Innovation-Led Transitions in Energy Supply*

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I generalize a benchmark model of directed technical change to allow for a non-unitary elasticity of substitution between machines and other factors of production, interpreted here as energy resources. I show that the economy becomes increasingly locked-in to the more advanced sector when machines and resources are gross substitutes, but a transition from a dominant sector to the other is possible when machines and resources are gross complements. In that case, research activity transitions before resource extraction does. In the presence of lock-in, temporary research subsidies are sufficient to permanently redirect the economy towards a path that avoids dangerous climate change, but emission taxes become critical in the absence of lock-in.

JEL: O33, O38, O44, Q48, Q55

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Energy historians have emphasized the multiple dramatic transformations in energy resource use that accompanied industrialization. According to Smil (2010, 2), the preindustrial era saw "only very slow changes" in resource use, "but the last two centuries have seen a series of remarkable energy transitions." Rosenberg (1994, 169) notes that "the diversity of energy inputs and the changing usage of those inputs over time is a central feature of the historical record." And in their seminal analysis, Marchetti and Nakicenovic (1979, 15) observe that the transitions have been so regular that "it is as though the system had a schedule, a will, and a clock." It is important to understand the economic drivers of these transitions. First, energy use is closely linked to the First Industrial Revolution (via coal), to the Second Industrial Revolution (via electricity and oil), and to the distribution of output across countries. Yet growth theory has largely abstracted from energy. Second, policy-makers around the world are currently attempting to induce a new transition to low-carbon resources in order to avoid dangerous climate change. Understanding the drivers of past transitions should improve policies that aim to stimulate and sustain a new transition.

Resource economists have long focused on how depletion or exhaustion can induce transitions between resources (e.g., Nordhaus, 1973; Chakravorty and Krulce, 1994; Chakravorty et al., 1997). For example, the Herfindahl (1967) rule holds that resources should be exploited in order of increasing cost. In contrast, energy and economic historians have argued that technological change, not depletion, has been critical to past transitions (e.g., Marchetti, 1977; Marchetti and Nakicenovic, 1979; Rosenberg, 1983; Grübler, 2004; Fouquet, 2010; Wilson and Grubler, 2011).[1] On this view, there has been a mismatch between economic theories of resource use and the patterns of the past two centuries.[2]

I develop a model of directed technical change in which innovation-led transitions occur endogenously. Here, a final good is produced from two types of energy services, which are gross substitutes. Each type of energy service is produced by combining an energy resource with specialized machines. For instance, coal is combined with steam engines to

[1] I give four examples. Marchetti and Nakicenovic (1979, 7–8) argue, "The causal importance of resource availability is weakened by the fact that oil successfully penetrated the energy market when coal still had an enormous potential, just as coal had previously penetrated the market when wood still had an enormous potential." Fouquet (2010, 6591) observes, "In all cases, cheaper or better services were the key to the switch [between sources of energy]. In a majority of cases, the driver was better or different services." Rosenberg (1994, 169) observes that "technological innovations are often not neutral with respect to their energy requirements." Finally, Grübler (2004, 170) writes, "It is important to recognize that these two major historical shifts [from biomass to coal, and then from coal to oil and natural gas] were not driven by resource scarcity or by direct economic signals such as prices, even if these exerted an influence at various times. Put simply, it was not the scarcity of coal that led to the introduction of more expensive oil. Instead, these major historical shifts were, first of all, technology shifts, particularly at the level of energy end use. Thus, the diffusion of steam engines, gasoline engines, and electric motors and appliances can be considered the ultimate driver, triggering important innovation responses in the energy sector and leading to profound structural change."

[2] This mismatch dates back to Jevons (1865), who analyzed the implications of the advancing depletion of British coal reserves. Madureira (2012) criticizes Jevons for underestimating the scope for innovation.

produce mechanical motion or electricity. A fixed measure of scientists works to improve these machines. Each scientist targets whichever type of machine provides a more valuable patent.

I show that the elasticity of substitution between resources and machines determines whether a transition in energy supply can occur in the absence of policy. Imagine that one sector initially dominates research and resource extraction. The technology in the sector that dominates research improves relative to the other sector's technology. I show that three forces determine how each sector's share of research and extraction change in the following period. First, as the dominant sector becomes more advanced, *market size effects* increase that sector's share of research and of extraction. The improvement in that sector's quality of machines expands the market for the energy resource, and the resulting increase in resource extraction raises the value of a patent by expanding the market for machines. This positive feedback between extraction and research works to lock in whichever sector is already dominant.[3] Second, a *patent quality effect* drives scientists to the sector where their patent will cover a higher quality machine. This effect draws additional scientists to the sector that dominated research in the previous period, which again works to lock in whichever sector is already dominant. Third, a *supply expansion effect* reduces the value of a patent as the average quality of a sector's machines increases. An improvement in the quality of a sector's machines shifts out the supply of machine services, which reduces the price of machine services and thus reduces the value of a patent. This force pushes scientists away from the sector that dominated research effort in the previous period. It is the only force that works against lock-in and in favor of a transition away from the dominant sector.

The elasticity of substitution between resources and machines determines the relative strengths of the patent quality and supply expansion effects. When that elasticity is strictly greater than 1 (machines are "energy-saving"), demand for machine services is elastic and the price of machine services does not fall by much as technology improves. The patent quality effect dominates the supply expansion effect. Whichever sector dominates research and extraction in some period then does so to an increasing degree in all later periods.[4] However, when that elasticity is strictly less than 1 (machines are "energy-using"), demand for machine services is inelastic and the price of machine services falls by a lot as technology improves. The supply expansion effect dominates the patent quality effect. In that case, as the dominant sector becomes more advanced, scientists can begin switching to the other sector. Eventually, their research output raises the quality of technology in the dominated sector, which begins increasing that sector's share of extraction via market size effects. The

[3] I actually show that a fourth force also matters. I label this a *machine substitution effect*. I abstract from it in this discussion because it works in the same direction as the market size effect.

[4] The forces generating lock-in are similar to those explored in a related literature on path dependency in technology adoption (e.g., David, 1985; Arthur, 1989; Cowan, 1990). That literature focuses on "dynamic increasing returns" as the source of path dependency, where the likelihood of using a technology increases in the number of times it was used in the past (perhaps through learning-by-doing or network effects). In the present setting, market size and patent quality effects both act like dynamic increasing returns.

shift in scientists away from the dominant sector can therefore generate a transition in energy supply. An elasticity of substitution strictly less than 1 is consistent with much empirical literature (e.g., Prywes, 1986; Manne and Richels, 1992; Chang, 1994; Koetse et al., 2008; van der Werf, 2008; Hassler et al., 2012; Stern and Kander, 2012), and we see that it is also consistent with historians' observations of innovation-led transitions.

My setting generalizes Acemoglu et al. (2012).[5] Their economy demonstrates a high degree of lock-in or path dependency: whichever sector initially dominates extraction and research effort will increase its dominance as time passes.[6] This result is not consistent with the history of energy transitions. I show that this result depends on their use of a Cobb-Douglas aggregator to combine resources and machines, which fixes the elasticity of substitution between resources and machines at unity. I show that a unit elasticity is the knife-edge case in which the patent quality and supply expansion effects exactly offset each other. The evolution of research and extraction in Acemoglu et al. (2012) is therefore determined entirely by market size effects, which we have seen generate positive feedbacks between research and extraction that lock in the dominant sector. I show that the assumption of Cobb-Douglas production has qualitatively important implications for their economy's dynamics.[7]

Numerical results on the optimal emission tax are in progress.

The next section describes the theoretical setting. Section 2 analyzes the relative incentive to research technologies in each sector. Section 3 describes the economy's laissez-faire dynamics. Section 4 argues that the case of energy-using machines best describes the world. Section 5 numerically explores the implications for climate change policies that aim to induce

[5]Formally, I analyze directed technical change (Acemoglu, 2002) when final good production has a nested constant elasticity of substitution structure that allows innovation and other inputs to be either substitutes or complements. A prominent strand of literature argues that complementarities have been a critical—and often overlooked—element of economic growth (Rosenberg, 1976; Matsuyama, 1995, 1999; Evans et al., 1998). Milgrom et al. (1991) show how complementarities between techniques and inputs can generate persistent patterns of technical change without needing to assume increasing returns. I similarly use complementarities to explain changes in energy technologies and supply without needing to assume increasing returns to innovation. We will see that increasing returns in fact here work to prevent transitions in research and resource extraction.

[6]An exception is when they model resources as exhaustible or depletable. Thus, when transitions arise in their setting, these transitions are driven by the same forces explored in the resource economics literature.

[7]Most analyses that combine directed technical change and energy have divided technologies between those that augment resources and those that augment other factors such as labor (Smulders and de Nooij, 2003; Di Maria and Valente, 2008; Grimaud and Rouge, 2008; Pittel and Bretschger, 2010; André and Smulders, 2012; Hassler et al., 2012). These studies have focused on the potential for technical change to enable long-run growth even when an exhaustible resource is essential to production. In contrast, the present paper and Acemoglu et al. (2012) both allow research effort to be directed between multiple types of resources in order to study questions about energy transitions. Recently, Acemoglu et al. (2016) developed a related setting in which two types of energy technologies compete in each of many product lines. Each product line's production function is Cobb-Douglas. As a result, this setting also generates strong path dependency or lock-in.

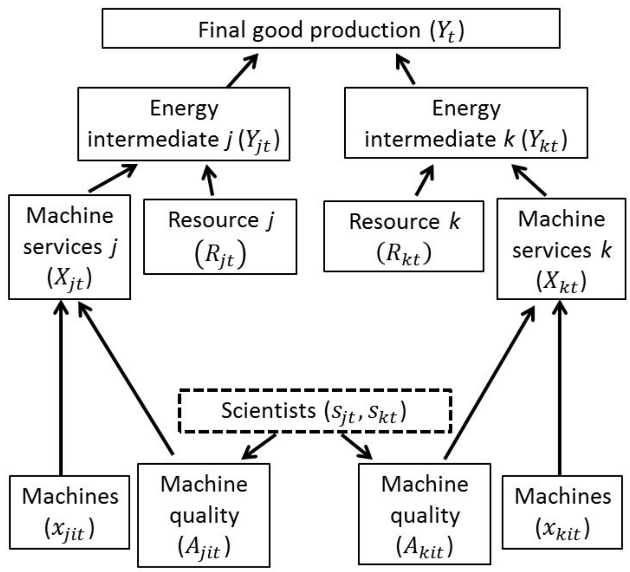


Figure 1: Overview of the theoretical setting.

a transition to clean energy. The final section concludes. The first appendix demonstrates that the equilibrium is tâtonnement-stable. The second appendix contains proofs and additional formal analysis.

1 Setting

Consider a discrete-time economy in which final-good production uses two types of energy intermediates. These energy intermediates are generated by combining energy resources with machines. Resources are supplied isoelastically. A fixed measure of households works as scientists, trying to improve the quality of machines used in producing the energy intermediates. Scientists decide which type of machine to work on. The equilibrium allocation of resources and scientists changes over time as technologies improve. Figure 1 illustrates the model setup, which we now formalize.

Begin with final good production. The time t final good Y_t is produced competitively from two energy intermediates Y_{jt} and Y_{kt}. The representative firm's production function takes the familiar constant elasticity of substitution (CES) form:

$$Y_t = A_Y \left(\nu \, Y_{jt}^{\frac{\epsilon-1}{\epsilon}} + (1-\nu) Y_{kt}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}.$$

The parameter $\nu \in (0,1)$ is the distribution (or share) parameter, and $A_Y > 0$ is a productivity parameter. We say that resource j is *higher quality* than resource k if and only if

$\nu > 0.5$. The parameter ϵ is the elasticity of substitution. The two energy intermediates are gross substitutes ($\epsilon > 1$).[8]

The energy intermediates Y_{jt} and Y_{kt} are the energy services produced by combining resource inputs with machines. Production of energy intermediates has the following CES forms:

$$Y_{jt} = \left(\kappa R_{jt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa) X_{jt}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \quad Y_{kt} = \left(\kappa R_{kt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa) X_{kt}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.$$

The parameter $\kappa \in (0,1)$ is the distribution (or share) parameter. I describe the resource inputs R and machine service inputs X below. The elasticity of substitution between these resource and machine inputs is σ. We call machines *energy-using* when resources and machines are gross complements ($\sigma < 1$), and we call machines *energy-saving* when resources and machines are gross substitutes ($\sigma > 1$). Resources and machines are less substitutable than are different types of energy intermediates ($\sigma < \epsilon$).

Machine services X_{jt} and X_{kt} are produced in a Dixit-Stiglitz environment of monopolistic competition from machines of varying qualities:

$$X_{jt} = \int_0^1 A_{jit}^{1-\alpha} x_{jit}^{\alpha} \, \mathrm{d}i \quad \text{and} \quad X_{kt} = \int_0^1 A_{kit}^{1-\alpha} x_{kit}^{\alpha} \, \mathrm{d}i,$$

where $\alpha \in (0,1)$. The machines x_{jit} and x_{kit} that work with a given resource at time t are divided into a continuum of types, indexed by i. The quality (or efficiency) of machine x_{jit} (or x_{kit}) is then given by A_{jit} (or A_{kit}). Machines of type i are produced by monopolists who each take the price of machine services (p_{jXt}, p_{kXt}) as given (each is small) but recognize their ability to influence the price (p_{jxit}, p_{kxit}) of machines of type i. The cost of producing a machine is $a > 0$ units of the final good, which we normalize to $a = \alpha^2$. The first-order condition for a producer of machine services yields the following demand curve for machines of type i in sector j (with analogous results for sector k):

$$x_{jit} = \left(\frac{p_{jXt}}{p_{jxit}} \alpha\right)^{\frac{1}{1-\alpha}} A_{jit}. \tag{1}$$

The monopolist producer of x_{jit} therefore faces an isoelastic demand curve and accordingly marks up its price by a constant fraction over marginal cost: $p_{jxit} = a/\alpha = \alpha$. In equilibrium, the producer of machine type i for use with resource j earns profits of:

$$\pi_{jxit} = (p_{jxit} - a) x_{jit} = \alpha(1-\alpha) p_{jXt}^{\frac{1}{1-\alpha}} A_{jit},$$

with analogous results for π_{kxit}.

[8] The restriction that $\epsilon > 1$ is consistent with recent evidence in Papageorgiou et al. (2016).

Scientists choose which resource they want to study (j or k) and are then randomly allocated to a machine type i. Each scientist succeeds in innovating with probability $\eta \in (0, 1]$. If they fail, scientists earn nothing and the quality of that type of machine is unchanged. Following Acemoglu et al. (2012), successful scientists receive a one-period patent to produce their type of machine, and they improve the quality of their machine type to $A_{jit} = A_{ji(t-1)} + \gamma A_{ji(t-1)}$ (using resource j as an example), where $\gamma > 0$. If a scientist succeeds in innovating at time t, she exercises her patent to obtain the monopoly profit π_{jxit}. Her expected reward to choosing to research machines that work with resource type j is therefore

$$\Pi_{jt} = \eta\, \alpha\, (1-\alpha) p_{jXt}^{\frac{1}{1-\alpha}} (1+\gamma) A_{j(t-1)}, \tag{2}$$

where $A_{j(t-1)}$ is the average quality of machines in sector j. This average quality evolves as

$$A_{jt} = \int_0^1 \left[\eta s_{jt}(1+\gamma)A_{ji(t-1)} + (1-\eta s_{jt})A_{ji(t-1)}\right] \mathrm{d}i = (1+\eta\gamma s_{jt})A_{j(t-1)}, \tag{3}$$

where s_{jt} is the measure of scientists working on resource j. All relationships for resource k are analogous. Scientists are of fixed measure, normalized to 1:

$$1 = s_{jt} + s_{kt}.$$

The resources R_{jt} and R_{kt} are supplied to each sector isoelastically:

$$R_{jt} = \Psi_j p_{jRt}^{\psi} \;\text{ and }\; R_{kt} = \Psi_k p_{kRt}^{\psi}, \tag{4}$$

where p_{jRt}, p_{kRt} are the prices received for each type of resource, $\psi > \alpha/(1-\alpha)$ is the price elasticity of resource supply, and $\Psi_j, \Psi_k > 0$ are supply shifters. We say that resource j is *more abundant* than resource k if and only if $\Psi_j > \Psi_k$. Requiring $\psi > \alpha/(1-\alpha)$ ensures that the own-price elasticity of resource supply is greater than the elasticity of machine services with respect to the resource price.

The economy's time t resource constraint is

$$Y_t \geq c_t + a \left[\int_0^1 x_{jit}\,\mathrm{d}i + \int_0^1 x_{kit}\,\mathrm{d}i\right],$$

where $c_t \geq 0$ is the composite consumption good. Households have strictly increasing utility for that consumption good. Scientists therefore each choose their resource type so as to maximize expected earnings.

We study equilibrium outcomes.

Definition 1. *An* equilibrium *is given by sequences of prices for energy intermediates (p^*_{jt}, p^*_{kt}), prices for machine services (p^*_{jXt}, p^*_{kXt}), prices for machines (p^*_{jxit}, p^*_{kxit}), prices* *for resources (p^*_{jRt}, p^*_{kRt}), demands for inputs ($Y^*_{jt}, Y^*_{kt}, R^*_{jt}, R^*_{kt}, X^*_{jt}, X^*_{kt}, x^*_{jit}, x^*_{kit}$), and factor allocations (s^*_{jt}, s^*_{kt}) such that, in each period t: (i) (Y^*_{jt}, Y^*_{kt}) maximizes profits of final good producers, (ii) ($R^*_{jt}, R^*_{kt}, X^*_{jt}, X^*_{kt}$) maximizes profits of energy intermediate producers, (iii) (p^*_{jxit}, x^*_{jit}) and (p^*_{kxit}, x^*_{kit}) maximize profits of the producers of machine i in sectors j and k, respectively, (iv) (s^*_{jt}, s^*_{kt}) maximizes expected earnings of scientists, (v) prices clear the factor and input markets, and (vi) average technologies evolve as in equation* (3).

The equilibrium prices clear all factor markets and all firms maximize profits. If scientists are employed in both sectors, they receive the same expected reward from both, and if they are employed in only one, they receive a greater expected reward in the sector with nonzero scientists. The first appendix establishes that the equilibrium is stable in a tâtonnement sense. Throughout, I use the price of the final good as the numeraire and drop the asterisks when clear.

2 The Direction of Research

We now consider the relative incentive to research technologies that work with resource j rather than technologies that work with resource k. From equation (2), we have

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)} + \gamma A_{j(t-1)}}{A_{k(t-1)} + \gamma A_{k(t-1)}} \left[\frac{p_{jXt}}{p_{kXt}}\right]^{\frac{1}{1-\alpha}}. \tag{5}$$

The intermediate-good producer's first-order conditions for profit-maximization yield

$$p_{jXt} = (1-\kappa) p_{jt} \left[\frac{X_{jt}}{Y_{jt}}\right]^{-1/\sigma} \quad \text{and} \quad p_{jRt} = \kappa\, p_{jt} \left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma}.$$

We see that the relative incentive to research technologies for use in sector j increases in the relative price of the intermediates and decreases in the machine-intensity of sector j's output. Combining the first-order conditions, we have

$$p_{jXt} = \frac{1-\kappa}{\kappa} \left[\frac{R_{jt}}{X_{jt}}\right]^{1/\sigma} p_{jRt}. \tag{6}$$

From equation (1) and the monopolist's markup, we have

$$x_{jit} = p_{jXt}^{\frac{1}{1-\alpha}} A_{jit}.$$

Substituting into the definition of X_{jt} and using the definition of A_{jt}, we have

$$X_{jt} = p_{jXt}^{\frac{\alpha}{1-\alpha}} A_{jt}. \tag{7}$$

Substitute into equation (6) and solve for equilibrium machine prices:

$$p_{jXt} = \left[p_{jRt} \frac{1-\kappa}{\kappa} \right]^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}} . \tag{8}$$

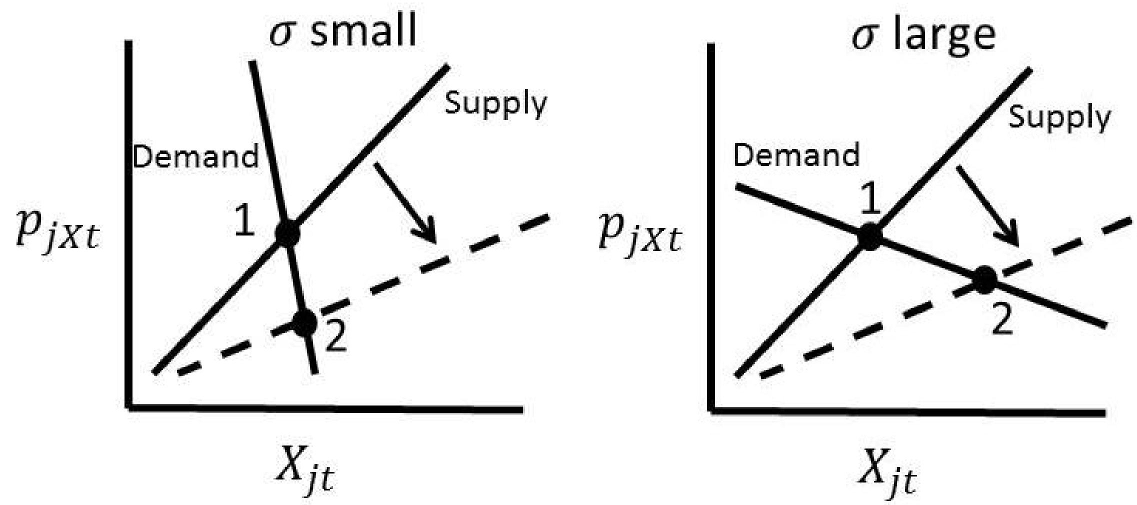
This yields

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \underbrace{\frac{A_{j(t-1)} + \gamma A_{j(t-1)}}{A_{k(t-1)} + \gamma A_{k(t-1)}}}_{\text{patent quality effect}} \underbrace{\left(\frac{A_{jt}}{A_{kt}} \right)^{\frac{-1}{\sigma+\alpha(1-\sigma)}}}_{\text{supply expansion effect}} \underbrace{\left(\frac{R_{jt}}{R_{kt}} \right)^{\frac{1}{\sigma+\alpha(1-\sigma)}}}_{\text{market size effect}} \underbrace{\left(\frac{p_{jRt}}{p_{kRt}} \right)^{\frac{\sigma}{\sigma+\alpha(1-\sigma)}}}_{\text{machine substitution effect}} . \tag{9}$$

We see four channels determining scientists' relative incentive to research machines. The first term directs research effort to the sector in which scientists will end up with the patent to better technology. This *patent quality effect* depends on the realized technology, not solely on the increment to technology produced by a scientist's efforts, which introduces a type of business-stealing distortion. Obtaining a patent to a sufficiently advanced technology is valuable even if the scientist does not improve the technology. If γ differed by sector and were very small in the more advanced sector, scientists could have a stronger incentive to research machines in the more advanced sector even though their efforts would not improve these machines. However, this business-stealing distortion vanishes under our assumption of identical γ because the ratio of the increments to technology ($\gamma A_{ji(t-1)}/\gamma A_{ki(t-1)}$) is identical to the ratio of the realized technologies (A_{jit}/A_{kit}). By attracting scientists to the more advanced sector, the patent quality effect here also attracts them to the sector where they make the greatest advance.

The other channels derive from the relative price of machine services: $(p_{jXt}/p_{kXt})^{1/(1-\alpha)}$ in equation (5). Figure 2 plots supply and demand for machine services X_{jt}, conditional on R_{jt}. The intersection of these supply and demand curves determines the equilibrium machine price p_{jXt}. The supply of X_{jt} follows from equation (7). It is steeper for smaller α. Demand for X_{jt} (conditional on R_{jt}) is given by equation (6), from the first-order conditions for the intermediate-good producers. The left panel of Figure 2 plots a case with less elastic demand (σ small), and the right panel of Figure 2 plots a case with more elastic demand (σ large).

Now consider the three machine price channels. We begin with the *supply expansion effect*, which pushes scientists away from the more advanced sector. From equation (7), the supply of X_{jt} shifts out when its machines' average quality A_{jt} increases, and it shifts out to an especially large degree when α is small. The dashed lines in Figure 2 plot the consequence of an increase in A_{jt}. When σ is small (machines are energy-using), the demand curve is steep because the marginal product of additional machines is constrained by the supply of R_{jt}. By shifting out supply, the increase in A_{jt} induces a relatively large decline in the equilibrium price p_{jXt}, from point 1 to point 2 in the left panel. However, when σ is large, machine are energy-saving and the demand curve is relatively flat. The increase in A_{jt} then induces a



(a) Strong supply expansion effect

(b) Weak supply expansion effect

Figure 2: Improving the average quality of the machines in sector j shifts the supply of machine services outward and changes the equilibrium from point 1 to point 2.

relatively small decline in the equilibrium price p_{jXt}. Improving technology therefore pushes scientists away to a greater degree when the demand curve is steep (σ is small) or the shift in supply is large (α is small) because it then reduces p_{jXt} more strongly.

Now consider the net effect of a relative improvement in sector j's average technology. We have seen that this relative improvement attracts scientists through the patent quality effect and repels scientists through the supply expansion effect. From equation (9), the supply expansion effect dominates the patent quality effect if and only if $\sigma < 1$. As $\sigma \to 0$, demand for machines becomes perfectly inelastic and the supply expansion effect becomes large. As $\sigma \to \infty$, demand for machines becomes perfectly elastic and the supply expansion effect vanishes. As $\sigma \to 1$, the two effects exactly cancel, so that the incentives to research machines in one sector or the other do not directly depend on the relative quality of technology in each sector. This result explains the absence of relative technology from the research incentives in Acemoglu et al. (2012): technology matters in their equation (17) via the same patent quality effect seen here (which they call a "direct productivity effect") and also through their "price effect", but substituting in for relative output prices from their equation (A.3) shows that these two effects exactly cancel. Conditional on market size, relative technology plays no role in steering research activity in their setting.[9] We will see that whether improving

[9] Relative technology ends up playing a role in their setting's equilibrium (see their equation (18)) because relative market size is proportional to the relative quality of technology (see their equation (A.5)). As we will discuss, this channel for relative technologies appears in our setting as well: our equation (14) will show that relative market size increases in the relative quality of technology. However, our use of a general CES

technology attracts or repels scientists determines whether a transition in energy supply is possible.

The final two machine price channels in equation (9) both make the relative incentive to research machines in sector j increase in sector j's share of resource production. The first of these two channels is a *market size effect*. From equation (6), an increase in R_{jt} shifts out demand for X_{jt}, and does so to an especially large degree as σ becomes small. Increasing the supply of one factor makes the other factor relatively scarce and thus increases demand for that other factor, and does so to an especially strong degree when the two factors are complements ($\sigma < 1$). The second of these channels is a *machine substitution effect*. It increases demand for X_{jt} when the price of resources increases. This channel is especially strong when the elasticity of substitution between resources and machines is large, and it vanishes as the elasticity goes to zero. Substituting for resource prices from (4), we see that the machine substitution effect amplifies the market size effect, and that it vanishes as the supply of resources becomes perfectly elastic.

3 Laissez Faire Dynamics

We now consider how each sector's share of research activity and resource extraction evolves over time. We then analyze transitions, lock-in, and balanced growth outcomes before providing a numerical example.

3.1 Evolution of Research and Extraction

Begin by considering how the equilibrium allocation of scientists changes over time. The appendix shows that, at an interior allocation of scientists,

$$s_{j(t+1)} - s_{jt} \propto \underbrace{\frac{\psi+\sigma}{\psi}\frac{R_{kt}}{R_{jt}}\left(\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}}\right)}_{\text{resource channel}} \underbrace{+2(\sigma-1)(1-\alpha)\frac{\eta\gamma}{1+\eta\gamma s_{kt}}\left(s_{jt}-\frac{1}{2}\right)}_{\text{innovation channel}}. \tag{10}$$

We see that the evolution of research effort depends on the evolution of extraction and of technology. The first term is a *resource channel*: scientists tend to move towards whichever sector sees its share of total resource extraction increase. This channel operates through the market size and machine substitution effects in equation (9). The second term is an *innovation channel*. If $s_{jt} > 0.5$, then sector j is becoming relatively more advanced as a result of time t research activity, and if $s_{jt} < 0.5$, then sector k is becoming relatively more advanced as a result of time t research activity. This channel pushes scientists towards whichever sector is becoming relatively more advanced if and only if $\sigma > 1$. Advancing technology affects relative research incentives through the patent quality and supply expansion effects in

aggregator means that the relationship between market size and relative technology is no longer linear.

equation (9), where the patent quality effect attracts scientists to the more advanced sector and the supply expansion effect repels scientists from the more advanced sector. We saw that the patent quality effect dominates the supply expansion effect if and only if $\sigma > 1$.

Now consider how sector j's share of extraction changes from time t to $t+1$. Combining the intermediate good producers' first-order condition for resources with the final good producers' first-order conditions, we find demand for each resource:

$$p_{jRt} = \kappa\, \nu \left[\frac{Y_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma} \quad \text{and} \quad p_{kRt} = \kappa\,(1-\nu) \left[\frac{Y_{kt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{kt}}{Y_{kt}}\right]^{-1/\sigma}. \tag{11}$$

Market-clearing for each resource then implies

$$\left[\frac{R_{jt}}{\Psi_j}\right]^{1/\psi} = \kappa\, \nu \left[\frac{Y_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{Y_{jt}}\right]^{-1/\sigma}, \tag{12}$$

$$\left[\frac{R_{kt}}{\Psi_k}\right]^{1/\psi} = \kappa\,(1-\nu) \left[\frac{Y_{kt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{kt}}{Y_{kt}}\right]^{-1/\sigma}. \tag{13}$$

Demand for sector j's resources (for example) shifts inward as the share of those resources in the production of intermediate good j increases and shifts inward as the share of intermediate good j in production of the final good increases.

Rearranging equations (12) and (13) and then dividing, we have:

$$\left[\frac{R_{jt}}{R_{kt}}\right]^{\frac{1}{\sigma}+\frac{1}{\psi}} = \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k}\right]^{1/\psi} \left[\frac{Y_{jt}}{Y_{kt}}\right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}}. \tag{14}$$

The change in sector j's share of resource extraction from time t to time $t+1$ therefore has the same sign as the change in sector j's share of intermediate good production. For any given quantity of resource extraction R_{jt}, increasing the average quality of technology A_{jt} increases production of the intermediate good Y_{jt}. The share of resources in production of intermediate good j falls as machine quality improves, which shifts demand for the resource R_{jt} outward in proportion to $1/\sigma$. However, intermediate good j's share of final good production increases, which shifts demand for R_{jt} inward in proportion to $1/\epsilon$. Because $\sigma < \epsilon$, the first effect dominates, so that increasing A_{jt} increases demand for R_{jt} (and does so more strongly when resources and machines are stronger complements). Thus, sector j's share of resource extraction increases from time t to $t+1$ when $s_{j(t+1)} = 1$, and it decreases from time t to $t+1$ when $s_{j(t+1)} = 0$.

When $s_{j(t+1)} \in (0,1)$, the average quality of technology improves in both sectors. The following proposition describes what happens in these intermediate cases, where we call a sector intensive in factor z if that factor's share of production in greater than half.

Proposition 1. *There exists a unique $\hat{s}_{t+1}$ such that sector j's share of resource extraction increases from time t to $t+1$ if and only if $s_{j(t+1)} \geq \hat{s}_{t+1}$. Let sector j be machine-intensive* *and sector k be resource-intensive. Then: $\hat{s}_{t+1}$ decreases in σ; as $\sigma \to 0$, $\hat{s}_{t+1} \to 1$; as $\sigma \to 1$, $\hat{s}_{t+1} \to 0.5$; and as $\sigma \to \infty$, $\hat{s}_{t+1} \to 0$.*

Proof. See appendix. □

There exists an intermediate value of $s_{j(t+1)}$, labeled $\hat{s}_{t+1}$, such that each resource's share of extraction is constant over time if and only if $s_{j(t+1)} = \hat{s}_{t+1}$. We will use this proposition to develop a graphical analysis in the next subsection.

The proof of Proposition 1 shows that $\hat{s}_{t+1}$ is

$$\hat{s}_{t+1} = \frac{\Sigma_{Y_{kt},X_{kt}} C_{jt}}{\Sigma_{Y_{jt},X_{jt}} C_{kt} + \Sigma_{Y_{kt},X_{kt}} C_{jt}}, \tag{15}$$

where $\Sigma_{w,z}$ is the elasticity of w with respect to z and where $C_{jt}, C_{kt} > 0$. $\Sigma_{Y_{jt},X_{jt}}$ large relative to $\Sigma_{Y_{kt},X_{kt}}$ means that increasing machine services in each sector has an especially strong effect on production of intermediate j. In this case, if scientists are divided equally between the two sectors at time $t+1$, then Y_j/Y_k increases from time t to $t+1$ and R_j/R_k therefore also increases from time t to $t+1$. Therefore, when $\Sigma_{Y_{jt},X_{jt}}$ is relatively large, the research allocation that holds R_j/R_k constant from time t to $t+1$ must have $s_{j(t+1)} < 0.5$.

Now consider what makes $\Sigma_{Y_{jt},X_{jt}}$ large or small relative to $\Sigma_{Y_{kt},X_{kt}}$. Note that $\Sigma_{Y_{jt},X_{jt}} = (1-\kappa)(X_{jt}/Y_{jt})^{\frac{\sigma-1}{\sigma}}$. Increasing machine services in each sector by the same percentage has an especially strong effect on intermediate good production in the relatively advanced, machine-intensive sector if and only if $\sigma > 1$. Intuitively, if machines substitute for resources ($\sigma > 1$), intermediate good producers respond more strongly to improving machines when machines are relatively abundant, but if machines complement resources ($\sigma < 1$), intermediate good producers respond more strongly to improving machines when machines are relatively scarce. Thus, if sector j is relatively machine-intensive, then the allocation of scientists that holds extraction constant must have $s_{j(t+1)} < 0.5$ if and only if $\sigma > 1$. Proposition 1 establishes this result formally.

Finally, note that as $\sigma \to 1$, the elasticity of intermediate good production with respect to machines is constant. Each sector responds to improved technology in the same way, regardless of how advanced they are. In this special case, $\hat{s}_{t+1} = 0.5$. Extraction then shifts towards whichever sector is advancing more rapidly.

3.2 Transitions and Lock-in

We are now ready to describe when a transition is possible. We define a transition as having occurred when one sector's share of research and extraction is at first increasing and later decreasing.

Definition 2. *A* transition *from sector j to sector k occurs between times t and $w > t$ when $R_{jt}/R_{kt} < R_{j(t+1)}/R_{k(t+1)}$, $s_{jt}/s_{kt} < s_{j(t+1)}/s_{k(t+1)}$, $R_{jw}/R_{kw} > R_{j(w+1)}/R_{k(w+1)}$, and*

$s_{jw}/s_{kw} > s_{j(w+1)}/s_{k(w+1)}$. *The economy is* locked-in *to sector j from time t onward if a transition does not occur between t and any time $w > t$.*

We now will see that whether a transition is possible depends on the elasticity of substitution σ and on the relative quality of technology in the two sectors.

Assumption 1. $\frac{A_{j(t-1)}}{A_{k(t-1)}} \geq \left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1}{(1-\alpha)(1+\psi)}}$.

Proposition 2. *Let $s_{jt} \geq 0.5$. If $\sigma < 1$ and $(1-\sigma)(1-\alpha) \leq 1/2$, then a transition from sector j to sector k can occur after time t only if Assumption 1 holds, in which case sector j's share of scientists begins decreasing before its share of extraction begins decreasing and $s_{jz} > 0.5$ at the last time z before sector j's share of extraction begins decreasing. If $\sigma \in (1, \epsilon)$ and Assumption 1 holds, then the economy is locked-in to sector j from time t onward.*

Proof. See appendix. □

Corollary 3. *Fix $\Psi_j = \Psi_k$. If $\sigma < 1$, $s_{jt} \geq 0.5$, and $(1-\sigma)(1-\alpha) \leq 1/2$, then a transition from sector j to sector k can occur only if $A_{j(t-1)}/A_{k(t-1)} > 1$. If $\sigma \in (1, \epsilon)$ and $\nu \geq 0.5$, then the economy is locked-in to sector j when $A_{j(t-1)} > A_{k(t-1)}$.*

Proof. See appendix. □

We see two cases.[10] First, if machines are energy-using ($\sigma < 1$), then a transition can occur from the relatively advanced sector to the relatively backwards sector. This transition is *innovation-led*: research activity begins switching to the relatively backwards sector even as the relatively advanced sector's share of extraction continues increasing. Second, if machines are energy-using ($\sigma \in (1, \epsilon)$), then a transition cannot happen. The economy is locked-in to the sector that is growing faster.

We now consider these cases in more detail. Figure 3 uses the results in Section 3.1 to represent the economy's dynamics. The horizontal axis plots the elasticity of substitution σ between resources and machines, and the vertical axis plots $s_{j(t+1)}$. In each region, the arrows describe whether $s_{j(t+1)}$ and R_{jt}/R_{kt} are increasing over the next period. The changes in R_{jt}/R_{kt} come from Proposition 1. The changes in $s_{j(t+1)}$ come from equation (10), assuming for the purpose of exposition that the change in relative resource extraction from period $t+1$ to $t+2$ has the same sign as the indicated change from period t to period $t+1$.

We first explore the dynamics of an innovation-led transition, which requires $\sigma < 1$. If sector j is increasing its share of scientists and extraction over time, then we must be moving up in the top left region in Figure 3, labeled region 1. As technology in sector j advances relative to that in sector k, the supply expansion effect pushes scientists towards sector k, but sector j's increasing share of resource extraction works to keep scientists in sector j.

[10]Note that $(1-\sigma)(1-\alpha) \leq 1/2$ holds for all $\sigma < 1$ if $\alpha \geq 1/2$ and holds for all α if $\sigma \geq 1/2$.

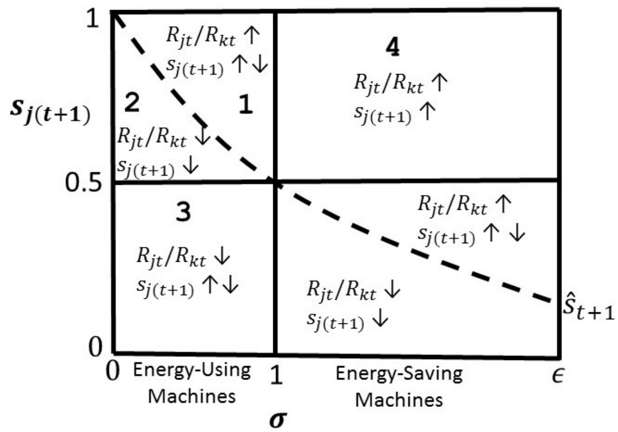


Figure 3: The evolution of research effort and extraction from times t to $t+1$ under the conditions of Proposition 1. The changes use the elasticity of substitution σ along the horizontal axis and the time $t+1$ research allocation along the vertical axis in Proposition 1 and equation (10). The dashed line plots $\hat{s}_{t+1}$, and the arrows indicate whether variables are increasing (↑) or decreasing (↓) in each region.

Small σ strengthens the supply expansion effect, but small σ also makes resources respond more strongly to advancing technology, which works against the supply expansion effect to keep scientists in sector j. As sector j's share of resource extraction increases, the supply expansion effect can eventually dominate, at which point research effort begins switching to sector k. In Figure 3, the arrow on s_{jt} switches from pointing up to pointing down and we begin moving down in region 1.

At this point, sector j is still advancing relative to sector k ($s_{j(t+1)} > \hat{s}_{t+1} > 0.5$), and sector j is still increasing its share of extraction as time passes. As we move down in region 1, we eventually reach $\hat{s}_{t+1}$ and enter region 2. This is the moment at which extraction begins shifting to sector k, and this is the moment when we say that the transition has occurred. The quality of technology in sector j is still advancing relative to that in sector k, but the improvements in the machine-scarce sector k are now increasing resource use at a faster rate than are the improvements in the machine-rich sector j.

As we keep moving down in Figure 3, we reach a point at which sector k begins attracting the majority of research effort and begins advancing faster than does sector j. Here, in region 3, the supply expansion effect begins pushing researchers towards sector j. Nonetheless, sector k's increasing share of extraction suffices to draw additional scientists to sector k for some further length of time.

The next proposition describes the moment at which resource extraction shifts from being dominated by one sector to being dominated by the other.

Proposition 4. *Assume that $\sigma < 1$ and $R_{jt} = R_{kt}$.*

1. *If* $\Psi_j = \Psi_k$*, then* $\nu > 0.5$ *if and only if* $A_{jt} < A_{kt}$*, in which case* $s_{jt} > 0.5$*.*

2. *If* $\nu = 0.5$*, then* $\Psi_j > \Psi_k$ *if and only if* $A_{jt} < A_{kt}$*, in which case* $s_{jt} > 0.5$*.*

Proof. See appendix. □

The relative quality of technology at the time of dominance-switching depends on the relative quality of each type of energy (determined by ν) and on the relative abundance of each type of resource (determined by Ψ_j and Ψ_k). If the two resources are equally abundant ($\Psi_j = \Psi_k$), then we can observe dominance switching away from the higher quality type of energy only if the lower quality type of energy has better technology (result 1). Something must be pulling extraction away from the higher-quality sector, and that something must be better technology in the less useful sector. Similarly, if the two intermediates are of equal quality ($\nu = 0.5$), then we can observe dominance switching away from the sector with the more abundant resource only if the other sector is relatively advanced (result 2). As before, something must be pulling extraction away from the sector favored by primitives, and that something must be better technology.

Putting these pieces together, we have seen from Corollary 3 that, when each resource is equally abundant, a transition from sector j to sector k is possible only if sector j is more advanced than sector k. For this transition to progress to the point where the formerly backward sector k becomes dominant, some primitives must be pulling activity towards sector k. In particular, the energy produced in sector k must be of higher quality. If sector k is relatively backward, produces lower quality energy, and has a less abundant resource, then it will never begin dominating resource extraction.[11] This conclusion might make us pessimistic about the likelihood of a laissez-faire transition to clean energy happening in the next decades.

Finally, consider the case of energy-saving machines. Here, $\sigma > 1$ and $\hat{s}_{t+1} < 0.5$, from Proposition 1. If sector j is relatively advanced, dominating research activity, and attracting a greater share of extraction over time, then we know from equation (10) that it must also be attracting more scientists over time. This case corresponds to the top right region (labeled 4) in Figure 3. Here sector j is becoming more advanced relative to sector k. Its increasingly improved relative technology attracts ever more scientists because the patent quality effect dominates the supply expansion effect in equation (9). Further, the continuing increase in sector j's share of resource extraction also works to attract more scientists to sector j. We therefore move up in Figure 3, towards a case in which all scientists work in sector j and sector j continues increasing its dominance of resource extraction. Once some sector begins dominating research effort and extraction, it continues increasing its share at all later times. In contrast to the case with $\sigma < 1$, the dominant energy resource is locked-in.

[11] Historically, transitions have indeed tended to occur in the direction of higher-quality forms of energy (Schurr, 1984; Grübler, 2004).

3.3 Long-Run Outcomes

Now consider long-run outcomes. We focus on a path with balanced growth in technology, along which A_{jt}/A_{kt} is constant. Begin by studying energy-using machines ($\sigma < 1$):

Proposition 5. *Assume $\sigma < 1$. The following are true along a path with balanced growth in technology:*

1. $s_{jt} = 0.5$.
2. R_{jt}/R_{kt} *is constant.*
3. $\Psi_j > \Psi_k$ *if and only if* $A_{jt} > A_{kt}$.
4. $A_{jt} > A_{kt}$ *if and only if* $R_{jt} > R_{kt}$.
5. *The following are equivalent:* $\Psi_j = \Psi_k$, $A_{jt} = A_{kt}$, $R_{jt} = R_{kt}$, *and* $\nu = 0.5$.
6. $R_{j(t+1)}/R_{jt}$ *and* $R_{k(t+1)}/R_{kt}$ *each equal* $\left(1 + \frac{1}{2}\eta\gamma\right)^{\frac{\psi}{\psi - \alpha/(1-\alpha)}}$, *which increases in* α *and decreases in* ψ.

Proof. See appendix. □

A_{jt}/A_{kt} is constant over time if and only if $s_{jt} = 0.5$ (result 1), and via equation (A-1), any equilibrium that keeps $s_{jt} = 0.5$ for multiple periods must have R_{jt}/R_{kt} constant over those periods (result 2). Thus, balanced growth in technology implies balanced growth in resource extraction.

The levels of relative technology and extraction are determined by the condition for relative extraction to be constant over time. Relative extraction is constant over time when $s_{j(t+1)} = \hat{s}_{t+1}$, as defined in Proposition 1. Balanced growth therefore requires $\hat{s}_{t+1} = 0.5$, which requires, from equation (15), that $\Sigma_{Y_{jt},X_{jt}} = \Sigma_{Y_{kt},X_{kt}}$ at $s_{jt} = 0.5$. For $\sigma < 1$, each elasticity becomes larger as that sector's resource becomes more abundant (i.e., as Ψ increases) and becomes smaller as that sector's average technology improves. Thus, we can have a balanced growth path in which $\Psi_j > \Psi_k$ if and only if $A_{jt} > A_{kt}$ (result 3). And recall that the relative innovation incentive in equation (9) is increasing in R_{jt}/R_{kt} and, for $\sigma < 1$, decreasing in A_{jt}/A_{kt}. A balanced growth path can therefore have $A_{jt} > A_{kt}$ if and only if $R_{jt} > R_{kt}$ (result 4). In sum, we have just seen that the sector with the more abundant resource has more advanced technology and a greater share of extraction along a balanced growth path.

Each type of extraction is increasing along a balanced growth path (result 6, using $\psi > \alpha/(1-\alpha)$). Resource use must increase at a rate that keeps the growth rate of intermediate good production constant. This in turn requires that resources and machine services grow at the same rate. Each technology is improving at rate $1 + 0.5\eta\gamma$, which works to increase the supply of machine services over time. From equation (7), the price of machine services is not sensitive to the supply of machine services when α is large. Therefore, when α is large, equilibrium production of machine services increases strongly as technology improves, which means that equilibrium resource use must also increase strongly as technology improves.

When ψ is small, resource supply is not sensitive to its price. It therefore takes a large increase in the resource price to make extraction grow as fast as machine services, but from equation (8), this increase in the resource price makes machine service production grow even faster. Small ψ therefore requires faster growth in resource extraction along a balanced growth path.

Finally, consider long-run outcomes when machines are energy-saving ($\sigma > 1$). There is a knife-edge case in which research is divided equally between the two sectors so that the relative quality of technology remains constant. However, in general, this economy will not approach a path with balanced growth in technology. Instead, it tends to be locked-in to the more advanced sector, in which case the quality of technology and the quantity of resource extraction always grow faster in the locked-in sector.

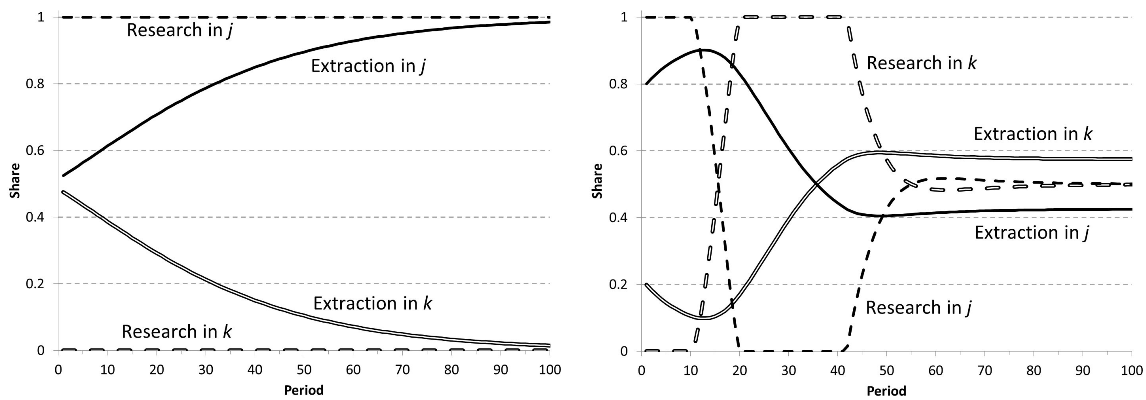
3.4 Numerical Example

In order to make these ideas more concrete, Figure 4 plots the evolution of sector j's share of extraction and of research activity, starting from a point at which sector j is more advanced.[12] Sector j begins with the majority of extraction and research activity, and its share of each is initially increasing. In the case of energy-saving technologies (left panel, $\sigma = 2$), research activity and extraction are locked-in to sector j, which attracts all research effort in all periods and increases its share of resource extraction over time. In the case of energy-using technologies (right panel, $\sigma = 0.5$), we see the type of innovation-led transition described above. Research activity begins shifting to sector k after 10 periods, and after an additional 3 periods, extraction also begins shifting to sector k. Sector k begins dominating research activity in period 18 and begins dominating extraction activity in period 36. Eventually, the supply expansion effect begins pushing scientists back towards sector j and we approach a balanced growth path with research activity and extraction divided equally between the two sectors (because $\Psi_j = \Psi_k$).

4 Are Machines Energy-Using or Energy-Saving?

We have seen that qualitatively different dynamics emerge when machines are energy-using rather than energy-saving. Three lines of evidence support the idea that, historically, innovation in energy-using machines was more important. First, as mentioned in the introduction,

[12] The example's parameters are $\epsilon = 3$, $\nu = 0.45$, $\alpha = 0.5$, $\kappa = 0.5$, $\psi = 3$, $\Psi_j = \Psi_k = 1$, $\eta = 1$, $\gamma = 0.5$, $A_{j0} = 0.5$, and $A_{k0} = 0.005$. The qualitative results are not sensitive to the choice of parameters.



(a) $\sigma = 2$ (b) $\sigma = 0.5$

Figure 4: An example of lock-in (left, with $\sigma = 2$), and an example of an innovation-led transition (right, with $\sigma = 0.5$)).

several energy and economic historians have observed that innovation, not depletion, appears to have been responsible for past transitions in energy supply. We have seen that innovation can indeed drive transitions when machines are energy-using. In contrast, when machines are energy-saving, then we see a counterfactually high degree of lock-in unless there are additional restrictions on resource supply.

Second, the endogenous dynamics of our setting with energy-using machines are qualitatively similar to historical patterns. Figure 5 plots resource shares since 1800. The patterns in these shares are similar to the patterns that emerge from our numerical simulations with energy-using machines (right panel of Figure 4) and nothing like the patterns that emerge from our simulations with energy-saving machines (left panel of Figure 4).

Third, Hassler et al. (2012) recently estimated the elasticity of substitution between energy and a capital-labor composite. They concluded that this elasticity is clearly less than unity and not statistically distinguishable from zero.[13] A Leontief production function for the intermediates Y_{jt} and Y_{kt} may describe the world better than does a Cobb-Douglas production function. This evidence again suggests that the case of energy-using machines is the more empirically relevant one.

[13] Much other empirical work has also suggested that the elasticity of substitution between energy and non-energy inputs is less than unity (Prywes, 1986; Manne and Richels, 1992; Chang, 1994; Koetse et al., 2008; van der Werf, 2008; Stern and Kander, 2012).

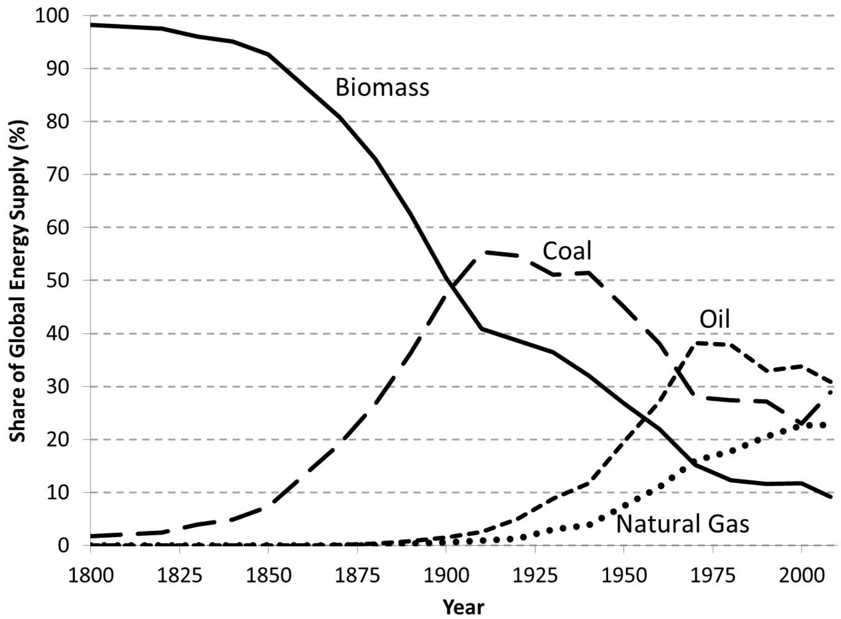


Figure 5: Shares of global energy supply, from Smil (2010). This plot excludes hydroelectricity and nuclear, whose shares never exceed 7%.

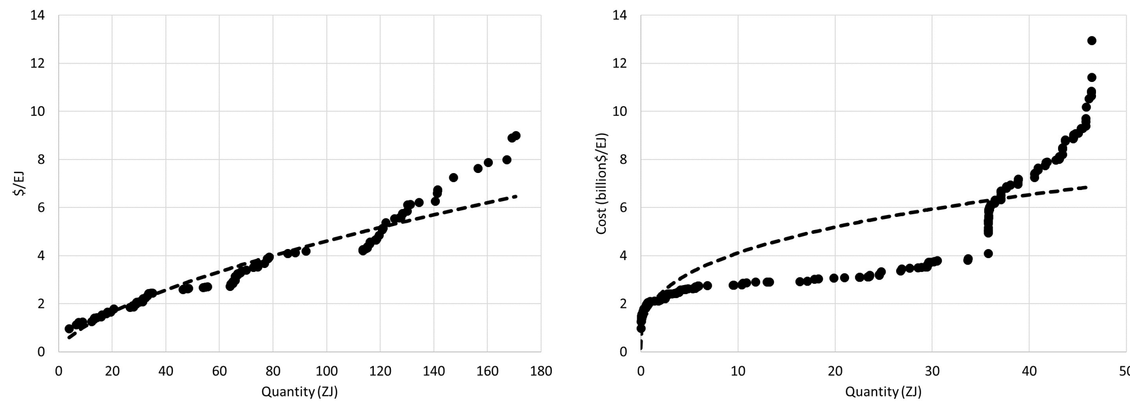
5 In Progress: Optimal Carbon Taxation

I now consider the implications of the present model for optimal policies to address climate change. Let resource j be carbon-intensive and resource k be carbon-free, so that consuming resource j generates emissions that drive climate change. The evolution of environmental quality, preferences over consumption and environmental quality, and intertemporal welfare follow Acemoglu et al. (2012): emissions contribute to a stock of carbon dioxide that decays over time, utility goes to negative infinity (with infinite marginal utility from a reduction in the stock) as that stock reaches a level consistent with a global temperature $\bar{T}$ that implies disaster, and the annual utility discount rate is 1.5%.[14] I here use $\bar{T} = 2°$C. I use a five-year timestep and a horizon of 300 years. I now describe the new aspects of my calibration before presenting the results.

First, consider the supply of the fossil resource. McCollum et al. (2014) developed supply curves for coal, oil, and gas for the MESSAGE energy model.[15] By measuring each resource in terms of energy and combining them into a single supply curve, I estimate ψ_j and Ψ_j as in equation (4). Doing this, we obtain $\psi_j = 1.58$ and $\Psi_j = 8981.28$, with the resource price in billion year 2014 dollars per EJ and resource quantity in EJ. The left panel of Figure 6 plots

[14] In ongoing work, I will instead use more realistic climate dynamics from Lemoine and Rudik (2014).

[15] I thank David McCollum for providing the data underlying the MESSAGE supply curve.



(a) Supply of the Fossil Resource (b) Supply of the Renewable Resource

Figure 6: The constructed supply curves (solid points) for the fossil resource (left) and the renewable resource (right), as well as the fitted curves (dashed lines) used in the simulations. Note that the scale of the horizontal axis differs between the panels.

the constructed supply curve as well as the fitted curve used in the present simulations.

Next consider the supply of the clean resource. Drawing in part on the work of others, Johnson et al. (2016) describe the supply of power from solar photovoltaics, concentrating solar power, onshore wind, and offshore wind available in each region of the world, and they do so separately for different qualities of resource. Costs are reported in dollars per unit power and resource potential is reported in units of energy. I convert costs to dollars per unit electrical energy by using the capacity factor reported for each resource quality bin in each region. This capacity factor adjusts for the fact that the power producible from renewable resources is not available throughout the day or throughout the year. And I convert dollars per unit of electrical energy to dollars per units of energy in the resource by using the efficiency of each type of generator.[16] Aggregating across resource types and regions, I estimate ψ_k and Ψ_k from equation (4), yielding $\psi_k = 3.00$ and $\Psi_k = 143.01$, with the resource price again in billion year 2014 dollars per EJ and resource quantity in EJ/year.[17] The fact that Ψ_k is so much smaller than Ψ_j reflects the relative abundance of fossil resources. The right panel of Figure 6 plots the constructed and fitted supply curves for the renewable resource.

The parameters of the innovation process are the same as in Acemoglu et al. (2012), and I also follow them in exploring cases with $\epsilon = 3$ and $\epsilon = 10$. I calibrate $1 - \kappa$ to match the share of machines in intermediate-good production in Acemoglu et al. (2012), which yields

[16] From the Energy Information Administration's Annual Energy Review 2011, the efficiencies are 12% for solar photovoltaic, 21% for solar thermal, and 26% for wind.

[17] The cost calculations assume that generators last for 20 years. All simulations assume that generators last for only a single timestep (i.e., five years).

$\kappa = 2/3$. I fix $\alpha = 0.5$, which has no analogue in Acemoglu et al. (2012). I explore values for σ of 0.5 and 2.

There are three differences with respect to Acemoglu et al. (2012) on the environmental side of the model. First, Acemoglu et al. (2012) have emissions arising from use of the dirty intermediate Y_{jt}, which implies that advancing technology A_{jt} in the fossil sector increases emissions. However, emissions should depend directly on dirty resource use, not on the quality of technology for using dirty resources. Second, and relatedly, Acemoglu et al. (2012) model their emission tax τ_t as being placed on the dirty intermediate and multiplying its price (so that the price becomes $(1 + \tau_t)\, p_{jt}$). I instead follow the more conventional approach of taxing emissions through an additive tax on resource use. The supply of the dirty resource j becomes $R_{jt} = \Psi_j \left[p_{jRt} - \tau_t\right]^{\psi_j}$. Third, I update the initial conditions to reflect more recent data: initial carbon dioxide becomes 400 ppm (based on observations in 2015), and multiplying year 2011 global carbon dioxide emissions from fossil fuel combustion (from the Carbon Dioxide Information Analysis Center) by the timestep length gives the initial increase in CO_2 as 20.6 ppm per timestep. I calculate the emission intensity of the fossil resource from the initial increase in CO_2 and initial consumption of the fossil resource.

We have four remaining free parameters: A_{j0}, A_{k0}, ν, and A_Y. I calibrate these so that the first period's equilibrium matches conditions on R_{j0}, R_{k0}, s_{k0}, and Y_0. Initial fossil resource consumption R_{j0} comes from summing the consumption of oil, gas, and coal from 2011–2015, as reported in the BP Statistical Review of World Energy. This yields $R_{j0} = 2333$ EJ. Using the analogous values for non-hydro renewables yields $R_{k0} = 112$ EJ.[18] The NSF's Business Research and Development and Innovation 2014 gives worldwide employment in fossil R&D as 19,000 people and in renewable R&D as 2,000 people.[19] This implies $s_{k0} = 0.0952$. Finally, World Bank data for global output implies $Y_0 = 64,750$ in billion year 2014 dollars. For $\sigma = 0.5$, this calibration yields $A_{j0} > A_{k0}$, with $\nu = 0.17$ for $\epsilon = 3$ and $\nu = 0.24$ for $\epsilon = 10$. For $\sigma = 2$ and $\epsilon = 10$, this calibration yields $A_{j0} < A_{k0}$ and $\nu = 0.54$.[20]

Figure 7 plots laissez-faire outcomes for cases with $\epsilon = 10$.[21] When machines are energy-using ($\sigma = 0.5$), we see clean technologies begin to dominate research activity within fifty years (top panel). The clean resource's share of resource use begins increasing as this research effort leads to improved clean energy technologies, but the clean resource's share

[18] To obtain the energetic content of renewables for the reported tonnes of oil equivalent, use BP's assumed thermal efficiency of 38% to obtain the equivalent electrical energy and then use a 20% generator efficiency to convert electrical energy to energy in the renewable resource (see footnote 16).

[19] For fossil R&D, I use NAICS 21 ("mining, extraction, and support activities"), and for renewable R&D, I use NAICS 22 ("utilities").

[20] There does not seem to be a way to choose A_{j0}, A_{k0}, ν, and A_Y to match the conditions on R_{j0}, R_{k0}, s_{k0}, and Y_0 when $\sigma = 2$ and $\epsilon = 2$.

[21] The case with $\sigma = 0.5$ and $\epsilon = 3$ has the clean resource's share of scientists (energy supply) increase at a slightly slower (faster) rate than in the plotted case with $\sigma = 0.5$ and $\epsilon = 10$. The resulting temperature trajectory for the case with $\sigma = 0.5$ and $\epsilon = 3$ is similar to the plotted trajectory for the case with $\sigma = 2$ and $\epsilon = 10$.

remains below 12% until after the next century (middle panel). The clean resource's increasing contribution to energy supply is not sufficient to prevent temperature from increasing rapidly (bottom panel). When machines are energy-saving ($\sigma = 2$), the fossil resource is more strongly locked-in. We see the clean resource's share of research effort quickly drop to zero (bottom panel), and we see its share of production decline monotonically (middle panel). Temperature therefore increases even faster than in the case of energy-using machines (bottom panel). The laissez-faire dynamics corresponding to $\sigma = 0.5$ appear more plausible than the laissez-faire dynamics corresponding to $\sigma = 2$ and are also consistent with the numerous projections that renewable resources will supply a greater share of energy over the next decades. For either value of σ, policy will be required to avoid exceeding the disaster temperature of 2°C.

In Acemoglu et al. (2012), a temporary subsidy to clean research can suffice to avoid an environmental disaster. However, we here see that research will quickly switch towards clean resources even in the absence of policy but that clean resources will nonetheless remain a small share of total energy production for many decades. A temporary research subsidy cannot suffice to avoid reaching the disaster temperature. An emission tax becomes critical once we weaken the lock-in generated by the assumption of $\sigma = 1$ in Acemoglu et al. (2012).

Figure 8 explores the optimal emission tax for cases with $\sigma = 0.5$. When $\epsilon = 3$, the tax begins around \$5 per tCO_2 and declines towards zero as the clean resource becomes sufficiently advanced (top panel). The emission tax immediately and permanently redirects all research effort towards the clean resource (not shown), and it increases the clean resource's share of energy supply above the laissez-faire level (middle panel). As a result, the world avoids the disaster temperature of 2°C.[22] When $\epsilon = 10$, it is easier to substitute renewable energy for fossil energy. A fairly small emission tax of just over \$14 per tCO_2 then suffices to redirect nearly all energy supply towards renewable resources, and we see the optimal tax indeed take such a path. As a result, temperature declines even more swiftly than in the case with $\epsilon = 3$.

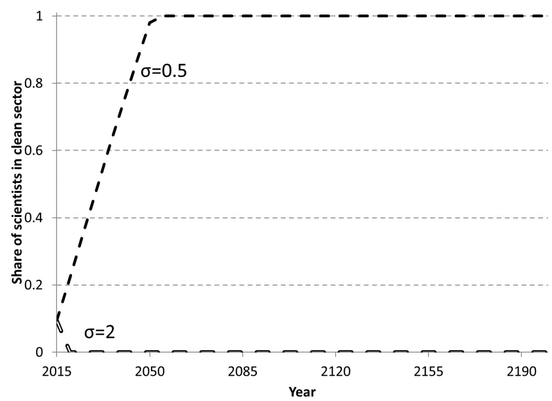
6 Conclusion

In progress.

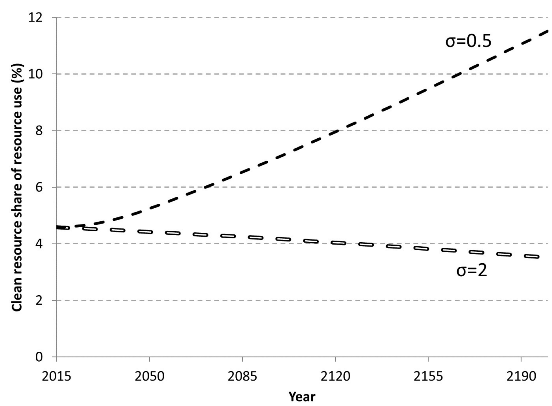
First Appendix: Tâtonnement Stability and Uniqueness

This first appendix considers the stability of each period's equilibrium. One may be concerned that interior equilibria are not "natural" equilibria in the presence of positive feedbacks from resource extraction to innovation and of potential complementarities. Indeed,

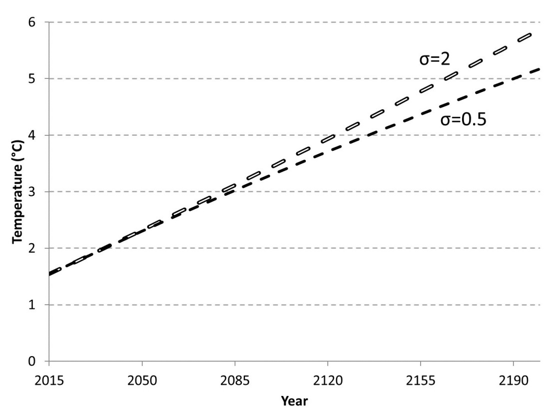
[22] In fact, we see temperature decline fairly quickly towards zero. These dynamics are unrealistic. In ongoing work, I will implement more realistic climate dynamics from Lemoine and Rudik (2014).



(a) Clean Resource Share of Research

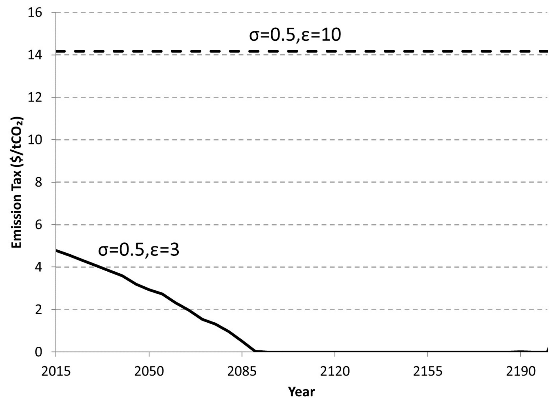


(b) Clean Resource Share of Resource Use

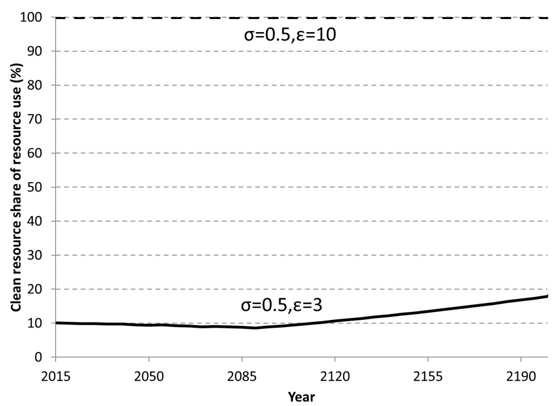


(c) Temperature

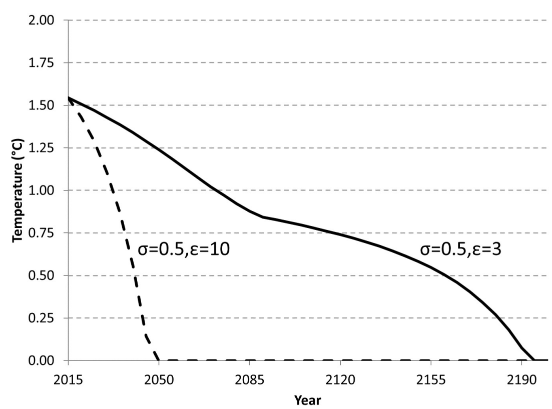
Figure 7: Laissez-faire outcomes. Plotted results use $\epsilon = 10$.



(a) Emission Tax



(b) Clean Resource Share of Resource Use



(c) Temperature

Figure 8: Outcomes under an optimized emission tax, with $\bar{T} = 2°C$.

Acemoglu (2002) and Hart (2012) have emphasized the role of knowledge spillovers in allowing interior research allocations to be stable in the long run. This appendix shows that interior equilibria are in fact "natural" equilibria in the present setting, and it also shows that the equilibrium is unique.

Substituting from the resource supply functions, we can rewrite equation (9) as

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)} + \gamma A_{j(t-1)}}{A_{k(t-1)} + \gamma A_{k(t-1)}} \left(\frac{A_{j(t-1)} + \eta\gamma s_{jt} A_{j(t-1)}}{A_{k(t-1)} + \eta\gamma s_{kt} A_{k(t-1)}} \right)^{\frac{-1}{\sigma+\alpha(1-\sigma)}} \left(\frac{R_{jt}}{R_{kt}} \right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k} \right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} . \tag{A-1}$$

Rearranging and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation.[23] Substituting into equations (12) and (13) then gives us two equations in two unknowns, which define the equilibrium R_{jt} and R_{kt} that clear the markets for each resource.

Define the tâtonnement adjustment process and stability as follows:

Definition 3. *A* tâtonnement adjustment process *increases R_{jt} if equation* (12) *is not satisfied and its right-hand side is greater, decreases R_{jt} if equation* (12) *is not satisfied and its left-hand side is greater, and obeys analogous rules for R_{kt} using equation* (13). *We say that an equilibrium (R^*_{jt}, R^*_{kt}) is* tâtonnement-stable *if and only if the tâtonnement adjustment process leads to (R^*_{jt}, R^*_{kt}) from (R_{jt}, R_{kt}) sufficiently close to (R^*_{jt}, R^*_{kt}).*

The tâtonnement process changes R_{jt} and R_{kt} so as to eliminate excess supply or demand, and tâtonnement stability requires that this adjustment process converge to an equilibrium point from values close to the equilibrium. We can show that our equilibrium is tâtonnement-stable:

Proposition A-1. *The equilibrium is tâtonnement-stable.*

Proof. See appendix. □

A Walrasian auctioneer would find our equilibrium at any time t.

Now use equations (12) and (13) to define R_{jt} and R_{kt} as functions of s_{jt},[24] and then restate equation (A-1) as a function only of s_{jt}:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)} + \gamma A_{j(t-1)}}{A_{k(t-1)} + \gamma A_{k(t-1)}} \left(\frac{A_{j(t-1)} + \eta\gamma s_{jt} A_{j(t-1)}}{A_{k(t-1)} + \eta\gamma (1-s_{jt}) A_{k(t-1)}} \right)^{\frac{-1}{\sigma+\alpha(1-\sigma)}} \left(\frac{R_{jt}(s_{jt})}{R_{kt}(s_{jt})} \right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k} \right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} . \tag{A-2}$$

The following corollary gives us the derivative of Π_{jt}/Π_{kt} with respect to s_{jt}:

[23] Technically, this function should be written to allow for corner solutions in the research allocation. The proof of stability will account for corner solutions.

[24] Rearrange equations (12) and (13) to put all terms on the right-hand side. For given s_{jt}, the Jacobian of this system in R_{jt} and R_{kt} is negative definite.

Corollary A-2. *The right-hand side of equation* (A-2) *strictly decreases in* s_{jt}.

Proof. See appendix. □

The supply expansion effect makes the relative incentive to research in sector j decline in the number of scientists working in sector j. However, when sector j's share of resource extraction increases in the relative quality of its technology, we have a positive feedback between research and extraction that maintains sector j's research incentives even as more scientists move to sector j. The proof shows, as is intuitive, that whether the relative incentive to research in sector j declines in the number of scientists working in sector j is identical to whether the equilibrium is tâtonnement-stable: tâtonnement-stability is not consistent with positive feedbacks that are strong enough to overwhelm the supply expansion effect. And we have already seen that interior equilibria are in fact tâtonnement-stable.

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Second Appendix: Proofs and Derivations

This second appendix derives results that will be useful in the proofs before providing proofs and derivations omitted from the main text.

Useful Lemmas

First, note that equations (7) and (8) imply

$$X_{jt} = \left[\frac{1-\kappa}{\kappa} p_{jRt}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}. \tag{A-3}$$

Rearranging equation (A-1) and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation:

$$s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right) = \frac{(1+\eta\gamma)\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi_j}}{[R_{kt}/\Psi_k]^{1/\psi_k}}\right]^{\sigma} - 1}{\eta\gamma + \eta\gamma\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi_j}}{[R_{kt}/\Psi_k]^{1/\psi_k}}\right]^{\sigma}}. \tag{A-4}$$

Let $\Sigma_{x,y}$ represent the elasticity of x with respect to y, and let $\Sigma_{x,y|z}$ represent the elasticity of x with respect to y holding z constant. The following lemma establishes signs and bounds for key elasticities:

Lemma A-3. *The following hold, with analogous results for sector k:*

1. $\Sigma_{Y_t,Y_{jt}}, \Sigma_{Y_t,Y_{kt}} \in [0,1]$ *and* $\Sigma_{Y_t,Y_{jt}} + \Sigma_{Y_t,Y_{kt}} = 1$.
2. $\Sigma_{Y_{jt},R_{jt}|X_{jt}}, \Sigma_{Y_{jt},X_{jt}} \in [0,1]$ *and* $\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} = 1$.
3. $\Sigma_{X_{jt},A_{jt}} = \frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha} \in (0,1)$
4. $\Sigma_{X_{jt},R_{jt}} = \frac{\alpha\sigma/\psi+\alpha}{\sigma(1-\alpha)+\alpha} \in (0,1)$ *if and only if* $\psi > \frac{\alpha}{1-\alpha}$
5. $\Sigma_{A_{jt},s_{jt}} = \frac{\eta\gamma s_{jt}}{1+\eta\gamma s_{jt}} \in [0,1)$
6. $\Sigma_{s_{jt},R_{jt}} = \frac{\psi+\sigma}{\psi}\frac{2+\eta\gamma}{\eta\gamma s_{jt}} Z_t > 0$, *where* $Z_t \in \left[\frac{1+\eta\gamma}{(2+\eta\gamma)^2}, \frac{1}{4}\right]$. $\Sigma_{s_{jt},R_{kt}} = -\Sigma_{s_{jt},R_{jt}}$.
7. $\Sigma_{s_{jt},A_{j(t-1)}} = -\frac{(1-\sigma)(1-\alpha)}{A_{j(t-1)}}\frac{(2+\eta\gamma)}{\eta\gamma} Z_t$, *which is* < 0 *if and only if* $\sigma < 1$. Z_t *is as above.* $\Sigma_{s_{jt},A_{k(t-1)}} = -\Sigma_{s_{jt},A_{j(t-1)}}$.
8. $\Sigma_{s_{jt},s_{kt}} = -s_{kt}/s_{jt} \leq 0$

Proof. All of the results follow by differentiation and the definition of an elasticity. #1 follows from differentiating the final-good production function $Y_t(Y_{jt}, Y_{kt})$, #2 follows from differentiating the intermediate-good production function $Y_{jt}(R_{jt}, X_{jt})$, #3 and #4 follow from differentiating equation (A-3), #5 follows from differentiating equation (3), #6 and #7 follow from differentiating equation (A-4), and #8 follows from the research constraint.

To derive #6 and #7, define

$$Z_t \triangleq \frac{\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi_j}}{[R_{kt}/\Psi_k]^{1/\psi_k}}\right]^{\sigma}}{\left[1+\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi_j}}{[R_{kt}/\Psi_k]^{1/\psi_k}}\right]^{\sigma}\right]^2}$$

and recognize that $s_{jt} \in (0, 1)$ implies

$$\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi_j}}{[R_{kt}/\Psi_k]^{1/\psi_k}}\right]^{\sigma} \in \left(\frac{1}{1+\eta\gamma},\ 1+\eta\gamma\right)$$

from equation (A-1).

□

Note that $\Sigma_{X,A}$ and $\Sigma_{X,R}$ are the same in each sector. We therefore often omit the sector subscripts on these terms.

Using $s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right)$, the equilibrium is defined by equations (12) and (13), which are functions only of R_{jt} and R_{kt}. Rewrite these equations as (suppressing the technology arguments)

$$1 = \kappa\, \nu \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{Y_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{Y_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi_j} \triangleq G_j(R_{jt}, R_{kt}),$$

$$1 = \kappa\,(1-\nu) \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{Y_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{Y_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi_k} \triangleq G_k(R_{jt}, R_{kt}).$$

Then we have:

Lemma A-4. *$\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$ and $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt} < 0$.*

Proof. Differentiating, we have:

$$
\begin{aligned}
\frac{\partial G_j(R_{jt}, R_{kt})}{\partial R_{jt}} =& G_j \Bigg\{ -\left(\frac{1}{\psi}+\frac{1}{\sigma}\right)\frac{1}{R_{jt}} + \left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\frac{1}{Y_{jt}}\left[\frac{\partial Y_{jt}}{\partial R_{jt}} + \frac{\partial Y_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right] \\
&+ \frac{1}{\epsilon}\frac{1}{Y_t}\left[\frac{\partial Y_t}{\partial Y_{jt}}\frac{\partial Y_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial Y_{jt}}\frac{\partial Y_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial Y_{kt}}\frac{\partial Y_{kt}}{\partial s_{kt}}\frac{\partial s_{kt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right]\Bigg\} \\
=& \frac{G_j}{R_{jt}}\Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\left[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
&- \frac{1}{\epsilon}\Bigg[\left(1-\Sigma_{Y_t,Y_{jt}}\right)\left(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
&- \Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg]\Bigg\}.
\end{aligned}
$$

If we are at a corner in s_{jt}, then $\Sigma_{s_{jt},R_{jt}} = 0$ and, using Lemma A-3, the above expression is clearly negative. So consider a case with interior s_{jt}. The final two lines are negative. So the overall expression is negative if the third-to-last line is negative, which is the case if and only if

$$
\begin{aligned}
0 \geq& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\left(\frac{\sigma+\psi}{\psi}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha}\right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[-1 + \frac{\sigma+\psi}{\psi}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha}\right],
\end{aligned} \tag{A-5}
$$

where we use results from Lemma A-3. Note that $\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t \leq 3/4$, which implies that $\Sigma_{Y_{jt},X_{jt}}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha} < 1$. Using this, inequality (A-5) holds if and only if

$$
\frac{\sigma}{\psi} \geq \Sigma_{Y_{jt},X_{jt}}\frac{-1+\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\alpha+\sigma(1-\alpha)}}{1-\Sigma_{Y_{jt},X_{jt}}\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\alpha+\sigma(1-\alpha)}}. \tag{A-6}
$$

$\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t \leq 3/4$ implies that $\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\alpha+\sigma(1-\alpha)} < 1$, which implies that the right-hand side of inequality (A-6) is negative. Thus, inequality (A-6) always holds and $\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$.

The analysis of $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt}$ is virtually identical.

□

Now define the matrix G:

$$G \triangleq \begin{bmatrix} \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}} \\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}} \end{bmatrix}.$$

Then we have:

Lemma A-5. *The determinant of G is positive.*

Proof. Analyze $det(G)$:

$$\begin{aligned}
det(G) \propto & \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \left[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right]\Bigg\} \\
& \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \left[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \left(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\right]\Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \bigg[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}} \left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
& \qquad - \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\bigg]\Bigg\} \\
& \Bigg\{\frac{1}{\epsilon}\bigg[\Sigma_{Y_t,Y_{kt}}\left(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right) \\
& \qquad + \Sigma_{Y_t,Y_{jt}}\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\bigg]\Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \bigg[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}} \left(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right) \\
& \qquad - \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\bigg]\Bigg\} \\
& \Bigg\{\frac{1}{\epsilon}\bigg[\Sigma_{Y_t,Y_{jt}}\left(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
& \qquad + \Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\bigg]\Bigg\} \\
& - \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)^2 \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}},
\end{aligned}$$

where we factored $G_j G_k / R_{jt} R_{kt}$. Use $\Sigma_{Y_t,Yjt} + \Sigma_{Y_t,Y_{kt}} = 1$ from Lemma A-3 and cancel terms with $1/\epsilon^2$ to obtain:

$$\begin{aligned}
det(G) \propto & \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\left[1 - \Sigma_{Y_{jt},R_{jt}|X_{jt}} - \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right]\right\} \\
& \quad \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\left[1 - \Sigma_{Y_{kt},R_{kt}|X_{kt}} - \Sigma_{Y_{kt},X_{kt}}\left(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\right]\right\} \\
& - \frac{1}{\sigma}\left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
& + \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}} \\
& \quad \left[-\left(\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}} + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\right. \\
& \quad \quad \left. + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right] \\
& + \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}} \\
& \quad \left[-\left(\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right. \\
& \quad \quad \left. + \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma}\left[\Sigma_{Y_{jt},R_{jt}|X_{jt}} + \Sigma_{Y_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
& \quad \left[\Sigma_{Y_{kt},R_{kt}|X_{kt}} + \Sigma_{Y_{kt},X_{kt}}\left(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\right].
\end{aligned} \tag{A-7}$$

All lines after the first three are positive by results from Lemma A-3. Expanding the products in those first three lines and rearranging, those first three lines become:

$$\begin{aligned}
& \frac{1}{\psi^2} \\
& + \frac{1}{\sigma^2}\left[1 - \Sigma_{X,R}\right]\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\left(1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\left[1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right] \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[1 - \Sigma_{X,R} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right] \\
& + \frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right),
\end{aligned} \tag{A-8}$$

where we write $\Sigma_{X,R}$ because this elasticity is the same in each sector. The term in parentheses on the second line becomes

$$\begin{aligned}
&1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\\
=&\frac{1}{\psi}\frac{\sigma}{\sigma(1-\alpha)+\alpha}\left\{\psi[1-\alpha] - \alpha - (1-\alpha)[\sigma+\psi]\frac{(2+\eta\gamma)^2}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})}Z_t\right\}.
\end{aligned} \tag{A-9}$$

Substituting for Z_t and using equation (A-1) at $\Pi_{jt}/\Pi_{kt} = 1$, we have

$$\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}.$$

Equation (A-9) then becomes

$$1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} = -\frac{\sigma}{\psi}.$$

Substituting into (A-8), we have that the first three lines of (A-7) are equal to

$$\begin{aligned}
&\frac{1}{\psi^2}\\
&-\frac{1}{\psi}\frac{1}{\sigma}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\\
&+\frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{kt},X_{kt}}\left[1-\Sigma_{X,R}-\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right]\\
&+\frac{1}{\psi}\frac{1}{\sigma}\Sigma_{Y_{jt},X_{jt}}\left[1-\Sigma_{X,R}-\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right]\\
&+\frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right).
\end{aligned}$$

The final line is positive. If the first through fourth lines are positive with nonzero $\Sigma_{s_{jt},R_{jt}}$ and nonzero $\Sigma_{s_{jt},R_{kt}}$, then they are also positive if s_{jt} is fixed (as at a corner solution). So $det(G) > 0$ if we are at a corner allocation in research. Factoring $1/\psi$, the first through fourth lines are positive if and only if:

$$\begin{aligned}
0 \leq & \frac{1}{\psi} + \frac{1}{\sigma}\Bigg[(1-\Sigma_{X,R}) \left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\
& \qquad - \Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} - \Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg] \\
= & \frac{1}{\psi} + \frac{1}{\sigma}\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\
& - \frac{1}{\sigma}\frac{\sigma+\psi}{\psi}\frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\
& \qquad + \sigma(1-\alpha)\Bigg(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Bigg)\frac{1}{2+\eta\gamma}\Bigg],
\end{aligned} \tag{A-10}$$

where we use $\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}$. Note that $\Sigma_{Y_{jt},X_{jt}}+\Sigma_{Y_{kt},X_{kt}}-\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}$ increases in $\Sigma_{Y_{jt},X_{jt}}$ and thus reaches a maximum at $\Sigma_{Y_{jt},X_{jt}} = 1$.

$$\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}} \leq 1 + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{kt},X_{kt}} = 1.$$

Also note that $\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})$ increases in each elasticity, and each elasticity is ≤ 1. Thus,

$$\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt}) \leq (1+\eta\gamma s_{kt}) + (1+\eta\gamma s_{jt}) = 2+\eta\gamma,$$

which implies

$$\Bigg(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Bigg)\frac{1}{2+\eta\gamma} \leq 1.$$

These results together imply that

$$\begin{aligned}
& \alpha + \sigma(1-\alpha) \\
\geq & \alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) + \sigma(1-\alpha)\Bigg(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Bigg)\frac{1}{2+\eta\gamma}.
\end{aligned} \tag{A-11}$$

Using this, we have that inequality (A-10) holds if and only if

$$\begin{aligned}\frac{\sigma}{\psi} \geq & \Bigg\{ -\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) + \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\ & \qquad + \sigma(1-\alpha)\Bigg(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Bigg)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\} \\ & \Bigg\{1 - \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\ & \qquad + \sigma(1-\alpha)\Bigg(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Bigg)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\}^{-1}. \end{aligned} \tag{A-12}$$

The denominator on the right-hand side is positive via inequality (A-11). The numerator on the right-hand side is equal to:

$$\begin{aligned}&\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\ &\left\{-1 + \frac{1}{\sigma(1-\alpha)+\alpha}\left[\alpha + \sigma(1-\alpha)\frac{\Bigg(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Bigg)}{(2+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right)}\right]\right\}\end{aligned} \tag{A-13}$$

Consider the fraction in brackets. If that fraction is ≤ 1, then the whole expression is negative. Assume that the fraction is > 1. Then:

$$\begin{aligned}&\Bigg(\Sigma_{Y_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{Y_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Bigg) > (2+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\right) \\ \Leftrightarrow & \eta\gamma s_{kt}\Sigma_{Y_{jt},X_{jt}} + \eta\gamma s_{jt}\Sigma_{Y_{kt},X_{kt}} \geq (1+\eta\gamma)\left(\Sigma_{Y_{jt},X_{jt}} + \Sigma_{Y_{kt},X_{kt}}\right) - (2+\eta\gamma)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\end{aligned}$$

Assume without loss of generality that $\Sigma_{Y_{jt},X_{jt}} > \Sigma_{Y_{kt},X_{kt}}$. Then the left-hand side of the last line attains its largest possible value when $s_{kt} = 1$. The inequality on the last line is then satisfied only if

$$0 > \Sigma_{Y_{jt},X_{jt}} + (1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}. \tag{A-14}$$

The right-hand side is monotonic in $\Sigma_{Y_{jt},X_{jt}}$. At $\Sigma_{Y_{jt},X_{jt}} = 1$, the right-hand side is

$$1 + (1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} = 1 - \Sigma_{Y_{kt},X_{kt}} \geq 0.$$

But this contradicts inequality (A-14). Now consider the other extremum: $\Sigma_{Y_{jt},X_{jt}} = 0$. The right-hand side of inequality (A-14) becomes:

$$(1+\eta\gamma)\Sigma_{Y_{kt},X_{kt}} \geq 0,$$

which again contradicts inequality (A-14). Because the right-hand side of inequality (A-14) was monotonic in $\Sigma_{Y_{jt},X_{jt}}$ and was not satisfied for either the greatest or smallest possible values for $\Sigma_{Y_{jt},X_{jt}}$, the inequality is not satisfied for any values of $\Sigma_{Y_{jt},X_{jt}}$. Thus, the fraction in brackets in (A-13) is ≤ 1, which means that the right-hand side of inequality (A-12) is ≤ 0 and inequality (A-12) is satisfied. As a result, the first three lines of (A-7) are positive, which means that $det(G) > 0$.

□

Derivation of Equation (10)

Equation (A-1) implicitly defines s_{jt} as a function of R_{jt}/R_{kt} and $A_{j(t-1)}/A_{k(t-1)}$ (for interior s_{jt}). To a first-order approximation, the total change in s_{jt} is

$$
\begin{aligned}
s_{j(t+1)} - s_{jt} =& \frac{\mathrm{d}s_{jt}}{\mathrm{d}[R_{jt}/R_{kt}]} \left[\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} \right] + \frac{\mathrm{d}s_{jt}}{\mathrm{d}[A_{j(t-1)}/A_{k(t-1)}]} \left[\frac{A_{jt}}{A_{kt}} - \frac{A_{j(t-1)}}{A_{k(t-1)}} \right] \\
=& - \frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial[R_{jt}/R_{kt}]}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}} \left[\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} \right] - \frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial[A_{j(t-1)}/A_{k(t-1)}]}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}} \left[\frac{A_{jt}}{A_{kt}} - \frac{A_{j(t-1)}}{A_{k(t-1)}} \right] \\
\propto& \frac{1+\sigma/\psi}{R_{jt}/R_{kt}} \left[\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} \right] - \frac{(1-\sigma)(1-\alpha)}{A_{j(t-1)}/A_{k(t-1)}} \left[\frac{A_{jt}}{A_{kt}} - \frac{A_{j(t-1)}}{A_{k(t-1)}} \right] \\
=& \frac{1+\sigma/\psi}{R_{jt}/R_{kt}} \left[\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} \right] - 2(1-\sigma)(1-\alpha) \frac{\eta\gamma}{1+\eta\gamma s_{kt}} \left(s_{jt} - \frac{1}{2} \right),
\end{aligned}
$$

where the second line uses the implicit function theorem and the third line factors $-\{\partial[\Pi_{jt}/\Pi_{kt}]/\partial s_{jt}\}^{-1}$, factors $[\sigma + \alpha(1-\sigma)]^{-1}$, and uses $\Pi_{jt} = \Pi_{kt}$ at an interior equilibrium.

From this we have the following lemma, which will be useful in proving Proposition 2:

Lemma A-6. *If $s_{jt} \geq 0.5$, $R_{jt}/R_{kt} \leq R_{j(t+1)}/R_{k(t+1)}$, and $(1-\sigma)(1-\alpha) \leq 0.5$, then $s_{j(t+1)} \geq 0.5$.*

Proof. Note that

$$
\begin{aligned}
-\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} =& \frac{1}{\sigma+\alpha(1-\sigma)} \frac{\Pi_{jt}}{\Pi_{kt}} \left(\frac{1+\eta\gamma s_{jt}}{1+\eta\gamma s_{kt}} \right)^{-1} \left(\frac{\eta\gamma}{1+\eta\gamma s_{kt}} + \frac{\eta\gamma(1+\eta\gamma s_{jt})}{(1+\eta\gamma s_{kt})^2} \right) \\
=& \frac{\eta\gamma}{\sigma+\alpha(1-\sigma)} \frac{\Pi_{jt}}{\Pi_{kt}} \frac{2+\eta\gamma}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})}.
\end{aligned}
$$

Restoring the factored terms in equation (10) and using $R_{jt}/R_{kt} \leq R_{j(t+1)}/R_{k(t+1)}$, we have

$$
s_{j(t+1)} - s_{jt} \geq -2(1-\sigma)(1-\alpha) \frac{1+\eta\gamma s_{jt}}{2+\eta\gamma} \left(s_{jt} - \frac{1}{2} \right) \geq -2(1-\sigma)(1-\alpha) \left(s_{jt} - \frac{1}{2} \right).
$$

Using the assumption that $(1-\sigma)(1-\alpha) \leq 0.5$, we have:

$$s_{j(t+1)} - s_{jt} \geq -\left(s_{jt} - \frac{1}{2}\right),$$

which implies $s_{j(t+1)} \geq 0.5$.

□

Proof of Proposition 1

The change in R_{jt}/R_{kt} from time t to $t+1$ is

$$\begin{aligned}\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} =& \frac{(R_{j(t+1)} - R_{jt})R_{kt} - (R_{k(t+1)} - R_{kt})R_{jt}}{R_{k(t+1)}R_{kt}} \\ \propto& \frac{R_{j(t+1)} - R_{jt}}{R_{jt}} - \frac{R_{k(t+1)} - R_{kt}}{R_{kt}},\end{aligned}$$

where the first equality adds and subtracts $R_{jt}R_{kt}$ in the numerator and the second line factors $R_{jt}/R_{k(t+1)}$. To a first-order approximation, this is

$$\frac{1}{R_{jt}}\left(\frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right) - \frac{1}{R_{kt}}\left(\frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right).$$

The derivatives follow from applying the implicit function theorem to the system of equations defining $G_j(R_{jt}, R_{kt})$ and $G_k(R_{jt}, R_{kt})$. Doing this yields:

$$\begin{aligned}&\frac{1}{R_{jt}}\left(\frac{-\frac{\partial G_j}{\partial A_{jt}}\frac{\partial G_k}{\partial R_{kt}} + \frac{\partial G_j}{\partial R_{kt}}\frac{\partial G_k}{\partial A_{jt}}}{det(G)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial G_j}{\partial A_{kt}}\frac{\partial G_k}{\partial R_{kt}} + \frac{\partial G_j}{\partial R_{kt}}\frac{\partial G_k}{\partial A_{kt}}}{det(G)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ &- \frac{1}{R_{kt}}\left(\frac{-\frac{\partial G_k}{\partial A_{jt}}\frac{\partial G_j}{\partial R_{jt}} + \frac{\partial G_k}{\partial R_{jt}}\frac{\partial G_j}{\partial A_{jt}}}{det(G)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial G_k}{\partial A_{kt}}\frac{\partial G_j}{\partial R_{jt}} + \frac{\partial G_k}{\partial R_{jt}}\frac{\partial G_j}{\partial A_{kt}}}{det(G)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ \propto& \left[-\frac{\partial G_j}{\partial A_{jt}}s_{j(t+1)}A_{jt} - \frac{\partial G_j}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial G_k}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial G_k}{\partial R_{jt}}\right] \\ &+ \left[\frac{\partial G_k}{\partial A_{jt}}s_{j(t+1)}A_{jt} + \frac{\partial G_k}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial G_j}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial G_j}{\partial R_{jt}}\right],\end{aligned} \tag{A-15}$$

where the first expression factors R_{jt}/R_{kt} and the second expression factors $\eta\gamma/det(G)$, which is positive by Lemma A-5. Differentiation and algebraic manipulations (including applying relationships from Lemma A-3) yield:

$$\begin{aligned}-\frac{\partial G_j}{\partial A_{jt}}s_{j(t+1)}A_{jt} - \frac{\partial G_j}{\partial A_{kt}}s_{k(t+1)}A_{kt} =& -G_j\left\{\frac{1}{\sigma} - \frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\right\}\Sigma_{Y_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}s_{j(t+1)} \\ &- G_j\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}(1-s_{j(t+1)}),\end{aligned}$$

$$\begin{aligned}\frac{\partial G_k}{\partial A_{jt}} s_{j(t+1)} A_{jt} + \frac{\partial G_k}{\partial A_{kt}} s_{k(t+1)} A_{kt} =& G_k \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \right\} \Sigma_{Y_{kt}, X_{kt}} \Sigma_{X_{kt}, A_{kt}} (1 - s_{j(t+1)}) \\ &+ G_k \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} s_{j(t+1)},\end{aligned}$$

$$\begin{aligned}\frac{1}{R_{jt}} \frac{\partial G_k}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial G_k}{\partial R_{jt}} =& \frac{G_k}{R_{jt} R_{kt}} \Bigg\{ - \frac{1}{\psi} - \frac{1}{\sigma} \left[1 - \Sigma_{Y_{kt}, R_{kt} | X_{kt}} - \Sigma_{Y_{kt}, X_{kt}} \Sigma_{X_{kt}, R_{kt}} \right] \\ &+ \frac{1}{\epsilon} \Sigma_{Y_t, Y_{jt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{Y_{jt}, X_{jt}} - \Sigma_{Y_{kt}, X_{kt}} \right] \Bigg\},\end{aligned}$$

$$\begin{aligned}\frac{1}{R_{jt}} \frac{\partial G_j}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial G_j}{\partial R_{jt}} =& \frac{G_j}{R_{jt} R_{kt}} \Bigg\{ - \frac{1}{\psi} - \frac{1}{\sigma} \left(1 - \Sigma_{Y_{jt}, R_{jt} | X_{jt}} - \Sigma_{Y_{jt}, X_{jt}} \Sigma_{X_{jt}, R_{jt}} \right) \\ &+ \frac{1}{\epsilon} \Sigma_{Y_t, Y_{kt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{Y_{kt}, X_{kt}} - \Sigma_{Y_{jt}, X_{jt}} \right] \Bigg\}.\end{aligned}$$

Using these in (A-15) and factoring $\Sigma_{X,A} G_j G_k / [R_{jt} R_{kt}]$ yields:

$$
\begin{aligned}
&\left\{ - s_{j(t+1)} \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \right\} \Sigma_{Y_{jt},X_{jt}} - (1 - s_{j(t+1)}) \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \right\} \\
&\qquad \left\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} \left[1 - \Sigma_{X,R} \right] \right\} \\
&+ \left\{ (1 - s_{j(t+1)}) \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \right\} \Sigma_{Y_{kt},X_{kt}} + s_{j(t+1)} \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} \right\} \\
&\qquad \left\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{Y_{jt},X_{jt}} \left(1 - \Sigma_{X,R} \right) \right\} \\
&+ \frac{1}{\epsilon} \left[1 - \Sigma_{X,R} \right] \left[\Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}} \right] \\
&\qquad \left\{ - s_{j(t+1)} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \right] \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} - (1 - s_{j(t+1)}) \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \right\} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \right\} \\
&- \frac{1}{\epsilon^2} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_t,Y_{kt}} \left[1 - \Sigma_{X,R} \right] \left[\Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}} \right] \left\{ (1 - s_{j(t+1)}) \Sigma_{Y_{kt},X_{kt}} + s_{j(t+1)} \Sigma_{Y_{jt},X_{jt}} \right\} \\
=& s_{j(t+1)} \Sigma_{Y_{jt},X_{jt}} \left\{ \frac{1}{\psi} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \right] \right. \\
&\qquad \left. + \frac{1}{\sigma} \left(1 - \Sigma_{X,R} \right) \left[\frac{1}{\sigma} \Sigma_{Y_{kt},X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} \right] \right\} \\
&- (1 - s_{j(t+1)}) \Sigma_{Y_{kt},X_{kt}} \left\{ \frac{1}{\psi} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \right] \right. \\
&\qquad\qquad \left. + \frac{1}{\sigma} \left(1 - \Sigma_{X,R} \right) \left[\frac{1}{\sigma} \Sigma_{Y_{jt},X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \right] \right\} \\
&+ \frac{1}{\epsilon} \left[1 - \Sigma_{X,R} \right] \left[\Sigma_{Y_{kt},X_{kt}} - \Sigma_{Y_{jt},X_{jt}} \right] \\
&\qquad \left\{ - s_{j(t+1)} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{kt}} \right] \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_{jt},X_{jt}} - (1 - s_{j(t+1)}) \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \right\} \Sigma_{Y_t,Y_{kt}} \Sigma_{Y_{kt},X_{kt}} \right. \\
&\qquad\quad \left. - \frac{1}{\epsilon} \Sigma_{Y_t,Y_{jt}} \Sigma_{Y_t,Y_{kt}} \left[(1 - s_{j(t+1)}) \Sigma_{Y_{kt},X_{kt}} + s_{j(t+1)} \Sigma_{Y_{jt},X_{jt}} \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
=&s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,Y_{kt}}\right]\Sigma_{Y_{kt},X_{kt}}\right\}\\
&-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,Y_{jt}}\right]\Sigma_{Y_{jt},X_{jt}}\right\}\\
&-s_{j(t+1)}\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,Y_{jt}}\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}+(1-s_{j(t+1)})\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,Y_{kt}}\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\right]\\
&+\frac{1}{\sigma^2}\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)-\frac{1}{\sigma}\frac{1}{\epsilon}\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{jt},X_{jt}}\Sigma_{Y_{kt},X_{kt}}\left(2s_{j(t+1)}-1\right)\\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\right]+\frac{1}{\sigma}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\left(1-\Sigma_{X,R}\right)\Sigma_{Y_{kt},X_{kt}}\Sigma_{Y_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)
\end{aligned}
$$

Substituting for $\Sigma_{X,R}$ and rearranging, we obtain

$$
\begin{aligned}
\frac{1}{\psi}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\Bigg[&s_{j(t+1)}\Sigma_{Y_{jt},X_{jt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{kt},X_{kt}}\right)\\
&-(1-s_{j(t+1)})\Sigma_{Y_{kt},X_{kt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{jt},X_{jt}}\right)\Bigg].
\end{aligned}
\tag{A-16}
$$

This expression is positive if and only if the term in brackets is positive. Define $\hat{s}_{t+1}$ as the $s_{j(t+1)}$ such that $R_{jt}/R_{kt} = R_{j(t+1)}/R_{k(t+1)}$. Then $\hat{s}_{t+1}$ is the root of the term in brackets. Solving for that root yields equation (15).

We now consider how $\hat{s}_{t+1}$ changes in σ. Holding the time t allocation of resource extraction and research fixed, the derivative of the right-hand side of equation (15) with respect to σ is proportional to

$$
\begin{aligned}
&-\left.\frac{\partial\Sigma_{Y_{jt},X_{jt}}}{\partial\sigma}\right|_{R_{jt},s_{jt}\text{ fixed}}\Sigma_{Y_{kt},X_{kt}}\left[1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{kt},X_{kt}}\right]\\
&\quad+\left.\frac{\partial\Sigma_{Y_{kt},X_{kt}}}{\partial\sigma}\right|_{R_{kt},s_{jt}\text{ fixed}}\Sigma_{Y_{jt},X_{jt}}\left[1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{Y_{jt},X_{jt}}\right].
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left.\frac{\partial\Sigma_{Y_{jt},X_{jt}}}{\partial\sigma}\right|_{R_{jt},s_{jt}\text{ fixed}}&\propto\frac{1}{\sigma^2}\kappa(1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}R_{jt}^{\frac{\sigma-1}{\sigma}}\left[\ln X_{jt}-\ln R_{jt}\right]\\
&>0\text{ iff }X_{jt}>R_{jt}.
\end{aligned}
$$

The results for sector k are analogous. $X_{jt} > R_{jt}$ when A_{jt} is sufficiently large. We have that the right-hand side of equation (15) (and thus also $\hat{s}_{t+1}$) decreases in σ if $X_{jt} > R_{jt}$ and $X_{kt} < R_{kt}$.

Consider what happens to the right-hand side of equation (15) as $\sigma \to 0$. The elasticity $\Sigma_{Y_{jt},X_{jt}}$ becomes $(X_{jt}/\min\{R_{jt}, X_{jt}\})^{\frac{\sigma-1}{\sigma}}$, and analogously for sector k. Thus, under the assumption that $X_{jt} > R_{jt}$ and $X_{kt} < R_{kt}$, $\Sigma_{Y_{jt},X_{jt}} \to 0$ as $\sigma \to 0$ and $\Sigma_{Y_{kt},X_{kt}} \to 1$ as $\sigma \to 0$. These imply that the right-hand side of equation (15) goes to 1 as $\sigma \to 0$.

As $\sigma \to 1$, the right-hand side of equation (15) goes to $1/2$.

Finally, as $\sigma \to \infty$, the elasticities $\Sigma_{Y_{jt},X_{jt}}$ and $\Sigma_{Y_{kt},X_{kt}}$ each go to either 1 or 0. Under the assumption that $X_{jt} > R_{jt}$ and $X_{kt} < R_{kt}$, $\Sigma_{Y_{jt},X_{jt}} \to 1$ while $\Sigma_{Y_{kt},X_{kt}} \to 0$. The right-hand side of equation (15) thus goes to 0 as $\sigma \to \infty$.

Proof of Proposition 2

The following lemma relates $\hat{s}_{t+1}$ and 0.5.

Lemma A-7. *If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \geq [\Psi_j/\Psi_k]^\theta$, for $\theta > 0$. If $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \leq [\Psi_j/\Psi_k]^\theta$, for $\theta > 0$.*

Proof. From equation (15),

$$\left\{\hat{s}_{t+1} \geq \frac{1}{2}\right\} \Leftrightarrow \left\{\Sigma_{Y_{kt},X_{kt}} \geq \Sigma_{Y_{jt},X_{jt}}\right\},$$

where the right-hand side is evaluated at $\hat{s}_{t+1}$. Using the explicit expressions for the elasticities, for intermediate-good production, and for X_{jt} and X_{kt} from equation (A-3), we have:

$$
\begin{aligned}
\Sigma_{Y_{kt},X_{kt}} \geq & \Sigma_{Y_{jt},X_{jt}} \\
\Leftrightarrow 0 \leq & \frac{(1-\kappa)X_{kt}^{\frac{\sigma-1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}Y_{kt}^{\frac{\sigma-1}{\sigma}}}{Y_{kt}^{\frac{\sigma-1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}}} \\
\Leftrightarrow 0 \leq & X_{kt}^{\frac{\sigma-1}{\sigma}}Y_{jt}^{\frac{\sigma-1}{\sigma}} - X_{jt}^{\frac{\sigma-1}{\sigma}}Y_{kt}^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 0 \leq & \kappa R_{jt}^{\frac{\sigma-1}{\sigma}}X_{kt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}X_{kt}^{\frac{\sigma-1}{\sigma}} - \kappa R_{kt}^{\frac{\sigma-1}{\sigma}}X_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa)X_{kt}^{\frac{\sigma-1}{\sigma}}X_{jt}^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq & \left(\frac{R_{jt}\left[\frac{1-\kappa}{\kappa}\left(\frac{R_{kt}}{\Psi_k}\right)^{1/\psi}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}}\left[\frac{R_{kt}}{A_{kt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}}A_{kt}}{R_{kt}\left[\frac{1-\kappa}{\kappa}\left(\frac{R_{jt}}{\Psi_j}\right)^{1/\psi}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}}\left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}}A_{jt}} \right)^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq & \left[\left(\frac{\Psi_j}{\Psi_k}\right)^{\frac{\alpha\,\sigma/\psi}{\sigma(1-\alpha)+\alpha}} \left(\frac{R_{jt}}{R_{kt}}\right)^{\frac{\sigma(1-\alpha-\alpha/\psi)}{\sigma(1-\alpha)+\alpha}} \left(\frac{A_{kt}}{A_{jt}}\right)^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \right]^{\frac{\sigma-1}{\sigma}} \\
\Leftrightarrow 1 \leq & \left(\frac{\Psi_j}{\Psi_k}\right)^{\chi\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]} \left(\frac{1+\eta\gamma s_{jt}}{1+\eta\gamma s_{kt}}\right)^{-\chi\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]} \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{\chi(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi)-(1+\sigma/\psi)]},
\end{aligned}
\tag{A-17}
$$

where the final line substitutes for R_{jt}/R_{kt} from equation (A-1) (which must hold for $\hat{s}_{t+1}$ interior) and where

$$\chi \triangleq \frac{\sigma-1}{[\sigma(1-\alpha)+\alpha][1+\sigma/\psi]} < 0 \text{ iff } \sigma < 1.$$

The right-hand side of inequality (A-17) is increasing in s_{jt} if and only if $\sigma < 1$. Therefore, if $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the strict version of the inequality does not hold at $s_{jt} = 0.5$, and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the inequality holds at $s_{jt} = 0.5$. If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \geq \left[\frac{\Psi_j}{\Psi_k}\right]^{\theta},$$

and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \leq \left[\frac{\Psi_j}{\Psi_k}\right]^{\theta},$$

where

$$\theta \triangleq \frac{-\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]}{(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi)-(1+\sigma/\psi)]} = \frac{1}{(1-\alpha)(1+\psi)} > 0.$$

□

Thus $\hat{s}_{t+1} \geq 0.5$ under the condition of the proposition.

First, consider the possibility of a transition after time t.

Assume that $\sigma < 1$, $s_{jt} \geq 0.5$, and $(1-\sigma)(1-\alpha) \leq 0.5$. And assume that $R_{jt}/R_{kt} < R_{j(t+1)}/R_{k(t+1)}$ and $s_{jt}/s_{kt} < s_{j(t+1)}/s_{k(t+1)}$ so that it makes sense to speak of potentially transitioning from sector j to sector k. At the time w where sector k's share of extraction begins increasing, $s_{j(w+1)} \leq \hat{s}_{w+1}$ and $s_{jw} > \hat{s}_w$. Imagine that a transition occurs over an interval for which Assumption 1 does not hold. Then $\hat{s}_w, \hat{s}_{w+1} < 0.5$ from Lemma A-7.

Assume that $s_{jw} \leq 0.5$. By Lemma A-6 and $R_{j(w-1)}/R_{k(w-1)} < R_{jw}/R_{kw}$, it cannot be true that $s_{j(w-1)} \geq 0.5$ when $(1-\sigma)(1-\alpha) \leq 0.5$. Therefore $s_{j(w-1)} < 0.5$. From equation (10), $s_{jw} > s_{j(w-1)}$. Using that resource extraction was increasing in every period from time t to w and proceeding backwards to time t by induction, we see that $s_{jw} \in (\hat{s}_w, 0.5]$ implies $s_{jt} < s_{jw}$, contrary to the assumption that $s_{jt} \geq 0.5$. A transition cannot occur if $s_{jw} \leq 0.5$.

So assume that $s_{jw} > 0.5$. A transition requires that $s_{j(w+1)} \leq \hat{s}_{w+1} < 0.5$. If $s_{j(w+1)} = \hat{s}_{w+1}$, then Lemma A-6 implies that $s_{j(w+1)} \geq 0.5$, which would be a contradiction. So a transition requires that $s_{j(w+1)} < \hat{s}_{w+1}$. Assume this to be the case. The right-hand side of equation (A-1) defines Π_{jt}/Π_{kt} as a function $f(s_{jt}, R_{jt}/R_{kt})$. Define $x \triangleq R_{jt}/R_{kt}$. Then $s_{j(w+1)} < \hat{s}_{w+1}$ implies that $f(\hat{s}_{w+1}, x) > 1$. And because $R_{j(t+1)}(\hat{s}_{w+1})/R_{k(w+1)}(\hat{s}_{w+1}) = x$ by the definition of $\hat{s}_{w+1}$, we also know that the (time $w+1$ version of the) right-hand side of equation (A-2) is > 1 when evaluated at $s_{j(w+1)} = \hat{s}_{w+1}$. From Corollary A-2, the equilibrium $s_{j(w+1)}$ must be strictly greater than $\hat{s}_{w+1}$. But this contradicts the assumption that $s_{j(w+1)} < \hat{s}_{w+1}$.

Combining the results of the last two paragraphs, we have seen that a transition cannot occur unless Assumption 1 holds.

Now consider whether the share of innovation or extraction begins decreasing first. At the time w where sector k's share of extraction begins increasing, $s_{jw} > \hat{s}_w$ which means that $s_{jw} > 0.5$. From equation (10), $s_{jw} > 0.5$ with $R_{jw}/R_{kw} \geq R_{j(w+1)}/R_{k(w+1)}$ implies that $s_{j(w+1)} < s_{jw}$. Innovation transitions no later than does extraction.

Because $\hat{s}_{w+1} > 0.5$ at the time w that sector k begins increasing its share of extraction, it must be true that $s_{jw} > 0.5$ at the last time w at which sector j's share of extraction has been increasing continuously since time t.

Now consider a case in which $\sigma > 1$ with $s_{jt} \geq 0.5$ and in which Assumption 1 holds at time t. By Lemma A-7, $\hat{s}_{t+1} < 0.5$ at time t. Assume that $R_{jt}/R_{kt} < R_{j(t+1)}/R_{k(t+1)}$ and $s_{jt}/s_{kt} < s_{j(t+1)}/s_{k(t+1)}$ so that it makes sense to speak of potentially transitioning from sector j to sector k. Because $s_{jt} \geq 0.5$, average technology in j (weakly) improves relative to that in k, so Assumption 1 will still hold at time $t+1$. Therefore $\hat{s}_{t+2} \leq 0.5$.

Consider s_{t+2}. It is easy to see from equations (A-1) and (14) that Π_{jt}/Π_{kt} increases in $A_{j(t-1)}/A_{k(t-1)}$, both directly and through the resource market's size. And we know that $A_{jt}/A_t < A_{j(t+1)}/A_{k(t+1)}$ because $s_{j(t+1)} > s_{jt}$ and $s_{jt} > 0.5$. From Corollary A-2, we have $s_{j(t+2)} > s_{j(t+1)}$. Therefore $s_{j(t+2)} > \hat{s}_{t+2}$ and $R_{j(t+1)}/R_{k(t+1)} < R_{j(t+2)}/R_{k(t+2)}$. Proceeding by induction, we see that sector j's shares of research and extraction increase forever. A transition cannot happen after time t.

Proof of Corollary 3

The case with $\sigma < 1$ follows directly from Proposition 2.

Consider the case with $\sigma \in (1, \epsilon)$ and assume that $A_{j(t-1)} > A_{k(t-1)}$ and $\nu \geq 0.5$. Then Assumption 1 holds. To satisfy equation (14) at $s_{jt} = 0.5$, it must be true that $R_{jt} > R_{kt}$. Using this in equation (A-1) with $A_{j(t-1)} > A_{k(t-1)}$, we see that $\Pi_{jt} > \Pi_{kt}$ at $s_{jt} = 0.5$. From Corollary A-2, we must have $s_{jt} > 0.5$. Applying Proposition 2, the economy is locked-in to sector j.

Proof of Proposition 4

Consider equation (14). The left-hand side is equal to unity when $R_{jt} = R_{kt}$, the right-hand side increases in A_{jt}, and the right-hand side decreases in A_{kt}.

If $\Psi_j = \Psi_k$ and $\nu > 0.5$, then the right-hand side of equation (14) can equal unity if and only if $A_{jt} < A_{kt}$. For $\sigma < 1$, $A_{kt} > A_{jt}$ with $R_{jt} = R_{kt}$ and $\Psi_j = \Psi_k$ implies from equation (A-1) that $\Pi_{jt} > \Pi_{kt}$ when evaluated at $s_{jt} = 0.5$, so in equilibrium $s_{jt} > 0.5$.

If $\nu = 0.5$ and $\Psi_j > \Psi_k$, then the right-hand side of equation (14) can equal unity if and only if $A_{jt} < A_{kt}$. For $\sigma < 1$, $A_{kt} > A_{jt}$ with $R_{jt} = R_{kt}$ and $\Psi_j > \Psi_k$ implies from equation (A-1) that $\Pi_{jt} > \Pi_{kt}$ when evaluated at $s_{jt} = 0.5$, so in equilibrium $s_{jt} > 0.5$.

Proof of Proposition 5

Along a path with balanced growth in technology, $A_{j(t+1)}/A_{k(t+1)} = A_{jt}/A_{kt}$. From equation (3), this holds if and only if $s_{jt} = 0.5$. For equation (A-1) to hold at $s_{jt} = 0.5$ along a balanced growth path, it must be the case that R_{jt}/R_{kt} is constant along this path. This, in turn, implies that $s_{j(t+1)} = \hat{s}_{t+1}$, which means that $\hat{s}_{t+1} = 0.5$.

Imposing equality in inequality (A-17) and using $\sigma < 1$ and $\hat{s}_{t+1} = 0.5$, we see that $\Psi_j > \Psi_k$ if and only if $A_{j(t-1)} > A_{k(t-1)}$ at all times along this path. Using $\Psi_j > \Psi_k$, $A_{j(t-1)} > A_{k(t-1)}$, and $s_{jt} = 0.5$ in equation (A-1) with $\sigma < 1$, we have that $R_{jt} > R_{kt}$. By similar logic, if $\Psi_j = \Psi_k$, then $A_{j(t-1)} = A_{k(t-1)}$ along this path and $R_{jt} = R_{kt}$, and from equation (14), $\nu = 0.5$.

Now consider the growth rate of R_{jt} and R_{kt} along the balanced growth path. Note that

$$\begin{aligned}
Y_{jt} =& \left(\kappa \left[R_{jt} \right]^{\frac{\sigma-1}{\sigma}} + (1-\kappa) \left\{ \left[\frac{1-\kappa}{\kappa} p_{jRt} \right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt} \right\}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\
=& A_{jt} \left(\kappa \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{\sigma-1}{\sigma}} + (1-\kappa) \left\{ \left[\frac{1-\kappa}{\kappa} \left(\frac{R_{jt}}{\Psi_j} \right)^{1/\psi} \right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} \right\}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\
\triangleq& A_{jt} \tilde{Y}_{jt}.
\end{aligned}$$

Substituting into equation (14), we have:

$$1 = \frac{\nu}{1-\nu} \left[\frac{\tilde{Y}_{jt}}{\tilde{Y}_{kt}} \right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}} \left[\frac{A_{jt}}{A_{kt}} \right]^{-\frac{1}{\epsilon}} \left[\frac{R_{jt}/A_{jt}}{R_{kt}/A_{kt}} \right]^{-1/\sigma} \left[\frac{p_{jRt}}{p_{kRt}} \right]^{-1}.$$

Equate to time $t+1$ variables:

$$\begin{aligned}
&\left[\frac{\tilde{Y}_{jt}}{\tilde{Y}_{kt}} \right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}} \left[\frac{A_{jt}}{A_{kt}} \right]^{-\frac{1}{\epsilon}} \left[\frac{R_{jt}/A_{jt}}{R_{kt}/A_{kt}} \right]^{-1/\sigma} \left(\frac{p_{jRt}}{p_{kRt}} \right)^{-1} \\
&\qquad = \left[\frac{\tilde{Y}_{j(t+1)}}{\tilde{Y}_{k(t+1)}} \right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}} \left[\frac{A_{j(t+1)}}{A_{k(t+1)}} \right]^{-\frac{1}{\epsilon}} \left[\frac{R_{j(t+1)}/A_{j(t+1)}}{R_{k(t+1)}/A_{k(t+1)}} \right]^{-1/\sigma} \left(\frac{p_{jR(t+1)}}{p_{kR(t+1)}} \right)^{-1}.
\end{aligned}$$

Recognizing that relative technology and relative resource extraction are constant along a balanced growth path, we have:

$$\frac{\tilde{Y}_{jt}}{\tilde{Y}_{kt}} = \frac{\tilde{Y}_{j(t+1)}}{\tilde{Y}_{k(t+1)}},$$

which implies

$$\frac{\tilde{Y}_{j(t+1)}}{\tilde{Y}_{jt}} = \frac{\tilde{Y}_{k(t+1)}}{\tilde{Y}_{kt}}.$$

Because this must hold for all time intervals once we reach the balanced growth path, each ratio must equal some constant, which we label χ. So we seek a constant χ such that

$\tilde{Y}_{j(t+1)} = \chi \tilde{Y}_{jt}$. Analyze:

$$\begin{aligned}
\tilde{Y}_{j(t+1)} =& \left(\kappa \left[\frac{R_{j(t+1)}}{A_{j(t+1)}} \right]^{\frac{\sigma-1}{\sigma}} + (1-\kappa) \left\{ \left[\frac{1-\kappa}{\kappa} \left(\frac{R_{j(t+1)}}{\Psi_j} \right)^{1/\psi} \right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{j(t+1)}}{A_{j(t+1)}} \right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} \right\}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\
=& \left(\kappa \left[\frac{R_{jt}}{A_{jt}} \frac{R_{j(t+1)}/R_{jt}}{A_{j(t+1)}/A_{jt}} \right]^{\frac{\sigma-1}{\sigma}} \right. \\
& \left. + (1-\kappa) \left\{ \left[\frac{1-\kappa}{\kappa} \left(\frac{R_{jt}}{\Psi_j} \right)^{1/\psi} \left(\frac{R_{j(t+1)}}{R_{jt}} \right)^{1/\psi} \right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \frac{R_{j(t+1)}/R_{jt}}{A_{j(t+1)}/A_{jt}} \right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} \right\}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} .
\end{aligned} \tag{A-18}$$

There exists χ such that this equals $\chi \tilde{Y}_{jt}$ if and only if

$$\begin{aligned}
& \frac{R_{j(t+1)}/R_{jt}}{A_{j(t+1)}/A_{jt}} = \left(\frac{R_{j(t+1)}}{R_{jt}} \right)^{\frac{1}{\psi} \frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left(\frac{R_{j(t+1)}/R_{jt}}{A_{j(t+1)}/A_{jt}} \right)^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} \\
\Leftrightarrow & \left(\frac{1}{1+\eta s_{jt}\gamma} \right)^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} = \left(\frac{R_{j(t+1)}}{R_{jt}} \right)^{\frac{\frac{1}{\psi}\alpha\,\sigma - \sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \\
\Leftrightarrow & \frac{R_{j(t+1)}}{R_{jt}} = (1+0.5\eta\gamma)^{\frac{\psi}{\psi - \frac{\alpha}{1-\alpha}}} ,
\end{aligned}$$

where the last line recognizes that $s_{jt} = 0.5$ along a balanced growth path. The same condition must hold for $R_{k(t+1)}/R_{kt}$.

Proof of Proposition A-1

The tâtonnement adjustment process generates, to constants of proportionality, the following system for finding the equilibrium within period t:

$$\begin{aligned}
\dot{R}_{jt} =& H_j \bigg(G_j(R_{jt}, R_{kt}) - 1 \bigg), \\
\dot{R}_{kt} =& H_k \bigg(G_k(R_{jt}, R_{kt}) - 1 \bigg),
\end{aligned}$$

where dots indicate time derivatives (where the fictional time for finding an equilibrium here flows within a period t), $H_i(0) = 0$, and $H_i'(\cdot) > 0$, for $i \in \{j, k\}$. The system's steady state occurs at the equilibrium values, which I denote with stars. Linearizing around the steady state, we have

$$\begin{bmatrix} \dot{R}_{jt} \\ \dot{R}_{kt} \end{bmatrix} \approx \begin{bmatrix} \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} H_j' & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}} H_j' \\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} H_k' & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}} H_k' \end{bmatrix} \begin{bmatrix} R_{jt} - R_{jt}^* \\ R_{kt} - R_{kt}^* \end{bmatrix} = G \begin{bmatrix} R_{jt} - R_{jt}^* \\ R_{kt} - R_{kt}^* \end{bmatrix}, \tag{A-19}$$

where G is the 2×2 matrix of derivatives, each evaluated at (R_{jt}^*, R_{kt}^*). The results of Lemma A-4 imply that the trace of G is negative, in which case at least one of the two eigenvalues must be negative. Lemma A-5 shows that $det(G) > 0$, which means that both eigenvalues must be strictly negative. The linearized system is therefore globally asymptotically stable, and, by Lyapunov's Theorem of the First Approximation, the full nonlinear system is locally asymptotically stable around the equilibrium. We have tâtonnement-stability if $det(G) > 0$.

Proof of Corollary A-2

Now treat equations (12) and (13) as functions of R_{jt}, R_{kt}, and s_{jt} (recognizing that $s_{kt} = 1 - s_{jt}$):

$$\begin{aligned} 1 &= \kappa\,\nu \left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{Y_{jt}(R_{jt}, s_{jt})}\right]^{1/\epsilon} \left[\frac{Y_{jt}(R_{jt}, s_{jt})}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi_j} &&\triangleq \hat{G}_j(R_{jt}, R_{kt}; s_{jt}), \\ 1 &= \kappa\,(1-\nu) \left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{Y_{kt}(R_{kt}, s_{jt})}\right]^{1/\epsilon} \left[\frac{Y_{kt}(R_{kt}, s_{jt})}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi_k} &&\triangleq \hat{G}_k(R_{jt}, R_{kt}; s_{jt}). \end{aligned}$$

This system of equations implicitly defines R_{jt} and R_{kt} as functions of the parameter s_{jt}. Define the matrix $\hat{G}$ analogously to the matrix G. Using the implicit function theorem, we have

$$\frac{\partial R_{jt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} \quad \text{and} \quad \frac{\partial R_{kt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})}.$$

Interpreting equation (A-1) as implicitly defining s_{jt} as a function of R_{jt} and R_{kt}, we have:

$$\frac{\partial s_{jt}}{\partial R_{jt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}} \quad \text{and} \quad \frac{\partial s_{jt}}{\partial R_{kt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}},$$

and thus

$$\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \quad \text{and} \quad \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}.$$

Using these expressions, consider how the right-hand side of equation (A-2) changes in s_{jt}:

$$\begin{aligned}
\frac{\mathrm{d}[\Pi_{jt}/\Pi_{kt}]}{\mathrm{d}s_{jt}} =& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}\frac{\partial R_{jt}}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}\frac{\partial R_{kt}}{\partial s_{jt}} \\
=& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} \\
& - \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} - \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})} \\
\propto& - \frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} \\
& - \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}} - \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}} \\
=& - \left(\frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\right) \\
& + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \\
=& - det(G)/H'.
\end{aligned}$$

The third expression factored $det(\hat{G})$, which is positive by the proof of Proposition A-1 for a corner solution in s_{jt}, and it also factored $\partial[\Pi_{jt}/\Pi_{kt}]/\partial s_{jt}$, which is negative. The final equality recognizes that the only difference between the equations with a hat and the equations without a hat are that the equations without a hat allow s_{jt} to vary with R_{jt} and R_{kt}. Lemma A-5 showed that $det(G) > 0$. Thus the right-hand side of equation (A-2) strictly decreases in s_{jt}.