Entangled Risks in Incomplete FX Markets[*]

Thomas Maurer[†]   Ngoc-Khanh Tran[‡]

December 31, 2016

Abstract

We study the implications of *risk entanglements* on international financial (FX) markets. Risk entanglement is a refinement of incomplete markets that some risks in asset markets cannot be *singly* traded. We show that in FX markets with entangled risks (i) there exist multiple pricing-consistent exchange rates, (ii) every exchange rate is affected by idiosyncratic risks, and (iii) exchange rates can be smooth while stochastic discount factors (SDFs) are volatile and almost uncorrelated. These results are in stark contrast to the case of complete markets or incomplete markets without risk entanglements.

JEL-Classification: F31, G15, G10.

Keywords: Exchange Rates, Entangled Risks, Jump Risks, Incomplete Markets, International Correlation Puzzle.

1 Introduction

In complete markets the exchange rate equals the ratio of the stochastic discount factors (SDF) of the two involved countries. The SDF growth in each country has a volatility of at least 50% to match the equity premium (Hansen and Jagannathan, 1991), while exchange rate growths have low volatilities of only 10% in the data. Therefore, the two SDF growths must have an extremely high implied correlation of $\rho = 1 - \frac{0.10^2}{2*0.5^2} \approx 98\%$: international risk sharing is almost perfect (Brandt et al., 2006). Inconsistent with this inference from asset prices, macroeconomic data implies a relatively low degree of international risk sharing; for instance Brandt et al. (2006) estimate cross-country correlations in consumption growth of only 30%. Turning to asset prices and assuming a correlation between SDF growths of only 30% and SDF growth volatilities of 50%, then the implied exchange rate growth volatility must be almost 60%, which is far from the volatility of 10% in the data. This irreconcilability of asset prices and macroeconomic data is known as the international correlation puzzle. The dashed black line in Figure 1 illustrates this: it is impossible to jointly have a smooth exchange rate and a low correlation between SDF growths.[1]

Given the correlation puzzle it is impossible to tell how much risk is actually shared in international markets, because the two approaches (based on asset prices versus macroeconomic data) deliver completely different results and we do not know which approach is more reliable. Resolving the puzzle is key to shed light on the important question of how efficient international risk sharing is, which in turn, has important implications for international policy. Moreover, a resolution of the puzzle is a step to a better understanding of how macroeconomic quantities affect asset prices.

How can we reconcile the correlation puzzle and how much risk is shared in international markets? As illustrated by the solid red line in Figure 1, we demonstrate that a smooth exchange rate and a low correlation between SDF growths can co-exist in a jump-diffusion setting with incomplete markets if risks are *entangled*. Consequently, the asset price implications for international risk sharing are weakened and can be reconciled with estimates based on macroeconomic quantities. Risk *entanglement* is a refinement of market incompleteness. We define risks as *entangled* if there exists at least one risk (diffusion or jump process) that affects the traded asset space but cannot be singly traded or replicated by a portfolio of traded assets. In contrast, we define risks as *completely disentangled* if there are sufficiently many non-redundant traded assets such that every risk that

---

[1]Details to Figure 1 are in section 4.

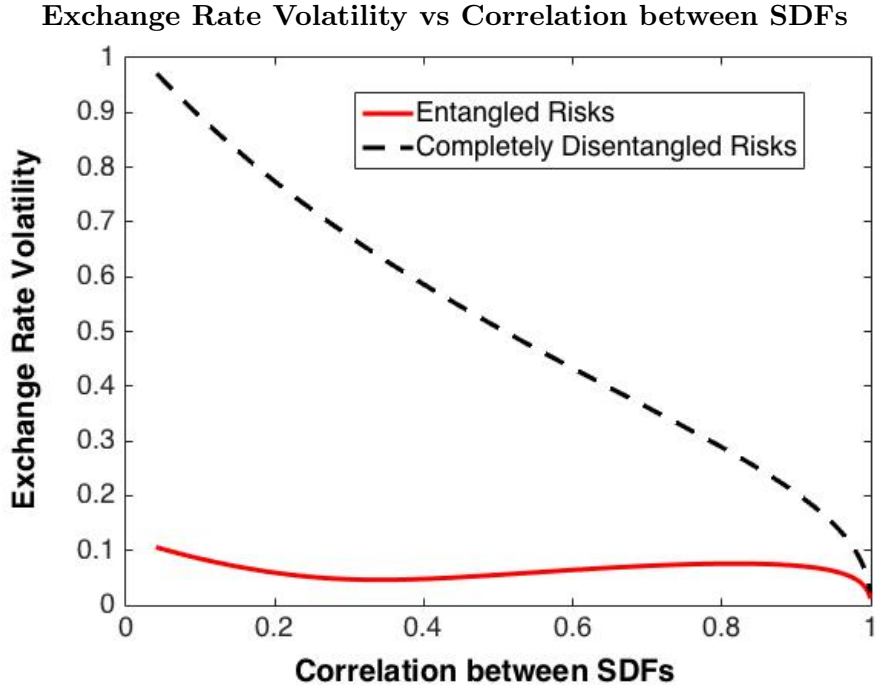**Exchange Rate Volatility vs Correlation between SDFs**

Figure 1: For details we refer to Section 4.

affects asset markets is singly traded by a portfolio.[2] It is unlikely that there are enough traded assets to completely span all risk sources in international financial markets and we expect risks to be entangled in reality.

Besides the implications for the international correlation puzzle, entangled risks yield further interesting and surprising results in FX markets. Most importantly, we find that if risks are entangled: (i) there exist multiple pricing-consistent exchange rates and (ii) every exchange rate loads on idiosyncratic risks if these idiosyncratic risks are entangled.[3] In stark contrast, if risks are completely disentangled: (i) there is always a unique pricing-consistent exchange rate and (ii) the exchange rate is only exposed to systematic risks.[4]

We start with a traded asset space denominated in the home currency and a home country SDF such that there is no arbitrage. We further take a foreign country SDF as given and endogenously solve for a pricing-consistent exchange rate – units of the foreign currency per unit of the home

---

[2]The requirement of complete disentanglement of risks is less stringent than the concept of complete markets. In contrast to complete markets, it does not require that investors can singly contract on every risk in the economy (i.e., all risks that affect traded assets or SDFs) but only the risks in the traded asset space.

[3]We define idiosyncratic risks as risks that are not priced in either country, i.e., no SDF loads on these risks.

[4]We define systematic risks as risks that are priced in at least one country, i.e., at least one SDF loads on these risks.

currency – such that the asset space denominated in the foreign currency does not permit arbitrage either. Home and foreign investors have access to the same traded assets but denominated in different currencies, i.e., gross returns to foreign investors are equal to gross returns to home investors multiplied by the gross return of the exchange rate.

Intuitively, given an exchange rate, the asset return space denominated in the home currency uniquely implies the return space denominated in the foreign currency. Moreover, in a general jump-diffusion setting with entangled risks, the two return spaces (denominated in the home and the foreign currency) are *distinct*. But if the exchange rate itself is endogenously implied by no-arbitrage pricing as in our analysis, it is also endogenous to, and jointly determined with, the asset return space denominated in the foreign currency. Because the presence of entangled jumps makes two return spaces distinct, there exist *multiple* exchange rates, each of which is endogenously pricing-consistent. With a numerical calibration, we illustrate that some of the pricing-consistent exchange rates are smooth while SDFs are hardly correlated.

We further show that in the special cases of either pure-diffusion risks or completely disentangled risks, the space spanned by assets denominated in the foreign currency always coincides with the space spanned by assets denominated in the home currency. Consequently, the exchange rate is unique in these two settings. Accordingly, the presence of both jumps and entanglement is essential to derive multiple pricing-consistent exchange rates. Several papers in the literature have pointed out the relevance of jump risks in international financial markets, though for different reasons than the novel concept of risk entanglement in our analysis (Backus et al., 2011; Brunnermeier et al., 2008; Burnside et al., 2011; Gavazzoni et al., 2013; Farhi et al., 2014; Farhi and Gabaix, 2014).

On a more technical note, we introduce a new portfolio approach to determine pricing-consistent exchange rates. We assume that risk-free bonds in both currencies are traded. Tradability of the foreign bond by home investors means that there exists a portfolio which replicates the return of the foreign bond denominated in the home currency. The foreign bond return denominated in the home currency is equal to the foreign risk-free gross return (denominated in the foreign currency) divided by the gross return of the exchange rate. Accordingly, the inverse of the exchange rate can be represented by a linear combination ("portfolio weights") of traded assets denominated in the home currency. It is then our task to solve for these portfolio weights after substituting the portfolio representation of the inverse of the exchange rate into the no-arbitrage pricing equation of foreign investors. In particular, we have one pricing equation and one unknown portfolio weight

associated with every risky asset. This yields a system of well-behaved polynomial equations with multiple solutions and thus, multiple pricing-consistent exchange rates. In general, the polynomial equations are of a higher order than 1 because the pricing equations are linear in the exchange rate but only the inverse of the exchange rate is linear in the unknown portfolio weights. Exceptions are the settings of completely disentangled risks in which the system becomes completely decoupled with a single solution, and the pure-diffusion setting in which by virtue of Itô's lemma the system becomes linear with a single solution.

The no-arbitrage pricing approach relating the exchange rate to the ratio of countries' SDFs has been employed in the international asset pricing literature at least since Saa-Requejo (1994). For incomplete market settings with pure-diffusion risks, this relationship holds with SDFs being replaced by their projections onto the traded asset space (e.g., Brandt et al. (2006), or Backus et al. (2001)), following the single-country projection approach of Hansen and Jagannathan (1991). Burnside and Graveline (2012) prove an intriguing impossibility result that, when markets are incomplete, the pricing-consistent exchange rate cannot be identified with the ratio of SDF projections in general. We observe that a reason giving rise to this impossibility result is the use of gross SDF growths in the projection construction. We instead relate our results to the projections of net SDF growths.[5]

Recent works by Bakshi et al. (2015) and Lustig and Verdelhan (2015) employ incomplete market settings to address the correlation puzzle by Brandt et al. (2006). These papers start with highly correlated SDFs, which price financial assets and imply a reasonable exchange rate volatility. They, then, add unspanned noises to the SDFs to reduce the correlation between them. Bakshi et al. (2015) discipline the amount of this unspanned noise by limiting the reward-to-risk ratio which a hypothetical asset written on this noise could earn. Lustig and Verdelhan (2015) conclude that the noise necessary to resolve the correlation puzzle is unreasonably large in their setting and is at odds with empirical currency risk premia. In contrast, we use risk entanglements to weaken the link between the exchange rate volatility and the correlation of SDFs. In addition, we provide two additional novel results that there exist multiple pricing-consistent exchange rates and idiosyncratic risks affect exchange rates if these risks are entangled.

There is a vibrant and large literature addressing the determination of exchange rates in structural equilibrium settings, in which real exchange rates are related to country-specific preference-

---

[5]Though the current paper's focus is not on SDF projectors (the paper's primary pricing objects are the exogenously given full SDFs), the distinction between projections of gross versus net SDF growths is important. In particular, by construction, whereas projections of gross SDF growths do not, projections of net SDF growths do price asset returns correctly in respective currencies.

based consumption baskets. Employing incomplete markets, to mention a few, Zapatero (1995) explicitly derives how exchange rates load on fundamental country-specific and global risks in a two country setting with fully integrated and independently incomplete markets, Sarkissian (2003) constructs an international version of Constantinides and Duffie (1996)'s incomplete-market asset pricing model, Dumas et al. (2003) and Chaieb and Errunza (2007) analyze the segmentation of country-specific asset markets, Pavlova and Rigobon (2007) feature demand shocks and multiple goods to relate exchange rates and asset prices in equilibrium, and Favilukis et al. (2015) employ market incompleteness to explain carry trade profits through imperfect risk sharing in equilibrium. Within a complete market framework, Colacito and Croce (2011), Colacito et al. (2015) and Bansal and Shaliastovich (2012) offer risk-based rationales for exchange rate movements and currency premia employing rich features of non-time separable preferences and long-run risk dynamics, and Stathopoulos (2016) addresses the international correlation puzzle employing habit formations. While the current paper takes full SDFs as exogenously given and primarily studies their possible constraints on the exchange rate, these full SDFs and their characteristics can only arise from explicit structural considerations of the literature. In Maurer and Tran (2016), we abstract from full (structural) SDFs and instead adopt market-based pricing kernels constructed purely from asset prices. Therein, the risk entanglement is also found to be highly relevant to exchange rate dynamics, but only when it affects assets in FX markets (not other markets). Structural settings incorporating risk entanglements offer novel perspectives in modeling risks and real exchange rates, and are subject of our future research.

The paper is organized as follows. Section 2 lays out a generic protocol to determine pricing-consistent exchange rates via a portfolio representation (no-arbitrage) approach. In a jump-diffusion incomplete-market setting, Section 3 delivers key results of the paper. In particular, Section 3.3 defines completely disentangled risks and demonstrates the uniqueness of the pricing-consistent exchange rate when incomplete markets are subject to these risks. Section 3.4 defines entangled risks and demonstrates the multiplicity of pricing-consistent exchange rates when incomplete markets are subject to these risks. Section 4 models risk entanglements to calibrate a smooth pricing-consistent exchange rate given volatile and modestly-correlated country-specific SDF growths. Section 5 concludes. Appendices A, B, C provide technical proofs and further supporting materials that have been omitted in the main text.

2 No-arbitrage Determination of Exchange Rates

In this section, we present a general no-arbitrage setting of international finance and provide assumptions needed to determine the exchange rate when asset (i.e., financial) markets are incomplete. The exchange rate is constructed to consistently price every traded asset in every denomination currency while preserving the law of one price at all time.

We consider the spot exchange rate e_t between any two countries H (home) and F (foreign). Our exchange rate convention is that,

$$e_t \text{ units of currency } F \text{ buy one unit of currency } H \text{ at time } t. \tag{1}$$

The absence of arbitrage implies the existence of stochastic discount factors (SDFs) $M_{H,t}$, $M_{F,t}$ which price assets in respective currencies H and F. Let Y denote a traded asset in international financial markets (also referred to as asset markets henceforth), and $\frac{Y_{t+dt}}{Y_t}$ the gross return *denominated in the home currency* H on asset Y. It then follows that $\frac{e_{t+dt}}{e_t} \frac{Y_{t+dt}}{Y_t}$ is the gross return denominated in the foreign currency F on the same asset Y. The asset can be priced in the home and the foreign currency.[6] Assuming frictionless international asset markets, either pricing route (denomination in the home or the foreign currency) must give the same price when express in the same currency. Therefore, the law of one price implies,

$$E_t \left[\frac{M_{Ht+dt}}{M_{Ht}} \frac{Y_{t+dt}}{Y_t} \right] = E_t \left[\frac{M_{Ft+dt}}{M_{Ft}} \frac{e_{t+dt}}{e_t} \frac{Y_{t+dt}}{Y_t} \right] = 1. \tag{2}$$

Assumptions: To specify the no-arbitrage determination of exchange rates in the approach above, we first explicitly state two customary but important assumptions.

1. **Assumption A1** – *Symmetric and fully integrated international asset markets: If an asset Y_t is traded in a country, it is traded in all countries without frictions.*

2. **Assumption A2** – *Tradability of country-specific risk-free bonds: Short-term risk-free bonds of every country are traded in international asset markets.*[7]

---

[6] In the latter case, we need to convert the asset's future payoffs (using the future exchange rate) into the foreign currency and price them using the foreign SDF. The price obtained is in the foreign currency, and can be converted to the home currency using the current exchange rate.

[7] Throughout the paper, bonds refer to country-specific risk-free money market accounts that pay short-term risk-free rates. A country I's bond price B_{It} (in I's currency) satisfies $dB_{It} = B_{It} r_{It} dt$, where r_{It} is the instantaneously risk-free rate (i.e., short rate) for the infinitesimal time period from t to $t + dt$.

Intuitively, Assumption A1 states that if a cash flow is originated and traded in a country, investors from all other countries can also trade this cash flow as long as they can convert it back and forth between the originated currency and their home currencies. The pricing of any traded asset Y_t in the home and the foreign currency is given by the respective Euler equation in (2). This is in difference with the recent literature studying effects of market incompleteness on the exchange rate dynamic; e.g., home and foreign investors do not trade identical sets of assets (due to some frictions) in Lustig and Verdelhan (2015). Note that our assumption on the international tradability of assets is not tantamount to an assumption of complete international asset markets, because the set of all traded assets might not span the space of innovations to SDFs M_{Ht}, M_{Ft} (see the characterization of market completeness below). In fact, the current paper features predominantly incomplete asset markets while maintaining Assumption A1 throughout. We view our assumption on the symmetric international tradability of assets as natural, in particular, when considering developed countries.

Assumption A2 simply models after another innocuous feature that investors in a country can participate in FX markets and, through them, in the short-term lending and borrowing of foreign currencies. Derivatives on exchange rates might also suffice to replicate plain-vanilla foreign short-term debt in case the latter is not directly accessible to home investors. In practice, it is this feature that underlies the viability and popularity of currency carry trades. In the current paper's no-arbitrage setting, Assumption A2 gives rise to an analytical representation of the exchange rate that can be adapted to any stochastic economic model to systematically determine pricing-consistent exchange rates.

Procedure: We now formalize the no-arbitrage procedure to determine the exchange rate.[8] Throughout, we assume that country-specific SDFs are distinctly specified, and given as exogenous pricing operators.[9]

Protocol 1 (No-arbitrage Determination of the Exchange Rate)

>   **Step 1**: *We first take as exogenously given, (i) the set of traded assets, as well as their return processes $\frac{Y_{t+dt}}{Y_t}$ denominated in the home currency,[10] and (ii) distinctly specified SDFs M_{Ht},*

---

[8]The no-arbitrage approach to determine the exchange rate described here is known and employed in the literature at least since Saa-Requejo (1994).

[9]Structural international asset pricing models may link M_{It} to the marginal utility of country I's representative investor. In the current paper, we do not address this linkage, but instead focus on the no-arbitrage international pricing of traded assets given country-specific SDFs.

[10]The choice of denomination currency is non-material, so we conventionally choose it to be the home currency.

M_{Ft} of countries involved.

Step 2: *We then determine the exchange rate process e_t endogenously on the requirements that, (a) it prices traded assets consistently across currency denominations (upholding law of one price (2)), and (b) a country's risk-free bond remains a traded asset to the other country's investors (after the bond's payoff is converted to the other currency).*

It turns out that Assumption A2 on the tradability of risk-free bonds can be exactly mapped into an analytical requirement on exchange rates as we show next.

Portfolio Representation of the Exchange Rate

For concreteness, we adopt the following notations for the traded assets to *home investors*,

$$\{Y\} \equiv \text{set of all } risky \text{ assets}, \qquad B_H \equiv \text{home bond with return } \frac{B_{H,t+dt}}{B_{H,t}} = 1 + r_H dt. \quad (3)$$

Therefore, the set of *all* traded assets to home investors is $\{B_H, Y\}$.[11] Assumption A2 within Protocol 1 of the exchange rate determination can be formalized by stipulating that the foreign bond, which is risky to home investors, is in the set $\{B_H, Y\}$ of *all* traded assets,

$$\frac{B_{F,t+dt}}{B_{F,t}} \frac{e_t}{e_{t+dt}} = \left(1 - \sum_{Y \in \{Y\}} \alpha_Y\right) \frac{B_{H,t+dt}}{B_{H,t}} + \sum_{Y \in \{Y\}} \alpha_Y \frac{Y_{t+dt}}{Y_t},$$

where $\frac{B_{F,t+dt}}{B_{F,t}} = 1 + r_F dt$ is the gross return on the foreign bond in the foreign currency. Hence, the above expression on the gross return on the foreign bond to home investors can be rewritten as a *portfolio representation* for the exchange rate,

$$\frac{e_t}{e_{t+dt}} = \frac{1}{1 + r_F dt} \left\{ \left(1 - \sum_{Y \in \{Y\}} \alpha_Y\right) \frac{B_{H,t+dt}}{B_{H,t}} + \sum_{Y \in \{Y\}} \alpha_Y \frac{Y_{t+dt}}{Y_t} \right\}. \quad (4)$$

The right-hand side of the above equation simply expresses the inverse of the exchange rate growth as a portfolio representation constructed from all traded returns under home investors' perspectives,

---

For the endogenous determination of the exchange rate from SDFs, we evidently cannot take, as exogenous inputs, the returns on the same assets in *multiple* currencies (due to an over-identification issue).

[11]To foreign investors, the set of all traded assets is identical to $\{B_H, Y\}$ (Assumption A1), though their returns differ when denominated in home and foreign currencies. In particular, the home bond is not risk-free to foreign investors and vice versa, because of uncertain exchange rate movements.

with weight α_Y on respective asset Y.[12] The representation (4) can also be seen as a differential but explicit formulation of the relationship between the exchange rate and ratio of countries' SDFs in incomplete markets (see Appendix B) . The gist of the no-arbitrage construction of the exchange rate (Protocol 1) now concretely turns into the determination of weights $\{\alpha_Y\}$ in (4) such that the Euler pricing equations (2) hold for every traded asset $Y \in \{Y\}$, and in every currency denomination.

Completeness of Asset Markets

The characterization of international asset market completeness in our study is as follows. Given countries' exogenous SDFs together with a set of traded assets (accessible to every investor in the world), markets are complete if every risk, which affects either SDFs or the traded asset payoffs, can be replicated by a portfolio of traded assets. In particular, all risks that matter for (and are priced by) investors, i.e., the risks impacting countries' SDFs, can be completely hedged by trading respective replicating portfolios.

Broadly defined, then, asset markets are incomplete in our study whenever the above complete market characterization does not hold. Thus, markets are incomplete when some risks affecting SDFs cannot be replicated by forming a portfolio of traded assets. Furthermore, markets are also incomplete when some risks affecting asset payoffs can not be individually replicated by any portfolio of traded assets because, e.g., these risks are coupled with one another or with other risks in asset markets. These refinements of market incompleteness turn out to be crucial for FX market settings in the presence of risks of discontinuous nature (jumps).[13]

3   No-arbitrage Determination of Exchange Rates

We now implement the no-arbitrage determination of exchange rates along Protocol 1 for generic incomplete-market settings. The market incompleteness arises from continuous and discontinuous movements (diffusion-jump) in both asset payoffs and SDFs. Jumps in exchange rates are not only an important feature documented in FX data, but also an eminent priced risk in FX markets as pointed out by Brunnermeier et al. (2008) and Burnside et al. (2011). For such generic settings, the portfolio representation of the exchange rate (4) proves to be a highly analytically convenient

---

[12]Note that the inverse of per-home-currency exchange rate's growth is the per-foreign-currency exchange rate's growth.

[13]Analysis and findings concerning jump risks in FX markets of various degrees of incompleteness are topics of Section 3.4.

approach.

3.1 Jump-Diffusion Setup

SDFs and Return Processes

We take return processes (denominated in the home currency) and SDFs as exogenously given. In a jump diffusion setting, SDF processes read,

$$\frac{M_{It+dt}}{M_{It}} = 1 - r_{It}dt - \eta_{It}^T dZ_t + \sum_{i \in \mathcal{J}_I} \left(e^{\Delta_i I \times d\mathcal{N}_{it}} - 1 \right) - \sum_{i \in \mathcal{J}_I} \lambda_i dt \left(e^{\Delta_i I} - 1 \right), \tag{5}$$

$$\text{with} \quad M_{I0} = 1, \quad t \in [0, \infty), \quad I \in \{H, F\}, \quad d\mathcal{N}_{it} \in \text{Poisson}(\lambda_i),$$

where d-dimensional standard Brownian motion Z_t captures d independent diffusion risks in our setting. There are j different and uncorrelated types of jump risks,[14] each denoted by an index $i \in \{1, \ldots, j\}$. A jump risk of particular type i is characterized by a discrete random variable $d\mathcal{N}_{it}$ having a Poisson distribution with arrival intensity λ_i. Accordingly, within an infinitesimal time interval $(t, t + dt)$, the respective Poisson counter $d\mathcal{N}_{it}$ takes value one with probability $\lambda_i dt$, and zero with probability $1 - \lambda_i dt$. Scalar $\Delta_i I$ denotes the discontinuous change (i.e., jump size) in country I's SDF growth when a jump of type i occurs. Vector η_{It} of d dimensions denotes the prices of diffusion risks in respective country I. The expected growth of SDF M_I (5) is the additive inverse of the risk-free rate $-r_I$ because M_I prices the risk-free bond of the respective country I.[15]

It suffices to demonstrate all key findings of this section in the simplest jump-risk setting with constant arrival intensities and constant jump sizes. In the above equation, $\mathcal{J}_I$ denotes the set of jump risks priced by country I's investors,

$$\mathcal{J}_I \equiv \{i : \text{ jump of type } i \text{ affects country } I\text{'s SDF } M_I\}. \tag{6}$$

Similarly, (cum-dividend) asset returns read,

$$\frac{Y_{t+dt}}{Y_t} = 1 + \mu_{Yt}dt + \sigma_{Yt}^T dZ_t + \sum_{i \in \mathcal{J}_Y} \left(e^{\Delta_i Y \times d\mathcal{N}_{it}} - 1 \right) - \sum_{i \in \mathcal{J}_Y} \lambda_i dt \left(e^{\Delta_i Y} - 1 \right), \tag{7}$$

---

[14]Correlated jump types can be decomposed into and constructed from uncorrelated jump types.
[15]Our analysis holds conditional on time t. Thus, r_{It}, η_{It}, $\Delta_i I$ and λ_i are adapted stochastic processes, i.e., we only need that they are known at time t.

where scalar μ_{Yt} and $d-$dimensional vector σ_{Yt} denote asset Y's expected return and return volatility respectively. Scalar $\Delta_i Y$ denotes the jump size in asset Y's return associated with a jump of type i.[16] Similar to $\mathcal{J}_I$ in (6),

$$\mathcal{J}_Y \equiv \{i : \text{ jump of type } i \text{ affects the return on asset } Y\}, \tag{8}$$

We conventionally include explicit compensation terms for each jump in SDFs and asset returns. Hence, drift terms of these processes fully reflect the effects of both continuous and discontinuous movements in the economy. Specifically, the drift μ_{Yt} in (7) incorporates asset Y's compensated return for its loadings on both diffusion and jump risks. Henceforth, we will drop time indices whenever such an omission does not create ambiguities.

The Exchange Rate

We determine the exchange rate process endogenously under the constraint of Assumption A2. Substituting asset returns (7) into the portfolio representation (4) of the exchange rate, then applying Itô's lemma for jump-diffusion processes yields an expression for the exchange rate growth,[17]

$$\frac{e_{t+dt}}{e_t} = 1 + \mu_e dt + \sigma_e^T dZ_t + \sum_{i \in \mathcal{J}_{\{Y\}}} \left(e^{\Delta_i e \times d\mathcal{N}_{it}} - 1 \right), \tag{9}$$

where drift μ_e, volatility σ_e, and jump size $\Delta_i e$ (associated with jumps of type i) of the exchange rate are respectively,

$$\mu_e = r_F - \left[1 - \sum_{Y \in \{Y\}} \alpha_Y \right] r_H + \sigma_e^T \sigma_e - \sum_{Y \in \{Y\}} \alpha_Y \left[\mu_Y - \sum_{i \in \mathcal{J}_Y} \lambda_i \left(e^{\Delta_i Y} - 1 \right) \right],$$

$$\sigma_e = - \sum_{Y \in \{Y\}} \alpha_Y \sigma_Y, \qquad e^{\Delta_i e \times d\mathcal{N}_{it}} - 1 \equiv \frac{1}{1 + \sum_{Y \in \mathcal{Y}_i} \alpha_Y \left(e^{\Delta_i Y \times d\mathcal{N}_{it}} - 1 \right)} - 1. \tag{10}$$

In the above expressions, $\{Y\}$ denotes the set of risky traded assets to home investors (3), $\mathcal{J}_Y$ the set of jump types pertaining to asset Y (8), and $\mathcal{J}_{\{Y\}}$ denotes the set of all jump types in the asset

---

[16]Similar to the moments of the SDFs, μ_{Yt}, σ_{Yt} and $\Delta_i Y$ are adapted stochastic processes and known at time t.

[17]Substituting (7) into (4) only yields an expression for the reciprocal of the exchange rate growth $\frac{e_t}{e_{t+dt}}$. We need to apply Itô's lemma to yield the multiplicative inverse of this ratio to obtain the proper exchange rate growth $\frac{e_{t+dt}}{e_t}$.

return space,

$$\mathcal{J}_{\{Y\}} \equiv \cup_{Y \in \{Y\}} \mathcal{J}_Y = \{i : \exists Y \in \{Y\} \text{ such that jump of type } i \text{ affects the return on asset } Y\},$$
(11)

and $\mathcal{Y}_i$ denotes the set of all assets affected by the jump of type i,

$$\mathcal{Y}_i = \{Y : \text{ such that asset } Y\text{'s return is affected by the jump of type } i\}.$$

Intuitively, because the exchange rate is being endogenously constructed from asset returns, the jump size $\Delta_i e$ in the exchange rate pertaining to a specific type i is a function of the jump sizes $\Delta_i Y$ of all assets pertaining to that type. Several observations are in order. First, by virtue of the portfolio representation of exchange rate volatility σ_e in (10), the tradability of countries' bonds (Assumption A2) clearly implies that diffusion risks impacting the exchange rate are identical to those impacting asset returns denominated in the home currency.[18] Consequently, because the volatility of asset Y's return in the foreign currency is $\sigma_{eY} = \sigma_e + \sigma_Y$, we have the following simple result pertaining exclusively to diffusion risks in the model.[19]

Remark 1 *The tradability of risk-free bonds (Assumption A2) assures that asset return spaces denominated in either currencies are subject to identical diffusion risks. As a results, the two return spaces have identical diffusion subspaces.*

Thus, in the absence of jump risks (i.e., pure diffusion), asset return spaces denominated in home and foreign currencies are identical. Second, any jump that affects returns on traded assets also enters the exchange rate dynamics in general as a result of the no-arbitrage determination of the exchange rate (4). Put differently, the set of jump types that affect the exchange rate is (11),

$$\mathcal{J}_e = \mathcal{J}_{\{Y\}}.$$
(12)

Third, the identity (10) that determines the exchange rate's jump size holds *almost surely*, i.e., for

---

[18]The statement of "identical diffusion risks" does not simply mean that the the two sets of Brownian motions (affecting asset returns in the home currency, and the exchange rate, respectively) are identical. It states a stronger result that the diffusion of the exchange rate growth $\sigma_e^T dZ_t$ can be perfectly replicated by the diffusion $\sigma_Y^T dZ_t$ of asset returns denominated in the home currency.

[19]A corresponding result concerning jump risk impacts on asset returns denominated in two currencies is markedly different and much richer. We defer a detailed analysis to Remarks 2 and 3, after the key concept of risk entanglement has been introduced.

the jump count being restricted to $d\mathcal{N}_{it} \in \{0, 1\}$.[20] We also note that, jumps in the exchange rate (9) do not appear in a compensated form. It is its inverse, $(1+r_F dt)\frac{e_t}{e_{t+dt}}$ (which is a portfolio return, according to (4)), that has a compensated form. That is, this inverse inherits the compensated form from the constituent asset returns in the replicating portfolio (4).

The Pricing of Risk-Free Bonds

We first observe that the pricing of the short-term risk-free bond B_I in the currency of the bond-issuing country I,

$$1 = E_t\left[\frac{M_{I,t+dt}}{M_{I,t}}\frac{B_{I,t+dt}}{B_{I,t}}\right] = E_t\left[\frac{M_{I,t+dt}}{M_{I,t}}(1 + r_I dt)\right], \qquad I \in \{H, F\},$$

is tantamount to the SDF M_I's expected growth rate being $-r_I$, and thus, is fully accounted for by process (5). Similarly, the pricing of the foreign risk-free bond B_F in the home currency,

$$1 = E_t\left[\frac{M_{H,t+dt}}{M_{H,t}}\frac{e_t}{e_{t+dt}}\frac{B_{F,t+dt}}{B_{F,t}}\right] = E_t\left[\frac{M_{H,t+dt}}{M_{H,t}}\frac{e_t}{e_{t+dt}}(1 + r_F dt)\right],$$

is automatically satisfied because the foreign bond's gross return to home investors, $\frac{e_t}{e_{t+dt}}(1+r_F dt)$, is a proper portfolio return (4), every constituent return of which satisfies a separate Euler equation of its own.

The pricing of the home risk-free bond B_H in the foreign currency, $1 = E_t\left[\frac{M_{F,t+dt}}{M_{F,t}}\frac{e_{t+dt}}{e_t}(1 + r_H dt)\right]$, can be rewritten as the premium to foreign investors on the home bond,

$$(\mu_e + r_H) - r_F = \sigma_e^T \eta_F \quad - \sum_{i\in\left(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_F\right)} \lambda_i\left(e^{\Delta_i F} - 1\right)\left(e^{\Delta_i e} - 1\right) - \sum_{i\in\mathcal{J}_{\{Y\}}} \lambda_i\left(e^{\Delta_i e} - 1\right), \quad (13)$$

where $\mathcal{J}_{\{Y\}}$ is the set of all jump types in the asset return space (11), and

$$\mathcal{J}_{\{Y\}} \cap \mathcal{J}_I \equiv \{i : \text{ jump of type } i \text{ affects both some asset } Y\text{'s return and country } I\text{'s SDF } M_I\}.$$
$$(14)$$

Evidently, the left-hand side of (13) is the excess return (premium) on the home bond to foreign investors, who earn a return on the home currency μ_e on top of bond interest r_H. This premium is intuitive, since in our exchange rate convention (1), a decrease in e_t is tantamount to an appreci-

---

[20]The probability that multiple Poisson jumps ($d\mathcal{N}_{it} \geq 2$) take place within any infinitesimal time period $(t, t+dt)$ is of order $\mathcal{O}\left(dt^2\right)$ or smaller, thus is identically zero in the mean-square convergence limit.

ation of the foreign currency. Foreign investors holding the home bond are exposed exclusively to exchange rate risks. Specifically, the return premium to foreign investors compensates for (i) loadings on diffusion risk (when the diffusion moves the foreign currency's value and the foreign SDF in the same direction, $\sigma_e^T \eta_F > 0$), and (ii) loadings on jump risk (when common jumps move the foreign currency's value and the foreign SDF in the same direction, $\left(e^{\Delta_i F} - 1\right)\left(e^{\Delta_i e} - 1\right) < 0$).[21] Furthermore, only jump risks associated with both the exchange rate and the foreign SDF is priced in the home bond premium (13) to foreign investors.[22]

The Pricing of Risky Assets

The pricing of risky asset Y (7) in the home currency, $1 = E_t\left[\frac{M_{H,t+dt}}{M_{H,t}}\frac{Y_{t+dt}}{Y_t}\right]$, implies the premium on this asset to home investors,

$$\mu_Y - r_H = \sigma_Y^T \eta_H \quad - \sum_{i \in (\mathcal{J}_Y \cap \mathcal{J}_H)} \lambda_i \left(e^{\Delta_i H} - 1\right)\left(e^{\Delta_i Y} - 1\right), \tag{15}$$

where $\mathcal{J}_Y \cap \mathcal{J}_H$ denotes the set of jump types common to both asset Y's returns and the home SDF (14). The intuition is similar to that underlying (13), with first and second terms on the right-hand side of (15) compensating home investors for taking diffusion and jump risks respectively. Again, only jump risks associated with both asset Y's returns and the home SDF is priced in the return premium (15).

The pricing of the same risky asset Y in the foreign currency, $1 = E_t\left[\frac{M_{F,t+dt}}{M_{F,t}}\frac{e_{t+dt}}{e_t}\frac{Y_{t+dt}}{Y_t}\right]$, can be rewritten as the excess return (premium) on this asset to foreign investors,

---

[21]We recall from expression (5) that M_{Ft} decreases with $\eta_F^T dZ_t$ and increases with $e^{\Delta_i F}$, and from (9) that e_t increases with both $\sigma_e^T dZ_t$ and $e^{\Delta_i e}$. Furthermore, jump types affecting the exchange rate, or the currency value, are the same as those affecting the asset return space, as seen in (12).

[22]We observe that the last term in (13) is a compensation term and is purely mechanical. It simply accounts for, and neutralizes, the non-material fact that the exchange rate's drift μ_e on the left-hand side of (13) also includes this identical compensation, see (9) and the discussion following (11).

$$(\mu_e + \mu_Y) - r_F = \left(\sigma_Y^T + \sigma_e^T\right)\eta_F - \sigma_Y^T\sigma_e - \sum_{i\in(\mathcal{J}_Y\cap\mathcal{J}_F)}\lambda_i\left(e^{\Delta_i F + \Delta_i Y + \Delta_i e} - 1\right)$$

$$- \sum_{i\in(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_F\setminus\mathcal{J}_Y)}\lambda_i\left(e^{\Delta_i F + \Delta_i e} - 1\right) - \sum_{i\in(\mathcal{J}_Y\setminus\mathcal{J}_F)}\lambda_i\left(e^{\Delta_i Y + \Delta_i e} - 1\right) \qquad (16)$$

$$+ \sum_{i\in(\mathcal{J}_{\{Y\}}\cap\mathcal{J}_F)}\lambda_i\left(e^{\Delta_i F} - 1\right) + \sum_{i\in\mathcal{J}_Y}\lambda_i\left(e^{\Delta_i Y} - 1\right) - \sum_{i\in(\mathcal{J}_{\{Y\}}\setminus\mathcal{J}_Y\setminus\mathcal{J}_F)}\lambda_i\left(e^{\Delta_i e} - 1\right),$$

where moments of the exchange rate μ_e and σ_e are given in (10), and $(\mathcal{J}_Y \setminus \mathcal{J}_F)$ denotes the set of jumps that affect the return on asset Y under consideration but not the foreign SDF M_F. Similarly, $\left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F \setminus \mathcal{J}_Y\right)$ denotes the set of jumps that affect the asset return space and the foreign SDF M_F but not the return on the particular asset Y under consideration, and $\left(\mathcal{J}_{\{Y\}} \setminus \mathcal{J}_Y \setminus \mathcal{J}_F\right)$ denotes the set of jumps that affect the asset return space but neither the return on the particular asset Y under consideration nor the foreign SDF M_F. It is reassuring that, in absence of jumps ($\lambda_i = 0$, $\forall i$, or $\Delta_i C = 0$, $\forall C \in \{H, F, Y, e\}$), excess return (16) reduces to a familiar premium (first two terms on the right-hand side) in pure-diffusion settings.

The premium to foreign investors on the risky asset Y (expressed by the terms on the right-hand side (16)) is intuitive. The term associated with η_F is the premium on (fundamental and exchange rate) diffusion risks ($\sigma_Y + \sigma_e$) borne by foreign investors holding the risky asset. The hedging benefit (i.e., when $\sigma_Y^T\sigma_e > 0$) between asset Y's return denominated in the home currency and the exchange rate reduces the risk of (and thus, the premium on) asset Y's return to foreign investors. When common jumps to all the SDF M_{Ft}, the asset payoff Y_t, and the exchange rate e_t are such that $\Delta_i F + \Delta_i Y + \Delta_i e > 0$ (the 3-simultaneous-jump term $\left(e^{\Delta_i F + \Delta_i Y + \Delta_i e} - 1\right)$ is positive), the risky asset is a net hedge to foreign investors, and acts to lower the premium.[23] Similar hedging benefits explain the reduction in the premium when $\Delta_i F + \Delta_i e > 0$, and $\Delta_i Y + \Delta_i e > 0$ (the 2-simultaneous-jump terms $\left(e^{\Delta_i F + \Delta_i e} - 1\right)$ and $\left(e^{\Delta_i Y + \Delta_i e} - 1\right)$ are positive). The remaining last three terms in (16) arise mechanically from the convention of compensated jumps.[24]

---

[23]For a simple illustration suppose that jump sizes associated with type i are such that, $\Delta_i F > 0$, $\Delta_i Y > 0$, $\Delta_i e > 0$. When the process of type i jumps, the asset payoff (in the home currency) increases, the home currency appreciates and the foreign SDF increases simultaneously. Hence, the payoff to foreign investors is high when these investors highly value this payoff, or the risky asset Y is a hedge to risk from jump i foreign investors are exposed to.

[24]While the convention's consequences do affect the specific expression of the exchange drift μ_e (9), they are non-material and need be neutralized in any pricing equation.

3.2 Exchange Rate Determination

We first observe that, in the absence of arbitrage and other frictions, the premium on any risky asset Y to foreign investors can be decomposed into (i) the premium on that risky asset to home investors and (ii) the premium on the home bond to foreign investors,

$$\underbrace{(\mu_e + \mu_Y) - r_F}_{\text{given in } (16)} = \underbrace{[\mu_Y - r_H]}_{\text{given in } (15)} + \underbrace{[(\mu_e + r_H) - r_F]}_{\text{given in } (13)}. \tag{17}$$

This identity is the basis of the no-arbitrage determination of the exchange rate in the jump-diffusion setting. Substituting into identity (17) the expressions (13), (15) and (16) for respective premia yields a specific expression for the above identity in the jump-diffusion setting. For *each* (and every) risky traded asset Y, this expression reads,

$$\sigma_Y^T \left(\eta_H - \eta_F + \sigma_e \right) + \sum_{i \in (\mathcal{J}_Y \cap \mathcal{J}_F)} \lambda_i e^{\Delta_i F + \Delta_i e} \left(e^{\Delta_i Y} - 1 \right) + \sum_{i \in (\mathcal{J}_Y \setminus \mathcal{J}_F)} \lambda_i e^{\Delta_i e} \left(e^{\Delta_i Y} - 1 \right)$$

$$\tag{18}$$

$$= \sum_{i \in (\mathcal{J}_Y \cap \mathcal{J}_H)} \lambda_i \left(e^{\Delta_i H} - 1 \right) \left(e^{\Delta_i Y} - 1 \right) + \sum_{i \in \mathcal{J}_Y} \lambda_i \left(e^{\Delta_i Y} - 1 \right), \qquad \forall Y \in \{Y\}.$$

Several observations on this key equation are in order. First, taking risk loadings in asset returns (σ_Y and $\Delta_i Y$) and market prices (η_H, η_F, $\Delta_i H$ and $\Delta_i F$) as exogenously given, (18) is an equation of portfolio weights $\{\alpha_Y\}$,[25] which quantify the effect of traded asset returns $\{Y_t\}$ on the constructed exchange rate e_t (9)-(10). If we have N non-redundant risky assets, we have N non-redundant equations of the type (18) (one equation per risky asset). Altogether, they form a system of N equations and N unknowns $\{\alpha_Y\}$ (with $Y \in \{Y\}$) that determines the exchange rate in the no-arbitrage approach (9).

Second, all jump types that affect SDFs M_{Ht} and M_{Ft} but not the asset return space $\{Y_t\}$, drop out from and do not contribute to the determination of the exchange rate, which is consistent with relationship (12). This is because such jumps represent risks unspanned by and decoupled from asset returns, to which the exchange rate belongs (via the tradability of short-term risk-free bonds). All other risks in the asset return space $\{Y\}$, either of continuous (diffusion) or discontinuous (jump) nature, *jointly* determine the solution weights $\{\alpha_Y\}$ in the portfolio representation (4), and thus, the exchange rate. Sections 3.3 and 3.4 below fully characterize the joint implications of the market incompleteness and nature of risks on the determination of the exchange rate.

---

[25] These weights are embedded in σ_e and $e^{\Delta_i e}$ (10).

Third, it is crucial to observe that, while $\sigma_e = -\sum_{Y \in \{Y\}} \alpha_Y \sigma_Y$ (10) is linear in portfolio weights $\{\alpha_Y\}$, the exchange rate's jump sizes $e^{\Delta_i e} = \frac{1}{1 + \sum_{Y \in \mathcal{Y}_i} \alpha_Y (e^{\Delta_i Y} - 1)}$ (10) are non-linear in $\{\alpha_Y\}$.[26] Therefore, the set (18) for all traded assets constitutes a system of N *nonlinear* equations. Consequently, there potentially exist *multiple* solution sets $\{\alpha_Y\}$, and hence, potentially *multiple* exchange rates e_t even when the exchange rate has been explicitly constructed within the traded asset return space. As we briefly discuss below the portfolio representation of the exchange rate (4), system (18) generalizes the standard approach relating the exchange rate with SDFs to incomplete market settings with jump risks.[27]

Our finding of potentially multiple pricing-consistent exchange rates is novel and surprising. Intuitively, jump risks generate distinct asset return spaces when returns are denominated in different currencies (Remarks 2 and 3 below). Therefore, when the exchange rate is an endogenous part pinning down the asset return space in the foreign currency (Protocol 1), there are potentially many exchange rate solutions. *Every* of these solutions is endogenously consistent. Indeed, given any exchange rate solution e, foreign SDF M_F price correctly all asset returns $\{eY_n\}$ denominated in the foreign currency using that exchange rate. The multiplicity, or equivalently the non-linearity of equation system (18) for all traded assets $Y \in \{Y\}$, requires as necessary conditions *both* incomplete markets *and* the presence of sizable jump risks. These conditions, however, are not sufficient to imply the multiplicity of the exchange rate. In this regard, we note that the exchange rate is *uniquely* determined in the no-arbitrage approach under either (i) incomplete markets and pure-diffusion risks (Appendix B), or (ii) complete international asset markets and jump-diffusion risks. Our formal analysis below establishes necessary and sufficient conditions for a multiplicity of the exchange rate to arise.

Fourth, the key equation (18) is implied by the foreign pricing of the home bond (13), and the foreign pricing of risky assets (16).[28] A natural question is whether a solution $\{\alpha_Y\}$ to the nonlinear system (18) satisfies *both* original pricing equations (13) *and* (16). This is an important consistency check. If at least one of these two pricing equations does not hold, the exchange rate

---

[26]This non-linearity is an important feature of disaster risks, for which movements in asset returns $\{\Delta_i Y\}$ are sizable and sudden (large jumps), and the linearity associated with mean-square convergence in pure-diffusion settings, e.g., $\frac{1}{1 + dt + dZ_t} = 1 - \frac{1}{2}dt - \frac{1}{2}dZ_t$, breaks down.

[27]In (incomplete-market) pure-diffusion settings, this relationship is known to simplify to a strict equality between the exchange rate and the ratio of projected SDFs (see Appendix B). In (incomplete-market) jump-diffusion settings, such equality does not hold when risks are entangled in FX markets Maurer and Tran (2016).

[28]We note that the home pricing of risky assets (15) does not involve the endogenous weights $\{\alpha_Y\}$, and thus, does not constitute a validation test for these solution weights. Rather, (15) is a constraint between exogenous asset pricing parameters μ_Y, σ_Y, $\Delta_i Y$.

e_t (9) constructed from portfolio weights $\{\alpha_Y\}$ fails to price the home bond or some other traded asset $Y \in \{Y\}$ correctly (even though this exchange rate e_t is able to deliver the compounded pricing equation (18)). In such a case, the constructed e_t would be evidently inconsistent with asset pricing. The following result rules out this inconsistency.

Proposition 1 *If a set of weights $\{\alpha_Y\}$, and the associated exchange rate e_t constructed from these weights via equations (9)-(10), are a solution to the system of equations (18) for all traded assets $Y \in \{Y\}$, then both pricing equations (13) and (16) hold for such $\{\alpha_Y\}$ and e_t.*

The proof of this proposition is relegated to Appendix A. The consistency result of Proposition 1 establishes the central role of the system (18) (one equation per traded asset) in the no-arbitrage determination of the exchange rate. We summarize the gist of constructing the exchange rate in the no-arbitrage approach, as well as our discussion following (18), in the next theorem.

Theorem 1 *On the premise of Assumption A1 (symmetric and frictionless asset markets to all investors) and Assumption A2 (risk-free bonds of all countries are traded),*

1. *the no-arbitrage construction of the exchange rate e_t (9)-(10) is feasible if and only if the weights $\{\alpha_Y\}$ in these equations solve the system (18) (for all traded assets $Y \in \{Y\}$),*

2. *furthermore, every exchange rate constructed using this approach is pricing-consistent.*

Specifically, in the no-arbitrage determination of the exchange rate, a pricing-consistent exchange rate *both* (i) prices all traded assets consistently across currency denominations *and* (ii) renders all bonds to remain traded assets by upholding the portfolio representation (4) of the exchange rate.

We next investigate the important issue of multiplicity of consistent exchange rates stemming from the non-linearity of the system of equations (18). It turns out that the multiplicity is crucially related to a subtle and novel characterization of asset market completeness in the presence of risks of discontinuous and sizable movements (i.e., jump risks).

3.3 Incomplete Markets with Completely Disentangled Risks

To gain a deeper insight into the pricing of jump risks in incomplete FX markets, we first refine the characterization of the market incompleteness with the following definition.

Definition 1 (Completely Disentangled Risks) *Incomplete markets with completely disentangled risks are an asset market setting, in which,*

1. *markets are incomplete,*[29]

2. *every single risk affecting the traded asset space can be* individually *replicated by a portfolio of traded assets (that is, asset risks are completely disentangled from one another in asset markets).*

Note that the first defining property pertains exclusively to market incompleteness (so it is related to the risk-loading configuration of SDFs), while the second pertains exclusively to disentangled asset risks (so it is not related to SDFs).

For the simplest illustration of a market setting with completely disentangled risks, assume that the asset return space is affected by (i) a d-dimensional diffusion risk $Z_t^T = (Z_{1t}, \ldots, Z_{dt})$, (ii) j types of (uncorrelated) jump risks defined by the respective Poisson counters $(d\mathcal{N}_{1t}, \ldots, d\mathcal{N}_{jt})$, and (iii) some other risks affecting only SDFs (but not asset returns). Then, an incomplete market setting with completely disentangled risks arises if there are $d + j$ primitive risky assets – each loads on one (and only one) of $d + j$ diffusion and jump risks – and a risk-free bond. Note however that such a collection of assets is not primitive. This is because there exist infinitely many equivalent, non-primitive, versions of these $d + j + 1$ risky and risk-free assets (Proposition (2) below).

Completely Disentangled Risks: Discussion and Properties

We emphasize that the "complete disentanglement" concept of asset markets underlying Definition 1 does not require the stringent level of completeness in complete markets (also specified at the end of section 2), but it is more refined than the effective completeness in effectively complete markets. On one hand, in the standard complete-market setting, an asset is available by construction to *singly* replicate every risk in the economy. Thus, the setting with completely disentangled risks is similar to complete markets in the sense that there are enough traded assets so that every risk source in asset markets can be singly replicated by a portfolio of assets. It is, however, less stringent than the setting of complete markets because it only requires the replication of risks in asset markets (but not necessarily risks affecting the SDFs), i.e., the concept of completely disentangled risks does not involve SDFs *à priori*. In other words, in the setting with completely disentangled risks,

---

[29]That is, per characterization of the market completeness given at the end of Section 2; some risks affecting either countries' SDFs or asset payoffs cannot be individually replicated (hedged) by any portfolio of traded assets.

markets are incomplete because some risks affecting SDFs do not affect asset markets. On the other hand, in the effectively complete-market setting, while risks may remain entangled in asset markets, entanglement patterns are identical in the asset return space and in the SDFs so that all risk "packages" that matter for investors' utilities can be effectively hedged. In comparison, the subtle "complete disentanglement" classification of Definition 1 is more detailed in the sense that there are enough traded assets to load singly on every risk, even when risks are entangled in SDFs and promiscuously priced by investors.

Intuitively, assuming incomplete markets with completely disentangled risks is more plausible than assuming the existence of complete markets. For instance, if a sufficient number of financial derivatives is traded, then it is possible that there are enough assets to construct portfolios to replicate every single risk in the traded asset space. An exception concerns the nature of rare disasters, in which case investors may find it too costly to fully account for and contract on such risks (e.g., disaster insurances like far out-of-the-money put options are not perfectly liquid but they are important assets if one would like to hedge jumps of large sizes). It is, however, rather implausible to assume that investors can contract on every single risk which affects the SDFs. There are many examples of labor income, human capital, health, taste or other preference shocks, which directly affect SDFs (i.e., η_{It} or $\Delta_i I$), or shocks to the investment opportunity set/conditional moments of asset returns (r_{It}, μ_{Yt}, σ_{Yt}, $\Delta_i Y$ or λ_i), which indirectly affect SDFs. Such shocks are likely to be unspanned by asset markets but affect SDFs and markets are incomplete.[30] Moreover, frictions can also be practical reasons why investors are unable to contract on important risks and markets are incomplete.

Completely disentangled risks are similar to pure-diffusion risks with respect to an important aspect that both settings foster identical risk space for asset returns denominated in different currencies. Indeed, complete risk disentanglement signifies that to each jump type i there is an asset Y that loads only on that jump risk. Therefore, for each type i, the corresponding jump component in the exchange rate movement (10) in this setting reduces to,

$$e^{\Delta_i e \times d\mathcal{N}_{it}} - 1 = \frac{1}{1 + \alpha_Y \left(e^{\Delta_i Y \times d\mathcal{N}_{it}} - 1 \right)} - 1, \qquad \forall i \in \mathcal{J}_{\{Y\}}.$$

---

[30]Notice that complete disentanglement concerns only risks affecting asset returns in $(t, t + dt)$ and not shocks affecting conditional moments. Thus, in a setting with completely disentangled risks, it may not be possible to contract on every shock to conditional asset return moments but at the same time it may be possible to contract on every possible asset return realization in $(t, t + dt)$ given the information at time t.

Consequently, the jump component of asset Y (which loads singly on type i) when denominated in the foreign currency is,

$$e^{\Delta_i(eY) \times d\mathcal{N}_{it}} - 1 = \frac{e^{\Delta_i Y \times d\mathcal{N}_{it}}}{1 + \alpha_Y \left(e^{\Delta_i Y \times d\mathcal{N}_{it}} - 1\right)} - 1, \qquad \forall i \in \mathcal{J}_{\{Y\}},$$

Similar to (10), above equalities hold almost surely (i.e., for $d\mathcal{N}_{it} \in \{0,1\}$). Clearly, in a market setting with completely disentangled risks, if asset Y's return in the home currency loads singly on the jump risk of type i, so does its return in the foreign currency. Therefore, the complete risk disentanglement is a property invariant to currencies of denomination as long as access to asset markets is symmetric to all investors and countries' risk-free bonds are traded.[31] The following remark recapitulates these results, which generalizes Remark 1 to the setting of completely disentangled risks.

Remark 2 *When risks are completely disentangled in asset markets (Definition 1), the tradability of bonds (Assumption A2) assures that asset return spaces denominated in either currency are subject to identical diffusion-jump risks. As a results, under these premises, the two return spaces are identical.*

Finally, we observe that there are infinitely many configurations of traded assets that can disentangle (i.e., individually replicate) every risk in asset markets. Proposition 2 below shows that all such configurations are equivalent. To be specific, suppose that asset markets collectively load on d uncorrelated diffusion risks and j uncorrelated types of jump risks. Consider the corresponding set $\mathcal{T}$ of $1+d+j$ traded *primitive* assets, which includes a risk-free bond B_I, d pure-diffusion assets $\{X_k\}$ (each loads exactly on one diffusion risk), and j pure-jump assets $\{W_i\}$ (each loads exactly on one type of jump risk). Without loss of generality, we formulate $\mathcal{T}$ from the perspective of investors in country I,

$$\mathcal{T} = B_I \cup \{Y\}, \quad \text{where the set of risky assets} \quad \{Y\} \equiv \{X_k, W_i : k \in \{1, \ldots, d\}; i \in \{1, \ldots, j\}\}. \tag{19}$$

---

[31]When some assets are traded only in certain countries as in Lustig and Verdelhan (2015), or when some countries' bonds are not available for trading, it is possible that asset markets in some countries feature completely disentangled risks, while they do not in other countries.

offers respective primitive returns,[32]

$$\frac{X_{kt+dt}}{X_{kt}} = 1 + \mu_k dt + \sigma_k dZ_{kt}, \qquad \mu_k = r_I + \sigma_k \eta_{Ik},$$

(20)

$$\frac{W_{it+dt}}{W_{it}} = 1 + \mu_i dt + \left(e^{\Delta_i W \times d\mathcal{N}_{it}} - 1\right) - \lambda_i dt \left(e^{\Delta_i W} - 1\right), \qquad \mu_i = r_I - \lambda_i \left(e^{\Delta_i I} - 1\right) \left(e^{\Delta_i W} - 1\right).$$

Obviously, according to Definition 1, risks in asset market $\mathcal{T}$ are completely disentangled. Moreover, we have the following result.

Proposition 2 *Every asset market configuration, in which these risks are completely disentangled, is isomorphic to the one spanned by the $1 + d + j$ primitive assets in set $\mathcal{T}$ (19).*

The proof of this proposition is relegated to Appendix A. The isomorphism formalizes the equivalence between two asset markets: every asset in the first space can be perfectly replicated by a portfolio of assets in the second market and vice versa.

We note that Proposition 2 places the concept of completely disentangled risks in asset markets (Definition 1) on a generic (base-independent) footing. In particular, any economic property that holds for the basic asset market configuration $\mathcal{T}$ remains equally valid for any other asset market configuration in which risks are completely disentangled.

Completely Disentangled Risks: Exchange Rate Determination

Without loss of generality, we now present a key necessary and sufficient condition for the uniqueness of the exchange rate pertaining to the basic set $\mathcal{T}$ (19) of traded assets.[33]

Theorem 2   *1. The system of nonlinear equations (18) has a unique solution $\{\alpha_Y\}$ if and only if risks in asset markets are completely disentangled (Definition 1).[34]*

    2. Accordingly, under complete risk disentanglement, there exists a unique exchange rate e_t (9) constructed according to the no-arbitrage approach specified in Protocol 1.

---

[32]All primitive assets are special cases of the generic asset (7). Therefore, their returns follow from (15), with the home index H replaced by a generic country specification I.

[33]All results remain intact for any other incomplete asset market configuration with completely disentangled risks by virtue of Proposition 2.

[34]Technically, the statement "if and only if" in Theorem 2 applies as long as equation system (18) remains genuinely non-linear. This specification rules out a special set of parametric values, under which equation system (18) reduces to either a degenerate non-linear system or a linear system (both of which possess a single, but special, solution).

3. *The unique exchange rate process,* $\frac{e_{t+dt}}{e_t} = 1 + \mu_e dt + \sigma_e^T dZ_t + \sum_{i \in \mathcal{J}_{\{Y\}}} \left(e^{\Delta_i e \times d\mathcal{N}_{it}} - 1 \right)$ (9) *is given as follows,*

$$
\textit{Jump components:} \quad
\begin{cases}
\textit{For } i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_H \cap \mathcal{J}_F \right) & : \quad \Delta_i e = \Delta_i H - \Delta_i F, \\
\textit{For } i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_H \right) \setminus \mathcal{J}_F & : \quad \Delta_i e = \Delta_i H, \\
\textit{For } i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F \right) \setminus \mathcal{J}_H & : \quad \Delta_i e = -\Delta_i F, \\
\textit{For } i \in \left(\mathcal{J}_{\{Y\}} \setminus \mathcal{J}_H \right) \setminus \mathcal{J}_F & : \quad \Delta_i e = 0.
\end{cases}
\tag{21}
$$

Diffusion and Drift components: $\qquad \sigma_{et} = \eta_{Ft\|} - \eta_{Ht\|},$

$$
\mu_{et} = r_{Ft} - r_{Ht} + \eta_{Ft\|}^T \sigma_{et} - \sum_{i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_H \right)} \lambda_i \left(e^{\Delta_i H} - 1 \right) - \sum_{i \in \left(\mathcal{J}_{\{Y\}} \cap \mathcal{J}_F \right)} \lambda_i \left(1 - e^{\Delta_i F} \right), \tag{22}
$$

where $\eta_{I\|}$ denotes the vector of country I's prices of diffusion risks projected onto the space of asset return risks (denominated in respective currency I).

We observe that the exchange rate does not load on idiosyncratic jump risks (which affect only asset payoffs but not SDFs, $i \in \mathcal{J}_{\{Y\}} \setminus \mathcal{J}_H \setminus \mathcal{J}_F$). The intuition is standard as follows. Because idiosyncratic risks are not priced in either country, returns denominated in either currency do not reflect these risks, and neither does the exchange rate constructed from asset returns across currencies. Important is that risks need be completely disentangled in asset markets so that we can decouple and isolate these idiosyncratic movements from any systematic movements (which affect both asset returns and SDFs), so that the constructed exchange rate only picks up the latter risks. In contrast, Section 3.4 below demonstrates that when risks are entangled in markets, idiosyncratic risks directly enter the exchange rate constructed in the no-arbitrage approach. Otherwise, the constructed exchange rate fully undoes ($\Delta_i e = -\Delta_i F$) every market jump risk priced by the foreign country ($i \in \mathcal{J}_{\{Y\}} \cap \mathcal{J}_F$), and fully adopts ($\Delta_i e = \Delta_i H$) every market jump risk priced by the home country ($i \in \mathcal{J}_{\{Y\}} \cap \mathcal{J}_H$). Therefore, for incomplete markets with completely disentangled risks, the unique exchange rate construct (21)-(22) appears to be a straightforward generalization of the pure-diffusion counterpart to include jump risks (see (34), Appendix B).

Intuitively, when there are enough assets to disentangle every individual risk affecting asset markets in a pair of countries, a respective pair of Euler equations (2) holds effectively for *every individual* priced risk in asset markets.[35] This no-arbitrage pricing requirement imposed individually

---

[35]Originally, Euler pricing equations hold for traded assets. But when risks are completely disentangled in the markets, each individual risk can be proxied by a traded asset (or a portfolio of traded assets), so that we can effectively associate the Euler equation (on the asset) with the individual risk that the asset loads on. Note that

on every market risk (across two denomination currencies) then allows us to individually determine each component of the exchange rate associated with each risk. Because the exchange rate is constructed endogenously within the traded asset market, the complete risk disentanglement in asset markets is all we need to uniquely pin down the exchange rate. In particular, diffusion risks and their pricing can be completely decoupled from jump risks and their pricing.

Proof: The key equation system determining the exchange rate is (18). We take asset markets with completely disentangled risks in the specific configuration $\mathcal{T}$ (19) without loss of generality. To isolate d diffusion risks, we apply equation (18) on each of the d pure-diffusion assets X_k ($k \in \{1, \ldots, d\}$). Because these assets do not load on jump risks ($\Delta_i X_k = 0$, $\forall k, i$), (18) reduces to a $d-$dimensional identity,

$$\eta_{H\|} - \eta_{F\|} + \sigma_e = 0,$$

where $v_\|$ denotes the projected components of a generic vector v onto the asset return space. The above equation delivers the familiar relationship between volatilities of SDFs and the exchange rate in a pure-diffusion setting (see also Appendix B). Combining this identity with the foreign pricing of the home bond (13) yields the exchange rate's expected growth μ_e in (22).

To isolate the j types of jump risks, we surgically apply equation (18) on each (and every) asset W_i that loads only on a single jump type i that affects both home and foreign SDFs ($i \in \mathcal{J}_Y \cap \mathcal{J}_H \cap \mathcal{J}_F$). Such an asset retains only the jump terms associated with type i in (18), or,

$$\lambda_i e^{\Delta_i F + \Delta_i e} \left(e^{\Delta_i W} - 1 \right) = \lambda_i \left(e^{\Delta_i H} - 1 \right) \left(e^{\Delta_i W} - 1 \right) + \lambda_i \left(e^{\Delta_i W} - 1 \right), \qquad i \in \mathcal{J}_Y \cap \mathcal{J}_H \cap \mathcal{J}_F.$$

Canceling common factor $\lambda_i \left(e^{\Delta_i W} - 1 \right)$ from both sides yields $e^{\Delta_i F + \Delta_i e} = e^{\Delta_i H}$, or equivalently the first identity in (21).

Similarly, surgically applying equation (18) on each (and every) asset W_i that loads only on a single jump type i that affects:

  a. the home but not the foreign SDF ($i \in \mathcal{J}_Y \cap \mathcal{J}_H \setminus \mathcal{J}_F$) yields the second identity in (21),

  b. the foreign but not the home SDF ($i \in \mathcal{J}_Y \cap \mathcal{J}_F \setminus \mathcal{J}_H$) yields the third identity in (21),

  c. neither the home nor the foreign SDF ($i \in \mathcal{J}_Y \setminus \mathcal{J}_H \setminus \mathcal{J}_F$) yields the last identity in (21).

Substituting this exchange rate jump size configuration into the home bond premium (13) in the

---

there are not necessarily enough assets to hedge all risks affecting SDFs, so that markets are still possibly incomplete.

foreign currency implies the exchange rate drift μ_e (22). The uniqueness of the exchange rate's components $(\{\Delta_i e\}, \sigma_e, \mu_e)$ satisfying equations (21)-(22) is self-evident. Proposition 1, then, qualifies this uniquely constructed e_t as a pricing-consistent exchange rate. ■

Next, we turn to a richer setting in which asset markets are not only incomplete, but also risks are not completely disentangled. Novel and surprising aspects of the exchange rate dynamics arise in this setting.

3.4 Incomplete Markets with Entangled Risks

To fully complement Definition 1 (completely disentangled market risks) of incomplete markets, we specify the following complementary characterization.

Definition 2 (Entangled Risks) *Incomplete markets with entangled risks are a market setting, in which,*

1. *asset markets are incomplete,*

2. *some risks affecting the traded asset space cannot be* individually *replicated by any portfolio of traded assets, that is, some asset risks are entangled in asset markets.*

Entangled Risks: Discussion and Properties

Intrinsically, the entanglement notion of risks stems from the nature of jump risks. This is because each jump type is unique and two or more jumps (either of same or different types) take place within an infinitesimal time interval with null probability. In contrast, if only diffusion risks are present in the economic setting, these risks are always completely disentangled in asset markets. This is because we can always combine and neatly partition the original diffusion shocks into a set of diffusion risks impacting asset returns and a second set of residual orthogonal risks. Consequently, the linear span of N non-redundant assets, which collectively are affected by d diffusion risks ($N < d$, as markets are supposedly incomplete), can always be transformed into an asset space governed by N redefined diffusion risks.[36]

In sharp contrast to either pure-diffusion risks (Remark 1) or completely disentangled risks (Remark 2), entangled risks give rise to distinct asset return spaces when returns are denominated in

---

[36]Specifically, we linearly combine and partition d original diffusion shocks into two orthogonal subsets. N-dimensional diffusion shocks that entirely span risks in the asset return space, and the residual $(d - N)$-dimensional orthogonal shocks.

different currencies. Intuitively, when risks are entangled in asset markets, the exchange rate's jump size (10) associated with some types i are necessarily proper (irreducible) combinations of jump sizes in multiple assets. As a result, at least some jump components of asset returns denominated in the foreign currency (i.e., returns on $\{eY\}$) are some (other) combinations of jump sizes in multiple assets. Precisely because there are not enough assets loading singly on each jump type (i.e., entanglement), these two sets of combinations are not equivalent. In other words, return spaces on $\{Y\}$ (i.e., the home currency denomination) and $\{eY\}$ (i.e., the foreign currency denomination) are distinct due to risk entanglement. More crucially, this analysis implies that when risk entanglement exists in international asset markets, how N non-redundant assets load on the set of the original j jump types is highly relevant to asset pricing. Similar intuition holds for the entanglement of diffusion and jump risks. Therefore, the concept of risk entanglement crucially matters in jump-diffusion settings. We recapitulate this discussion in the following remark, which corresponds to, but extends beyond, Remark 1 (for pure diffusion) and Remark 2 (for complete risk disentanglement).

Remark 3 *When risks are entangled in asset markets (Definition 2), assuming the tradability of risk-free bonds (Assumption A2), asset return spaces denominated in different currencies are subject to different risk configurations. As a results, under these premises, the two return spaces are distinct.*

Entangled Risks: Exchange Rate Determination

As a corollary to Theorem 2 (by flipping the necessary and sufficient condition therein), the next result follows immediately.

Theorem 3   *1. The system of nonlinear equations (18) has multiple solutions $\{\alpha_Y\}$ if and only if risks are entangled in asset markets (Definition 2).*

2. *Accordingly, under risk entanglement, there exist multiple exchange rates e_t (9) constructed according to the no-arbitrage approach specified in Protocol 1.*

3. *The multiplicity as well as the expressions of pricing-consistent exchange rates are endogenous not only to the assets available to trade, but also to the specific entanglement configuration of risks in asset markets.*

Three novel insights follow from this Theorem, and all of them are crucially related to the entanglement of risks in asset markets (Definition 2).

First, there exist multiple exchange rates that can price assets and their risks consistently across currency denominations. While the explicit inclusion of jump risks with sizable discontinuous movements in our setting unveils a non-linear dynamic for the exchange rate determination, asset markets must be sufficiently incomplete for this non-linear dynamic to result in a multiplicity of pricing-consistent exchange rates. The degree of market incompleteness is exactly quantified by the notion of risk entanglement. Reassuringly, the latter is both a necessary and sufficient condition for the multiplicity of exchange rate solutions. The intuition follows directly from Remark 3. By no-arbitrage (in particular, the portfolio representation (4)), the exchange rate is related to the differential of SDF growths that are in the spaces of asset returns denominated in respective currencies. These asset return spaces are related to one another by the exchange rate, and are distinct in the presence of risk entanglement. Therefore, when the exchange rate is endogenous in this process, there exist potentially many exchange rate solutions, every of which prices all traded assets consistently across currency denominations. In contrast, the exchange rate solution is unique for either pure-diffusion or completely disentangled risks because in such settings, asset return spaces are identical across currency denominations.

Second, idiosyncratic risks (which impact asset payoffs but not the SDF of either country) may impact *all* solutions of pricing-consistent exchange rates. This is a surprising result, given that these idiosyncratic risks are not priced in expected asset returns in either currency, and the exchange rate is constructed from these returns. The reason is that idiosyncratic risks are entangled with systematic risks (which impact asset payoffs as well as SDFs) when markets are sufficiently incomplete and fall into the classification of Definition 2. As a results, there are not enough assets to disentangle each (idiosyncratic as well as systemic) risk affecting asset markets (though idiosyncratic risks are not priced in any country). Therefore, both systematic and idiosyncratic risks, enter the constructed exchange rates via their entanglement. Such a scenario is intuitive and plausible because investors do not have incentives – idiosyncratic risks do not affect their marginal utilities – to create a new market and trade trade idiosyncratic risks.

Third, the set of available traded assets and how risks are entangled in asset markets are critical to the determination of the exchange rate. Plausibly, the exchange rate is endogenous to the risk structure embedded in the space of available traded assets. This feature goes beyond the dependence between asset markets and the exchange rate in pure-diffusion or completely disentangled risk settings. In those settings, the exchange rate is the same regardless of how risks are packaged in traded assets (see Proposition 2). In the presence of risk entanglement, however, the specificity of

this risk packaging into asset markets is highly relevant because it endogenously shapes the asset return space in the foreign currency (which now is distinct from that in the home currency, Remark 3). This endogenous dependence on the risk structure in asset markets then translates into the specificity of all exchange rate solutions.

We illustrate these insights in the following specific settings, in which (for simplicity) there are only two traded assets. In all settings, risks (either pure jump, or diffusion-jump) are entangled. As a result, there are two pricing-consistent exchange rates in each setting. Furthermore, some of our specific examples are constructed to demonstrate explicitly that idiosyncratic risks (either jump or diffusion) affect all solutions of the exchange rate in the no-arbitrage approach.

Scenario 1: Diffusion-Jump Risk Entanglements

We consider a market setting with two countries $I \in \{H, F\}$, and two traded assets. Home investors can trade the home bond B_H and a single risky asset Y. A single diffusion risk (characterized by Brownian motion Z_{1t}) and a single type of jump risk (characterized by Poisson counter $d\mathcal{N}_{1t}$) affect asset markets. From home investors' perspective, these risks are embedded in the traded asset space as follows,[37]

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_{Y1} dZ_{1t} + \left(e^{\Delta_1 Y \times d\mathcal{N}_{1t}} - 1\right) - \lambda_1 dt \left(e^{\Delta_1 Y} - 1\right), \qquad \frac{dB_{Ht}}{B_{Ht}} = r_H dt. \qquad (23)$$

The countries' exogenous SDFs are also affected by the same diffusion and jump risks $\{dZ_{1t}, d\mathcal{N}_{1t}\}$ (thus, $\mathcal{J}_H, \mathcal{J}_F \supset \{1\}$), as well as by other unspanned (diffusion and jump) risks,[38]

$$\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_I^T dZ_t + \sum_{i \in \mathcal{J}_I} \left(e^{\Delta_i I \times d\mathcal{N}_{it}} - 1\right) - \sum_{i \in \mathcal{J}_I} \lambda_i dt \left(e^{\Delta_i I} - 1\right), \qquad I \in \{H, F\}.$$

We observe that this setting features entangled risks of Definition 2 because markets are incomplete and the diffusion risk dZ_{1t} is always entangled with the jump risk of type one $d\mathcal{N}_{1t}$ in asset markets.[39]

We look for the exchange rate of the form (4), with weights α_Y on asset Y and $(1 - \alpha_Y)$ on the

---

[37]The specific return belongs to metaphor (7) adopted in our generic consideration. In (23), $\sigma_{Y1} \in \mathbf{R}$ is a scalar because the diffusion dZ_{1t} affecting asset markets is one-dimensional.

[38]These specific SDFs belong to metaphor (5) adopted in our generic consideration.

[39]From home investors' perspective, B_H does not load on either dZ_{1t} or $d\mathcal{N}_{1t}$. Thus, the home bond cannot be used in a portfolio to separate these two risks from the risky asset Y (23) available to home investors.

bond B_H. Equivalently, quantities in (10) in the current setting simplify respectively to,

$$\sigma_{e1} = -\alpha_Y \sigma_{Y1}, \qquad e^{\Delta_1 e} = \frac{1}{1 + \alpha_Y \left(e^{\Delta_1 Y} - 1\right)}. \tag{24}$$

Substituting the above expressions into (18) yields an equation determining the exchange rate according to our no-arbitrage approach,

$$\sigma_{Y1} \left(\eta_{H1} - \eta_{F1} - \alpha_Y \sigma_{Y1}\right) \; + \; \frac{\lambda_1 e^{\Delta_1 F} \left(e^{\Delta_1 Y} - 1\right)}{1 + \alpha_Y \left(e^{\Delta_1 Y} - 1\right)} \;\; = \;\; \lambda_1 e^{\Delta_1 H} \left(e^{\Delta_1 Y} - 1\right). \tag{25}$$

This quadratic equation in the portfolio weight α_Y has exactly two distinct solutions, $\alpha_Y^{(1)}$, $\alpha_Y^{(2)}$. Notice that weight α_Y can take positive or negative values without restrictions. We make several observations. First, each of the two solutions $\alpha_Y^{(1)}$, $\alpha_Y^{(2)}$ yields one exchange rate process according to (24). It is important to notice that there are exactly two possible exchange rate processes; any linear combination $\hat{\alpha}_Y = \theta \alpha_Y^{(1)} + (1 - \theta) \alpha_Y^{(2)}$ for $\theta \in \mathbf{R} \setminus \{0, 1\}$ is not a valid solution. The reason lies in the non-linearity of equation (25). Second, the set of parameters is robust in the sense that the solutions $\alpha_Y^{(1)}$, $\alpha_Y^{(2)}$ are real numbers.[40]

Third, in general, none of the two exchange rate solutions in the entangled risk setting (solutions to (24), (25)) coincides with the (unique) solution in a completely disentangled risk setting. For illustration, the latter arises when, e.g., there is a second risky asset X loading only on the diffusion risk, $\frac{dX_t}{X_t} = \mu_X dt + \sigma_{X1} dZ_{1t}$. Recall that (Theorem 2), if risks are completely disentangled, then the unique exchange rate has the diffusion term $\sigma_{e1} = \eta_{F1} - \eta_{H1}$ and the jump size $\Delta_1 e = \Delta_1 H - \Delta_1 F$. In order for the exchange rate process in the entangled risk case to take this form, we would need α_Y (that is, either $\alpha_Y^{(1)}$ or $\alpha_Y^{(2)}$) to satisfy the following two equations at the same time,

$$\alpha_Y = \frac{\eta_{H1} - \eta_{F1}}{\sigma_Y} \quad \text{and} \quad \alpha_Y = \frac{e^{\Delta_1 F - \Delta_1 H} - 1}{e^{\Delta_1 Y} - 1}.$$

In general, this knife-edge restriction does not hold (no such α_Y exists) and the right hand sides of the two equations are only identical under very special and rather unlikely conditions on the market prices of risks and the risk loadings of the risky asset.

Finally, the risk entanglement notion universally applies for both systematic (i.e., priced) and idiosyncratic (i.e., non-priced) risks, which allows idiosyncratic risks to influence the cross-country

---

[40]It is, though, not certain that both solutions always yield an economically reasonable exchange rate, i.e., reasonable size of drift and diffusion terms and jump size.

pricing dynamic through its entanglement with systematic risks.[41] We define diffusion dZ_{1t} (respectively, jump $d\mathcal{N}_{1t}$) as systematic if, at least for one country $I \in \{H, F\}$, the market price η_{I1} (respectively, $\Delta_1 I$) is non-zero. We define diffusion dZ_{1t} (respectively, jump $d\mathcal{N}_{1t}$) as idiosyncratic if their market prices η_{I1} (respectively, $\Delta_1 I$) are identically zero for both $I \in \{H, F\}$. Clearly, equations (24) and (25) are robust to whether we set η_{I1} or $\Delta_1 I$ for any $I \in \{H, F\}$ to zero or a non-zero value. In particular, even in the case of entangled idiosyncratic risk we obtain multiple consistent exchange rates, and none of these coincides with the (unique) exchange rate obtained in a market with completely disentangled risks. This is important because it emphasizes the importance of the market structure (that is, the risk entanglement configuration, or how risks are embedded in asset markets) for the determination of exchange rates. Moreover, it is surprising that the exchange rate (which after all is the relative valuation of consumption baskets between countries) may depend on risks which are orthogonal to investors marginal utilities, or in other words, risks which investors do not seem to care about.

Scenario 2: Pure Jump Risk Entanglements

We consider again a market setting with two countries, a risk-free bond and a risky asset (B_H, Y respectively) in home currency, and two risk sources affecting asset Y. In contrast to the previous scenario, here we assume that both risk sources (affecting Y) are jump processes $J_Y = \{1, 2\}$, characterized respectively by Poisson counters $\{d\mathcal{N}_{1t}, d\mathcal{N}_{2t}\}$. There are no diffusion risks in the traded asset space. The dynamics of Y's and B_H's values are as follows,

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sum_{i \in \{1,2\}} \left(e^{\Delta_i Y \times d\mathcal{N}_{it}} - 1 \right) - \sum_{i \in \{1,2\}} \lambda_i dt \left(e^{\Delta_i Y} - 1 \right), \qquad \frac{dB_{Ht}}{B_{Ht}} = r_H dt. \qquad (26)$$

The countries' exogenous SDFs are also affected by these asset market risks $\{d\mathcal{N}_{1t}, d\mathcal{N}_{2t}\}$ (thus $\mathcal{J}_H, \mathcal{J}_F \supset \{1, 2\}$), as well as other unspanned (diffusion and jump) risks,

$$\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_I^T dZ_t + \sum_{i \in \mathcal{J}_I} \left(e^{\Delta_i I \times d\mathcal{N}_{it}} - 1 \right) - \sum_{i \in \mathcal{J}_I} \lambda_i dt \left(e^{\Delta_i I} - 1 \right), \qquad I \in \{H, F\}.$$

---

[41]On a related note, it is important to recall that, in our characterization given below, idiosyncratic risks are not priced in either country.

This setting features entangled risks of Definition 2 because markets are incomplete and two types $\{d\mathcal{N}_{1t}, d\mathcal{N}_{2t}\}$ of jump risks are always entangled with one another in asset markets.[42]

With α_Y and $(1 - \alpha_Y)$ denoting corresponding weights on asset Y and bond B_H as in portfolio representation (4), quantities in (10) in the current setting simplify respectively to,

$$\sigma_e = 0, \qquad e^{\Delta_i e} = \frac{1}{1 + \alpha_Y \left(e^{\Delta_i Y} - 1\right)}, \quad i \in \{1, 2\}.$$

Substituting above expressions into (18) yields an equation determining the exchange rate,

$$\sum_{i \in \{1,2\}} \frac{\lambda_i e^{\Delta_i F} \left(e^{\Delta_i Y} - 1\right)}{1 + \alpha_Y \left(e^{\Delta_i Y} - 1\right)} = \sum_{i \in \{1,2\}} \lambda_i e^{\Delta_i H} \left(e^{\Delta_i Y} - 1\right).$$

This quadratic equation in portfolio weight α_Y yields exactly two distinct solutions, and thus, there are exactly two consistent exchange rate processes in this setting. Again, none of the two exchange rates (in general) coincides with the (unique) exchange rate in an equivalent economy with completely disentangled risks (the proof is in the same spirit as in the jump-diffusion scenario). Finally, rich dynamics of possible consistent exchange rates are robust to whether risks are systematic or idiosyncratic, as long as they are entangled.

4 Entangled Risks and the International Correlation Puzzle

We provide a numerical example which shows that smooth exchange rates and low correlations between SDFs can co-exist in a jump-diffusion setting with entangled risks. Moreover, we illustrate that the exchange rate is distinct from the ratio of SDFs and expected carry trade returns are of an order of 5% (as in the data) in our example of entangled risks. We further contrast these results with an example of complete markets (and completely disentangled risks), where the exchange rate is always equal to the ratio of country-specific SDFs and the exchange rate volatility and carry trade returns are unreasonably large when the correlation between home and foreign SDFs is moderate. Jumps and entanglement of risks are important in our case because they introduce interesting non-linearities in the system of equations (18) determining the exchange rate. In particular, the exchange rate is no longer determined by only priced risks (or SDF risk loadings) but depends on the specific entanglement configuration of risks in asset markets.

---

[42]From home investors' perspective, B_H does not load on either jump type, and thus, it cannot be used in a portfolio to separate the two jump types from the risky asset Y (26).

We contrast two economies: (I) an economy with complete risk disentanglement, and (II) an economy with entangled jump and diffusion risks. In economy (I) with completely disentangled risks we assume there are two diffusion processes dZ_{1t} and dZ_{2t}. In economy (II) with risk entanglement we introduce a single Poisson jump process $d\mathcal{N}_{1t}$ in addition to the two diffusion risks. Hence, the difference between the two economies stems from the additional jump risk in economy (II). SDFs M_H and M_F are exposed to all risk sources in each respective economy,

$$\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_{I1} dZ_{1t} - \eta_{I2} dZ_{2t} + \left(e^{\Delta_1 I \times d\mathcal{N}_{1t}} - 1\right) - \lambda_1 dt \left(e^{\Delta_1 I} - 1\right), \qquad I \in \{H, F\},$$

where in economy (I) with completely disentangled risks we set $\Delta_1 I = 0$, $\forall I \in \{H, F\}$. In both economies we assume that home investors can trade one risk-free bond B_H and three risky assets Y_1, Y_2 and the foreign bond (which is risky when denominated in the home currency, i.e., $\frac{B_F}{e}$). Notice that only two of these three risky assets are non-redundant according to equation (4). Foreign investors trade the same assets. For simplicity we assume that all risks impacting the economy are also in the traded asset space, that is, Y_1 and Y_2 are loading on all risks,

$$\begin{aligned}
\frac{dY_{jt}}{Y_{jt}} &= \mu_{Y_j} dt + \sigma_{Y_j 1} dZ_{1t} + \sigma_{Y_j 2} dZ_{2t} + \left(e^{\Delta_1 Y_j \times d\mathcal{N}_{1t}} - 1\right) - \lambda_1 dt \left(e^{\Delta_1 Y_j} - 1\right), \qquad j \in \{1, 2\} \\
\frac{dB_{Ht}}{B_{Ht}} &= r_H dt,
\end{aligned}$$

where in economy (I) with completely disentangled risks we set $\Delta_1 Y_j = 0$, $\forall j \in \{1, 2\}$. Therefore, economy (I) is a complete market economy (2 diffusion risks and 2 non-redundant risky assets and a risk-free bond). Assuming additional risks which affect the SDFs but not the traded assets is a straightforward extension but does not conceptually change our numerical illustration.

Table 1 and 2 report the parameters in the two economies ("Exogenous Quantities") and the endogenously determined exchange rates ("Endogenous Results"). Table 1 summarizes the results in the diffusion setting with completely disentangled risks (economy (I)). Table 2 contains the values in the jump-diffusion setting with entangled risks (economy (II)). We choose the market prices η_{I1}, η_{I2} and $\Delta_1 I$ $\forall I \in \{H, F\}$ such that the total volatilities of M_H and M_F are identical (just under 60%), and the total correlation between the two SDF growths is 30%.[43] The large SDF volatilities are consistent with the Hansen and Jagannathan (1991) bound, and the modest SDF correlation matches the correlation between consumption growths across developed economies

---

[43] All variances, covariances and correlations in this section are total variances, covariances and correlations, i.e., they include diffusion and jump risks.

Table 1: Exchange Rate in Economy with Completely Disentangled Risk

SDFs M_H,M_F: $\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_{I1} dZ_{1t} - \eta_{I2} dZ_{2t}$

Risky Assets Y_1,Y_2: $\frac{dY_{jt}}{Y_{jt}} = \mu_{Y_j} dt + \sigma_{Y_j 1} dZ_{1t} + \sigma_{Y_j 2} dZ_{2t}$

	Exogenous Quantities			
	M_H	M_F	Y_1	Y_2
Diffusion dZ_{1t}	$\eta_{H1} = 0.04$	$\eta_{F1} = 0.556$	$\sigma_{Y_1 1} = 0.099$	$\sigma_{Y_2 1} = 0.097$
Diffusion dZ_{2t}	$\eta_{H2} = 0.57$	$\eta_{F2} = 0.13$	$\sigma_{Y_1 2} = 0.113$	$\sigma_{Y_2 2} = 0.114$
Volatility	57.1%	57.1%	15%	15%
Risk Premium	NA	NA	7%	6.9%

	Endogenous Results
	Exchange Rate e
Diffusion dZ_{1t}	$\sigma_{e1} = 0.516$
Diffusion dZ_{2t}	$\sigma_{e2} = -0.44$
Volatility	67.8%
$Corr\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$	30%
$Corr\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right)$	1.5%

Notes: Given the exogenous quantities specifying the market prices of risk (risk loadings of SDFs M_H and M_F) and the risk exposures of the two traded assets Y_1 and Y_2, we endogenously determine the exchange rate according to the system of two equations (18). $Corr\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \frac{Cov\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)}{Vol\left(\frac{dM_H}{M_H}\right)Vol\left(\frac{dM_F}{M_F}\right)}$ is the correlation between the two SDFs, where $Cov\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \sum_{i=1}^{2} \eta_{Hi}\eta_{Fi} dt$ is the covariance between the two SDFs and $Vol\left(\frac{dM_I}{M_I}\right) = \sqrt{\sum_{i=1}^{2} \eta_{Ii}^2 dt}$ is the volatility of SDF I. $Corr\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right) = \frac{Cov\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right)}{Vol\left(\frac{dY_1}{Y_1}\right)Vol\left(\frac{de}{e}\right)}$ is the correlation between the stock market Y_1 and the exchange rate, where $Cov\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right) = \sum_{i=1}^{2} \sigma_{Y_1 i}\sigma_{ei} dt$ is the covariance between Y_1 and e and $Vol\left(\frac{dY_1}{Y_1}\right) = \sqrt{\sum_{i=1}^{2} \sigma_{Y_1 i}^2 dt}$ and $Vol\left(\frac{de}{e}\right) = \sqrt{\sum_{i=1}^{2} \sigma_{ei}^2 dt}$ are volatilities of Y_1 and e.

SDFs M_H,M_F: $\frac{dM_{It}}{M_{It}} = -r_I dt - \eta_{I1} dZ_{1t} - \eta_{I2} dZ_{2t} + \left(e^{\Delta_1 I \times d\mathcal{N}_{1t}} - 1\right) - \lambda_1 dt \left(e^{\Delta_1 I} - 1\right)$

Risky Assets Y_1,Y_2: $\frac{dY_{jt}}{Y_{jt}} = \mu_{Y_j} dt + \sigma_{Y_j 1} dZ_{1t} + \sigma_{Y_j 2} dZ_{2t} + \left(e^{\Delta_1 Y_j \times d\mathcal{N}_{1t}} - 1\right) - \lambda_1 dt \left(e^{\Delta_1 Y_j} - 1\right)$

	Exogenous Quantities			
	M_H	M_F	Y_1	Y_2
Diffusion dZ_{1t}	$\eta_{H1} = 0.04$	$\eta_{F1} = 0.556$	$\sigma_{Y_11} = 0.099$	$\sigma_{Y_21} = 0.097$
Diffusion dZ_{2t}	$\eta_{H2} = 0.57$	$\eta_{F2} = 0.13$	$\sigma_{Y_12} = 0.113$	$\sigma_{Y_22} = 0.114$
Jump $d\mathcal{N}_{1t}$	$\Delta_1 H = 0.04$	$\Delta_1 F = 0.04$	$\Delta_1 Y_1 = -0.03$	$\Delta_1 Y_2 = 0$
Total Volatility	57.4%	57.4%	15.4%	15%
Risk Premium	NA	NA	7%	6.9%
Jump Intensity $\lambda_1 = 1.5$				

	Endogenous Results
	Exchange Rate e
Diffusion dZ_{1t}	$\sigma_{e1} = 0.002$
Diffusion dZ_{2t}	$\sigma_{e2} = -0.004$
Jump $d\mathcal{N}_{1t}$	$\Delta_1 e = -0.039$
Total Volatility	4.7%
$Corr\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$	30%
$Corr\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right)$	20%

Notes: Given the exogenous quantities specifying the market prices of risk (risk loadings of SDFs M_H and M_F) and the risk exposures of the two traded assets Y_1 and Y_2, we endogenously determine the exchange rate according to the system of two equations (18). $Corr\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \frac{Cov\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)}{Vol\left(\frac{dM_H}{M_H}\right) Vol\left(\frac{dM_F}{M_F}\right)}$ is the total correlation between the two SDFs, where $Cov\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right) = \sum_{i=1}^{2} \eta_{Hi}\eta_{Fi} dt + \lambda_1 dt \left(e^{\Delta_1 H} - 1\right)\left(e^{\Delta_1 F} - 1\right)$ is the total covariance between the two SDFs and $Vol\left(\frac{dM_I}{M_I}\right) = \sqrt{\sum_{i=1}^{2} \eta_{Ii}^2 dt + \lambda_1 dt \left(e^{\Delta_1 I} - 1\right)^2}$ is the total volatility of SDF I. $Corr\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right) = \frac{Cov\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right)}{Vol\left(\frac{dY_1}{Y_1}\right) Vol\left(\frac{de}{e}\right)}$ is the total correlation between the stock market Y_1 and the exchange rate, where $Cov\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right) = \sum_{i=1}^{2} \sigma_{Y_1 i}\sigma_{ei} dt + \lambda_1 dt \left(e^{\Delta_1 Y_1} - 1\right)\left(e^{\Delta_1 e} - 1\right)$ is the total covariance between Y_1 and e and $Vol\left(\frac{dY_1}{Y_1}\right) = \sqrt{\sum_{i=1}^{2} \sigma_{Y_1 i}^2 dt + \lambda_1 dt \left(e^{\Delta_1 Y_1} - 1\right)^2}$ and $Vol\left(\frac{de}{e}\right) = \sqrt{\sum_{i=1}^{2} \sigma_{ei}^2 dt + \lambda_1 dt \left(e^{\Delta_1 e} - 1\right)^2}$ are total volatilities of Y_1 and e.

(Brandt et al., 2006). Jump sizes in the two SDFs (in the jump-diffusion economy (II)) are symmetric ($\Delta_1 H = \Delta_1 F = 4\%$). We interpret Y_1 as the stock market (denominated in the home currency). Therefore, we choose the diffusion risk loadings $\sigma_{Y_1 1}$ and $\sigma_{Y_1 2}$ such that the (diffusion) volatility of Y_1 is about 15%, which roughly matches the unconditional volatility of the US stock market. We set the jump size $\Delta_1 Y_1$ of asset Y_1 equal to -3% and the jump intensity $\lambda_1 = 1.5$, which match the estimation by Backus et al. (2011). The total volatility increases only marginally after adding the jump. Moreover, we choose the risk loadings of SDFs such that the risk premium on Y_1 is 7%, which matches the average excess return of the US stock market. Y_2 is an additional generic asset, which helps to complete markets in economy I. When risks are completely disentangled (economy I), then the specific risk loadings of Y_2 are irrelevant (as long as they are distinct from Y_1) because the unique pricing consistent exchange rate is fully determined by the home and foreign SDFs and is independent of the asset market configuration. In contrast, when risks are entangled (economy II) the specification of Y_2 matters. The foreign bond and thus the inverse of the exchange rate is a portfolio of the risk-free home bond and assets Y_1 and Y_2 according to equation (4), and thus, the specification of Y_2 is not innocuous to the determination of the pricing consistent exchange rates. Home and foreign risk-free rates do not affect the system of equations (18), and thus, we do not specify them in our analysis.

Once the SDFs and asset price processes are specified, we solve for the exchange rate according to the system of two equations (18). In economy (I) with completely disentangled risks we have a unique exchange rate according to Theorem 2. While the model is set up to produce a reasonably low correlation between the two SDFs (30%, which is similar to correlations in consumption growths across developed countries), it produces an unreasonably large exchange rate volatility of 67.8% (Table 1). Not surprisingly this finding confirms the puzzle posed by Brandt et al. (2006). We further report the correlation between the stock market Y_1 and the exchange rate (denoted by $Corr\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right)$ in Table 1), which is close to zero as in the data.

In contrast, in economy (II) with entangled risks we have multiple exchange rates which are consistent with no-arbitrage pricing. In particular, in the case of a single jump type, two-dimensional diffusion and two non-redundant risky assets, the system (18) boils down to two quadratic equations and there exist multiple solutions (Theorem 3). We only report one consistent solution in Table 2, the solution we regard as the economically most reasonable one. In a more general and realistic setting with many more assets and diffusion and jump processes there are many more possible exchange rate candidates. The solution reported in Table 2 addresses the puzzle posed by

Brandt et al. (2006): the exchange rate is smooth (total volatility of 4.7%) and at the same time the total correlation between the two SDF growths is a modest 30%. The total correlation between the exchange rate and the stock market, $Corr\left(\frac{dY_1}{Y_1}, \frac{de}{e}\right)$ is 20%, which is close to the data (Brandt et al., 2006). We emphasize that none of the exchange rates in the jump-diffusion setting with entangled risks (neither the one reported in Table 2 nor the other less reasonable ones) coincides with the exchange rate obtained in the diffusion model with completely disentangle risks. That is, none of the solutions in economies (I) and (II) overlap.

Finally, we vary η_{H1} and η_{H2} to change the total correlation $Corr\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$ between the two SDF growths.[44] Figure 1 in the Introduction plots the total volatility of the endogenously determined exchange rate in economies (I) and (II) against $Corr\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$. The solid red line in Figure 1 plots the exchange rate volatility in economy (II) with entangled jump-diffusion risks, while the dashed black line represents the exchange rate volatility in economy (I) with completely disentangled diffusion risks. The dashed black line illustrates the correlation puzzle: a reasonably low correlation between the two SDF growths implies an unreasonably large variation in the exchange rate, or an reasonably smooth exchange rate comes with almost perfectly correlated SDFs. In contrast, the solid red line shows that independent of the total correlation between the two SDFs, the total volatility of the exchange rate is reasonably small in the case of entangled jump-diffusion risks. Therefore, an incomplete market setting with entangled risks is able to reconcile a smooth exchange rate and a low cross-country correlation in SDF growths.

Furthermore, we calculate the expected carry trade return to the home investor of borrowing the foreign currency and lending the home currency,

$$E\left[CT^H_{-F/+H}\right] = \sigma_{e1}\eta_{H1} + \sigma_{e2}\eta_{H2} + \lambda_1\left(e^{\Delta_1 H} - 1\right)\left(e^{-\Delta_1 e} - 1\right).$$

The solid red line in the top panel of Figure 2 shows that the expected carry trade return is almost 5% when the correlation between the SDFs is low in the economy with entangled risks (economy (II)). This value is similar to the average carry trade return of borrowing low and lending high interest rate currencies in the data. In contrast, the expected carry trade return is unreasonably large when the correlation between SDFs is moderate in the setting of completely disentangled risks (economy (I)) as illustrated by the dashed black line.

---

[44]In particular, we fix the parameters $\eta_{F2} = 0.13$, $\sigma_{Y_11} = 0.099$, $\sigma_{Y_12} = 0.113$, $\sigma_{Y_21} = 0.097$, $\sigma_{Y_22} = 0.114$, while we vary $\eta_{H1} \in [-0.1, 0.5]$ and simultaneously adjust η_{H2} to keep the equity premium of the stock market Y_1 equal to 7% and set η_{F1} such that variance of SDF growths of H and F are identical (i.e., $\eta^2_{F1} + \eta^2_{F2} = \eta^2_{H1} + \eta^2_{H2}$).

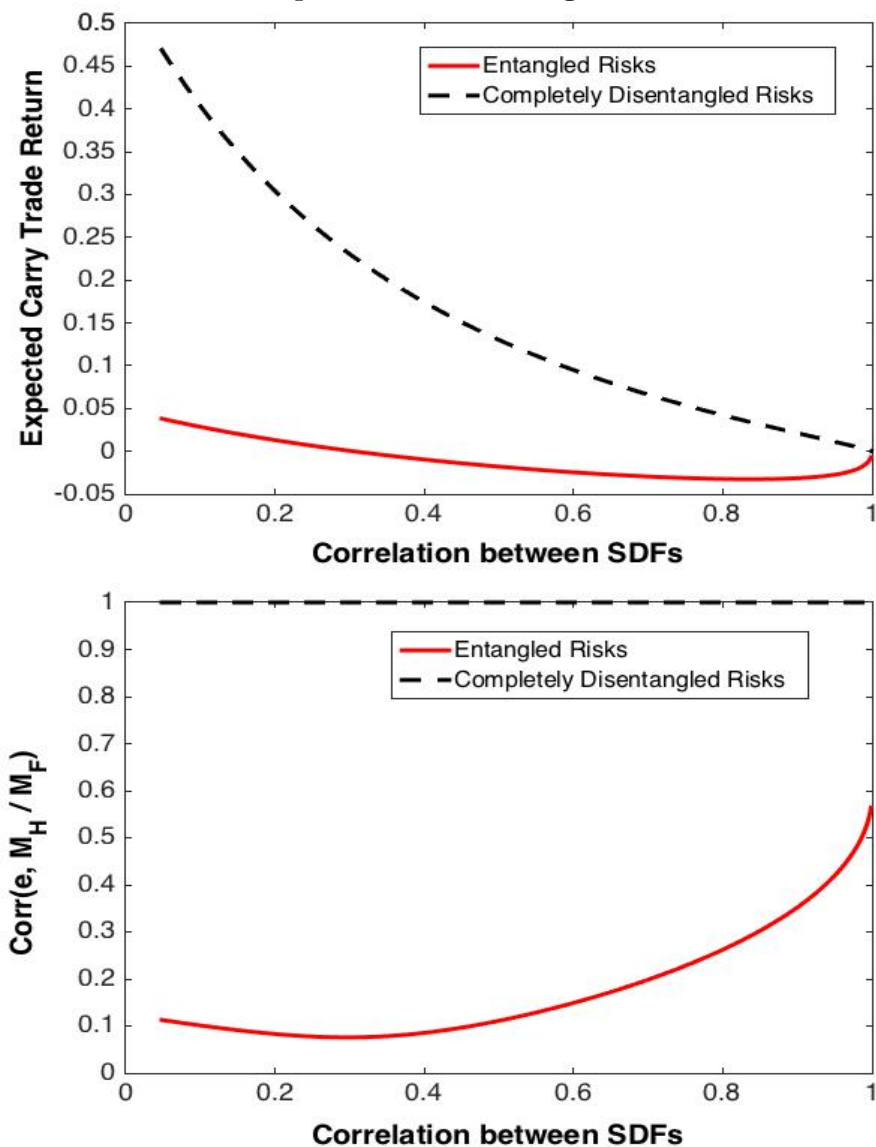**Carry Trade and Relationship between Exchange Rate and the Ratio of SDFs**

Figure 2: Top Panel: Expected carry trade return if risks are entangled (solid red line) or completely disentangled (dashed black line). Bottom Panel: Correlation between exchange rate growth and change in ratio of home and foreign SDFs if risks are entangled (solid red line) or completely disentangled (dashed black line). We fix the parameters $\eta_{F2} = 0.13$, $\sigma_{Y_11} = 0.099$, $\sigma_{Y_12} = 0.113$, $\sigma_{Y_21} = 0.097$, $\sigma_{Y_22} = 0.114$, while we vary $\eta_{H1} \in [-0.1, 0.5]$ in order to vary $Corr\left(\frac{dM_H}{M_H}, \frac{dM_F}{M_F}\right)$. We simultaneously adjust η_{H2} to keep the equity premium of the stock market Y_1 equal to 7% and set η_{F1} such that variance of SDF growths of H and F are identical (i.e., $\eta_{F1}^2 + \eta_{F2}^2 = \eta_{H1}^2 + \eta_{H2}^2$). In the case of entangled risks (economy (II)) we set $\Delta_1 H = \Delta_1 F = 0.04$, $\Delta_1 Y_1 = -0.03$, $\Delta_1 Y_2 = 0$ and $\lambda = 1.5$. In the case of completely disentangled risks (economy (I) there are no jumps, i.e., all jump parameters are zero.

The bottom panel in Figure 2 further shows the correlation between changes in the exchange rate and changes in the ratio of country-specific SDFs. In the case of complete markets (economy (I), dashed black line), the exchange rate is equal to the ratio of SDFs and the correlation is 1. In the case of entangled risks (economy (II), solid red line), the exchange rate is distinct from the ratio of SDFs and the correlation between the two quantities is particularly low when the correlation between the SDFs is low.

5   Conclusion

We discuss the concept of *risk entanglements* in incomplete FX markets. We define risks as *completely disentangled* if there is a sufficient number of non-redundant traded assets such that for every risk (diffusion or jump process) that affects the traded asset space there exists a portfolio which is solely exposed to this particular risk. The requirement of complete disentanglement of risks is less stringent than the concept of complete markets, because in contrast to complete markets, it does not require that investors can contract on every risk that affects the SDF but only the risks in the traded asset space. On the other hand, we define risks as *entangled* if there exists at least one risk affecting asset markets, which cannot be singly replicated by a portfolio of traded assets.

We, then, investigate how entangled risks affect exchange rates and find several interesting and surprising results. First, in incomplete markets with completely disentangled risks, we show that there exists a unique exchange rate, which only loads on systematic risk. This results is in accordance with the current literature. In contrast, in incomplete markets with entangled (jump-diffusion) risks, we show that multiple exchange rates may arise, all of which are consistent with no-arbitrage pricing. This is because the system of equations which pins down the pricing-consistent exchange rate is non-linear.[45] Moreover, we show that in general, none of the possible exchange rates coincides with the (unique) exchange rate in an "equivalent" economy with completely disentangled risks. Interestingly, in the case of entangled risks, even idiosyncratic risks, i.e., risks that affect the traded assets but not the SDFs, may affect every pricing-consistent exchange rate. Therefore, exchange rates, which measure the relative valuation of consumption baskets across countries,

---

[45]In a diffusion (continuous) setting, the non-linearity disappears because increments of continuous processes are infinitesimally small in any infinitesimal time interval. Applying Itô's lemma on non-linear functions renders a system linear in increments and thus a unique solution. In contrast, in a setting of prominent discontinuous processes (sizable jumps), these features are not warranted in general. Non-linear functions render a system non-linear in jump sizes. Interestingly, in the case of completely disentangled jump risks, the non-linear system is completely decoupled, i.e., each equation in the system concerns only a single jump risk. As a result, there exists again a unique exchange rate solution.

can be affected by risks which are otherwise unimportant to investors, i.e., idiosyncratic risks are orthogonal to investors' marginal utilities. Finally, we address the international correlation puzzle (Brandt et al., 2006), and provide a robust and simple numerical calibration to demonstrate that in a jump-diffusion setting with entangled risks, a smooth exchange rate and volatile country specific SDF growths with a modest correlation can co-exist.

References

Backus, David, Mikhail Chernov, and Ian Martin, 2011, Disasters Implied by Equity Index Options, *Journal of Finance* 66, 1969–2012.

Backus, David, Silverio Foresi, and Chris Telmer, 2001, Affine Models of Currency Pricing: Accounting for the Forward Premium Anomaly, *Journal of Finance* 56, 279–304.

Bakshi, Gurdip, Mario Cerrato, and John Crosby, 2015, Risk Sharing in International Economies and Market Incompleteness, Working paper, University of Maryland.

Bansal, Ravi, and Ivan Shaliastovich, 2012, A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets, *Review of Financial Studies* 26, 1–33.

Brandt, Michael W., John H. Cochrane, and Pedro Santa-Clara, 2006, International Risk-Sharing is Better Than You Think (or Exchange Rates are Much Too Smooth), *Journal of Monetary Economics* 53, 671–698.

Brunnermeier, Markus K, Stefan Nagel, and Lasse H. Pedersen, 2008, Carry Trades and Currency Crashes, in Kenneth Rogoff, Michael Woodford, and Daron Acemoglu, eds., *NBER Macroeconomics Annual 2008*, volume 23, 313–347.

Burnside, Craig, Martin Eichenbaum, Isaac Kleshchelski, and Sergio Rebelo, 2011, Do Peso Problems Explain the Returns to the Carry Trade?, *Review of Financial Studies* 24, 853–891.

Burnside, Craig, and Jeremy Graveline, 2012, On the Asset Market View of Exchange Rates, working paper no. 18646, NBER.

Chaieb, Ines, and Vihang Errunza, 2007, International Asset Pricing Under Segmentation and PPP Deviations, *Journal of Financial Economics* 86, 543–578.

Colacito, R., and M. M. Croce, 2011, Risks for the Long Run and the Real Exchange Rate, *Journal of Political Economy* 119, 153–182.

Colacito, R., M. M. Croce, F. Gavazzoni, and R. Ready, 2015, Currency Risk Factors in a Recursive Multi-Country Economy, Working paper.

Constantinides, G., and D. Duffie, 1996, Asset Pricing with Heterogeneous Consumers, *Journal of Political Economy* 104, 219–240.

Dumas, Bernard, Campbell R. Harvey, and Pierre Ruiz, 2003, Are correlations of stock returns justified by subsequent changes in national outputs?, *Journal of International Money and Finance* 22, 777–811.

Farhi, Emmanuel, Samuel Fraiberger, Xavier Gabaix, Romain Ranciere, and Adrien Verdelhan, 2014, Crash Risk in Currency Markets, Working paper, Harvard University.

Farhi, Emmanuel, and Xavier Gabaix, 2014, Rare Disasters and Exchange Rates, Working paper, Harvard University and New York University.

Favilukis, Jack, Lorenzo Garlappi, and Sajjad Neamati, 2015, The Carry Trade and Uncovered Interest Parity when Markets are Incomplete, Working paper, University of British Columbia.

Gavazzoni, Federico, Batchimeg Sambalaibat, and Chris Telmer, 2013, Currency Risk and Pricing Kernel Volatility, Working paper, ssrn no. 2179424.

Hansen, Lars Peter, and Ravi Jagannathan, 1991, Implications of Security Market Data for Models of Dynamic Economies, *Journal of Political Economy* 99, 225–262.

Lustig, Hano, and Adrien Verdelhan, 2015, Does incomplete spanning in international financial markets help to explain exchange rates?, Working paper, NBER.

Maurer, Thomas, and Ngoc-Khanh Tran, 2016, Incomplete Asset Market View of the Exchange Rate Determination, Working paper, Olin Business School, Washington University in St. Louis.

Pavlova, Anna, and Roberto Rigobon, 2007, Asset Prices and Exchange Rates, *Review of Financial Studies* 20, 1139–1181.

Saa-Requejo, Jesus, 1994, The dynamics and the term structure of risk premia in foreign exchange markets, Working paper, INSEAD.

Sarkissian, Sergei, 2003, Incomplete Consumption Risk Sharing and Currency Risk Premiums, *Review of Financial Studies* 16, 983–1005.

Stathopoulos, Andreas, 2016, Asset Prices and Risk Sharing in Open Economies, *Review of Financial Studies* forthcoming.

Zapatero, Fernando, 1995, Equilibrium asset prices and exchange rates , *Journal of Economic Dynamics and Control* 19, 787811.

Appendices

A   Proofs and Derivations

Proof of Proposition 1: It suffices to show that any solution to the system (18) is also a solution to the foreign pricing of the home bond (13). The other pricing equation (16) follows immediately from identity (17).

Suppose $\{\alpha_Y\}$, and the associated exchange rate e_t (9)-(10), solve system (18). Because (18) is the explicit expression of the no-arbitrage pricing relationship (17), $\{\alpha_Y\}$ and e_t must also satisfy the latter, and its equivalent version (in terms of Euler equations),

$$
\underbrace{E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{Y_{t+dt}}{Y_t}\right]-1}_{\text{Euler pricing eq. (16)}} = \underbrace{E_t\left[\frac{M_{Ht+dt}}{M_{Ht}}\frac{Y_{t+dt}}{Y_t}\right]-1}_{\text{Euler pricing eq. (15)}} + \underbrace{E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{B_{Ht+dt}}{B_{Ht}}\right]-1}_{\text{Euler pricing eq. (13)}}.
$$

By rearranging terms, the above equation can be rewritten as

$$
E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\frac{Y_{t+dt}}{Y_t} - \frac{B_{Ht+dt}}{B_{Ht}}\right\}\right] = E_t\left[\frac{M_{Ht+dt}}{M_{Ht}}\frac{Y_{t+dt}}{Y_t}\right]-1.
$$

Note that the right-hand side is identically zero (implied from the Euler equation on the traded asset Y) – a property that has nothing to do with the solution of system (18). Consequently, multiplying both sides by weight α_Y, then summing over all Y in the traded risky asset space $\{Y\}$ yields,

$$
E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\left(\sum_{Y\in\{Y\}}\alpha_Y\frac{Y_{t+dt}}{Y_t}\right) - \frac{B_{Ht+dt}}{B_{Ht}}\sum_{Y\in\{Y\}}\alpha_Y\right\}\right] = 0,
$$

or equivalently,

$$
E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\left(\sum_{Y\in\{Y\}}\alpha_Y\frac{Y_{t+dt}}{Y_t}\right) + \frac{B_{Ht+dt}}{B_{Ht}}\left(1 - \sum_{Y\in\{Y\}}\alpha_Y\right)\right\}\right] = E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{B_{Ht+dt}}{B_{Ht}}\right].
$$

The portfolio representation (4) of the exchange rate, which underlies the no-arbitrage determination of the exchange rate (Protocol 1), then transforms above equation into,

$$
E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\left\{\frac{B_{F,t+dt}}{B_{F,t}}\frac{e_t}{e_{t+dt}}\right\}\right] = E_t\left[\frac{M_{Ft+dt}}{M_{Ft}}\frac{e_{t+dt}}{e_t}\frac{B_{Ht+dt}}{B_{Ht}}\right],
$$

After the cancellation of the exchange rate, the left-hand side of above equation is identically one (as an Euler equation associated with the foreign pricing of the foreign bond) – a property that has nothing to do with the solution of system (18). Hence, the above equation reduces to $1 = E_t \left[\frac{M_{Ft+dt}}{M_{Ft}} \frac{e_{t+dt}}{e_t} \frac{B_{Ht+dt}}{B_{Ht}} \right]$, which yields (13) ■

Proof of Proposition 2: Without loss of generality, the proof is from the perspective of investors in country I. By construction, asset markets spanned by primitive assets in set $\mathcal{T}$ (19) can completely disentangle any single risk in the set of d diffusion risks $\{dZ_{kt}\}_{k=1}^{d}$ and j types of jump risks $\{d\mathcal{N}_{it}\}_{i=1}^{j}$. To prove Proposition 2 we then just need to show that primitive assets in $\mathcal{T}$ can span any *arbitrary* asset return that bears these (and only these) $d+j$ risks in any possible way. This is because the set $\mathcal{A}$ of all these arbitrary assets is the most complete possible set as long as the $d+j$ asset market risks are concerned, and thus, these risks must be completely disentangled in the set $\mathcal{A}$.[46]

Let's consider an arbitrary gross realized return $\frac{A_{t+dt}}{A_t}$ from $\mathcal{A}$,

$$\frac{A_{t+dt}}{A_t} = 1 + \mu_A dt + \sigma_A^T dZ_t + \sum_{i=1}^{j} \left(e^{\Delta_i A \times d\mathcal{N}_{it}} - 1 \right) - \sum_{i=1}^{j} \lambda_i dt \left(e^{\Delta_i A} - 1 \right).$$

We now explicitly construct a portfolio P of weights $\{\beta_B, \beta_k, \beta_i\}_{k=1,i=1}^{d,j}$, respectively associated with primitive assets $\{B_I, X_k, W_i\}_{k=1,i=1}^{d,j}$ in $\mathcal{T}$ (19)-(20),

$$\frac{P_{t+dt}}{P_t} = 1 + \beta_B r_I dt + \sum_{k=1}^{d} \beta_k \left[\mu_k dt + \sigma_k dZ_{kt} \right] + \sum_{k=1}^{j} \beta_i \left[\mu_i dt + \left(e^{\Delta_i W \times d\mathcal{N}_{it}} - 1 \right) - \lambda_i dt \left(e^{\Delta_i W} - 1 \right) \right],$$

$$\text{with portfolio normalization:} \qquad \beta_B + \sum_{k=1}^{d} \beta_k + \sum_{k=1}^{j} \beta_i = 1, \tag{27}$$

that perfectly replicates the arbitrary return $\frac{A_{t+dt}}{A_t}$. In order, we match diffusion, jump, and free (drift) components of $\frac{A_{t+dt}}{A_t}$ and $\frac{P_{t+dt}}{P_t}$.

Matching diffusion components: Because primitive asset X_k loads on a single diffusion component dZ_{kt} (20), the respective weight β_k in the replicating portfolio P is immediate and unique,

$$\sigma_{Ak} = \beta_k \sigma_k \implies \beta_k = \frac{\sigma_{Ak}}{\sigma_k}, \qquad \forall k \in \{1, \ldots, d\}.$$

---

[46]Though markets are still possibly incomplete because some risks affecting the SDFs are not in $\mathcal{A}$.

Matching jump components: Similarly, because primitive asset W_i loads on a single type of jump $d\mathcal{N}_{it}$ (20), the matching equation is simple. Crucially, we note that because jumps of two (or more) different types almost surely do not jump together within an infinitesimal time span of dt. Therefore, we need to match the *changes* in returns $\frac{A_{t+dt}}{A_t}$ and $\frac{P_{t+dt}}{P_t}$ induced by each (and every) jump type i separately.[47] When a jump takes place, respective counter $d\mathcal{N}_{it}$ increases from 0 to 1 (while all other counters $\{d\mathcal{N}_{lt}\}_{l \neq i}$ remain at 0), so the matching of jump-induced *changes* in returns implies the respective weight β_i in the replicating portfolio P,

$$\left(e^{\Delta_i A} - 1\right) = \beta_i \left(e^{\Delta_i W} - 1\right) \implies \beta_i = \frac{e^{\Delta_i A} - 1}{e^{\Delta_i W} - 1}, \qquad \forall i \in \{1, \ldots, j\}.$$

It is important to observe that, by forming a portfolio (e.g., of a risk-free bond with an asset W_i sensitive to jump type i), one can replicate and transform the original asset's jump size $\Delta_i W$ to an arbitrary jump size $\Delta_i A$ associated with the same jump type i.

Matching free components: the weight associated with the risk-free bond is implied from weights $\{\beta_k, \beta_i\}$ found earlier via the normalization (27). Then, by virtue of no-arbitrage, the free terms (no-jump terms associated with dt while setting all jump counters $\{d\mathcal{N}_{it}\}$ to zero) are automatically matched,

$$\mu_A - \sum_{i=1}^{j} \lambda_i \left(e^{\Delta_i A} - 1\right) = \beta_B r_I + \sum_{k=1}^{d} \beta_k \mu_k + \sum_{i=1}^{j} \beta_i \left[\mu_i - \lambda_i \left(e^{\Delta_i W} - 1\right)\right],$$

This is because once the risk terms of two traded portfolios are matched, their expected returns (i.e., free terms) must also match by no arbitrage.[48] ■

B   Exchange Rate in Pure-Diffusion Incomplete-Market Settings

The primary objective of this appendix is to illustrate, in a pure-diffusion risk settings, the equivalent relationship between two approaches to determine the exchange rate, namely (i) the portfolio representation approach (4) of this paper, and (ii) the more familiar SDF ratio of the pure-diffusion literature (see (36)), in either complete or incomplete market setting. Our analysis also clarifies

---

[47]The reason we care primarily about the *changes* (of two returns to be matched) induced by jumps is that the no-jump (base) levels are accounted for in, and including in the matching of, the free components. See next.

[48]We can also directly verify the matching of the free terms using the returns (20), the expression for the bond weight β_B, and the fact that as arbitrary asset A is traded, its expected return μ_A must satisfy the generic relationship (15) applied to A.

the appropriate construction of SDF projections, which concern net growth of original SDFs. The full analysis on the relationship between these two approaches in generic settings (beyond diffusion risks) is beyond the scope of this appendix.[49] Therefore, the illustrating results in the current Appendix apply technically for diffusion settings.

Let the quartet $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ denote the standard filtered probability space, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration associated with $d-$dimensional standard Brownian motion Z_t (representing d independent diffusion risks in the economy). Net stochastic discount factor (SDF) growths of country $I \in \{H, F\}$ and gross asset returns respectively are,

$$
\begin{aligned}
m_{It+dt} \equiv \frac{M_{It+dt}-M_{It}}{M_{It}} &= -r_{It}dt - \eta_{It}^T dZ_t, & M_{I0} = 1,\ t \in [0, \infty), \\
r_{It} \in \mathcal{F}_t,\ \eta_{It} \in \mathcal{F}_t, & & I \in \{H, F\}, \\
\frac{Y_{t+dt}}{Y_t} = 1 + \mu_{Yt}dt + \sigma_{Yt}^T dZ_t, & & \mu_{Yt} \in \mathcal{F}_t,\ \sigma_{Yt} \in \mathcal{F}_t,\ \ t \in [0, \infty),
\end{aligned}
\tag{28}
$$

where r_{It}, $d-$vector η_{It}, and Y_t respectively denote the (instantaneously) risk-free rate, country I's prices of the d diffusion risks, and asset Y's price in H's currency. Substituting the return on asset Y above into the portfolio representation (4) of the exchange rate yields,

$$
\frac{e_{t+dt}}{e_t} = 1 + \mu_{et}dt + \sigma_{et}^T dZ_t, \qquad\qquad \text{with,} \tag{29}
$$

$$
\mu_{et} = r_{Ft} - \left[1 - \sum_{Y \in \{Y\}} \alpha_{Yt}\right] r_{Ht} - \sum_{Y \in \{Y\}} \alpha_{Yt}\mu_{Yt} + \sigma_{et}^T\sigma_{et}, \qquad \sigma_{et} = -\sum_{Y \in \{Y\}} \alpha_{Yt}\sigma_{Yt}.
$$

The home pricing of asset Y, and the foreign pricing of the home bond and asset Y, produce respective premia,

$$
\begin{aligned}
\mu_Y - r_H &= \eta_H^T\sigma_Y, & r_H + \mu_e - r_F &= \sigma_e^T\eta_F, \\
\mu_Y + \mu_e - r_F &= \left(\sigma_Y^T + \sigma_e^T\right)\eta_F - \sigma_e^T\sigma_Y,
\end{aligned}
\tag{30}
$$

which then imply a no-arbitrage relationship for every asset Y,[50]

$$
\left(\eta_F^T - \eta_H^T - \sigma_e^T\right)\sigma_Y = 0, \qquad \forall Y \in \{Y\}, \tag{31}
$$

where μ_e and σ_e are given in (29). Reassuringly, this is the pure-diffusion version of the general

---

[49]A general incomplete-market analysis on SDF projectors and their possible relationship to the exchange rate in presence of entangled jump risks is intricate but offers novel insights Maurer and Tran (2016).

[50]Summing premium $\mu_Y - r_H = \eta_H^T\sigma_Y$ on asset Y to investors H and premium $r_H + \mu_e - r_F = \sigma_e^T\eta_F$ on the home bond B_H to investors F yields, $\mu_Y + \mu_e - r_F = \eta_H^T\sigma_Y + \sigma_e^T\eta_F$. Comparing this to the premium $\mu_Y + \mu_e - r_F = \left(\sigma_Y^T + \sigma_e^T\right)\eta_F - \sigma_e^T\sigma_Y$ on asset Y to investors F yields identity (31).

identity (16) in the main text.

Complete Markets: In complete-market settings, there exists a traded *primitive* risky asset Y that loads singly on each (and every) diffusion risk Z_{it} in Z_t. Applying (31) on each (and every) primitive asset Y yields $\eta_{Fi} - \eta_{Hi} - \sigma_{ei} = 0$, $\forall i \in \{1, \dots, d\}$, or after being stacked in a vector form, $\eta_F - \eta_H - \sigma_e = 0$. (29) then implies,

$$\sigma_e = \eta_F - \eta_H, \qquad \mu_e = r_F - r_H + \left(\eta_F^T - \eta_H^T\right)\eta_F. \tag{32}$$

By virtue of (28), we observe that the SDF ratio process $d\left(\frac{M_{Ht}}{M_{Ft}}\right)$ has drift and diffusion respectively identical to μ_e and σ_e in (32). Therefore, under mild regularity conditions for the unique solution of the stochastic differential equation (SDE) (29), its solution coincides with the ratio of SDFs in complete-market settings,

$$e_t = \frac{M_{Ht}}{M_{Ft}}, \qquad t \in [0, \infty). \tag{33}$$

This is the known complete-market relationship between the real exchange rate and countries' SDFs.[51]

Incomplete Markets – Portfolio Representation Approach: In pure-diffusion incomplete-market settings, only parts of the risks affecting investors' utilities can be replicated by assets returns. Therefore, at each time t, the linear space generated by the asset volatility vectors $\{\sigma_{Yt}\}$ does not span the one generated by the price-of-risk vectors $\{\eta_{It}\}$. Accordingly, we explicitly partition the systematic volatility space (spanned by $\{\eta_I\}$ of all countries I) into two components, (i) a subspace spanned by all traded asset volatilities $\{\sigma_Y\}$ (denoted by subscript $\parallel$) and (ii) the residual orthogonal subspace (denoted by subscript $\perp$). Similarly, the diffusion risk space is partitioned into risks affecting asset markets and unspanned risks that investors face in incomplete markets, $Z_t = Z_{t\parallel} \oplus Z_{t\perp}$. The pricing of the foreign bond in the home currency (Assumption A2) implies that exchange rate volatility σ_e is spanned by the asset return volatilities.[52] Consequently, in the incomplete-market pure-diffusion setting, (32) becomes,

$$\sigma_e = \eta_{F\parallel} - \eta_{H\parallel}, \qquad \mu_e = r_F - r_H + \left(\eta_{F\parallel}^T - \eta_{H\parallel}^T\right)\eta_{F\parallel}. \tag{34}$$

For every country I, we *construct* the projected net SDF growth processes from the respective

---

[51]In the literature, the standard derivation of (33) is typically less explicit. It first equalizes sides of (2) for a traded asset Y_t. Then exhausting a complete set of traded assets Y_t (in complete markets) implies (33).

[52]That is, σ_e belongs to the traded asset (parallel) space as seen from (29) $\sigma_e = -\sum_Y \alpha_Y \sigma_Y$.

projected prices of risk as the unique solution of the following SDE,

$$\frac{dM_{It+dt\|}}{M_{It\|}} \equiv \frac{M_{It+dt\|} - M_{It\|}}{M_{It\|}} = -r_{It}dt - \eta_{It\|}^T dZ_{t\|}, \quad t \in [0, \infty), \quad M_{I0\|} = 1, \quad I \in \{H, F\}. \quad (35)$$

Several observations are in order. First, this definition gives a specific construction of the "projector" of the SDF onto the traded asset space. It is crucial to note that the construction starts out with projecting the price-of-risk vector η_{It} onto the space of traded asset returns to obtain $\eta_{It\|}$, from which the projected *net* SDF growth, $\frac{dM_{It+dt\|}}{M_{It\|}}$, is uniquely constructed. The projected SDF level $M_{It\|}$, then, follows uniquely from the construction of the projected net SDF growth $\frac{dM_{It+dt\|}}{M_{It\|}}$.[53] Second, the particular definition of the drift and volatility of $M_{It\|}$ in (35) is not accidental. These moments are constructed to enable the resulting stochastic process $M_{It\|}$ to price the risk-free bond and all other traded assets Y_t by no arbitrage.[54] The combination of SDF (35) and exchange rate moments (34), (29) implies a key no-arbitrage identity relating the exchange rate and SDFs,

$$\text{in pure-diffusion incomplete-market settings:} \quad e_t = \frac{M_{Ht\|}}{M_{Ft\|}}, \quad t \in [0, \infty). \quad (36)$$

As (36) generalizes the complete-market relationship (33), versions of (36) have been employed to assess the effects of market incompleteness on the dynamics of real exchange rates in the literature. Our portfolio representation approach to the exchange rate, via a stochastic analysis, points to an explicit interpretation and construction of the projected SDFs in (36). Specifically, $M_{It\|}$ is obtained via the stochastic differential equation (35), which itself evolves from the original projection of the price-of-risk vector η_{It} (or equivalently, the projection of *net* SDF growth $\frac{dM_{It+dt\|}}{M_{It\|}}$) onto the space of traded asset returns.[55]

Incomplete Markets – Projection Analysis: In the literature, object $M_{I\|}$ in identity (36) arises from a projection analysis. In diffusion settings, this section offers an alternative derivation of (36) using projection formalism. Therefore, we connect that approach with the portfolio representation of the exchange rate adopted in this paper. Further supporting technical details can be found in

---

[53]It is important to note that the *gross* SDF growth, which is $\frac{M_{It+dt\|}}{M_{It\|}} = 1 + \frac{dM_{It+dt\|}}{M_{It\|}}$ is not the subject of our projection.

[54]Indeed, by virtue of Y's return (28), we have,

$$E_t\left[\frac{M_{It+dt\|}}{M_{It\|}} \exp(r_I dt)\right] = E_t\left[\left(1 + \frac{dM_{It+dt\|}}{M_{It\|}}\right)\exp(r_I dt)\right] = 1,$$
$$E_t\left[\frac{M_{It+dt\|}}{M_{It\|}} \frac{Y_{t+dt} + D_{Yt}dt}{Y_t}\right] = E_t\left[\left(1 + \frac{dM_{It+dt\|}}{M_{It\|}}\right)\left(1 + \mu_{Yt}dt + \sigma_{Yt}^T dZ_t\right)\right] = 1.$$

[55]Therefore, importantly, in (36), $M_{It\|}$ is neither the projected SDF level nor the projected gross SDF growth onto the traded asset space.

the Online Appendix C, also attached at the end of this paper.

First, the projection of the home *net* SDF growth $m_{Ht+dt} = -r_H dt - \eta_H^T dZ_t$ (28) onto the traded asset space spanned by *net* asset returns (in the home currency) yields,[56]

$$\widehat{m}_{Ht+dt} = \widehat{\beta}_{HN+1} r_{Ht} dt + \sum_{i=1}^{N} \widehat{\beta}_{Hi} x_{Hit+dt}.$$

A standard minimization of squared errors associated with this projection yields (see Online Appendix C),

$$\widehat{m}_{Ht+dt} = -\widehat{\mu}_{mH} dt - \widehat{\sigma}_{mH}^T dZ_t, \quad \text{with}, \quad \widehat{\mu}_{mH} = r_H, \quad \widehat{\sigma}_{mH} = \sigma_H \left(\sigma_H^T \sigma_H \right)^{-1} \sigma_H^T \eta_H \qquad (37)$$

The drift $\widehat{\mu}_{mH} = r_H$ arises from the fact that the home bond is traded and priced by projection $\widehat{m}_H$. The volatility $\widehat{\sigma}_{mH}$ possesses the exact expression of an OLS estimate. This clearly demonstrates that the diffusion $\widehat{\sigma}_{mH}$ of the projected net SDF growth $\widehat{m}_{Ht+dt}$ is precisely the projection $\eta_{H\parallel}$ (35) of the price-of-risk vector η_H (28) onto the space spanned by the asset return volatility vectors $\{\sigma_{Hi}\}_{i=1}^{N}$ (34). Therefore, $\widehat{\sigma}_{mH} = \eta_{H\parallel}$.

Similarly, the projection of the foreign *net* SDF growth $m_{Ft+dt} = -r_F dt - \eta_F^T dZ_t$ (28) onto the traded asset space spanned by *net* asset returns (in the foreign currency) produces,

$$\widehat{m}_{Ft+dt} = -\widehat{\mu}_{mF} dt - \widehat{\sigma}_{mF}^T dZ_t, \quad \text{with}, \quad \widehat{\mu}_{mF} = r_F, \quad \widehat{\sigma}_{mF} = \sigma_F \left(\sigma_F^T \sigma_F \right)^{-1} \sigma_F^T \eta_F = \eta_{F\parallel}. \qquad (38)$$

Substituting $\eta_{I\parallel}$, $I \in \{H, F\}$ obtained above into (34) implies, $\widehat{\sigma}_{mF} - \widehat{\sigma}_{mH} = \eta_{F\parallel} - \eta_{H\parallel} = \sigma_e$. As a consequence, the incomplete-market construction of the exchange rate $e_t = \frac{M_{Ht\parallel}}{M_{Ft\parallel}}$ (36) in diffusion settings is reconfirmed using an explicit projection approach, together with our interpretation of how "projected" SDFs $M_{It\parallel}$, $I \in \{H, F\}$ are constructed.[57] This result demonstrates the equivalence of the two approaches to construct the unique SDF within the asset return space. Either approach implies the key incomplete-market identity (36): while the portfolio representation of the exchange rate *identifies* $M_{I\parallel}$ with the solution of SDE (35), the projection analysis features the *net* SDF

---

[56] We denote the gross and net return on asset i denominated in currency I by X_{Iit+dt} and x_{Iit+dt} respectively,

$$\begin{cases} 1 \le i \le N: & x_{Iit+dt} = X_{Iit+dt} - 1 = \mu_{Ii} dt + \sigma_{Ii}^T dZ_t, \\ i = N+1: & x_{Iit+dt} = X_{Iit+dt} - 1 = r_H dt, \end{cases} \qquad t \in [0, \infty).$$

[57] Specifically, the identification of the respective moments of the projected SDF growths in the two approaches (portfolio representation and projection analysis) to the exchange rate, $\widehat{\mu}_{mI} = r_{I\parallel}$ and $\widehat{\sigma}_{mI} = \eta_{I\parallel}$, $I \in \{H, F\}$, demonstrates that these SDFs are identical.

growth projected onto the space of *net* asset returns.

Revisiting Burnside and Graveline (2012): Projecting the gross SDF growth $\mathcal{M}_{It+dt} \equiv \frac{M_{It+dt}}{M_{It}}$ onto the space of gross assets returns $\{X_{Iit+dt}\}$ (footnote 56) respectively for $I \in \{H, F\}$,

$$\widehat{\mathcal{M}}_{Ht+dt} = \sum_{i=1}^{N+1} \widehat{\gamma}_{Hi} X_{Hit+dt}, \quad \widehat{\mathcal{M}}_{Ft+dt} = \sum_{i=1}^{N+1} \widehat{\gamma}_{Fi} X_{Fit+dt} = \frac{e_{t+dt}}{e_t} \sum_{i=1}^{N+1} \widehat{\gamma}_{Fi} X_{Hit+dt}, \quad (39)$$

Burnside and Graveline (2012) derive an impossibility result, $\frac{\widehat{\mathcal{M}}_{Ht+dt}}{\widehat{\mathcal{M}}_{Ft+dt}} \neq \frac{e_{t+dt}}{e_t}$, i.e., the ratio of SDF projections does not equal the exchange rate in general.[58] This impossibility result does not contradict key identity (36) because the latter concerns *net* growth quantities as we discussed earlier. To see this, note that a standard minimization of squared errors associated with the projection (39) yields an expression for the projected gross SDF growth, and an extra constraint for the estimates, (see Online Appendix C),

$$\widehat{\mathcal{M}}_{It+dt} = 1 - r_I dt - \eta_{I\|}^T dZ_t, \quad \frac{1}{2} \eta_{I\|}^T \eta_{I\|} = r_I, \quad I \in \{H, F\}, \quad (40)$$

where r_I and $\eta_{I\|}$ denote respectively I's risk-free rate and projected prices of risks. Clearly, for every country I, projected prices of risk $\eta_{I\|}$ are not a sufficient statistics for I's risk-free rate r_I. Therefore, the above constraints strongly tying these two quantities are *spurious* and can neither be presumed in a generic no-arbitrage international asset pricing setting nor be expected to hold universally in the data. Equivalently, this result simply indicates that the object $\widehat{\mathcal{M}}_{It+dt}$ – constructed as the projection (39) of gross SDF growths onto the space of asset gross returns – are inconsistent with its prerequisite ability to price traded assets in either currency. This is a restatement of Burnside and Graveline (2012)'s impossibility result. While the net and gross quantities differ by a mere constant of 1, this difference has a profound impact on the associated projection. Intuitively, the implementation of the projection of gross quantities is constrained by three separate matchings (of respective terms of order 1, dt, and dZ_t). Whereas, the implementation of the projection of net quantities is constrained only by two separate matchings (of terms of order dt, and dZ_t). More matchings are tantamount to more constraints, which incapacitate the ability of the projected gross SDF growth $\widehat{\mathcal{M}}_{It+dt}$ (39) to price exchange rate risks as implied by the impossibility result.

---

[58] Burnside and Graveline (2012) give a simple proof by contradiction. Contrary to the impossibility result, assume that $\frac{\widehat{\mathcal{M}}_{Ht+dt}}{\widehat{\mathcal{M}}_{Ft+dt}} = \frac{e_{t+dt}}{e_t}$. Linear projections (39) then imply, $\frac{e_t}{e_{t+dt}} \sum_{i=1}^{k+1} \widehat{\gamma}_{Hi} X_{Hit+dt} = \frac{e_{t+dt}}{e_t} \sum_{i=1}^{k+1} \widehat{\gamma}_{Fi} X_{Hit+dt}$. Given arbitrary asset returns $\{X_{Hit+dt}\}$, this equality is non-linear (in $\frac{e_{t+dt}}{e_t}$), and therefore, is generally violated.

C Online Appendix: Further Technical Derivations

This appendix provides omitted technical details concerning projected SDFs in diffusion settings. Markets are incomplete with $d-$dimensional diffusion risk Z_t, and $N+1$ traded assets ($N+1 < d$). From country I's perspective ($I \in \{H, F\}$), first N assets are risky, last (or $N+1$-th) asset is I's bond. Gross and net asset returns (respectively $\{X_{Ii}\}$ and $\{x_{Ii}\}$, $i \in \{1, \ldots, N+1\}$) are specified in footnote 56.

Home SDF projection: First explicitly construct the projection $\widehat{m}_{Ht+dt}$ of the home *net* SDF growth $m_{Ht+dt} = -r_H dt - \eta_H^T dZ_t$ (28) onto the net asset return space spanned by $\{x_{Hit+dt}\}$. From an explicit minimization of squared errors,

$$\min_{\{\beta_{Hi}\}} \left| \left(-r_H dt - \eta_H^T dZ_t\right) - \left(\beta_{HN+1} r_H dt + \sum_{i=1}^N \beta_{Hi} x_{Hit+dt}\right) \right|^2,$$

follow N first-order optimality conditions,

$$\sigma_{Hi}^T \left(\eta_H + \sum_{j=1}^N \widehat{\beta}_{Hj} \sigma_{Hj} \right) = 0, \quad i \in \{1, \ldots, N\}.$$

This system of N linear equations yields a unique solution of ordinary least-squares (OLS) type,

$$\begin{bmatrix} \widehat{\beta}_{H1} \\ \vdots \\ \widehat{\beta}_{HN} \end{bmatrix} = -\left(\sigma_H^T \sigma_H\right)^{-1} \sigma_H^T \eta_H, \quad \text{with} \quad \sigma_H \equiv \begin{bmatrix} \sigma_{H11} & \cdots & \sigma_{HN1} \\ \vdots & \ddots & \vdots \\ \sigma_{H1d} & \cdots & \sigma_{HNd} \end{bmatrix}, \quad \eta_H \equiv \begin{bmatrix} \eta_{H1} \\ \vdots \\ \eta_{Hd} \end{bmatrix}$$

and the projection of net SDF growth, $\widehat{m}_{Ht+dt} = \widehat{\beta}_{HN+1} r_H dt + \sum_{i=1}^N \widehat{\beta}_{Hi} x_{Hit+dt}$. Because the home risk-free bond is traded, therefore is priced by the SDF projection $\widehat{m}_{Ht+dt}$, we also have, $\widehat{m}_{Ht+dt} = -r_H dt - \widehat{\sigma}_{mH}^T dZ_{t+dt}$. Identifying the diffusion and drift terms of the two above expressions proves (37),

$$\widehat{\sigma}_{mH} = \sigma_H \left(\sigma_H^T \sigma_H\right)^{-1} \sigma_H^T \eta_H, \qquad \widehat{\beta}_{HN+1} = -1 - \sum_{i=1}^N \frac{\mu_{Hi}}{r_H} \widehat{\beta}_{Hi},$$

where $\widehat{\beta}_{Hi}$ for $i \in \{1, \ldots, N\}$ has been obtained earlier in in the projection.

Foreign SDF projection: Itô's lemma relates net return dynamics (on the same assets) across

currencies, (recall exchange rate process (29)),

$$x_{Fit+dt} \equiv X_{Fit+dt} - 1 = \frac{e_{t+dt}}{e_t} X_{Hit+dt} - 1 = \left(\mu_{Hi} + \mu_e + \sigma_{Hi}^T \sigma_e\right) dt + \left(\sigma_{Hi}^T + \sigma_e^T\right) dZ_t, \forall i.$$

Then repeating the above projection of the foreign net SDF growth onto the space of net asset returns $\{x_{Fit+dt}\}$ yields parallel results, $\sum_{i=1}^N \widehat{\beta}_{Fi} x_{Fit+dt} + \widehat{\beta}_{FN+1} r_F dt = \widehat{m}_{Ft+dt} \equiv -\widehat{\mu}_{mF} dt - \widehat{\sigma}_{mF}^T dZ_t$, with,

$$\widehat{\mu}_{mF} = r_F, \qquad \widehat{\sigma}_{mF} = \sigma_F \left(\sigma_F^T \sigma_F\right)^{-1} \sigma_F^T \eta_F = \eta_{F\|},$$

$$\widehat{\beta}_{FN+1} = -1 - \sum_{i=1}^N \frac{\mu_{Fi}}{r_F} \widehat{\beta}_{Fi}, \qquad \begin{bmatrix} \widehat{\beta}_{F1} \\ \vdots \\ \widehat{\beta}_{FN} \end{bmatrix} = -\left(\sigma_F^T \sigma_F\right)^{-1} \sigma_F^T \begin{bmatrix} \eta_{F1} \\ \vdots \\ \eta_{Fd} \end{bmatrix}.$$

Gross SDF projection: To explicitly implement the projection (39) of the gross SDF growth onto the gross return space, we consider an associated minimization of squared errors for every country $I \in \{H, F\}$,

$$\min_{\{\gamma_{Ii}\}} \left| \left(1 - r_I dt - \eta_I^T dZ_t\right) - \left(\gamma_{IN+1}[1 + r_I dt] + \sum_{i=1}^N \gamma_{Ii} X_{Iit+dt}\right) \right|^2,$$

Using representation for gross returns X_{Iit+dt} (see footnote 56) and grouping terms transform the objective function above into,

$$\left| \left(1 - \sum_{i=1}^{N+1} \gamma_{Ii}\right) - \left(r_I + r_I \gamma_{IN+1} + \sum_{i=1}^N \mu_{Ii} \gamma_{Ii}\right) dt - \left(\eta_I^T + \sum_{i=1}^N \gamma_{Ii} \sigma_{Ii}^T\right) dZ_t \right|^2,$$

Evidently, it is necessary that the free term vanish for this objective function to attain a minimum, which generates an optimality constraint for the slope coefficients, $\sum_{i=1}^{N+1} \gamma_{Ii} = 1$, $I \in \{H, F\}$. The substitution of this constraint into the above objective function reduces it further to,

$$\left| \left(r_I + r_I \gamma_{IN+1} + \sum_{i=1}^N \mu_{Ii} \gamma_{Ii}\right) dt + \left(\eta_I^T + \sum_{i=1}^N \gamma_{Ii} \sigma_{Ii}^T\right) dZ_t \right|^2, \quad I \in \{H, F\},$$

which has identical structure of the previous minimization. As a result, the optimal solution of the

current minimization reads,

$$\begin{bmatrix} \widehat{\gamma}_{I1} \\ \vdots \\ \widehat{\gamma}_{IN} \end{bmatrix} = - \left(\sigma_I^T \sigma_I \right)^{-1} \sigma_I^T \eta_I, \quad \text{with} \quad \sigma_I \equiv \begin{bmatrix} \sigma_{I11} & \cdots & \sigma_{IN1} \\ \vdots & \ddots & \vdots \\ \sigma_{I1d} & \cdots & \sigma_{INd} \end{bmatrix}, \quad \eta_I \equiv \begin{bmatrix} \eta_{I1} \\ \vdots \\ \eta_{Id} \end{bmatrix}, \qquad (41)$$

and the projection of gross SDF growth $\mathcal{M}_{It}$,

$$\widehat{\mathcal{M}}_{It+dt} = \widehat{\gamma}_{IN+1}(1 + r_I dt) + \sum_{i=1}^{N} \widehat{\gamma}_{Ii} X_{Iit+dt} = 1 + \left(r_I + \sum_{i=1}^{N} \widehat{\gamma}_{Ii} \left[\mu_{Ii} - r_I \right] \right) dt + \left(\sum_{i=1}^{N} \widehat{\gamma}_{Ii} \sigma_{Ii}^T \right) dZ_t.$$

In the second equality above we have used the parametrization in footnote 56 for returns and the optimality constraint $\sum_{i=1}^{N+1} \gamma_{Ii} = 1$. Furthermore, by virtue of solution (41), $\sum_{i=1}^{N} \widehat{\gamma}_{Ii} \sigma_{Ii} = -\sigma_I \left(\sigma_I^T \sigma_I \right)^{-1} \sigma_I^T \eta_I$. Therefore, the diffusion of the projection $\widehat{\mathcal{M}}_{It+dt}$ is precisely the projection $\eta_{I\parallel}$ (37)-(38) of prices of risk η_I onto space spanned by volatilities $\{\sigma_{Ii}\}$. Thus, we have,

$$\widehat{\mathcal{M}}_{It+dt} = 1 + \left[r_I + \sum_{i=1}^{N} \widehat{\gamma}_{Ii} \left(\mu_{Ii} - r_I \right) \right] dt - \eta_{I\parallel}^T dZ_t, \qquad \eta_{I\parallel} = \sigma_I \left(\sigma_I^T \sigma_I \right)^{-1} \sigma_I^T \eta_I. \qquad (42)$$

The prerequisite that this projection be able to price N risky assets $E_t \left[\widehat{\mathcal{M}}_{It+dt} X_{Iit+dt} \right] = 1$, as well as I's risk-free bond $E_t \left[\widehat{\mathcal{M}}_{It+dt} \exp \left(r_I dt \right) \right] = 1$, together with projection of gross SDF (42) yields respectively risk premia and the drift term for $\widehat{\mathcal{M}}_{It+dt}$,

$$\mu_{Ii} - r_I = \eta_{I\parallel}^T \sigma_{Ii}, \quad i \in \{1, \ldots, N\}, \qquad r_I + \sum_{i=1}^{N} \widehat{\gamma}_{Ii} \left[\mu_{Ii} - r_I \right] = -r_I. \qquad (43)$$

As a result, the projection (42) can be written as, $\widehat{\mathcal{M}}_{It+dt} = 1 - r_I dt - \eta_{I\parallel}^T dZ_t$. Now substituting the first set of identities of (43) into the last identity of that same equation, and using solution (41) for slope coefficients $\{\widehat{\gamma}_{Ii}\}$ proves constraints (40), $\frac{1}{2} \eta_{I\parallel}^T \eta_{I\parallel} = r_I, \forall I \in \{H, F\}$, where projected prices of risk $\eta_{I\parallel}$ are given in (41).