Dynamic Reserves in Matching Markets: Theory and Applications*

Orhan Aygün† and Bertan Turhan‡

November, 2016

Abstract

Indian engineering school admissions, which draw more than 500,000 applications per year, suffer from an important market failure: Through their affirmative action program, a certain number of seats are reserved for different castes and tribes. However, when some of these seats are unfilled, they are not offered to other groups, and the system is vastly wasteful. Moreover, since students care not only about the school they are assigned to but also whether they are assigned through reserves or not, they may manipulate the system both by not revealing their privilege type and by changing their preferences over programs. In this paper, we propose a new matching model with the ability to release vacant seats to the use of other students by respecting certain affirmative action objectives. We design a new choice function for schools that respects affirmative action objectives, and increases efficiency. We propose a mechanism that is stable, strategy proof, and respects test-score improvements with respect to these choice functions. Moreover, we show that some distributional objectives that can be achieved by capacity-transfers cannot be achieved by slot-specific priorities.

*We thank Dilip Abreu, M. Oğuz Afacan, Peter Biro, Rahul Deb, Umut Dur, Federico Echenique, Isa Hafalır, Andrei Gomberg, Yuichiro Kamada, Navin Kartik, Fuhito Kojima, Ran Shorrer, Tayfun Sönmez, Utku Ünver, Alexander Westkamp, and Bumin Yenmez for helpful comments and the seminar audiences at the Boston College, University of St. Andrews, ITAM, North American Meeting of the Econometric Society 2016, GAMES 2016, the 13th meeting of Social Choice and Welfare 2016, Conference on Economic Design 2015, 9th Meeting of Matching in Practice, and Workshop on Market Design in memory of Dila Meryem Hafalır at ITAM for their helpful suggestions. All errors are ours.

†orhan.aygun@boun.edu.tr , Department of Economics, Boğaziçi University

‡bertan.turhan@itam.mx, Centro de Investigacion Economica and Department of Economics, ITAM

1 Introduction

Engineering school admissions in India function through a centralized matching market in which students with differently privileged backgrounds, such as, different caste and tribe membership, are treated with different admission criteria. Students have different preferences over which admissions criteria they are admitted under. Therefore, students may prefer not to reveal their caste and tribe information in the application process. Besides this strategic calculation burden on students, the current system suffers from a crucial market failure which is the main focus of this paper: The centralized assignment mechanism fails to transfer some unfilled seats reserved for under-privileged castes and tribes to the use of remaining students. Hence, it is vastly wasteful.

In this paper, we propose a remedy to this problem through a new matching model with contracts and the ability to utilize vacant seats of certain types for other students[1]. Moreover, our remedy removes the strategic manipulation burden, about which seat types they should apply for at an engineering program, from students' shoulders[2]. We propose a strategy proof and stable mechanism that respects test-score improvements in this framework.

More specifically, our model addresses the real-life applications as follows: There are schools and students to be matched. A given student may possibly match with a given school under more than one type. Each school has a pre-specified order[3] in which these different privilege groups are to be considered. Different schools might have different orders. Each student is a member of at least one privilege group.[4] Each student has a preference over school-privilege type pairs since students care not only about which institution they are matched to but also about the contractual terms (or privilege type) under which they are admitted. Each school has a target distribution[5] of its slots over privilege types, but we do not consider these target distributions as hard bounds.[6] If there is less demand from at least one privilege type, schools are given opportunity to utilize these vacant seats by transferring them over to other privilege groups. Schools might have preferences over how to redistribute

[1]The idea of *"reserves"* is first introduced in a school choice framework by Hafalir, Yenmez and Yildirim (2013). Capacity-transfer is first introduced by Westkamp (2013) in a matching problem with complex constraints to study German university admissions system.

[2]Our remedy makes it weakly dominant strategy for each student to report their caste and tribe information.

[3]We will call this order a *precedence order,* following the terminology of Kominers and Sönmez (2016).

[4]In the Indian engineering school admissions, students who do not belong to certain castes and tribes can obtain school seats only through general category seats. General category is actually "no-privilege" category that each student belongs to. However, for our modeling purposes we consider general category as one of the privilege types by abusing the meaning of the word "privilege". Hence, this is why we can say that each student is a member of at least one privilege group.

[5]In India, this target distribution is dictated by law in the form of reserves.

[6]See Hafalir et. al (2013) and Ehlers et al. (2014).

these unassigned slots[7].

We design choice functions for schools that allows them to transfer capacities from low-demand privilege types to high-demand privilege types. Each school respects an exogenously given, possibly different, precedence order when it fills its slots. For each privilege type there is an associated choice function, which we call a *"sub-choice function"*. Given the target distribution of the school and the set of contracts, the first privilege type of the school fills its slots according to its sub-choice function. Then, it moves to the second privilege type. Sub-choice functions are linked to each other by two components. First, since we take a pre-specified precedence order, the choice in each privilege group depends on what has been chosen by the privilege groups that are considered earlier. Given the chosen contracts from the first privilege type, the remaining set of contracts for the second privilege type can be found as follows: if a student has one of her contracts chosen by the first privilege type, then all of her contracts are removed (rejected) for the rest of the choice process. The second component that links sub-choice functions of different privilege types is that the capacity of a privilege type changes dynamically according to the number of unassigned slots in the privilege types considered earlier in the choice procedure, i.e., the possible transfer of unassigned slots from privilege groups to other privilege groups. The idea here is that the capacity of the privilege type following the first privilege type is a function of the number of unassigned seats in the first privilege type. The capacity of the third privilege type is a function[8] of the numbers of unassigned seats in the first and second privilege types, and so on. In short, each sub-choice function has two inputs: the set of remaining contracts to consider, which depends on the choices of the privilege types considered before it, and its capacity, which changes dynamically according to the number of unassigned slots in the privilege types considered earlier. The overall choice of an institution is the union of sub-choices by its different privilege types.

In our main application of engineering college admissions at the Indian Institutes of Technology (IITs) , which we describe in part 2, students are strictly ranked according to test scores.[9] Then, for each privilege type, students with that privilege type are ordered according to their test-score ranking. Sub-choice functions for each privilege type, then, are induced from these strict rankings. These types of choice functions are common in practice

[7]Currently, in many states in India, if there is less demand from the OBC category and some of the seats that are reserved for OBC students are vacant, these seats can be utilized by general category applicants. Transfers from the SC and ST categories even in the case of vacancies are not allowed by law.

[8]We require each such function to be monotonic. Monotonicity of a capacity transfer scheme is a very mild requirement that is introduced by Westkamp (2013).

[9]In the cadet-branch matching problem, cadets are also ranked according to test scores, i.e., the order of merit list. See Sönmez and Switzer (2013) and Sönmez (2013).

and are called *q-responsive*.[10]

We present the cumulative offer algorithm as an allocation rule with overall functions of schools described above. Our overall choice functions fail to satisfy unilateral substitutability and the law of aggregate demand.[11] We are, however, able to show that the cumulative offer mechanism yields stable outcome, is strategy proof, and respects improvements in test scores. The main purpose of introducing dynamic reserves is to increase efficiency. We show that the outcome of the cumulative offer process under any monotonic capacity-transfer scheme Pareto dominates the outcome of the cumulative offer mechanism outcome without a capacity-transfer.

1.1 Literature Review

Affirmative action in school choice, the so-called *controlled choice problem*, is first introduced by the seminal paper of *Abdulkadiroğlu and Sönmez (2003).* Authors approach the school choice problem from a mechanism design perspective and extend their analysis to accommodate a simple affirmative action policy with type-specific quotas. *Kojima (2012)* investigates the consequences of these proposed affirmative action policies on students' welfare in a setup where there are two student types, minorities and majorities, and quotas for majority students only. He finds examples in which all minority students are made worse off under these affirmative action policies, and he concludes that authorities should be cautious when implementing such affirmative action policies.

Hafalir, Yenmez and Yildirim (2013) is the first paper suggesting dynamic reserves, in a simpler setup. To circumvent inefficiencies caused by majority quotas, authors offer *minority reserves.* Schools assign minority reserves such that if the number of minority students in a school is less than its minority reserves, then any minority student is preferred to any majority student in that school. If there are not enough minority students to fill the reserves, majority students can still be admitted to fill up that school's reserved seats so that unfilled minority seats are allowed to be transferred to majorities.

Westkamp (2013) studies a matching problem with complex constraints in the context of German university admissions. The author develops a choice protocol for schools in a way that transfers between different groups of seats within a university is possible. In his model, a student might get assigned to the same school under different admissions criterion. He assumes that students are indifferent between these different admissions criterion and the ties in students' preferences are broken according to the precedence order of the choice

[10] See Roth and Sotomayor (1990) and Chambers and Yenmez (2015).

[11] These two conditions on choice functions are sufficient for the cumulative offer mechanism to be strategy proof. See Hatfield and Kojima (2010) and Aygün and Sönmez (2012).

protocol he defines. The main difference of our work from his is that in our model (and also in the Indian Engineering College admissions problem) students are not indifferent between different admission criterion for the same school. The preference domain of students in our setup is larger than his preference domain. The choice protocol he defines satisfies the substitutes condition and also the law of aggregate demand whereas both of them fail in our framework.

Our work is in the line of research that focuses on the real life applications of matching models with contracts started with *Sönmez and Switzer (2013)* and *Sönmez (2013)*. Both papers consider the cadet-branch matching problem at US Army and provide the first practical application of matching problems with contracts in which choice functions of branches fail to satisfy substitutability condition but satisfy the unilateral substitutability condition. The priority ranking of cadets is known as *order-of-merit-list*, and is the same in every branch. Each cadet is able to "*buy*" priority in some branches if he/she is willing to serve some extra years. Cadet-matching problem is reminiscent of the engineering schools admissions problem in India. In the former problem there are two ways for a cadet to obtain a certain branch: the baseline contract and the extended service of years contract while there are three different privileged groups in the latter: SC, ST, and OBC and also each student is able to obtain a seat through merit slots. Their choice functions are special cases of ours when there is no further utilization of vacant seats in the case of low demand. The solution with dynamic reserves improves upon their solution with regard to efficiency. Their choice functions satisfy the unilateral substitutability and the law of aggregate demand where our family of choice functions may fail them both.

Matching with slot-specific priorities of *Kominers and Sönmez (2016)* is another work that is closely related to ours. They study a many-to-one matching problem with contracts where each student has a unit demand and schools may have multiple slots available. In their model, each school slot has its own linear priority order over contracts and each school chooses contracts by filling its slots sequentially according to an order of precedence. The lexicographic choice function they develop may not satisfy the substitutability and the law of aggregate demand conditions.[12] Despite these difficulties, they show that the cumulative offer mechanism is stable and strategy proof. It also respects unambiguous improvements in priority. The main advantage of the choice functions we develop over theirs, as we show in Section 7, some distributional objectives that can be achieved by using our choice functions cannot be achieved by using the lexicographic choice functions. However, for the type of problems we consider, every distributional objective that can be achieved by lexicographic

[12] As it is the case for our choice functions, their choice functions might fail the unilateral substitutes condition.

choice functions can also be achieved by our choice functions.

In a recent work, *Hatfield and Kominers (2015)* develop a nice theory in which, if a choice function has a *completion* that is substitutable and satisfies the irrelevance of rejected contracts conditions then the stable outcome exists under these choice functions. Also, if this completion satisfies the law of aggregate demand condition, then the cumulative offer algorithm becomes strategy-proof. We show that the family of choice functions we design is *substitutably completable* and the completion satisfies the law of aggregate demand. Therefore, we provide an important practical application for their theory.

Kamada and Kojima (2015) is another related paper to ours and is in the context of Japanese medical residency matching market. In this market, each region consists of a number of hospitals and is assigned a regional cap. Each hospital has a physical capacity. However, total number of doctors assigned to hospitals in the same region cannot exceed the regional cap. By defining a certain stability concept that is tailored to a particular government goal to equalize the number of doctors across hospitals beyond target capacities, they utilize specific capacity-transfer schemes between hospitals in the same region. In Kamada and Kojima (2015b), the authors provides a general theory of matching under distributional constraints to accommodate a wide range of policy goals. They define a preference relation for regions over the possible capacity-transfers. They require this preference to be *substitutable* and *acceptant.* In their novel proof they create a hypothetical matching problem between doctors and *regions* by regarding each region as a hypothetical consortium of hospitals that acts as one agent. They define a region's choice function over *contracts* rather than doctors, where a contract specifies a doctor-hospital pair to be matched. The main deviation of our work from theirs is that we do not require schools' preferences over possible capacity-transfer schemes to be substitutable and acceptant. To make this point clear, currently, in Indian engineering school admissions in many states, only vacant OBC seats are allowed to be transferred to general category. If there are vacant seats in SC and ST categories these seats must remain unfilled by law. We can incorporate this policy restriction into our model by defining the stability notion with regard to the choice functions that implements this specific policy. However, this type of choice function is not allowed in their setting. Regions' choice functions over contracts satisfy the substitutability and the law of aggregate demand conditions in their model, however, schools' choice functions in our model may fail both.

The rest of the paper is organized as follows: In section 2 we introduce our main application of engineering college admissions in IITs in India. We describe the shortcomings of the current admissions procedure in the State of Maharashtra. In section 3 the model is presented and the choice procedures of schools are designed. In section 4 conditions on preferences and (sub)choice functions are described. In section 5 the allocation procedure we advocate, the cumulative offer process, is defined. In section 6 we present our main results. In section 7 we explain the difference between our choice procedure and the lexicographic choice procedures of Kominers and Sönmez (2015). Section 8 concludes. The technical details and all of the lengthy proofs are in the Appendices.

2 Admissions to Engineering Schools (IITs) in India

Countries in which minority groups have suffered from historic discrimination are commonly characterized by considerable schooling inequalities between these groups and the majority of the population. Particularly when the inequality is great, governments have adopted strong *affirmative action* policies in higher education to remedy it, eschewing a voluntary preferential system in favor of a "*reservation system*" that reserves a fixed percentage of seats in higher education institutions for the relevant groups. The fundamental assumption underlying the imposition of a reservation system is that minority students gain admission to selective programs, they would otherwise not have access to, and such gains generate social return, in the near future.[13]

India is one of the few countries that practices affirmative action on a large scale. *"Reservation in India"* is the process of setting aside a certain percentage of seats in government institutions for members of underrepresented communities, defined primarily by castes and tribes. *Scheduled castes* (SC), *Scheduled Tribes* (ST), and *Other Backward Classes* (OBC) are the primary beneficiaries of the reservation policies under the constitution, which have the objective of ensuring to level the playing field.[14]

Among all higher education institutions in India, engineering schools are the most prestigious. The admission procedure in engineering schools is organized and regulated by the Indian Institutes of Technology (IITs). The IITs practice affirmative action and offer reservation to minority sectors of society. The following table shows the reservation structure of engineering schools in the State of Maharashtra:[15]

[13]See Bertrand et al. (2010). They argue that affirmative action successfully targets the financially disadvantaged in India. The authors find that, despite poor entrance exam scores, lower-caste entrants obtain a positive return for admission.

[14]For a brief history of affirmative action policies in India, see Bertrand et al. (2010) and Weisskopf (2004).

[15]See "Rules for Admissions to First year of Degree Courses in Engineering/Technology in Government, Govt. Aided and Unaided Engineering institutes in Maharashtra State-Academic year 2014-2015".

Category	*Reservation*
Scheduled Castes (SC)	13%
Scheduled Tribes (ST)	7%
Other Backward Classes (OBC)	30%
General Category	50%

As shown in the above table, the reservation system sets aside a proportion of all possible positions for members of a specific group. Those not belonging to the designated communities can compete only for general-category positions (merit slots), while members of the designated communities can compete for both reserved seats and general-category seats. However, a student who belongs to one of the designated groups is given an opportunity to use his or her caste (or tribe) background as a privilege. If students from designated communities do not use their caste or tribe privileges, then they are considered only for general-category seats. Claiming a reserved seat for students from designated communities is *optional*. If they state their privilege and get accepted to a program with a reserved seat in that category, they have to prove their membership in the group by providing a legal document.

2.1 Engineering School Admission Procedure and the DTE Mechanism

In the Maharashtra engineering school admission procedure, students are ranked based on their total scores in the "Maharashtra Common Entrance Test (MT CET)." This ranking is used to assign students to general-category seats. Rankings for privilege types SC, ST, and OBC are derived as follows: For each category, the relative rankings of the same-category students are preserved, and the students from other categories are removed. For students with the same score, students are ranked first by their math scores, then by chemistry scores, and finally by physics scores. In the circumstance that students have the same three scores in each field, age determines the priority, i.e., the older student is given priority. As such, each student has a unique ranking. Each student submits his or her preferences over engineering programs. They can rank at most 100 programs. Together with their program rankings, they can also submit their privilege type if they are coming from SC, ST, or OBC communities and want to use this privilege.

The Directorate of Technical Education (DTE), the institution in charge of admissions to engineering schools in Maharashtra, uses the following mechanism to allocate seats to students in the centralized admission process (CAP):

Step 1 : Each student applies to his or her top choice. Each school considers the applications for the general-seat category first, following the ranking $\succ$. Students are assigned general-category seats one by one following $\succ$ up to the capacity of general-category seats. If there are more students than allowed by the capacity of the general-seat category, the remaining students are considered for the reserved categories depending on their submitted privilege type. For each reserved category *SC, ST,* and *OBC*, students are assigned seats one by one following the priority order of privilege type up to the capacity of that category. The remaining students are rejected.

In general, at step n:

Step n : Each student who was rejected in the previous step applies to his or her next-choice school. Each school fills its general seats first following $\succ$ from the tentatively held students and new applicants. Students are assigned general-category seats one at a time following $\succ$ up to the capacity of the general-category seats. If there are more students than allowed by the capacity of the general-seat category, the remaining students are considered for the reserved categories depending on their stated privilege type. For each reserved category *SC, ST,* and *OBC*, students are assigned seats one at a time following the priority ranking in each privilege type up to the capacity of that category. The remaining students are rejected.

This algorithm ends in finitely many steps. When outcomes are announced, all students learn their program assignments *together with the privilege type under which they were accepted*. DTE announces privilege types together with the program assignment for each student to show the public that reservations are actually respected.

After the above centralized admissions process is done, if there are empty seats in *OBC* category, then these seats are converted into general seats[16] and filled by general-category applicants according to test score rankings. This process is called the "counseling" process.[17] In our proposed solution we argue that transferring otherwise vacant seats from OBC category to general category in the main admission round by designing a suitable choice functions for schools preserves desirable strategic, fairness and efficiency properties.

2.2 The Shortcomings of the DTE Mechanism

The mechanism used by the DTE has significant shortcomings. Two main problems with their admission procedure are listed below. The Indian authorities either are not aware of the first problem or they find it insignificant; however, they realize that the second problem

[16]These types of assignment procedures are called "sequential" and are proven to have adverse fairness, efficiency, and strategic properties. See Dur and Kesten (2014).

[17]In many of the other states in India, none of the vacant seats that are reserved for ST, SC, and OBC categories are transferred into general-category seats.

which is the main concern of our work is important and are trying to solve it.

(**i**) Students are asked to state their preferences over the set of programs, even though their assignments specify a program name together with a seat type. The preference domain is narrower than the allocation domain in that students' preferences over seat types are not investigated but are assumed in a specific way. For example, suppose a student, say from OBC background, submits two schools in his preference list, school A and school B, such that he prefers school A to school B. However, when his assignment is announced, it is going to be in the following form: "general-category seat from school B" or "OBC-category seat from school A." The DTE assignment procedure simply *assumes* that students only care about which program they are admitted to. They assume that for each program a student ranks in his or her preference list, he or she prefers the general-category seat type of that program over the reserved-type seat if the student submitted any privilege along with his application. However, for several reasons, which we will discuss below, students may actually care about what type of seats they receive with their program assignments. Their true preferences might be over program name-seat type pairs, not just program names. As in the problem of narrower preference domain of the cadet-branch problem in USMA and ROTC,[18] the DTE assumes each student prefers the general seats over the reserved seats given a program. Hence, given a preference relation over schools, the DTE generates a new preference profile such that the relative ranking of schools is the same, and in each school the general seat is preferred over the reserved-category seat for every student. However, across different programs with different types of seats, students might have more complicated preferences.

- Some students might not want to reveal their caste and tribe information and hence would prefer general-category seats over type-specific seats. One of the main reasons for this is the fact that students who obtain a seat from a reserved category are discriminated against in some universities. Opponents of the reservation policies in India argue that the policy is anti-meritocratic and decreases the average quality of Indian engineering schools. As a result, many students who obtain reserved seats feel discriminated against, as the following item illustrates:

> "*A survey among first year students (2013-14 batch) belonging to various SC, ST and OBC categories, has revealed that an alarming 56% of them feel discriminated against in the institution, albeit in a discreet manner. Nearly 60% of those in the reserved category also said they experienced more academic pressure than*

[18]See Switzer and Sönmez (2013) and Sönmez (2013).

those in the general category."[19]

Because of this pressure, some SC, ST, and OBC students might prefer general-category seats rather than reserved-category seat for personal reasons such as pride and dignity.[20] However, mechanisms currently in use do not let students express these concerns in their preferences.

If a student from a designated community uses her privilege and is assigned to a reserved seat, then she might be exempt from school fees or will pay very low fees, will receive book grants, and will be able to live for free in college housing. Because of financial reasons, a high-score, poor student from a designated community would prefer a reserved seat over a general seat. Financial difficulties that *SC, ST* and *OBC* students might face is illustrated in the following quote from an online education forum:

> *"It's estimated that 70% of Below Poverty Line in India comprises of Scheduled caste people. It's very difficult for an SC/ST/OBC student to crack JEE advanced and once they crack this exam, they have to face even a bigger problem. How will they afford at least 1.20 lakh Rupees per year for this technical education?... So what we conclude from all this is that it's not an easy task for reserved category students to get education in IITs. I do agree that there are some reserved category students who take advantage of all this. I guess at least 30% of reserved category students are economically well and they can afford all this on their own. This is a flaw in the system and we have to accept it."*[21]

- General-category seats are regarded as more prestigious. Students from designated communities who care about obtaining prestigious seats have more complicated preferences than preferences simply over programs. Also, some give political reasons for arguing against the reservation policy. Many students from designated communities are against caste-based reservation policies and do not claim caste or tribe privileges. In that case, they are considered for only general seats.

Example 1. Suppose that student i who has privilege ST submits the following preference over schools: $s_1 P_i s_2 P_i s_3$. The DTE generates the following preference relation from the

[19] http://www.dnaindia.com/mumbai/report-caste-discrimination-in-india-s-elite-institutions-students-2016745

[20] http://www.quora.com/How-does-it-feel-to-be-an-SC-ST-category-student-in-IIT

[21] http://www.quora.com/Reservation-in-India/As-per-the-policy-of-my-institute-IIT-Roorkee-SC-ST-students-are-provided-with-concession-and-waiver-in-fee-irrespective-of-their-family-income

stated preference: $s_1^{Gen} P_i s_1^{ST} P_i s_2^{Gen} P_i s_2^{ST} P_i s_3^{Gen} P_i s_3^{ST}$. However, student i's true preference might be as follows: $s_1^{Gen} P_i s_2^{Gen} P_i s_3^{Gen} P_i s_1^{ST} P_i s_2^{ST} P_i s_3^{ST}$.

This student can manipulate the DTE mechanism by misrepresenting her preferences. Also, the mechanism may create an adverse incentive to have lower test scores if a student from a designated category wants to gain admission only through reserved-category seats; i.e., in the above example, a student from an ST community might have the following true preference: $s_1^{ST} P_i s_2^{ST} P_i s_3^{ST} P_i s_1^{Gen} P_i s_2^{Gen} P_i s_3^{Gen}$.

As such, it is obvious that the DTE mechanism is not fair, does not respect improvements, and is manipulable. Furthermore, it is actually very easy to manipulate the DTE mechanism. In our model, we expand the preference domain to program-seat type pairs to fully alleviate this problem. Every preference profile over only schools can be represented when preferences are defined over program type-seat type pairs.

The second problem, which is the main focus of this paper regarding the DTE mechanism, is that every year, many reserved seats remain vacant and the public (especially general-category applicants) react negatively to this fact.

(**ii**) The capacities of reserved seats in the SC and ST categories are taken to be hard bounds. In other words, if there are not enough applications for one of the privilege types SC or ST, some of the seats will remain empty. In Maharashtra, the data show that most years applications from ST students have been low. Hence, some seats reserved for the ST students have remained vacant.[22] However, if there is any vacant seat from the OBC category, the DTE converts that seat into a general-category seat.[23] Also, the number of applications from designated communities is volatile over time. Due to insufficient demand from some of these communities, every year many seats that are reserved for SC and ST students remain vacant:

> *"As admissions to engineering colleges across the state closed, seats in some of the finest institutes that charge almost nothing have gone abegging. Not only are seats open in some of the most prestigious colleges of the state, slots are vacant in some of the top streams too: 69 in electronics, 38 in mechanical engineering, 27 in civil engineering, 23 in computer science and 10 in electrical engineering... 269 seats are yet to be filled."*[24]

In our model, we introduce a choice procedure with dynamic reserves such that capacity can

[22]See Weisskopf (2004). See also Bertrand et al. (2010).

[23]For the details of the admission procedure for engineering schools, see Weisskopf (2004), Kochar (2009), and Bertrand et al. (2010).

[24]http://timesofindia.indiatimes.com/city/mumbai/Prestigious-government-engineering-colleges-still-have-vacant-seats/articleshow/39833944.cms

be transferred from one group of seats to another in a matching-with-contracts framework. Allowing capacity-transfer increases efficiency by utilizing slots that would otherwise remain vacant.

3 The Model

In a matching problem with dynamic reserves, there is a set of students $I = \{i_1, ..., i_n\}$, a set of schools $S = \{s_1, ..., s_m\}$, a set of privileges $\Theta = \{t_1, ...t_k\}$, and a (finite) set of contracts $X = I \times S \times \Theta$. Each student $i \in I$ has a set of privileges $\tau(i) \subseteq \Theta$ he or she can claim, where $\tau : I \Rightarrow \Theta$ is a privilege correspondence. Each contract $x \in X$ is between a student $i(x) \in I$ and a school $s(x) \in S$, and states the privilege $t(x) \in \tau(i(x))$. We extend the notations $i(\cdot)$, $s(\cdot)$, and $t(\cdot)$ to sets of contracts by setting $i(Y) \equiv \cup_{y \in Y}\{i(y)\}$, and $s(Y) \equiv \cup_{y \in Y}\{s(y)\}$ for any $Y \subseteq X$. For $Y \subseteq X$, we denote $Y_i \equiv \{y \in Y : i(y) = i\}$; analogously, we denote $Y_s \equiv \{y \in Y : s(y) = s\}$ and $Y_t \equiv \{y \in Y : t(y) = t\}$.

Each student $i \in I$ has a (linear) preference order P^i (with weak order R^i) over contracts in $X_i = \{x \in X : i(x) = i\}$. For ease of notation, we assume that each $i \in I$ also ranks a "null contract" $\emptyset_i$, which represents remaining unmatched (and hence is always available), so that we may assume that s ranks all the contracts in X_i.[25] We say that the contracts $x \in X_i$ for which $\emptyset_i P^i x$ are *unacceptable to* i. Let $\mathcal{P}$ denote the set of all preferences over $S \times \Theta$. A preference profile of students is denoted by $P = (P^{i_1}, ..., P^{i_n}) \in \mathcal{P}^n$. A preference profile of all students except student i_l is denoted by $P_{-i_l} = (P^{i_1}, ..., P^{i_{l-1}}, P^{i_{l+1}}, ..., P^{i_n}) \in \mathcal{P}^{n-1}$.

An *allocation* $X' \subset X$ is a set of contracts such that each student appears in at most one contract and no school appears in more contracts than its capacity allows. Let $\mathcal{X}$ denote the set of all allocations. Given a student $i \in I$ and an allocation X' with $(i, s, t) \in X'$, we refer to the pair (s, t) as the *assignment* of student i under allocation X'. Student preferences over allocations are induced by their assignments under these allocations.

Definition 1. *(Pareto dominance) An allocation $Y \subseteq X$ Pareto dominates allocation $Z \subseteq X$ if $Y_i R^i Z_i$ for all $i \in I$ and $Y_i P^i Z_i$ for at least one $i \in I$.*

A *direct mechanism* is a mechanism where the strategy space is the set of preferences $\mathcal{P}$ for each agent i. Hence a direct mechanism is simply a function $\psi : \mathcal{P}^n \longrightarrow \mathcal{X}$ that selects an allocation for each preference profile.

[25] We use the convention that $\emptyset_i P^i x$ if $x \in X \setminus X_i$.

Choice Procedure of Schools

Each school $s \in S$ reserves certain parts of its capacity for special student groups to achieve some distributional objectives. These sorts of constraints are encoded in the choice procedure of school s. First of all, each school pre-specifies a linear order in which privilege types are considered. We assume that for each $s \in S$, the privileges are ordered to a (linear) *order of precedence* $\rhd^s$. The interpretation of $\rhd^s$ is that if $t \rhd^s t'$, then, whenever possible, the slots reserved for students with privilege t are filled before the slots reserved for students with privilege t'. Note that a student might have multiple privileges, so that a set of students I may not be partitioned into disjoint sets of students with different privileges. In particular, a given student may be considered multiple times by a choice procedure.

School s initially has a target distribution of its seats over different student groups with different privileges. Let $\overline{q}_s$ denote the total capacity of school s. The number of reserved seats for students with privilege t_j is denoted by $\overline{q}^s_{t_j}$. Then, we have $\overline{q}_s = \sum_{j=1}^{k} \overline{q}^s_{t_j}$. School s has a strict preference for filling these slots according to its target distribution. If the target distribution cannot be achieved because too few agents from one or more of the k privilege groups apply, then school s can express its preferences over possible alternative distributions of privilege types by specifying how its capacity is to be redistributed.

For a given school $s \in S$, $C^s(.) : 2^X \longrightarrow 2^X$ denotes the *overall choice function* of school s. Without loss of generality, assume that the precedence order is $t_1 \rhd^s t_2 \rhd^s ... \rhd^s t_k$[26]. Given a set of contracts $Y \subseteq X$, $C^s(Y)$ is determined as follows:

- Given $\overline{q}^s_{t_1}$ and $Y = Y^0 \subseteq X$, let $Y_1 \equiv C^s_{t_1}(Y^0, \overline{q}^s_{t_1})$ be the set of chosen contracts with privilege t_1. Then, let $r_1 = \overline{q}^s_{t_1} - \mid Y_1 \mid$ be the number of vacant seats that were initially reserved for students with privilege t_1. Define $\widetilde{Y}_1 \equiv \{y \in Y : i(y) \in i(Y_1)\}$. This is the set of all contracts of students whose contract is chosen by the sub-choice function $C^s_{t_1}(\cdot, \cdot)$. If a contract of a student with privilege t_1 is chosen, then all of the contracts naming that student shall be removed from the set of available contracts for the rest of the procedure. The set of remaining contracts is then $Y^1 = Y^0 \setminus \widetilde{Y}_1$.

- Given the set of remaining contracts Y^1 and the capacity $q^s_{t_2} = q^s_{t_2}(r_1) \geq \overline{q}^s_{t_2}$, let $Y_2 = C^s_{t_2}(Y^1, q^s_{t_2})$ be the set of chosen contracts with privilege t_2, where the capacity of the group of seats for students with privilege t_2 is a function of the number of unused seats from the first group. Let $r_2 = q^s_{t_2} - \mid Y_2 \mid$ be the number of unused seats that were reserved for students with privilege t_2. Define $\widetilde{Y}_2 \equiv \{y \in Y^1 : i(y) \in i(Y_2)\}$. If a

[26] It is important to note that our model allows for a privilege type to appear multiple times in the order of precedence. For example, if there is too much demand from the students in the first privilege type, after all privilege types are considered the first privilege type might be re-considered. Also,

contract of a student with privilege t_2 is chosen by the sub-choice function $C^s_{t_2}(\cdot,\cdot)$, then all of the contracts belonging to that student will be removed from the set of available contracts. Then, the remaining set of contracts is $Y^2 = Y^1 \setminus \widetilde{Y}_2$.

- In general, let $Y_j = C^s_{t_j}(Y^{j-1}, q^s_{t_j})$ be the set of chosen contracts with privilege t_j from the set of available contracts Y^{j-1}, where $q^s_{t_j} = q^s_{t_j}(r_1, ..., r_{j-1}) \geq \bar{q}^s_{t_j}$ is the capacity of the group of seats for students with privilege t_j as a function of the vector of the number of unused seats $(r_1, ..., r_{j-1})$ that are initially reserved for students with privileges $t_1, ..., t_{j-1}$, respectively. Let $r_j = q^s_{t_j} - \mid Y_j \mid$ be the number of vacant seats that were reserved for students with privilege t_j. Define $\widetilde{Y}_j = \{y \in Y^{j-1} : i(y) \in i(Y_j)\}$. The set of remaining contracts is then $Y^j = Y^{j-1} \setminus \widetilde{Y}_j$.

- Given the set of contracts $Y = Y^0$ and the capacity $q^s_{t_1}$ of the group of seats reserved for students with privilege t_1, which comes first in the precedence order, we define the *overall choice function*[27] of school s as $C^s(Y) = \bigcup_{j=1}^{k} C^s_{t_j}(Y^{j-1}, q^s_{t_j}(r_1, ..., r_{j-1}))$.

It is important to note that the choice procedure defined above creates a family of choice functions. Different capacity-transfer schemes yield different choice functions. The class of choice functions defined include slot-specific choice functions of Kominers and Sönmez (2016), choice functions defined for cadet-branch matching problems in Sönmez and Switzer (2013) and Sönmez (2013), choice protocols defined by Westkamp (2013), and choice functions defined in Kamada and Kojima (2015).

Stability

An *outcome* is a set of contracts $Y \subseteq X$. We follow the Gale and Shapley (1962) tradition in focusing on match outcomes that are *stable* in the sense that

- neither students nor schools wish to unilaterally walk away from their assignments, and

- no student desires a slot at which she has a justified claim, with some desirable contract, under the precedence structure and choice procedures.

Definition 2. We say that an outcome Y is **stable** if it is

(i) **individually rational**- $C^i(Y) = Y_i$ for all $i \in I$ and $C^s(Y) = Y_s$ for all $s \in S$, and

(ii) **unblocked**- there does not exist a school $s \in S$ and **blocking set** $Z \neq C^s(Y)$ such that $Z = C^s(Y \cup Z)$ and $Z_i = C^i(Y \cup Z)$ for all $i \in i(Z)$.

[27] If there is no transfer of seats across different privilege groups this choice procedure collapses the choice procedure Alva (2016) defines.

Note that if the first condition fails, then there is either a student or a school who prefers rejecting a contract that involves him/it. If the second condition fails, then there exists an unselected contract x where not only student $i(x)$ prefers $(s(x), t(x))$ over his assignment but also contract x can be selected by school $s(x)$ given its composition.

The standard stability definition stated above makes use of the choice functions we develop. It is important to note that since different capacity-transfer scheme yield a different choice function for a school, the stability definition is tailored to the capacity-transfer schemes used. In a sense, the above definition gives us a family of stability notions. The normative meaning of a blocking coalition depends on the capacity-transfer schemes that are used.

Monotone Capacity-Transfers

The idea behind the class of problems we study is that each school is required to reserve certain parts of its capacity for different privilege types and may prefer making, or be required to make, some of these reserved seats available to other privilege types if its capacity cannot be filled by the first privilege types. Each institution has a pre-specified order in which different privileges are considered while filling its slots and also has a target capacity distribution over these privilege groups. If its target distribution cannot be achieved because too few students from one or more privilege types apply, the institution would like to have an alternative distribution over privilege types. To guarantee the existence of stable matchings along with many other possibility results under capacity-transfers, in our framework we require the capacity-transfer scheme to be monotonic.

Definition 3. A capacity-transfer scheme is **monotonic**, if for all $j \in \{2, ..., k\}$ and all pairs of sequences $(r_l, \widetilde{r}_l)_{l=1}^{j-1}$ such that $\widetilde{r}_l \geqslant r_l$ for all $l \leqslant j-1$, $q^s_{t_j}(\widetilde{r}_1, ..., \widetilde{r}_{j-1}) \geqslant q^s_{t_j}(r_1, ..., r_{j-1})$, and $\sum\limits_{m=1}^{j} [q^s_{t_m}(\tilde{r}_1, ..., \tilde{r}_{m-1}) - q^s_{t_m}(r_1, ..., r_{m-1})] \leq \sum\limits_{m=1}^{j} [\tilde{r}_m - r_m]$.

Monotonicity of capacity-transfer schemes requires that, whenever weakly more seats are left unassigned in *every* privilege type from t_1 to t_{j-1}, weakly more seats should be available for privilege type t_j. Notice that *no capacity-transfer* trivially satisfies this definition, so it is considered a monotonic capacity-transfer. The definition also requires that the difference of the capacities of the same group of seats under two different vectors of number of vacant seats cannot exceed the summation of the differences between the number of vacant seats up to that group.

If the reserve structure is defined as imposing hard bounds, then there is no capacity-transfer. In this paper, we propose that the control constraints be interpreted as *soft bounds* or flexible capacities rather than hard bounds. For example, transferring all of the unassigned seats from privilege types (other than general category) that have empty slots to only general category satisfies the monotonic capacity-transfer definition and can be considered a flexible-capacity scheme. Even though transferring all of the unassigned seats from other privilege types to the general category might seem more likely to occur in real-life merit-based assignment procedures to promote competition among agents (students), in our framework, the capacity-transfer schemes that institutions can implement are very flexible because different institutions might have different distributional concerns. As long as the capacity-transfer scheme is monotonic, each institution can express its preferences over different capacity-transfers where it prefers to fill its slots according to its initial target distribution.

4 Conditions on Choice Functions

Let X be the set of contracts. $\mathcal{P}(X) = 2^X$ is the power set of X. A choice function is $C : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ such that for every $Y \subseteq X$, $C(Y) \subseteq Y$. We now discuss the extent to which schools' choice functions and sub-choice functions satisfy the conditions that have been key to previous analyses of matching-with-contracts models.

Definition 4. A choice function C^s satisfies **substitutability** if for all $z, z' \in X$ and $Y \subseteq X$, $z \notin C^s(Y \cup \{z\}) \Longrightarrow z \notin C^s(Y \cup \{z, z'\})$.

Hatfield and Milgrom (2005) introduce this substitutability condition, which generalizes the earlier *gross substitutes* condition of Kelso and Crawford (1982). Hatfield and Milgrom (2005) also show that substitutability is sufficient to guarantee the existence of stable outcomes. However, their analysis implicitly assumes the **irrelevance of rejected contracts** (**IRC**)[28] condition defined below:

Definition 5. Given a set of contracts X, a choice function $C^s : 2^X \longrightarrow 2^X$ satisfies IRC if $\forall Y \subset X$, $\forall z \in X \setminus Y$, $z \notin C^s(Y \cup \{z\}) \Longrightarrow C^s(Y) = C^s(Y \cup \{z\})$.

Aygün and Sönmez (2013) show that the substitutability condition together with the IRC condition assures the existence of a stable allocation.

Substitutability of choice functions is necessary in the maximal domain sense for guaranteed existence of stable outcomes in a variety of settings. However, substitutability is *not* necessary for the guaranteed existence of stable outcomes in settings where agents have unit demand.[29] Hatfield and Kojima (2010) show that the following condition, which is

[28] Alkan (2002) refers to it as "consistency."

[29] See Hatfield and Kojima (2008) and (2010).

weaker than substitutability, not only suffices for the existence of stable outcomes but also guarantees that there is no conflict of interest among agents.[30]

Definition 6. A choice function C^s satisfies **unilateral substitutability** (US) if $z \notin C^s(Y \cup \{z\}) \implies z \notin C^s(Y \cup \{z, z'\})$ for all $z, z' \in X$ and $Y \subseteq X$ for which $i(z) \notin i(Y)$ (i.e., no contracts in Y are associated with student $i(z)$).

Unilateral substitutability has been proven to be crucial in market design applications. The choice functions of branches in the cadet-branch problem do not satisfy substitutability. However, they do satisfy unilateral substitutability. Unilateral substitutability, together with the law of aggregate demand, guarantees the existence of an agent-optimal stable allocation, and under them the agent-proposing deferred-acceptance mechanism is strategy proof.

Definition 7. A choice function C^s satisfies **bilateral substitutability** (BS) if $z \notin C^s(Y \cup \{z\}) \implies z \notin C^s(Y \cup \{z, z'\})$ for all $z, z' \in X$ and $Y \subseteq X$ for which $i(z), i(z') \notin i(Y)$.

Bilateral substitutability of a choice function is implied by unilateral substitutability. The BS together with the IRC of overall choice functions guarantees the existence of a stable allocation in a matching-with-contracts framework under no capacity-transfer. However, BS and IRC together are weak conditions (even under no capacity-transfer) in the sense that many well-known properties of stable allocations in the standard matching problem do not carry over to the matching-with-contracts setting. For instance, the agent-optimal stable allocation fails to exist. Strengthening BS to US restores most of these well-known properties.[31]

The choice functions C^s do satisfy substitutability whenever each agent offers at most one contract to school s.

Definition 8. A choice function $C^s(\cdot)$ satisfies **weak substitutability** (WS) if $z \notin C^s(Y \cup \{z\}) \implies z \notin C^s(Y \cup \{z, z'\})$ for all $z, z' \in X$ and $Y \subseteq X$ for which $\mid Y \cup \{z, z'\} \mid = \mid i(Y \cup \{z, z'\}) \mid$.

The WS condition, first introduced by Hatfield and Kojima (2008), is in general necessary (in the maximal domain sense) for the guaranteed existence of stable outcomes (Proposition 1 of Hatfield and Kojima, 2008). Notice that if every student has only one privilege type, WS corresponds to substitutability.

Definition 9. A choice function $C^s(\cdot)$ satisfies **the law of aggregate demand** (LAD) if $Y \subseteq Y' \implies \mid C^s(Y) \mid \leq \mid C^s(Y') \mid$.

[30] As in the work of Hatfield and Milgrom (2005), IRC is implicitly assumed throughout the work of Hatfield and Kojima (2010). See Aygün and Sönmez (2012) for details.

[31] See Afacan and Turhan (2015) for the axiomatization of the gap between US and BS.

That is, the size of the chosen set never shrinks as the set of contracts grows under the law of aggregate demand.[32] Hatfield and Milgrom (2005) introduce the LAD condition in a matching-with-contracts framework, and it has proven to be critical. Hatfield and Kojima (2010) show that if choice functions of schools all satisfy US and LAD, every student and school signs the same number of contracts at every stable allocation (i.e., the rural-hospital theorem holds). Moreover, the cumulative offer mechanism becomes strategy proof and weakly Pareto efficient for agents. If schools do not have preferences that generate their choices, then all of these results are obtained under the additional IRC condition of Aygün and Sönmez (2012).

Definition 10. A choice function $C^s(\cdot)$ satisfies **quota monotonicity** (QM) if for any $q, q^{'} \in \mathbb{Z}_+$ such that $q < q^{'}$, for all $Y \subseteq X$

$$C^s(Y, q) \subseteq C^s(Y, q^{'}),\ \textit{and}$$

$$\mid C^s(Y, q^{'}) \mid - \mid C^s(Y, q) \mid \leq q^{'} - q.$$

We introduce the above quota-monotonicity condition, which requires choice functions to satisfy two conditions: First, given any set of contracts, if there is an increase in the capacity, then we require the choice function to select every contract it was choosing before increasing its capacity. It might choose some additional contracts. Second, if, say, the capacity of a privilege type is increased by 2, then the difference between the number of contracts chosen after and before the capacity increase cannot exceed 2. Since we allow capacities of privilege types to change dynamically during the choice procedure by exogenously given monotonic capacity-transfer schemes, quota monotonicity will be a crucial regulative condition on privileges' sub-choice functions to obtain positive results. However, it will be trivially satisfied if the sub-choice functions are derived from strict priority rankings induced by test scores in merit-based allocation problems.

4.1 Conditions on Sub-choice Functions for Merit-Based Applications

In the Indian engineering school admission problem (and also in the cadet-branch matching problem in USMA and ROTC), each sub-choice function (one for each privilege type) is induced from a strict ranking of students according to test scores. Since each student from a particular privilege type is acceptable for the privilege types she announces at every school, the sub-choice functions of every privilege type are acceptant.

[32] In a different setting, Alkan (2002) refers to the LAD as "cardinal monotonicity."

Definition 11. A sub-choice function $C^s_{t_j}(\cdot, q)$ is $q-acceptant$ if $\mid C(Y) \mid = min\{q, \mid Y \mid\}$ for every $Y \subseteq X$. A sub-choice function is *acceptant* if it is $q-acceptant$ for some q.

This definition basically says that if the number of applicants is less than the capacity of the privilege type, every contract (each is associated with a different student/cadet) must be chosen, and if the number of applicants is more than the capacity of the privilege type, then the capacity must be filled.

The following is the standard responsiveness definition presented in the literature.

Definition 12. *(Responsive priorities (Roth, 1985))* The preferences of school s are responsive with capacity q if *(i)* for any $i, j \in I$, if $\{i\} \succ_s \{j\}$, then for any $I' \subseteq I \setminus \{i, j\}$, $I' \cup \{i\} \succ_s I' \cup \{j\}$, *(ii)* for any $i \in I$, if $\{i\} \succ_s \emptyset$, then for any $I' \subseteq I$ such that $\mid I' \mid < q$, $I' \cup \{i) \succ_s I'$, *(iii)* $\emptyset \succ_s I'$ for any $I' \subseteq I$ with $\mid I' \mid > q$.

In our framework, we can state both acceptance and responsiveness in a single condition, following Chambers and Yenmez (2015). Note that each agent (cadet) has only one contract with a given privilege type in our framework.[33] Let $\succ$ be the strict ranking of agents according to test scores. For privilege type t_j, the priority ranking associated with it, $\succ_{t_j}$, is obtained from $\succ$ as follows: for every $i, j \in I$ such that $t_j \in \tau(i) = \tau(j)$, $i \succ_{t_j} j$ if and only if $i \succ j$, and for every $k \in I$ such that $\tau(k) \neq t$, $\emptyset \succ_{t_j} k$.

Definition 13. A sub-choice function $C^s_{t_j}(\cdot, q)$ of institution s for privilege type t_j is q-*responsive* if there exists a strict priority ordering $\succ_{t_j}$ on the set of contracts naming privilege type t_j, X_{t_j}, and a positive integer q such that for any $Y \subseteq (X_s \cap X_{t_j})$,

$$C^s_{t_j}(Y, q) = \bigcup_{i=1}^{q} \{y^*_i\},$$

where y^*_i is defined as $y^*_1 = \underset{Y}{max} \succ_{t_j}$ and, for $2 \leq i \leq q$, $y^*_i = \underset{Y \setminus \{y^*_1, \dots, y^*_{i-1}\}}{max} \succ_{t_j}$.

Responsiveness and acceptance are both crucial for matching applications where admissions are merit-based. A sub-choice function $C^s_{t_j}(\cdot, q)$ is $q-responsive$ if there is a strict priority ordering over the agents for which the sub-choice function always selects the highest-ranked available agents.[34] If a school's sub-choice functions are q-responsive, then for each privilege type the school acts as if it has preferences over contracts with a capacity constraint, and the school takes the highest-ranking students available to that privilege type up to its capacity.

[33]This is not necessarily the case in Kominers and Sönmez (2015). In their slot-specific priorities setting, an agent may have multiple contracts with a privilege type for a given institution.

[34]See Chambers and Yenmez (2015).

4.2 Respect for Unambiguous Improvements

One of the most important parameters of the Indian engineering school admission problem is the strict ranking of agents according to test scores. Let $\succ$ be the strict ranking of students. For each school $s \in S$ the strict ranking of contracts in privilege type t_j is obtained from $\succ$ as follows: $x \succ^s_{t_j} y$ if and only if $i(x) \succ i(y)$ and $t(x) = t(y) = t_j$. If $t_j \notin \tau(i)$, then $\emptyset_s \succ^s_{t_j} x$ for all x such that $i(x) = i$. The choice function for each privilege type is obtained from these strict rankings, i.e., $C^s_{t_j}(Y, q^s_{t_j}) = C^s_{t_j}(Y, q^s_{t_j} \mid \succ^s_{t_j})$, which is q-responsive.

Clearly, a reasonable mechanism would never penalize a student as a result of an improvement in his standing in the strict ordering according to test scores. Given two strict rankings of students according to test scores $\succ$ and $\succ'$, we say that $\succ'$ is an *unambiguous improvement* for student i over $\succ$ if

1. the relative ranking between all students except student i remains exactly the same between $\succ$ and $\succ'$, although

2. the standing of student i is strictly better under $\succ'$ than under $\succ$.

Definition 14. A mechanism *respects improvements* if a student never receives a strictly worse assignment as a result of an unambiguous improvement of his priority ranking.[35]

Violation of this condition may create adverse incentives for some agents to lower their test scores to obtain a better outcome according to their true preferences, as in the current application procedure for engineering school admissions in India.[36]

5 The Cumulative Offer Process

The cumulative offer algorithm, which is the generalization of the agent-proposing deferred acceptance algorithm of Gale and Shapley, is the central allocation mechanism used in a matching-with-contracts framework. We now introduce the cumulative offer process for matching with contracts (see Hatfield and Kojima (2010); Hatfield and Milgrom (2005); Kelso and Crawford (1982)).

Here, we provide an intuitive description of this algorithm; we give a more technical statement in Appendix A.

Definition 15. In the **cumulative offer process**, students propose contracts to schools in a sequence of steps $l = 1, 2, ...$:

[35]This property was first formulated by Balinski and Sönmez (1999).

[36]See Sönmez (2013), where the author discusses how cadets intentionally lower their OML to obtain better outcomes.

Step 1 : Some student $i^1 \in I$ proposes his most-preferred contract, $x^1 \in X_{i^1}$. School $s(x^1)$ holds x^1 if $x^1 \in C^{s(x^1)}(\{x^1\})$, and rejects x^1 otherwise. Set $A^2_{s(x^1)} = \{x^1\}$, and set $A^2_{s'} = \emptyset$ for each $s' \neq s(x^1)$; these are the sets of contracts available to schools at the beginning of Step 2.

Step 2 : Some student $i^2 \in I$ for whom no contract is currently held by any school proposes his most-preferred contract that has not yet been rejected, $x^2 \in X_{i^2}$. School $s(x^2)$ holds the contract in $C^{s(x^2)}(A^2_{s(x^2)} \cup \{x^2\})$ and rejects all other contracts in $A^2_{s(x^2)} \cup \{x^2\}$; schools $s' \neq s(x^2)$ continue to hold all contracts they held at the end of Step 1. Set $A^3_{s(x^2)} = A^2_{s(x^2)} \cup \{x^2\}$, and set $A^3_{s'} = A^2_{s'}$ for each $s' \neq s(x^2)$.

Step l : Some student $i^l \in I$ for whom no contract is currently held by any school proposes his most-preferred contract that has not yet been rejected, $x^l \in X_{i^l}$. School $s(x^l)$ holds the contract in $C^{s(x^l)}(A^l_{s(x^l)} \cup \{x^l\})$ and rejects all other contracts in $A^l_{s(x^l)} \cup \{x^l\}$; schools $s' \neq s(x^l)$ continue to hold all contracts they held at the end of Step $l-1$. Set $A^{l+1}_{s(x^l)} = A^l_{s(x^l)} \cup \{x^l\}$, and set $A^{l+1}_{s'} = A^l_{s'}$ for each $s' \neq s(x^l)$.

If at any time no student is able to propose a new contract, that is, if all students for whom no contracts are on hold have proposed all contracts they find acceptable, then the algorithm terminates. The *outcome of the cumulative offer process* is the set of contracts held by schools at the end of the last step before termination.

In the cumulative offer process, students propose contracts sequentially. Schools accumulate offers, choosing at each step (according to C^s) a set of contracts to hold from the set of all previous offers. The process terminates when no student wishes to propose a contract.

Remark 1. Note that we do not explicitly specify the order in which students make proposals. Hirata and Kasuya (2014) show that in the matching-with-contracts model, the outcome of the cumulative offer process is *order-independent* if the overall choice function of every school satisfies the bilateral substitutability and the irrelevance of rejected contracts conditions. In our setup, the overall choice function of every school satisfies BS and IRC, and hence, the order-independence result holds for our mechanism.

6 Main Results

We now develop our general theoretical results. Overall choice functions of schools were defined in Section 4 as the union of choices by sub-choice functions. Sub-choices are linked by both their choices and the monotonic capacity-transfer scheme. Each sub-choice function has two inputs: the set of remaining (rejected) contracts by the sub-choice functions that precede it and the capacity of the privilege type as a function of the number of unassigned seats from all of the privilege types considered before it. For the overall choice function, to guarantee the existence of a stable allocation under monotonic capacity-transfer schemes, we impose certain conditions on sub-choice functions. As shown by Aygün and Sönmez (2012), the IRC condition is needed for the overall choice functions of institutions to guarantee the existence of a stable allocation. To achieve this, we require that every sub-choice function satisfies IRC. Since sub-choice functions are linked by their two inputs in our framework, we need to impose further axioms on top of BS and IRC, namely, the law of aggregate demand and quota monotonicity under monotonic capacity-transfer schemes, in order to achieve an overall choice function that satisfies the BS and IRC.

6.1 The Existence of a Stable Allocation under Monotonic Capacity-Transfers

To ensure that overall choice functions satisfy IRC, it suffices to impose IRC on sub-choice functions for any capacity-transfer scheme (not necessarily monotonic).

Proposition 1. *Suppose that all sub-choice functions satisfy IRC. Then, the overall choice function satisfies IRC.*

Proof. See Appendix C. □

Remark 2. For the rest of the paper we always assume that sub-choice functions satisfy IRC so that the overall choice functions of schools satisfy it as well.

When each student has only one contract associated with a school, then substitutability becomes identical to weak substitutability (WS). To obtain an overall choice function that satisfies WS, it suffices for sub-choice functions to satisfy WS, LAD, and QM.

Proposition 2. *Suppose that all sub-choice functions satisfy WS, LAD, and QM. If the capacity-transfer scheme is monotonic, then the overall choice function also satisfies WS and IRC.*

Proof. See Appendix C. □

The following proposition is key to guarantee the existence of a stable allocation. The BS condition on overall choice functions, together with IRC, is sufficient to guarantee the existence of stable outcomes.

Proposition 3. *Suppose that sub-choice functions satisfy BS, LAD, and QM. If the capacity-transfer scheme is monotonic, then the overall choice function satisfies BS and IRC.*

Proof. See Appendix C. □

If overall choice functions of institutions satisfy BS and IRC, then by Hatfield and Kojima (2010) and Aygün and Sönmez (2012), a stable allocation exists.

Theorem 1. *Suppose that all sub-choice functions satisfy BS, LAD, and QM. If the capacity-transfer scheme is monotonic, then there exists a stable allocation.*

Proof. By Proposition 1 and Proposition 3, we know that the overall choice function of each school satisfies BS and IRC. Then, by Theorem 1 of Hatfield and Kojima (2010), together with Theorem 1 of Aygün and Sönmez (2012), the set of stable outcomes is non-empty. □

In the Indian engineering school admission problem, sub-choice functions are derived from strict priority rankings according to exam scores. These type of sub-choice functions trivially satisfy BS, IRC, LAD, and QM. By Theorem 1, we have existence of a stable allocation under these type of sub-choice functions. Also, the outcome of the cumulative offer process is a stable allocation. We state them as a corollaries below:

Corollary 1. *Suppose that all sub-choice functions are q-responsive. Then, under a monotonic capacity-transfer scheme, there exists a stable allocation.*

Corollary 2. *Suppose that all sub-choice functions are q-responsive. Then, the cumulative offer algorithm outcome under any monotone capacity-transfer scheme is stable.*

6.2 Incentives

If the overall choice functions of schools satisfy US and LAD, then the cumulative offer mechanism is (group) strategy proof.[37] Even though US and LAD are sufficient for strategy-proofness, they are not necessary in some frameworks. For example, Kominers and Sönmez (2015) provide a choice function that violates both US and LAD, but they show that the cumulative offer mechanism is strategy-proof in their slot-specific priorities setup. Later, Hatfield and Kominers (2015) developed a theory in which a condition, *substitutable completion*, plays a key role. They show that if schools' choice functions have substitutable completions so that this completion satisfies the LAD, then the cumulative offer process becomes strategy-proof under these choice functions. In our problem, if we set the capacity of each privilege type equal to 1 and do not allow capacity-transfer, our problem collapses to a *specific version* of the slot-specific priorities model of Kominers and Sönmez (2015). In our Indian school choice application, each sub-choice function is induced from a strict priority

[37] See Hatfield and Kojima (2010) and Aygün and Sönmez (2012).

ranking of contracts (also students since each student can have at most one contract for a certain privilege type) that is obtained from a strict ranking of students according to test scores.

Overall choice functions fail to satisfy the US in our setup:

Proposition 4. *Suppose that sub-choice functions are q-responsive and the capacity-transfer scheme is monotonic. The overall choice functions of schools may fail to satisfy unilateral substitutability.*

Proof. Consider $X = \{x_1, x_2, y_2\}$ with $S = \{s\}$, $I = \{i, j\}$, and $\theta = \{t_1, t_2\}$ where $i(x_1) = i(x_2) = i$, $i(y_2) = j$ and $t(x_1) = t_1$, $t(x_2) = t(y_2) = t_2$. Student i has a higher test score than student j. Also, $s(x_1) = s(x_2) = s(y_2) = s$. The school has two slots, where the first one is reserved for type t_1 students and the second one is reserved for type t_2 students and $s_1 \rhd^s s_2$ with the following priorities:

$\Pi^{s_1} : x_1 \succ \emptyset_{s_1}$ and $\Pi^{s_2} : x_2 \succ y_2 \succ \emptyset_{s_2}$.

Suppose that the school set the following capacity-transfer scheme: $\bar{q}_{s_1} = 1$ is given. $q_{s_2}(r_1) = 1$ for both $r_1 = 0$ and $r_1 = 1$, i.e., even if the first slot remains empty, there will be no transfer of this empty seat. Note that the monotonicity of a capacity-transfer scheme is satisfied when there is no capacity-transfer.

Then, C^s fails to satisfy unilateral substitutability. To see why, consider $C^s(\{x_2, y_2\}) = \{x_2\}$ and $C^s(\{x_1, x_2, y_2\}) = \{x_1, y_2\}$. Note that $y_2 \notin C^s(\{x_2, y_2\})$ but $y_2 \in C^s(\{x_1, x_2, y_2\})$. □

Furthermore, overall choice functions in our setting might not satisfy the LAD.

Proposition 5. *Suppose that sub-choice functions are q-responsive and the capacity-transfer scheme is monotonic. The overall choice functions of schools may fail to satisfy the law of aggregate demand.*

Proof. Consider $X = \{x_1, x_2, y_1\}$ with $S = \{s\}$, $I = \{i, j\}$, and $\theta = \{t_1, t_2\}$ where $i(x_1) = i(x_2) = i$, $i(y_1) = j$, and $t(x_1) = t(y_1) = t_1$, $t(x_2) = t_2$. Also, $s(x_1) = s(x_2) = s(y_1) = s$. The school has two slots, where the first one is reserved for type t_1 students and the second one is reserved for type t_2 students and $s_1 \rhd^s s_2$ with the following priorities:

$\Pi^{s_1} : x_1 \succ y_1 \succ \emptyset_{s_1}$ and $\Pi^{s_2} : x_2 \succ \emptyset_{s_2}$

Suppose that the school sets the following capacity-transfer scheme: $\bar{q}_{s_1} = 1$ is given. $q_{s_2}(r_1) = 1$ for both $r_1 = 0$ and $r_1 = 1$, i.e., even if the first slot remains empty there will be no transfer of this empty seat. Then, C^s fails to satisfy the law of aggregate demand. Consider $C^s(\{x_2, y_1\}) = \{x_2, y_1\}$ and $C^s(\{x_1, x_2, y_1\}) = \{x_1\}$. □

Even though overall choice functions fail to satisfy US and LAD, if a contract is rejected at any step of the cumulative offer algorithm, then that contract cannot be held at any further step. In other words, there is no renegotiation of a rejected contract.

Proposition 6. *Suppose that sub-choice functions are q-responsive. If a contract z is rejected by school s at any step of the cumulative offer algorithm, then it cannot be held by school s in any subsequent step.*

Proof. See Appendix C. □

When no renegotiation occurs in the cumulative offer process, the algorithm coincides with the standard student-proposing deferred-acceptance algorithm.[38]

The standard definition of strategy-proofness is as follows:

Definition 16. A direct mechanism φ is *strategy-proof* if $\nexists i \in I$, $P_{-i} \in \mathcal{P}_{-i}$, P^i, $\tilde{P}^i \in \mathcal{P}$ such that $\varphi(\tilde{P}^i, P_{-i}) P^i \varphi(P)$.

That is, no matter which student we consider, no matter what her true preferences P^i are, no matter what other preferences P_{-i} other students report (true or not), and no matter which potential "misrepresentation" $\tilde{P}^i$ student i considers, truthful preference revelation is in her best interests. Hence, students can never benefit from "gaming" the mechanism φ.

Theorem 2. *Suppose that all sub-choice functions are q-responsive and that the capacity-transfer scheme is monotonic. Then, the cumulative offer mechanism Φ as a direct mechanism is strategy-proof.*

Proof. See Appendix B. □

The proof of Theorem 2 makes use of the theory developed by Hatfield and Kominers (2015). We show that the choice function we design has a substitutable completion that satisfies the LAD. In that regard, engineering school admissions in IITs provides an important real-life application for the theory developed by Hatfield and Kominers (2015).

6.3 Student-Optimal Stable Outcomes

In our framework, a student-optimal stable outcome might not exist.

Proposition 7. *A student-optimal stable outcome might not exist.*

[38] See Hatfield and Kojima (2010).

Proof. Consider $X = \{x_1, x_2, y_2\}$ with $S = \{s\}$, $I = \{i, j\}$, and $\theta = \{t_1, t_2\}$, where $i(x_1) = i(x_2) = i$, $i(y_2) = j$, and $t(x_1) = t_1$, $t(x_2) = t(y_2) = t_2$. Also, $s(x_1) = s(x_2) = s(y_2) = s$. The school has two slots, each with a different privilege type t_1 and t_2. The precedence order is $t_1 \rhd^s t_2$. The priorities of each privilege type is as follows: $\Pi^{t_1}: x_1 \succ \emptyset_{t_1}$ and $\Pi^{t_2}: x_2 \succ y \succ \emptyset_{t_2}$. Student preferences are $x_2 P_i x_1 P_i \emptyset$ and $y_2 P_j \emptyset$. Without capacity-transfer, the cumulative offer algorithm outcome is $\{x_2\}$. However, the outcome $\{x_1, y_2\}$ is also stable. Since there is no Pareto-domination relationship between the two outcomes $\{x_2\}$ and $\{x_1, y_2\}$, and they are the only stable outcomes, there is no student-optimal stable outcome in this example. □

Even when student-optimal stable outcomes *do* exist, the cumulative offer process might not select them.

Proposition 8. *The cumulative offer algorithm outcome might be Pareto dominated by the student-optimal stable outcome.*

Proof. See Example 4 in Kominers and Sönmez (2015). □

6.4 Respect for Unambiguous Improvements

The failure to respect improvements hurts the mechanism not only from a normative perspective but also via the adverse incentives it creates if students' efforts play any role in determining the strict ranking of students according to test scores. As in most merit-based resource-allocation problems, this is the case for engineering school admissions in India.

Theorem 3. *The cumulative offer mechanism Φ respects unambiguous improvements under any monotonic capacity-transfer scheme.*

Proof. See Appendix C. □

6.5 Increasing Efficiency through Monotonic Capacity-Transfer Schemes

The following example illustrates the idea that the outcome of the cumulative offer algorithm under monotonic capacity-transfers Pareto dominates the outcome of the cumulative offer algorithm under no capacity-transfers.

Example 2. Consider $X = \{x_1, x_2, y_1, y_3, z_1, z_2, w_2, w_3\}$ with $S = \{s\}$, and $I = \{i, j, k, l\}$ where $i(x_1) = i(x_2) = i$, $i(y_1) = i(y_3) = j$, $i(z_1) = i(z_2) = k$, and $i(w_2) = i(w_3) = l$. All the contracts are with school s. $\Theta = \{t_1, t_2, t_3\}$ where $t(x_1) = t(y_1) = t(z_1) = t_1$, $t(x_2) = t(z_2) = t(w_2) = t_2$, and $t(y_3) = t(w_3) = t_3$. School s has three seats, one for each type of student, i.e., $\bar{q}_{t_1} = \bar{q}_{t_2} = \bar{q}_{t_3} = 1$ is the target distribution of the school. Students are ranked according to test scores from highest to lowest as: $i - j - k - l$. Hence, the following priorities for each type are derived:

$$\Pi^{t_1} : \; x_1 \succ y_1 \succ z_1 \succ \emptyset_{t_1}$$
$$\Pi^{t_2} : \; x_2 \succ z_2 \succ w_2 \succ \emptyset_{t_2}$$
$$\Pi^{t_3} : \; y_3 \succ w_3 \succ \emptyset_{t_3}$$

The student preferences over contracts naming them are as follows:

$$P_i : \; x_2 P_i x_1 P_i \emptyset_i$$
$$P_j : \; y_3 P_j y_1 P_j \emptyset_j$$
$$P_k : \; z_2 P_k z_1 P_k \emptyset_k$$
$$P_l : \; w_2 P_l w_3 P_l \emptyset_l$$

If there is no capacity-transfer, then the cumulative offer algorithm outcome is $\{x_2, y_3, z_1\}$. Now, suppose that the school has the following monotonic capacity-transfer scheme: $\bar{q}_{t_1} = 1$. If $r_1 = 0$, then $q_{t_2} = 1$. If $r_1 = 1$, then $q_{t_2} = 2$. If $r_1 = 0$ and $r_2 = 0$, then $q_{t_3} = 1$. If $r_1 = 1$ and $r_2 = 0$, then $q_{t_3} = 1$. If $r_1 = 0$ and $r_2 = 1$, then $q_{t_3} = 2$. If $r_1 = 1$ and $r_2 = 1$, then $q_{t_3} = 2$. Under this capacity-transfer scheme the outcome of the cumulative offer process is $\{x_2, y_3, z_2\}$. The important observation here is that the outcome of the cumulative offer algorithm under a monotonic capacity-transfer scheme Pareto dominates the outcome of the cumulative offer algorithm under no capacity-transfers. Even though agents i and j obtain the same assignment, agent k obtains a strictly better assignment under the monotonic capacity-transfer described above.

Now, we generalize the observation obtained from the example above:

Theorem 4. *If the sub-choice functions are derived from an underlying strict ranking of students $\succ$ according to test scores, then the outcome of the cumulative offer algorithm under any monotonic capacity-transfer, $\Phi_{\succ}(P, q)$, Pareto dominates the outcome of the cumulative offer algorithm under no capacity-transfer, $\Phi_{\succ}(P, \bar{q})$.*

Proof. See Appendix C. □

Hence, introducing monotonic capacity-transfer increases efficiency by utilizing seats that would remain unassigned without capacity-transfer.

7 Capacity-Transfers versus Slot-specific Priorities[39]

At a first glance, distributional objectives that can be achieved by capacity-transfers might seem to be handled by the slot-specific priorities model of *Kominers and Sönmez (2015).* Below we provide a simple example of a distributional objective that can easily be achieved by capacity-transfers but cannot be achieved using slot-specific priorities.

Proposition 9. *Some distributional objectives that can be achieved by capacity-transfers cannot be achieved by slot-specific priorities.*

Proof. Consider the following problem with $I = \{i, j, k, l\}$ and $S = \{s\}$ with $q_s = 2$. There are three different types of students, i.e., $\Theta = \{t_1, t_2, t_3\}$. Student i only has type t_1 and hence a single contract x_1. Student j only has type t_2 and a single contract y_2. Student k has two types: type t_2 and type t_3; and two contracts related to these types z_2 and z_3, respectively. Finally, student l has two types: type t_1 and type t_3; and two contracts related to these types w_1 and w_3. Hence, the set of contracts for this problem is $X = \{x_1, y_2, z_2, z_3, w_1, w_3\}$. Students have the following test score ordering from highest to lowest: $i - j - k - l$.

The school reserves the first seat for type t_1 students, and the second seat for type t_2 students. If either the first seat or the second seat cannot be filled with students whom the seats are reserved for, they are filled with type t_3 student(s). The precedence order is such that first seat is filled first with a type t_1 student if possible, and then the second seat is filled with type t_2 student if possible. If any of these seats cannot be filled with the intended student types, all of the vacant seats are filled with type t_3 student(s) at the very end, if possible.

We can represent the distributional objective described above by capacity-transfers as follows: Initially $\overline{q}_{t_1} = \overline{q}_{t_2} = 1$ and $\overline{q}_{t_3} = 0$. If either of the first two seats cannot be filled, $q_{t_3} = r_1 + r_2$ where $r_1, r_2 \in \{0, 1\}$. Some of the choice situations under the capacity-transfer described above are given below:

[39] We would like to thank Andrei Gomberg, Fuhito Kojima, and Tayfun Sönmez, who recommended we analyze the relationship between our model and the model of slot-specific priorities.

Y	$C(Y)$
$\{x_1, y_2, z_2, z_3, w_1, w_3\}$	$\{x_1, y_2\}$
$\{y_2, z_2, z_3\}$	$\{y_2, z_3\}$
$\{x_1, z_2, z_3\}$	$\{x_1, z_2\}$
$\{y_2, w_1, w_3\}$	$\{y_2, w_1\}$
$\{x_1, w_1, w_3\}$	$\{x_1, w_3\}$
$\{z_2, z_3\}$	$\{z_2\}$
$\{w_1, w_3\}$	$\{w_1\}$

In order to implement the choices above with slot-specific priorities, we need to find a strict ranking of the contracts in X for both of the slots. Since $\{x_1, y_2\}$ is chosen from the grand set, from one of the slots x_1 and from the other slot y_2 must be chosen.

Case 1: x_1 is chosen from slot 1 and y_2 is chosen from slot 2. Then, x_1 is the highest priority contract in slot 1. Since $C(\{x_1, z_2, z_3\}) = \{x_1, z_2\}$, then z_2 must have higher priority than z_3 in the strict priority ranking of slot 2 because x_1 will be chosen from the first slot. Notice that both z_2 and z_3 must have lower priority than y_2 in the strict ranking of slot 2. Also, since $C(\{y_2, z_2, z_3\}) = \{y_2, z_3\}$, then it must be the case that z_3 has higher priority than z_2 in the strict priority of the first slot. Notice that z_3 can not be chosen from the second slot as z_2 has higher priority. However, $C(\{z_2, z_3\}) = \{z_2\}$. Contradiction.

Case 2: y_2 is chosen from slot 1 and x_1 is chosen from slot 2. Then, y_2 has the highest priority in slot 1. Since $C(\{y_2, w_1, w_3\}) = \{y_2, w_1\}$, then in the ranking of slot 2 w_1 must have higher priority than w_3. Also, since $C(\{x_1, w_1, w_3\}) = \{x_1, w_3\}$, it follows that in the ranking of slot 1 w_3 must have higher priority than w_1 because w_3 cannot be chosen from slot 2 as it has a lower priority than w_1 there. However, $C(\{w_1, w_3\}) = \{w_1\}$. Contradiction.

Hence, the distributional objective described above can be achieved by capacity-transfers and cannot be implemented by slot-specific priorities.

However, as long as the ranking of contracts that have the same privilege type respects the test score ordering in each slot, it is easy to see that for every diversity objective that can be achieved by slot-specific priorities, there exists an initial target distribution of seats to privilege types and a monotonic capacity-transfer scheme that achieves the same diversity objective. $\square$

8 Conclusion

In this paper we have studied a matching problem with distributional concerns where agents care not only about the institution they are matched with but also about the contractual terms of the contract with the institution. In other words, we expand the preference domain of agents from institutions only to institutions-contractual terms pairs. Each institution can be thought as a union of different divisions, where each division is associated with exactly one contractual term. Institutions have target distributions over their divisions in the form of reserves. If these reserves are considered to be hard bounds, then in the case that demand for a particular division is less than its target capacity, some slots will remain empty. To overcome this problem and to increase efficiency, we introduce capacity-transfers across divisions when one or more of the divisions is not able to fill to its target capacity. The capacity-transfer scheme is embedded into divisions' choice functions, i.e., sub-choice functions. The overall choice function of an institution can be thought of as the union of choices with these sub-choice functions.

We offer the cumulative offer mechanism under monotonic capacity-transfers as an allocation rule in merit-based object allocation problems where agents are ranked strictly according to certain test scores. When each privilege has a q-responsive choice function obtained from a strict priority ranking, the cumulative offer mechanism is stable and strategy proof. Moreover, the cumulative offer mechanism respects improvement in test scores, i.e., improvement in the ranking of an agent. By introducing monotonic capacity-transfers in the matching-with-contracts framework, we obtain efficiency gain in the sense that the outcome of the cumulative offer algorithm under monotonic capacity-transfer Pareto dominates the outcome of the cumulative offer algorithm without capacity-transfer.

9 Appendices

A. Formal Description of the Cumulative Offer Process

The **cumulative offer process associated to proposal order** Γ is the following algorithm

1. Let $l = 0$. For each $s \in S$, let $D_s^0 \equiv \emptyset$, and let $A_s^1 \equiv \emptyset$.

2. For each $l = 1, 2, ...$
   Let i be the $\Gamma_l-maximal$ agent $i \in I$ such that $i \notin i(\underset{s\in S}{\cup} D_s^{l-1})$ and $\underset{P^i}{max}(X \setminus (\underset{s\in S}{\cup} A_s^l))_i \neq \emptyset_i-$ that is, the agent highest in the proposal order who wants to propose a new contract$-$ if such agent exists. (If no such agent exists, then proceed to Step 3, below.)
   (**a**) Let $x = \underset{P^i}{max}(X \setminus (\underset{s\in S}{\cup} A_s^l))_i$ be i's most preferred contract that has not been proposed.
   (**b**) Let $s = s(x)$. Set $D_s^l = C^s(A_s^l \cup \{x\})$ and set $A_s^{l+1} = A_s^l \cup \{x\}$. For each $s^{'} \neq s$, set $D_{s^{'}}^l = D_{s^{'}}^{l-1}$ and set $A_{s^{'}}^{l+1} = A_{s^{'}}^l$.

3. Return the outcome

$$Y \equiv (\bigcup_{s\in S} D_s^{l-1}) = (\bigcup_{s\in S} C^s(A_s^l))$$

consisting of contracts held by institutions at the point when no agents want to propose additional contracts.

Here, the sets D_s^{l-1} and A_s^l denote the set of contracts **held by** and **available to** institution s at the beginning of the cumulative offer process step l. We say that a contract z is **rejected** during the cumulative offer process if $z \in A_{s(z)}^l$ but $z \notin D_{s(z)}^{l-1}$ for some l.

B. Substitutably Completable Choice Functions

Definition 17. *(Hatfield and Kominers, 2015)* A *completion* of a many-to-one choice function C^s of school $s \in S$ is a choice function $\overline{C}^s$ such that for all $Y \subseteq X$, either $\overline{C}^s(Y) = C^s(Y)$ or there exists a distinct $z, z' \in \overline{C}^s(Y)$ such that $i(z) = i(z')$.

If a choice function C^s has a completion that is *substitutable* and satisfies the *irrelevance of rejected contracts* condition, then we say that C^s is *substitutably completable.* If every choice function in a profile of choice functions C is substitutably completable, then we say that C is *substitutably completable.*

Theorem 5. *(Theorem 1 of Hatfield and Kominers, 2015) If C is substitutably completable, then there exists an outcome that is stable with respect to C.*

The following is useful in proving our results as well.

Theorem 6. *(Theorem 2 of Hatfield and Kominers, 2015) If C is substitutably completable, then the outcome of the student-proposing deferred-acceptance algorithm under any substitutable completion of C is the same as the outcome of the student-proposing deferred-acceptance algorithm under C; moreover, that outcome is stable under C.*

Our proof for strategy proofness of the cumulative offer mechanism with respect to the choice functions with capacity-transfers we developed is heavily based on the following result from Hatfield and Kominers, 2015.

Theorem 7. *(Theorem 3 of Hatfield and Kominers, 2015) Let $\varphi(C^I, C^S)$ be the mechanism that implements the student-proposing deferred-acceptance algorithm given choice profile (C^I, C^S). If, for each $s \in S$, the choice function C^s has a substitutable completion that satisfies the law of aggregate demand, then $\varphi(C^I, C^D)$ is (group) strategy-proof for students.*

Substitutable Completion of the Choice Function We Introduced

Let $\overline{C}^s$ be a choice function of school $s \in S$ in the following way:

- Given $\overline{q}^s_{t_1}$ and $Y = \overline{Y}^0 \subseteq X$, let $\overline{Y}_1 \equiv C^s_{t_1}(\overline{Y}^0, \overline{q}^s_{t_1})$ be the set of chosen contracts with privilege t_1. Then, let $\tilde{r}_1 = \overline{q}^s_{t_1} - \mid \overline{Y}_1 \mid$ and $\overline{Y}^1 = \overline{Y}^0 \setminus \overline{Y}_1$.

- For $j \in \{2, ..., k\}$, given the set of remaining contracts $\overline{Y}^{j-1}$ and the capacity $\tilde{q}^s_{t_j} = q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}) \geq \overline{q}^s_{t_j}$, let $\overline{Y}_j \equiv C^s_{t_j}(\overline{Y}^{j-1}, \tilde{q}^s_{t_j})$ be the set of chosen contracts with privilege t_j, where the capacity of the group of seats for agents with privilege t_j is a function of the number of unused seats from the previous groups. Let $\tilde{r}_j = \tilde{q}^s_{t_j} - \mid \overline{Y}_j \mid$ be the number of unused seats that were reserved for agents with privilege t_j. Then, $\overline{Y}^j = \overline{Y}^{j-1} \setminus \overline{Y}_j$.

- Given the set of contracts $Y = \overline{Y}^0$ and the capacity $\overline{q}^s_{t_1}$ of the group of seats reserved for students with privilege t_1, we define the overall choice function of school $s \in S$ as $\overline{C}^s(Y) = \bigcup_{j=1}^{k} C^s_{t_j}(\overline{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}))$.

The lemma below shows that the choice function above is the completion of the choice function we define in section 3.2.

Lemma 1. *$\overline{C}^s$ is a completion of C^s.*

Proof. For a given $Y = \overline{Y}^0 \subseteq X$, assume there is no pair of contracts $z, z' \in X$ such that $i(z) = i(z')$ and $z, z' \in \overline{C}^s(Y)$. If $\overline{C}^s$ is a completion of C^s, then $\overline{C}^s(Y) = C^s(Y)$ must be satisfied.

Given $\overline{q}^s_{t_1}$ and $Y = \overline{Y}^0 \subseteq X$, we have $\overline{Y}_1 = C^s_{t_1}(\overline{Y}^0, \overline{q}^s_{t_1}) = Y_1$ due to the construction of $\overline{C}^s$. Moreover, $\tilde{r}_1 = r_1$ and $\tilde{q}^s_{t_2} = q^s_{t_2}$.

For t_2, given the set of remaining contracts $\overline{Y}^1$ and the capacity $\tilde{q}^s_{t_2}$, we have $\overline{Y}_2 = C^s_{t_2}(\overline{Y}^1, \tilde{q}^s_{t_2})$. Since there are no two contracts of a student chosen by $\overline{C}^s$, we can deduce that all of the remaining contracts of agents whose contracts are chosen by $C^s_{t_1}(\overline{Y}^0, \overline{q}^s_{t_1})$ are rejected by $C^s_{t_2}(\overline{Y}^1, \tilde{q}^s_{t_2})$. Therefore, equality of capacities and the IRC of sub-choice functions imply $C^s_{t_2}(\overline{Y}^1, \tilde{q}^s_{t_2})$=$C^s_{t_2}(\overline{Y}^1, q^s_{t_2})$=$C^s_{t_2}(Y^1, q^s_{t_2})$. Hence, we have $\overline{Y}_2 = Y_2$, $\tilde{r}_2 = r_2$, and $\tilde{q}^s_{t_3} = q^s_{t_3}$.

For t_j where $j \in \{3, ..., k\}$, given the set of remaining contracts $\overline{Y}^{j-1}$ and the capacity $\tilde{q}^s_{t_j}$, we have $\overline{Y}_j = C^s_{t_j}(\overline{Y}^{j-1}, \tilde{q}^s_{t_j})$. Since there are no two contracts of an agent chosen by $\overline{C}^s$, one can deduce that all of the remaining contracts of agents whose contracts are chosen by previous sub-choice functions are rejected by $C^s_{t_j}(\overline{Y}^{j-1}, \tilde{q}^s_{t_j})$. Therefore, equality of capacities and the IRC of sub-choice functions imply that $C^s_{t_j}(\overline{Y}^{j-1}, \tilde{q}^s_{t_j})$=$C^s_{t_j}(\overline{Y}^{j-1}, q^s_{t_j})$=$C^s_{t_j}(Y^{j-1}, q^s_{t_j})$. Hence, we have $\overline{Y}_j = Y_j$, $\tilde{r}_j = r_j$, and $\tilde{q}^s_{t_{j+1}} = q^s_{t_{j+1}}$.

Given the set of contracts Y and the capacity $\bar{q}^s_{t_1}$ of the group of seats reserved for students with privilege t_1, we have $C^s(Y) = \overset{k}{\underset{j=1}{\cup}} Y_j = \overset{k}{\underset{j=1}{\cup}} \overline{Y}_j = \overline{C}^s(Y)$. $\square$

The completion defined above satisfies the IRC as stated in the following lemma.

Lemma 2. *$\overline{C}^s$ satisfies the IRC.*

Proof. For any $Y \subseteq X$ such that $Y \neq \overline{C}^s(Y)$, let x be one of the rejected contracts, i.e., $x \in Y \setminus \overline{C}^s(Y)$. To show that the IRC is satisfied, we need to prove that $\overline{C}^s(Y) = \overline{C}^s(Y \setminus \{x\})$. Let $\tilde{Y} = Y \setminus \{x\}$. Suppose $(\overline{Y}_j, \bar{r}_j, \overline{Y}^j)$ and $(\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j)$ denote the sequence of chosen contracts, the number of vacant slots, and the remaining set of contracts for each privilege type from Y and $\tilde{Y}$, respectively.

For privilege t_1, since the sub-choice functions satisfy the IRC, we have $\overline{Y}_1 = \tilde{Y}_1$. Moreover, $\bar{r}_1 = \tilde{r}_1$ and $\overline{Y}^1 \setminus \{x\} = \tilde{Y}^1$. By induction, for each t_j where $j \in \{2, ..., k\}$, since each sub-choice function is assumed to satisfy the IRC, we have $\overline{Y}_j = \tilde{Y}_j$, $\bar{r}_j = \tilde{r}_j$, and $\overline{Y}^j \setminus \{x\} = \tilde{Y}^j$. Since for all $j \in \{1, ..., k\}$, $\overline{Y}_j = \tilde{Y}_j$, we have $\overline{C}^s(Y) = \overline{C}^s(\tilde{Y})$. Hence, $\overline{C}^s$ satisfies the IRC. $\square$

The completion we define satisfies the substitutability condition, as well.

Lemma 3. *$\overline{C}^s$ is a substitutable choice function.*

Proof. For any $Y \subseteq X$ such that $Y \neq \overline{C}^s(Y)$, let x be one of the rejected contracts and let z be an arbitrary contract in $X \setminus Y$, i.e., $x \in Y \setminus \overline{C}^s(Y)$ and $z \in X \setminus Y$. To show substitutability, we need to show that $x \notin \overline{C}^s(Y \cup \{z\})$. Let $\tilde{Y} = Y \cup \{z\}$. Suppose $(\overline{Y}_j, \bar{r}_j, \overline{Y}^j)$ and $(\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j)$ denote the sequence of chosen contracts, the number of vacant slots, and the remaining set of contracts for each privilege type from Y and $\tilde{Y}$, respectively.

If $z \in \tilde{Y} \setminus \overline{C}^s(\tilde{Y})$, then by the IRC of $\overline{C}^s$, $\overline{C}^s(\tilde{Y}) = \overline{C}^s(Y)$. Therefore, $x \notin \overline{C}^s(\tilde{Y})$.

Now, let $z \in \overline{C}^s(\tilde{Y})$. First, let j' be the privilege type such that $z \in \tilde{Y}_{j'}$. For any $j < j'$, by the IRC of sub-choice functions, $x \notin \tilde{Y}_j = \overline{Y}_j$. For the privilege $t_{j'}$, by the substitutability of sub-choice functions, $x \notin \tilde{Y}_{j'}$. Also, since sub-choice functions satisfy the LAD, we have $\tilde{r}_{j'} \leq \bar{r}_{j'}$. By the monotone capacity-transfer scheme, $q^s_{t_{j'+1}}(\tilde{r}_1, ..., \tilde{r}_j) \leq q^s_{t_{j'+1}}(\bar{r}_1, ..., \bar{r}_j)$. Moreover, by substitutability of sub-choice functions, $\overline{Y}^{j'} \subseteq \tilde{Y}^{j'}$.

For privilege $t_{j'+1}$, lower capacity and QM imply that $x \notin C^s_{t_{j'+1}}(\tilde{Y}^{j'}, q^s_{t_{j'+1}}(\tilde{r}_1, ..., \tilde{r}_{j'})) \subseteq \overline{Y}_{j'+1}$. Also, by the substitutability of sub-choice functions,

$$\overline{Y}^{j'} \subseteq \tilde{Y}^{j'} \Longrightarrow x \notin C^s_{t_{j'+1}}(\tilde{Y}^{j'}, q^s_{j'+1}(\tilde{r}_1, ..., \tilde{r}_{j'})) = \tilde{Y}_{j'+1}.$$

Also, by the QM and LAD, $\tilde{r}_{j'+1} \leq \bar{r}_{j'+1}$. By the monotone capacity-transfer scheme, $q^s_{t_{j'+2}}(\tilde{r}_1, ..., \tilde{r}_{j'+1}) \leq q^s_{j'+2}(\bar{r}_1, ..., \bar{r}_{j'+1})$. Moreover, by the substitutability of sub-choice functions, $\overline{Y}^{j'+1} \subseteq \tilde{Y}^{j'+1}$.

By induction, for t_j where $j \in \{j'+1, ..., k\}$, lower capacity and QM imply that $x \notin C^s_{t_j}(\overline{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-$ $\overline{Y}_j$. Also, by substitutability of sub-choice functions,

$$\overline{Y}^{j-1} \subseteq \tilde{Y}^{j-1} \Longrightarrow x \notin C^s_{t_j}(\tilde{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1})) = \tilde{Y}_j.$$

Since, for all $j \in \{1, ..., k\}$, $x \notin \tilde{Y}_j$, we have $x \notin \overline{C}^s(\tilde{Y})$. Hence, $\overline{C}^s$ is a substitutable choice function. $\square$

The last lemma in this section states that the completion defined satisfies the LAD.

Lemma 4. *$\overline{C}^s$ satisfies the LAD.*

Proof. For any $Y \subseteq \tilde{Y} \subseteq X$, let $(\overline{Y}_j, \bar{r}_j, \overline{Y}^j)$ and $(\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j)$ denote the sequence of chosen contracts, the number of vacant slots, and the remaining set of contracts for each privilege type from Y and $\tilde{Y}$, respectively. To prove the LAD, we need to show that $| \overline{C}^s(Y) | \leq | \overline{C}(\tilde{Y}) |$.

For privilege t_1, since the sub-choice functions satisfy the LAD, we have $| \overline{Y}_1 | \leq | \tilde{Y}_1 |$. Therefore, $\tilde{r}_1 \leq \bar{r}_1$. Moreover, substitutability of sub-choice functions implies $\overline{Y}^1 \subseteq \tilde{Y}^1$.

Suppose that for some privilege t_j, $\overline{Y}^{j-1} \subseteq \tilde{Y}^{j-1}$ and $\tilde{r}_l \leq \bar{r}_l$ for all $l \leq j$. We need to show that the same statements hold for the privilege t_{j+1} as well. For privilege t_j, by the inductive assumption, $q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}) \leq q^s_{tj}(\bar{r}_1, ..., \bar{r}_{j-1})$. Moreover, by the LAD and QM of sub-choice functions, the following inequalities hold:

$$| \overline{Y}_j | - | \tilde{Y}_j | \leq | \overline{Y}_j | - | C^s_{t_j}(\overline{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1})) | \leq q^s_{t_j}(\bar{r}_1, ..., \bar{r}_{j-1}) - q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}) \leq \bar{r}_j - \tilde{r}_j.$$

Therefore, $\tilde{r}_2 = q^s_{t_2}(\tilde{r}_1) - | \tilde{Y}_2 | \leq q^s_{t_2}(\bar{r}_1) - | \overline{Y}_2 | = \bar{r}_2$. The first inequality above is due to the LAD, and the second and the third inequalities follow from the QM. Also, by the QM, we can deduce:

$$C^s_{t_j}(\overline{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1})) \subseteq \overline{Y}_j, \text{ or}$$

$$(\overline{Y}^{j-1} \setminus \overline{Y}^j) \subseteq (\overline{Y}^{j-1} \setminus C^s_{t_j}(\overline{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}))),$$

and by the substitutability of sub-choice functions

$$(\overline{Y}^{j-1} \setminus C^s_{t_j}(\overline{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}))) \subseteq (\tilde{Y}^{j-1} \setminus \tilde{Y}^j).$$

So, we have $\overline{Y}^j \subseteq \tilde{Y}^j$. Hence, by our induction, $\tilde{r}_j \leq \bar{r}_j$, for all j.

Now, let the sequence k_j be the difference in vacant slots, i.e., $\bar{r}_j = \tilde{r}_j + k_j$ for all j. The definitions of sequences $\bar{r}_j$ and $\tilde{r}_j$ imply $\mid \tilde{Y}_j \mid = q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q^s_{t_j}(\bar{r}_1, ..., \bar{r}_{j-1}) + \mid \overline{Y}_j \mid + k_j$. Hence,

$$\sum_{l=1}^{j} \mid \tilde{Y}_l \mid = \sum_{l=1}^{j} \mid \overline{Y}_l \mid + \sum_{l=1}^{j} [k_l - (q^s_{t_l}(\bar{r}_1, ..., \bar{r}_{l-1}) - q^s_{t_l}(\tilde{r}_1, ..., \tilde{r}_{l-1}))],$$

$$\sum_{l=1}^{j} \mid \tilde{Y}_l \mid \geq \sum_{l=1}^{j} [\mid \overline{Y}_l \mid + q^s_{t_j}(\bar{r}_1, ..., \bar{r}_{l-1}) - q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{l-1})],$$

$$\sum_{l=1}^{j} \mid \tilde{Y}_j \mid \geq \sum_{l=1}^{j} \mid \overline{Y}_j \mid .$$

where the first inequality follows from the second part of the monotonicity definition, and the second inequality follows from the first part of the definition of monotonicity. Since j is arbitrary, the above inequality implies $\mid \overline{C}^s(Y) \mid \leq \mid \overline{C}^s(\tilde{Y}) \mid$, which completes the proof. □

C. Proofs Omitted from the Main Text

- *Proof of Proposition 1:*

Proof. Take a set of contracts $Y \subseteq X$ and a contract $z \in X \setminus Y$ such that $z \notin C^s(Y \cup \{z\})$. We need to prove that $C^s(Y) = C^s(Y \cup \{z\})$. Suppose that $t(z) = t_j$. Then the contract z is not chosen by the sub-choice function of the privilege types t_l, $l = 1, ..., j-1$. Note that if any other contract of the agent $i(z)$ is chosen by the sub-choice functions of privileges $t_1, ..., t_{j-1}$, the proof is done because when another contract of agent $i(z)$ is chosen at any step, the contract z is removed from the process for the remaining steps. So we will consider the non-trivial case where none of the contracts of agent $i(z)$ are chosen up to the privilege type t_j. Since all the sub-choice functions satisfy IRC up to privilege type t_j, the same contracts will be chosen from the sets Y and $Y \cup \{z\}$ by the sub-choice functions $C^s_{t_1}(\cdot, \cdot), ..., C^s_{t_{j-1}}(\cdot, \cdot)$, respectively. Let us denote the number of unused seats for privilege type t_l from the initial contract sets Y and $Y \cup \{z\}$ as r_l and $\tilde{r}_l$, respectively. Since $t(z) = t_j$, we have $r_l = \tilde{r}_l$ for $l = 1, ..., j-1$. This implies that $q^s_{t_j}(r_1, ..., r_{j-1}) = q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1})$. Let us denote the remaining set of contracts after the choice by the choice function of privilege type t_l from the initial contract sets Y and $Y \cup \{z\}$ as $Y^l$and $\tilde{Y}^l$, respectively.

By our assumption, we know that $z \notin C^s_{t_j}(\tilde{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}))$ and $\tilde{Y}^{j-1} = Y^{j-1} \cup \{z\}$. By the IRC of the sub-choice function $C^s_{t_j}(\cdot,\cdot)$, we obtain $C(\tilde{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1})) = C^s_{t_j}(Y^{j-1}, q^s_{t_j}(r_1, ..., r_{j-1}))$. Also, $r_j = \tilde{r}_j$. If $i(z) \in i[C^s_{t_j}(\tilde{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}))]$, then the contract z is removed from the process. Otherwise, we have $\tilde{Y}^j = Y^j \cup \{z\}$. Since $q^s_{t_{j+1}}(r_1, ..., r_j) = q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j)$ the same argument holds for the privilege type t_{j+1}. By proceeding in the same fashion, we obtain $C(\tilde{Y}^l, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_l)) = C^s_{t_j}(Y^l, q^s_{t_j}(r_1, ..., r_l))$ for all $l = 1, ..., k$. Hence, we have $C^s(Y) = C^s(Y \cup \{z\})$. □

- *Proof of Proposition 2:*

Proof. Since all sub-choice functions satisfy IRC, the overall choice function satisfies IRC as well by Proposition 1. In order to prove that the overall choice function satisfies WS, we take a set of contracts $Y \subseteq X$ and two contracts $x, z \in X \setminus Y$ such that $\mid Y \cup \{x,z\} \mid = \mid i(Y \cup \{x,z\}) \mid$. Suppose that $z \notin C^s(Y \cup \{z\})$. We need to show that $z \notin C^s(Y \cup \{x,z\})$. We consider two cases:

Case 1: $x \notin C^s(Y \cup \{x,z\})$. Since the overall choice function satisfies IRC, we then have $C^s(Y \cup \{x,z\}) = C^s(Y \cup \{z\})$. Hence, by our assumption, we have $z \notin C^s(Y \cup \{x,z\})$.

Case 2: $x \in C^s(Y \cup \{x,z\})$. Let the privilege type of agent $i(x)$ be $t(x) = t_j$ where $j \in \{1, ..., k\}$. Then for each $l \notin \{1, ..., j-1\}$, neither x nor z are chosen by sub-choice functions. By IRC of sub-choice functions, since x is not chosen by the sub-choice functions of privileges $t_1, ..., t_{j-1}$, sub-choices from the sets $(Y \cup \{z\})$ and $(Y \cup \{x,z\})$ for privilege types $t_1, ..., t_{j-1}$ are identical. Hence, the number of unused seats of privilege types $t_1, ..., t_{j-1}$ from the sets $(Y \cup \{z\})$ and $(Y \cup \{x,z\})$ are the same, i.e., $r_l = \tilde{r}_l$ for every $l \in \{1, ..., j-1\}$. This implies that the capacity of privilege type t_j, $q^s_{t_j}(r_1, ..., r_{j-1})$, is equal to $q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1})$. Let Y^l be the set of remaining contracts after sub-choice for privilege type t_j from the set $(Y \cup \{z\})$ and $\tilde{Y}^l$ be the set of remaining contracts after sub-choice for privilege t_j from the set $(Y \cup \{x,z\})$. Note that $\tilde{Y}^l = Y^l \cup \{x\}$ for all $l \in \{1, ..., j-1\}$.

Let Y_j and $\tilde{Y}_j$ be the set of chosen contracts by sub-choice functions for privilege t_j from the sets $(Y \cup \{z\})$ and $(Y \cup \{x,z\})$, respectively. By the weak substitutability of sub-choice functions for privilege t_j, we have $z \notin \tilde{Y}_j$. It is easy to see that $Y^j \subseteq \tilde{Y}^j$ because otherwise there exists a contract $y \in Y^j$ (means $y \notin Y_j$) but $y \notin \tilde{Y}^j$ (means $y \in \tilde{Y}_j$). Since each agent has only one contract, we have contradiction with the fact that sub-choice functions satisfy weak substitutability (WS). By the LAD of the sub-choice functions, we have $\mid Y_j \mid \leq \mid \tilde{Y}_j \mid$. Hence, we have $q^s_{t_{j+1}} = q^s_{t_{j+1}}(r_1, ..., r_j) \geq \tilde{q}^s_{t_{j+1}} = q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j)$ by monotonicity of the capacity-transfer scheme as $r_j \geq \tilde{r}_j$ and $r_l = \tilde{r}_l$ for every $l \in \{1, ..., j-1\}$.

By our assumption, we know that $z \notin C^s_{t_{j+1}}(Y^j, q^s_{t_{j+1}})$. By QM of sub-choice functions, we have $z \notin C^s_{t_{j+1}}(Y^j, \tilde{q}^s_{t_{j+1}})$. Then, WS and IRC of sub-choice functions imply that $z \notin \tilde{Y}_{j+1} =$

$C^s_{t_{j+1}}(\tilde{Y}^j, \tilde{q}^s_{t_{j+1}})$. By the LAD of sub-choice functions, we have $\mid Y_{j+1} \mid \leq \mid \tilde{Y}_{j+1} \mid$. This implies that $r_{j+1} \geq \tilde{r}_{j+1}$, and, hence, $q^s_{t_{j+2}}(r_1, ..., r_{j+1}) \geq q^s_{t_{j+2}}(\tilde{r}_1, ..., \tilde{r}_{j+1})$ by the monotonicity of the capacity-transfer scheme. Also, it is easy to see that $Y^{j+1} \subseteq \tilde{Y}^{j+1}$. Repeating the same arguments for the rest of the privileges gives us $z \notin C^s(Y \cup \{x, z\})$ and completes the proof. □

Lemma 5. *Take $Y \subseteq X$ and $x, z \in X \setminus Y$ such that $i(x) \neq i(y)$ and $i(x), i(z) \notin i(Y)$. Suppose that $z \notin C^s(Y \cup \{z\})$. Set $Y^0 = Y \cup \{z\}$ and $\tilde{Y}^0 = Y^0 \cup \{x\}$. Suppose also that $x \in \tilde{Y}_j = C^s_{t_j}(\tilde{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}))$. Let $Y^j = Y^{j-1} \setminus \{x \in Y^{j-1} : \ i(x) \notin i(Y_j)\}$ and $\tilde{Y}^j = \tilde{Y}^{j-1} \setminus \{x \in \tilde{Y}^{j-1} : \ i(x) \notin i(\tilde{Y}_j)\}$. Then, $Y^j \subseteq \tilde{Y}^j$.*

Proof. Assume not. Then there exists a contract $y \in Y^j$ such that $y \in \tilde{Y}_j$ (hence, $y \notin \tilde{Y}^j$) and $i(y) \notin i(Y_j)$. Since none of the contracts of agent $i(y)$ are chosen from Y^{j-1}, removing them from Y^{j-1} does not change the set of chosen contracts by IRC of the sub-choice function, i.e., construct the set $A = Y^{j-1} \setminus \{y' \in Y^{j-1} : \ i(y') = i(y)\}$, and we have $C^s_{t_j}(A, q) = C^s_{t_j}(Y^{j-1}, q)$. Now consider the choice from the sets $A \cup \{y\}$ and $\tilde{Y}^{j-1}$. We have $y \notin C^s_{t_j}(A \cup \{y\}$,q). Notice that y is the only contract of agent $i(y)$ in $A \cup \{y\}$. Now consider the set $A \cup \{x, y\}$. Since $y \in \tilde{Y}_j$, by the IRC of the sub-choice function, we have $y \in C^s_{t_j}(A \cup \{x, y\}, q)$. This contradicts with the BS of the sub-choice function because $y \notin C^s_{t_j}(A \cup \{y\}$,q) and yet $y \in C^s_{t_j}(A \cup \{x, y\}, q)$. This completes the proof. □

- *Proof of Proposition 3:*

Proof. Since all sub-choice functions satisfy IRC, the overall choice function satisfies IRC as well by Proposition 1. To prove that the overall choice function also satisfies bilateral substitutability, consider a set of contracts $Y \subseteq X$ and contracts $x, z \in X \setminus Y$ such that $i(x), i(z) \notin i(Y)$. Suppose that $z \notin C^s(Y \cup \{z\})$. We need to show that $z \notin C^s(Y \cup \{x, z\})$. There are two cases to consider:

Case 1 : $x \notin C^s(Y \cup \{x, z\})$.

Since the overall choice function satisfies IRC, we then have $C^s(Y \cup \{x, z\}) = C^s(Y \cup \{z\})$. Hence, by our assumption, we have $z \notin C^s(Y \cup \{x, z\})$.

Case 2: $x \in C^s(Y \cup \{x, z\})$.

There exist $j \in \{1, ..., k\}$ such that $x \in \tilde{Y}_j = C^s_{t_j}(\tilde{Y}^{j-1}, q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1}))$. For all $i \in \{1, ..., j-1\}$, we know that $x \notin \tilde{Y}_i$ and $z \notin Y_i$ by our assumptions. Then, by the BS of sub-choice functions of the privileges $t_1, ..., t_{j-1}$, we have $z \notin \tilde{Y}_i$. Also note that $\tilde{Y}^i = Y^i \cup \{x\}$ and $z \in Y^i$ for all $i \in \{0, 1, .., j-1\}$. By Lemma 1, we know that $Y^j \subseteq \tilde{Y}^j$. Also, since $r_1 = \tilde{r}_1$,...,$r_{j-1} = \tilde{r}_{j-1}$ we have $q^s_{t_j}(r_1, ..., r_{j-1}) = q^s_{t_j}(\tilde{r}_1, ..., \tilde{r}_{j-1})$. By the LAD, we know that

$\mid Y_j \mid \leq \mid \tilde{Y}_j \mid$. Hence we have $q^s_{t_{j+1}}(r_1, ..., r_j) \geq q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j)$ by the monotonicity of the capacity-transfer scheme.

We need to prove that $z \notin C^s_{t_{j+1}}(\tilde{Y}^j, q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j))$. We know, by our assumption, that $z \notin C^s_{t_{j+1}}(Y^j, q^s_{t_{j+1}}(r_1, ..., r_j))$ where $Y^j \subseteq \tilde{Y}^j$ and $q^s_{t_{j+1}}(r_1, ..., r_j) \geq q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j)$. Also, notice that $i(\tilde{Y}^j \backslash Y^j) \cap i(Y^j) = \emptyset$. By QM of the sub-choice functions $z \notin C^s_{t_{j+1}}(Y^j, q^s_{t_{j+1}}(r_1, ..., r_j))$ implies $z \notin C^s_{t_{j+1}}(Y^j, q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j))$. If $i(\tilde{Y}^j \setminus Y^j) \notin i(\tilde{Y}_{j+1})$, then by the IRC of the sub-choice function we have $z \notin \tilde{Y}_{j+1}$. Otherwise, there must exist $y' \in \tilde{Y}^j \setminus Y^j$ such that $y' \in \tilde{Y}_{j+1} = C^s_{t_{j+1}}(\tilde{Y}^j, q^s_{t_{j+1}}(\tilde{r}_1, ...\tilde{r}_j))$. Note that $i(y') \notin i(Y^j)$. Let $\{y', ..., w'\}$ be the set of contracts in $\tilde{Y}^j \setminus Y^j$ such that each of them is chosen by $\tilde{Y}_{j+1}$. By the IRC of the sub-choice function, removing the other contracts of the doctors $i(\{y', ..., w'\})$ from the set $\tilde{Y}^j$ does not change the chosen set. Therefore, $C^s_{t_{j+1}}(\tilde{Y}^j, q^s_{t_{j+1}}(\tilde{r}_1, ...\tilde{r}_j)) = C^s_{t_{j+1}}(Y^j \cup \{y', ..., w'\}, q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j))$. The BS of the sub-choice function implies $z \notin C^s_{t_{j+1}}(Y^j \cup \{y', ..., w'\}, q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j))$. Hence, $z \notin \tilde{Y}_{j+1}$.

We now need to prove $Y^{j+1} \subseteq \tilde{Y}^{j+1}$. Take $y \in Y^{j+1}$. We know that $y \notin Y_{j+1}$. Then, by QM, this implies that $y \notin C^s_{t_{j+1}}(Y^j, q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j))$. Finally, BS and IRC imply that $y \notin \tilde{Y}_{j+1} = C^s_{t_{j+1}}(\tilde{Y}^j, q^s_{t_{j+1}}(\tilde{r}_1, ..., \tilde{r}_j))$, i.e., $y \in \tilde{Y}^{j+1}$.

To finish the proof, we need to show that $\tilde{r}_{j+1} \leq r_{j+1}$, i.e., $q_{j+1}(\tilde{r}) - \mid \tilde{Y}_{j+1} \mid \leq q_{j+1}(r) - \mid Y_{j+1} \mid$. By the monotonicity of the capacity-transfer scheme, we have $\tilde{q}_{j+1} \leq q_{j+1}$. By the LAD, this implies $\mid C^s_{t_{j+1}}(Y^{j+1}, q^s_{j+1}(r)) \mid - \mid C^s_{t_{j+1}}(Y^{j+1}, q^s_{j+1}(\tilde{r})) \mid \leq q^s_{j+1}(r) - q^s_{j+1}(\tilde{r})$. Again by the LAD, we obtain $\mid C^s_{t_{j+1}}(\tilde{Y}^{j+1}, q^s_{j+1}(\tilde{r})) \mid \geq \mid C^s_{t_{j+1}}(Y^{j+1}, q^s_{j+1}(\tilde{r})) \mid$. The last two inequalities together imply that
$\mid Y_{j+1} \mid - \mid \tilde{Y}_{j+1} \mid = \mid C^s_{t_{j+1}}(Y^{j+1}, q^s_{j+1}(r)) \mid - \mid C^s_{t_{j+1}}(\tilde{Y}^{j+1}, q^s_{j+1}(\tilde{r})) \mid \leq q^s_{j+1}(r) - q^s_{j+1}$.

Since the same observations apply to all of the remaining privileges after t_{j+1}, this observation ends the proof. □

- *Proof of Proposition 6:*

Proof. Towards a contradiction, let t' be the first step a school s holds a contract z it previously rejected at step $t < t'$. Since z is rejected by school s at step t, there are two cases to consider:

(i) z was on hold at step $(t-1)$, i.e., $z \in C^s(A_s(t-1))$, or

(ii) z was offered to school s at step t, i.e., $z = A_s(t) \setminus A_s(t-1)$.

In either case no other contract of student $i(z)$ could be on hold by school s at Step $(t-1)$. But then, since z is the first contract to be held after an earlier rejection, school s cannot have held another contract by student $i(z)$ at Step t. That is,

$$i(z) \notin i[C^s(A_s(t))],$$

Then by IRC $z \in A_s(t) \setminus C^s(A_s(t))$ implies that

$$z \notin C^s(C^s(A_s(t)) \cup \{z\}),$$

and yet

$$z \in C^s(A_s(t^{'})).$$

Consider every step $t^{''}$ in the cumulative offer algorithm where $t < t^{''} \leq t^{'}$. In each stage one of the following cases occurs:

(i) a new contract, x, from another student with the same privilege type as $t(z)$ is offered, i.e., $i(x) \neq i(z)$ but $t(x) = t(z) = t_j$,

(ii) a new contract, x, from another student with a different privilege type than $t(z)$ is offered, i.e., $i(x) \neq i(z)$ and $t(x) \neq t(z) = t_j$, or

(iii) a new contract from student $i(z)$, $z^{'}$, with a different privilege type than $t(z)$ is offered, i.e., $i(z^{'}) = i(z)$ but $t(z^{'}) \neq t(z) = t_j$.

In each case and for each step of the cumulative offer algorithm between steps t and $t^{'}$, we will show that z is not going to be recalled.

(i) In this case note that both r_l and Y_l for $l = 1, ..., j-1$ remain unchanged. Hence the capacity of the privilege type t_j will be as same as the capacity before receiving the offer x. Since $\succ^{t_j}$ is responsive with capacity $q^s_{t_j}$, z will be rejected as it was before the arrival of the contract x, because now competition for slots is higher.

(ii) There are several sub-cases to consider in this case. If the contract x is chosen by a sub-choice function of a privilege t_l where $l > j$, then the contract z will be rejected again since the capacity of the privilege type t_j and all the chosen contracts Y_k where $k < j$ will be the same. If the contract x is chosen by any privilege type t_l where $l < j$, the number of unused seats for all the privileges after the privilege t_l will be weakly smaller. By the monotonicity of capacity-transfer scheme, the capacity of the privilege type t_j will be weakly smaller. Note that the contract x cannot be the contract of any student whose contract is on hold at the privilege type t_j by the dynamics of the cumulative offer algorithm. Finally, if the contract x is not chosen by any of privileges, then by the IRC of the overall choice function z will be rejected.

(iii) For this case there are several cases to consider as well. If $z^{'}$ is not chosen by any privileges by the IRC of the overall choice function, z will be rejected. If $z^{'}$ is chosen by the privilege $t(z^{'}) = t_l$ where $l < j$, then the contract z will be removed from the process by the definition of our choice function, and, hence, z will be rejected again. If $z^{'}$ is chosen by a privilege $t(z^{'}) = t_l$ where $l > j$, then neither the number of unused seats r_k where $k < j$ nor the set of chosen contracts Y_k where $k < j$ changes. Privilege type t_j will have the same capacity as it had before the arrival of z'. Therefore, z will be rejected.

Hence it contradicts with $z \in C^s(A_s(t'))$. □

- *Proof of Proposition 7:*

Proof. Let Y be the outcome of the cumulative offer algorithm. Since agents/students only offer their acceptable contracts during the cumulative offer process, we have $C^i(Y) = Y_i$ for all $i \in I$. Towards a desired contradiction, suppose that Y is not stable. Then, there must exist a school $s \in S$ and a set of blocking contracts $Z \neq C^s(Y)$ such that $Z = C^s(Y \cup Z)$ and $Z_i = C^i(Y \cup Z)$ for all $i \in i(Z)$. Consider an agent/student $j \in i(Z)$ where $Z_j P^j Y_j$. By the definition of the cumulative offer algorithm, agent j must have offered contract Z_j before offering the contract Y_j. Since $Z_j \notin Y$, then Z_j must have been rejected at some step of the cumulative offer process. It holds for every agent whose more preferred contract in Y compared to their contract in Z. So there is a step t of the cumulative offer process in which $(Y \cup Z) \subseteq A_s(t)$. By Proposition 6, a rejected contract during the cumulative offer algorithm cannot be on hold at a further step under the monotone capacity-transfer scheme, i.e., there is no renegotiation. It contradicts with our assumption that $Z = C^s(Y \cup Z)$. □

- *Proof of Theorem 2:*

Proof. Lemmas 1-3 show that C^s is substitutably completable. Moreover, Lemma 4 shows that the substitutable completion of C^s, $\overline{C}^s$, satisfies the LAD. Therefore, our Theorem 2 is a corollary of Theorem 3 in Hatfield and Kominers, 2015. □

- *Proof of Theorem 3:*

Proof. Fix a student i and let $\succ'$ be an unambiguous improvement for student i over $\succ$.

We will first consider the outcome of the cumulative offer mechanism under a monotone capacity-transfer when the sub-choice functions for each school are induced from strict priority rankings $\succ'_{t_1}, \succ'_{t_2}, ..., \succ'_{t_k}$, respectively. Recall that, by Remark 1, the order of students making offers has no impact on the outcome of the cumulative offer algorithm. Therefore, we can obtain the outcome of the cumulative offer algorithm when the strict ranking of students according to test scores is $\succ'$: First, entirely ignore student i and run the cumulative offer algorithm until it stops. Let X' be the resulting set of contracts. At this point, student i makes an offer for her first-choice contract x^1. Her offer may cause a chain of rejections, which may eventually cause contract x^1 to be rejected as well. If that happens, student i makes an offer for her second choice x^2, which may cause another chain of rejections, and so on. Let this process terminate after student i makes an offer for her lth choice contract

x^l. There may still be a chain of rejections after this offer, but it does not reach student i again. Hence, student i receives her lth choice under $\Phi^{COM}(\succ')$.

Next consider the outcome of the cumulative offer mechanism under the same monotone capacity-transfer when the sub-choice functions for each school are induced from strict priority rankings $\succ_{t_1}, \succ_{t_2}, ..., \succ_{t_k}$, respectively. Initially entirely ignore student i and run the cumulative offer algorithm until it stops. Since the only difference between the two scenarios is the standing of student i in the priority list, X' will again be the resulting set of contracts. Next, student i makes an offer for her first-choice contract x^1. Since $\succ'$ is an unambiguous improvement for student i over $\succ$, precisely the same sequence of rejections will take place until she makes an offer for her lth choice contract x^l. Therefore, student i cannot receive a better contract than her lth choice under $\Phi^{COM}(\succ)$, even though she can receive a worse contract than her lth choice if the rejection chain returns back to her. □

- *Proof of Theorem 4:*

Proof. Consider two problems $(I, S, P^{|I|}, \succ, (\bar{q}^s_{t_j})_{s\in S})$ and $(I, S, P^{|I|}, \succ, (q^s_{t_j}(r_1, ..., r_{j-1}))_{s\in S})$ in which the first one has no capacity-transfer while the second one allows monotone capacity-transfer across different privilege types; everything else is the same in both problems. Note that for every institution $s \in S$ and all privilege types t_j, $j = 1, ..., k$, we have $q^s_{t_j}(r_1, ..., r_{j-1}) \geq \bar{q}^s_{t_j}$.

We need to show that each agent $i \in I$ obtains a weakly better outcome in the cumulative offer algorithm with monotone capacity-transfer than she obtains in the cumulative offer algorithm without capacity-transfer. Consider the following proposal order $\succ$, the strict ranking of agents according to test scores. Let $i_1 - i_2 - ... - i_n$ be the enumeration of agents according to $\succ$ where i_1 has the highest test score, i_2 has the second highest test score, and so on. Let $I'_{i_l} \equiv \{i_j \in I\colon j < l\}$ be the set of agents who have higher test scores than agent i_l. We are going to prove the theorem by induction on students following the proposal order $\succ$.

The first ranked student according to $\succ$ obtains the same outcome under both a monotone capacity-transfer scheme and no capacity-transfer. Hence, he weakly prefers the assignment from the second problem over the assignment from the first problem. Suppose that x'_l is the contract agent i_l obtains in the cumulative offer algorithm with monotone capacity-transfer and x_l is the contract she obtains from the cumulative offer algorithm with no capacity-transfer. Assume that for all $l \leq L$, $x'_l R^{i_l} x_l$. We need to show that this also holds for agent i_{L+1}, i.e., $x'_{L+1} R^{i_{L+1}} x_{L+1}$. Assume not. Suppose that agent i_{L+1} obtains a contract y in the cumulative offer algorithm with monotone capacity-transfer such that $x_{L+1} P^{i_{L+1}} y$ where x_{L+1} is the contract she obtains in the cumulative offer algorithm without capacity-transfer. We know that $q^{s(x_{L+1})}_{t(x_{L+1})}(r) \geq \bar{q}^{s(x_{L+1})}_{t(x_{L+1})}$ by the monotone capacity-transfer. Also, by our inductive hypothesis, the set of agents in $(I^{'}_{i_{L+1}} \cap X_{s(x_{L+1})} \cap X_{t(x_{L+1})})$ whose contracts are not on hold in the cumulative offer algorithm with monotone capacity-transfer at the step where agent i_{L+1} offers her contract x_{L+1} is contained by the set of agents in $(I^{'}_{i_{L+1}} \cap X_{s(x_{L+1})} \cap X_{t(x_{L+1})})$ whose contracts are not on hold in the cumulative offer algorithm without capacity-transfer at the step where agent i_{L+1} offer her contract x_{L+1}. Then, this means that when there are weakly more seats available and there are fewer agents whose scores are higher than agent i_{L+1} in the privilege type $t(x_{L+1})$ at institution $s(x_{L+1})$, her contract x_{L+1} is rejected, while it is accepted when there are weakly more students whose scores are higher than i_{L+1} vying for a seat in the same institution and for the same privilege type, and there are weakly less seats available. This contradicts with the construction of our sub-choice functions, which are q-responsive. Hence, $x^{'}_{L+1} R^{i_{L+1}} x_{L+1}$ completes the proof.

□

References

[1] Abdulkadiroğlu, A., "College admission with affirmative action," *International Journal of Game Theory, 2005, 33, 535-549.*

[2] Abdulkadiroğlu, A., and Tayfun Sönmez, "School Choice: A mechanism design approach," *American Economic Review, 2003, 93, 729-747.*

[3] Abdulkadiroğlu, A., Parag A. Pathak, Alvin E.Roth, and Tayfun Sönmez, "The Boston public school match," *American Economic Review, 2005, 95, 368-371.*

[4] Abdulkadiroğlu, A., Parag A. Pathak, and Alvin E. Roth, "The New York City high school match," *American Economic Review, 2005, 95, 364-367.*

[5] Abdulkadiroğlu, A., Parag A. Pathak, and Alvin E. Roth, "Strategy proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match," *American Economic Review, 2009, 99 (5), 1954-1978.*

[6] Afacan, M. O., and Bertan Turhan, "On relationships between substitutes conditions," *Economics Letters, 2015, 126: 10-12*

[7] Alkan, A., "A class of multi partner matching markets with a strong lattice structure," *Economic Theory, 2002, 19, 737-746.*

[8] Alva, S., "Stability and matching with aggregate actors", 2016, Working Paper.

[9] Aygün, O., and Inacio Bo, "College admissions with multidimensional reserves: The Brazilian affirmative action case", 2016, Working Paper.

[10] Aygün, O., and Tayfun Sönmez, "The importance of irrelevance of rejected contracts in matching under weakened substitutes conditions," 2012. Boston College Working Paper.

[11] Aygün, O., and Tayfun Sönmez, "Matching with contracts: Comment," *American Economic Review, 103 (5), 2050-2051.*

[12] Balinski, M., and Tayfun Sönmez, "A tale of two mechanisms: Student placement," *Journal of Economic Theory, 1999, 84, 73-94.*

[13] Bertrand, M., Rema Hanna, and Sendhil Mullainathan, "Affirmative action in education: Evidence from engineering college admissions in India," *Journal of Public Economics, 2010, 94, 16-29.*

[14] Chambers, C. P., and Bumin Yenmez, "Choice and matching," 2015, Carnegie Mellon University Working Paper.

[15] Deshpande, A., " Affirmative action in India and the United States," *2005, World Development Report Background Paper, World Bank, Washington D.C.*

[16] Dur, U., and Onur Kesten, "Sequential versus Simultaneous Assignment Processes and Two Applications", 2014, Working Paper.

[17] Dur, U., Scott Duke Kominers, Parag Pathak, and Tayfun Sönmez, "The demise of walk zones in Boston: Priorities vs. precedence in school choice", *NBER Working Paper 18981.*

[18] Echenique, F., and Bumin Yenmez, "How to control controlled school choice", *American Economic Review, 2015, 105 (8): 2679-94.*

[19] Ehlers, L., Isa Emin Hafalir, M. Bumin Yenmez, and Muhammed A. Yildirim, " School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory, 2014, 153, 648-683.*

[20] Gale, D., and Lloyd S. Shapley, "College admissions and the stability of marriage," *American Mathematical Monthly, 1962, 69, 9-15.*

[21] Hafalir, I. E., M. Bumin Yenmez, and Muhammed Ali Yildirim, "Effective affirmative action in school choice," *Theoretical Economics, 2013, 8, 325-363.*

[22] Hatfield, J. W., and Fuhito Kojima, "Matching with contracts: Comment," *American Economic Review, 2008, 98, 1189-1194.*

[23] Hatfield, J. W., and Fuhito Kojima, "Substitutes and stability for matching with contracts," *Journal of Economic Theory, 2010, 145, 1704-1723.*

[24] Hatfield, J. W., and Paul Milgrom, "Matching with contracts," *American Economic Review, 2005, 95, 913-935.*

[25] Hatfield, J. W., and Scott Kominers, "Hidden Substitutes," Proceedings of the 16th ACM Conference on Electronic Commerce, 2015.

[26] Hirata, D., and Yusuke Kasuya, "Cumulative offer process is order-independent," *Economics Letters,* 2014, 124, 37-40.

[27] Kamada, Y. and Fuhito Kojima, "Efficient matching under distributional constraints: Theory and applications*," American Economic Review, 2015, 105, 67-99.*

[28] Kamada, Y. and Fuhito Kojima, " General theory of matching under distributional constraints", 2015b, mimeo.

[29] Kelso, A. S., Vincent P. Crawford, "Job matching, coalition formation, and gross substitutes," *Econometrica, 1982, 50, 1483-1504.*

[30] Kochar, A., "Affirmative action through quotas: The effect on higher education in India," 2009. Stanford Center International Development Working Paper.

[31] Kojima, F., "School choice: Impossibilities for affirmative action," *Games and Economic Behavior, 2012, 75, 685-693.*

[32] Kominers, S. D., and Tayfun Sönmez, "Matching with slot-specific priorities: Theory," *Theoretical Economics, forthcoming, 2016.*

[33] Manjunath, V., and Bertan Turhan, "Two school systems, one district: What to do when a unified admissions process is impossible," *Games and Economic Behavior, 2016, 95, 25-40.*

[34] "Rules for admission to first year of degree courses in engineering/technology in government, govt. aided and unaided engineering institutes in Maharashtra state: Academic year 2014-2015." http://fileserver.mkcl.org/FE2014/OasisModules_Files/Files/51.pdf?did=1056

[35] Roth, A. E., and Marilda Sotomayor, "Two-sided Matching: A Study in Game-Theoretic Modeling and Analysis," Vol. 18 of Econometric Society Monographs, Cambridge University Press, Cambridge England, 1990.

[36] Sönmez, T., "Bidding for army career specialities: Improving the ROTC branching mechanism," *Journal of Political Economy, 2013, 121, 186-219.*

[37] Sönmez, T., and Tobias B. Switzer, "Matching with (branch-of-choice) contracts at United States Military Academy," *Econometrica, 2013, 81, 451-488.*

[38] Weisskopf, T. E., "Impact of reservation on admissions to higher education in India," 2004, *Economic and Political Weekly,* Special Articles.

[39] Westkamp, A., “Market structure and matching with contracts,” *Journal of Economic Theory, 2010, 145 (5), 1724-1738.*

[40] Westkamp, A., “An Analysis of the German university admissions system,” *Economic Theory, 2013, 53, 561-589.*