algorithm based on interval arithmetic methods that finds all equilibria for our economy. As this algorithm is much slower, we have only been able to run it for about 2 thousand economies, again finding a unique equilibrium in all cases.

In the second half of the paper we use our unified framework to study the implications of scale economies for the welfare effects of trade. We first establish that all countries gain from trade as long as the product of the trade and scale elasticities is weakly lower than one in all industries. This is so even if the scale elasticity differs across industries — for example, because of cross-industry variation in the strength of Marshallian externalities in the Ricardian model. This is an important finding in light of previous results with this type of model where countries could lose from trade.

We extend the "sufficient statistics approach" to the quantification of the gains from trade in Arkolakis, Costinot and Rodríguez-Clare (2012) to multi-industry models with scale economies. The isomorphism across models still applies in this setting in the sense that, for the same industry-level trade and scale elasticities, the different models we consider deliver the same gains from trade and the same counterfactual implications given industry-level data.[6] Given trade and scale elasticities, we now have a general formula for the gains from trade in terms of industry-level trade, expenditure and

[6]As in Arkolakis *et al.* (2012), the gains from trade are defined as the negative of the welfare change caused by a move back to autarky.

revenue shares. This formula can be used to explore the way in which scale economies affect the gains from trade.

For the simple case in which the scale elasticity is the same across industries, we show that the gains from trade are *lower* with scale economies than without. This result may seem counterintuitive, but the reader should keep in mind that the gains from trade are defined as *conditional on trade shares*, so that we can compare the gains from trade implied by different models that are consistent with the same data (Costinot and Rodríguez-Clare, 2014). Thus, the intuition that scale economies should lead to larger gains from trade through deeper industry-level specialization and larger trade flows is simply not operative here, although as we explain below this intuition is relevant for the *gains from trade liberalization*.

So why do scale economies lead to lower gains from trade? The move back to autarky implies a reallocation of labor across industries that, in the presence of scale economies, leads to productivity gains in expanding industries and productivity losses in contracting industries. Since the industries that expand are those where the country has positive net imports, it must be that they have a high expenditure share or a low employment share. A higher expenditure share implies that a given productivity gain matters more for welfare, whereas a low employment share implies that a given absolute increase in employment leads to a higher proportional expansion and a higher productivity gain. As a result, a move back to autarky generates a positive expenditure-weighted average productivity change, and this implies lower welfare losses. A corollary of this reasoning is that the decline in the gains from trade from the presence of economies of scale is stronger for economies that exhibit a higher degree of industry-level specialization.

The previous results are specific to the case in which the scale elasticity is the same across industries. If the scale elasticity varies across industries, the implication of economies of scale depends not only on the *degree* of industry-level specialization, but also on its *pattern*. Everything else equal, countries that happen to specialize in industries with high scale elasticities will gain more than countries that specialize in sectors with low scale elasticities. Countries that specialize in industries with high scale elasticities may even gain more from trade in the presence of economies of scale compared to the standard model without them.

Computing the gains from trade is made easy by the fact that one can write closed-form solutions for labor allocations in the autarky counterfactual equilibrium. In contrast, computing the gains from trade liberalization (i.e., the welfare effect from a decline in trade costs) generally requires solving the model numerically to characterize changes in industry-level trade and revenue shares. We consider two simple cases for which we can do so analytically: first, a case of two mirror-image countries, and, second, a case with exogenous wages. In the first case we find that the gains from trade liberalization are higher with scale economies than without, a reflection of the magnified response of industry-level specialization and trade to the decline in trade costs in the presence of economies of scale. In the second case we find that countries lose from unilateral trade liberalization and from a foreign technological improvement if the product of the trade and scale elasticities is above a threshold value that is a function of industry-level import and export shares. We think of this as a generalization of the results in Venables (1987), which were specific to the Krugman (1980) model.

We complement our exploration of the effect of scale economies on the gains from trade and the gains from trade liberalization by applying our framework to data on 31 industries from the World Input Output Database (WIOD, Timmer *et al.*, 2015) in 2008. We start by focusing on the case with common trade and scale elasticities across industries. As explained above, the presence of scale economies leads to a decline in the gains from trade, and this decline is more pronounced in countries that have a higher degree of specialization across industries. Thus, for example, for the country with the highest degree of industry specialization, Korea, the gains from trade decrease from 6.6% to 4.1%, while they barely decrease for the country with the lowest degree of industry specialization, Brazil.

We then study how the gains from trade are affected by scale economies when the scale elasticity varies across industries. We consider two possibilities. The first is that scale economies are present only in manufacturing industries — a typical case considered in the literature (see, for instance, Ethier, 1982a,b). Relative to the case with no scale economies, gains from trade increase for countries that specialize in manufacturing and the opposite happens for countries that specialize away from manufacturing. For example, gains from trade increase from 3 to 3.5% for China, while they decrease from 5.7 to 3.5% for Greece. The second possibility we consider is that scale elasticities are inversely proportional to trade elasticities (as in the standard multi-industry Krugman or Melitz-Pareto models), with trade elasticities varying across industries and calibrated to those estimated by Caliendo and Parro (2015).[7] In this case, countries that happen to specialize in industries with high trade elasticities (and hence lower implied scale elasticities) experience bigger declines in gains from trade, while the opposite happens for countries specializing in industries with low trade elasticities. The effects are sizable — for example, moving from a model without to a model with scale economies leads to a decline in the gains from trade in Greece from 14.5 to 5.5% but an increase in the gains from trade in Japan from 2.4 to 6.1%.

We next use the model to quantify the welfare implications of unilateral trade liberalization and foreign productivity gains in an environment with economies of scale, comparing the results to those in an environment without economies of scale. To link this exercise to the theoretical analysis inspired by Venables (1987), we assume again that the manufacturing sector exhibits scale economies while all other sectors do not. We find that gains from unilateral trade liberalization in manufacturing decrease as we allow for scale economies in that sector, but the gains are always positive. We show that this arises because of wage adjustments that are ruled out in the Venables-type analysis. On the other hand, we find that most countries experience losses from an improvement in Chinese manufacturing productivity.

Our final quantitative exercise is to explore the role of economies of scale in explaining trade flows and industry-level specialization in the data. We find that if scale economies are as strong as those in the Krugman model then most of the industry-level specialization that we see in the data is due to economies of scale rather than pure Ricardian comparative advantage.

There are two papers that study the question of uniqueness of equilibrium in the multi-industry Krugman model: Hanson and Xiang (2004) consider the case of two countries and a continuum of industries while Behrens *et al.* (2009) consider the case of many countries and one industry with exogenous wages. They both show uniqueness under the assumption that there are no corner allocations. We extend their result to a more general environment, emphasize the key condition that the product of the

[7] This corresponds to the case considered by Costinot and Rodríguez-Clare (2014), see column 3 in their Table 4.1.

trade and scale elasticities is weakly lower than one in all industries, allow for corner allocations, and provide extensive simulation analysis that strongly suggests that the uniqueness result extends to the case with more than two countries.

Our welfare analysis is related to that in Costinot and Rodríguez-Clare (2014), who compute gains from trade and gains from trade liberalization for multi-industry economies under perfect and monopolistic competition. Compared to that paper, we further establish analytically that all countries gain from trade as long as the product of the trade and scale elasticities is weakly lower than one in all industries, we connect a country's decline in the gains from trade to its degree of industry specialization, we analyze how varying scale elasticities across industries interact with a country's inter-industry trade pattern to affect its gains from trade, and we connect the results for the gains from trade liberalization to the insights in Venables (1987).

Finally, our paper is also related to Somale (2014), who introduces sector-specific innovation into a multi-sector Eaton and Kortum (2002) model (via mechanisms from Eaton and Kortum, 2001) to quantify its implications for welfare. Interestingly, although the model in Somale (2014) is dynamic, the balanced growth path is also characterized by the same system of equations as all the models that we consider in this paper, and so our results extend to this case as well. Somale (2014) also studies the quantitative importance of scale economies in determining industry-level specialization, but whereas he focuses on the variance of comparative advantage, we compare direct measures of trade and specialization between the data and those that would arise in a counterfactual world where everything is the same except that there are no economies of scale.

## 2. A Multi-Industry Gravity Model with Scale Economies

We first present the key equilibrium equations of the model and then discuss how these equations arise in three different settings: (i) our multi-industry Ricardian model with Marshallian externalities; (ii) the multi-industry Krugman (1980) model with possibly different elasticities of substitution between varieties from the same and different countries; and (iii) the multi-industry Melitz (2003) model with Pareto-distributed productivity as in Chaney (2008) and also allowing for different elasticities of substitution between varieties from the same and different countries.

There are $N$ countries indexed by $n$, $i$ and $l$, and $K$ industries or sectors indexed by $k$. The only factor of production is labor, which is immobile across countries and perfectly mobile across industries within a country. We use $\bar{L}_i$ and $w_i$ to denote the inelastic labor supply and the wage level in country $i$, respectively. Each country has a representative consumer with upper-tier Cobb-Douglas preferences with industry-level expenditure shares $\beta_{i,k} \in (0,1)$ for all $(i,k)$ with $\sum_{k=1}^{K} \beta_{i,k} = 1$ for all $i$. Trade costs are of the standard iceberg type, so that delivering a unit of any industry-$k$-good from country $i$ to country $n$ requires shipping $\tau_{ni,k} \geq 1$ units of the good, with $\tau_{ii,k} = 1$ for all $i$ and all $k$ and $\tau_{nl,k} \leq \tau_{ni,k}\tau_{il,k}$ for all $n$, $l$, $i$ and $k$.

Let $X_{n,k}$ denote country-$n$'s total expenditure on industry $k$ and let $\lambda_{ni,k}$ denote the share of this expenditure devoted to imports from country $i$. Balanced trade implies $X_{n,k} = \beta_{n,k} w_n \bar{L}_n$.

We focus on models that generate industry-level economies of scale and a log-linear gravity equation for industry-level trade shares. Below we show that our Ricardian model with industry-level *external economies of scale* as well as Krugman (1980) and Melitz (2003) satisfy this criteria. Economies of scale are captured by an industry-level productivity shifter that can vary with total industry employment according to $\widetilde{S}_{i,k} L_{i,k}^{\psi_k}$, where $\widetilde{S}_{i,k}$ is a constant, $L_{i,k}$ denotes total employment in industry $(i,k)$, and $\psi_k$ is the *scale elasticity* in industry $k$, which is assumed to be common across countries. Industry-level trade shares are given by

$$\lambda_{ni,k} = \frac{\left(w_i \tau_{ni,k} / \widetilde{S}_{i,k} L_{i,k}^{\psi_k}\right)^{-\varepsilon_k}}{\sum_l \left(w_l \tau_{nl,k} / \widetilde{S}_{l,k} L_{l,k}^{\psi_k}\right)^{-\varepsilon_k}},$$

where $\varepsilon_k$ is the *trade elasticity* in industry $k$, defined formally by $\varepsilon_k \equiv -\frac{\partial \ln(\lambda_{ni,k}/\lambda_{nn,k})}{\partial \ln \tau_{ni,k}}$. Letting $\alpha_k \equiv \varepsilon_k \psi_k$ and $S_{i,k} \equiv \widetilde{S}_{i,k}^{\varepsilon_k}$, we rewrite trade shares more conveniently as

$$\lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{L}_k) = \frac{S_{i,k} L_{i,k}^{\alpha_k} \left(w_i \tau_{ni,k}\right)^{-\varepsilon_k}}{\sum_l S_{l,k} L_{l,k}^{\alpha_k} \left(w_l \tau_{nl,k}\right)^{-\varepsilon_k}}, \tag{1}$$

where $\boldsymbol{w} \equiv (w_1, ..., w_N)$ is the vector of wages and $\boldsymbol{L}_k \equiv (L_{1,k}, \ldots, L_{N,k})$ is the vector of labor allocations to industry $k$ across all countries. In turn, the price index for industry

$k$ in country $n$ is

$$P_{n,k} = \mu_{n,k} \left( \sum_{l} S_{l,k} L_{l,k}^{\alpha_k} \left( w_l \tau_{nl,k} \right)^{-\varepsilon_k} \right)^{-1/\varepsilon_k}, \tag{2}$$

and the aggregate price index is $P_n = \widetilde{\beta}_n \prod_{k=1}^{K} P_{n,k}^{\beta_{n,k}}$, where $\mu_{n,k}$ and $\widetilde{\beta}_n$ are some constants.[8]

We now introduce industry and labor market clearing conditions. In contrast to multi-industry gravity models without scale economies (e.g., Donaldson (2016), Costinot *et al.* (2012)), here we can have equilibria with corner allocations (i.e., $L_{i,k} = 0$ for some $k$ and for some, but not all, $i$), so we need to be careful when formulating the market clearing conditions. With this in mind, we specify the market clearing condition for any industry $(i, k)$ as a set of *complementary slackness conditions*,

$$L_{i,k} \geq 0, \quad G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right) \geq 0, \quad L_{i,k} G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right) = 0, \tag{3}$$

where

$$G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right) \equiv w_i - \frac{1}{L_{i,k}} \sum_{n} \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{L}_k) \beta_{n,k} w_n \bar{L}_n \tag{4}$$

is the excess of the wage over revenue per worker in industry $(i, k)$. Note that for positive labor allocations equation (3) implies $G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right) = 0$, which can be reformulated as $w_i L_{i,k} = \sum_n \lambda_{ni,k} \beta_{n,k} w_n \bar{L}_n$, a standard industry clearing condition.[9]

Finally, the labor-market clearing condition for any country $i$ is simply

$$\sum_{k} L_{i,k} = \bar{L}_i. \tag{5}$$

Denote by $\boldsymbol{L} \equiv (\boldsymbol{L}_1, \ldots, \boldsymbol{L}_k)$ the vector of labor allocations across industries. The equilibrium of the economy is a wage vector and labor allocation $(\boldsymbol{w}, \boldsymbol{L}) \in \mathbb{R}_{++}^{N} \times$

[8]The constant $\mu_{n,k}$ will be specified below for each model, while $\widetilde{\beta}_n$ is the standard Cobb-Douglas term $\widetilde{\beta}_n \equiv \prod_k \beta_{n,k}^{-\beta_{n,k}}$.

[9]A subtle issue arises here with the evaluation of $G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right)$ and $L_{i,k} G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right)$ at points with $L_{i,k} = 0$. If we think of the codomain of these functions as the set of real numbers then $\boldsymbol{L}_k$ with $L_{i,k} = 0$ (for at least some, but not all $i$) is not in their domain. To avoid this, we define the codomain as the extended real number line $\mathbb{R} \cup \{-\infty, +\infty\}$ and we define $G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right)$ and $L_{i,k} G_{i,k} \left( \boldsymbol{w}, \boldsymbol{L}_k \right)$ for $L_{i,k} = 0$ by $\lim_{\boldsymbol{x} \to \boldsymbol{L}_k} \left[ w_i - \frac{1}{x_i} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}) \beta_{n,k} w_n \bar{L}_n \right]$ and $\lim_{\boldsymbol{x} \to \boldsymbol{L}_k} x_i \left[ w_i - \frac{1}{x_i} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}) \beta_{n,k} w_n \bar{L}_n \right]$, respectively. (Of course, for any point with $L_{i,k} > 0$ these alternative definitions are perfectly consistent with the ones in the text.) For each $k$, we still leave the point $\boldsymbol{L}_k$ with $L_{i,k} = 0$ for all $i$ outside the domain. The formal definitions are in Appendix A.

$\left(\mathbb{R}_+^{NK} \setminus \mathbb{Z}_0^{NK}\right)$ such that (3) holds for all $(i,k)$ and (5) holds for all $i$, where

$$\mathbb{Z}_0^{NK} \equiv \left\{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K) \in \mathbb{R}_+^{NK} : \boldsymbol{x}_k = \mathbf{0} \text{ for some } k\right\}$$

is the set of labor allocations with zero total labor (across countries) devoted to some industries.

## 2.1. A Ricardian Model with Marshallian Externalities

We now show how the multi-industry Eaton and Kortum (2002, henceforth EK) model as developed by Costinot, Donaldson and Komunjer (2012, henceforth CDK), but extended to allow for Marshallian externalities leads to the equilibrium conditions presented above.

Each industry is composed of a continuum of goods $\omega \in [0,1]$. Preferences are Cobb-Douglas across industries with weights $\beta_{i,k}$, and CES across goods within each industry $k$ with elasticity of substitution $\sigma_k$.

The production technology exhibits constant or increasing returns to scale due to national external economies of scale at the industry level (i.e., Marshallian externalities). In particular, labor productivity for good $\omega$ in industry $(i,k)$ is $z_{i,k}(\omega) L_{i,k}^{\phi_k}$, where $z_{i,k}(\omega)$ is an exogenous productivity parameter, $L_{i,k}$ is the total labor allocated to industry $(i,k)$, and $\phi_k$ is the industry specific parameter that governs the strength of Marshallian externalities. We model $z_{i,k}(\omega)$ as in EK: $z_{i,k}(\omega)$ is independently drawn from a Fréchet distribution with shape parameter $\theta_k$ and scale parameter $T_{i,k}$, and we assume that $\theta_k > \sigma_k - 1$.

There is perfect competition, and the positive effect of industry size on productivity, $L_{i,k}^{\phi_k}$, is external to the firm. Thus, firms take as given both prices and unit costs, which are given by $c_{ni,k}(\omega) = \frac{\tau_{ni,k} w_i}{z_{i,k}(\omega) L_{i,k}^{\phi_k}}$. This implies that $p_{ni,k}(\omega) = c_{ni,k}(\omega)$. Since consumers can shop for the best deal around the world, prices must satisfy $p_{n,k}(\omega) = \min_{1 \le i \le N} \left\{p_{ni,k}(\omega)\right\}$. Following the same procedure as in EK, trade shares can be shown to satisfy

$$\lambda_{ni,k} = \frac{T_{i,k} L_{i,k}^{\theta_k \phi_k} \left(w_i \tau_{ni,k}\right)^{-\theta_k}}{\sum_l T_{l,k} L_{l,k}^{\theta_k \phi_k} \left(w_l \tau_{nl,k}\right)^{-\theta_k}}$$

with price indices given by

$$P_{n,k} = \mu_k^{Ric} \left( \sum_l T_{l,k} L_{l,k}^{\theta_k \phi_k} \left( w_l \tau_{nl,k} \right)^{-\theta_k} \right)^{-1/\theta_k},$$

where $\mu_k^{Ric} \equiv \Gamma\left(\frac{1-\sigma_k+\theta_k}{\theta_k}\right)^{\frac{1}{1-\sigma_k}}$, with $\Gamma$ being the Gamma function which typically arises in this Ricardian setting. These two equations collapse to the expressions for trade shares and industry price indexes in equations (1) and (2) by setting with $S_{i,k} = T_{i,k}$, $\varepsilon_k = \theta_k$, $\psi_k = \phi_k$ and $\mu_{n,k} = \mu_k^{Ric}$. See the first row of Table 1.

Finally, the equilibrium condition (3) can be seen as capturing the standard complementary slackness condition in the Ricardian model requiring the price to be weakly lower than the unit cost, with equality if there is positive production in the industry. Multiplying both the price and the unit cost by labor productivity (adjusted by trade costs), this is the same as requiring that revenue per worker be weakly lower than the wage, with equality if there is positive employment in the industry.

Table 1: Mapping to Different Models

| Model | Trade elasticity, $\varepsilon_k$ | Scale elasticity, $\psi_k$ | $\alpha_k$ |
|---|---|---|---|
| CDK with ME | $\theta_k$ | $\phi_k$ | $\theta_k\phi_k$ |
| Multi-Sector Krugman | $\sigma_k - 1$ | $\frac{1}{\sigma_k-1}$ | 1 |
| Multi-Sector Melitz-Pareto Model | $\theta_k$ | $\frac{1}{\theta_k}$ | 1 |
| Generalized Multi-Sector Krugman | $\eta_k - 1$ | $\frac{1}{\sigma_k-1}$ | $\frac{\eta_k-1}{\sigma_k-1}$ |
| Generalized Multi-Sector Melitz-Pareto | $\frac{\theta_k}{1+\theta_k\left(\frac{1}{\eta_k-1}-\frac{1}{\sigma_k-1}\right)}$ | $\frac{1}{\theta_k}$ | $\frac{1}{1+\theta_k\left(\frac{1}{\eta_k-1}-\frac{1}{\sigma_k-1}\right)}$ |

## 2.2. A Krugman Model with Two-Tier CES preferences

Here we present a multi-industry Krugman model with an added layer of product differentiation so that the elasticity of substitution across varieties from different countries is allowed to differ from the elasticity of substitution across varieties from the same country (with nested CES preferences). We again show that this model leads to the equilibrium conditions in equations (3) and (5).

There is a continuum of differentiated varieties within each industry. Preferences are multi-tiered: Cobb-Douglas across industries with weights $\beta_{i,k}$, CES across country bundles within an industry with elasticity $\eta_k$, and CES across varieties within a country bundle with elasticity of substitution $\sigma_k > 1$.

Let $A_{i,k}$ be the exogenous productivity in $(i,k)$ which is common across firms in that industry, let $F_{i,k}$ denote the fixed cost (in terms of labor) associated with the production of any variety in $(i,k)$, and let $M_{i,k}$ the measure of goods produced in $(i,k)$. There is monopolistic competition and trade shares are $\lambda_{ni,k} = (P_{ni,k}/P_{n,k})^{1-\eta_k}$, where $P_{ni,k} = M_{i,k}^{1/(1-\sigma_k)}\left(\bar{\sigma}_k w_i \tau_{ni}/A_{i,k}\right)$ is the price index in country $n$ of country $i$ varieties of industry $k$, $\bar{\sigma}_k \equiv \sigma_k/\left(\sigma_k-1\right)$ is the mark-up, and $P_{n,k} = \left(\sum_i P_{ni,k}^{1-\eta_k}\right)^{1/(1-\eta_k)}$.

We now solve for equilibrium variety $M_{i,k}$ as a function of industry employment $L_{i,k}$ and then use the result to derive an expression for trade shares for this model. Variable profits in $(i,k)$ are simply total industry revenues divided by $\sigma_k$. Letting $\Pi_{i,k}$ be total profits net of fixed costs in industry $(i,k)$, we then have $\Pi_{i,k} = \sum_n \lambda_{ni,k}X_{n,k}/\sigma_k - w_i M_{i,k}F_{i,k}$. If $L_{i,k} > 0$ then free entry implies zero profits so total revenues must equal total wage payments in industry $(i,k)$, $\sum_n \lambda_{ni,k}X_{n,k} = w_i L_{i,k}$. Combined with $\Pi_{i,k} = 0$ we then have $M_{i,k} = L_{i,k}/\sigma_k F_{i,k}$. Trade shares are then

$$\lambda_{ni,k} = \frac{A_{i,k}^{\eta_k-1} F_{i,k}^{-\frac{\eta_k-1}{\sigma_k-1}} L_{i,k}^{\frac{\eta_k-1}{\sigma_k-1}} \left(w_i \tau_{ni,k}\right)^{-(\eta_k-1)}}{\sum_l A_{l,k}^{\eta_k-1} F_{l,k}^{-\frac{\eta_k-1}{\sigma_k-1}} L_{l,k}^{\frac{\eta_k-1}{\sigma_k-1}} \left(w_l \tau_{nl,k}\right)^{-(\eta_k-1)}}$$

with price indices given by

$$P_{n,k} = \mu_k^{Krug}\left(\sum_l A_{l,k}^{\eta_k-1} F_{l,k}^{-\frac{\eta_k-1}{\sigma_k-1}} L_{l,k}^{\frac{\eta_k-1}{\sigma_k-1}} \left(w_l \tau_{nl,k}\right)^{-(\eta_k-1)}\right)^{-1/(\eta_k-1)},$$

where $\mu_k^{Krug} = \sigma_k^{\frac{1}{\sigma_k-1}}\bar{\sigma}_k$. These two equations collapse to the expressions for trade shares and industry price indexes in equations (1) and (2) by setting $S_{i,k} = A_{i,k}^{\eta_k-1}F_{i,k}^{-\frac{\eta_k-1}{\sigma_k-1}}$, $\psi_k = (\sigma_k-1)^{-1}$, $\varepsilon_k = (\eta_k-1)$ and $\mu_{n,k} = \mu_k^{Krug}$. Note also that if we set $\sigma_k = \eta_k$ for all $k$ then this is just the standard multi-industry Krugman model, while if $\sigma_k \to \infty$, then $(\eta_k-1)/(\sigma_k-1) \to 0$ and we obtain the multi-industry Armington model. See rows 2 and 4 of Table 1.[10]

To deal with the possibility of corner labor allocations under monopolistic competition, we require that profits per firm in industry $(i, k)$ be weakly lower than zero, with strict equality if $L_{i,k} > 0$, exactly as captured by the complementary slackness conditions in (3).

### 2.3. A Melitz-Pareto Model with Two-Tier Preferences

We now briefly present a model à la Melitz (2003) with Pareto distributed productivity and the same preferences as in the Krugman model above and show that it leads to the same equilibrium conditions (3) and (5).[11]

After paying a fixed "entry" cost $F_{i,k}$ in units of labor in country $i$, firms are able to produce a variety in industry $(i, k)$ with labor productivity drawn from a Pareto distribution with shape parameter $\theta_k > \sigma_k - 1$ and location parameter $b_{i,k}$. Firms from $i$ can then pay a fixed "marketing" cost $f_{n,k}$ in units of labor of $n$ to serve that market.[12,13] In Appendix B we show that this leads to trade shares

$$\lambda_{ni,k} = \frac{b_{i,k}^{\theta_k \xi_k} F_{i,k}^{-\xi_k} L_{i,k}^{\xi_k} \left(w_i \tau_{ni,k}\right)^{-\theta_k \xi_k}}{\sum_l b_{l,k}^{\theta_k \xi_k} F_{l,k}^{-\xi_k} L_{l,k}^{\xi_k} \left(w_l \tau_{nl,k}\right)^{-\theta_k \xi_k}}$$

and price indices

$$P_{n,k} = \mu_{n,k}^{Mel} \left( \sum_l b_{l,k}^{\theta_k \xi_k} F_{l,k}^{-\xi_k} L_{l,k}^{\xi_k} \left(w_l \tau_{nl,k}\right)^{-\theta_k \xi_k} \right)^{-1/\theta_k \xi_k}$$

where $\xi_k \equiv \frac{1}{1+\theta_k\left(\frac{1}{\eta_k - 1} - \frac{1}{\sigma_k - 1}\right)}$, $\mu_{n,k}^{Mel} \equiv \bar{\mu}_k^{Mel} \left(\frac{f_{n,k}}{\beta_{n,k} L_n}\right)^{\left(\frac{1}{\sigma_k - 1} - \frac{1}{\theta_k}\right)}$ and $\bar{\mu}_k^{Mel}$ is some con-

[10] Is straightforward to incorporate Marshallian externalities into the multi-industry Krugman model presented above. For instance, letting $A_{i,k} \equiv \tilde{A}_{i,k} L_{i,k}^{\phi_k}$ yields a setting with scale and trade elasticities $\psi_k = (\sigma_k - 1)^{-1} + \phi_k$ and $\varepsilon_k = \eta_k - 1$, respectively.

[11] Feenstra *et al.* (2014) also consider a multi-industry Melitz-Pareto model with possibly different elasticities of substitution across varieties from different countries and across varieties from the same country.

[12] To simplify the analysis, we assume that the fixed marketing cost to serve destination $n$ does not vary across origins $i$. Allowing these fixed costs to vary across country pairs would imply that instead of a term $S_{i,k}$ we would have a term $S_{ni,k}$ that varies across country pairs, but this would not change any of our main conclusions below.

[13] The assumption that fixed marketing costs are paid in units of labor of the destination country is critical for the result that this model collapses to the general structure introduced above. This is related to the discussion in ACR about how their macro-level restriction R3' obtains in the Melitz-Pareto model if and only if the fixed cost is paid in units of labor of the destination country.

stant defined in Appendix B. These two equations collapse to the expressions for trade shares and industry price indexes in equations (1) and (2), respectively, by setting $S_{i,k} = b_{i,k}^{\theta_k \xi_k} F_{i,k}^{-\xi_k}$, $\psi_k = 1/\theta_k$, $\varepsilon_k = \theta_k \xi_k$ and $\mu_{n,k} = \mu_{n,k}^{Mel}$. Note also that if we set $\sigma_k = \eta_k$ for all $k$ then $\xi_k = 1$ and this model is just a multi-industry version of the Melitz-Pareto model in Arkolakis *et al.* (2008). See rows 3 and 5 in Table 1.

## 3. Characterizing Equilibrium

To characterize the equilibrium we proceed in two steps: we first characterize the equilibrium labor allocations given wages, and then we characterize wages that satisfy labor market clearing given the corresponding equilibrium labor allocations.

**Two-Step Equilibrium Definition**. The equilibrium labor allocations for some wage vector $\boldsymbol{w} \in \mathbb{R}_{++}^N$ are given by $\boldsymbol{L} \in \mathbb{R}_{+}^{NK} \setminus \mathbb{Z}_0^{NK}$ that satisfy (3) for all $(i, k)$. Let $\mathcal{L}(\boldsymbol{w})$ be the set of such equilibrium allocations. A wage vector $\boldsymbol{w} \in \mathbb{R}_{++}^N$ is an equilibrium wage vector if there exists an element $\boldsymbol{L} \in \mathcal{L}(\boldsymbol{w})$ such that $\boldsymbol{L}$ also satisfies (5) for all $i$.

Note that given wages, for each industry $k$ we have a system of $N$ nonlinear complementary slackness conditions in $L_{i,k}$ for $i = 1, ..., N$ specified by (3). For the first step we exploit the fact that this system is independent across $k$. We now introduce some additional notation and definitions.

**Interior, Corner and Complete Specialization Allocations**. An allocation $\boldsymbol{L}_k$ is an *interior* allocation if $L_{i,k} > 0$ for all $i$; an allocation $\boldsymbol{L}_k$ is a *corner* allocation if $L_{i,k} = 0$ for at least one $i$; and an allocation $\boldsymbol{L}_k$ is a *complete specialization* allocation if there is a unique $i^*(k)$ such that $L_{i,k} = 0$ for all $i \neq i^*(k)$.[14]

**Industry-Level Equilibrium Labor Allocations**. Given wage $\boldsymbol{w}$, $\mathcal{L}_k(\boldsymbol{w})$ denotes the set of equilibrium labor allocations in industry $k$, i.e., for any $\boldsymbol{L}_k \in \mathcal{L}_k(\boldsymbol{w})$ , $\boldsymbol{L}_k$ satisfies complementary slackness conditions (3) for industry $k$.

[14]Note that there are many complete specialization allocations. For instance, it could be the case that production in industry 1 is concentrated solely in country 1 ($i^*(1) = 1$) or in country 2 ($i^*(1) = 2$), and so on.

### 3.1. Step 1: Equilibrium Labor Allocations

Before we proceed, let us introduce an additional assumption on the matrix of trade costs which we employ to prove our results in the case of $\alpha_k = 1$ for some $k$:

**Assumption 1.** *Matrix*

$$\begin{pmatrix} \tau_{11,k}^{-\varepsilon_k} & \dots & \tau_{1N,k}^{-\varepsilon_k} \\ \vdots & & \vdots \\ \tau_{N1,k}^{-\varepsilon_k} & \dots & \tau_{NN,k}^{-\varepsilon_k} \end{pmatrix}$$

*is non-singular.*

We discuss the role of this assumption as well as sufficient conditions which guarantee it below. For now, note that this assumption is violated if trade is free (i.e., $\tau_{ni,k} = 1$ for all $n$ and $i$).

Given the previous definitions, we are now ready to state our first Proposition.

**Proposition 1.** *If either (a) $0 \leq \alpha_k < 1$, or (b) $\alpha_k = 1$ and Assumption 1 holds, then the set $\mathcal{L}_k(\boldsymbol{w})$ is a singleton; if $\alpha_k > 1$, then the set $\mathcal{L}_k(\boldsymbol{w})$ contains multiple allocations, including (but not necessarily limited to) one for each complete specialization allocation. Moreover, the unique allocation in $\mathcal{L}_k(\boldsymbol{w})$ is an interior allocation if $0 \leq \alpha_k < 1$, while it may be an interior or a corner allocation if $\alpha_k = 1$.*

This proposition states conditions under which, given any vector of positive wages and any industry $k$, the system (3) of $N$ non-linear complementary slackness conditions in $L_{i,k}$ for $i = 1, ..., N$ has a unique solution, with $L_{i,k} > 0$ for all $i$ if $0 \leq \alpha_k < 1$. The case with $\alpha_k = 0$ is trivial: given wages, labor allocations are explicitly obtained from the conditions $L_{i,k} G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k) = 0$. Below we focus on the case with $\alpha_k > 0$.

Before proving the proposition, we simplify notation by suppressing the sub-index $k$ and letting $a_{ni} \equiv S_i \left(w_i \tau_{ni}\right)^{-\varepsilon} w_i^{-\alpha}$ and $b_n \equiv \beta_n w_n \bar{L}_n$. Combining equations (1) and (4) we then have

$$\frac{G_i\left(\boldsymbol{w}, \boldsymbol{L}\right)}{w_i} = 1 - \frac{1}{w_i L_i} \sum_n \frac{a_{ni}(w_i L_i)^\alpha}{\sum_l a_{nl}(w_l L_l)^\alpha} b_n.$$

Transforming variables with $x_i \equiv w_i L_i$, letting $\boldsymbol{x} \equiv (x_1, ..., x_N)$, and with a slight abuse of notation, we can write

$$G_i\left(\boldsymbol{x}\right) = 1 - \sum_n \frac{a_{ni} x_i^{\alpha-1}}{\sum_l a_{nl} x_l^{\alpha}} b_n.^{15}$$

The system in (3) can now be written as a non-linear complementarity problem (NCP) in $\boldsymbol{x}$:

$$x_i \geq 0, \quad G_i(\boldsymbol{x}) \geq 0, \quad x_i G_i(\boldsymbol{x}) = 0, \quad i = 1, \ldots, N. \tag{6}$$

Note that if $\boldsymbol{x}$ solves (6) then $\sum_i x_i G_i(\boldsymbol{x}) = 0$ and hence $\sum_i x_i = \sum_i b_i$. This implies that the solution to (6) satisfies $\boldsymbol{x} \in \Gamma \equiv \left\{\boldsymbol{x} \in \mathbb{R}^N \,|\, x_i \geq 0,\ i = 1, \ldots, N;\ \sum_i x_i = \sum_i b_i\right\}$.

To prove Proposition 1 we follow a popular approach in the economics literature that consists of characterizing equilibria of general equilibrium models as solutions to optimization problems.[16] Doing this is possible if, for example, the function $\boldsymbol{G}(\boldsymbol{x}) \equiv (G_1(\boldsymbol{x}), \ldots, G_N(\boldsymbol{x}))$ has a Jacobian that is symmetric at all points in its domain, since in this case the function $\boldsymbol{G}$ is the gradient of some other function $F$ that we can use in the optimization problem.[17] Fortunately, our function $\boldsymbol{G}$ satisfies this symmetry condition. In fact, it is easy to see that $\boldsymbol{G}$ is the gradient of function $F : \mathbb{R}_+^N \setminus \{\boldsymbol{0}\} \to \mathbb{R}$ defined by

$$F(\boldsymbol{x}) \equiv \alpha \sum_n x_n - \sum_n b_n \ln\left(\sum_i a_{ni} x_i^{\alpha}\right). \tag{7}$$

As we establish formally below, this makes it possible to solve the NCP in (6) by way of solving $\arg\min_{\boldsymbol{x}\in\Gamma} F(\boldsymbol{x})$, where $\Gamma$ is the compact set defined above.[18]

We now focus on the characterization of the optimization problem $\arg\min_{\boldsymbol{x}\in\Gamma} F(\boldsymbol{x})$ and then establish formally the connection between this problem and the NCP in (6).

Existence of a solution to $\arg\min_{\boldsymbol{x}\in\Gamma} F(\boldsymbol{x})$ follows immediately from the fact that $\Gamma$ is a compact set and $F(\cdot)$ is continuous on $\Gamma$. To establish uniqueness, we show in

[15] Analogously to our treatment of the original functions $G_{i,k}(\boldsymbol{w}, \boldsymbol{L}_k)$ and $L_{i,k} G_i(\boldsymbol{w}\boldsymbol{L}_k)$ at $L_{i,k} = 0$ (see footnote 9), we define values of $G_i(\boldsymbol{x})$ and $x_i G_i(\boldsymbol{x})$ at $x_i = 0$ by their limits.

[16] Negishi (1960) is probably the most well-known example of this approach in which market equilibria are characterized as solutions to a social planner's problem. Kehoe, Levine and Romer (1992) describe a more general framework in which the optimization problem does not necessarily have an economic interpretation. Our case fits into their general framework.

[17] A classical result in mathematics states that a vector function is a gradient map if and only if its Jacobian is symmetric in the domain of the function (see, for example, Theorem 4.1.16 on page 95 in Ortega and Rheinboldt, 2000).

[18] We thank Anca Ciurte and Ioan Rasa for pointing us in this direction.

Appendix C that under the conditions of Proposition 1 function $F(\cdot)$ is strictly convex on $\Gamma$. Thus, since $\Gamma$ is a convex set, $F(\cdot)$ has at most one global minimum on $\Gamma$. This establishes the following result:

**Lemma 1.** *If either (a) $0 < \alpha < 1$, or (b) $\alpha = 1$ and Assumption 1 holds, then $F(\cdot)$ has a unique global minimum on $\Gamma$.*

Let us denote the unique global minimum of $F(\cdot)$ on $\Gamma$ by $\boldsymbol{x}^*$. In Appendix C we prove the following result:

**Lemma 2.** *If $0 < \alpha < 1$ then $x_i^* > 0$ for all $i = 1, \ldots, N$.*

Finally, we prove the part of Proposition 1 concerning the case of $\alpha \leq 1$ by combining the two previous lemmas with the following equivalence result:

**Lemma 3.** *If either (a) $0 < \alpha < 1$, or (b) $\alpha = 1$ and Assumption 1 holds, then $\boldsymbol{x}$ is a global minimum of $F(\cdot)$ on $\Gamma$ if and only if $\boldsymbol{x}$ is a solution to (6).*

The proof of this result is almost trivial, because the conditions in (6) are just the first-order conditions for the minimization of $F(\cdot)$ on $\Gamma$. The only complication is that to invoke the first-order conditions, we need to have differentiability of $F(\cdot)$ on $\Gamma$ which is understood as differentiability of $F(\cdot)$ on some open set containing $\Gamma$. In case of $\alpha < 1$ any such open set necessarily includes points $\boldsymbol{x}$ with $x_i \leq 0$, at which $F(\cdot)$ is not differentiable. We deal formally with this complication in Appendix C.

One might wonder if the equilibrium labor allocation is continuous in $\alpha$ as we approach $\alpha = 1$ from below. Economically speaking, one would expect this to be the case, so that if at $\alpha = 1$ we have a corner allocation with $x_i = 0$ for some country $i$ then $x_i(\alpha) > 0$ for all $\alpha < 1$ but $\lim_{\alpha \uparrow 1} x_i(\alpha) \to 0$. Mathematically, however, this result is not trivial because the function $\boldsymbol{G}$ is not jointly continuous in $\boldsymbol{x}$ and $\alpha$ for $\alpha = 1$ and points $\boldsymbol{x}$ with $x_i = 0$ for some $i$. Still, thanks to the optimization approach followed in the previous lemmas, we can establish the left continuity of $\boldsymbol{x}(\alpha)$ by invoking the Theorem of the Maximum (see Theorem 3.6 in Stokey, Lucas and Prescott, 1989) (see Appendix C for details).

**Lemma 4.** *If Assumption 1 holds, then $\boldsymbol{x}(\alpha)$ is continuous as a function of $\alpha$ for all $\alpha \in (0, 1]$. In particular, $\lim_{\alpha \uparrow 1} \boldsymbol{x}(\alpha) = \boldsymbol{x}(1)$.*

Consider now the case with $\alpha > 1$. We can easily show that there are many solutions to (6). To see this, choose some $i^*$ and set $x_{i^*} = \sum_n b_n$ and $x_i = 0$ for $i \neq i^*$. It is easy to check that this satisfies $G_{i^*}(\boldsymbol{x}) = 0$ and $G_i(\boldsymbol{x}) = 1 \geq 0$ for all $i \neq i^*$, so that all conditions in (6) are satisfied. Of course, this is only one example and there are many other possible equilibrium allocations for this case. However, characterizing the complete set of equilibria is not the focus of our analysis.

We finish this subsection by commenting on the role of Assumption 1 in Proposition 1 as well as on sufficient conditions under which it holds. While Assumption 1 plays no role in the proof of uniqueness when $\alpha_k < 1$, we cannot rule out multiplicity of equilibria if it is violated when $\alpha_k = 1$. As mentioned above, this assumption is violated if trade is frictionless. In fact, it is violated if there is at least one pair of countries with frictionless trade (in both directions) between them. Suppose, for instance, that there were no trade costs between two countries $i$ and $j$. The triangular inequality implies the trade costs between $i$ and $j$ and all other countries are the same (i.e., $\tau_{ni,k} = \tau_{nj,k}$ and $\tau_{in,k} = \tau_{jn,k}$ for all $n \neq i, j$). It then follows that the $i$ and $j$ row in the matrix of Assumption 1 are the same, so the non-singularity requirement is violated. Notice, however, that the multiplicity that arises in this case is that at most the overall labor allocation $L_{i,k} + L_{j,k}$ is determined, but not $L_{i,k}$ or $L_{j,k}$. This type of non-uniqueness is similar to the one that can arise in Ricardian models under frictionless trade and is irrelevant for welfare: real wages are the same across any two equilibria in this set. Moreover, with any small trade costs between $i$ and $j$ the non-uniqueness disappears, rendering these cases non-generic.

While it is easy to check if Assumption 1 is satisfied for a particular parametrization, we can say a little bit more about the conditions which guarantee that this assumption holds. Behrens, Lamorgese, Ottaviano and Tabuchi (2004) invoke classical results by Schoenberg (1938) to show that, if trade costs $\tau_{ni}$ correspond to the Euclidian distance between countries $n$ and $i$, then the matrix in Assumption 1 is positive definite (and, hence, non-singular) as long as all countries are at distinct locations. In fact, any three distinct numbers that satisfy the triangle inequality can be mapped to lengths of sides of a triangle in $\mathbb{R}^2$, which means that any such numbers correspond to Euclidean distances between vertices of a triangle in $\mathbb{R}^2$. Together with the results from Schoenberg (1938), this observation implies that for $N = 3$ the matrix in Assumption 1 is positive definite if (i) the iceberg trade costs are symmetric, (ii) greater than $1$ for different countries, and (iii) satisfy the triangle inequality. For $N > 3$, conditions (i)-(iii) do not generally imply that the iceberg trade costs correspond to distances in an Euclidean space. Still, extensive simulations for trade-freeness matrices for $N = 4, 5, 6$ lead us to conjecture that conditions (i)-(iii) guarantee that the matrix in Assumption 1 is positive definite. Moreover, we conjecture that we can even dispense with the symmetry condition (i) — in this case it is the sum of the matrix in Assumption 1 with its transpose that is positive definite.

### 3.2. Step 2: Equilibrium Wages

In what follows, we restrict the analysis to the case $0 \leq \alpha_k \leq 1$. For this case, Proposition 1 establishes that the solution of the system of complementary slackness conditions (3) determines a function from wages to labor allocations, $\boldsymbol{L}(\boldsymbol{w})$, for $\boldsymbol{w} \in \mathbb{R}^N_{++}$. Letting

$$Z_i(\boldsymbol{w}) \equiv \sum_k L_{i,k}(\boldsymbol{w}) - \bar{L}_i \tag{8}$$

be the excess labor demand in country $i$ defined for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$ and letting $\boldsymbol{Z}(\boldsymbol{w}) \equiv (Z_1(\boldsymbol{w}), ..., Z_N(\boldsymbol{w}))$, the labor-market clearing conditions for all countries can be written simply as

$$\boldsymbol{Z}(\boldsymbol{w}) = 0. \tag{9}$$

To establish existence of a solution to this system of equations, we will invoke continuity of $\boldsymbol{L}_k(\boldsymbol{w})$. We again exploit the equivalence between the system in (3) and a constrained optimization problem and invoke Theorem of the Maximum from Stokey, Lucas and Prescott (1989) to establish that $\boldsymbol{L}_k(\boldsymbol{w})$ is a continuous function for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$ (see Appendix C for details).

**Lemma 5.** *If either (a) $0 \leq \alpha_k < 1$, or (b) $\alpha_k = 1$ and Assumption 1 holds, then the function $\boldsymbol{L}_k(\boldsymbol{w})$ is continuous for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$.*

We now state our result for existence of equilibrium.

**Proposition 2.** *Assume that for all $k$ either (a) $0 \leq \alpha_k < 1$, or (b) $\alpha_k = 1$ and Assumption 1 holds. Then there exists a vector of wages $\boldsymbol{w} \in \mathbb{R}^N_{++}$ that satisfies (9).*

*Proof.* The case with $\alpha_k = 0$ is a simple extension of the existence proof by Alvarez and Lucas (2007) to the case of multiple industries. Here we focus on the case with $0 < \alpha_k \leq 1$. To establish existence of a solution to (9), we invoke Proposition 17.C.1 in Mas-Colell, Whinston and Green (1995, MWG) and show that the following properties are satisfied: (i) $\boldsymbol{Z}(\boldsymbol{w})$ is continuous; (ii) $\boldsymbol{Z}(\boldsymbol{w})$ is homogeneous of degree zero; (iii) $\boldsymbol{w} \cdot \boldsymbol{Z}(\boldsymbol{w}) = 0$ for all $\boldsymbol{w}$ (Walras' law); (iv) there is an $A > 0$ such that $Z_i(\boldsymbol{w}) > -A$ for all $i$ and $\boldsymbol{w}$; (v) if $\boldsymbol{w}^s \to \boldsymbol{w}$ as $s \to \infty$, where $\boldsymbol{w} \neq 0$ and $w_i = 0$ for some $i$, then $Max\{Z_1(\boldsymbol{w}^s), ..., Z_N(\boldsymbol{w}^s)\} \to \infty$ as $s \to \infty$. Property (i) follows from Lemma 5, while properties (ii)-(iv) are immediate. The proof of (v) is in Appendix C. □

In what follows, we first provide sufficient conditions for a unique equilibrium in the case of two countries ($N = 2$), and in the case of multiple countries with free trade. After that we discuss additional complexities that arise in the general setting with multiple countries and costly trade.

**Proposition 3.** *Assume that $N = 2$ and that for all $k$ either (a) $0 \leq \alpha_k < 1$, or (b) $\alpha_k = 1$ and Assumption 1 holds. Then there exists a unique (normalized) vector of wages $\boldsymbol{w} \in \mathbb{R}^N_{++}$ that satisfies* (9).

**Proposition 4.** *Assume that $0 \leq \alpha_k < 1$ for all $k$ and trade is frictionless in all industries, i.e., that $\tau_{ni,k} = 1$ for all $n$, $i$, and $k$. Then there exists a unique (normalized) vector of wages $\boldsymbol{w} \in \mathbb{R}^N_{++}$ that satisfies* (9).

We prove both Propositions 3 and 4 by showing that the labor excess demand function $\boldsymbol{Z}(\boldsymbol{w})$ has the gross substitutes property under the assumptions of these propositions. Uniqueness of solution then follows from Proposition 17.F.3 from MWG.

The previous results establish that if there are two countries, or if there are many countries but no trade costs, or if there are many countries and positive trade costs but wages are pinned down by an outside good, then the equilibrium exists and is unique.[19]

[19] If we assume that there is a freely traded "outside good" industry in which production exhibits constant returns to scale and assume that all countries produce a positive amount of this good, as is typically done in the literature, then wages are exogenous and the proof from Proposition 1 — which is valid for any finite $N$ — implies a unique allocation of labor across industries. Note, however, that we need to assume that all countries produce the outside good — if some countries do not produce that good then wages are not pinned down and we do not have a proof of uniqueness for more than two countries for this case.

With positive trade costs and more than two countries, our excess labor demand system does not, in general, satisfy the gross-substitutes property, and so this property can no longer be invoked for establishing a unique vector of wages for $N > 2$ and costly trade. While scale economies act to reinforce the gross substitutes property when there are two countries, the same is not necessarily true for three or more countries. For instance, a rise in the wage in one country, say country 1, may reduce the demand for labor there, while at the same time raising the demand for labor in another country, say country 2, which is so far consistent with the gross substitutes property. The complexities arise from the fact that the increased labor demand in country 2 can generate productivity effects that can lead to increased exports to a third country, say country 3, which can, in turn, result in a fall in the demand for labor there. In other words, a rise in wages in country 1 can result in a fall in the demand for labor in country 3, thereby, violating the gross substitutes property.

A more powerful approach for proving that the equilibrium is unique is the Index Theorem, which – roughly speaking – says that if the determinant of the negative of the Jacobian of the normalized excess labor demand system (i.e., the Jacobian of $-\boldsymbol{Z}(\boldsymbol{w})$ after removing the last column and the last row) is positive at any equilibrium wage vector then there is at most one equilibrium (Kehoe, 1980). The challenge here is that the Jacobian of the aggregate labor demand is the sum of the Jacobians of the labor demand coming from each sector (i.e., $D\boldsymbol{Z}(\boldsymbol{w}) = \sum_k D\boldsymbol{L}_k(\boldsymbol{w})$), and establishing conditions on the determinant of a sum of matrices is extremely difficult. Indeed, we have a proof that the determinant of the negative of the normalized sector-level Jacobians (i.e., $-D\boldsymbol{L}_k(\boldsymbol{w})$ after removing the last column and the last row) is always positive, but we have not been able to extend this property to the sum of those modified Jacobians. It is natural to look for some property of matrices that implies a positive determinant and that survives under summation and check that this property is satisfied by the negative of the normalized sector-level Jacobians. Gross substitutes is one such property, but as mentioned above it doesn't hold for more than two countries.[20] Another such property is positive definiteness (including its extension to non-symmetric matrices), but unfortunately we have examples in which the negative of the normalized sector-level

[20]Mathematically, gross substitutes implies that the negative of the normalized Jacobian is a diagonally dominant Z matrix. This property survives under summation and implies a positive determinant.

Jacobians are not positive definite.

An entirely different approach is to transform the equilibrium system into a mapping whose fixed point is a solution of that system, and then show that this mapping is a contraction mapping. In principle, if there is a unique equilibrium, then such a contraction mapping exists, but finding the right transformation is of course very challenging. We view the techniques explored in Allen, Arkolakis and Li (2015) as a variant of this approach. In Appendix C we show how the equilibrium system of our economy with $\alpha_k = \alpha$ for all $k$ can be transformed to the kind of system for which Allen, Arkolakis and Li (2015) establish sufficient conditions for uniqueness. Unfortunately, we find that those sufficient conditions are not satisfied in our economy whenever $\alpha \geq 0$.

Lacking a proof, we have systematically looked for counter-examples via numerical analysis, but we have found none. We report on this work in the next subsection.

## 3.3. Computation of Equilibrium

### 3.3.1. A Practical Algorithm for Finding an Equilibrium

The preceding analysis suggests two alternative approaches to numerically compute an equilibrium. First, one can use an algorithm that properly deals with the complementary slackness conditions in the system of Equations (3) and (5) for $(\boldsymbol{w}, \boldsymbol{L})$. This requires an algorithm for non-linear complementarity problems, such as the PATH solver (Ferris and Munson, 1999). Second, one can follow the approach used above to prove existence and uniqueness of equilibrium and break the problem in two steps: first, for each wage vector $\boldsymbol{w}$ find $\boldsymbol{L}_k(\boldsymbol{w})$ for each $k$ by solving the optimization problem associated with (7), and second, find the wage vector such that the excess labor demand $\boldsymbol{Z}(\boldsymbol{w}) \equiv \sum_k \boldsymbol{L}_k(\boldsymbol{w}) - \bar{\boldsymbol{L}}$ is zero using the *tatonnement* iterative procedure proposed by Alvarez and Lucas (2007).

It turns out, however, that a third approach does best. Consider the function $\boldsymbol{w}(\boldsymbol{T})$ that one would get simply by solving for wages in the standard multi-sector model with no scale economies and technology parameters $\boldsymbol{T} = \{T_{i,k}\}$, and let $L^d_{i,k}(\boldsymbol{T}, \boldsymbol{w})$ be labor demand as a function of technology parameters and wages also in that model. Let $\boldsymbol{T}(\boldsymbol{L})$ be defined by $T_{i,k}(\boldsymbol{L}) = S_{i,k} L_{i,k}^{\alpha_k}$ and let $\boldsymbol{H}(\boldsymbol{L}) \equiv \boldsymbol{L}^d(\boldsymbol{T}(\boldsymbol{L}), \boldsymbol{w}(\boldsymbol{T}(\boldsymbol{L})))$. By definition of $\boldsymbol{w}(\boldsymbol{T})$ we must have $\sum_k L^d_{i,k}(\boldsymbol{T}(\boldsymbol{L}), \boldsymbol{w}(\boldsymbol{T}(\boldsymbol{L}))) = \bar{L}_i$ for all $i$ and $\boldsymbol{L}$. Thus, if $\boldsymbol{L}^*$ is an interior fixed point of the mapping $\boldsymbol{H}(\boldsymbol{L})$ then $(\boldsymbol{w}^*, \boldsymbol{L}^*) = (\boldsymbol{w}(\boldsymbol{T}(\boldsymbol{L}^*), \boldsymbol{L}^*)$ is an equilibrium of our economy with economies of scale. Since $\boldsymbol{H}(\boldsymbol{L})$ is a continuous mapping from the compact set $\Lambda \equiv \{\boldsymbol{L} | \sum_k L_{i,k} = \bar{L}_i\}$ to itself, then we can use the iterative procedure given by $\boldsymbol{L_{t+1}} = \boldsymbol{H}(\boldsymbol{L}_t)$ to compute the equilibrium points.

We have used this algorithm for counterfactual analysis with many countries and sectors (see Subsection 4.4 and Section 5) and found that it can easily handle corners and that it is very robust. We have also used this algorithm on economies with three or four countries, two sectors, $\alpha = 0.9$ and randomly chosen values for all other parameters. Compared to a standard Newton method, it is slower but way more robust. We randomly generated more than a million economies with three countries and two sectors, and more than half a million economies with four countries and two sectors. In all cases the algorithm using the iterative procedure with $\boldsymbol{L_{t+1}} = \boldsymbol{H}(\boldsymbol{L}_t)$ found a solution, whereas the Newton method found a solution only for some initial conditions.

Because the Newton method is faster, we used it in combination with our iterative procedure in an effort to find examples with multiple equilibria. For each of the random economies mentioned above, we computed the equilibrium with the iterative procedure, and also with the Newton method with 400 different starting points. If there were multiple equilibria, we would likely have one of the solutions of the Newton method be different than the one found by the iterative procedure, but this never happened.[21] For the case with $\alpha = 0.9$ we also computed the sign of the determinant of the (negative of the) normalized excess labor demand evaluated at the equilibrium we found. By the Index Theorem, a negative value would imply multiplicity. We always found this sign to be positive.

#### 3.3.2. A Rigorous Algorithm for Finding All Equilbria

The fact that we do not find different equilibria when starting from different initial conditions does not prove that there is no multiplicity. To prove uniqueness of equilibrium for a particular parameterization of our economy, we need a procedure that guarantees to find all solutions to our system of complementary slackness conditions. One approach for doing so is to transform our equilibrium system into a system of polyno-

[21]To check that this procedure delivers multiplicity when we know they exist, we used the same code for $\alpha = 2$. We find that this leads to multiple equilibria for randomly generated parameters with three economies and two sectors.

mial equations, and then apply methods from algebraic geometry to find all solutions of that transformed system. This is feasible in our case as long as all trade and scale elasticities are rational numbers.[22] An alternative approach is to exploit interval arithmetic, which is a method that allows direct computation of outer bounds for the range of values of a function evaluated in some interval. This approach has the benefit that it can deal with the complementary slackness conditions that become relevant when some $\alpha \geq 1$. Moreover, interval-arithmetic based algorithms for finding all solutions of a system of non-linear equations are natural extensions of the bisection and Newton methods that are well-known in economics.

The key idea of interval arithmetic is that instead of working with functions defined over variables, we work with extensions of those functions that are defined over intervals. Consider a function $F : A \to \mathbb{R}^N$, with $A \subseteq \mathbb{R}^N$. Given that $F$ satisfies certain conditions (which are most likely to be satisfied for functions typically encountered in economics), interval arithmetic can be used to construct a function $\mathbb{F}$ operating directly on intervals $\mathbb{X} \subseteq A$ with the following properties: (i) for any interval $\mathbb{X} \subseteq A$ and any vector $x \in \mathbb{X}$, if $y = F(x)$, then $y \in \mathbb{F}(\mathbb{X})$;[23] (ii) if $[x, x]$ is the degenerate interval with unique element $x$, then $\mathbb{F}([x, x]) = F(x)$; and (iii) the "smaller" is an interval $\mathbb{X}$, the "tighter" is the enclosure of the range of values of $F$ over $\mathbb{X}$ by interval $\mathbb{F}(\mathbb{X})$.

Such interval extensions of regular functions are useful for characterizing solutions of $F(x) = 0$. Indeed, property (i) guarantees that if $0 \notin \mathbb{F}(\mathbb{X})$ then there is no solution to $F(x) = 0$ in $\mathbb{X}$. Imagine then that $0 \in \mathbb{F}(\mathbb{X}_0)$ for $\mathbb{X}_0 \subseteq A$. We can partition $\mathbb{X}_0$ into $I$ subintervals $\mathbb{X}_i$ for $i = 1, \ldots, I$ and check whether $0 \in \mathbb{F}(\mathbb{X}_i)$. If $0 \in \mathbb{F}(\mathbb{X}_i)$ for some $i$, we can further partition the interval $\mathbb{X}_i$ and iterate, at each point discarding intervals $\mathbb{X}$ for which $0 \notin \mathbb{F}(\mathbb{X})$. Up to a level of precision supplied in the algorithm, we can then find all "small" intervals $\mathbb{X}$ such that $0 \in \mathbb{F}(\mathbb{X})$. Property (iii) from above guarantees that, if such intervals $\mathbb{X}$ are "small" enough, then they contain solutions to $F(x) = 0$. This procedure is a multi-dimensional bisection algorithm and is essentially a rigorous

[22]For a review of the use of algebraic geometry methods to find all solutions to systems of polynomial equations, and the use of these techniques in economics, see Kubler, Renner and Schmedders (2014).

[23]The converse is not required: we may have $y \in \mathbb{F}(\mathbb{X})$ with $y \neq F(x)$ for all $x \in \mathbb{X}$. To guarantee this converse property, we essentially need to solve global minimization and maximization problems, $\underline{F} \equiv \min_{x \in \mathbb{X}} F(x)$ and $\overline{F} = \max_{x \in \mathbb{X}} F(x)$, and assign $\mathbb{F}(\mathbb{X}) = \left[\underline{F}, \overline{F}\right]$. Solving such optimization problems for each interval $\mathbb{X}$ is, of course, prohibitively expensive. One of the purposes of interval arithmetic is construction of estimated bounds on the range of values of function $F$ over an interval $\mathbb{X}$ with acceptable tolerance and with as little effort as possible.

way of doing what economists sometimes call “grid search”. And, just as Newton-based methods are preferable to bisection/“grid search” for finding solutions of systems of equations whenever the relevant functions are differentiable, here too it is possible and preferable to use interval analogs of Newton-based algorithms if $F$ is differentiable.

# 4. Scale Economies and the Welfare Effects of Trade

In this section we explore the implications of scale economies for the welfare effects of trade. We restrict the analysis to the case in which $0 \leq \alpha_k \leq 1$ for all $k$. We first study how scale economies affect the gains from trade and the welfare effects from trade liberalization, and we conclude by quantifying the different effects using counterfactual analysis when the model is made to be perfectly consistent with the data.

## 4.1. Gains from Trade

In principle, countries that specialize in industries with weak economies of scale could even lose from trade — the premise of Frank Graham's argument for protection. It turns out, however, that this cannot happen if $0 \leq \alpha_k \leq 1$ for all $k$. The formal proof is in the Appendix D.1, but the basic idea can be understood by the following simple argument. Setting $w_i = 1$ by choice of numeraire, industry level price indices can be written as

$$P_{i,k}^{-\varepsilon_k} = \mu_k^{-\varepsilon_k} S_{i,k} L_{i,k}^{\alpha_k} / \lambda_{ii,k}. \tag{10}$$

Without scale economies, gains from trade are assured by the fact that $\lambda_{ii,k} < 1$ implies $P_{i,k} < \mu_k S_{i,k}^{-1/\varepsilon_k}$, where the RHS term is the price index under autarky. Scale economies imply that $L_{i,k}$ could fall with trade, and so now from Equation (10) we see that $P_{i,k}$ could be *higher* with trade relative to autarky. But note that in equilibrium we must have $L_{i,k} > \lambda_{ii,k}\beta_{i,k}\bar{L}_i$, since the RHS is just the total labor cost associated with *domestic* sales. Combining this with Equation (10) yields $P_{i,k}^{-\varepsilon_k} > \mu_k^{-\varepsilon_k} S_{i,k} \left(\beta_{i,k}\bar{L}_i\right)^{\alpha_k} \lambda_{ii,k}^{\alpha_k - 1}$. Thus, if $0 \leq \alpha_k \leq 1$ then $P_{i,k} < \mu_k S_{i,k}^{-1/\varepsilon_k} \left(\beta_{i,k}\bar{L}_i\right)^{-\alpha_k/\varepsilon_k}$, which is the price under autarky when there are scale economies. As we can also see, if $\alpha_k > 1$ then one could have higher prices in some industries with trade than without, leading to the possibility of losses from trade. This argument establishes the following Proposition.

**Proposition 5.** *If* $0 \leq \alpha_k \leq 1$ *then all countries gain from trade.*

*Proof.* See Appendix D.1. □

This result can be seen as a generalization of Proposition 1 in Venables (1987), which states that in a Krugman (1980) model with an "outside good" all countries gain from trade. Formally, the model in Venables (1987) is isomorphic to ours when we consider two countries and two industries, one having no trade costs, no scale economies, and an infinite trade elasticity (the "outside good"), and the other having trade costs, scale economies, and a finite trade elasticity, with $\alpha_k = 1$. Proposition 5 shows that this generalizes to a case without an "outside good", with multiple sectors and arbitrary scale economies as long as $\alpha_k \leq 1$ for all $k$.

To further explore the implications of scale economies for the magnitude of the gains from trade, we assume that the equilibrium is interior so that all trade shares and labor allocations are strictly positive. This allows us to derive an expression for the gains from trade as a function of industry-level data and the *trade* and *scale* elasticities that extend the multi-sector expressions in Arkolakis *et al.* (2012) (henceforth ACR). [24]

Real wages in the model with scale economies can be written as

$$w_n/P_n = \mu_n \prod_k \left( S_{n,k} L_{n,k}^{\varepsilon_k \psi_k} \lambda_{nn,k}^{-1} \right)^{\beta_{n,k}/\varepsilon_k},$$

where $\mu_n \equiv \prod_k \mu_{n,k}^{-1}$. Using the hat notation $\hat{x} = x'/x$, given some foreign shock (i.e., a shock that does not affect the exogenous variables in country $n$), the change in welfare in country $n$ is

$$\hat{w}_n/\hat{P}_n = \prod_k \hat{\lambda}_{nn,k}^{-\beta_{n,k}/\varepsilon_k} \cdot \prod_k \hat{L}_{n,k}^{\beta_{n,k}\psi_k}. \tag{11}$$

The first term on the RHS of this expression is the standard multi-industry formula for gains from trade (with upper-tier Cobb-Douglas preferences), while the second term is an adjustment for scale economies.

To better understand this expression, we can use the fact that the welfare effect of an infinitesimally small change in wages and prices is $d \ln W_n = d \ln w_n - \sum_k \beta_{n,k} d \ln P_{n,k}$. Totally log-differentiating $\lambda_{nn,k} = (L_{n,k}^{-\psi_k} w_n / P_{n,k})^{-\varepsilon_k}$ and substituting into the previous

[24] Given Proposition 1, the assumption that labor allocations are strictly positive is not restrictive for the case with $0 \leq \alpha_k < 1$ for all $k$.

equation yields

$$d\ln W_n = -\sum_k \beta_{n,k}\frac{d\ln\lambda_{nn,k}}{\varepsilon_k} + \sum_k \beta_{n,k}\psi_k d\ln L_{n,k}. \tag{12}$$

The first term term on the right hand side captures the welfare effect of an infinitesimally small foreign shock taking home productivity as given, while the second term captures the welfare effect of that shock through home productivity changes caused by changing industry employment levels in the presence of scale effects. Of course, integrating the first term over some discrete shock yields $\prod_k \hat{\lambda}_{nn,k}^{-\beta_{n,k}/\varepsilon_k}$, while integrating the second term yields $\prod_k \hat{L}_{n,k}^{\beta_{n,k}\psi_k}$.

Following ACR, we define the gains from trade as the negative of the percentage change in real income as we move from the observed equilibrium to autarky,

$$GT_n \equiv -\left(1 - \frac{w_n^A/P_n^A}{w_n/P_n}\right).$$

We compute $GT_n$ by applying (11) and noting that for the move back to autarky we have $\hat{\lambda}_{nn,k} = 1/\lambda_{nn,k}$, and $\hat{L}_{n,k} = \beta_{n,k}/r_{n,k}$, where $r_{n,k} \equiv L_{n,k}/\bar{L}_n$ denotes the industry revenue (or employment) shares in the observed equilibrium. Using $e_{n,k} \equiv X_{n,k}/X_n$ for observed industry expenditure shares (of course, $e_{n,k} = \beta_{n,k}$ in the model), this leads to a formula for the gains from trade that depends only on the country's observables $\lambda_{nn,k}$, $e_{n,k}$ and $r_{n,k}$ as well as the trade and scale elasticities, $\varepsilon_k$ and $\psi_k$,

$$GT_n = 1 - \Delta_n \prod_k \lambda_{nn,k}^{e_{n,k}/\varepsilon_k}, \tag{13}$$

where

$$\Delta_n \equiv \prod_k \left(e_{n,k}/r_{n,k}\right)^{e_{n,k}\psi_k}.$$

The expression for the gains from trade in the standard perfectly competitive model with no scale economies obtains from (13) by setting $\psi_k = 0$ for all $k$, thereby implying $\Delta_n = 1$. The implication of scale economies for the gains from trade then depends on whether $\Delta_n \gtrless 1$.

Consider first the case in which the scale elasticity is the same across industries

($\psi_k = \psi$ for all $k$) and note that

$$\Delta_n^{1/\psi} = \exp D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n),$$

where $\boldsymbol{r}_n \equiv (r_{n1}, ..., r_{nK})$, $\boldsymbol{e}_n \equiv (e_{n1}, ..., e_{nK})$, and

$$D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n) \equiv \sum_k e_{n,k} \ln(e_{n,k}/r_{n,k}) \tag{14}$$

is the Kullback-Leibler divergence of $\boldsymbol{r}_n$ from $\boldsymbol{e}_n$. We can think of $D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n)$ as a measure of industry specialization in country $n$ — in autarky we would have $\boldsymbol{r}_n = \boldsymbol{e}_n$ and $D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n) = 0$, while if $\boldsymbol{r}_n \neq \boldsymbol{e}_n$ then $D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n) > 0$. This implies that $\Delta_n > 1$ (except if $\boldsymbol{r}_n = \boldsymbol{e}_n$, in which case $\Delta_n = 1$) so that, given trade shares, scale economies actually *reduce* the gains from trade, with a larger decline for higher values of $\psi$ and for countries that exhibit higher levels of specialization.[25]

We can gain intuition about this result by going back to equation (12) and noting that if $\psi_k = \psi$ for all $k$ then the second term on the RHS of that equation can be written as $\psi \sum_k e_{n,k} \frac{dL_{n,k}}{L_{n,k}}$. A move back to autarky implies the expansion of industries with net imports and hence a high expenditure share or low employment. In either case, the expenditure-weighted productivity gain in expanding industries will be on average higher than the expenditure-weighted productivity loss in contracting industries, and hence $\psi \sum_k e_{n,k} \frac{dL_{n,k}}{L_{n,k}} > 0$. After integration, this leads to $\Delta_n > 1$.

In the more general case in which $\psi_k$ varies across $k$, $\Delta_n$ can be rewritten as

$$\Delta_n = \exp \bar{\psi} \left[ D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n) - \sum_k \frac{\psi_k - \bar{\psi}}{\bar{\psi}} \ln \left( \frac{r_{n,k}}{e_{n,k}} \right)^{e_{n,k}} \right], \tag{15}$$

where $\bar{\psi} \equiv (1/K) \sum_k \psi_k$. Notice that there are now two (possibly competing) forces: the first measures the *degree of specialization* (*DS*) in country $n$ as represented by $D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n)$, while the second measures the *pattern of specialization* (*PS*), i.e., the tendency of country $n$ to specialize in industries with either higher or lower than average scale eco-

[25]The opposite result would hold if instead of economies of scale we had diseconomies of scale. For example, in a setting with $\psi = 0$ and worker-level heterogeneity, Galle *et al.* (2015) show that $GT_n = 1 - \prod_k \lambda_{nn,k}^{e_{n,k}/\varepsilon_k} (e_{n,k}/r_{n,k})^{-e_{n,k}/\kappa}$, where $\kappa$ is a parameter that determines the degree of heterogeneity. The argument above now implies that the gains from trade are *higher* than in the case with no scale economies, which obtains here in the limit as $\kappa \to \infty$, and corresponds to the case in which workers are homogeneous.

nomies as represented by $\sum_k \frac{\psi_k - \bar{\psi}}{\bar{\psi}} \ln \left(r_{n,k}/e_{n,k}\right)^{e_{n,k}}$. Since *DS* always pushes towards lower gains relative to the case with no scale economies, the overall effect of scale economies on gains from trade depends on the direction and magnitude of *PS*. Countries that tend to specialize in industries with lower than average scale economies — so that *PS* is negative — gain less from trade with scale economies than without. However, in countries that tend to specialize in industries with higher than average scale economies — so that *PS* is positive — the effect of scale economies on the gains from trade is ambiguous. If for a country's *PS* is strong enough to overcome its *DS*, then such a country could have higher gains with than without scale economies (we explore a decomposition of these effects in the quantitative section below).

## 4.2. Welfare Effects from Trade Liberalization: Two Cases

In this section we consider two simple cases for which we can derive analytical results for the welfare gains from a decline in trade costs. The goal is to understand how the presence of scale economies affects the gains from trade liberalization.

### 4.2.1. Mirror-Image Countries

Our first example entails two industries and two mirror-image countries. Of course, for mirror-image countries we know that wages will be the same, so we can just normalize wages to one ($w = 1$) in both countries. For ease of exposition we index countries $i = H, F$, where $H$ and $F$ represent Home and Foreign, respectively. Let $\bar{L} = 2$, $\beta_{i,k} = 1/2$ for all $(i, k)$, and let $S_{H,1} = S_{F,2} = S$ and $S_{H,2} = S_{F,1} = 1$, for $S > 1$. Hence, Home has the comparative advantage in industry 1, and Foreign in industry 2. We assume that $\varepsilon_k = \varepsilon$ and $\psi_k = \psi$ for $k = 1, 2$.

To establish a link with the results in the previous subsection, we first illustrate that the gains from trade are decreasing in $\psi$. We then show that the conclusion is reversed once we consider a trade liberalization exercise in which trade shares respond endogenously as we lower trade costs. There we find that the gains from trade liberalization are increasing in $\psi$.

Home's gains from trade are simply

$$GT_H = 1 - \left( \frac{1/2}{r_{H,1}} \cdot \frac{1/2}{1 - r_{H,1}} \right)^{\psi/2} \left( \lambda_{HH,1}^{\frac{1}{2\varepsilon}} \cdot \lambda_{HH,2}^{\frac{1}{2\varepsilon}} \right). \tag{16}$$

The term $\left(\frac{1/2}{r_{H,1}} \cdot \frac{1/2}{1-r_{H,1}}\right)^{\psi/2}$ corresponds to $\Delta_H$ in Equation (13) and is higher than one as long as there is industry-level specialization.[26] Thus, given trade shares, gains are lower with scale effects ($\psi > 0$) than without ($\psi = 0$). It is also easy to see that these gains are decreasing in $\psi$.

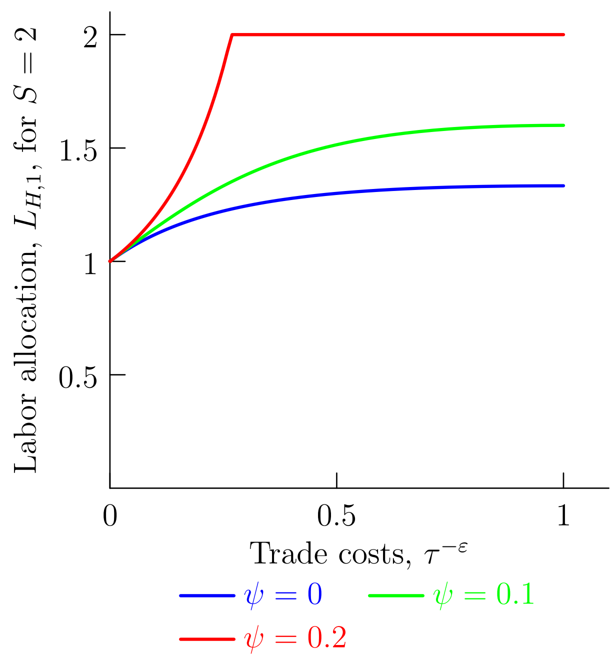
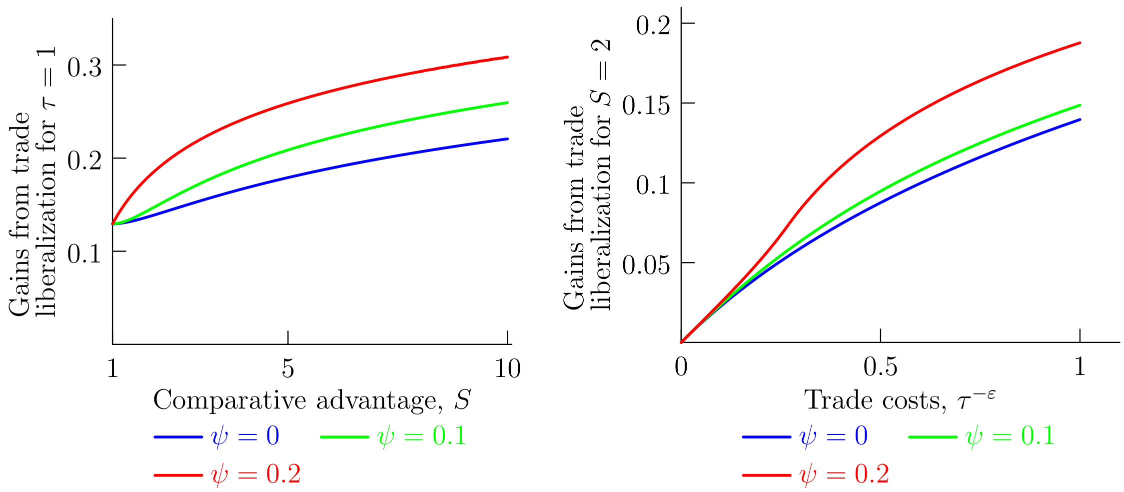



Figure 1: Industry specialization and economies of scale

Next we study the gains from trade liberalization, allowing for endogenous responses of trade shares to trade costs.[27] We set $\varepsilon = 5$ and $\psi = \{0, 0.1, 0.2\}$. The case $\psi = 0.2$ implies $\alpha = 1$, as in the standard multi-industry Krugman or Melitz-Pareto models whereas the case $\psi = 0$ corresponds to the standard multi-industry gravity model without scale economies. Setting $\psi = 0.1$ allows for an intermediate case with $0 < \alpha < 1$.

[26] The term $\frac{1/2}{r_{H,1}} \cdot \frac{1/2}{1-r_{H,1}}$ is minimized at $r_{H,1} = 1/2$, and specialization according to comparative advantage implies $r_{H,1} > 1/2$, hence we must have $\left(\frac{1/2}{r_{H,1}} \cdot \frac{1/2}{1-r_{H,1}}\right)^{\psi/2} > 1$.

[27] In the terminology of ACR, this corresponds to an "ex-ante analysis" whereas the results for the gains from trade above correspond to an "ex-post analysis."




(a) Scale economies and comparative advantage

(b) Scale economies and gravity

Figure 2: Gains from trade liberalization

Note that in all these cases $L_{i,k} = 1$ for all $(i, k)$ under autarky (i.e., when $\tau = \infty$). As $\tau$ falls from $\infty$, country $H$ specializes in industry 1 and country $F$ specializes in industry 2, but the extent of specialization will be stronger with $\psi = 0.2$ than $\psi = 0.1$, and with $\psi = 0.1$ than $\psi = 0$, as illustrated in Figure 1. Figure 2 shows the implications for the gains from trade liberalization for each of these three cases. We see that the gains from trade liberalization increase with $\psi$. The intuition is simple: countries gain by specializing according to comparative advantage, and the concentration of production also allows for a greater exploitation of scale economies, which, in turn, generates additional efficiency gains.[28]

### 4.2.2. Outside Good

The fact that, in the region of uniqueness, countries always gain from trade (relative to autarky) does not necessarily imply that there are always gains from *further* trade liberalization. In fact, our model nests the model considered by Venables (1987) and so we know that a decline in inward trade costs may decrease welfare. To see this more

[28]The gains from trade liberalization in Home can be seen as the increase in $GT_H$ in (16) as $\tau$ falls. The decline in $\tau$ leads to deeper industry-level specialization, as captured by a higher $r_{H,1}$, and thus increases $\left(\frac{1/2}{r_{H,1}} \cdot \frac{1/2}{1-r_{H,1}}\right)^{\psi/2}$ and lowers $GT_H$. But there is also a change in trade shares, and this decreases $\lambda_{HH,1}^{\frac{1}{2\varepsilon}} \cdot \lambda_{HH,2}^{\frac{1}{2\varepsilon}}$, which more than offsets the previous effect.

explicitly, consider a case with two countries and two industries, with $\varepsilon_1 = \infty > \varepsilon_2$, $\psi_1 = 0 < \psi_2 \leq 1/\varepsilon_2$ (so that $\alpha_2 \leq 1$), and with no trade costs in industry 1, $\tau_{12,1} = \tau_{21,1} = 1$. If we start with an interior equilibrium (i.e., $L_{i,k} > 0$ for $i = 1, 2$ and $k = 1, 2$) then wages are pinned down by (exogenous) productivities in industry 1 (the *outside good*), and — suppressing the industry sub-index — the labor allocation in industry 2 is given by $(L_1, L_2)$ that solves

$$w_i L_i = \sum_n S_i L_i^{\alpha} (w_i \tau_{ni})^{-\varepsilon} P_n^{\varepsilon} \beta_n w_n \bar{L}_n, \tag{17}$$

for $i = 1, 2$, with $P_n^{-\varepsilon} = \sum_j S_j L_j^{\alpha} (w_j \tau_{nj})^{-\varepsilon}$. The case considered by Venables (1987) entails $\alpha = 1$, in which case the previous system can be rewritten as a system in $(P_1, P_2)$,

$$w_i = \sum_n S_i (w_i \tau_{ni})^{-\varepsilon} P_n^{\varepsilon} \beta_n w_n \bar{L}_n. \tag{18}$$

It is then easy to see that a decline in $\tau_{12}$ leads to an increase in $P_1$ and a decrease in $P_2$, exactly as in Venables (1987). Of course, if $\alpha = 0$ then $P_n^{-\varepsilon} = \sum_j S_j^{\alpha} (w_j \tau_{nj})^{-\varepsilon}$ and so $P_1$ would decrease while there would be no change in $P_2$.

We can understand these results by noting that $w_n/P_n = S_n^{1/\varepsilon} L_n^{\psi} \lambda_{nn}^{-1/\varepsilon}$. If $\psi = 0$ then a decline in inward trade costs decreases the domestic trade share $\lambda_{nn}$ and increases the real wage. But with $\psi > 0$ there is an offsetting productivity effect arising from the decline in $L_n$. If $\alpha = 1$ then the net effect is negative.

More generally, we can use Equation (17) to show that $\partial P_n / \partial \tau_{ni} < 0$, as with $\alpha = 1$, if and only if $\alpha \in (\bar{\alpha}^{n,\tau}, 1]$, where $\bar{\alpha}^{n,\tau} \in (0, 1)$ is a function of import and export shares in industry 2 — see Appendix D.2. A similar result holds for the effect of a foreign productivity increase in industry 2: this lowers welfare (i.e., $\partial P_n / \partial S_i > 0$ for $n \neq i$) if and only if $\alpha \in (\bar{\alpha}^{n,S}, 1]$, where $\bar{\alpha}^{n,S}$ is different from $\bar{\alpha}^{n,\tau}$ because of additional effects associated with a productivity increase. Formally,

**Proposition 6.** *Assume there are two countries and two industries, with industry 1 playing the role of an outside good (i.e., $\varepsilon_1 = \infty$, $\psi_1 = 0$ and $\tau_{12,1} = \tau_{21,1} = 1$) and industry 2 having scale economies with $\alpha > 0$. Assume that the initial equilibrium is interior (i.e., $L_{i,k} > 0$ for $i = 1, 2$ and $k = 1, 2$). There exists a threshold $\bar{\alpha}^{n,\tau} \in (0, 1)$ such that country $n$ loses from a small unilateral trade liberalization in industry 2 if and only if*

$\alpha \in (\bar{\alpha}^{n,\tau}, 1]$. *Similarly, there exists a threshold* $\bar{\alpha}^{n,S} \in (0,1)$ *such that country* $n$ *loses from a small foreign productivity improvement in industry* $2$ *if and only if* $\alpha \in (\bar{\alpha}^{n,S}, 1]$.

*Proof.* See Appendix D.2. □

These results generalize the propositions of immiserizing inward trade liberalization and foreign productivity improvements in 4.2.2 in two ways. First, the result holds as long as scale economies are strong enough, with the threshold for $\alpha$ depending on import and export shares in the industry with scale economies. Second, and more broadly, the result is shown to be a manifestation of the more general idea that a shock that pushes a country to specialize in an industry with weak economies of scale (here the outside good) may lower the gains from trade.

## 4.3. Gains from Trade: Numbers Using Data

In this subsection we continue our exploration of the gains from trade using actual data. Using (13) as our reference point, we follow CR and compute measures of $\lambda_{nn,k}$, $e_{n,k}$, and $r_{n,k}$ using data on 31 sectors from the WIOD in 2008.[29] We start by assuming a common trade elasticity of 5 for all industries (i.e., $\varepsilon_k = 5$ for all $k$) and consider two cases for the scale elasticity.[30] For the first case we assume a common scale elasticity across all industries and consider three subcases: (i) no scale economies, $\psi_k = 0$ for all $k$; (ii) intermediate scale economies, $\psi_k = 0.1$ for all $k$; and (iii) strong scale economies, $\psi_k = 0.2$ for all $k$. Note that these three subcases correspond to assuming $\alpha_k = 0$ for all $k$, $\alpha_k = 0.5$ for all $k$, and $\alpha_k = 1$ for all $k$, respectively. For the second case we assume no scale economies for all non-manufacturing industries and strong scale economies for all manufacturing industries, i.e., $\psi_k = 0$ for all $k \notin \mathbb{M}$ and $\psi_k = 0.2$ for all $k \in \mathbb{M}$, where $\mathbb{M}$ is the set of manufacturing industries. This case is used below to understand how specialization in industries with weak or strong scale economies affects the gains from trade.

[29] Equation (13) ignores trade deficits. As discussed in CR, this implies that our results in this subsection capture the change in real income rather than the change in real expenditure caused by shutting down trade and closing any trade deficits that exist in the data.

[30] We choose a value of 5 for the trade elasticity as this is a typical value used in the literature — see Head and Mayer (2014).

Table 2: Gains from Trade, Common Elasticities

| | Gains from Trade | | | | Degree of specialization | |
|---|---|---|---|---|---|---|
| | $\psi_k = 0$ common | $\psi_k = 0.1$ common | $\psi_k = 0.2$ common | $\psi_k = 0.2$ manuf. | All industries | Manuf. only |
| Country | (1) | (2) | (3) | (4) | (5) | (6) |
| AUS | 3.0% | 2.7% | 2.3% | 2.1% | 0.036 | 0.048 |
| BEL | 13.8% | 13.2% | 12.5% | 14.0% | 0.072 | -0.012 |
| BRA | 1.6% | 1.6% | 1.5% | 1.5% | 0.004 | 0.004 |
| CHN | 3.0% | 2.8% | 2.7% | 3.5% | 0.016 | -0.025 |
| GRC | 5.7% | 4.6% | 3.6% | 3.5% | 0.110 | 0.114 |
| JPN | 2.4% | 2.0% | 1.6% | 2.9% | 0.041 | -0.023 |
| KOR | 6.6% | 5.3% | 4.1% | 7.3% | 0.133 | -0.037 |
| USA | 2.1% | 2.0% | 1.9% | 1.7% | 0.007 | 0.020 |
| **Average** | 6.3% | 5.9% | 5.6% | 6.1% | 0.040 | 0.008 |

Columns 1, 2 and 3 in Table 2 report the gains from trade for the first case with scale elasticities as in subcases (i), (ii) and (iii), respectively, while column 5 reports the degree of industry specialization as represented by the term $D_{KL}(\boldsymbol{e}_n \parallel \boldsymbol{r}_n)$. This table as well as the ones below present results for a select set of countries – tables with the full set of countries in the WIOD are presented in Appendix D.5. Consistent with Proposition 5, gains from trade decrease as we allow for stronger scale economies, and this decline is stronger for countries that have a higher degree of industry specialization. This is illustrated in Figure 3, which plots the gains from trade net of the standard ACR gains (i.e., the gains that would arise in the absence of scale economies) for each subcase.

Turning to the second case, note that here we have

$$GT_n = 1 - \left( \prod_k (\lambda_{nn,k})^{e_{n,k}} \exp\left[ \sum_{k \in \mathbb{M}} e_{n,k} \ln\left(e_{n,k}/r_{n,k}\right) \right] \right)^{0.2}.$$

The term $\sum_{k\in\mathbb{M}} e_{n,k} \ln\left(e_{n,k}/r_{n,k}\right)$ is no longer the Kullback-Leibler divergence because we are only adding across $k \in \mathbb{M}$, so both $\sum_{k\in\mathbb{M}} e_{n,k}$ and $\sum_{k\in\mathbb{M}} r_{n,k}$ can be lower than $1$. This implies that this term can be negative (capturing specialization in manufac-

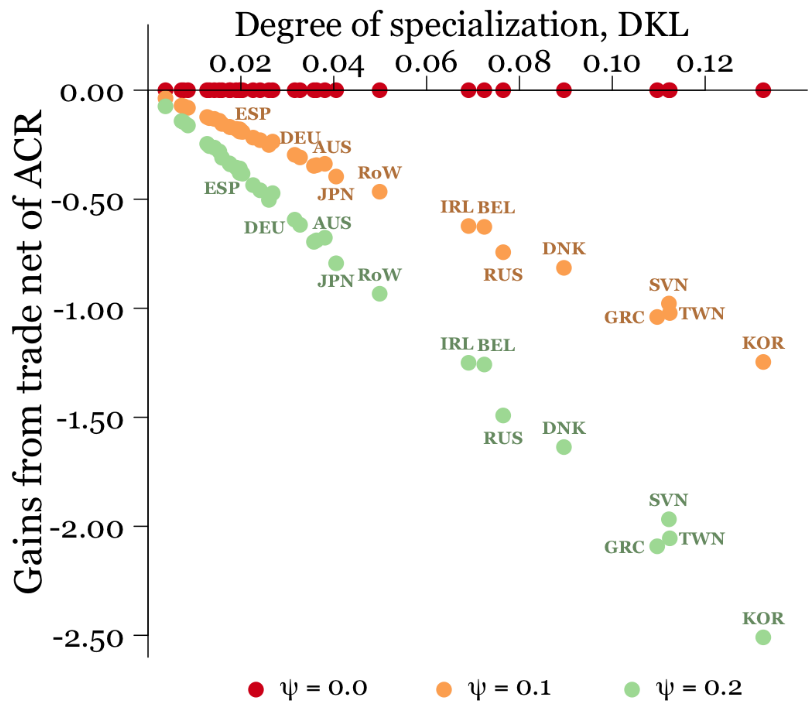



Figure 3: Degree of specialization and gains from trade

turing) and exert a positive effect on the gains from trade. Column 4 in Table 2 reports the gains from trade associated with this case, while column 6 reports the term $\sum_{k\in\mathbb{M}} e_{n,k} \ln\left(e_{n,k}/r_{n,k}\right)$. As expected, countries that specialize in manufacturing industries have higher gains from trade than in the case without scale economies. Figure 4 illustrates this by plotting the *pattern of specialization* (*PS*) — measuring in this case the tendency to specialize in manufacturing — against the the *degree of specialization* (*DS*) for selected countries as defined in (15). For each point we also report the name of the country, the standard ACR gains and the gains with $\alpha_k = 1$ in all manufacturing industries, respectively.

Next, we allow for heterogeneous trade elasticities across industries. Trade elasticities for agriculture and manufacturing industries are from Caliendo and Parro (2015), while for service industries we assume a trade elasticity of 5.[31] For each industry we maintain the assumption that the scale elasticity is inversely proportional to the trade elasticity and, consistent with our first exercise, separately consider three cases: (i)

[31] Since there is little trade in services the value of that elasticity has negligible effects.

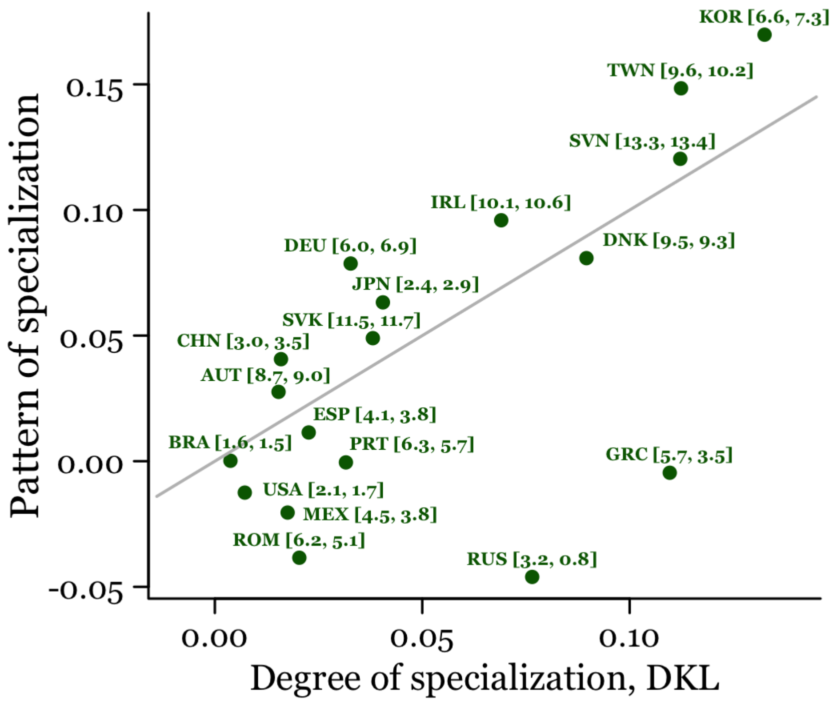



Figure 4: Pattern of specialization and gains from trade

$\psi_k = 0$ for all $k$, (ii) $\psi_k = 0.5/\varepsilon_k$ for all $k$, and (iii) $\psi_k = 1/\varepsilon_k$ for all $k$.

The gains corresponding to these three cases are reported in columns 1, 2 and 3 of Table 3.[32] While cross-country average gains from trade are not very different across the three cases, we do see that for some countries the gains can be substantially larger with economies of scale, while for others these gains are much smaller. For example, relative to the case with no scale economies, Japan's gains almost double in the intermediate case, and almost triple in the case with the highest scale elasticity (to be precise, gains are $2.4\%$, $4.3\%$ and $6.1\%$, respectively), whereas for Greece the reverse is true (gains are $14.5\%$, $10.1\%$ and $5.5\%$, respectively). Still for other countries these gains do not change much. For example, in Austria's case gains are $29.1\%$, $29.3\%$ and $29.4\%$.

[32]The results in columns 1 and 3 are are similar to those reported in columns 2 and 3 of Table 4.1 in CR. The difference is due to the way expenditure shares, $\beta_{n,k}$, are inferred from the data. CR calculate $\beta_{n,k}$ as shares of final consumption, while in this paper we calculate $\beta_{n,k}$ as shares of total expenditure. This distinction is not important for our model because it does not feature intermediates.

Table 3: Gains from Trade, Caliendo-Parro elasticities

| | Gains from Trade | | | Degree and Pattern of Specialization | | |
|---|---|---|---|---|---|---|
| | $\psi_k = 0$ | $\psi_k = \frac{0.5}{\varepsilon_k}$ | $\psi_k = \frac{1}{\varepsilon_k}$ | *DS* | *PS* | Overall |
| Country | (1) | (2) | (3) | (4) | (5) | (6) |
| AUS | 7.6% | 5.5% | 3.5% | 0.036 | -0.111 | 0.147 |
| BEL | 29.8% | 29.6% | 29.3% | 0.072 | 0.051 | 0.021 |
| BRA | 3.4% | 3.6% | 3.9% | 0.004 | 0.019 | -0.015 |
| CHN | 4.1% | 4.0% | 4.0% | 0.016 | 0.013 | 0.003 |
| GRC | 14.5% | 10.1% | 5.5% | 0.110 | -0.230 | 0.340 |
| JPN | 2.4% | 4.3% | 6.1% | 0.041 | 0.173 | -0.133 |
| KOR | 6.2% | 8.7% | 11.2% | 0.133 | 0.318 | -0.185 |
| USA | 4.5% | 4.4% | 4.3% | 0.007 | 0.000 | 0.007 |
| **Average** | 14.6% | 14.2% | 13.7% | 0.040 | 0.008 | 0.032 |

To better understand these results, Table 3 reports the *DS*, *PS* and overall effect on gains (i.e., the term inside the square parenthesis on the RHS of Equation (15)) in columns 4, 5 and 6, respectively. On closer examination we now see that Japan's tendency to specialize in industries with higher than average scale economies (*PS* is positive) is dominant enough to overpower *DS*, thereby implying larger gains from trade relative to the standard framework without scale economies. In contrast, for Greece the tendency to specialize in industries with lower than average scale economies (*PS* is negative) reinforces *DS*, implying lower gains from trade. Interestingly, while the results also reveal China's tendency to specialize in industries with higher than average scale economies (*PS* is higher than zero), the extent of this pattern of specialization is not sufficient to imply larger overall gains from trade. Finally, notice that these results generalize insights from the previous case in which scale economies are assumed only in manufacturing industries.

### 4.4. Gains from Trade Liberalization and Foreign Productivity Gains

We now use the exact hat algebra approach popularized by Dekle, Eaton and Kortum (2008) to explore the implications of a decline in inward trade costs or productivity gains abroad. This technique works as long as we start from an equilibrium that does not have corners, which is the case in our data as there are no $(i,k)$ pairs with $L_{i,k}=0$. As we show in Appendix D.3, the system to compute the hat changes in labor allocations and wages is

$$\hat{L}_{i,k}\geq 0, \quad G'_{i,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)\geq 0, \quad \hat{L}_{i,k}G'_{i,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)=0, \quad \text{for all} \quad (i,k),$$

$$\sum_k \hat{L}_{i,k}Y_{i,k}=Y_i, \quad \text{for all} \quad i,$$

where $Y_i\equiv w_i\bar{L}_i$ and $Y_{i,k}\equiv w_iL_{i,k}$, and

$$G'_{i,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)\equiv \hat{w}_i-\frac{1}{\hat{L}_{i,k}Y_{i,k}}\sum_n \frac{\hat{S}_{i,k}\hat{L}_{i,k}^{\alpha_k}\left(\hat{w}_i\hat{\tau}_{ni,k}\right)^{-\varepsilon_k}\lambda_{ni,k}}{\sum_l \hat{S}_{l,k}\hat{L}_{l,k}^{\alpha_k}\left(\hat{w}_l\hat{\tau}_{nl,k}\right)^{-\varepsilon_k}\lambda_{nl,k}}\beta_{n,k}(\hat{w}_nY_n+D_n),$$

where $D_n$ are trade imbalances in the data. Note that this still allows the counterfactual equilibrium to exhibit corner allocations. Therefore, we need to calculate changes in welfare explicitly by using the formula $\hat{w}_n/\hat{P}_n$, where changes in price indices are given by

$$\hat{P}_n=\prod_k\left(\sum_l \hat{S}_{l,k}\hat{L}_{l,k}^{\alpha_k}\left(\hat{w}_l\hat{\tau}_{nl,k}\right)^{-\varepsilon_k}\lambda_{nl,k}\right)^{-\beta_{n,k}/\varepsilon_k}.$$

We use the algorithm outlined in Section 3.3 to compute $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{L}}$ given some exogenous shock to trade costs or technology levels. A detailed description of the algorithm is in Appendix D.4. Theoretical results from Section 4.2.2 suggest that there might be losses from unilateral trade liberalization or foreign productivity improvements. To explore these theoretical results quantitatively, we again divide all sectors into manufacturing ($\mathbb{M}$) and non-manufacturing. We assume that non-manufacturing sectors do not exhibit economies of scale ($\alpha_k=0$ for these sectors), while for the manufacturing sectors we consider three possible scenarios: "low" $\alpha$ ($\alpha_M=0$), "intermediate" $\alpha$ ($\alpha_M=1/2$), and "high" $\alpha$ ($\alpha_M=1$).

Table 4: Inward Trade Liberalization and Foreign Productivity Improvement

| | $\widehat{\tau}_{ni,k} = 0.9 \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{S}_{CHN,k} = 1.01^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | |
|---|---|---|---|---|---|---|
| | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ |
| Country | (1) | (2) | (3) | (4)* | (5)* | (6)* |
| AUS | 0.64% | 0.51% | 0.25% | 0.94% | 0.31% | -0.76% |
| BEL | 1.89% | 1.81% | 1.45% | 0.16% | -0.11% | -0.44% |
| BRA | 0.35% | 0.26% | 0.16% | 0.32% | 0.10% | -0.13% |
| CHN | 0.63% | 0.54% | 0.43% | 48.87% | 50.99% | 53.08% |
| GRC | 0.90% | 0.64% | 0.08% | 0.62% | 0.27% | -0.22% |
| JPN | 0.30% | 0.25% | 0.20% | 0.00% | -0.15% | -0.17% |
| KOR | 0.80% | 0.68% | 0.53% | -0.00% | -0.55% | -0.97% |
| USA | 0.39% | 0.30% | 0.17% | 0.28% | 0.05% | -0.17% |

* The numbers specified in the column have been multiplied by 100.

In the first three columns of Table 4 we present results of the exercise where a country unilaterally lowers trade costs with all other countries by $10\%$ in manufacturing industries. Each row in these columns presents welfare implications of this exercise for a particular country. For example, the row corresponding to the United States presents welfare changes for that country given with $\hat{\tau}_{ni,k} = 0.9$ for $n = US$ and all $k \in \mathbb{M}$ and with all other parameters unchanged. In the last three columns of Table 4 we present the welfare implications of productivity increase of 1% in all manufacturing sectors in China, $\hat{S}_{ik} = 1.01^{\varepsilon_k}$ for $i =$ China and all $k \in \mathbb{M}$.[33]

The results in Table 4 reveal that the gains from unilateral trade liberalization in the sectors with scale economies get smaller as we consider a higher scale elasticity. For example, the gains for the United States decline from $0.39\%$ when $\alpha_M = 0$ to $0.17\%$ for $\alpha_M = 1$. Still, in contrast to the results of Section 4.2.2, even for $\alpha_M = 1$ the gains are positive for all countries. One key difference between the theoretical exercise in Section 4.2.2 and the quantitative exercise in Table 4 is that in the theoretical exercise we have an outside good that pins down wages, whereas in the quantitative exercise wages fully adjust in response to the shock. In Table 5 we considered the same change in trade costs as in the first three columns of Table 4 but holding wages fixed. As we see in the last column of Table 5, now Japan and Korea experience losses from unilateral trade

[33] Holding wages fixed, the hat change in the domestic price index of country $i$ in sector $k$ given some $\hat{S}_{ik}$ is $\hat{S}_{i,k}^{1/\varepsilon_k}$. That is why, to think of an x% productivity increase in China's sector $k$, we set $\hat{S}_{i,k} = (1+x)^{\varepsilon_k}$.

liberalization in manufacturing if $\alpha_M = 1$. To get even closer to the theoretical exercise in Section 4.2.2, we aggregated all sectors into manufacturing and non-manufacturing and all countries into the liberalizing country and the rest of the world. Columns $4$-$6$ of Table 5 report the results associated with a shock $\hat{\tau}_{ni,M} = 0.9$, whereas columns $1$-$3$ report those associated with a smaller shock $\hat{\tau}_{ni,M} = 0.99$. Column $6$ shows that now most countries lose from unilateral trade liberalization when $\alpha_M = 1$, while in column $3$ we see that all countries lose. The reason that some countries still gain in the case of the larger shock is that the shock pushes them to a corner equilibrium in which they no longer allocate any labor to manufacturing.

Table 5: Unilateral Trade Liberalization with Fixed Wages

| | $\widehat{\tau}_{ni,k} = 0.99 \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{\tau}_{ni,k} = 0.9 \quad \forall\, k \in \mathbb{M}$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $2 \times 2$ | | | | | | $N \times S$ | | |
| | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ |
| Country | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| AUS | 0.08% | 0.05% | -0.00% | 0.91% | 0.69% | -0.00% | 0.89% | 0.71% | 0.22% |
| BEL | 0.19% | 0.18% | -0.01% | 2.12% | 2.08% | 1.44% | 2.10% | 2.05% | 1.75% |
| BRA | 0.05% | 0.03% | -0.00% | 0.60% | 0.36% | -0.00% | 0.59% | 0.38% | 0.03% |
| CHN | 0.08% | 0.05% | -0.01% | 1.04% | 0.68% | -0.09% | 1.01% | 0.79% | 0.91% |
| GRC | 0.12% | 0.08% | -0.00% | 1.40% | 1.08% | -0.00% | 1.37% | 1.12% | 0.51% |
| JPN | 0.04% | 0.03% | -0.00% | 0.55% | 0.35% | -0.02% | 0.54% | 0.36% | -0.01% |
| KOR | 0.11% | 0.07% | -0.00% | 1.35% | 0.97% | -0.02% | 1.34% | 0.98% | -0.06% |
| USA | 0.06% | 0.04% | -0.00% | 0.70% | 0.48% | -0.04% | 0.68% | 0.55% | 0.47% |

Turning to the gains from foreign productivity improvements in the industries with scale economies, the last three columns of Table 4 show that some countries lose when China's productivity increases in manufacturing. Negative values in column $4$ simply reflect terms of trade effects, which worsen for countries that compete more directly with China. Columns $5$ and $6$ show additional effects arising from the mechanism outlined in Section 4.2.2. In particular, China's productivity increase in manufacturing implies a contraction in manufacturing employment in other countries, which in turn leads to a decline in productivity given the presence of scale economies in that sector. Thus, we see more and more countries losing from China's productivity increase as the strength of scale economies increases to $\alpha_M = 1/2$ and then to $\alpha_M = 1$.

Table 6: Foreign Productivity Improvement with Fixed Wages

| | $\widehat{S}_{RoW,k} = 1.01^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{S}_{RoW,k} = 2^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{S}_{CHN,k} = 1.01^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $2 \times 2$ | | | | | | $N \times S$ | | |
| | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ |
| Country | (1)* | (2)* | (3)* | (4) | (5) | (6) | (7)* | (8)* | (9)* |
| AUS | 7.47% | 3.05% | -7.11% | 10.31% | 10.14% | 10.15% | 1.66% | 1.13% | -0.62% |
| BEL | 18.97% | 14.73% | -85.11% | 16.81% | 16.79% | 16.81% | 1.05% | 0.30% | -3.95% |
| BRA | 4.61% | 0.46% | -5.33% | 11.09% | 10.27% | 10.15% | 0.88% | 0.36% | 0.25% |
| CHN | 8.02% | -0.67% | -16.28% | 19.14% | 18.96% | 20.14% | 40.93% | 47.40% | 74.24% |
| GRC | 11.74% | 6.95% | -4.39% | 14.25% | 14.09% | 14.09% | 0.87% | 0.28% | 0.48% |
| JPN | 4.22% | -0.68% | -8.31% | 10.38% | 9.75% | 9.87% | 1.48% | 0.25% | -3.19% |
| KOR | 10.66% | 0.46% | -20.93% | 19.66% | 19.05% | 19.13% | 3.45% | 0.74% | -11.19% |
| USA | 5.53% | 1.98% | -4.81% | 9.36% | 9.37% | 9.68% | 1.36% | 0.86% | -0.02% |

* The numbers specified in the column have been multiplied by 100.

We explore these findings further in Table 6, where (as in Table 5) we fix wages to the baseline equilibrium (i.e., $\hat{w}_i = 1$). With no terms of trade changes, now all countries gain from China's productivity increase if there are no scale economies (column 7), and more and more countries lose as scale economies become stronger. To link these findings with the theoretical results in Section 4.2.2, the first three columns of Table 6 show analogous results but after aggregating to two sectors (manufacturing and non-manufacturing) and two countries (the country in each row and the rest of the world). Columns 1 and 2 show that almost all countries gain from a productivity increase in manufacturing in the rest of the world if $\alpha_M \leq 1/2$ but all countries lose when $\alpha_M = 1$, a reflection of the fact that in most cases the actual threshold $\bar{\alpha}^S$ derived in Section 4.2.2 is high and close to 1. A final set of results is presented in columns 4-6, which show analogous results to those in columns 1-3 but for a doubling of foreign productivity. Interestingly, this now leads to gains in all countries. The reason is that the shock is so large that manufacturing employment falls to zero in all countries, thus truncating the mechanism in Section 4.2.2 since a sector cannot keep shrinking after it vanishes.

## 5. Scale Economies and Trade Flows

In this subsection we quantify the role of scale economies in determining industry-level specialization and trade flows. In particular, we ask how these variables would change if we shut down scale economies but leave all other exogenous variables unchanged.

We rely on the fact that if $\boldsymbol{L}$ is an equilibrium of the actual economy with scale economies then it is also an equilibrium of the economy with no scale economies given by

$$w_i L_{i,k} = \sum_{n=1}^{N} \frac{T_{i,k}\left(w_i \tau_{ni,k}\right)^{-\varepsilon_k}}{\sum_{l=1}^{N} T_{l,k}\left(w_l \tau_{nl,k}\right)^{-\varepsilon_k}} \beta_{n,k}(w_n \bar{L}_n + D_n)$$

and

$$\sum_{k=1}^{K} L_{i,k} = \bar{L}_i,$$

where $T_{i,k} \equiv S_{i,k} L_{i,k}^{\alpha_k}$ and where $D_n$ are trade deficits satisfying $\sum_n D_n = 0$. Thus, if we want to know the counterfactual allocation for the economy with $\alpha_k = 0$ for all $k$ but everything else equal, we can use the exact hat algebra approach in the economy with no scale effects subjected to a shock to productivities $T_{i,k}$ given by $\hat{T}_{i,k} = S'_{i,k} / \left(S_{i,k} L_{i,k}^{\alpha_k}\right) = \hat{S}_{i,k} L_{i,k}^{-\alpha_k}$.[34] To focus on the interaction between specialization and scale economies, we assume that $\hat{S}_{i,k}$ is such that if country $i$ was in autarky then the shock would have no effect on productivity — since in autarky $L_{i,k} = \beta_{i,k} \bar{L}_i$ this requires $\hat{S}_{i,k} = (\beta_{i,k} \bar{L}_i)^{\alpha_k}$. Using $L_{i,k} = r_{i,k} \bar{L}_i$ and measuring $\beta_{i,k}$ by $e_{i,k}$, this implies that $\hat{T}_{i,k} = (e_{i,k}/r_{i,k})^{\alpha_k}$. Combining the two previous equations and using $Y_i \equiv w_i \bar{L}_i$, we get a system in wage changes given by

$$\hat{w}_i Y_i = \sum_{k=1}^{K} \sum_{n=1}^{N} \frac{(e_{i,k}/r_{i,k})^{\alpha_k} \left(\hat{w}_i\right)^{-\varepsilon_k} \lambda_{ni,k}}{\sum_{l=1}^{N} (e_{l,k}/r_{l,k})^{\alpha_k} \left(\hat{w}_l\right)^{-\varepsilon_k} \lambda_{nl,k}} e_{n,k} \left(\hat{w}_n Y_n + D_n\right).$$

The solution for $\hat{w}_i$ can then be used to get the implied hat change in the labor allocation from

$$\hat{L}_{i,k} = \frac{1}{\hat{w}_i Y_{i,k}} \sum_{n=1}^{N} \frac{(e_{i,k}/r_{i,k})^{\alpha_k} \left(\hat{w}_i\right)^{-\varepsilon_k} \lambda_{ni,k}}{\sum_{l=1}^{N} (e_{l,k}/r_{l,k})^{\alpha_k} \left(\hat{w}_l\right)^{-\varepsilon_k} \lambda_{nl,k}} e_{n,k} \left(\hat{w}_n Y_n + D_n\right),$$

[34] We can ignore corner solutions because the data has no zeros at the industry level (i.e., $r_{i,k} > 0$ for all $i, k$) and the shock that we consider moves us away from corners.

where $Y_{i,k} \equiv w_i L_{i,k}$. Finally, we can get the implied change in trade flows from

$$\hat{X}_{ni,k} = \frac{1}{X_{ni,k}} \cdot \frac{(e_{i,k}/r_{i,k})^{\alpha_k} (\hat{w}_i)^{-\varepsilon_k} \lambda_{ni,k}}{\sum_{l=1}^{N} (e_{l,k}/r_{l,k})^{\alpha_k} (\hat{w}_l)^{-\varepsilon_k} \lambda_{nl,k}} e_{n,k} (\hat{w}_n Y_n + D_n).$$

Table 7 presents the results of this exercise for each of the countries in our sample. Column 1 reports the degree of specialization ($DS$) in the data as defined in (14). Columns 2 and 3 report, respectively, the implied percentage change in $DS$ and in total exports for each country when $\alpha_k = 1/2$ and $\varepsilon_k = 5$ for all $k$. Columns 4 and 5 do the same but for $\alpha_k = 1$.

Table 7: Scale and Trade Flows

| | | $\alpha = 0.5$ | | $\alpha = 1$ | |
|---|---|---|---|---|---|
| | $DS_i$ | $\widehat{DS}_i$ | $\widehat{EX}_i$ | $\widehat{DS}_i$ | $\widehat{EX}_i$ |
| Country | (1) | (2) | (3) | (4) | (5) |
| AUS | 0.04 | -60.80% | -9.44% | -93.03% | -16.61% |
| BEL | 0.07 | -64.83% | -0.86% | -98.20% | -0.36% |
| BRA | 0.00 | -27.81% | -3.99% | -49.37% | -7.73% |
| CHN | 0.02 | -50.09% | -5.71% | -77.27% | -10.19% |
| GRC | 0.11 | -80.81% | -34.11% | -92.28% | -36.71% |
| JPN | 0.04 | -64.05% | -7.84% | -91.77% | -16.35% |
| KOR | 0.13 | -64.98% | -7.19% | -94.25% | -17.12% |
| USA | 0.01 | -47.24% | -4.85% | -71.86% | -8.63% |

As expected, the removal of economies of scale implies a decline in the degree of specialization and total trade. Column 2 shows that the decline in $DS$ is quite strong even in the case in which the initial economies of scale are given by $\alpha_k = 1/2$ for all $k$, with $DS$ falling by more than $50\%$ in almost all cases. Column 4 shows that this decline is much sharper in the case of $\alpha_k = 1$ for all $k$, with $DS$ falling by more than $70\%$ in almost all cases, and in several cases by more than $90\%$. Columns 3 and 5 show that total exports fall but less dramatically: the decline in total world exports corresponding to these two columns is $5.4\%$ and $10\%$.

These results imply that if the data is generated by a gravity model with scale economies as strong as in the Krugman model (i.e., $\alpha_k = 1$ for all $k$), then most of the industry-level specialization that we observe is due to scale economies rather than Ricardian comparative advantage. In other words, without scale economies amplifying Ricardian productivity differences, industry-level specialization would be, in many cases, more than 90% less than what we see in the data. In contrast, since most of world trade is intra-industry trade, scale economies as strong as in the Krugman model would be responsible for only around 10% of total world trade.

## 6. Concluding Remarks

For over a century since Alfred Marshall's initial exposition, economists have been intrigued with the implications of industry-level external economies of scale for trading economies. Despite such interest, however, the discomfort with the plethora of equilibria and counter-intuitive implications in early work relegated Marshallian externalities to the "back-burner" of the recent trade literature. In this paper we show how one can add Marshallian externalities to the Eaton and Kortum (2002) framework while retaining the property that the equilibrium is unique. The resulting model has exactly the same mathematical structure as generalized versions of the multi-industry Krugman and Melitz-Pareto models and so our uniqueness result applies to these well-known models as well. The key condition for uniqueness is simple and intuitive: the scale elasticity must be weakly lower than the inverse of the trade elasticity for all industries.

The model is rich in welfare implications. Most importantly, if parameters are in the region of uniqueness then all countries gain from trade. Economies of scale tend to make gains from trade lower and gains from trade liberalization higher relative to models without scale economies, with results more positive for countries that specialize in industries with stronger than average scale economies.

Finally, our exploration of the role of economies of scale in explaining trade flows and industry-level specialization in the data yields very definitive results: if scale economies are as strong as those in the Krugman or Melitz-Pareto models then most of the industry-level specialization that we see in the data is due to economies of scale rather than pure Ricardian comparative advantage.

Many questions remain open for future research. How can we extend our uniqueness results for endogenous wages beyond the cases of $N = 2$ or frictionless trade? How can we robustly estimate the strength of industry-level scale economies? Is the condition necessary for uniqueness likely to be satisfied? Are scale economies stronger in some industries than others? If so, which ones? How does the presence of industry-level scale economies affect country-level and world-level optimal trade policies?

# A. Definitions

It is clear that both functions $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ which appear in the non-linear complementarity problem (3) are well-defined for all positive wages and positive labor allocations, i.e., for all $\boldsymbol{w}\in\mathbb{R}^N_{++}$ and $\boldsymbol{L}_k\in\mathbb{R}^N_{++}$. We are interested in extending the definitions of $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ to the set of all non-negative labor allocations excluding the point with $L_{i,k}=0$ for all $i$, i.e., to the set $\mathbb{R}^N_{+}\setminus\{\boldsymbol{0}\}$. To this end, we allow for function $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ to take infinite values. Formally, we consider the function $G_{i,k}:\mathbb{R}^N_{++}\times\mathbb{R}^N_{+}\setminus\{\boldsymbol{0}\}\to\mathbb{R}\cup\{-\infty,+\infty\}$,[35] and for each given vector of wages $\boldsymbol{w}\in\mathbb{R}^N_{++}$ and vector of labor allocations $\boldsymbol{L}_k\in\mathbb{R}^N_{+}\setminus\{\boldsymbol{0}\}$ we formally define $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ by the limits

$$G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)\equiv\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}\left[w_i-\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)\beta_{n,k}w_n\bar{L}_n\right]$$

and

$$L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)\equiv\lim_{\boldsymbol{x}^t\to\boldsymbol{L}_k}x_i^t\left[w_i-\frac{1}{x_i^t}\sum_n\lambda_{ni,k}(\boldsymbol{w},\boldsymbol{x}^t)\beta_{n,k}w_n\bar{L}_n\right],$$

where $\left\{\boldsymbol{x}^t\right\}_{t=1}^{\infty}$ is any sequence converging to $\boldsymbol{L}_k$ such that $\boldsymbol{x}^t\in\mathbb{R}^N_{++}$ for $t=1,2,\ldots$.

Let us verify that functions $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ are well-defined. Since for all $\boldsymbol{L}_k\in\mathbb{R}^N_{++}$ functions $G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ and $L_{i,k}G_{i,k}\left(\boldsymbol{w},\boldsymbol{L}_k\right)$ are well-defined and continuous, the above limits coincide with the values of these functions in the corresponding points.

Next, consider any sequence $\left\{\boldsymbol{x}^t\right\}_{t=1}^{\infty}$ with $\boldsymbol{x}^t\in\mathbb{R}^N_{++}$ for $t=1,2,\ldots$ and converging

[35]Here the set $\mathbb{R}\cup\{-\infty,+\infty\}$ is the extended real number system with symbols $-\infty$ and $+\infty$ following the standard conventions (see, for example, p. 11-12 in Rudin, 1976). In particular, for any $x\in\mathbb{R}$, $-\infty<x<+\infty$.

to $\boldsymbol{L}_k$. We have

$$\lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} \frac{1}{x_i^t} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}^t) = \lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} \left[x_i^t\right]^{\alpha_k - 1} \sum_n \frac{S_{i,k} \left(w_i \tau_{ni,k}\right)^{-\varepsilon_k}}{\sum_l S_{l,k} \left[x_l^t\right]^{\alpha_k} \left(w_l \tau_{nl,k}\right)^{-\varepsilon_k}} \beta_{n,k} w_n \bar{L}_n.$$

Then, since, $\boldsymbol{L}_k \neq \boldsymbol{0}$,

$$\lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} \sum_l S_{l,k} \left[x_l^t\right]^{\alpha_k} \left(w_l \tau_{nl,k}\right)^{-\varepsilon_k} = \sum_l S_{l,k} L_{l,k}^{\alpha_k} \left(w_l \tau_{nl,k}\right)^{-\varepsilon_k} > 0 \text{ for all } n.$$

Hence, $\lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} \frac{1}{x_i^t} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}^t) = \infty$ if $L_{i,k} = 0$ and $0 \leq \alpha_k < 1$, and $\lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} \frac{1}{x_i^t} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}^t)$ is a positive number if $L_{i,k} > 0$ or if $\alpha_k \geq 1$. This, in turn, implies that

$$\begin{aligned} G_{i,k}\left(\boldsymbol{w}, \boldsymbol{L}_k\right) &= \lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} \left[w_i - \frac{1}{x_i^t} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}^t) \beta_{n,k} w_n \bar{L}_n\right] \\ &= \begin{cases} -\infty, & \text{if } L_{i,k} = 0 \text{ and } 0 \leq \alpha_k < 1, \\ \text{finite number}, & \text{if } L_{i,k} > 0 \text{ or } \alpha_k \geq 1. \end{cases} \end{aligned}$$

So, the limit always exists and is either $-\infty$ or a finite number. Hence, function $G_{i,k}$ is well-defined with its codomain given by the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$.

Similarly, it is easy to verify that $\lim_{\boldsymbol{x}^t \to \boldsymbol{L}_k} x_i^t \left[w_i - \frac{1}{x_i^t} \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{x}^t) \beta_{n,k} w_n \bar{L}_n\right]$ always exists. Moreover, this limit is always a finite number. Hence, function $L_{i,k} G_{i,k}$ is also well-defined.

## B. Linking Melitz and Marshall

We consider a generalized Melitz-Pareto framework, in which the elasticity of substitution between varieties from different countries, $\eta_k$, is different from the elasticity of substitution between varieties from the same country, $\sigma_k$.

Let us ignore the industry subscript for a moment. Let $\Omega_{ni}$ denote the set of varieties that $i$ sells to $n$. The price index of these goods is $P_{ni} \equiv \left(\int_{\omega \in \Omega_{ni}} p_{ni}(\omega)^{1-\sigma} d\omega\right)^{\frac{1}{1-\sigma}}$. Let $M_i$ denote total entry in country $i$ and $\varphi_{ni}^*$ denote the cutoff productivity such that $i$

exports to $n$ all goods with productivity higher than $\varphi^*_{ni}$. We have

$$\begin{aligned} P_{ni}^{1-\sigma} &= M_i \int_{\varphi^*_{ni}}^{\infty} [p_{ni}(\varphi)]^{1-\sigma}\, dG_i(\psi) \\ &= \theta b_i^{\theta} M_i \left[\frac{\sigma}{\sigma-1} w_i \tau_{ni}\right]^{1-\sigma} \int_{\varphi^*_{ni}}^{\infty} \varphi^{\sigma-\theta-2} d\varphi \\ &= M_i \left[\frac{\sigma}{\sigma-1} w_i \tau_{ni}\right]^{1-\sigma} \frac{b_i^{\theta}\left(\varphi^*_{ni}\right)^{\sigma-\theta-1}}{\theta-(\sigma-1)}. \end{aligned}$$

The condition that determines the cutoff $\varphi^*_{ni}$ is

$$\frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1} \cdot \frac{w_i \tau_{ni}}{\varphi^*_{ni}}\right)^{1-\sigma} P_{ni}^{\sigma-1} \left(\frac{P_{ni}}{P_n}\right)^{1-\eta} X_n = w_n f_n.$$

This implies that

$$\varphi^*_{ni} = \frac{w_i \tau_{ni}}{P_{ni}} \left(\frac{\sigma}{\widetilde{\sigma}} \cdot \frac{w_n f_n}{X_n}\right)^{\frac{1}{\sigma-1}} \left(\frac{P_{ni}}{P_n}\right)^{\frac{1-\eta}{1-\sigma}},$$

where $\widetilde{\sigma} \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}$. Plugging this expression into the expression for the price index yields

$$P_{ni} = w_i \tau_{ni} \left(\frac{w_i \tau_{ni}}{P_{ni}} \left(\frac{X_n}{w_n f_n}\right)^{\frac{1}{1-\sigma}} \left(\frac{P_{ni}}{P_n}\right)^{\frac{1-\eta}{1-\sigma}}\right)^{\frac{\theta}{\sigma-1}-1} M_i^{\frac{1}{1-\sigma}} \bar{\kappa}^o_{i,k},$$

where

$$\bar{\kappa}^o_{i,k} \equiv \frac{\sigma}{\sigma-1} \left(\frac{b_i^{\theta}\left[\frac{\sigma}{\widetilde{\sigma}}\right]^{1-\frac{\theta}{\sigma-1}}}{\theta-(\sigma-1)}\right)^{\frac{1}{1-\sigma}}.$$

Bringing back the $k$ subindex and using the well known result that in this model equilibrium entry in each industry must satisfy $M_{i,k} = \frac{\sigma_k-1}{\sigma_k \theta_k} \frac{L_{i,k}}{F_{i,k}}$, we then have

$$P_{ni,k} = w_i \tau_{ni,k} \left(\frac{w_i \tau_{ni,k}}{P_{ni,k}} \left(\frac{X_{n,k}}{w_n f_n}\right)^{\frac{1}{1-\sigma_k}} \left(\frac{P_{ni,k}}{P_{n,k}}\right)^{\frac{1-\eta_k}{1-\sigma_k}}\right)^{\frac{\theta_k}{\sigma_k-1}-1} L_{i,k}^{\frac{1}{1-\sigma_k}} \kappa^o_{i,k}.$$

where $\kappa^o_{i,k} \equiv \left(\frac{\sigma_k-1}{\sigma_k\theta_k}F^{-1}_{i,k}\right)^{\frac{1}{1-\sigma_k}}\bar{\kappa}^o_{i,k}$. Using $X_{n,k} = \beta_{n,k}w_n\bar{L}_n$ we get

$$P_{ni,k} = w_i\tau_{ni,k}\left(\frac{w_i\tau_{ni,k}}{P_{ni,k}}\left(\frac{P_{ni,k}}{P_{n,k}}\right)^{\frac{1-\eta_k}{1-\sigma_k}}\right)^{\frac{\theta_k}{\sigma_k-1}-1} L_{i,k}^{\frac{1}{1-\sigma_k}}\kappa^d_{n,k}\kappa^o_{i,k},$$

where

$$\kappa^d_{n,k} \equiv \left[\frac{\beta_{n,k}\bar{L}_n}{f_{n,k}}\right]^{\frac{1}{1-\sigma_k}\left(\frac{\theta_k}{\sigma_k-1}-1\right)}.$$

Solving for $P_{ni,k}$ we get

$$P^{1-\eta_k}_{ni,k} = \left[(w_i\tau_{ni,k})^{-\theta_k}P_{n,k}^{\frac{1-\eta_k}{1-\sigma_k}(\theta_k-\sigma_k+1)}L_{i,k}(\kappa^d_{n,k}\kappa^o_{i,k})^{1-\sigma_k}\right]^{\xi_k}.$$

where $\xi_k$ is as defined in the text. Using $P^{1-\eta_k}_{n,k} = \sum_i P^{1-\eta_k}_{ni,k}$ and $\lambda_{ni,k} = \left(\frac{P_{ni,k}}{P_{n,k}}\right)^{1-\eta_k}$ wet get the expression for trade shares in the text. Finally, the expression for the price index in the text is obtained by combining $P^{1-\eta_k}_{n,k} = \sum_i P^{1-\eta_k}_{ni,k}$ with the result above for $P_{ni,k}$, with $\bar{\mu}^{Mel}_k \equiv \left(\frac{\sigma_k-1}{\sigma_k}\right)^{\frac{\sigma_k}{\theta_k}}\left(\frac{\frac{1}{\theta_k}\left(\frac{\sigma_k}{\bar{\sigma}_k}\right)^{1-\frac{\theta_k}{\sigma_k-1}}}{\theta_k-(\sigma_k-1)}\right)^{\frac{1}{\theta_k}}$.

# C. Existence and Uniqueness

## C.1. Proof of Proposition 1

**Proof of Lemma 1.** Here we prove that function $F(\cdot)$ defined in (7) is strictly convex on $\Gamma$. The uniqueness of solution of the minimization problem $\min_{\boldsymbol{x}\in\Gamma} F(\boldsymbol{x})$ then follows from the fact that $\Gamma$ is a convex set, and a strictly convex function can have at most one global minimum on a convex set.

If $0 < \alpha < 1$, then for any $n$ function $\sum_i a_{ni}x_i^\alpha$ is a strictly concave function. And since the logarithm is a strictly concave function, $F(\cdot)$ is strictly convex.

If $\alpha = 1$, then we need to make sure that for any two vectors $\boldsymbol{x} \neq \boldsymbol{y}$ we cannot have that $\sum_i a_{ni}x_i = \sum_i a_{ni}y_i$ for all $n$. Otherwise, we would have $F(\gamma\boldsymbol{x} + (1-\gamma)\mathbf{y}) = \gamma F(\boldsymbol{x}) + (1-\gamma)F(\boldsymbol{y})$ for any $\gamma \in [0,1]$ and strict convexity would be violated. Assumption 1 guarantees that matrix $A = (a_{ni})$ is non-singular. Hence, for any $\boldsymbol{x} \neq \boldsymbol{y}$ we have

$\sum_i a_{ni}x_i \neq \sum_i a_{ni}y_i$ for at least one $n$. Hence, in case of $\alpha = 1$, $F(\cdot)$ is also strictly convex under Assumption 1. $\square$

**Proof of Lemma 2.** Suppose, without loss of generality, that $x_1^* = 0$. Since $\sum_n x_n^* = \sum_n b_n > 0$, we can also suppose without loss of generality that $x_2^* \neq 0$. Consider the vector $\boldsymbol{x}(\varepsilon) = (\varepsilon, x_2^* - \varepsilon, x_3^*, \ldots, x_N^*)$, where $\varepsilon \in [0, x_2^*]$. Clearly, $\boldsymbol{x}(\varepsilon) \in \Gamma$. Define

$$\tilde{F}(\varepsilon) \equiv F\left(\boldsymbol{x}\left(\varepsilon\right)\right) = \alpha \sum_n x_n^* - \sum_n b_n \ln\left(a_{n1}\varepsilon^\alpha + a_{n2}(x_2^* - \varepsilon)^\alpha + \sum_{i=2}^N a_{ni}\left(x_i^*\right)^\alpha\right).$$

We now show that $\tilde{F}\left(\varepsilon\right) < \tilde{F}(0)$ for small enough $\varepsilon > 0$. We have

$$\frac{\partial\left(a_{j1}\varepsilon^\alpha + a_{j2}(x_2^* - \varepsilon)^\alpha\right)}{\partial\varepsilon} = \alpha a_{j1}\varepsilon^{\alpha-1} - \alpha a_{j2}(x_2^* - \varepsilon)^{\alpha-1},$$

which, given $\alpha \in (0,1)$, is positive for small enough $\varepsilon$. This implies that $a_{j1}\varepsilon^\alpha + a_{j2}(x_2^* - \varepsilon)^\alpha + \sum_{i=2}^N a_{ji}\left[x_i^*\right]^\alpha > a_{j2}\left[x_2^*\right]^\alpha + \sum_{i=2}^N a_i\left[x_i^*\right]^\alpha$ for small enough $\varepsilon$. Since $\ln(\cdot)$ is a strictly increasing function, we then get that $\tilde{F}\left(\varepsilon\right) < \tilde{F}\left(0\right)$ for small enough $\varepsilon$, a contradiction. This implies that $\boldsymbol{x}^*$ cannot be a global minimum of $F\left(\cdot\right)$ on $\Gamma$. Hence, in the case of $\alpha \in (0,1)$ we must have $x_i^* > 0$ for all $i$. $\square$

**Proof of Lemma 3.** Let us start with the simpler case of $\alpha = 1$. We can take the set $D \equiv \left\{\boldsymbol{x} \in \mathbb{R}^N \mid \sum_i a_{ni}x_i > 0 \text{ for all } n\right\}$ as the domain of $F(\cdot)$ and use $\Gamma$ as the constraint set. Clearly, $\Gamma \subset D \cap \left\{\boldsymbol{x} \in \mathbb{R}^N \mid \sum_i x_i = \sum_i b_i\right\}$ and $F(\cdot)$ is differentiable on $D$ for $\alpha = 1$. Consider the minimization problem

$$\begin{aligned} & \min_{\boldsymbol{x}\in D} F(\boldsymbol{x}) \\ \text{s.t.} & \\ & x_i \geq 0, \quad i = 1, \ldots, N; \\ & \sum_i x_i = \sum_i b_i. \end{aligned} \tag{19}$$

Its first-order conditions, after some manipulations, can be written as

$$
\begin{aligned}
&x_i \geq 0, \quad \frac{\partial F(\boldsymbol{x})}{\partial x_i} \geq 0, \quad x_i \frac{\partial F(\boldsymbol{x})}{\partial x_i} = 0, \quad i = 1, \ldots, N, \\
&\sum_i x_i = \sum_i b_i.
\end{aligned}
$$

These conditions are also sufficient, because $F(\cdot)$ is strictly convex and the set that satisfies the constraints of the minimization problem is a convex set. Then, since $G_i(\boldsymbol{x}) = \frac{\partial F(\boldsymbol{x})}{\partial x_i}$, we get that any solution of the minimization problem (19) is also a solution of the NCP in (6) and vice versa.

Let us now turn to the case with $0 < \alpha < 1$. Let $\boldsymbol{x}^*$ be a minimum of $F(\cdot)$ on $\Gamma$. By Lemma 2, $x_i^* > 0$ for all $i$. Let $\delta > 0$ be some number such that $\delta \leq x_i^*$ for all $i = 1, \ldots, N$. Define the domain of $F$ by $\tilde{D} \equiv \left\{\boldsymbol{x} \in \mathbb{R}^N \,|\, x_i > \delta/2, \; i = 1, \ldots, N\right\}$. Consider the minimization problem:

$$
\begin{aligned}
&\min_{\boldsymbol{x} \in \tilde{D}} F(\boldsymbol{x}) \\
&\text{s.t.} \\
&\sum_i x_i = \sum_i b_i.
\end{aligned} \tag{20}
$$

Since $\tilde{D} \cap \left\{\boldsymbol{x} \in \mathbb{R}^N \mid \sum_i x_i = \sum_i b_i\right\} \subset \Gamma$, if $\boldsymbol{x}^*$ minimizes $F$ on $\Gamma$, it also solves the minimization problem (20). Since $x_i^* > \delta/2$ for all $i$, the first-order conditions for (20) are given by

$$
\frac{\partial F(\boldsymbol{x}^*)}{\partial x_i} = 0, \; i = 1, \ldots, N, \quad \text{and} \quad \sum_i x_i^* = \sum_i b_i. \tag{21}
$$

Hence, $\boldsymbol{x}^*$ solves NCP in (6). Conversely, if $\boldsymbol{x}^*$ solves NCP, then $x_i^* > 0$ for all $i$ because the condition $G_i(\boldsymbol{x}^*) \geq 0$ cannot be satisfied for $x_i^* = 0$ if $0 < \alpha < 1$. Hence, $\boldsymbol{x}^*$ satisfies conditions (21), which are the first order conditions for an interior solution of (20) with an appropriately chosen $\delta > 0$. Since these first-order conditions are also sufficient, $\boldsymbol{x}^*$ solves (20). Now, suppose by contradiction that the minimum of $F(\cdot)$ on $\Gamma$ is some $\boldsymbol{x}^{**} \neq \boldsymbol{x}^*$. Then, by Lemma 2, $x_i^{**} > 0$ for all $i$. Therefore we can extend the open set on which $F(\cdot)$ is differentiable to include both $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$. Then both $\boldsymbol{x}^*$ and $\boldsymbol{x}^{**}$ satisfy the first-order conditions (21), which gives a contradiction given that $F(\cdot)$ is strictly convex and the constraint set is convex. Hence, $\boldsymbol{x}^*$ is the minimum of $F(\cdot)$ on $\Gamma$. □

**Proof of Lemma 4.** Let us formally bring argument $\alpha$ into the notation of function $F$ defined in (7), i.e., consider the function $F(\boldsymbol{x};\alpha)$. Lemma 1 establishes that under Assumption 1 the solution to the optimization problem $\min_{x\in\Gamma} F(\boldsymbol{x};\alpha)$ defines a function $\boldsymbol{x} : (0,1] \to \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$. Clearly, $F(\boldsymbol{x};\alpha)$ is continuous for all $\boldsymbol{x} \in \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$ and $\alpha \in (0,1]$. $\Gamma$ is a compact set which is the same for all $\alpha \in (0,1]$. Thus, all conditions for Theorem of the Maximum (Theorem 3.6) from Stokey, Lucas and Prescott (1989) are statisfied, and $\boldsymbol{x}(\alpha)$ is continuous for all $\alpha \in (0,1]$. □

## C.2. Proof of Proposition 2

**Proof of Lemma 5.** The case with $\alpha_k = 0$ is trivial because labor allocations are explicitly obtained from the goods market clearing conditions $L_{i,k}G_{i,k}(\boldsymbol{w},\boldsymbol{L}_k) = 0$, and the resulting expression for $L_{i,k}(\boldsymbol{w})$ are obviously continuous. Below we focus on the case with $\alpha_k \in (0,1]$.

Define a multi-valued correspondence $\Gamma_k : \mathbb{R}^N_{++} \to \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$ by

$$\Gamma_k(\boldsymbol{w}) = \left\{ \boldsymbol{L}_k \in \mathbb{R}^N \mid L_{i,k} \geq 0, \sum_i w_i L_{i,k} = \sum_i \beta_{i,k} w_i \bar{L}_i \right\}.$$

Define function $F_k : \left(\mathbb{R}^N_+ \setminus \{\mathbf{0}\}\right) \times \mathbb{R}^N_{++} \to \mathbb{R}$ by

$$F_k(\boldsymbol{L}_k;\boldsymbol{w}) = \alpha_k \sum_n w_n L_{n,k} - \sum_n \beta_{n,k} w_n \bar{L}_n \ln\left(\sum_i S_{i,k} L_{i,k}^{\alpha_k} (w_i \tau_{ni,k})^{-\varepsilon_k}\right).$$

First, Lemma 1 establishes that under conditions (a) or (b) the solution of the optimization problem $\min_{\boldsymbol{L}_k\in\Gamma(\boldsymbol{w})} F_k(\boldsymbol{L}_k;\boldsymbol{w})$ determines a function from wages to labor allocations, $\boldsymbol{L}_k(\boldsymbol{w})$, for any $\boldsymbol{w} \in \mathbb{R}^N_{++}$.

Next, it is straightforward to show that $\Gamma_k(\boldsymbol{w})$ is both lower hemi-continuous and upper hemi-continuous for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$ (see the corresponding definitions in Stokey, Lucas and Prescott, 1989). Hence, $\Gamma_k(\boldsymbol{w})$ is continuous for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$. Clearly, $\Gamma_k(\mathbf{w})$ is also compact-valued for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$. Finally, $F_k(\boldsymbol{L}_k;\boldsymbol{w})$ is continuous for all $\boldsymbol{L}_k \in \mathbb{R}^N_+ \setminus \{\mathbf{0}\}$ and $\boldsymbol{w} \in \mathbb{R}^N_{++}$. Thus, all conditions for Theorem 3.6 (Theorem of the Maximum) from Stokey, Lucas and Prescott (1989) are statisfied, and $\boldsymbol{L}_k(\boldsymbol{w})$ is a continuous function for all $\boldsymbol{w} \in \mathbb{R}^N_{++}$. □

**Proof of Proposition 2, property (v).** Consider any wage sequence $\{\boldsymbol{w}^s\}_{s=1}^{\infty}$ such that $\boldsymbol{w}^s \to \boldsymbol{w}$ as $s \to \infty$, where $\boldsymbol{w} \neq 0$ is a finite vector of wages such that $w_i = 0$ for some $i$. If there exists some industry $(i,k)$ such that $L_{i,k}(\boldsymbol{w}^s) \to \infty$, then $Z_i(\boldsymbol{w}^s) \to \infty$ and so $\underset{i}{Max}\,\{Z_1(\boldsymbol{w}^s),...,Z_N(\boldsymbol{w}^s)\} \to \infty$. Now suppose that there is no such industry and that for any industry $(i,k)$ the limit $\lim_{s\to\infty} L_{i,k}(\boldsymbol{w}^s)$ is a finite (non-negative) number.

Consider index $j$ such that wage $w_j^s$ converges to $0$ weakly "faster" than other wages. Formally, index $j$ is such that for any $j' \neq j$ the limit $\lim_{s\to\infty} w_j^s/w_{j'}^s$ is finite. Such index always exists because there is a finite number of indices. We have that, for any $s$, $L_{j,k}(\boldsymbol{w}^s)$ satisfies the nonlinear complementary slackness condition (3). In particular, $L_{j,k}(\boldsymbol{w}^s)$ satisfies the inequality

$$w_j^s \geq \sum_n \frac{S_{j,k} L_{j,k}(\boldsymbol{w}^s)^{\alpha_k - 1}\left[w_j^s \tau_{nj,k}\right]^{-\varepsilon_k}}{\sum_l S_{l,k} L_{l,k}(\boldsymbol{w}^s)^{\alpha_k}\left[w_l^s \tau_{nl,k}\right]^{-\varepsilon_k}} \beta_{n,k} w_n \bar{L}_n.$$

This inequality can be equivalently rewritten as

$$w_j^s \geq \sum_{n=1}^{N} \frac{S_{j,k} \tau_{nj,k}^{-\varepsilon_k} L_{j,k}(\boldsymbol{w}^s)^{\alpha_k - 1}}{S_{j,k} L_{j,k}(\boldsymbol{w}^s)^{\alpha_k} \tau_{nj,k}^{-\varepsilon_k} + \sum_{l \neq j} S_{l,k} L_{l,k}(\boldsymbol{w}^s)^{\alpha_k} \tau_{nl,k}^{-\varepsilon_k}\left[w_j^s/w_l^s\right]^{\varepsilon_k}} w_n^s \bar{L}_n. \tag{22}$$

Clearly, the denominator of any term in the above summation (22) converges to a finite number (which can be either positive or zero). The numerator of any term in the summation (22) converges to either a finite positive number or to infinity. Also, there exists at least one index $n$ such that $\lim_{s\to\infty} w_n^s > 0$. Then, for this index $n$ the corresponding term in the summation (22) converges to either a finite positive number or to $\infty$. This, in turn, implies that the whole sum in (22) converges to either a finite positive number or to $\infty$. At the same time, the left-hand side of inequality (22) converges to $0$. A contradiction. Hence, there is at least one industry $(i,k)$ such that $L_{i,k}(\boldsymbol{w}^s) \to \infty$, and so $\max\{Z_1(\boldsymbol{w}^s),...,Z_N(\boldsymbol{w}^s)\} \to \infty$. □

## C.3. Proof of Proposition 3

This proof proceeds by showing that $\boldsymbol{Z}(\boldsymbol{w})$ satisfies the gross substitutes property (GSP). Uniqueness of wages then follows from Proposition 17.F.3 from MWG.

Consider any particular industry $k$. Let us separately analyze the two possibilities $0 \leq \alpha_k < 1$ and $\alpha_k = 1$.

If $0 \leq \alpha_k < 1$, then for any $i$ we have that $L_{i,k}(\boldsymbol{w}) > 0$ for any wage vector $\boldsymbol{w} \in \mathbb{R}^N_{++}$, and $L_{i,k}(\boldsymbol{w})$ solves:

$$w_i L_{i,k}(\boldsymbol{w}) = \sum_n \lambda_{ni,k}(\boldsymbol{w}, \boldsymbol{L}_k(\boldsymbol{w}))\beta_{n,k} w_n \bar{L}_n.$$

By differentiating both sides of this expression w.r.t. wages, we can get a linear system of equations which determines the effect of wages on labor allocations. Let us introduce additional notation to write in matrix form this effect. Denote $x_{ij,k} \equiv \dfrac{d \ln L_{i,k}(\boldsymbol{w})}{d \ln w_j}$, $q_{i,k} \equiv w_i L_{i,k}(\boldsymbol{w})$, and $b_{i,k} \equiv \beta_{i,k} w_i \bar{L}_i$. Let $B_k$ denote the diagonal matrix with elements $b_{i,k}$ along the diagonal, $Q_k$ the diagonal matrix with elements $q_{i,k}$ along the diagonal, $\Lambda_k$ the matrix of sector level expenditure shares $\lambda_{ij,k}$, and $X_k$ the matrix of partials $x_{ij,k}$. Finally, let $U_k \equiv \left((1-\alpha_k) Q_k + \alpha_k \Lambda_k^T B_k \Lambda_k\right)$ and $V_k \equiv \left(\Lambda_k^T B_k + \varepsilon_k \Lambda_k^T B_k \Lambda_k - (1+\varepsilon_k) Q_k\right)$. In this notation the effect of wages on labor allocations is obtained from the system:

$$U_k X_k = V_k.$$

It straightforward to check that matrix $U_k$ is a positive definite matrix with all positive elements, and matrix $V_k$ has negative diagonal and positive off-diagonal elements. Since $U_k$ is positive definite, the inverse exists and its determinant is positive. Moreover, $U_k^{-1} = \frac{1}{\det(U_k)} C_k^T$, where $C_k^T$ is the transpose of the matrix of cofactors $C_k$ of $U_k$. Since all the elements of $U_k$ are positive, then for $N = 2$, $C_k$ is a $2 \times 2$ matrix consisting of positive diagonal elements and negative off-diagonal elements.[36] Therefore, $U_k^{-1}$ has this property as well. One can then readily verify that $U_k^{-1} V_k$ is a matrix with the same properties as $V_k$ — it has negative diagonal and positive off diagonal elements. Thus the Jacobian matrix of wages effects on labor allocations in industry $k$ with $0 \leq \alpha_k < 1$ satisfies the GSP.

If $\alpha_k = 1$, then $L_{i,k}(\boldsymbol{w})$ can be equal to $0$ for some $i$, and we cannot establish differentiability of labor allocations in that region. We are going to check directly what happens to labor allocations as wages change. To that end, assume without loss of generality that $\boldsymbol{w}'$ and $\boldsymbol{w}''$ are such that $w_1'' > w_1'$ and $w_2'' = w_2' = 1$. Let us show that

[36]This is no longer true with $N > 2$.

$L_{2,k}(\boldsymbol{w''}) \geq L_{2,k}(\boldsymbol{w'})$ for all $k$ and there is some industry $\tilde{k}$ such that $L_{2,\tilde{k}}(\boldsymbol{w''}) > L_{2,\tilde{k}}(\boldsymbol{w'})$.

In general, given wage $\boldsymbol{w'}$ there are three cases: (a) $L_{1,k}(\boldsymbol{w'}) = 0$ and $L_{2,k}(\boldsymbol{w'}) = \beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2$; (b) $L_{i,k}(\boldsymbol{w'}) > 0$ for $i = 1, 2$; (c) $L_{2,k}(\boldsymbol{w'}) = 0$ and $L_{1,k}(\boldsymbol{w'}) = \frac{1}{w_1'} \left( \beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2 \right)$.

Let us consider these different cases.

**Case (a).** In this case we have $G_{1,k}(\boldsymbol{w'}) \geq 0$ and $G_{1,k}(\boldsymbol{w'})$ simplifies to:

$$G_{1,k}(\boldsymbol{w'}) = w_1 - \frac{S_{1,k} \left[ w_1' \right]^{-\varepsilon_k}}{S_{2,k} L_{2,k}(\boldsymbol{w'}) \tau_{12,k}^{-\varepsilon_k}} \beta_{1,k} w_1' \bar{L}_1 - \frac{S_{1,k} \left( w_1' \tau_{21,k} \right)^{-\varepsilon_k}}{S_{2,k} L_{2,k}(\boldsymbol{w'})} \beta_{2,k} \bar{L}_2.$$

After substituting $L_{2,k}(\boldsymbol{w'}) = \beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2$ into the above expression for $G_{1,k}(\boldsymbol{w'})$, and dividing both sides of this expression by $w_1'$, we get:

$$\frac{G_{1,k}(\boldsymbol{w'})}{w_1'} = 1 - \frac{S_{1,k}}{S_{2,k} \tau_{12,k}^{-\varepsilon_k}} \cdot \frac{\left[ w_1' \right]^{-\varepsilon_k} \beta_{1,k} \bar{L}_1}{\beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2} - \frac{S_{1,k} \tau_{21,k}^{-\varepsilon_k}}{S_{2,k}} \cdot \frac{\left[ w_1' \right]^{-1-\varepsilon_k} \beta_{2,k} \bar{L}_2}{\beta_{1,k} w_1' \bar{L}_1 + \beta_{2,k} \bar{L}_2}.$$

Clearly, the right-hand side of this expression is increasing in $w_1'$. Hence, $G_{1,k}(\boldsymbol{w''})/w_1'' > G_{1,k}(\boldsymbol{w'})/w_1' \geq 0$, which in turn implies that $G_{1,k}(\boldsymbol{w''}) > 0$. Therefore, $L_{1,k}(\boldsymbol{w''}) = 0$ and $L_{2,k}(\boldsymbol{w''}) = \beta_{1,k} w_1'' \bar{L}_1 + \beta_{2,k} \bar{L}_2$ solve the complementary slackness problem (3). In other words, we still remain in case (a) after we increase the wage of the country 1 from $w_1'$ to $w_1''$. Clearly, in this case $L_{2,k}(\cdot)$ is a strictly increasing function of the wage of the first country, $L_{2,k}(\boldsymbol{w''}) > L_{2,k}(\boldsymbol{w'})$.

**Case (b).** We know that, as long as we are in case (b), $L_{1,k}(\cdot)$ is a decreasing function and $L_{2,k}(\cdot)$ is an increasing function of $w_1$. Therefore, starting in case (b) with $\boldsymbol{w'}$ and gradually increasing $w_1$ from $w_1'$ to $w_1''$, we either remain in case (b) or switch to case (a) at some point. The above argument for case (a) implies that, once we switch to case (a), we will remain in case (a) as we keep increasing $w_1$. Thus, for $\boldsymbol{w''}$ we can either be in case (a) or in case (b), but not in case (c), and since in both cases (a) and (b) $L_{2,k}(\cdot)$ is a strictly increasing function of $w_1$, we must have $L_{2,k}(\boldsymbol{w''}) > L_{2,k}(\boldsymbol{w'})$.

**Case (c).** In this case, we can be in any of the cases (a)-(c) for $\boldsymbol{w''}$. If we are in cases (a) or (b) for $\boldsymbol{w''}$, then $L_{2,k}(\boldsymbol{w''}) > L_{2,k}(\boldsymbol{w'}) = 0$. If we are in case (c) for $\boldsymbol{w''}$, then $L_{2,k}(\boldsymbol{w''}) = L_{2,k}(\boldsymbol{w'}) = 0$, but there must exist some industry $\tilde{k}$, for which we are in case (a) or (b) for $\boldsymbol{w'}$ (regardless of the value of $\alpha_{\tilde{k}}$ in this industry). Applying the arguments above, for any such industry we have $L_{2,\tilde{k}}(\boldsymbol{w''}) > L_{2,\tilde{k}}(\boldsymbol{w'})$.

Since the effect of changes in wages on $\boldsymbol{Z}(\boldsymbol{w})$ consists of the sum (across industries) of effects on industry-level labor allocations, we conclude that $\boldsymbol{Z}(\boldsymbol{w})$ satisfies the GSP. $\square$

## C.4. Proof of Proposition 4

In this proof we will use matrices $B_k$, $Q_k$, $\Lambda_k$, $U_k$, and $V_k$ defined in the proof of Proposition 3 in Appendix C.3. In addition to that, let $L_k$ be a diagonal matrix with elements $L_{i,k}$ along the diagonal; $W$ be a diagonal matrix with elements $w_i$ along the diagonal; $D\boldsymbol{Z}(\boldsymbol{w})$ be the Jacobian matrix of the excess demand system, $Z(w)$, with elements $\partial Z_i(\boldsymbol{w})/\partial w_j$; and $D\boldsymbol{L}_k(\boldsymbol{w})$ be the Jacobian matrix of industry-level labor allocations with elements $\partial L_{i,k}(\boldsymbol{w})/\partial w_j$.

We have

$$D\boldsymbol{Z}(\boldsymbol{w}) = \sum_k D\boldsymbol{L}_k(\boldsymbol{w}) = \sum_k L_k U_k^{-1} V_k W^{-1}.$$

Matrix $V_k$ has the following properties: (i) entries in each row add up to $0$; (ii) diagonal entries are negative; (iii) off-diagonal entries are positive. For all industries $k$ with $\alpha_k = 0$ matrix $U_k$ reduces to diagonal matrix $Q_k$ with positive diagonal elements. Therefore, we can immediately conclude that for all such industries $D\boldsymbol{L}_k(\boldsymbol{w})$ has properties (i)-(iii) as well. The rest of this appendix section is devoted to proving that for all industries $k$ with $0 < \alpha_k < 1$ matrix $D\boldsymbol{L}_k(\boldsymbol{w})$ also has properties (i)-(iii) under free trade. Since summation of matrices with properties (i)-(iii) again gives a matrix with these properties, the whole Jacobian of the excess demand system, $D\boldsymbol{Z}(\boldsymbol{w})$, has properties (i)-(iii) under free trade. This means that the excess demand system $\boldsymbol{Z}(\boldsymbol{w})$ has the gross substitutes property. Hence, there is at most one normalized vector of wages such that $\boldsymbol{Z}(\boldsymbol{w}) = \boldsymbol{0}$.

Consider any industry $k$ with $0 < \alpha_k < 1$. For brevity of notation we drop the industry index $k$ in the rest of this proof. According to Proposition 1 all industry-level labor allocations are interior, and so, $L_i > 0$, $\lambda_{ii} > 0$, $q_i > 0$ for all $i$. We start with three lemmas which apply to the general case of costly trade.

**Lemma 6.** *Let $\mu_1, \ldots, \mu_N$ be eigenvalues of matrix $Q^{-1}\Lambda^T B \Lambda$. Then $\mu_i$ is real and $0 \leq$*

$\mu_i \leq 1$ *for each* $i$.

*Proof.* Consider matrix $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2}$, and let $\mu$ be any eigenvalue of this matrix with the corresponding eigenvector $v$. By definition of an eigenvalue, $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2} v = \mu v$. This is equivalent to $Q^{-1}\Lambda^T B\Lambda \left(Q^{-1/2}v\right) = \mu \left(Q^{-1/2}v\right)$. Hence, $\mu$ is an eigenvalue of $Q^{-1}\Lambda^T B\Lambda$ with the corresponding eigenvector $Q^{-1/2}v$. Therefore, matrices $Q^{-1}\Lambda^T B\Lambda$ and $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2}$ have the same eigenvalues, and so $\mu_1, \ldots, \mu_N$ are eigenvalues of $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2}$.

Clearly, matrix $Q^{-1/2}\Lambda^T B\Lambda Q^{-1/2}$ is positive semi-definite. Hence, all its eigenvalues are real and nonnegative, i.e., $\mu_i$ is real and $\mu_i \geq 0$ for each $i$. Next, matrix $Q^{-1}\Lambda^T B\Lambda$ is a positive stochastic matrix (its entries in each row add up to 1). Therefore, the Perron-Frobenius theorem implies that 1 is its eigenvalue with algebraic multiplicity one and $|\mu_i| < 1$ for any $|\mu_i| \neq 1$. Since $\mu_i \geq 0$ for all $i$, we have the statement of the lemma. □

**Lemma 7.** $\lim_{t\to\infty} \alpha^t \left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t = 0$.

*Proof.* Eigenvalues of matrix $I_N - Q^{-1}\Lambda^T B\Lambda$ are $1-\mu_1, \ldots, 1-\mu_N$, where $\mu_1, \ldots, \mu_N$ are eigenvalues of $Q^{-1}\Lambda^T B\Lambda$. Lemma 6 implies that $0 \leq 1-\mu_i \leq 1$ for all $i$. Then, since eigenvalues of matrix $\alpha\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)$ are $\alpha\left(1-\mu_1\right), \ldots, \alpha\left(1-\mu_1\right)$, we have that $\rho\left(\alpha\left[I_N - Q^{-1}\Lambda^T B\Lambda\right]\right) < 1$, where $\rho\left(\cdot\right)$ is the spectral radius of a matrix. Therefore, $\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t \to 0$ as $t \to \infty$ (see, for example, Theorem 5.6.12 in Horn and Johnson, 2013). □

**Lemma 8.** $U^{-1}V = \varepsilon\alpha^{-1}I_N - \sum_{t=0}^{\infty} \alpha^t \left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t \left[\left(1+\varepsilon\alpha^{-1}\right) I_N - Q^{-1}\Lambda^T B\right]$.

*Proof.* Consider $U^{-1}$:

$$\begin{aligned} U^{-1} &= \left[\left(1-\alpha\right)Q + \alpha\Lambda^T B\Lambda\right]^{-1} = \left[\left(1-\alpha\right)I_N + \alpha Q^{-1}\Lambda^T B\Lambda\right]^{-1} Q^{-1} \\ &= \left[I_N - \alpha\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)\right]^{-1} Q^{-1}. \end{aligned}$$

Lemma 7 implies that we can write

$$\left[I_N - \alpha\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)\right]^{-1} = \sum_{t=0}^{\infty} \alpha^t \left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t$$

(see, for example, Corollary 5.6.15 in Horn and Johnson, 2013). Then

$$
\begin{aligned}
U^{-1}V &= -\sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t\left[(1+\varepsilon)\,I_N - Q^{-1}\Lambda^T B - \varepsilon Q^{-1}\Lambda^T B\Lambda\right] \\
&= -\varepsilon\sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^{t+1} - \sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t\left[I_N - Q^{-1}\Lambda^T B\right] \\
&= \varepsilon\alpha^{-1}I_N - \sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda^T B\Lambda\right)^t\left[\left(1+\varepsilon\alpha^{-1}\right)I_N - Q^{-1}\Lambda^T B\right].
\end{aligned}
$$

□

Let us now consider the case of frictionless trade. In this case the matrix of trade shares, $\Lambda$, has the same entries in each column:

$$
\Lambda \equiv \begin{pmatrix} \lambda_{11} & \dots & \lambda_{NN} \\ \vdots & & \vdots \\ \lambda_{11} & \dots & \lambda_{NN} \end{pmatrix}.
$$

So, it can be represented (with a slight abuse of notation) as $O\Lambda$ where $O$ is an $N \times N$ matrix of ones (i.e., $O = \iota \cdot \iota^T$ with $\iota^T \equiv (1, \dots, 1)$) and

$$
\Lambda \equiv \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}.
$$

Then, in this notation

$$
U = (1-\alpha)\,Q + \alpha\Lambda OBO\Lambda \quad \text{and} \quad V = \Lambda OB + \varepsilon\Lambda OBO\Lambda - (1+\varepsilon)\,Q.
$$

Denote $b \equiv \sum_n b_n$ and observe that $OBO = bO$. Also, since $L_i$ satisfies the goods market clearing condition, $w_iL_i = \sum_n \lambda_i b_n = \lambda_i \sum_n b_n = b\lambda_i$. Then, since in our notation $q_{ii} = w_iL_i$, we have that $Q = WL = b\Lambda$. These equalities together with Lemma 8 allow us to write:

$$
\begin{aligned}
U^{-1}V &= \varepsilon\alpha^{-1}I_N - \sum_{t=0}^{\infty}\alpha^t\left(I_N - Q^{-1}\Lambda OBO\Lambda\right)^t\left[\left(1+\varepsilon\alpha^{-1}\right)I_N - Q^{-1}\Lambda OB\right] \\
&= \varepsilon\alpha^{-1}I_N - \sum_{t=0}^{\infty}\alpha^t\left(I_N - O\Lambda\right)^t\left[\left(1+\varepsilon\alpha^{-1}\right)I_N - b^{-1}OB\right].
\end{aligned}
$$

Using the fact that $\sum_i \lambda_i = 1$ and, hence, $O\Lambda O = O$, we get:

$$\begin{aligned}(I_N - O\Lambda)\left[\left(1+\varepsilon\alpha_k^{-1}\right)I_N - b^{-1}OB\right] &= \left(1+\varepsilon\alpha^{-1}\right)I_N - b^{-1}OB \\ &\quad - \left(1+\varepsilon\alpha^{-1}\right)O\Lambda + b^{-1}O\Lambda OB \\ &= \left(1+\varepsilon\alpha^{-1}\right)(I_N - O\Lambda),\end{aligned}$$

and

$$(I_N - O\Lambda)(I_N - O\Lambda) = I_N - O\Lambda - O\Lambda + O\Lambda O\Lambda = I_N - O\Lambda.$$

Therefore,

$$\begin{aligned}U^{-1}V &= \varepsilon\alpha^{-1}I_N - \left[\left(1+\varepsilon\alpha^{-1}\right)I_N - b^{-1}OB\right] - \left(1+\varepsilon\alpha^{-1}\right)\sum_{t=1}^{\infty}\alpha^t (I_N - O\Lambda)^t \\ &= -\left(I_N - b^{-1}OB\right) - \left(1+\varepsilon\alpha^{-1}\right)(I_N - O\Lambda)\sum_{t=1}^{\infty}\alpha^t \\ &= \left(b^{-1}OB - I_N\right) + \frac{\alpha+\varepsilon}{1-\alpha}(O\Lambda - I_N).\end{aligned}$$

Observe that both matrices $\left(b^{-1}OB - I_N\right)$ and $(O\Lambda - I_N)$ have properties (i)-(iii) listed at the beginning of this appendix section. Hence, matrix $U^{-1}V$ has properties (i)-(iii) as well. This, in turn, implies that matrix $\Lambda U^{-1}VW^{-1}$ also has properties (i)-(iii). This concludes our proof. □

### C.5. Applying Uniqueness Results in Allen, Arkolakis and Li (2015)

With a slight change of notation, the equilibrium system in $\{w_i\}$, $\{L_{ik}\}$, $\{P_{ik}\}$ can be written as

$$w_i L_{ik} = \sum_n \nu_{nik} w_i^{-\varepsilon_k} L_{ik}^{\alpha_k} P_{nk}^{\varepsilon_k} \beta_{nk} w_n \bar{L}_n \tag{23}$$

$$P_{nk}^{-\varepsilon_k} = \sum_j \nu_{njk} w_j^{-\varepsilon_k} L_{jk}^{\alpha_k} \bar{L}_i \tag{24}$$

$$\bar{L}_i = \sum_k L_{ik} \tag{25}$$

Let $\kappa > 1$ be some parameter and let $\tilde{\alpha}_k \equiv \frac{\varepsilon_k + \kappa\alpha_k}{\kappa - 1}$. Changing variables to $w_{ik} = w_i \left(L_{ik}/\bar{L}_i\right)^{1/\kappa}$ and $y_i = w_i$ the system (23)- (25) now becomes

$$y_i^{1+\varepsilon_k} \left(w_{ik}/y_i\right)^{\kappa} \bar{L}_i = \sum_n \nu_{nik} \left(\left(w_{ik}/y_i\right)^{\kappa} \bar{L}_i\right)^{-\frac{\varepsilon_k}{\kappa} + \tilde{\alpha}_k\left(\frac{\kappa-1}{\kappa}\right)} P_{nk}^{\varepsilon_k} \beta_{nk} y_n \bar{L}_n \tag{26}$$

$$P_{nk}^{-\varepsilon_k} = \sum_j \nu_{njk} y_j^{-\varepsilon_k} \left(\left(w_{jk}/y_j\right)^{\kappa} \bar{L}_j\right)^{-\frac{\varepsilon_k}{\kappa} + \tilde{\alpha}_k\left(\frac{\kappa-1}{\kappa}\right)} y_i^{\kappa} = \sum_k w_{ik}^{\kappa} \tag{27}$$

$$y_i^{\kappa} = \sum_k w_{ik}^{\kappa} \tag{28}$$

This is a system in $\{y_i\}$, $\{w_{ik}\}$, $\{P_{ik}\}$. It is clear that the system (23)-(25) has a unique solution if and only if the system (26)-(28) with $\kappa > 1$ and $\tilde{\alpha}_k = \frac{\varepsilon_k + \kappa\alpha_k}{\kappa - 1}$ has a unique solution. So we can do the analysis of uniqueness of our equilibrium system with the transformed system (26)-(28). Also note that this transformed system corresponds to the equilibrium system of equations for an economy where labor is heterogeneous as in a Roy-Frechet model as in Galle *et al.* (2015), with scale elasticity $\tilde{\alpha}_k$.

If we assume that $\tilde{\alpha}_k = \tilde{\alpha}$ and $\varepsilon_k = \varepsilon$, then we can map the transformed system (26)-(28) into the system in Equation (1) in Allen *et al.* (2015) (henceforth AAL) and explore if their Theorem 1 can be invoked to establish uniqueness. Letting $x_{is}^1 \equiv w_{is}$, $x_{is}^2 \equiv P_{is}$, and $x_{is}^3 \equiv y_i$, the system can be rewritten as

$$\left(x_{is}^1\right)^{1+\varepsilon+(1-\tilde{\alpha})(\kappa-1)} \left(x_{is}^3\right)^{-(1-\tilde{\alpha})(\kappa-1)} = \sum_{jk} K_{is,jk}^1 \left(x_{jk}^2\right)^{\varepsilon} x_{jk}^3, \tag{29}$$

$$\left(x_{is}^2\right)^{-\varepsilon} = \sum_{jk} K_{is,jk}^2 \left(x_{jk}^1\right)^{-\varepsilon+\tilde{\alpha}(\kappa-1)} \left(x_{jk}^3\right)^{-\tilde{\alpha}(\kappa-1)}, \tag{30}$$

$$\left(x_{is}^{3}\right)^{\kappa} = \sum_{jk} K_{is,jk}^{3} \left(x_{jk}^{1}\right)^{\kappa}. \tag{31}$$

This maps into Equation (1) in AAL with each "location" being an $(i, s)$ pair. Following AAL's notation, we have

$$\Gamma = \begin{pmatrix} 1+\varepsilon+(1-\tilde{\alpha})(\kappa-1) & 0 & -(1-\tilde{\alpha})(\kappa-1) \\ 0 & -\varepsilon & 0 \\ 0 & 0 & \kappa \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & \varepsilon & 1 \\ -\varepsilon+\tilde{\alpha}(\kappa-1) & 0 & -\tilde{\alpha}(\kappa-1) \\ \kappa & 0 & 0 \end{pmatrix}.$$

Assuming that $\tilde{\alpha}$ is in between 0 and 1 and that $\kappa > 1+\varepsilon/\tilde{\alpha}$ then

$$A \equiv \left|B\Gamma^{-1}\right| = \begin{pmatrix} 0 & 1 & \frac{1}{\kappa} \\ \frac{\tilde{\alpha}(\kappa-1)-\varepsilon}{(1-\tilde{\alpha})(\kappa-1)+\varepsilon+1} & 0 & \frac{\kappa-1}{\kappa}(1-\tilde{\alpha})\left[\frac{\tilde{\alpha}}{1-\tilde{\alpha}} - \frac{\tilde{\alpha}(\kappa-1)-\varepsilon}{(1-\tilde{\alpha})(\kappa-1)+\varepsilon+1}\right] \\ \frac{\kappa}{(1-\tilde{\alpha})(\kappa-1)+\varepsilon+1} & 0 & \frac{(1-\tilde{\alpha})(\kappa-1)}{(1-\tilde{\alpha})(\kappa-1)+\varepsilon+1} \end{pmatrix}$$

In simulations we see that the spectral radius $\rho(A)$ is always higher than one, which implies that we cannot invoke AAL's Theorem 1 to establish uniqueness.

There are parameter restrictions under which we can show that $\rho(A) \leq 1$. In particular, in simulations we always find that if $\kappa \leq 1+\varepsilon/\tilde{\alpha}$ then $\rho(A) \leq 1$. Unfortunately, however, the case $\kappa \leq 1+\varepsilon/\tilde{\alpha}$ corresponds to one in which $\alpha = -\frac{\varepsilon}{\kappa} + \tilde{\alpha}(\frac{\kappa-1}{\kappa}) \leq 0$, so this does not correspond to our system with scale economies.

Finally, it is interesting to explore how the AAL approach can be used to establish uniqueness for labor allocations given wages. That would correspond to the case in which we take $y_i$ as given and ignore equation (28) in the system (26)- (28). Then the system can be written as

$$w_{ik}^{\kappa} = \sum_{n} \mu_{nik} w_{ik}^{-\varepsilon_k+\tilde{\alpha}_k(\kappa-1)} \left(P_{nk}\right)^{\varepsilon_k}$$

and

$$(P_{nk})^{-\varepsilon_k} \equiv \sum_j \eta_{njk} w_{jk}^{-\varepsilon_k+\tilde{\alpha}_k(\kappa-1)}$$

for some positive parameters $\mu_{nik}$ and $\eta_{njk}$. With $\varepsilon_k = \varepsilon$ and $\tilde{\alpha}_k = \tilde{\alpha}$, this maps into Equation (1) in AAL as

$$\left(x_{ik}^1\right)^{1+\varepsilon+(1-\tilde{\alpha})(\kappa-1)} = \sum_{ns} K_{ik,ns}^1 \left(x_{ns}^2\right)^{\varepsilon}$$

and

$$\left(x_{ik}^2\right)^{-\varepsilon} = \sum_{ns} K_{ik,ns}^2 \left(x_{ns}^1\right)^{-\varepsilon+\tilde{\alpha}(\kappa-1)}.$$

This entails

$$\Gamma = \begin{pmatrix} 1+\varepsilon+(1-\tilde{\alpha})(\kappa-1) & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon+\tilde{\alpha}(\kappa-1) & 0 \end{pmatrix}.$$

For $\tilde{\alpha} \in [0,1]$ and $\kappa > 1+\varepsilon/\tilde{\alpha}$ we have

$$A \equiv \left|B\Gamma^{-1}\right| = \begin{pmatrix} 0 & 1 \\ \frac{-\varepsilon+\tilde{\alpha}(\kappa-1)}{1+\varepsilon+(1-\tilde{\alpha})(\kappa-1)} & 0 \end{pmatrix}.$$

We have $\rho(A) = \left(\frac{-\varepsilon+\tilde{\alpha}(\kappa-1)}{1+\varepsilon+(1-\tilde{\alpha})(\kappa-1)}\right)^{1/2}$, which is lower than one if $1+2\varepsilon+(1-2\tilde{\alpha})(\kappa-1) > 0$. If $\tilde{\alpha} \leq 1/2$ then AAL's Theorem 1 implies uniqueness for all $\kappa > 1+\varepsilon/\tilde{\alpha}$. If this latter condition is not satisfied then for $\tilde{\alpha} \in [0,1]$ and $\kappa < 1+\varepsilon/\tilde{\alpha}$ we have

$$A = \begin{pmatrix} 0 & 1 \\ \frac{\varepsilon-\tilde{\alpha}(\kappa-1)}{1+\varepsilon+(1-\tilde{\alpha})(\kappa-1)} & 0 \end{pmatrix}.$$

The spectral radius of this is $\left(\frac{\varepsilon-\tilde{\alpha}(\kappa-1)}{1+\varepsilon+(1-\tilde{\alpha})(\kappa-1)}\right)^{1/2}$. This is lower than one if $\varepsilon-\tilde{\alpha}(\kappa-1) < 1+\varepsilon+(1-\tilde{\alpha})(\kappa-1)$ which is always true. We conclude that if $0 \leq \tilde{\alpha} \leq 1/2$ (and $\kappa > 1$) then there is uniqueness. This translates into the condition that $\alpha_k = \alpha < 1/2$ for all $k$ for a unique labor allocation given wages in the original model with homogeneous labor. This condition is more stringent condition than the one in Proposition 1.

# D. Gains from Trade

## D.1. Proof of Proposition 5

It is sufficient to show that for any industry $k$ the price index is lower with trade. We need to consider two cases: (a) $\alpha_k \in [0,1)$ and (b) $\alpha_k = 1$. Suppressing subindex $k$, first note that for case (a) we have

$$\sum_n \lambda_{ni}(\boldsymbol{L})\beta_n w_n \bar{L}_n = w_i L_i,$$

where

$$\lambda_{ni}(\boldsymbol{L}) \equiv \frac{S_i L_i^\alpha (w_i \tau_{ni})^{-\varepsilon}}{\sum_l S_l L_l^\alpha (w_l \tau_{nl})^{-\varepsilon}}.$$

The expression for the price index, $P_n = \mu_n \left(\sum_l S_l L_l^\alpha (w_l \tau_{nl})^{-\varepsilon}\right)^{-1/\varepsilon}$, implies

$$\sum_n S_i L_i^\alpha (w_i \tau_{ni})^{-\varepsilon} \mu_n^{-\varepsilon} P_n^\varepsilon \beta_n w_n \bar{L}_n = w_i L_i,$$

and

$$\frac{w_i}{P_i} = \mu_i^{-1} \left(\frac{S_i L_i^\alpha}{\lambda_{ii}}\right)^{1/\varepsilon}.$$

Using labor in country $i$ as numeraire $(w_i = 1)$, we can rewrite the above two expressions as

$$\mu S_i \mu_i^{-\varepsilon} P_i^\varepsilon \beta_i \bar{L}_i + \sum_{n \neq i} S_i \tau_{ni}^{-\varepsilon} \mu_n^{-\varepsilon} P_n^\varepsilon \beta_n w_n \bar{L}_n = L_i^{1-\alpha},$$

and

$$P_i^{-\varepsilon} = \mu_i^{-\varepsilon} \frac{S_i L_i^\alpha}{\lambda_{ii}}.$$

These two equations further imply

$$P_i^{\varepsilon/\alpha} = \mu_i^{\varepsilon/\alpha} \frac{(1-\Delta)\lambda_{ii}^{(1-\alpha)/\alpha}}{S_i^{1/\alpha} \beta_i \bar{L}_i},$$

where $\Delta \equiv L_i^{\alpha-1} \sum_{n \neq i} S_i \tau_{ni}^{-\varepsilon} \mu_n^{-\varepsilon} P_n^\varepsilon \beta_n w_n \bar{L}_n$.

Since with trade we have $0 < \Delta < 1$ and $\lambda_{ii} < 1$, while with autarky we have $\Delta = 0$ and $\lambda_{ii} = 1$, it follows $P_i$ (welfare) is lower (higher) under trade than in autarky.

For case (b), the complementary slackness condition implies that (recall for this case a corner equilibrium is a possibility)

$$\sum_n S_i (w_i \tau_{ni})^{-\varepsilon} \mu_n^{-\varepsilon} P_n^{\varepsilon} \beta_n w_n \bar{L}_n \leq w_i.$$

Setting $w_i = 1$ by choice of numeraire, this implies

$$P_i^{\varepsilon} \leq \mu_i^{\varepsilon} \frac{1 - \Delta'}{S_i \beta_i L_i},$$

where $\Delta' \equiv \sum_{n \neq i} S_i \tau_{ni}^{-\varepsilon} \mu_n^{-\varepsilon} P_n^{\varepsilon} \beta_n w_n \bar{L}_n$. With trade we have $0 < \Delta' < 1$, whereas in autarky we have $\Delta' = 0$. Moreover, in autarky the above inequality turns into an equality. Hence $P_i$ is lower with trade than in autarky. □

## D.2. Derivation of Threshold for Outside Good

Letting $x_i \equiv w_i L_i$, $y_i \equiv P^{\varepsilon}$, $a_{ni} \equiv S_i w_i^{-\alpha-\varepsilon} \tau_{ni}^{-\varepsilon}$, $b_n \equiv \beta_n w_n \bar{L}_n$ and log-differentiating the system in (17) around an equilibrium point for some change in $a_{ni}$ we get

$$d \ln x_i = \frac{1}{1-\alpha} \sum_n \chi_{ni} \left( d \ln a_{ni} + d \ln y_n \right) \quad \text{for} \quad i = 1, ..., N,$$

$$d \ln y_i = -\sum_j \lambda_{ij} \left( d \ln a_{ij} + \alpha d \ln x_j \right) \quad \text{for} \quad n = 1, ..., N,$$

where $\lambda_{ni} \equiv a_{ni} x_i^{\alpha} y_n$ are import shares and $\chi_{ij} \equiv \dfrac{a_{ij} x_j^{\alpha} y_i b_i}{\sum_n a_{nj} x_j^{\alpha} y_n b_n}$ are export shares (i.e. $\chi_{ij}$ is the share of total country $j$ exports directed to country $i$).

Let $\mathcal{X}$ be the matrix of export shares with elements $\chi_{ni}$, $\Lambda$ be the matrix of import shares with elements $\lambda_{ni}$, let $X$ and $Y$ be column vectors with elements $d \ln x_i$ and $d \ln y_i$, let $A$ be the matrix with typical element $d \ln a_{ni}$, and let matrix $\mathbf{1}$ be a column vector whose entries are all ones. We can rewrite the system in matrix form as

$$X = \frac{1}{1-\alpha} \left( \left[ \mathcal{X}^T \circ A^T \right] \mathbf{1} + \mathcal{X}^T Y \right),$$

$$Y = -\left[ \Lambda \circ A \right] \mathbf{1} - \alpha \Lambda X,$$

where the symbol "$\circ$" denotes the Hadamard product. Substituting the first equation into the second and rearranging we get

$$\left(\gamma I + \Lambda \mathcal{X}^T\right) Y = -\left(\gamma \left[\Lambda \circ A\right] \mathbf{1} + \Lambda \left[\mathcal{X}^T \circ A^T\right] \mathbf{1}\right), \tag{32}$$

where $\gamma = \dfrac{1-\alpha}{\alpha}$.

Since $\sum_n \lambda_{ni} b_n = x_i$ implies $\chi_{ij} = \dfrac{\lambda_{ij} b_i}{x_j}$, we can write $\mathcal{X} = B \Lambda L^{-1}$ and by extension $\Lambda \mathcal{X}^T = \Lambda L^{-1} \Lambda^T B$, where $L$ is a diagonal matrix with elements $x_i$ on the diagonal and $B$ is a diagonal matrix with elements $b_i$ on the diagonal. Observe that matrices $\Lambda L^{-1} \Lambda^T B$ and $\left(B^{\frac{1}{2}} \Lambda\right) L^{-1} \left(B^{\frac{1}{2}} \Lambda\right)^T$ have the same eigenvalues, and that matrix $\left(B^{\frac{1}{2}} \Lambda\right) L^{-1} \left(B^{\frac{1}{2}} \Lambda\right)^T$ is positive semidefinite. It then follows that all eigenvalues of $\Lambda L^{-1} \Lambda^T B$ are real and nonnegative, which, in turn, implies that eigenvalues of $\gamma I + \Lambda \mathcal{X}^T$ are real and positive for any $\gamma > 0$, and so $\det\left(\gamma I + \Lambda \mathcal{X}^T\right) > 0$ for $\gamma > 0$. Since we are interested only in the signs of entries of $Y$ in expression (32), we can then focus on

$$-\det\left(\gamma I + \Lambda \mathcal{X}^T\right) Y = adj\left(\gamma I + \Lambda \mathcal{X}^T\right) \left(\gamma \left[\Lambda \circ A\right] \mathbf{1} + \Lambda \left[\mathcal{X}^T \circ A^T\right] \mathbf{1}\right), \tag{33}$$

where $adj\left(\cdot\right)$ is the adjugate of a matrix.

Consider now the case $N = 2$ and without loss of generality consider a unilateral trade liberalization for country 1. We are then interested in the sign of $\partial \ln y_1 / \partial \ln a_{12}$, and so for this case we have $d \ln a_{11} = d \ln a_{22} = d \ln a_{21} = 0$ and $d \ln a_{12} \neq 0$. Using the facts $adj\left(\gamma I + \Lambda \mathcal{X}^T\right) = \gamma I + adj\left(\Lambda \mathcal{X}^T\right)$ (this is true only in the case of $2 \times 2$ matrices), $adj\left(\Lambda \mathcal{X}^T\right) = adj\left(\mathcal{X}^T\right) adj\left(\Lambda\right)$ and $adj\left(\Lambda\right) \Lambda = \det\left(\Lambda\right)$, and applying the result in (33) together with some manipulation we have

$$-\frac{\det\left(\gamma I + \Lambda \mathcal{X}\right)}{\lambda_{12} d \ln a_{12}} Y = \gamma^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} \lambda_{21}\chi_{21} + \lambda_{22}\chi_{22} + \chi_{12} \\ -\lambda_{21}\chi_{11} - \lambda_{22}\chi_{12} + \dfrac{\lambda_{22}}{\lambda_{12}}\chi_{12} \end{pmatrix} + \frac{\chi_{12}}{\lambda_{12}} \det\left(\Lambda\right) \begin{pmatrix} -\chi_{21} \\ \chi_{11} \end{pmatrix}.$$

Using the expression above together with some algebra one can then show that there exists $\bar{\gamma}^{1,\tau} > 0$ such that for any $\gamma \in (0, \infty)$ we have that $\partial \ln y_1 / \partial \ln a_{12}$ is negative if and only if $\gamma > \bar{\gamma}^{1,\tau}$, with $\bar{\gamma}^{1,\tau}$ given by

$$\bar{\gamma}^{1,\tau} = \frac{\sqrt{D^\tau} - \left(\lambda_{21}\chi_{21} + \lambda_{22}\chi_{22} + \chi_{12}\right)}{2} > 0,$$

where

$$D^{1,\tau} \equiv (\lambda_{21}\chi_{21} + \lambda_{22}\chi_{22} + \chi_{12})^2 + 4(\chi_{11}\chi_{22} - \chi_{12}\chi_{21})\lambda_{21}$$

is always positive.[37] Since $\gamma > \bar{\gamma}^{1,\tau} \Leftrightarrow \alpha < \bar{\alpha}^{1,\tau} = 1/(1+\bar{\gamma}^{1,\tau})$ and since

$$-\partial \ln P_1/\partial \ln \tau_{12} = \partial \ln P_1^{\varepsilon}/\partial \ln \tau_{12}^{-\varepsilon} = \partial \ln y_1/\partial \ln a_{12},$$

the result in the text immediately follows.

Consider now a productivity increase in country 2. Here we are interested in the sign of $\partial \ln y_1/\partial \ln a_{22}$, and so for this case we have $d\ln a_{11} = d\ln a_{21} = 0$ and $d\ln a_{22} \neq 0$. Note also that we have $d\ln a_{12} = d\ln \tau_{12}^{-\varepsilon} + d\ln a_{22} = d\ln a_{22}$ since $d\ln \tau_{12}^{-\varepsilon} = 0$. Analogous to the trade liberalization exercise above one can readily show that $\partial \ln P_1/\partial \ln S_2 = \partial \ln P_1/\partial \ln a_{22} < 0$ if and only if $\bar{\gamma}_n^S \Leftrightarrow \alpha < \bar{\alpha}_n^S = 1/(1+\bar{\gamma}^{1,S})$, with $\bar{\gamma}^{1,S}$ given by

$$\bar{\gamma}^{1,S} \equiv \frac{\sqrt{D^{1,S}} - \left(\lambda_{21}\chi_{21} + \chi_{12} - \lambda_{11}\lambda_{22}\lambda_{12}^{-1}\chi_{21}\right)}{2} > 0$$

where

$$\begin{aligned} D^{1,S} \equiv & \left(\lambda_{21}\chi_{21} + \chi_{12} - \lambda_{11}\lambda_{22}\lambda_{12}^{-1}\chi_{21}\right)^2 + 4(\chi_{11}\chi_{22} - \chi_{12}\chi_{21})\lambda_{21} \\ & + 4\left(\lambda_{11}\lambda_{22}\lambda_{12}^{-1}\chi_{21} + \lambda_{22}\chi_{22}\right)\chi_{22}. \end{aligned}$$

The result in the text then immediately follows.

## D.3. Derivation of Algebra in Section 4.4

To derive the system of equations in Section 4.4, we start by writing down the system of equations for the counterfactual equilibrium:

$$L'_{i,k} \geq 0, \quad G'_{i,k}\left(\boldsymbol{w'}, \boldsymbol{L'}_k\right) \geq 0, \quad L'_{i,k}G'_{i,k}\left(\boldsymbol{w'}, \boldsymbol{L'}_k\right) = 0,$$

$$\sum_k L'_{i,k} = \bar{L}_i,$$

[37] In particular, it is straight forward to verify that $\tau_{12}\tau_{21} \geq 1$ implies $\chi_{11}\chi_{22} - \chi_{12}\chi_{21} \geq 0$.

with

$$G'_{i,k}\left(\boldsymbol{w'},\boldsymbol{L'}_k\right) \equiv w'_i - \frac{1}{L'_{i,k}}\sum_n \lambda'_{ni,k}(\boldsymbol{w'},\boldsymbol{L'}_k)\beta_{n,k}\left(w'_n\bar{L}_n + D_n\right),$$

and

$$\lambda'_{ni,k}(\boldsymbol{w'},\boldsymbol{L'}_k) = \frac{S'_{i,k}(L'_{i,k})^{\alpha_k}\left(w'_i\tau'_{ni,k}\right)^{-\varepsilon_k}}{\sum_l S'_{l,k}(L'_{l,k})^{\alpha_k}\left(w'_l\tau'_{nl,k}\right)^{-\varepsilon_k}}.$$

This can be rewritten in hat notation as

$$\hat{L}_{i,k}L_{i,k} \geq 0, \quad \frac{G'_{i,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)}{w_i} \geq 0, \quad \hat{L}_{i,k}w_iL_{i,k}\frac{G'_{i,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)}{w_i} = 0,$$

$$\sum_k \hat{L}_{i,k}w_iL_{i,k} = w_i\bar{L}_i,$$

with

$$G'_{i,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right) = \hat{w}_iw_i - \frac{1}{\hat{L}_{i,k}L_{i,k}}\sum_n \lambda'_{ni,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right)\beta_{n,k}\left(\hat{w}_nw_n\bar{L}_n + D_n\right),$$

and

$$\lambda'_{ni,k}\left(\hat{\boldsymbol{w}},\hat{\boldsymbol{L}}_k\right) = \frac{\hat{S}_{i,k}\hat{L}_{i,k}^{\alpha_k}\left(\hat{w}_i\hat{\tau}_{ni,k}\right)^{-\varepsilon_k}\lambda_{ni,k}}{\sum_l \hat{S}_{l,k}\hat{L}_{l,k}^{\alpha_k}\left(\hat{w}_l\hat{\tau}_{nl,k}\right)^{-\varepsilon_k}\lambda_{nl,k}}.$$

Substitutions then yield the system in the text.

## D.4. Algorithm for Counterfactuals in Subsection 4.4 and Section 5

We now proceed to describe the algorithm used in the quantitative analyses in Subsection 4.4 and Section 5.[38] For some given $\hat{\boldsymbol{L}} = \left\{\hat{L}_{ik}\right\}$, we can use the tatonnement process proposed by Alvarez and Lucas (2007) to find the wages that clear labor markets. This entails[39]

$$w_i^{(t+1)} = w_i^{(t)} + \nu\frac{\sum_k X_{ik}^{(t)} - w_i^{(t)}\bar{L}_i}{\bar{L}_i},$$

[38]Note that for Section 5 we only need to work with the model for $\alpha_k = 0$ for all $k$ (the standard multi-sector Eaton and Kortum (2002)), and so for this case the algorithm is more straightforward.

[39]For the exercises with fixed wages we simply set $v = 0$.

where $\nu$ is an arbitrary constant satisfying $\nu \in (0,1]$, $X_{ik}^{(t)} \equiv \sum_n \lambda_{nik}^{(t)} \beta_{nk} w_n^{(t)} \bar{L}_n$ and

$$\lambda_{nik}^{(t)} = \frac{S_{ik} L_{ik}^{\alpha_k} \left(w_i^{(t)}\right)^{-\varepsilon_k} \tau_{nik}^{-\varepsilon_k}}{\sum_j S_{jk} L_{jk}^{\alpha_k} \left(w_j^{(t)}\right)^{-\varepsilon_k} \tau_{njk}^{-\varepsilon_k}}.$$

By Walras Law we can show that this is a mapping that takes wages satisfying the restriction $\sum_i w_i \bar{L}_i = 1$ to this same set. Using hat notation $\hat{w}_i^{(t)} \equiv w_i^{(t)}/w_i^{(0)}$ with the superindex $(0)$ denoting variables in the baseline equilibrium (corresponding by assumption to the data), the algorithm can be written as

$$\hat{w}_i^{(t+1)} w_i^{(0)} = \hat{w}_i^{(t)} w_i^{(0)} + \nu \frac{\sum_k \hat{X}_{ik}^{(t)} X_{ik}^{(0)} - \hat{w}_i^{(t)} w_i^{(0)} \bar{L}_i}{\bar{L}_i},$$

where now $\hat{X}_{ik}^{(t)} X_{ik}^{(0)} = \sum_n \hat{\lambda}_{nik}^{(t)} \lambda_{nik}^{(0)} \beta_{nk} \hat{w}_n^{(t)} w_n^{(0)} \bar{L}_n$ and

$$\hat{\lambda}_{nik}^{(t)} \lambda_{nik}^{(0)} = \frac{\hat{S}_{ik} \hat{L}_{ik}^{\alpha_k} \left(\hat{w}_i^{(t)}\right)^{-\varepsilon_k} \hat{\tau}_{nik}^{-\varepsilon_k} \lambda_{nik}^{(0)}}{\sum_j \hat{S}_{jk} \hat{L}_{jk}^{\alpha_k} \left(\hat{w}_j^{(t)}\right)^{-\varepsilon_k} \hat{\tau}_{njk}^{-\varepsilon_k} \lambda_{njk}^{(0)}}$$

Finally, using $Y_i^{(t)} \equiv w_i^{(t)} \bar{L}_i$, this can be rewritten as

$$\hat{w}_i^{(t+1)} = \hat{w}_i^{(t)} + \nu \frac{\sum_k \hat{X}_{ik}^{(t)} X_{ik}^{(0)} - \hat{w}_i^{(t)} Y_i^{(0)}}{Y_i^{(0)}},$$

For given $\hat{\boldsymbol{L}}$ we can implement this algorithm to find $\hat{w}_i(\hat{\boldsymbol{L}})$.

Now we iterate on $\hat{\boldsymbol{L}}$ (outer loop) using labor demand (in value) for sector $ik$. More explicitly, the algorithm is as follows. Define

$$\hat{\lambda}_{nik}\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}\right) \equiv \frac{\hat{S}_{ik} \hat{L}_{ik}^{\alpha_k} \hat{w}_i^{-\varepsilon_k} \hat{\tau}_{nik}^{-\varepsilon_k}}{\sum_j \hat{S}_{jk} \hat{L}_{jk}^{\alpha_k} \hat{w}_j^{-\varepsilon_k} \hat{\tau}_{njk}^{-\varepsilon_k} \lambda_{njk}^{(0)}}$$

and

$$\hat{X}_{ik}\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}\right) \equiv \frac{1}{X_{ik}^{(0)}} \sum_n \hat{\lambda}_{nik}\left(\hat{\boldsymbol{w}}, \hat{\boldsymbol{L}}\right) \lambda_{nik}^{(0)} \beta_{nk} \hat{w}_n Y_n^{(0)}.$$

First, for a given $\hat{\boldsymbol{L}}^{(l)}$, get $\hat{\boldsymbol{w}}(\hat{\boldsymbol{L}}^{(l)})$ using the following iterative procedure (inner loop):

$$\hat{w}_i^{(t+1)} = \hat{w}_i^{(t)} + \nu \frac{\sum_k \hat{X}_{ik} \left( \hat{\boldsymbol{w}}^{(t)}, \hat{\boldsymbol{L}}^{(l)} \right) X_{ik}^{(0)} - \hat{w}_i^{(t)} Y_i^{(0)}}{Y_i^{(0)}}.$$

Then, use the following iterative procedure (outer loop) to get $\hat{\boldsymbol{L}}$:

$$\hat{L}_{ik}^{(l+1)} = \frac{1}{\hat{w}_i(\hat{\boldsymbol{L}}^{(l)}) Y_{ik}^{(0)}} \sum_n \hat{\lambda}_{nik} \left( \hat{\boldsymbol{w}}(\hat{\boldsymbol{L}}^{(l)}), \hat{\boldsymbol{L}}^{(l)} \right) \lambda_{nik}^{(0)} \beta_{nk} \hat{w}_n \left( \hat{\boldsymbol{L}}^{(l)} \right) Y_n^{(0)}.$$

## D.5. Complete Tables for Quantitative Results in Sections 4 and 5

Table 8: Gains from Trade, Common Elasticities

| | Gains from Trade | | | | Degree of specialization | |
|---|---|---|---|---|---|---|
| | $\psi_k = 0$ common | $\psi_k = 0.1$ common | $\psi_k = 0.2$ common | $\psi_k = 0.2$ manuf. | All industries | Manuf. only |
| Country | (1) | (2) | (3) | (4) | (5) | (6) |
| AUS | 3.0% | 2.7% | 2.3% | 2.1% | 0.036 | 0.048 |
| AUT | 8.7% | 8.6% | 8.5% | 9.0% | 0.015 | -0.012 |
| BEL | 13.8% | 13.2% | 12.5% | 14.0% | 0.072 | -0.012 |
| BRA | 1.6% | 1.6% | 1.5% | 1.5% | 0.004 | 0.004 |
| CAN | 5.2% | 5.0% | 4.7% | 4.5% | 0.024 | 0.037 |
| CHN | 3.0% | 2.8% | 2.7% | 3.5% | 0.016 | -0.025 |
| CZE | 8.0% | 7.8% | 7.7% | 8.4% | 0.014 | -0.022 |
| DEU | 6.0% | 5.7% | 5.4% | 6.9% | 0.033 | -0.046 |
| DNK | 9.5% | 8.6% | 7.8% | 9.3% | 0.090 | 0.009 |
| ESP | 4.1% | 3.8% | 3.6% | 3.8% | 0.023 | 0.011 |
| FIN | 5.5% | 5.1% | 4.8% | 6.3% | 0.036 | -0.042 |
| FRA | 4.0% | 3.8% | 3.7% | 4.2% | 0.020 | -0.006 |
| GBR | 4.6% | 4.5% | 4.5% | 4.2% | 0.008 | 0.022 |
| GRC | 5.7% | 4.6% | 3.6% | 3.5% | 0.110 | 0.114 |
| HUN | 12.3% | 12.0% | 11.8% | 11.9% | 0.027 | 0.018 |
| IDN | 3.5% | 3.4% | 3.3% | 3.2% | 0.013 | 0.018 |
| IND | 3.1% | 3.0% | 2.9% | 2.8% | 0.013 | 0.014 |
| IRL | 10.1% | 9.5% | 8.8% | 10.6% | 0.069 | -0.027 |
| ITA | 3.7% | 3.4% | 3.2% | 3.9% | 0.026 | -0.015 |
| JPN | 2.4% | 2.0% | 1.6% | 2.9% | 0.041 | -0.023 |
| KOR | 6.6% | 5.3% | 4.1% | 7.3% | 0.133 | -0.037 |
| MEX | 4.5% | 4.3% | 4.1% | 3.8% | 0.018 | 0.038 |
| NLD | 9.4% | 9.3% | 9.1% | 9.8% | 0.020 | -0.018 |
| POL | 6.0% | 5.9% | 5.8% | 5.9% | 0.009 | 0.006 |
| PRT | 6.3% | 6.0% | 5.7% | 5.7% | 0.032 | 0.032 |
| ROM | 6.2% | 6.0% | 5.8% | 5.1% | 0.020 | 0.059 |
| RUS | 3.2% | 2.5% | 1.7% | 0.8% | 0.076 | 0.123 |
| SVK | 11.5% | 11.2% | 10.9% | 11.7% | 0.038 | -0.011 |
| SVN | 13.3% | 12.3% | 11.3% | 13.4% | 0.112 | -0.008 |
| SWE | 6.7% | 6.5% | 6.4% | 7.2% | 0.019 | -0.026 |
| TUR | 4.0% | 3.9% | 3.7% | 4.1% | 0.018 | -0.004 |
| TWN | 9.6% | 8.6% | 7.5% | 10.2% | 0.112 | -0.036 |
| USA | 2.1% | 2.0% | 1.9% | 1.7% | 0.007 | 0.020 |
| RoW | 6.9% | 6.4% | 6.0% | 5.8% | 0.050 | 0.061 |
| **Average** | 6.3% | 5.9% | 5.6% | 6.1% | 0.040 | 0.008 |

Table 9: Gains from Trade, Caliendo-Parro elasticities

| | Gains from Trade | | | Degree and Pattern of Specialization | | |
|---|---|---|---|---|---|---|
| | $\psi_k = 0$ | $\psi_k = \frac{0.5}{\varepsilon_k}$ | $\psi_k = \frac{1}{\varepsilon_k}$ | *DS* | *PS* | Overall |
| Country | (1) | (2) | (3) | (4) | (5) | (6) |
| AUS | 7.6% | 5.5% | 3.5% | 0.036 | -0.111 | 0.147 |
| AUT | 29.1% | 29.3% | 29.4% | 0.015 | 0.027 | -0.012 |
| BEL | 29.8% | 29.6% | 29.3% | 0.072 | 0.051 | 0.021 |
| BRA | 3.4% | 3.6% | 3.9% | 0.004 | 0.019 | -0.015 |
| CAN | 16.7% | 15.7% | 14.7% | 0.024 | -0.056 | 0.081 |
| CHN | 4.1% | 4.0% | 4.0% | 0.016 | 0.013 | 0.003 |
| CZE | 16.6% | 18.4% | 20.2% | 0.014 | 0.166 | -0.151 |
| DEU | 12.9% | 15.1% | 17.2% | 0.033 | 0.204 | -0.171 |
| DNK | 24.6% | 22.4% | 20.1% | 0.090 | -0.104 | 0.194 |
| ESP | 9.4% | 9.7% | 10.0% | 0.023 | 0.047 | -0.025 |
| FIN | 10.3% | 9.9% | 9.5% | 0.036 | 0.007 | 0.029 |
| FRA | 9.8% | 10.5% | 11.3% | 0.020 | 0.076 | -0.056 |
| GBR | 11.7% | 11.4% | 11.1% | 0.008 | -0.018 | 0.025 |
| GRC | 14.5% | 10.1% | 5.5% | 0.110 | -0.230 | 0.340 |
| HUN | 31.1% | 31.9% | 32.7% | 0.027 | 0.103 | -0.076 |
| IDN | 6.3% | 5.3% | 4.3% | 0.013 | -0.058 | 0.070 |
| IND | 4.6% | 4.5% | 4.4% | 0.013 | 0.006 | 0.008 |
| IRL | 18.3% | 15.4% | 12.5% | 0.069 | -0.163 | 0.232 |
| ITA | 8.3% | 8.4% | 8.5% | 0.026 | 0.036 | -0.010 |
| JPN | 2.4% | 4.3% | 6.1% | 0.041 | 0.173 | -0.133 |
| KOR | 6.2% | 8.7% | 11.2% | 0.133 | 0.318 | -0.185 |
| MEX | 11.5% | 12.0% | 12.5% | 0.018 | 0.057 | -0.039 |
| NLD | 22.7% | 22.0% | 21.2% | 0.020 | -0.044 | 0.064 |
| POL | 18.0% | 18.6% | 19.3% | 0.009 | 0.064 | -0.055 |
| PRT | 19.0% | 18.1% | 17.1% | 0.032 | -0.050 | 0.081 |
| ROM | 13.8% | 12.4% | 11.0% | 0.020 | -0.089 | 0.110 |
| RUS | 15.9% | 9.0% | 1.6% | 0.076 | -0.457 | 0.534 |
| SVK | 30.3% | 31.7% | 33.1% | 0.038 | 0.176 | -0.138 |
| SVN | 33.1% | 33.2% | 33.3% | 0.112 | 0.120 | -0.008 |
| SWE | 13.7% | 14.5% | 15.4% | 0.019 | 0.084 | -0.065 |
| TUR | 11.3% | 11.9% | 12.5% | 0.018 | 0.064 | -0.047 |
| TWN | 10.2% | 9.8% | 9.5% | 0.112 | 0.084 | 0.028 |
| USA | 4.5% | 4.4% | 4.3% | 0.007 | 0.000 | 0.007 |
| RoW | 14.8% | 11.1% | 7.2% | 0.050 | -0.241 | 0.290 |
| **Average** | 14.6% | 14.2% | 13.7% | 0.040 | 0.008 | 0.032 |

Table 10: Inward Trade Liberalization and Foreign Productivity Improvement

| | $\widehat{\tau}_{ni,k} = 0.9 \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{S}_{CHN,k} = 1.01^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | |
|---|---|---|---|---|---|---|
| | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ |
| Country | (1) | (2) | (3) | (4)* | (5)* | (6)* |
| AUS | 0.64% | 0.51% | 0.25% | 0.94% | 0.31% | -0.76% |
| AUT | 1.61% | 1.51% | 0.96% | 0.19% | -0.15% | -0.25% |
| BEL | 1.89% | 1.81% | 1.45% | 0.16% | -0.11% | -0.44% |
| BRA | 0.35% | 0.26% | 0.16% | 0.32% | 0.10% | -0.13% |
| CAN | 1.18% | 1.05% | 0.69% | 0.50% | 0.13% | -0.28% |
| CHN | 0.63% | 0.54% | 0.43% | 48.87% | 50.99% | 53.08% |
| CZE | 1.85% | 1.76% | 1.50% | 0.28% | 0.18% | 0.32% |
| DEU | 1.26% | 1.18% | 0.83% | 0.08% | -0.11% | -0.27% |
| DNK | 1.32% | 1.21% | 0.82% | 0.37% | -0.05% | -0.58% |
| ESP | 0.70% | 0.59% | 0.43% | 0.03% | -0.21% | -0.47% |
| FIN | 0.98% | 0.89% | 0.76% | -0.11% | -0.29% | -0.28% |
| FRA | 0.82% | 0.74% | 0.64% | 0.08% | -0.09% | -0.21% |
| GBR | 0.85% | 0.75% | 0.58% | 0.24% | 0.02% | -0.18% |
| GRC | 0.90% | 0.64% | 0.08% | 0.62% | 0.27% | -0.22% |
| HUN | 2.48% | 2.35% | 1.91% | 0.36% | 0.15% | 0.01% |
| IDN | 0.77% | 0.64% | 0.48% | 0.52% | 0.09% | -0.19% |
| IND | 0.61% | 0.50% | 0.28% | 0.35% | 0.15% | 0.13% |
| IRL | 1.19% | 1.06% | 0.74% | -0.20% | -0.50% | -0.66% |
| ITA | 0.69% | 0.61% | 0.53% | -0.04% | -0.18% | -0.21% |
| JPN | 0.30% | 0.25% | 0.20% | 0.00% | -0.15% | -0.17% |
| KOR | 0.80% | 0.68% | 0.53% | -0.00% | -0.55% | -0.97% |
| MEX | 1.12% | 1.00% | 0.73% | 0.39% | 0.29% | 0.26% |
| NLD | 1.32% | 1.22% | 0.80% | 0.10% | -0.27% | -0.86% |
| POL | 1.31% | 1.22% | 0.87% | 0.33% | 0.17% | 0.04% |
| PRT | 1.14% | 1.00% | 0.61% | 0.31% | -0.04% | -0.39% |
| ROM | 1.27% | 1.11% | 0.81% | 0.15% | -0.10% | -0.24% |
| RUS | 0.87% | 0.65% | 0.24% | 0.70% | 0.31% | -0.24% |
| SVK | 2.31% | 2.22% | 1.72% | 0.12% | -0.03% | -0.13% |
| SVN | 2.11% | 2.01% | 1.41% | 0.08% | -0.08% | -0.27% |
| SWE | 1.22% | 1.11% | 0.81% | -0.09% | -0.48% | -0.81% |
| TUR | 0.73% | 0.65% | 0.27% | 0.05% | -0.02% | -0.05% |
| TWN | 1.66% | 1.51% | 0.73% | -0.13% | -0.29% | -0.70% |
| USA | 0.39% | 0.30% | 0.17% | 0.28% | 0.05% | -0.17% |
| RoW | 1.14% | 0.93% | 0.56% | 0.90% | 0.14% | -1.12% |

* The numbers specified in the column have been multiplied by 100.

Table 11: Unilateral Trade Liberalization with Fixed Wages

| | $\widehat{\tau}_{ni,k} = 0.99 \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{\tau}_{ni,k} = 0.9 \quad \forall\, k \in \mathbb{M}$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $2 \times 2$ | | | | | | $N \times S$ | | |
| | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ |
| Country | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| AUS | 0.08% | 0.05% | -0.00% | 0.91% | 0.69% | -0.00% | 0.89% | 0.71% | 0.22% |
| AUT | 0.17% | 0.15% | -0.00% | 1.97% | 1.82% | 0.31% | 1.94% | 1.81% | 1.12% |
| BEL | 0.19% | 0.18% | -0.01% | 2.12% | 2.08% | 1.44% | 2.10% | 2.05% | 1.75% |
| BRA | 0.05% | 0.03% | -0.00% | 0.60% | 0.36% | -0.00% | 0.59% | 0.38% | 0.03% |
| CAN | 0.13% | 0.11% | -0.00% | 1.54% | 1.32% | -0.03% | 1.51% | 1.34% | 0.68% |
| CHN | 0.08% | 0.05% | -0.01% | 1.04% | 0.68% | -0.09% | 1.01% | 0.79% | 0.91% |
| CZE | 0.21% | 0.18% | -0.00% | 2.39% | 2.15% | -0.03% | 2.36% | 2.15% | 1.05% |
| DEU | 0.14% | 0.12% | -0.01% | 1.71% | 1.50% | -0.18% | 1.69% | 1.54% | 1.51% |
| DNK | 0.14% | 0.13% | -0.00% | 1.56% | 1.48% | 0.62% | 1.55% | 1.46% | 0.90% |
| ESP | 0.09% | 0.06% | -0.00% | 1.13% | 0.85% | -0.01% | 1.10% | 0.89% | 0.24% |
| FIN | 0.11% | 0.09% | -0.00% | 1.36% | 1.13% | -0.01% | 1.33% | 1.14% | 0.22% |
| FRA | 0.10% | 0.08% | -0.00% | 1.18% | 0.98% | -0.04% | 1.17% | 1.01% | 0.74% |
| GBR | 0.09% | 0.08% | -0.00% | 1.10% | 0.96% | -0.03% | 1.08% | 0.97% | 0.68% |
| GRC | 0.12% | 0.08% | -0.00% | 1.40% | 1.08% | -0.00% | 1.37% | 1.12% | 0.51% |
| HUN | 0.26% | 0.24% | -0.00% | 2.93% | 2.79% | 1.27% | 2.89% | 2.73% | 2.07% |
| IDN | 0.09% | 0.06% | -0.00% | 1.16% | 0.80% | -0.00% | 1.11% | 0.86% | 0.50% |
| IND | 0.08% | 0.05% | -0.00% | 1.01% | 0.67% | -0.01% | 0.97% | 0.73% | 0.49% |
| IRL | 0.13% | 0.11% | -0.00% | 1.50% | 1.39% | -0.01% | 1.49% | 1.37% | 0.77% |
| ITA | 0.09% | 0.06% | -0.00% | 1.10% | 0.82% | -0.03% | 1.09% | 0.86% | 0.21% |
| JPN | 0.04% | 0.03% | -0.00% | 0.55% | 0.35% | -0.02% | 0.54% | 0.36% | -0.01% |
| KOR | 0.11% | 0.07% | -0.00% | 1.35% | 0.97% | -0.02% | 1.34% | 0.98% | -0.06% |
| MEX | 0.13% | 0.10% | -0.00% | 1.61% | 1.26% | -0.02% | 1.56% | 1.31% | 0.37% |
| NLD | 0.14% | 0.13% | -0.00% | 1.54% | 1.47% | 0.66% | 1.52% | 1.45% | 1.11% |
| POL | 0.15% | 0.12% | -0.00% | 1.77% | 1.50% | -0.02% | 1.71% | 1.53% | 0.80% |
| PRT | 0.14% | 0.11% | -0.00% | 1.60% | 1.36% | -0.01% | 1.57% | 1.38% | 0.52% |
| ROM | 0.15% | 0.11% | -0.00% | 1.82% | 1.47% | -0.00% | 1.76% | 1.53% | 1.10% |
| RUS | 0.10% | 0.06% | -0.00% | 1.23% | 0.84% | -0.00% | 1.17% | 0.90% | 0.66% |
| SVK | 0.24% | 0.23% | -0.00% | 2.75% | 2.63% | 1.23% | 2.73% | 2.60% | 1.71% |
| SVN | 0.23% | 0.21% | -0.00% | 2.55% | 2.42% | 1.08% | 2.52% | 2.38% | 1.40% |
| SWE | 0.13% | 0.11% | -0.00% | 1.58% | 1.40% | -0.02% | 1.55% | 1.40% | 0.90% |
| TUR | 0.09% | 0.06% | -0.00% | 1.14% | 0.81% | -0.01% | 1.07% | 0.88% | 0.47% |
| TWN | 0.19% | 0.16% | -0.00% | 2.22% | 1.95% | -0.04% | 2.17% | 1.88% | 0.94% |
| USA | 0.06% | 0.04% | -0.00% | 0.70% | 0.48% | -0.04% | 0.68% | 0.55% | 0.47% |
| RoW | 0.14% | 0.11% | -0.01% | 1.64% | 1.40% | -0.20% | 1.61% | 1.43% | 1.11% |

Table 12: Foreign Productivity Improvement with Fixed Wages

| | $\widehat{S}_{RoW,k} = 1.01^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{S}_{RoW,k} = 2^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | | $\widehat{S}_{CHN,k} = 1.01^{\varepsilon_k} \quad \forall\, k \in \mathbb{M}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $2 \times 2$ | | | | | | $N \times S$ | | |
| | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ | $\alpha_M = 0$ | $\alpha_M = 0.5$ | $\alpha_M = 1$ |
| Country | (1)* | (2)* | (3)* | (4) | (5) | (6) | (7)* | (8)* | (9)* |
| AUS | 7.47% | 3.05% | -7.11% | 10.31% | 10.14% | 10.15% | 1.66% | 1.13% | -0.62% |
| AUT | 16.99% | 9.50% | -47.78% | 17.80% | 17.70% | 17.72% | 0.79% | -0.25% | 0.61% |
| BEL | 18.97% | 14.73% | -85.11% | 16.81% | 16.79% | 16.81% | 1.05% | 0.30% | -3.95% |
| BRA | 4.61% | 0.46% | -5.33% | 11.09% | 10.27% | 10.15% | 0.88% | 0.36% | 0.25% |
| CAN | 12.96% | 6.23% | -21.34% | 15.10% | 14.99% | 15.03% | 1.86% | 1.31% | -2.77% |
| CHN | 8.02% | -0.67% | -16.28% | 19.14% | 18.96% | 20.14% | 40.93% | 47.40% | 74.24% |
| CZE | 20.30% | 9.64% | -51.27% | 23.36% | 23.18% | 23.20% | 2.46% | 1.80% | 7.75% |
| DEU | 14.26% | 5.27% | -40.92% | 17.89% | 17.92% | 18.17% | 1.56% | 0.61% | -3.74% |
| DNK | 13.71% | 8.82% | -43.30% | 13.16% | 13.11% | 13.11% | 0.97% | 0.23% | 1.52% |
| ESP | 9.17% | 3.06% | -11.18% | 13.45% | 13.23% | 13.27% | 0.94% | 0.43% | 0.26% |
| FIN | 11.19% | 2.38% | -28.54% | 15.34% | 15.08% | 15.09% | 1.28% | 0.36% | -0.16% |
| FRA | 9.82% | 3.56% | -18.07% | 12.62% | 12.52% | 12.60% | 0.86% | 0.29% | 0.62% |
| GBR | 9.38% | 4.83% | -16.24% | 10.33% | 10.28% | 10.31% | 1.05% | 0.63% | 0.33% |
| GRC | 11.74% | 6.95% | -4.39% | 14.25% | 14.09% | 14.09% | 0.87% | 0.28% | 0.48% |
| HUN | 25.66% | 17.03% | -76.88% | 25.62% | 25.52% | 25.53% | 2.46% | 1.62% | 16.01% |
| IDN | 9.17% | 2.00% | -11.34% | 16.20% | 15.66% | 15.64% | 2.23% | 1.19% | -0.22% |
| IND | 7.97% | 1.93% | -8.05% | 15.21% | 14.64% | 14.65% | 2.00% | 1.40% | 3.22% |
| IRL | 12.83% | 5.61% | -46.63% | 14.15% | 14.04% | 14.05% | 1.39% | 0.54% | 1.87% |
| ITA | 8.85% | 1.69% | -15.11% | 14.18% | 13.91% | 14.01% | 0.77% | -0.04% | 0.13% |
| JPN | 4.22% | -0.68% | -8.31% | 10.38% | 9.75% | 9.87% | 1.48% | 0.25% | -3.19% |
| KOR | 10.66% | 0.46% | -20.93% | 19.66% | 19.05% | 19.13% | 3.45% | 0.74% | -11.19% |
| MEX | 13.24% | 5.02% | -16.90% | 18.32% | 18.06% | 18.11% | 2.12% | 1.62% | -1.57% |
| NLD | 13.52% | 8.79% | -52.95% | 12.88% | 12.84% | 12.86% | 1.27% | 0.88% | -0.41% |
| POL | 14.78% | 6.18% | -27.07% | 18.17% | 17.99% | 18.01% | 1.63% | 1.14% | 0.54% |
| PRT | 13.55% | 6.66% | -19.55% | 15.77% | 15.62% | 15.62% | 0.43% | -0.37% | 0.96% |
| ROM | 15.19% | 7.30% | -15.50% | 19.18% | 18.95% | 18.95% | 0.80% | 0.00% | 2.79% |
| RUS | 9.90% | 4.71% | -4.31% | 15.60% | 15.26% | 15.29% | 1.66% | 1.06% | 0.02% |
| SVK | 24.11% | 16.06% | -80.89% | 23.77% | 23.67% | 23.68% | 1.85% | 0.70% | -5.44% |
| SVN | 22.32% | 14.67% | -69.27% | 22.01% | 21.91% | 21.91% | 1.29% | 0.27% | 11.58% |
| SWE | 13.29% | 5.41% | -35.17% | 15.54% | 15.40% | 15.42% | 0.70% | -0.32% | 1.25% |
| TUR | 9.08% | 1.96% | -12.45% | 15.21% | 14.79% | 14.79% | 1.23% | 0.53% | 1.84% |
| TWN | 18.58% | 6.98% | -50.21% | 22.95% | 22.74% | 22.78% | 3.93% | 1.15% | -21.74% |
| USA | 5.53% | 1.98% | -4.81% | 9.36% | 9.37% | 9.68% | 1.36% | 0.86% | -0.02% |
| RoW | 13.79% | 7.49% | -18.09% | 16.65% | 16.83% | 17.20% | 3.01% | 2.17% | -7.27% |

* The numbers specified in the column have been multiplied by 100.

Table 13: Scale and Trade Flows

| | | $\alpha = 0.5$ | | $\alpha = 1$ | |
|---|---|---|---|---|---|
| | $DS_i$ | $\widehat{DS}_i$ | $\widehat{EX}_i$ | $\widehat{DS}_i$ | $\widehat{EX}_i$ |
| Country | (1) | (2) | (3) | (4) | (5) |
| AUS | 0.04 | -60.80% | -9.44% | -93.03% | -16.61% |
| AUT | 0.02 | -61.85% | -0.85% | -83.99% | -1.38% |
| BEL | 0.07 | -64.83% | -0.86% | -98.20% | -0.36% |
| BRA | 0.00 | -27.81% | -3.99% | -49.37% | -7.73% |
| CAN | 0.02 | -62.92% | -1.84% | -93.10% | -2.57% |
| CHN | 0.02 | -50.09% | -5.71% | -77.27% | -10.19% |
| CZE | 0.01 | -64.56% | -2.24% | -89.40% | -4.07% |
| DEU | 0.03 | -57.46% | -3.45% | -86.77% | -6.47% |
| DNK | 0.09 | -72.81% | -7.52% | -96.89% | -10.50% |
| ESP | 0.02 | -65.03% | -2.08% | -92.65% | -3.93% |
| FIN | 0.04 | -56.97% | -5.17% | -84.00% | -9.23% |
| FRA | 0.02 | -65.84% | -2.65% | -94.39% | -5.50% |
| GBR | 0.01 | -52.64% | -2.09% | -80.93% | -3.93% |
| GRC | 0.11 | -80.81% | -34.11% | -92.28% | -36.71% |
| HUN | 0.03 | -61.97% | -2.43% | -86.21% | -4.16% |
| IDN | 0.01 | -39.76% | -3.14% | -73.45% | -7.05% |
| IND | 0.01 | -56.57% | -3.09% | -78.23% | -4.30% |
| IRL | 0.07 | -62.27% | -4.05% | -92.27% | -6.43% |
| ITA | 0.03 | -67.40% | -3.66% | -95.44% | -6.92% |
| JPN | 0.04 | -64.05% | -7.84% | -91.77% | -16.35% |
| KOR | 0.13 | -64.98% | -7.19% | -94.25% | -17.12% |
| MEX | 0.02 | -47.52% | -3.45% | -75.86% | -6.32% |
| NLD | 0.02 | -58.22% | -2.80% | -84.96% | -5.20% |
| POL | 0.01 | -64.82% | -2.10% | -89.59% | -3.65% |
| PRT | 0.03 | -64.28% | -1.68% | -90.12% | -2.58% |
| ROM | 0.02 | -50.04% | -1.92% | -74.73% | -2.60% |
| RUS | 0.08 | -50.66% | -13.67% | -79.56% | -25.18% |
| SVK | 0.04 | -68.64% | -2.56% | -95.38% | -4.71% |
| SVN | 0.11 | -59.87% | -1.38% | -97.67% | -0.71% |
| SWE | 0.02 | -56.61% | -2.67% | -85.26% | -5.07% |
| TUR | 0.02 | -56.67% | -7.46% | -81.44% | -13.15% |
| TWN | 0.11 | -64.59% | -8.37% | -93.34% | -16.58% |
| USA | 0.01 | -47.24% | -4.85% | -71.86% | -8.63% |
| RoW | 0.05 | -53.75% | -7.86% | -84.78% | -15.04% |

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