Mortgage Choices in Equilibrium*

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Abstract

This paper identifies the optimal mortgage choice of homeowners, who are subject to counter-cyclical income volatility and idiosyncratic income risk. The market equilibrium of investors with similar income risk exposures, leads to pro-cyclical equilibrium interest rates. The key determinant of the optimal mortgage choice is the correlation of income with the business cycle. Homeowners with high (low) correlation of income choose an ARM (FRM) and the more risk-averse homeowners are more likely to choose an ARM. Mortgage choices of homeowners have an impact on the financial market equilibrium, where interest rates become less volatile for larger shares of ARMs.

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1 Introduction

The single largest liability of any household is the mortgage they take out when purchasing a house. They are allowed to do so if their future income is sustainable enough to cover the interest and principal payments required by the mortgage provider. The contribution of this paper is to show how the income of homeowners are linked to the state of the economy, impacting the choice of mortgage, when investors providing the mortgages are exposed to the same aggregate shocks. There are many aspects to mortgages which could be considered, but I focus on the interest-rate frequency choice, i.e. either fixed-rate (FRMs) or adjustable-rate (ARMs). The interest rates of these products are found in the equilibrium of investors and not exogenously specified. This ensures that the model is internally consistent, as opposed to a partial equilibrium framework where asset prices are specified exogenously and not necessarily representing the same economy for homeowners and investors. In the general equilibrium the model predicts how the mortgage choices of homeowners impact the financial market equilibrium, a very important question considering the size of the mortgage market in developed countries, currently 14 trillion dollars outstanding in the US.[1]

To motivate my model I use the empirical fact documented in Bloom (2009) that aggregate income volatility is counter-cyclical, that is, high during recessions and low during expansions. Any asset-pricing model with this assumption generates pro-cyclical interest rates due to the precautionary-savings motive of investors. In Figure 1.1 we see that this can be observed in the data, during recent recessions interest rates decline. These interest rate dynamics are an important factor in which type of mortgage a homeowner will optimally choose and it is the link between this empirical fact and mortgage choices which motivates this paper. Homeowners whose income is driven more by the aggregate economy finds it optimal to choose an ARM. As interest rates are low in recessions, where the homeowner's income is low, the ARM acts as a natural hedge to income risk. The

[1] Board of Governors of the Federal Reserve System (US), Mortgage Debt Outstanding, All holders [MDOAH], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/MDOAH, June 16, 2017.

FRM is optimal when the homeowner's income has a larger idiosyncratic component and the hedge from the ARM disappears. The more risk-averse homeowners value the hedging ability of the ARM higher and are therefore more likely to choose an ARM.

The mortgage choices of homeowners affect the aggregate risk the investors must bear and, thus, the equilibrium interest rates and market prices of risk. For higher shares of ARMs this leads to less volatile equilibrium interest rates and higher market prices of risk. As ARMs are pro-cyclical investment products, when their supply increases the investors are subject to less favorable interest payments. That is, during recessions the investors who hold the ARMs receive lower premiums when their marginal utility is high. This increases the market price of risk, while the less volatile interest rates stem from the shifting demand of the investors who prefer ARMs during expansions and FRMs during recessions. The main contribution of this paper are these general equilibrium effects obtained by modelling both the homeowners and investors.

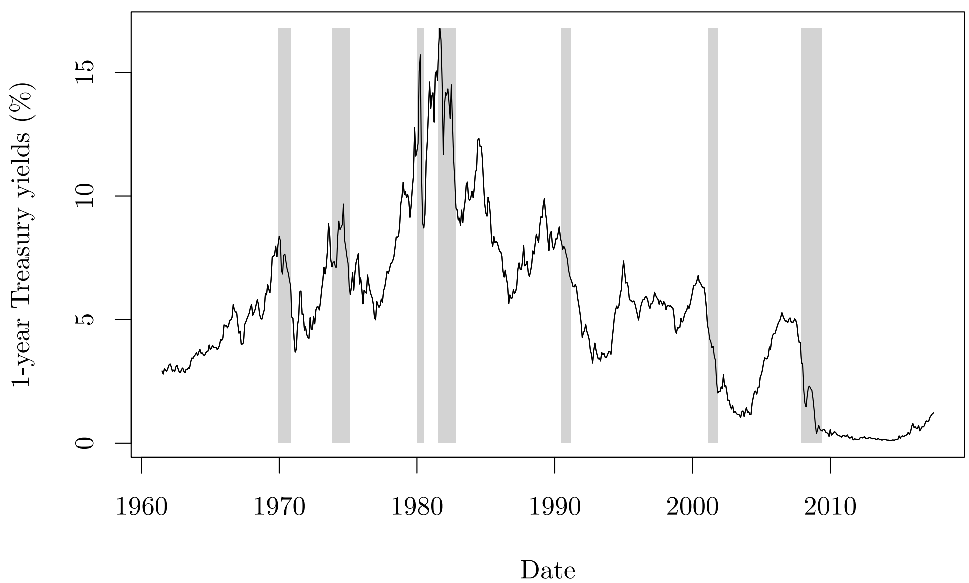


Figure 1.1: 1-year US treasury yields and NBER recessions (shaded areas). Interest rates decline during recessions.

Prior leading research on optimal mortgage choice include Campbell and Cocco (2003), Koijen, Hemert, and Nieuwerburgh (2009) and Campbell and Cocco (2015). In these models, the bond market dynamics are exogenously specified without relation to the investors' income dynamics and the homeowners' mortgage choices. In this paper, the bond market dynamics are endogenously derived from the equilibrium consequences of the optimal decisions of investors and homeowners based on exogenous income dynamics. Thus, the model is better suited to answer the mortgage choice question based on the income process of the homeowner, as the economic state is the same for both the homeowner and through the investors, the bond market.

Contrary to Koijen et al. (2009), who find that the more risk-averse homeowners are more likely to choose an FRM, this paper finds that the more risk-averse homeowners choose an ARM. As noted above, this relies on the fact that the bond market and income processes are uncorrelated in Koijen et al. (2009) and, therefore, a risk-averse homeowner is more likely to choose the fixed interest payments instead of the uncertain payments of an ARM. In this paper, the bond market and income processes are correlated, and the more risk-averse homeowners seek the income hedge of an ARM.

Campbell and Cocco (2003) and Koijen et al. (2009) also find benefits of ARMs in terms of smoothing consumption. Their results rely on introducing inflation risk, such that the fixed payments of a nominal FRM are risky in real terms, whereas the floating payments of a nominal ARM are more stable in real terms. As the nominal interest rate is equal to the real interest rate plus a premium for expected inflation, this means that during periods of low expected inflation, nominal interest rates are low, other things equal. In periods with low expected inflation the consumption good today is expensive compared to periods with high expected inflation. Therefore, the real net income after interest rate payments of a homeowner with an ARM is therefore relatively stable when including inflation, as the number of consumption goods the homeowner can purchase is stabilized. This is not the case for a homeowner with an FRM as her nominal interest payments are fixed based on the expected inflation through the duration of the mortgage and, thus, the number of consumption goods she can buy in the future depends highly on the realized future inflation.

This paper also investigates the mortgage choice for homeowners who are subject to negative skewness and high kurtosis in income as documented by Guvenen, Karahan, Ozkan, and Song (2015). Modelling the income to include a small probability of a sudden decline captures both negative skewness and high kurtosis in income. The probability of such a shock is higher during recessions and therefore the ARM provides a hedge against unemployment risk. This shows that whenever a homeowner's income is highly impacted by the state of the overall economy, either through higher volatility or increased probability of negative shocks during recessions, the ARM is the better choice to hedge the income risk of the homeowner.

Vayanos and Vila (2009) show how demand shocks of preferred-habitat investors affect interest rates, most profound for longer maturities through changes in risk premia (the short rate is modelled exogenously). This is in line with my model, where the homeowners play the role as preferred-habitat investors who choose their mortgage depending on the term structure of interest rates. The investors in the financial market acts as arbitrageurs when setting the equilibrium interest rates and market prices of risk. Therefore, the choices of homeowners affect the financial market equilibrium. In this paper, the supply of mortgage-backed securities are found from the mortgage choices of homeowners and thereby affects market quantities such as interest rates and market prices of risk as the bonds must be held by the market investors. Greenwood and Vayanos (2014) show how the supply and maturity structure of government bonds affects returns and yields in a parsimonious equilibrium model. They find that an increase in the supply of long-term government bonds leads to higher instantaneous returns and higher yields on bonds, and the results are stronger for the bonds of long maturity. Malkhozov, Mueller, Vedolin, and Venter (2016) find similar results when looking at the duration of mortgage-backed securities. The results of these papers rely on risk-averse arbitrageurs which must be compensated for an increase in duration risk and this result is also present in this general equilibrium model where the choices of homeowners depend on the market equilibrium. Including institutions or non-myopic investors into an equilibrium model, where the objective function and subsequent holdings of these investors differ from the representative agent is present in Basak and Pavlova (2012), Koijen and Yogo (2015) and others. This paper contributes to the literature of asset pricing through the optimality of homeowners affecting the market equilibrium.

For simplicity, this paper does not include inflation and real house prices are assumed to be constant over time. This is done in order to focus on the interest payments of the homeowner and, furthermore, it is unclear wether ARMs or FRMs are best suited to hedge changes in house prices. If a downturn is associated with falling house prices a homeowner with a fixed-rate mortgage may not be able to utilize her prepayment option as the new loan which needs to be originated comes with a collateral (house) of lower value. This means that she is stuck with a higher interest rate on her mortgage compared to an ARM which gets a lower rate automatically. This was observed during the great recession as documented by Agarwal, Amromin, Chomsisengphet, Piskorski, Seru, and Yao (2015), where the US government found it optimal to bail out insufficiently collateralized mortgages with a reduction of around 140 basis points in interest rates. This would not have been necessary if the mortgages were adjustable-rate as their interest rates would follow the lowered market rates. However, not all downturns are associated with falling house prices. In fact, since 1975 it has only happened during the recent crisis where we have seen a drop in the US House Price Index compared to the value one year before.[2] Finally, I do not allow for a prepayment option in the FRMs in order to solve the model in closed form, so one might suggest that my results would not hold with realistic FRMs. In the robustness section I address this using the setup of the model and the main point is that, if we would include an option in the FRM the homeowner would have to pay a premium for this and the needed compensation for the investors in equilibrium is too high.

The rest of the paper is organized as follows: In Section 2, the model is set up; in Section 3, the equilibrium with homeowners is analyzed in three steps: First the market equilibrium with FRMs only, second the optimal mortgage choice for given asset prices, and finally how is the market equilibrium affected by the introduction of ARMs; Section 4 provides additional considerations. Finally, Section 5 offers some concluding remarks.

[2] US. Federal Housing Finance Agency, All-Transactions House Price Index for the United States [USSTHPI], Federal Reserve Bank of St. Louis `https://research.stlouisfed.org/fred2/series/USSTHPI`

2 The Model

Consider a setting with $I < \infty$ investors and $H < \infty$ homeowners living on the time interval $[0, T]$. Both the investors and homeowners receive an exogenously given income process. The investors can trade continuously in a financial market consisting of two basic financial assets, while the homeowners must choose at time 0 between a fixed-rate mortgage (FRM) and an adjustable-rate mortgage (ARM). The mortgages are assumed to be interest-only and the homeowners are not allowed to switch mortgage after time 0.

Homeowners and investors do not directly interact with each other by introducing a housing institution. Homeowner h purchases her home from the housing institution at price F_h using either a fixed-rate mortgage or an adjustable-rate mortgage. House prices are constant over time such that the house is sold back to the institution at exactly the same price as the original face value of the mortgage. The institution then participates in the financial markets in order to pass on the interest payments of the homeowners to the investors.

2.1 Exogenous income processes

The income process Y_i of investor i and $\bar{Y}_h$ of homeowner h is given by the following dynamics

$$dY_{it} = (\mu_i + \kappa_i v_t)\, dt + \sqrt{v_t}\,(\sigma_i dW_t + \beta_i dZ_{it}), \qquad i = 1, 2, ..., I, \tag{2.1}$$

$$d\bar{Y}_{ht} = (\bar{\mu}_h + \bar{\kappa}_h v_t)\, dt + \sqrt{v_t}\left(\bar{\sigma}_h dW_t + \bar{\beta}_h d\bar{Z}_{ht}\right), \qquad h = 1, 2, ..., H, \tag{2.2}$$

where $(Z_i)_{i=1}^{I}$ and $\left(\bar{Z}_h\right)_{h=1}^{H}$ are independent Brownian motions, each independent of the common Brownian motion W. The process v captures the stochastic income volatility of all individuals and is assumed to follow a Feller process

$$dv_t = (\mu_v + \kappa_v v_t)\, dt + \sigma_v \sqrt{v_t} dW_t, \qquad v_0 > 0. \tag{2.3}$$

with $\mu_v \geq \frac{1}{2}\sigma_v^2, \sigma_v < 0$, and $\kappa_v < 0$. This ensures that $v_t > 0, \forall t$ and the process is mean-reverting. By modeling the income dynamics to depend on the process v both in the drift and in the volatility along with the same Brownian motion affecting v allows for several empirical observations. Bloom (2009) finds that aggregate income volatility is counter-cyclical. This means that when income is high, the volatility of income is low and when income is low, the volatility is high. In order to match the observed counter-cyclicality of aggregate income volatility, consider the dynamics of aggregate income of the investors $\mathcal{E}_t = \sum_{i=1}^{I} Y_{it}$:[3]

$$d\mathcal{E}_t = (\mu_{\mathcal{E}} + \kappa_{\mathcal{E}} v_t)\, dt + \sqrt{v_t}\left(\sigma_{\mathcal{E}} dW_t + \sum_{i=1}^{I} \beta_i dZ_{it}\right),$$

where

$$\mu_{\mathcal{E}} = \sum_{i=1}^{I} \mu_i, \quad \kappa_{\mathcal{E}} = \sum_{i=1}^{I} \kappa_i, \quad \sigma_{\mathcal{E}} = \sum_{i=1}^{I} \sigma_i. \tag{2.4}$$

Assume without loss of generality that $\sigma_{\mathcal{E}}$ is positive. Counter-cyclical aggregate income volatility is then obtained with

$$\sigma_v < 0.$$

This can be seen by looking at what happens to v and $\mathcal{E}$ when shocks in W are positive. In this case, aggregate income is affected positively since $\sigma_{\mathcal{E}} > 0$, while aggregate income volatility, v, is affected negatively. This means that aggregate income volatility is counter-cyclical as Bloom (2009) observes in the data.

Another observation made by Bloom (2009) is that after a large negative shock to aggregate income, income growth is impacted negatively. To model this, assume that $\kappa_{\mathcal{E}} < 0$, such that after a negative shock to aggregate income, income volatility increases by the counter-cyclicality and expected aggregate income growth is reduced. In order to keep expected income growth from being negative at each point in time, assume $\mu_{\mathcal{E}} > 0$. Since $\kappa_v < 0$, v is mean-reverting and income growth bounces back when volatility reverts back from high levels after aggregate shocks to income, as observed by Bloom (2009).

[3]This can readily be extended to total aggregate income.

To show that the market equilibrium is in fact feasible, consider the following condition on the parameters.

Condition 2.1. *There exists $p > 1$ such that the following ordinary differential equation has a non-exploding solution in* $[0, T]$,

$$b_p'(x) = \frac{-p}{\tau_\Sigma}\left(\kappa_{\mathcal{E}} - \frac{1}{2}\frac{\sigma_{\mathcal{E}}^2}{\tau_\Sigma} - \frac{1}{2}\sum_{i=1}^{I}\frac{\beta_i^2}{\tau_i}\right) + \left(\kappa_v - \sigma_v\frac{\sigma_{\mathcal{E}}}{\tau_\Sigma}\right)b_p(x) + \frac{1}{2}\sigma_v^2 b_p(x)^2, \qquad b_p(0) = 0,$$

where $\tau_\Sigma = \sum_{i=1}^{I}\tau_i$. *The above ODE has a solution if*

$$\left(\kappa_v - \sigma_v\frac{\sigma_{\mathcal{E}}}{\tau_\Sigma}\right)^2 + 2\sigma_v^2\frac{p}{\tau_\Sigma}\left(\kappa_{\mathcal{E}} - \frac{1}{2}\frac{\sigma_{\mathcal{E}}^2}{\tau_\Sigma} - \frac{1}{2}\sum_{i=1}^{I}\frac{\beta_i^2}{\tau_i}\right) > 0.$$

The above condition holds for many realistic parameter sets described above.

2.2 Preferences

Each investor and homeowner has a CARA-utility function of the following form

$$U_i(x) = -e^{-\frac{1}{\tau_i}x},$$
$$\bar{U}_h(x) = -e^{-\frac{1}{\bar{\tau}_h}x},$$

where $\frac{1}{\tau}$ is the absolute risk aversion and τ is the risk tolerance of the individual. Both investors and homeowners receive utility from continuous consumption only. Furthermore, their life-time utility is time-additive with time-preference rate δ. As in Christensen and Larsen (2014), model the consumption strategy c_i of the investors in excess of their income, such that investor i's life-time utility is written as

$$\int_0^T e^{-\delta_i t}U_i(Y_{it} + c_{it})\,dt.$$

For the homeowners, their life-time utility depends on their choice of mortgage. After choosing a mortgage type, it is fixed until time T and the homeowner receives utility from consumption equal to the income less the interest payment. That is, the life-time utility of homeowner h is

$$\int_0^T e^{-\bar{\delta}_h t} \bar{U}_h \left(\bar{Y}_{ht} - I_{ht}\right) dt, \tag{2.5}$$

where the interest payments I_{ht} are determined by the mortgage choice of the homeowner.

2.3 Traded assets

Investors can trade continuously in two assets each in zero-net supply, which will form the basis of the mortgages available to the homeowners:

- A money market account, $S^{(0)}$, paying the stochastic risk-free rate, r, such that

$$dS_t^{(0)} = r_t S_t^{(0)} dt, \qquad S_0^{(0)} = 1. \tag{2.6}$$

- An annuity, S, paying a constant continuous unit dividend with maturity at time T, such that

$$S_t = \mathbb{E}_t \left[\int_t^T \frac{\xi_s^\psi}{\xi_t^\psi} ds \right], \qquad S_T = 0, \tag{2.7}$$

  for any state price deflator ξ^ψ.

The dynamics of r and ξ^ψ are determined in equilibrium by the optimal choices of homeowners and investors, and market clearing conditions.

2.4 Mortgage designs

To construct the mortgages using the traded assets, introduce now the redundant asset, a zero-coupon-bond maturing at time T with characteristics:

$$B(t,T) = \mathbb{E}\left[\frac{\xi_T^\psi}{\xi_t^\psi}\right] \qquad B(T,T) = 1,$$

for any state-price deflator ξ^ψ. As the zero-coupon bond is redundant, it can be removed from the investors' optimization problem.

The mortgages provided by the housing institution to the homeowners are interest-only. This means that the financial wealth of homeowner h, X_h satisfies the following

$$X_{h0} = -F_h \quad \text{and} \quad X_{hT-} = -F_h, \tag{2.8}$$

where X_{T-} denotes the financial wealth just before time T where the outstanding principal is repaid. A homeowner h with an FRM pays continuous interest payments equal to $F_h r^f$, where r^f is the fixed rate of the mortgage. If homeowner h instead uses an ARM to finance their home, then their interest payments at time t is $F_h r_t$, where r_t is the instantaneous risk-free interest rate in the financial market. For these two types of mortgages the interest payment schemes homeowners can choose from to maximize their expected utility in (2.5) is

$$I_{ht} = r_t F_h \quad or \quad I_{ht} = r^f F_h.$$

The constant interest rate r^f is set such that the fixed-rate mortgage is at par value. That is, as an interest-only FRM can be replicated by a portfolio consisting of $F_h r^f$ annuities and F_h zero-coupon bonds maturing at time T, the following must hold

$$F_h = F_h r^f S_0 + F_h B\left(0,T\right).$$

Therefore, the fixed interest rate must satisfy

$$r^f = \frac{1 - B\left(0,T\right)}{S_0}. \tag{2.9}$$

This resembles the usual expression for the swap rate, defined as the fixed interest rate of an interest rate swap with value equal to zero. This is intuitive, as the mortgages are replicated by the traded assets and is fairly priced.

The housing institution uses the financial markets in order to pass on the interest payments of the homeowners to the investors. How the housing institution constructs its portfolio holdings of the three traded assets $\theta_{\mathcal{H}t} = \left(\theta_{\mathcal{H}t}, \theta^{(0)}_{\mathcal{H}t}, \theta^{(T)}_{\mathcal{H}t}\right)$ is illustrated in Figure 2.1 and explained as follows. Let $\mathbf{F}^f$ be the total face value of all FRMs and $\mathbf{F}^a$ be the total face value of all ARMs. At time t the housing institution then receives fixed interest payment rates equal to $\mathbf{F}^f r^f$ from the FRMs and $\mathbf{F}^a r_t$ from the ARMs. The fixed interest payments are sold to the investors by writing $\mathbf{F}^f r^f$ annuities at time 0, $\left(\theta_{\mathcal{H}t} = -\mathbf{F}^f r^f\right)$, while the floating interest payments are sold by having a short position in the money market account of value $\mathbf{F}^a$, $\left(\theta^{(0)}_{\mathcal{H}t} = \frac{-\mathbf{F}^a}{S^{(0)}_t}\right)$. The floating interest payments of the homeowners ensure that the position can remain fixed. In order to negate the short position in the money market at time T, the financial institution holds a long position in the zero-coupon bond maturing at time T equal to $\mathbf{F}^a$, $\left(\theta^{(T)}_{\mathcal{H}t} = \mathbf{F}^a\right)$. That is, at time T, the financial wealth of the housing institution needs to be zero.

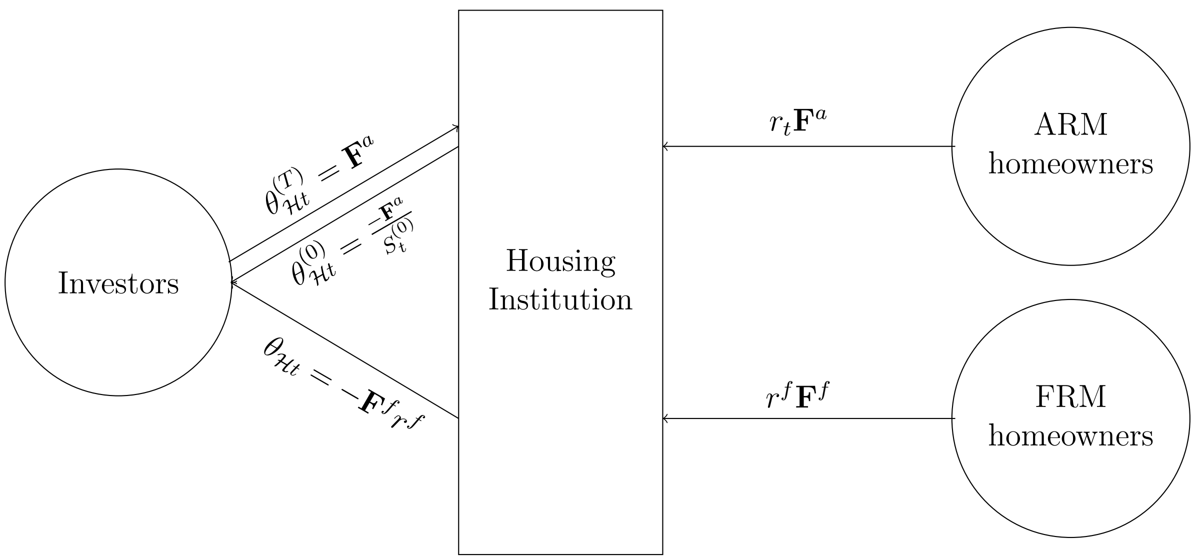


Figure 2.1: An illustration of how the housing institution sells the interest payments to the investors.

At all times the housing institution's portfolio consists of

$$\theta_{\mathcal{H}t} = -\mathbf{F}^f r^f, \quad \theta^{(0)}_{\mathcal{H}t} = -\frac{\mathbf{F}^a}{S^{(0)}_t}, \quad \theta^{(T)}_{\mathcal{H}t} = \mathbf{F}^a, \tag{2.10}$$

meaning that the financial wealth of the housing institution is

$$X_{\mathcal{H}t} = -\mathbf{F}^f r^f S_t - \mathbf{F}^a + \mathbf{F}^a B\left(t, T\right). \tag{2.11}$$

Given a mortgage choice distribution of the homeowners $(\mathbf{F}^a, \mathbf{F}^f)$, the market clearing condition is

$$\sum_{i=1}^{I} \theta_{it} = \mathbf{F}^f r^f, \qquad \sum_{i=1}^{I} \theta_{it}^{(0)} = \frac{\mathbf{F}^a}{S_t^{(0)}}, \qquad \sum_{i=1}^{I} \theta_{it}^{(T)} = -\mathbf{F}^a. \tag{2.12}$$

That is, the homeowners supply a positive number of annuities and a negative balance in the money market account to construct their mortgages. The positive demand for the zero-coupon bond is generated by the investors with no incentive to leave a positive balance in the money market account at time T.

2.5 Market Equilibrium without homeowners

The above model without homeowners is solved in Christensen and Larsen (2014) and the following quantities are important for when we include homeowners.

Optimal consumption dynamics for each investor are found as a function of the risk-free rate, r_t, the market price of aggregate risk, λ_t, and the state variable v_t, such that $d\hat{c}_{it} = (...)dt + (...)dW_t$. In equilibrium, $\sum_{i=1}^{I} \hat{c}_{it} = 0$ and the equilibrium interest rate and market price of risk must satisfy

$$r_t = \frac{\sum_{i=1}^{I} \tau_i \delta_i}{\tau_\Sigma} + \frac{1}{\tau_\Sigma}\left(\mu_\mathcal{E} + \kappa_\mathcal{E} v_t\right) - \frac{1}{2}\frac{1}{\tau_\Sigma^2}\left(\sigma_\mathcal{E}^2 v_t + \tau_\Sigma \sum_{i=1}^{I} \frac{\beta_{Y_i}^2}{\tau_i} v_t\right). \tag{2.13}$$

$$\lambda_t = \sqrt{v_t}\frac{\sigma_\mathcal{E}}{\tau_\Sigma}, \tag{2.14}$$

where $\tau_\Sigma = \sum_{i=1}^{I} \tau_i$ and $\mu_\mathcal{E}, \kappa_\mathcal{E}, \sigma_\mathcal{E}$ are defined in (2.4). As the risk-free rate is a decreasing function of v_t, with $\kappa_\mathcal{E} < 0$, this means that interest rates are pro-cyclical, that is, low in recessions and high in expansions. This is exactly what financial markets experience with significant drops in interest rates during recessions as noted in Figure 1.1. The market price of risk is, however, an increasing function of income volatility making it counter-cyclical. This is in line with investors requiring additional expected return on risky assets, in times of high uncertainty.

The price of a zero-coupon bond and the annuity are given in the following Lemma.

Lemma 2.1. *The time t price of a zero-coupon bond with a dividend payment of 1 at time s is*

$$B(t,s) = e^{b(s-t)v_t - a(s-t)} \tag{2.15}$$

where a and b must satisfy the ordinary differential equations

$$\begin{aligned} a'(s-t) &= \left(\frac{\sum_{i=1}^{I} \tau_i \delta_i}{\tau_\Sigma} + \frac{1}{\tau_\Sigma} \mu_\mathcal{E} \right) - b(s-t)\mu_v \\ b'(s-t) &= \frac{-1}{\tau_\Sigma} \left(\kappa_\mathcal{E} - \frac{1}{2}\frac{\sigma_\mathcal{E}^2}{\tau_\Sigma} - \frac{1}{2}\sum_{i=1}^{I} \frac{\beta_i^2}{\tau_i} \right) + b(s-t)\left(\kappa_v - \sigma_v \frac{\sigma_\mathcal{E}}{\tau_\Sigma} \right) + \frac{1}{2} b(s-t)^2 \sigma_v^2 \end{aligned}$$

with initial conditions $a(0) = b(0) = 0$.

The time t annuity price is

$$\begin{aligned} S_t &= \int_t^T B(t,s)ds. \\ dS_t + dt &= (r_t S_t + \sigma_{St}\lambda_t)\, dt + \sigma_{St} dW_t \\ \sigma_{St} &= \sigma_v \sqrt{v_t} \int_t^T B(t,u)\, b(u-t)\, du. \end{aligned} \tag{2.16}$$

As the dynamics of the risky asset, the annuity, only depends on W, the market is incomplete as the investors cannot trade their individual income risk stemming from Z_i. One can show that the function b is always positive, meaning that the volatility of the annuity, σ_{St}, is negative, as $\sigma_v < 0$. This means that after large negative shocks to aggregate income, the value of the annuity goes up and a long position in the annuity then provides a hedge against the systematic income risk of the investors. This is a consequence of the pro-cyclical interest rates as the annuity is a fixed income asset. It also means that the risk premium on the annuity is negative, as the investors are willing to give up expected returns to get access to the income hedge of the annuity.

The optimal financial wealth process of investor i is given as

$$\widehat{X}_{it} = \mathbb{E}\left[\int_t^T \frac{\xi_T^{min}}{\xi_t^{min}} \widehat{c}_{iu} du \right]. \tag{2.17}$$

It is well-defined and admits a feasible trading strategy $\widehat{\boldsymbol{\theta}}$ satisfying the following budget constraint.

$$d\widehat{X}_{it} = \left(\widehat{X}_{it} r_t - \widehat{c}_{it} + \widehat{\theta}_{it} \sigma_{St} \lambda_t\right) dt + \widehat{\theta}_{it} \sigma_{St} dW_t. \tag{2.18}$$

That is the optimal consumption and investment strategy of investor i leads to the wealth process above in (2.17). This result states that the wealth of investor i is just the risk-adjusted present value of the consumption strategy $\widehat{c}_i$.

Note that the found equilibrium is not necessarily unique. That is, the risk-free rate r_t in (2.13), the market price of risk λ_t in (2.14) and the dynamics of the annuity in (2.16) do not necessarily constitute the only equilibrium, as the market is incomplete.

Nonetheless, an equilibrium is attained where the risk-free rate and the market price of risk are simple functions of the common state variable v. Interest rates are shown to be pro-cyclical and the market price of risk is counter-cyclical. This means that interest rates are low during recessions, while the market price of risk is high. This is in line with the increased demand for savings during times of high uncertainty. In the appendix I calibrate the parameters of this baseline model which will be used later in the analysis.

3 The Mortgage Equilibrium

3.1 Market Equilibrium with FRMs

In this section, the equilibrium is found when homeowners are only allowed fixed-rate mortgages. Therefore, the housing institution's portfolio with $\mathbf{F}^a = 0$ in (2.10) only consists of

$$\theta_{\mathcal{H}t} = -\mathbf{F}^f r^f.$$

This creates a positive supply of annuities for the investors as the housing institution uses the annuity to pass on the interest payments to the investors.

The only difference to the basic model is then a positive supply of annuities seen by the investors, so in equilibrium the financial market clearing conditions are

$$\sum_{i=1}^{I} \theta_{it} = \mathbf{F}^f r^f, \quad \text{and} \quad \sum_{i=1}^{I} \theta_{it}^{(0)} = 0.$$

To find the equilibrium condition for the goods market, consider the total financial wealth in the economy which should be equal to zero at all times

$$0 = \sum_{i=1}^{I} X_{it} + X_{\mathcal{H}t}.$$

By Itô's Lemma and the financial wealth of the housing institution given in (2.11)

$$\begin{aligned}
0 &= \sum_{i=1}^{I} dX_{it} + dX_{\mathcal{H}t} \\
&= \sum_{i=1}^{I} \theta_{it} \left(dS_t + dt\right) + \sum_{i=1}^{I} \theta_{it}^{(0)} dS_t^{(0)} - \sum_{i=1}^{I} c_{it} dt - \mathbf{F}^f r^f dS_t \\
&= \left(\sum_{i=1}^{I} \theta_{it} - \mathbf{F}^f r^f \right) \left(dS_t + dt\right) + \sum_{i=1}^{I} \theta_{it}^{(0)} dS_t^{(0)} - \left(\sum_{i=1}^{I} c_{it} - \mathbf{F}^f r^f \right) dt.
\end{aligned}$$

So from the equilibrium conditions above, total consumption of the investors in excess of their income should satisfy

$$\sum_{i=1}^{I} c_{it} = \mathbf{F}^f r^f.$$

That is, the consumption of the investors extracted from financial wealth must equal the total interest payments from homeowners, an intuitive result. As interest payments of FRMs are constant over time, the total consumption level must also be constant meaning

$$d \sum_{i=1}^{I} c_{it} = 0$$

as in the basic model. This now means that the interest rate r_t and market price of volatility risk λ_t are the same, and all the quantities found in the basic model are the same.

What remains to be shown is that the optimal investment strategies of the investors result in an equilibrium. This can be shown similarly to the proof in Christensen and Larsen (2014). The main difference is the financial wealth of investor i which must satisfy (2.17), so the total financial wealth of the investors must satisfy

$$\begin{aligned}
\sum_{i=1}^{I} \widehat{X}_{it} &= \mathbb{E}\left[\int_t^T \frac{\xi_T^{min}}{\xi_t^{min}} \sum \widehat{c}_{iu} du\right] \\
&= \mathbb{E}\left[\int_t^T \frac{\xi_T^{min}}{\xi_t^{min}} \mathbf{F}^f r^f du\right] \\
&= \mathbf{F}^f r^f S_t
\end{aligned}$$

Following the same steps as in their paper all the markets clear.

From the above analysis, we see that by including homeowners with fixed-rate mortgages, the equilibrium market quantities do not change. This is not surprising since the interest payments of the homeowners are known at all times. This allows the investors to trade away their added exposure, stemming from the added supply, using a strategy known at time 0.

3.2 Optimal Mortgage Choice

Now, the individual homeowner's optimal mortgage decision between either a fixed-rate or an adjustable-rate mortgage is found. Fixed-rate mortgages do not affect the market quantities as shown above, so assume that all homeowners finance their house using a fixed-rate mortgage. Now, consider whether or not it is optimal for a given household to change their fixed-rate mortgage into an adjustable-rate mortgage.

The first thing to consider is the price of such a swap. As an FRM can be replicated by $-r^f F_h$ annuities and $-F_h$ zero-coupon bonds maturing at time T, and an ARM is a constant balance of $-F_h$ in the money market account then the price of a swap must be

$$P_h^{swap} = r^f F_h S_0 + F_h B(0,T) - F_h = F_h \left(r^f S_0 + B(0,T) - 1\right). \tag{3.1}$$

As the fixed rate was determined to be $r^f = \frac{1-B(0,T)}{S_0}$, the price of a swap is equal to zero.

This shows that r^f is indeed a swap rate as previously argued. In Section 4.1, the case where r^f is different from the swap rate is analyzed, but for now, assume that the price of the swap is zero.

The optimal mortgage choice is determined simply by comparing the life-time expected utility of each mortgage, i.e., it is optimal to switch to an ARM if

$$\mathbb{E}\left[\int_0^T e^{-\bar{\delta}_h t}\bar{U}_h\left(\bar{Y}_{ht} - r_t F_h\right) dt\right] > \mathbb{E}\left[\int_0^T e^{-\bar{\delta}_h t}\bar{U}_h\left(\bar{Y}_{ht} - r^f F_h\right) dt\right], \tag{3.2}$$

for the individual homeowner's income process $\bar{Y}_h$ defined in (2.2) and utility function $\bar{U}_h$. To calculate the life-time expected utility, consider the expected utility function of homeowner h at time t defined as

$$J_{ht} \equiv J\left(\bar{Y}_h, v, t\right) = \mathbb{E}_t\left[\int_t^T -e^{-\bar{\delta}_h(s-t) - \frac{1}{\bar{\tau}_h}\left(\bar{Y}_{hs} - I_{hs}\right)} ds\right], \tag{3.3}$$

where I_h denotes the interest payment scheme.

Theorem 3.1. *The expected utility function of homeowner h with a fixed-rate mortgage is*

$$J_{ht}^{FRM} = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - r^f F_h\right)} \int_t^T e^{-\bar{\delta}_h(s-t) + a_F(s-t) + b_F(s-t)v_t} ds,$$

where the functions a_F and b_F solve the ordinary differential equations in (A.7) *and* (A.6), *respectively, with initial conditions $a_F(0) = b_F(0) = 0$.*

The expected utility function of homeowner h with an adjustable-rate mortgage is

$$J_{ht}^{ARM} = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - r_t F_h\right)} \int_t^T e^{-\bar{\delta}_h(s-t) + a_A(s-t) + b_A(s-t)v_t} ds,$$

where the functions a_A and b_A solve the ordinary differential equations in (A.12) *and* (A.11), *respectively, with initial conditions $a_A(0) = b_A(0) = 0$.*

This means that the life-time expected utility of either type of mortgage can now be determined as

$$J_{h0}^{FRM} = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{h0} - r^f F_h\right)} \int_0^T e^{-\bar{\delta}_h s + a_F(s) + b_F(s) v_0} ds, \tag{3.4}$$

$$J_{h0}^{ARM} = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{h0} - F_h r_0\right)} \int_0^T e^{-\bar{\delta}_h s + a_A(s) + b_A(s) v_0} ds. \tag{3.5}$$

where the integrals can be approximated numerically.

Koijen, Hemert, and Nieuwerburgh (2009) find that the homeowner's current income level does not have an effect on the optimal mortgage choice. This is also the case in this model as the inequality (3.2) can be simplified to

$$-e^{\frac{1}{\bar{\tau}_h} F_h r_0} \int_0^T e^{-\bar{\delta}_h s + a_A(s) + b_A(s) v_0} ds > -e^{\frac{1}{\bar{\tau}_h} F_h r^f} \int_0^T e^{-\bar{\delta}_h s + a_F(s) + b_F(s) v_0} ds, \tag{3.6}$$

independent of $\bar{Y}_{h0}$. It is a result of the choice of utility function with a constant absolute risk aversion, where the aversion to risk is the same for all levels of consumption and, hence, levels of income. Another utility function such as the power utility function may lead to an impact of the current income level on the optimal mortgage choice, but it complicates the model with default probabilities when the homeowner's income is below the required interest payment. Rewriting the inequality further yields

$$r^f - r_0 > \frac{\bar{\tau}_h}{F_h} \log\left(\frac{\int_0^T e^{-\bar{\delta}_h s + a_A(s) + b_A(s) v_0} ds}{\int_0^T e^{-\bar{\delta}_h s + a_F(s) + b_F(s) v_0} ds}\right).$$

That is, the homeowner chooses an ARM if the spread between the fixed rate of the FRM and the current rate of an ARM is above some threshold. With the complexity of the integrals, it is not possible to analytically conclude any determinants of the optimal mortgage choice.

The homeowner-specific *utility-equivalent rate* (UE-rate) is the fixed rate of an FRM where the homeowner is indifferent between the two types of mortgages. This is readily solved as

$$\tilde{r}^f = r^f + \frac{\bar{\tau}_h}{F_h} \log\left(\frac{J_{h0}^{ARM}}{J_{h0}^{FRM}}\right). \tag{3.7}$$

That is, if the homeowner prefers the ARM,

$$J_{h0}^{ARM} > J_{h0}^{FRM} \Leftrightarrow \frac{J_{h0}^{ARM}}{J_{h0}^{FRM}} < 1,$$

then she would need to be compensated with a lower rate on the FRM in order not to switch to the ARM. Conversely, if the homeowner prefers the FRM with the market fixed rate, she would be willing to pay an additional interest and keep the FRM. The difference between the UE-rate and the market fixed rate is a universal measure of the homeowner's preference between the two mortgages and will be used in the analysis of the mortgage choice in the following sections.

3.2.1 Sensitivity analysis

To illustrate how homeowner-specific parameters and the state of the economy affects the optimal mortgage choice this numerical example is considered. This is done by calculating the difference between the utility-equivalent rate in (3.7) and the market fixed rate in (2.9).

Table 1 summarizes the calibrated parameters from Section A.1. The base case parameters of the homeowners are equal to those of the investors. The mortgage face value is assumed to be $F_h = 10$ with time-to-maturity $T = 30$, corresponding to 10 times her current income which is set at 1 such that the income volatility is $V_i = 16\%$ of her income.

One of the main determinants of the optimal mortgage choice is the correlation coefficient between the homeowner's income and the state variable v. Therefore, consider homeowners with different correlation coefficients

$$\bar{\rho}_h = \frac{\bar{\sigma}_h}{\sqrt{\bar{\sigma}_h^2 + \bar{\beta}_h^2}}.$$

Note that this is the negated correlation between dv_t and dY_{ht} and is positive to ensure

State variable	
μ_v	0.3062
κ_v	-0.3062
Long-term mean of v	1
σ_v	-0.1603
Initial volatility level, v_0	1

Investors	
Risk-aversion, $1/\tau_i$	2
Time-preference, δ_i	0.01
μ_i	0.0442
κ_i	-0.005
σ_i	0.0477
β_i	0.1516
Correlation between state variable and income, $-\rho_i$	0.3
Long-term overall volatility, $V_i = \sqrt{\sigma_i^2 + \beta_i^2}$	0.1589

Homeowners	
Income and preference parameters, identical to investors	-
T	30
F_h	10

Table 1: Baseline parameter values.

counter-cyclical income volatility and can, therefore, be seen as the correlation between the homeowner's income and the business cycle.

The difference between the utility-equivalent fixed rate for a homeowner with the base case parameters and the current swap rate is

$$\tilde{r}^f_{base} - r^f = 0.12\%.$$

That is, the homeowner prefers an FRM as she is willing to pay a premium on the fixed rate instead of switching to an ARM.

In Figure 3.1, we see how the correlation between the homeowner's income and the business cycle impacts the mortgage choice of the homeowner. When the homeowner's income is more exposed to the business cycle, the ARM becomes more attractive. That

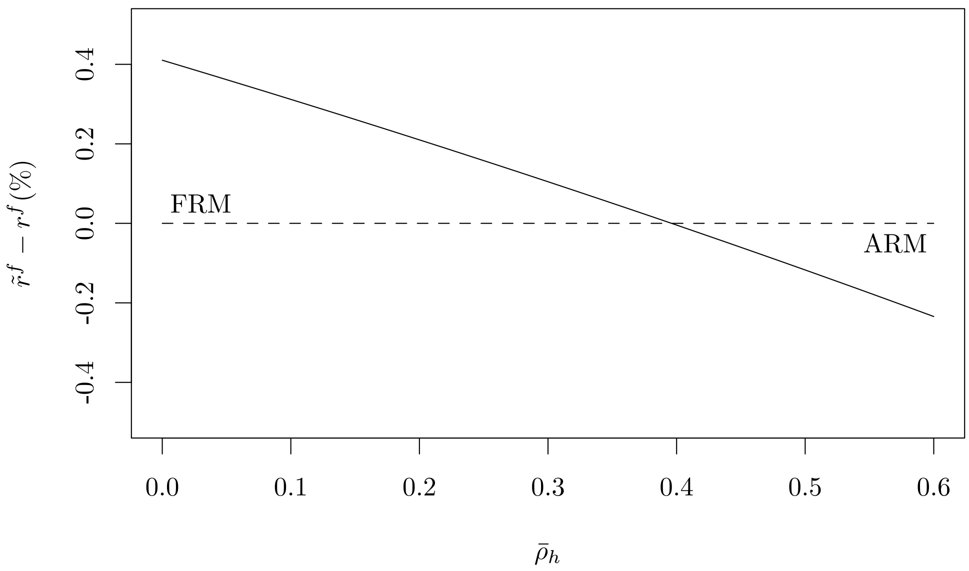


Figure 3.1: The difference between the utility-equivalent fixed rate and the current swap rate for different correlation coefficients of the homeowner.

is, when the UE-rate is lower than the market fixed-rate, the homeowner demands a compensation in the fixed rate of the FRM in order not to switch. This is because the ARM is better at hedging her income risk when the income movements co-vary more with the business cycle. We note that for the base case parameters the cutoff level is approximately 40% where the homeowner is indifferent between the ARM and the FRM.

In Figure 3.2, the optimal mortgage choice can be seen for homeowners with different risk aversions. For more risk-averse homeowners the ARM becomes more attractive as we can see that such homeowners demand a compensation in the fixed rate of the FRM in order not to switch to the ARM. This clearly shows that the ARM is able to hedge income risk as risk-averse homeowners have an increasing incentive to hedge their consumption. This also means that for the more risk-averse homeowners the needed correlation coefficient is lower for the ARM to be optimal. This will be analysed more clearly in Section 3.3, for a distribution of homeowners with different risk aversions and income correlations.

In Figure 3.3 we see the impact of the homeowner's income volatility. When the homeowner's income volatility is higher, the ARM becomes more attractive. The homeowner values the income hedge of the ARM more when she is exposed to larger income shocks.

To complete the analysis we see in Figure 3.4 that the time-preference, expected income

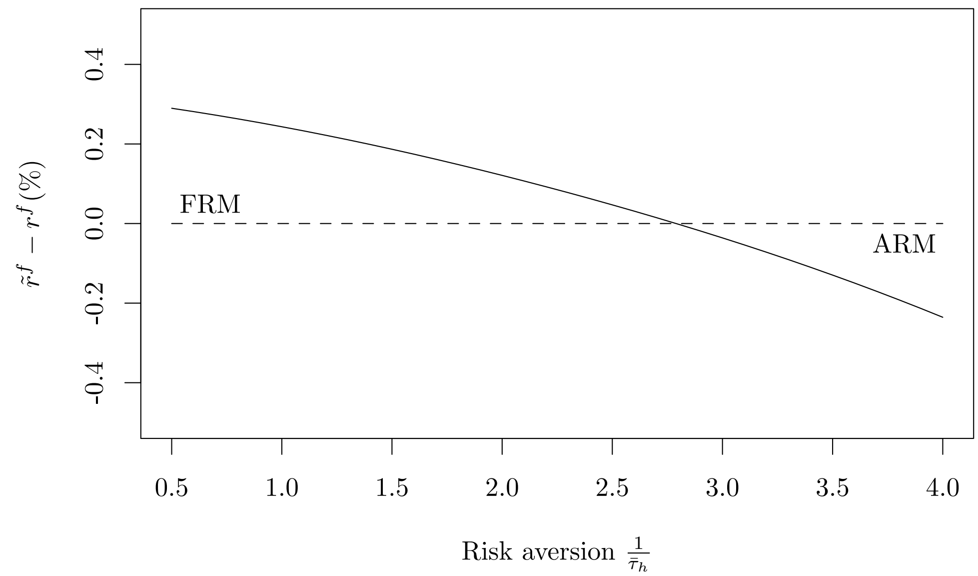


Figure 3.2: The difference between the utility-equivalent fixed rate and the current swap rate for different risk aversions of the homeowner.

parameters and debt-to-income ratio have little effect on the optimal mortgage choice, although for homeowners with a large mortgage compared to their income, they seem to be less likely to find the ARM optimal.

Now, consider how the state of the economy has an impact on the mortgage choice of the homeowner. In Figure 3.5 we see how the initial state v_0 impacts the mortgage choice for homeowners with different risk aversions (Panel A), and homeowners with different income risk (Panel B). Common for both is that when v_0 is low, that is, during expansions with decreasing yield curves and high short rate, the homeowner prefers the ARM. As v_t is mean-reverting, the short rate is mean-reverting and the homeowners expect the interest rate to decrease and therefore prefer the ARM compared to the FRM with a higher fixed rate than the steady-state short rate. The opposite is true for recessions, where the interest rates are low and expected to increase. Here, the FRM is more attractive with a lower fixed rate. In Panel A of Figure 3.5 we see how the risk aversion impacts the trade-off between expectations of future interest rates and the income hedge of an ARM. For the more risk-averse homeowners ($\frac{1}{\bar{\tau}_h} = 3$) the ARM is still optimal in the steady state ($v_0 = 1$) even though interest rates are not expected to decrease. This is also present in Panel B for different correlations between the homeowner's income and the business cycle.

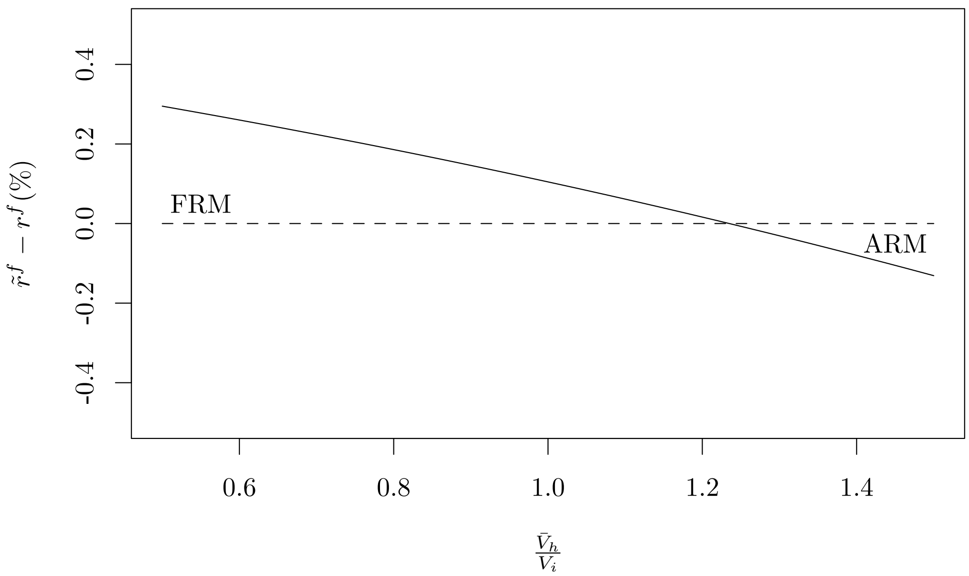


Figure 3.3: The difference between the utility-equivalent fixed rate and the current swap rate for different income volatilities.

For higher correlations the ARM is preferred even if v_0 is greater, but close to one.

Figure 3.5 shows that the optimal mortgage choice is state-dependent, that is, in a multi-period model with options to switch mortgage at several points in time, these choices will depend on the state of the economy. In Section 4.1, I consider a homeowner with an FRM who enters with a fixed-rate different from the current swap rate, and this is shown to have no impact on the optimal mortgage choice. That is, only the state of the economy and not the individual homeowner's financial state has an impact on the optimal mortgage choice.

3.3 Market and Mortgage Choice Equilibrium

In this section, the equilibrium interest rates and market price of risk are affected by the choices of the homeowners in the setting where homeowners are subject to similar income dynamics as the investors. As seen in the previous section, if a homeowner chooses a fixed-rate mortgage, it does not affect these quantities as the payments to the investors are then known and constant. If, however, a homeowner chooses to finance her home using an adjustable-rate mortgage it does have an effect. The financial quantities r_t and

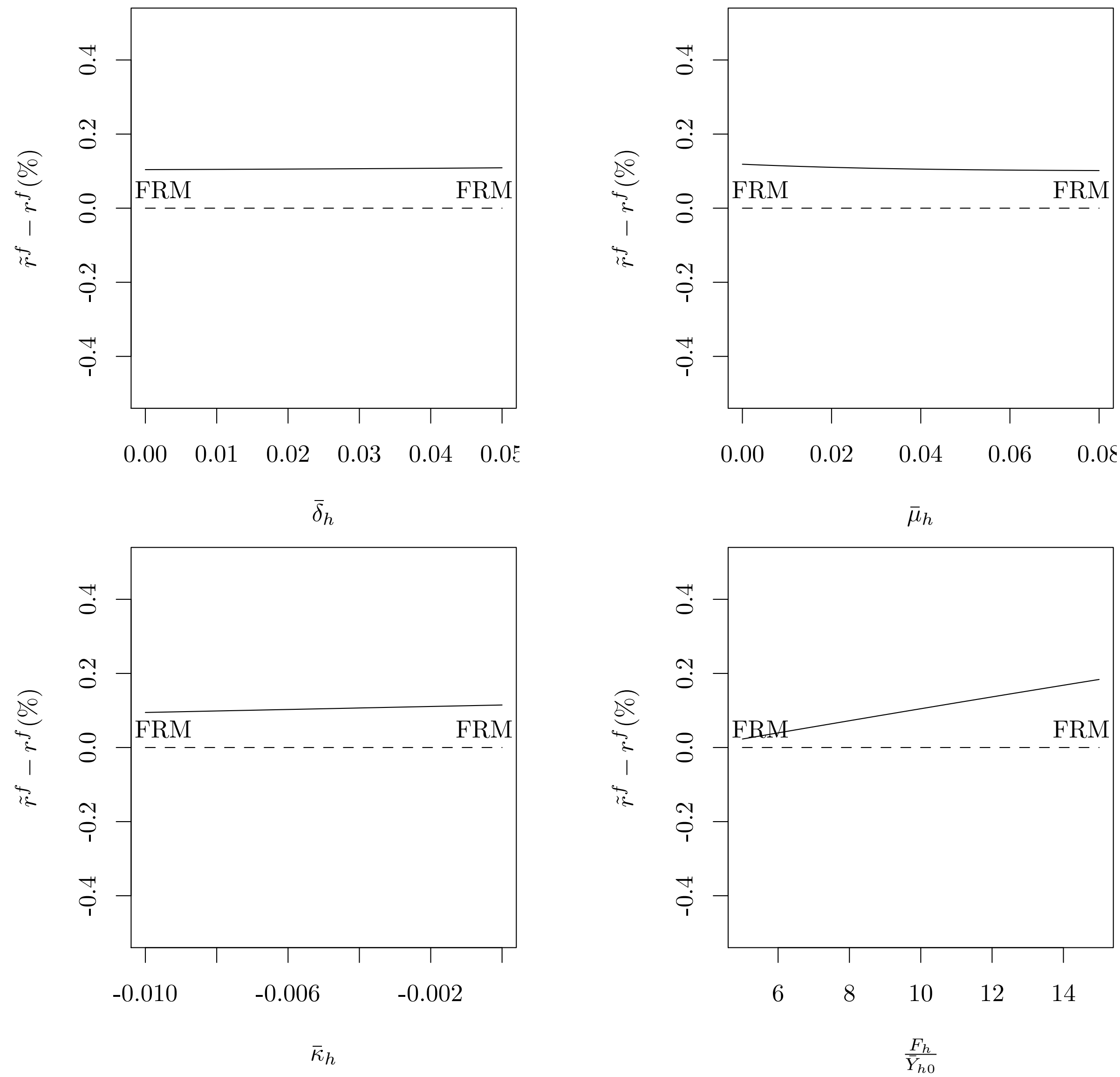


Figure 3.4: The difference between the utility-equivalent fixed rate and the current swap rate for different time-preferences, $\bar{\delta}_h$, expected income change parameters, $\bar{\mu}_h$ and $\bar{\kappa}_h$, and debt-to-income ratio $\frac{F_h}{Y_0}$.

λ_t are found for any mortgage choice and the mortgage choice equilibrium is pinned down using the optimality conditions found for the homeowners in the previous section. I define a mortgage choice equilibrium as follows.

Definition 3.1. *A mortgage distribution* $\left\{\mathbf{F}^a, \mathbf{F}^f\right\}$ *is an equilibrium if, the resulting market quantities lead to an equilibrium in the financial markets and no homeowner finds it optimal to switch mortgage type.*

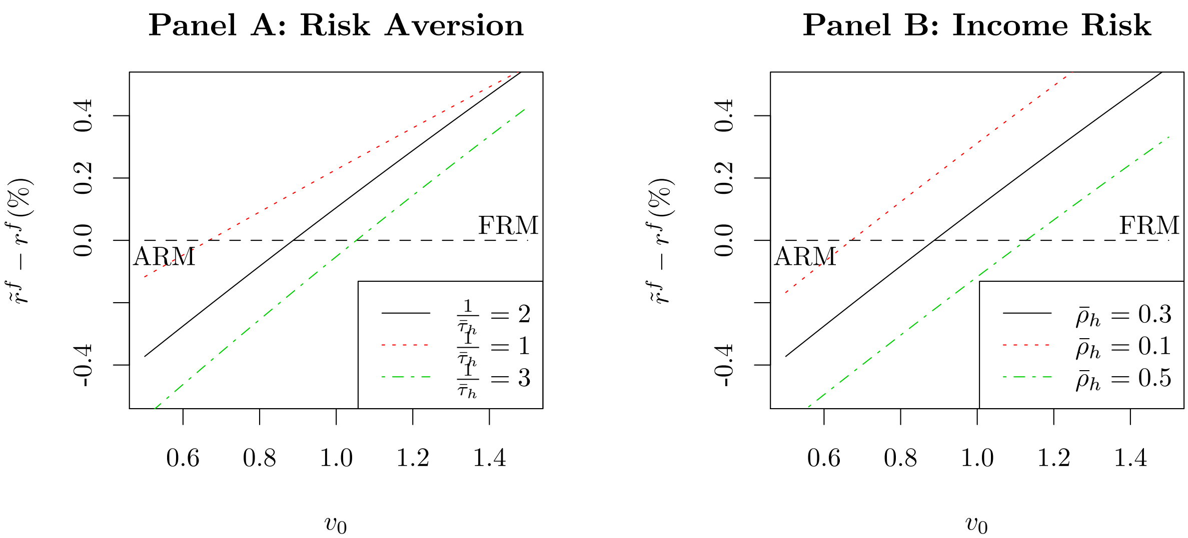


Figure 3.5: The optimal mortgage choice of homeowners in different initial states. Panel A depicts the effect for different risk aversions. Panel B depicts the effect for different income correlations.

Recall the market clearing conditions in (2.12)

$$\sum_{i=1}^{I} \theta_{it} = \mathbf{F}^f r^f, \quad \sum_{i=1}^{I} \theta_{it}^{(0)} = \frac{\mathbf{F}^a}{S_t^{(0)}}, \quad \sum_{i=1}^{I} \theta_{it}^{(T)} = -\mathbf{F}^a. \tag{3.8}$$

Now, consider the total financial wealth in the economy which must be equal to 0 at all times

$$0 = \sum_{i=1}^{I} X_{it} + X_{\mathcal{H}t}.$$

By Itô's Lemma, the dynamics of this must also be zero and by the financial wealth of the housing institution in (2.11)

$$\begin{aligned} 0 &= \sum_{i=1}^{I} dX_{it} + dX_{\mathcal{H}t} \\ &= \left(\sum_{i=1}^{I} \theta_{it} - \sum_{h \in \mathcal{H}_F} r^f F_h \right) (dS_t + dt) + \left(\sum_{i=1}^{I} \theta_{it}^{(T)} + \mathbf{F}^a \right) dB(t,T) \\ &\quad + \left(\sum_{i=1}^{I} \theta_{it}^{(0)} r_t S_t^{(0)} + \mathbf{F}^f r^f - \sum_{i=1}^{I} c_{it} \right) dt. \end{aligned}$$

From the market clearing conditions in (3.8), the market clearing condition in the goods market must be

$$\sum_{i=1}^{I} c_{it} = \mathbf{F}^f r^f + r_t \mathbf{F}^a.$$

This is a similar result as found in the case, where homeowners could only choose an FRM. That is, the total excess consumption of the investors is equal to the total interest payments of the homeowners. This difference in the market clearing conditions leads to different equilibrium interest rates and market price of risk given in the following theorem.

Lemma 3.1. *Given a mortgage distribution $\{\mathbf{F}^a, \mathbf{F}^f\}$, the equilibrium interest rate and market price of risk is given as*

$$r_t \equiv R_0(\mathbf{F}^a) - R_1(\mathbf{F}^a) v_t, \tag{3.9}$$

$$\lambda_t \equiv L(\mathbf{F}^a)\sqrt{v_t} = \left(L(0) - \frac{\sigma_v}{\tau_\Sigma}\mathbf{F}^a R_1(\mathbf{F}^a)\right)\sqrt{v_t}, \tag{3.10}$$

where the functions $R_0(\mathbf{F}^a), R_1(\mathbf{F}^a)$ solve the following system of equations

$$R_1(\mathbf{F}^a) - R_1(0) = R_1(\mathbf{F}^a)\frac{\mathbf{F}^a}{\tau_\Sigma}(\kappa_v - L(0)\sigma_v) + \frac{1}{2}\left(\frac{\sigma_v}{\tau_\Sigma}\right)^2 (\mathbf{F}^a)^2 R_1(\mathbf{F}^a)^2, \tag{3.11}$$

$$R_0(\mathbf{F}^a) = R_0(0) - \mathbf{F}^a R_1(\mathbf{F}^a)\frac{\mu_v}{\tau_\Sigma}, \tag{3.12}$$

and the initial conditions $R_0(0), R_1(0), L(0)$ are given in (2.13) *and* (2.14).

It is clear that if $\kappa_v < L(0)\sigma_v + \frac{\tau_\Sigma}{\mathbf{F}^a}$, then the second-order linear equation (3.11) has two positive roots if

$$\left(\frac{\mathbf{F}^a}{\tau_\Sigma}(\kappa_v - L(0)\sigma_v) - 1\right)^2 - 4R_1(0)\frac{1}{2}\left(\frac{\sigma_v}{\tau_\Sigma}\right)^2 (\mathbf{F}^a)^2 > 0.$$

That is, if these two conditions hold, we still have a negative relationship between aggregate volatility and interest rates. In most cases, these solutions are on each side of the case with only FRMs $R_1(0)$. To see which solution leads to the best economic interpretation consider the solution of (3.12) which is always lower for positive $R_1(\mathbf{F}^a)$. The solution $R_1(\mathbf{F}^a) > R_1(0)$, therefore, leads to lower interest rates in all states of the economy compared to the case with only FRMs. The solution $0 < R_1(\mathbf{F}^a) < R_1(0)$ leads to higher interest rates when the income volatility is sufficiently high compared to the case with only FRMs. As ARMs create a positive supply in the money market account, the latter solution provides the best economic interpretation as investors require a high rate of interest to invest additional financial wealth in the money market account.[4] As $R_0(\mathbf{F}^a)$ is lower when some homeowners switch to ARMs, this means that the upper level of the risk-free rate decreases when more and more homeowners choose ARMs.

We can now see that the market price of risk with homeowners choosing ARMs (3.10) is higher compared to no ARMs (as $\sigma_v < 0$). That is, the market price of risk increases when homeowners are allowed to choose ARMs. The reason for this is the increased supply of floating interest rates stemming from ARMs. As the investors receive less interest payments during recessions when homeowners are allowed to choose ARMs (r_t is low), this increases the market price of W-risk as assets positively depending on W become less attractive.

This shows that when allowing homeowners to use an ARM to finance their homes, the only difference to the market quantities r_t and λ_t is a change in parameter values for a given mortgage distribution $\{\mathbf{F}^a, \mathbf{F}^f\}$. This means that all of the steps in showing that it is indeed an equilibrium are exactly identical to what happened in Section 2.5. Note that the introduced demand for zero-coupon bonds from the housing institution is redundant for the investors, as they can negate their supply by continuously trading in the annuity and the money market account. The optimal mortgage choice for each homeowner can thus be solved in the same manner as in Section 3.2 by simply changing the parameters. This leads to the mortgage choice equilibrium when no homeowner is better off changing their type of mortgage.

[4] The solution $R_1(\mathbf{F}^a) > R_1(0)$ also leads to extremely sensitive interest rates towards income volatility with $R_1(\mathbf{F}^a) > 1$ and volatility reverting around 1.

3.3.1 Sensitivity analysis

To illustrate the effects of mortgages choices in equilibrium, this numerical example considers a distribution of homeowners and analyzes the effect on the yield curve of introducing ARMs.

Again, assume the same baseline parameters in Table 1 and the base case parameters, except for two dimensions. Let homeowners differ in income risk composition and risk-aversion of the homeowners, according to a uniform distribution over the space

$$\left(\bar{\rho}_h, \frac{1}{\bar{\tau}_h}\right) \in [0, 0.6] \times [0.5, 4]\,. \tag{3.13}$$

I only analyze heterogeneity over these two parameters as the individual mortgage choice was not affected significantly by changes in the other parameters. As the number of investors and number of homeowners have an impact on the equilibrium in this section, assume that the homeowner to investor ratio is 1:8.

From the analysis of the individual homeowner's mortgage choice we know that there is a cutoff level in both correlation and risk aversion where the mortgage choice changes. Now, consider how the cutoff level of one of these parameters is affected by the other. This is done by finding the optimal mortgage choice for different risk aversion and different risk exposures $\bar{\rho}_h$. This leads to the cutoff levels in Figure 3.6. Below and to the left of the cutoff levels, an FRM is optimal, and to the right and above, an ARM is optimal.

The more risk-averse homeowners have a cutoff level for the correlation to the market, $\bar{\rho}_h$, lower than less risk-averse homeowners for switching to an ARM. The reason for this is that the ARM acts as a natural hedge against the systematic income risk and the need for this hedge is higher for more risk-averse homeowners. The figure shows that homeowners with a risk aversion below 1.3 only find it optimal to finance their home with an FRM. As the fixed rate on the FRM is equal to 3.48%, significantly below the long-term level of the risk-free rate, which is 3.80%, this explains how the FRM seems attractable for less risk-averse investors; in most periods their interest payments are lower than that of the ARM and they are not too worried about times of recession.

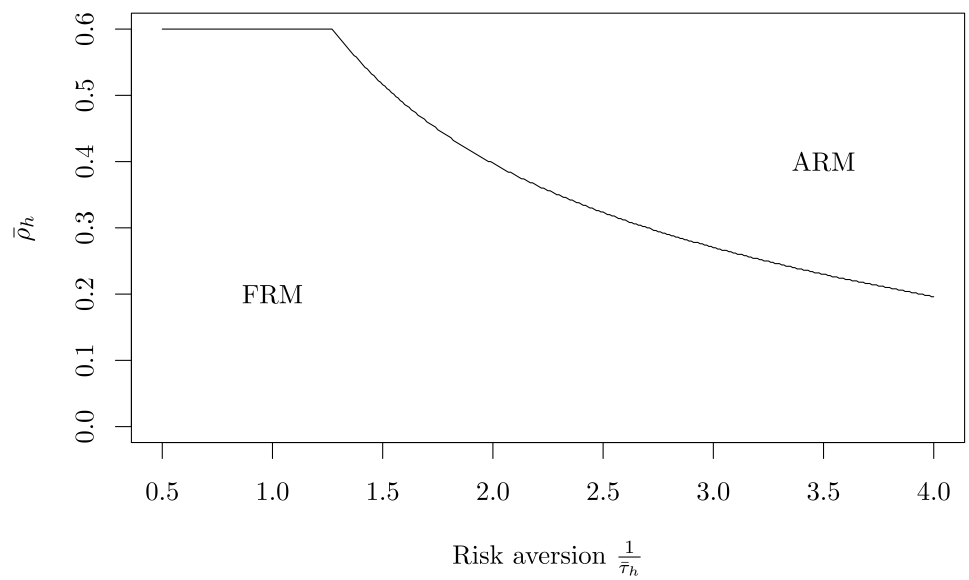


Figure 3.6: The line represents the initial cutoff level for choosing an ARM vs. FRM when all homeowners use an FRM. If the homeowner is above the line then it is optimal to use an ARM.

Knowing that the choices of homeowners affect the market equilibrium quantities, r_t and λ_t, we can now analyze how many homeowners find it optimal to switch after the ARM share has increased from the above homeowners switching to an ARM. The optimal mortgage choice of each homeowner is found using the sign of

$$\tilde{r}^f - r^f = \frac{\bar{\tau}_h}{F_h} \log\left(\frac{J_{h0}^{ARM}}{J_{h0}^{FRM}}\right),$$

where the expected utilities of each mortgage type J_h^{ARM}, J_h^{FRM} are modified according to the parameter shift stemming from an increased ARM share. If the ARM share found by these optimality checks is different from the previous share, then the guess is updated to the newly found share, and the procedure is repeated until the found share is the same as the share used to find the market quantities r_t, λ_t.

By doing this recursive search for an equilibrium ARM share, the equilibrium mortgage choice is illustrated in Figure 3.7. We see that the cutoff level of correlation decreases in equilibrium and even more so for the more risk-averse homeowners. That is, the ARM is more attractive because the downside of the ARM, the high interest rates during ex-

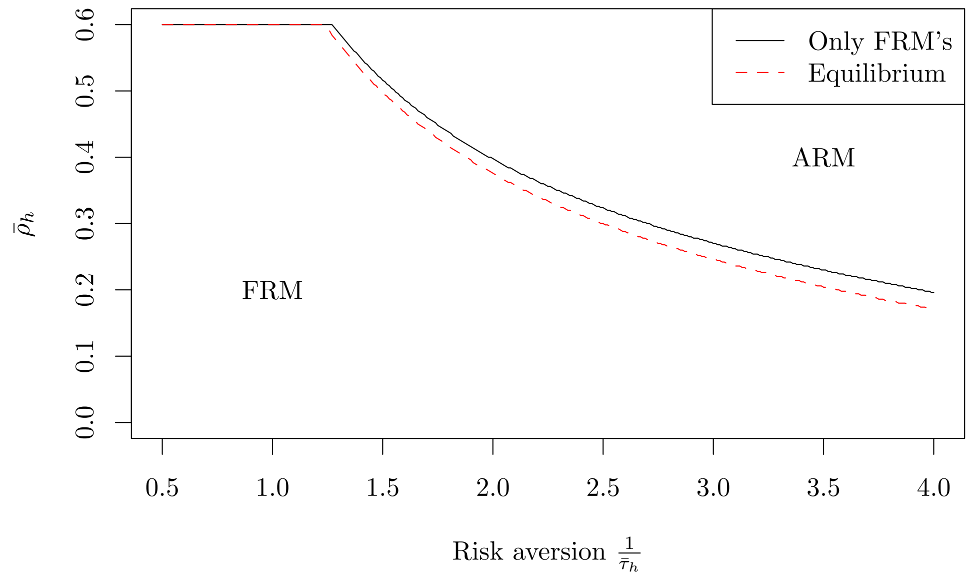


Figure 3.7: The red dashed line shows the cutoff for choosing an ARM vs. FRM in equilibrium. If the homeowner is above the line then it is optimal to use an ARM. The black solid line shows the initial cutoff level in the case that no homeowner uses an ARM corresponding to Figure 3.6 and equivalently found using $\mathbf{F}^a = 0$.

pansions, is lowered. However, when more homeowners choose the ARM, interest rates become less volatile. This makes the ARM less effective at hedging systematic income risk. This means that at some point the tradeoff between the lowered upper limit and the less effective hedging of the ARM rates equalize and no homeowner finds it optimal to switch.

The equilibrium is also analyzed in different initial states. In Panel A of Figure 3.8 we see that when the mortgages are issued in expansions ($v_0 = 0.8$), a much larger portion of homeowners choose the ARM. This is not surprising from the result on the individual mortgage choice in Figure 3.5. Now, as there are many homeowners choosing to switch to an ARM in order to gain from interest rates expected to decrease, the intuitive result would be that the short rate increases due to the increased supply. This is, however, not the case. As the homeowners are moving away from the FRM, which the investors can use to hedge their own income risk, the investors are willing to compensate the homeowners to stay with the FRM by lowering the fixed rate. As the fixed rate is set using no arbitrage, the only way to do so (in this one-factor model) is to lower the entire yield curve and actually lower the current rate on the ARM. Now, the less risk-averse homeowners do not gain as much from taking out the ARM so they stick with the FRM. The more risk-averse homeowners value the ARM more when the risk of high interest rates decreases and find it optimal to switch when sufficiently enough homeowners switch.

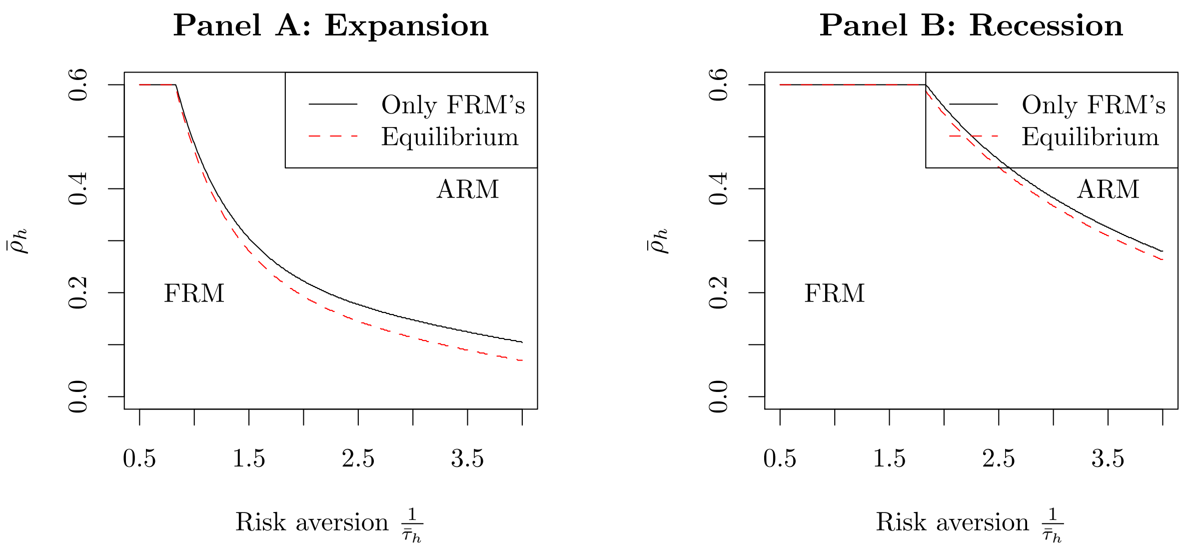


Figure 3.8: The mortgage choice equilibria in different initial states of the economy. Panel A corresponds to $v_0 = 0.8$, an expansion. Panel B corresponds to $v_0 = 1.2$, a recession.

In Panel B of Figure 3.8, the initial state of the economy is a recession ($v_0 = 1.2$). Now, as we saw in Figure 3.5, the ARM is less attractive even though it offers a lower rate today, the benefits of choosing the FRM with a significantly lower rate than what future short rates are expected to be are simply too large. As fewer homeowners find it optimal to switch, the equilibrium does not change significantly.

As previously noted, the investor-to-homeowner ratio has an impact on the equilibrium market quantities. The mortgage choice equilibrium does not change significantly, but if we look at the differences in the potential yield curves, the changes are quite significant. This is seen in Figure 3.9. For an economy with a small mortgage market the resulting equilibrium yield curves are seen in Panel A, where we as expected find a narrower band in which the yield curve operates compared to when $\mathbf{F}^a = 0$ in Figure A.2. Panel B shows the equilibrium yield curves of an economy with a large mortgage market. We see that the effect on the yield curve is much larger for the large mortgage market even though approximately the same fraction of homeowners find it optimal to switch mortgage. For the base case of a homeowner-to-investor ratio of 1:8 the resulting yield curves are somewhere in between the two. This clearly shows how different economies are affected by changes in their mortgage market and begs the empirical question wether or not this is present in the data.

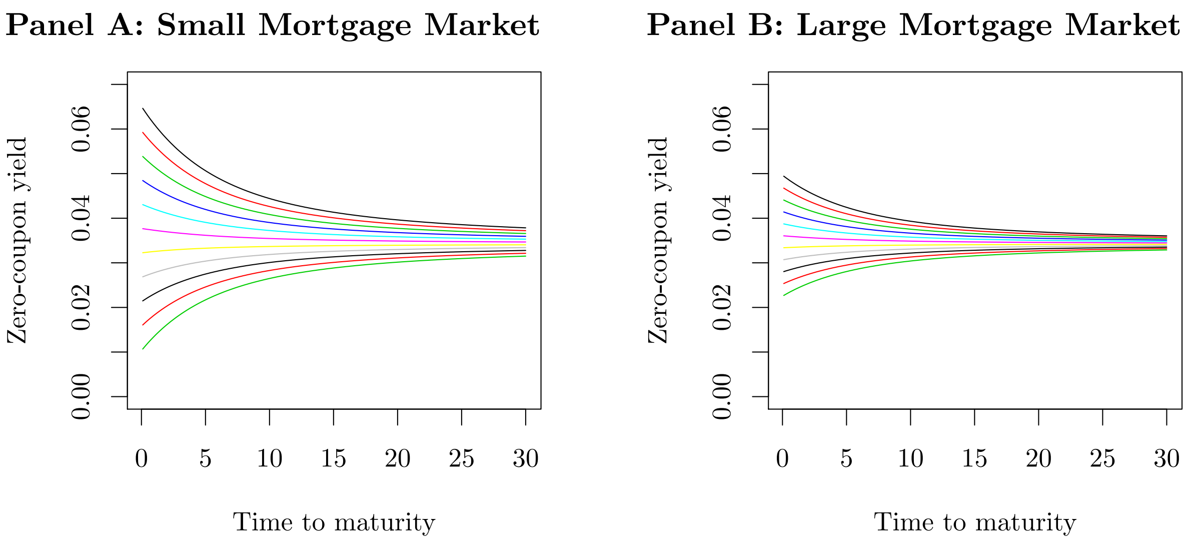


Figure 3.9: The yield curves in the mortgage choice equilibrium for different sizes of the mortgage market. Panel A corresponds to a homeowner-to-investor ratio of 1:2. Panel B corresponds to a homeowner-to-investor ratio of 1:20.

4 Additional Considerations

In this section, I show that swapping a fixed-rate mortgage with an adjustable-rate mortgage does not depend on the rate of the fixed-rate mortgage as long as the swap is financed using a premium on the ARM. I also consider the partial equilibrium of mortgage choices for a different model of income dynamics. The main determinants for choosing between an adjustable- versus a fixed-rate mortgage remain unchanged, that is, for interest rates determined in the partial equilibrium of the investors, the more risk-averse homeowners choose an ARM and the correlation between the homeowners income and the overall economy intensifies this result. Finally I have a note on how the prepayment option should be introduced in the model and argue why the results would not change significantly.

4.1 Mortgage swap for non-market fixed rate

Now, consider homeowners who enter with a fixed-rate mortgage, where the fixed rate is different from the current swap rate. This means that in order to switch from an FRM to an ARM, the homeowner needs to pay for the swap. The price of the swap is calculated in equation (3.1) as

$$P_h^{swap} = F_h \left(r^f S_0 + B\left(0, T\right) - 1 \right).$$

The homeowner is allowed to finance the swap by paying a constant interest rate premium on the ARM. The fair premium is then $x = \frac{P_h^{swap}}{S_0}$, as the price of x annuities replicating the constant premium payment is P_h^{swap}. This can be rewritten as

$$\begin{aligned} x &= \frac{F_h \left(r^f S_0 + B\left(0, T\right) - 1 \right)}{S_0} \\ &= F_h \left(r^f - \frac{1 - B\left(0, T\right)}{S_0} \right) \\ &= F_h \left(r^f - r^{f,swap} \right). \end{aligned}$$

After a switch to an ARM, the homeowner pays the floating interest payment $r_t F_h$ and constant premium x. This means that the life-time expected utility of an ARM can be calculated as

$$\begin{aligned} J_{h0}^{ARM,r^f} &= \mathbb{E}\left[\int_0^T -e^{-\bar{\delta}_h t - \frac{1}{\tau_h}\left(\bar{Y}_{ht} - r_t F_h - F_h\left(r^f - r^{f,swap}\right)\right)} dt \right] \\ &= e^{\frac{1}{\tau_h} F_h\left(r^f - r^{f,swap}\right)} \mathbb{E}\left[\int_0^T -e^{-\bar{\delta}_h t - \frac{1}{\tau_h}\left(\bar{Y}_{ht} - r_t F_h\right)} dt \right] \\ &= e^{\frac{1}{\tau_h} F_h\left(r^f - r^{f,swap}\right)} J_{h0}^{ARM}. \end{aligned}$$

At the same time, the life-time expected utility of an FRM with fixed rate $r^f \neq r^{f,swap}$ is calculated as

$$\begin{aligned} J_{h0}^{FRM,r^f} &= \mathbb{E}\left[\int_0^T -e^{-\bar{\delta}_h t - \frac{1}{\tau_h}\left(\bar{Y}_{ht} - r^f F_h\right)} dt\right] \\ &= e^{\frac{1}{\tau_h} F_h\left(r^f - r^{f,swap}\right)} \mathbb{E}\left[\int_0^T -e^{-\bar{\delta}_h t - \frac{1}{\tau_h}\left(\bar{Y}_{ht} - r^{f,swap} F_h\right)} dt\right] \\ &= e^{\frac{1}{\tau_h} F_h\left(r^f - r^{f,swap}\right)} J_{h0}^{FRM}. \end{aligned}$$

This means that the life-time expected utility of either mortgage type is multiplied by the same constant, $e^{\frac{1}{\tau_h} F_h\left(r^f - r^{f,swap}\right)}$, compared to the case, where the fixed rate is set equal to the current swap rate. Therefore, the rate on a fixed-rate mortgage does not have an impact on the optimal mortgage choice. This is not surprising as the swap is fairly priced and in cases, where the fixed rate is higher than the swap rate the price of the swap is high. When homeowners switch mortgage type in the real economy, it often entails a different repayment scheme. In an extended model one could allow for different ways of financing the swap and in turn find the optimal payment scheme. For example, the linear payment scheme should possibly be replaced by a scheme with lower payments in the beginning of the mortgage and larger payments at the end, where the income level most likely has increased. Additional homeowners may then find it optimal to switch not only to get access to the natural hedge of an ARM, but also a better suited payment scheme.

4.2 Pro-cyclical income skewness

To show that the main results hold for more realistic income processes, I now consider an alternative income process for the homeowner motivated by the observations made in Guvenen, Karahan, Ozkan, and Song (2015). They find that on an individually basis it is not the second moment of income dynamics which moves over time, but instead the third and fourth moment. To capture this fact, consider modelling the income dynamics as follows,

$$d\bar{Y}_{ht} = \mu_h(t)dt + \eta_h^+ dN_{ht}^+ - \eta_h^- dN_{ht}^-, \tag{4.1}$$

where μ_h is a deterministic function capturing the life-cycle of income movements, $\eta_h^+, \eta_h^- > 0$ capture small positive jumps and large negative jumps in income after shocks to the generalized Poisson jump processes N_h^+ and N_h^- respectively. The generalized Poisson processes are assumed to have stochastic intensity, also known as Cox processes, see Lando (1998) and more recently Wachter (2013) for other finance application of such processes. To model pro-cyclical negative skewness of income, the intensity of N_h^+ and N_h^- are assumed to be affine functions of income volatility of the investors v,

$$\lim_{\Delta t \to 0} \mathbb{E}_t \left[N_{ht+\Delta t}^+ - N_{ht}^+ \right] = \tilde{a}_h^+ - \tilde{b}_h^+ v_t$$
$$\lim_{\Delta t \to 0} \mathbb{E}_t \left[N_{ht+\Delta t}^- - N_{ht}^- \right] = \tilde{a}_h^- + \tilde{b}_h^- v_t,$$

where $\tilde{b}_h^+, \tilde{b}_h^- > 0$ capture a lower (higher) probability of increases (decreases) in income during recessions where v_t is high and r_t is low. The expected utilities of each type of mortgage are found in the following corollary.

Corollary 4.1. *The expected utility function of homeowner h with income dynamics* (4.1) *and a fixed-rate mortgage is*

$$J_{ht}^{FRM} = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - r^f F_h\right)} \int_t^T e^{-\bar{\delta}_h(s-t) - \frac{1}{\bar{\tau}_h}\int_t^s \mu_h(u)du + a_{F,N}(s-t) + b_{F,N}(s-t)v_t} ds,$$

where the functions $a_{F,N}$ and $b_{F,N}$ solve some ordinary differential equations similar to Theorem 3.1, with initial conditions $a_{F,N}(0) = b_{F,N}(0) = 0$.

The expected utility function of homeowner h with an adjustable-rate mortgage is

$$J_{ht}^{ARM} = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - r_t F_h\right)} \int_t^T e^{-\bar{\delta}_h(s-t) - \frac{1}{\bar{\tau}_h}\int_t^s \mu_h(u)du + a_{A,N}(s-t) + b_{A,N}(s-t)v_t} ds,$$

where the functions $a_{A,N}$ and $b_{A,N}$ solve some ordinary differential equations similar to

Theorem 3.1, with initial conditions $a_{A,N}(0) = b_{A,N}(0) = 0$.

To analyze the mortgage choice of homeowners with pro-cyclical skewness, the function $\mu_h(t)$ is chosen to match calibration of individual income over the life-cycle in Cocco et al. (2005) described as a parabola

$$Y(age) = -4.3148 + 0.3194age - \frac{0.0577age^2}{10} + \frac{0.0033age^3}{100}.$$

Assuming that homeowners take out their mortgage at the age of 30, the corresponding function μ_h is

$$\mu_h(t) = 0.3194 - \frac{0.1154(t+30)}{10} + \frac{0.0099(t+30)^2}{100},$$

with initial income 0.9652. In the following analysis, base case parameters of the investors are as in Section A.1 and the initial state is $v_0 = 1$. The preference parameters are the same as the investors

$$\bar{\tau}_h = \frac{1}{2}, \qquad \bar{\delta}_h = 0.01.$$

The shock parameters are set according to two rules.

- The steady state expected increase in income is $\mu_h(t)$.
- Negative shocks are 4 times in magnitude of positive shocks.

These two rules ensure that the function $\mu_h(t)$ does indeed capture the ex-ante expected life-cycle of income and negative shocks are more severe than positive shocks capturing the negative skewness. The rules correspond to the restrictions

$$\eta_h^+(\tilde{a}_h^+ - \tilde{b}_h^+) = \eta_h^-(\tilde{a}_h^- + \tilde{b}_h^-)$$
$$\eta_h^- = 4\eta_h^+.$$

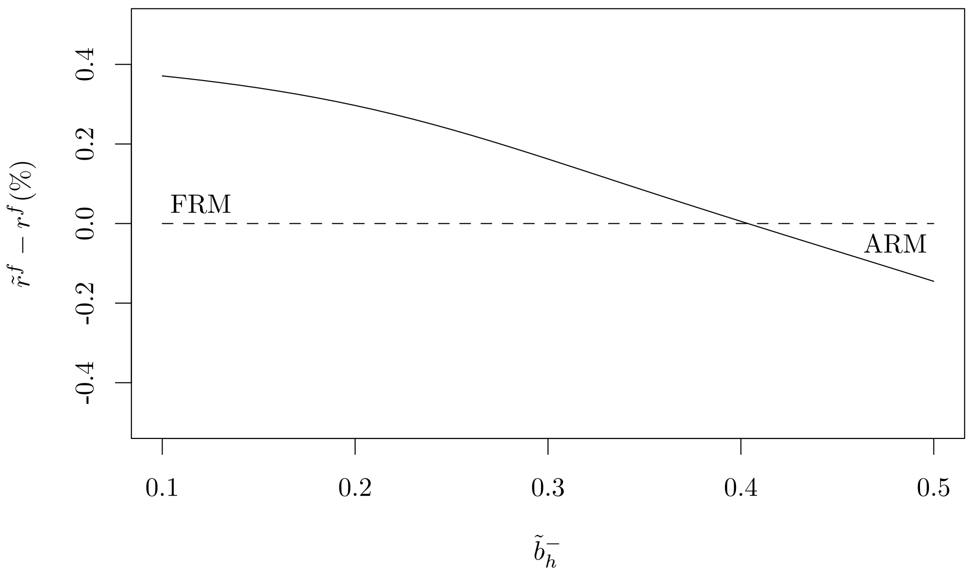


Figure 4.1: The optimal mortgage choice of a homeowner with different intensities of income shocks.

The base case parameters for the shocks are

$$\eta_h^+ = 0.05, \quad \eta_h^- = 0.2, \quad a_h^+ = 3.6, \quad b_h^+ = 2.4, \quad a_h^- = -0.1, \quad b_h^- = 0.4.$$

This corresponds to negative shocks of approximately 20% every 3.33 years and positive shocks of 5% every 1.2 years in the steady state $v_t = 1$. During recessions the intensity of positive shocks decreases as $v_t > 1$ while the intensity of negative shocks increases. The mortgage is again 10 times the homeowners initial income $F_h = 10 * Y_0 = 9.652$, and the time-to-maturity is 30 years.

In Figure 4.1, we see how the optimal mortgage choice is impacted by the intensity exposure of the homeowner. That is, when $\tilde{b}_h^-$ is high, the intensity of negative jumps in income are affected more by the state of the economy. The other shock parameters are chosen to ensure the same steady state intensity of negative shocks and the same ratio between the intensities of the two types of shocks in all states. As the homeowner is more prone to negative shocks during recessions and experiences less positive shocks, the ARM is better at hedging the homeowner's income risk for higher $\tilde{b}_h^-$. Again, the ARM is preferred for homeowners whose income are affected more by the business cycle,

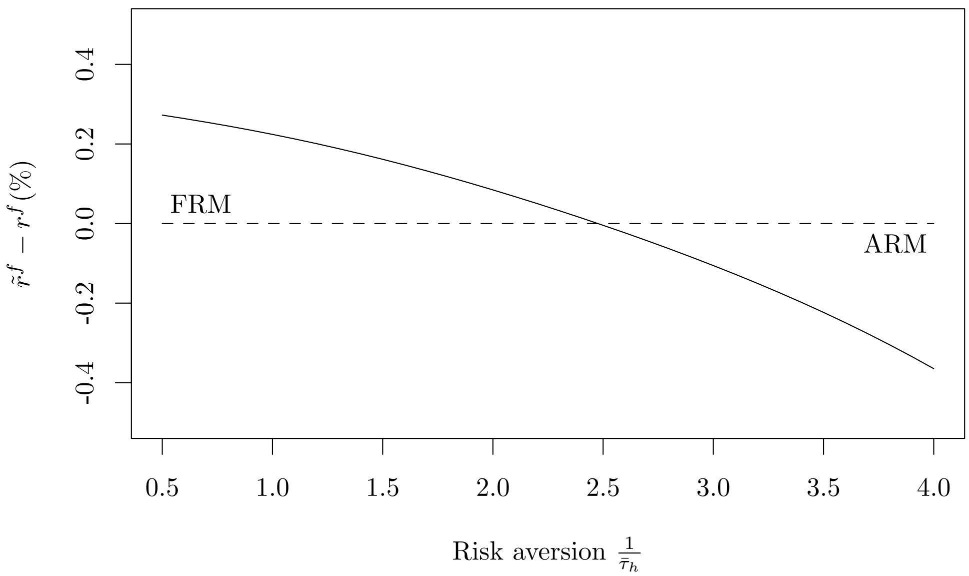
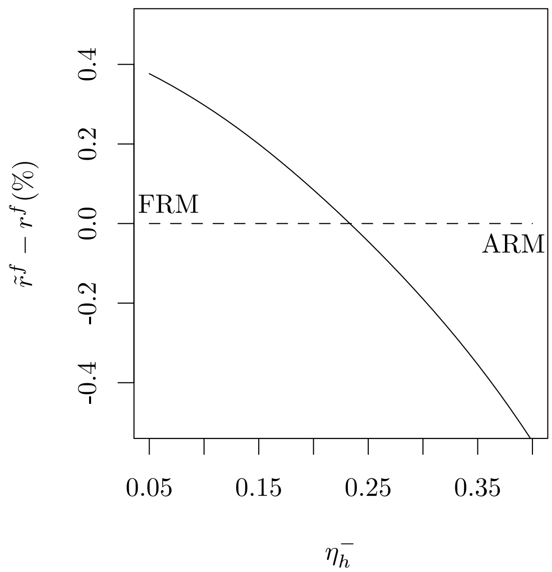


Figure 4.2: The optimal mortgage choice of a homeowner as a function of risk aversion.

in this setting through higher probabilities of decrease in income during recessions where the interest rate is low. The same picture is present in Figure 4.2, where we see that the more risk-averse homeowners seek the income hedge of the ARM while the less risk-averse find the ARM too expensive compared to the FRM with a lower fixed rate than the long term floating interest rate.

In Figure 4.3, we see that the size effect of the income shocks are as expected. When income drops are higher, the homeowner has increased incentive to buy the income hedge of the ARM. Also, as we saw for homeowners with counter-cyclical income volatility, the initial state of the economy has an impact on the optimal mortgage choice and the explanation is the same. During expansions, low v_0 and high r_0, as v_t is mean-reverting, the short rate is mean-reverting and the homeowners expect the interest rate to decrease and therefore prefer the ARM compared to the FRM with a higher fixed rate.

This section shows that the claim of this paper - ARMs are able to hedge homeowners' income risk - is robust to different kinds of income processes, here capturing pro-cyclical negative skewness and counter-cyclical kurtosis as documented by Guvenen et al. (2015).



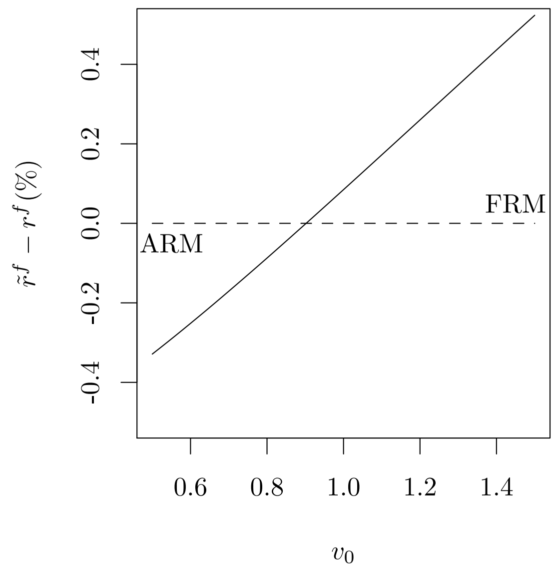


Figure 4.3: The difference between the utility-equivalent fixed rate and the current swap rate for different size effects, η_h, and initial states, v_0.

4.3 Prepayment option

The FRMs in this paper do not have the prepayment option incorporated and the results might not be strong enough to help homeowners choose between ARMs and FRMs with the prepayment option. Therefore, I consider a small example to address this.

Assume that homeowners must choose between an ARM and an FRM where the FRM gives the option to repay the balance of the loan F_h, financed by issuing a new mortgage at time $T/2$ with the prevailing asset prices after paying a proportional prepayment fee CF_h. First consider the switch from an FRM to a new FRM. The value of this option at time $T/2$ is

$$
\begin{aligned}
V^{F\to F} &= F_h r^f S_{T/2} - F_h r^f_{T/2} S_{T/2} - F_h C \\
&= F_h \left[(r^f - r^f_{T/2}) S_{T/2} - C \right],
\end{aligned}
$$

where $r^f_{T/2}$ is the swap rate at time $T/2$. Therefore, the homeowner will repay the mortgage if

$$r^f > r^f_{T/2} - \frac{C}{S_{T/2}} = \frac{1 - B(T/2, T) + C}{S_{T/2}}.$$

That is, there exists a cutoff-level for the short-rate such that, when $r_{T/2} < r^*$ the homeowner is better off swapping to a lower fixed-rate mortgage. Second, consider whether to switch to an ARM at time $T/2$. This is the case when

$$\begin{aligned} E_{T/2}\left[\int_{T/2}^{T} -e^{-\bar{\delta}_h t - \frac{1}{\bar{\tau}_h}(Y_{ht} - r^f F_h)} dt\right] &< E_{T/2}\left[\int_{T/2}^{T} -e^{-\bar{\delta}_h t - \frac{1}{\bar{\tau}_h}\left(Y_{ht} - F_h\left(r_t + \frac{C}{S_{T/2}}\right)\right)} dt\right] \\ \Leftrightarrow e^{\frac{F_h}{\bar{\tau}_h}(r^f - r^f_{T/2})} &> \frac{J^{ARM}_{hT/2}}{J^{FRM}_{hT/2}} \\ \Leftrightarrow r^f &> \tilde{r}^f_{T/2} + \frac{C}{S_{T/2}}, \end{aligned}$$

where $\tilde{r}^f_{T/2}$ is the UE-rate at time $T/2$. The right-hand side of the above inequality is increasing in the prevailing short rate $r_{T/2}$, so homeowner h finds it optimal to switch to an ARM if $r_{T/2} < r^{**}_h$ for some r^{**}_h.

In total, homeowners find it optimal to prepay their mortgage if interest rates are sufficiently small. This is in states of the world where volatility is high. Bearing that in mind lets consider how the prepayment option is valued. To compensate the investors for the prepayment risk, an FRM comes with a premium p, such that the interest payments are

$$I_{ht} = (r^f + p)F_h, \quad t \le T/2, \qquad I_{ht} = r^f F_h, \quad t > T/2.$$

The premium will be found using the state-price deflator of the investors

$$\begin{aligned} E\left[\int_0^{T/2} \xi_t p \mathbf{F}^f dt\right] &= E\left[\xi_{T/2}\mathbf{F}^f x_{T/2}(r^f S_{T/2} + B(T/2, T) - 1)\right] \\ \Leftrightarrow p &= \frac{E\left[\xi_{T/2} x_{T/2}(r^f - r^f_{T/2})S_{T/2}\right]}{E\left[\int_0^{T/2} \xi_t dt\right]}, \end{aligned}$$

where $x_{T/2}$ is the fraction of the homeowners prepaying their mortgage conditional on the state at time $T/2$ and $(r^f - r^f_{T/2})S_{T/2}$ is the value of the swap that the homeowners have at time $T/2$. The equilibrium premium should then be substantial as the random variables $\left\{x_{T/2}, \left(r^f - r^f_{T/2}\right), S_{T/2}, \xi_{T/2}\right\}$ are all positively correlated, since homeowners prepay when interest rates are low and marginal utility of investors is high. This might actually push homeowners away from fixed-rate mortgages if the premium is too high, and some homeowners might even prefer to not have the option. This should be explored in future research.

5 Conclusion

This paper, solves both the individual homeowner's mortgage decision and on an aggregate level the mortgage equilibrium of a distribution of homeowners. Both the investors' and homeowners' income are subject to counter-cyclical income volatility. The counter-cyclical income volatility of the investors result in high equilibrium interest rates during expansions, where income is high and volatility is low, and low equilibrium interest rates during recessions, where income is low and volatility is high. By pricing the two types of mortgages, fixed-rate (FRM) and adjustable-rate (ARM) using the financial markets, the mortgages are fairly priced and linked to the state of the economy. This interest rate process with low rates during recessions and high rates during expansions, makes ARMs attractive to homeowners with counter-cyclical income volatility, as it provides a natural hedge against systematic income risk.

The key determinant to the optimal mortgage choice is the level of idiosyncratic income risk. For homeowners subject to increased levels of idiosyncratic income risk, the ability of an ARM to hedge their income risk decreases. There is then an upper level of idiosyncratic income risk, for which the homeowner finds it optimal to use an adjustable-rate mortgage compared to a fixed-rate mortgage. This upper level is an increasing function of the homeowner's risk aversion, which shows that ARMs do in fact provide a hedge against systematic income risk. That is, a more risk-averse homeowner uses an ARM for lower levels of systematic income risk.

The mortgage equilibrium is determined recursively by adjusting the risk-free rate and market price of risk given a number of homeowners using ARMs to finance their home. More risk-averse homeowners find it optimal to switch from an FRM to an ARM as the ARM share increases. This is mainly caused by the main downside of an ARM decreasing. That is, the possible high interest payments during expansions are lowered, because even though the economy is expanding it does not necessarily mean that the homeowner's income increases as well. Interest rates become less volatile and the hedging ability of the ARM diminishes as the interest rate does not adjust enough to provide a stable income net of interest payments. Therefore, at some point the trade-offs cancel and the mortgage equilibrium is reached once no homeowner finds it optimal to switch in the market equilibrium for the given ARM share.

In this paper, the mortgage equilibrium is solved only at time 0. As the individual mortgage choice is state-dependent, a dynamic model in which homeowners can switch at pre-specified points in time could be examined. This is, however, difficult to solve as future decisions needs to be taken into account. In addition, as FRMs often include a prepayment option, future research should enlighten whether such an option is more valuable to the homeowner than the hedging ability of an ARM. I have given a short motivation as to how this should be introduced to the model and argued why it would not change the choice between ARM and FRM significantly. This is because prepayments occur when interest rates are low, and this is costly to the investors with high marginal utilities in these states of the economy. Therefore, investors must be compensated with high premiums. In terms of the homeowners income risk this would only create a "steeper" decision rule in terms of the picture in Figure 3.1, while risk aversion is more ambiguous.

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A Appendix

A.1 Calibration

In this section, the model without homeowners is calibrated using data on the US Federal Funds Rate and the OECD Consumer Confidence Indicator for the United States.[5] The calibration is used to analyze the interest rate structure attained in the found equilibrium.

The Consumer Confidence Indicator (CCI) released by the OECD is used to construct an appropriate process of v. The index typically ranges in the interval $(95, 105)$ where periods above 100 indicates that the economy is in expansion. Now, impose that v is given as

$$v_t = 1 - 15\frac{(CCI_t - 100)}{100}.$$

That is, consumer confidence is high, v is low, during expansions and consumer confidence is low, v is high, during recessions. The constant 15 is chosen to generate sufficient variation in v.

The time-series using this transformation can be seen in Figure A.1. The spikes can be identified to several economic events, e.g., the oil crisis of the 1970s, the recession following the fight against inflation in the early 1980s, and the great recession in the late 2000s.

To estimate the income volatility parameters, $\mu_v, \kappa_v, \sigma_v$, consider the Euler scheme of (2.3)

$$v_{t+\Delta t} - v_t = \kappa_v \left(\frac{\mu_v}{\kappa_v} + v_t\right)\Delta t + \sigma_v\sqrt{v_t}\sqrt{\Delta t}\varepsilon_{t+\Delta t},$$

where $\varepsilon_{t+\Delta t} \sim N(0,1)$. κ_v is then calibrated using the sample analogue of

$$\kappa_v = \frac{1}{\Delta t}\frac{\mathbb{E}\left[(v_{t+\Delta t} - v_t)(v_t - \mathbb{E}[v_t])\right]}{\mathbb{E}\left[(v_t - \mathbb{E}[v_t])^2\right]},$$

[5]Source: `https://research.stlouisfed.org/fred2/series/FEDFUNDS` and `https://research.stlouisfed.org/fred2/series/CSCICP03USM665S`

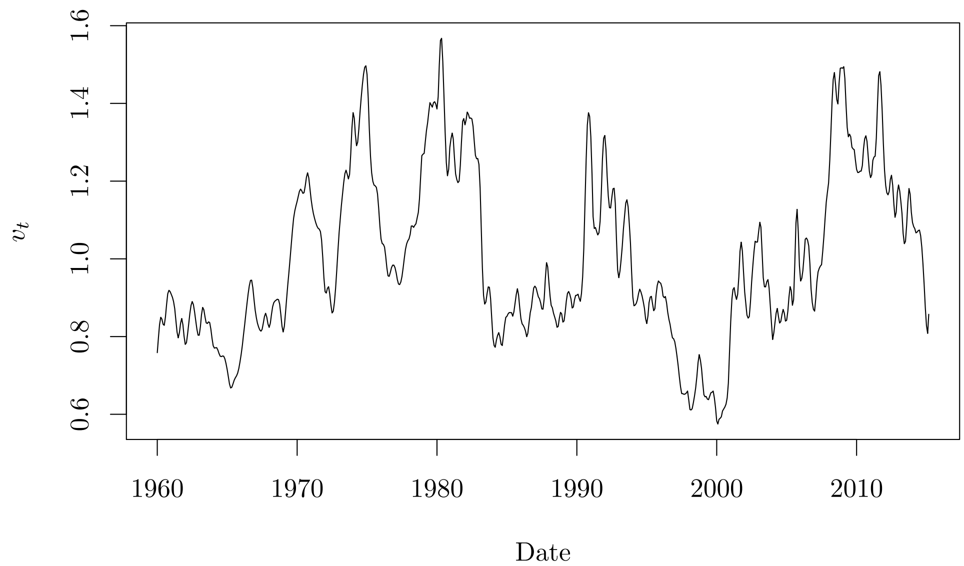

Figure A.1: Time-series of $v_t = 1 - \frac{CCI_t}{100}$.

as $\mathbb{E}[v_t] = -\frac{\mu_v}{\kappa_v}$. To ensure a long-term expected level of v equal to one, set $\mu_v = -\kappa_v$. After these parameters have been calibrated σ_v is calibrated as the sample analogue of,

$$\begin{aligned}\sigma_v &= -\sqrt{\sigma_v^2} \\ &= -\sqrt{\mathbb{E}\left[\frac{(v_{t+\Delta t} - v_t - \kappa_v(v_t - 1)\Delta t)^2}{v_t \Delta t}\right]}.\end{aligned}$$

Now comes the choice of Δt. For monthly sampling the process is nearly a random walk with $\kappa_v = -0.13$, so to ensure a reasonable level of mean-reversion to match the picture in Figure A.1, the parameters are calibrated using quarterly sampling as

$$\mu_v = 0.3062, \quad \kappa_v = -0.3062, \quad \sigma_v = -0.1603.$$

The dynamics of v are now mean-reverting and when $\mu_v = -\kappa_v$, the long-term mean of v is one. The income dynamics of the investors are calibrated to match the observed Federal Funds rates using equation (2.13). Assume that all investors are identical with the following preference parameters often used in the literature

$$\tau_i = \frac{1}{2}, \quad \delta_i = 0.01.$$

Bloom (2009, Figure 3) shows how expected growth is impacted by a shock to volatility and $\kappa_i = -0.005$ is set to match this fact. The income parameters are calibrated to match the interest rate dynamics over the period 1985-2015 to avoid periods with extreme inflation.

$$\mu_i = 0.04425, \qquad V_i = \sqrt{\sigma_i^2 + \beta_i^2} = 0.1589,$$

where V_i is the steady-state income volatility of the investors when income is scaled to 1. This is in line with Cocco, Gomes, and Maenhout (2005), who find that labor income volatility of the average household is on average equal to 10% and income in this paper includes financial income so it is bound to be higher. They also find practically no correlation between labor income and the stock market. As Y_i in this model includes both financial and labor income, assume a correlation coefficient between Y_i and the state variable v of $\rho_i = -0.3$ which implies from above that

$$\sigma_i = -\rho_i V_i = 0.048, \qquad \beta_i = 0.1516.$$

The levels of μ_i and κ_i ensure that the drift of the investor's income is positive for levels of v around 1 and decreasing in v as desired. With σ_v and σ_i having different sign, the income volatility of the investor is counter-cyclical. This means for positive changes in Y_i coming from W, the income volatility v decreases.

The real yield curve can then be plotted for different levels of v_t as

$$y(t,s) = -\frac{1}{s-t}\ln(B(t,s)), \qquad s > t.$$

This can be seen in Figure A.2. The different levels of $v \in \{0.5, 0.6, ..., 1.5\}$ correspond to current levels of the risk-free interest rate of $6.8\%, 6.2\%, ..., 0.8\%$ as calculated with relation (2.13). Note that the yield curve converges to the short rate for time to maturity going towards zero. In times of high uncertainty, the risk-free rate is low which can be explained by the investors' desire for pre-cautionary savings increasing demand for the money market account and, hence, driving the interest rate down.

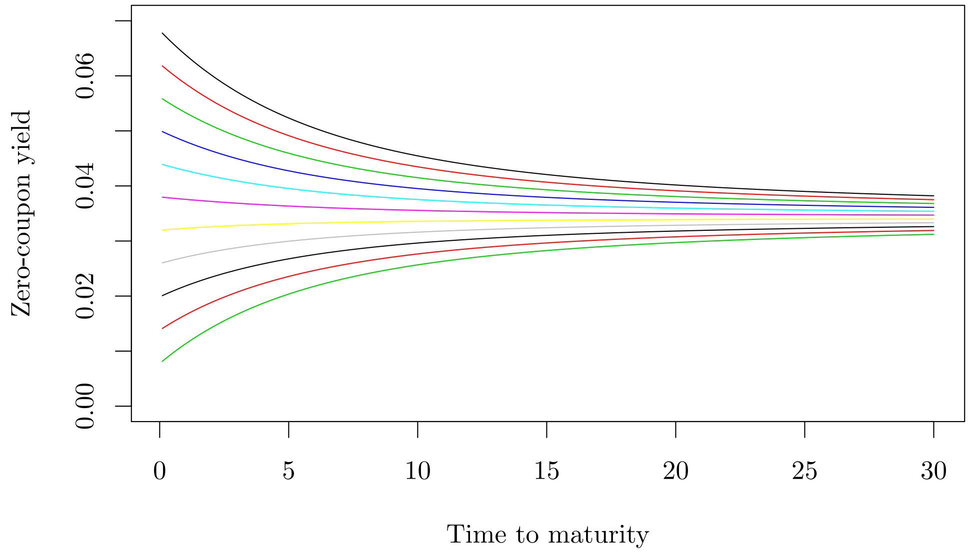

Figure A.2: Yield curves for levels of $v_t \in \{0.5, 0.6, ..., 1.5\}$ from top curve to bottom.

Because of the mean-reverting behavior of income volatility, the yield curves converge to a level just around the long-term level of r when $v = 1$. As $v > 0$ and r is decreasing in v there is an upper level to the risk-free interest rate, in the calibration 9.85%. There is, however, no lower limit to the interest rate, so unlike the CIR-model, this model allows for negative interest rates (as even certain nominal government bond markets have experienced in recent years).

A.2 Solutions of ODE's

Consider the following ODE of the function b

$$b'(x) = d_1 - d_2 b(x) + d_3 b(x)^2, \quad b(0) = 0. \tag{ODE Type I}$$

If $d_2^2 - 4d_1d_3 > 0$ then the solution of (ODE Type I) is

$$b(x) = \frac{2d_1(e^{\omega x} - 1)}{(\omega + d_2)(e^{\omega x} - 1) + 2\omega},$$

where $\omega = \sqrt{d_2^2 - 4d_1 d_3}$. Consider now the following ODE of the function a

$$a'(x) = d_4 + d_5 b(x), \qquad a(0) = 0, \tag{ODE Type II}$$

where the function b solves (ODE Type I). The solution of (ODE Type II) can be found as

$$\begin{aligned} a(x) &= a(0) + \int_0^x a'(s)\, ds \\ &= d_4 x + d_5 \int_0^x b(s)\, ds. \end{aligned}$$

The integral over the function b is

$$\int_0^x b(s)\, ds = \frac{1}{d_3} \left\{ \frac{1}{2} (\omega + d_2) x + \log \left(\frac{2\omega}{(\omega + d_2)(e^{\omega x} - 1) + 2\omega} \right) \right\}.$$

So the solution of (ODE Type II) is

$$a(x) = d_4 x + \frac{d_5}{d_3} \left\{ \frac{1}{2} (\omega + d_2) x + \log \left(\frac{2\omega}{(\omega + d_2)(e^{\omega x} - 1) + 2\omega} \right) \right\}.$$

A.3 Proof of Theorem 3.1

The proof of Theorem 3.1 relies on solving a partial differential equation of J with terminal conditions. These terminal conditions arise from the fact that homeowners receive utility from consumption only

$$J\left(\bar{Y}_h, v, T\right) = 0 \tag{A.1}$$

$$\frac{\partial J\left(\bar{Y}_h, v, T\right)}{\partial t} = -\bar{U}_h\left(\bar{Y}_{ht} - I_{hT}\right). \tag{A.2}$$

Often the last condition is not stated as well, but for my solution it is needed. It states that the indirect utility just before time T decreases by the instantaneous utility of terminal consumption. By considering (3.3) in discrete time with time points $t = t_0 < t_1 < ... <$

$t_N = T$

$$
\begin{aligned}
J\left(\bar{Y}_h, v, t\right) &= \mathbb{E}_t\left[\sum_{n=0}^{N-1} -e^{-\bar{\delta}_h n\Delta t - \frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht_n} - I_{ht_n}\right)}\Delta t\right] \\
&= \mathbb{E}_t\left[-e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - I_{ht}\right)}\Delta t + e^{-\bar{\delta}_h\Delta t}\mathbb{E}_{t+\Delta t}\left[\sum_{n=1}^{N-1} -e^{-\bar{\delta}_h(n-1)\Delta t - \frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht_n} - I_{ht_n}\right)}\Delta t\right]\right] \\
&= -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - I_{ht}\right)}\Delta t + e^{-\bar{\delta}_h\Delta t}\mathbb{E}_t\left[J\left(\bar{Y}_h, v, t+\Delta t\right)\right].
\end{aligned}
$$

Rewriting the above, we get the following equation

$$
\frac{e^{\bar{\delta}_h\Delta t} - 1}{\Delta t} J\left(\bar{Y}_h, v, t\right) = -e^{\bar{\delta}_h\Delta t - \frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - I_{ht}\right)} + \frac{\mathbb{E}_t\left[J\left(\bar{Y}_h, v, t+\Delta t\right)\right] - \mathbb{E}_t\left[J\left(\bar{Y}_h, v, t\right)\right]}{\Delta t}.
$$

By letting $\Delta t \to 0$, the fraction on the right-hand side approaches the drift of the process $J_{ht} = J\left(\bar{Y}_h, v, t\right)$, and $\frac{e^{\bar{\delta}_h\Delta t} - 1}{\Delta t} \to \bar{\delta}_h$ by l'Hôpitals. This means that J must solve the following partial differential equation

$$
\bar{\delta}_h J\left(\bar{Y}_h, v, t\right) = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - I_{ht}\right)} + (\text{drift of } J) \tag{A.3}
$$

with terminal conditions in (A.1) and (A.2).

Before finding the function J with the homeowner's income process, we need to know how an educated guess of the function J looks like. Therefore, assume that the homeowner income dynamics follow a generalized Brownian motion

$$
d\bar{Y}_{ht} = \bar{\mu}_h dt + \bar{\sigma}_h dW_t,
$$

where $\bar{\mu}_h$ and $\bar{\sigma}_h$ are constants. With these dynamics we can calculate the life-time expected utility of a fixed-rate mortgage. Since the process v does not affect the utility of the household in this case, the drift of J is easily determined as

$$
\frac{\partial J}{\partial t} + J_Y\bar{\mu}_h + \frac{1}{2}J_{YY}\bar{\sigma}_h^2.
$$

So in this case equation $(A.3)$ is

$$e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}-r^f F_h\right)} = \frac{\partial J}{\partial t} + J_Y \bar{\mu}_h + \frac{1}{2} J_{YY} \bar{\sigma}_h^2 - \bar{\delta}_h J \tag{A.4}$$

with terminal condition $J\left(\bar{Y}_h, T\right) = 0$. In this case, however, we can find the function J by calculating the integral explicitly

$$\begin{aligned} J\left(\bar{Y}_{ht}, t\right) &= \mathbb{E}_t\left[\int_t^T -e^{-\bar{\delta}_h(s-t)-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{hs}-r^f F_h\right)} ds\right] \\ &= -e^{\frac{1}{\bar{\tau}_h} r^f F_h} \int_t^T \mathbb{E}_t\left[e^{-\bar{\delta}_h(s-t)-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}+\bar{\mu}_h(s-t)+\bar{\sigma}_h(W_s-W_t)\right)}\right] ds \\ &= -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}-r^f F_h\right)} \int_t^T \mathbb{E}_t\left[e^{-\bar{\delta}_h(s-t)-\frac{1}{\bar{\tau}_h}\left(\bar{\mu}_h(s-t)+\bar{\sigma}_h(W_s-W_t)\right)}\right] ds. \end{aligned}$$

As $W_s - W_t$ is normally distributed then

$$\begin{aligned} J\left(\bar{Y}_{ht}, t\right) &= -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}-r^f F_h\right)} \int_t^T e^{-\bar{\delta}_h(s-t)-\frac{1}{\bar{\tau}_h}\bar{\mu}_h(s-t)+\frac{1}{2\bar{\tau}_h^2}\bar{\sigma}_h^2(s-t)} ds \\ &= -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}-r^f F_h\right)} \frac{1}{-\bar{\delta}_h - \frac{1}{\bar{\tau}_h}\bar{\mu}_h + \frac{1}{2\bar{\tau}_h^2}\bar{\sigma}_h^2}\left[e^{\left(-\bar{\delta}_h-\frac{1}{\bar{\tau}_h}\bar{\mu}_h+\frac{1}{2\bar{\tau}_h^2}\bar{\sigma}_h^2\right)(T-t)} - 1\right]. \end{aligned}$$

By calculating the relevant derivatives of this function and plugging it into equation $(A.4)$, we see that it indeed solves the equation.

Now, consider the income process of the homeowner in (2.2)

$$d\bar{Y}_{ht} = \left(\bar{\mu}_h + \bar{\kappa}_h v_t\right) dt + \sqrt{v_t}\left(\bar{\sigma}_h dW_t + \bar{\beta}_h dZ_{ht}\right).$$

The drift of J is now

$$\frac{\partial J}{\partial t} + J_Y\left(\bar{\mu}_h + \bar{\kappa}_h v_t\right) + \frac{1}{2} J_{YY}\left(\bar{\sigma}_h^2 + \bar{\beta}_h^2\right) v_t + J_v\left(\mu_v + \kappa_v v_t\right) + \frac{1}{2} J_{vv} \sigma_v^2 v_t + J_{Yv} \sigma_v \bar{\sigma}_h v_t,$$

so equation $(A.3)$ becomes

$$e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}-I_{ht}\right)} = \frac{\partial J}{\partial t} + J_Y\left(\bar{\mu}_h + \bar{\kappa}_h v_t\right) + \frac{1}{2}J_{YY}\left(\bar{\sigma}_h^2 + \bar{\beta}_h^2\right)v_t + J_v\left(\mu_v + \kappa_v v_t\right) + \frac{1}{2}J_{vv}\sigma_v^2 v_t + J_{Yv}\sigma_v\bar{\sigma}_h v_t - \bar{\delta}_h J \tag{A.5}$$

with terminal conditions in (A.1) and (A.2). An educated guess of J comes from the previous analysis

$$J\left(\bar{Y}_{ht}, v_t, t\right) = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}-I_{ht}\right)}\int_t^T e^{-\bar{\delta}_h(s-t)+a_j(s-t)+b_j(s-t)v_t}ds,$$

where the deterministic functions a_j and b_j, $(j \in \{F, A\})$, must solve some ordinary differential equations. The terminal condition of J in (A.2) translates to initial conditions for the functions $a_j(0) = b_j(0) = 0$.

For an FRM, meaning $I_{ht} = r^f F_h$, consider the short-hand notation

$$J^{FRM}\left(\bar{Y}_{ht}, v_t, t\right) = -C\int_t^T D ds,$$

where $C = e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht}-r^f F_h\right)}$ and $D = e^{-\bar{\delta}_h(s-t)+a_F(s-t)+b_F(s-t)v_t}$. The relevant derivatives are

$$\begin{aligned}
\frac{\partial J}{\partial t} &= -C\left(\int_t^T\left(\bar{\delta}_h - a_F'(s-t) - b_F'(s-t)v_t\right)Dds - e^{a_F(0)+b_F(0)v_t}\right)\\
&= C\int_t^T\left(-\bar{\delta}_h + a_F'(s-t) + b_F'(s-t)v_t\right)Dds + C\\
J_Y &= \frac{1}{\bar{\tau}_h}C\int_t^T Dds, \qquad J_{YY} = -\frac{1}{\bar{\tau}_h^2}C\int_t^T Dds\\
J_v &= -C\int_t^T b_F(s-t)Dds, \qquad J_{vv} = -C\int_t^T b_F(s-t)^2 Dds\\
J_{Yv} &= \frac{1}{\bar{\tau}_h}C\int_t^T b_F(s-t)Dds.
\end{aligned}$$

By plugging these derivatives into equation $(A.5)$, we get the following equation

$$
\begin{aligned}
C = {} & C \int_t^T \left(-\bar{\delta}_h + a_F'(s-t) + b_F'(s-t)v_t\right) D ds + C + (\bar{\mu}_h + \bar{\kappa}_h v_t) \frac{1}{\bar{\tau}_h} C \int_t^T D ds \\
& - \frac{1}{2}\left(\bar{\sigma}_h^2 + \bar{\beta}_h^2\right) v_t \frac{1}{\bar{\tau}_h^2} C \int_t^T D ds - (\mu_v + \kappa_v v_t) C \int_t^T b_F(s-t) D ds \\
& - \frac{1}{2}\sigma_v^2 v_t C \int_t^T b_F(s-t)^2 D ds + \sigma_v \bar{\sigma}_h v_t \frac{1}{\bar{\tau}_h} C \int_t^T b_F(s-t) D ds + \bar{\delta}_h C \int_t^T D ds,
\end{aligned}
$$

that is,

$$
\begin{aligned}
0 = {} & \int_t^T D \left[a_F'(s-t) + b_F'(s-t)v_t + (\bar{\mu}_h + \bar{\kappa}_h v_t)\frac{1}{\bar{\tau}_h} - \frac{1}{2}\left(\bar{\sigma}_h^2 + \bar{\beta}_h^2\right) v_t \frac{1}{\bar{\tau}_h^2} \right. \\
& \left. - (\mu_v + \kappa_v v_t) b_F(s-t) - \frac{1}{2}\sigma_v^2 v_t b_F(s-t)^2 + \sigma_v \bar{\sigma}_h v_t \frac{1}{\bar{\tau}_h} b_F(s-t) \right] ds \\
= {} & \int_t^T D \left(a_F'(s-t) + \frac{\bar{\mu}_h}{\bar{\tau}_h} - \mu_v b_F(s-t) + \left[b_F'(s-t) + \frac{\bar{\kappa}_h}{\bar{\tau}_h} - \frac{1}{2}\frac{\left(\bar{\sigma}_h^2 + \bar{\beta}_h^2\right)}{\bar{\tau}_h^2} \right.\right. \\
& \left.\left. - \kappa_v b_F(s-t) - \frac{1}{2}\sigma_v^2 b_F(s-t)^2 + \sigma_v \bar{\sigma}_h \frac{1}{\bar{\tau}_h} b_F(s-t) \right] v_t \right) ds.
\end{aligned}
$$

As this must hold for all t and all levels of v_t, the integrand must equal zero and we get the following ODE's for the functions a_F and b_F

$$
b_F'(s-t) = \frac{1}{2}\frac{\left(\bar{\sigma}_h^2 + \bar{\beta}_h^2\right)}{\bar{\tau}_h^2} - \frac{\bar{\kappa}_h}{\tau_h} - \left(\sigma_v \bar{\sigma}_h \frac{1}{\bar{\tau}_h} - \kappa_v\right) b_F(s-t) + \frac{1}{2}\sigma_v^2 b_F(s-t)^2 \tag{A.6}
$$

$$
a_F'(s-t) = -\frac{\bar{\mu}_h}{\bar{\tau}_h} + \mu_v b_F(s-t). \tag{A.7}
$$

These types of ODE's are identical to the ODE's in Lemma 2.1 with homeowner specific parameters instead of the aggregate investors', so the solutions are equivalently by Appendix A.2

$$
b_F(s-t) = \frac{2\frac{1}{2}\sigma_v^2\left(e^{\omega_F(s-t)} - 1\right)}{\left(\omega_F + \sigma_v \bar{\sigma}_h \frac{1}{\bar{\tau}_h} - \kappa_v\right)\left(e^{\omega_F(s-t)} - 1\right) + 2\omega_F},
$$

where

$$
\omega_F = \sqrt{\left(\sigma_v \bar{\sigma}_h \frac{1}{\bar{\tau}_h} - \kappa_v\right)^2 - 4\frac{1}{2}\sigma_v^2\left(\frac{1}{2}\frac{\left(\bar{\sigma}_h^2 + \bar{\beta}_h^2\right)}{\bar{\tau}_h^2} - \frac{\bar{\kappa}_h}{\bar{\tau}_h}\right)},
$$

and the function a_F is

$$a_F(s-t) = -\frac{\bar{\mu}_h}{\bar{\tau}_h}(s-t) + \mu_v \frac{1}{\frac{1}{2}\frac{\left(\bar{\sigma}_h^2+\bar{\beta}_h^2\right)}{\bar{\tau}_h^2} - \frac{\bar{\kappa}_h}{\bar{\tau}_h}} \left[\frac{1}{2}\left(\omega_F + \sigma_v \bar{\sigma}_h \frac{1}{\bar{\tau}_h} - \kappa_v\right)(s-t) + \log\left(\frac{2\omega_F}{\left(\omega_F + \sigma_v \bar{\sigma}_h \frac{1}{\bar{\tau}_h} - \kappa_v\right)\left(e^{\omega_F(s-t)} - 1\right) + 2\omega_F}\right)\right].$$

Similarly can be done for an adjustable-rate mortgage. Now, the interest payment is not constant, but a function of v as determined in equilibrium

$$r_t = \frac{\sum_{i=1}^I \tau_i \delta_i}{\tau_\Sigma} + \frac{1}{\tau_\Sigma}\mu_\mathcal{E} + \frac{1}{\tau_\Sigma}\left(\kappa_\mathcal{E} - \frac{1}{2}\frac{\sigma_\mathcal{E}^2}{\tau_\Sigma} - \frac{1}{2}\sum_{i=1}^I \frac{\beta_i^2}{\tau_i}\right) v_t \tag{A.8}$$

$$= R_0 - R_1 v_t \tag{A.9}$$

by defining the constant R_1 for notational purposes. An educated guess of J for an adjustable-rate mortgage is

$$J^{ARM}\left(\bar{Y}_{ht}, v_t, t\right) = -e^{-\frac{1}{\bar{\tau}_h}\left(\bar{Y}_{ht} - R_0 F_h + R_1 F_h v_t\right)} \int_t^T e^{-\bar{\delta}_h(s-t) + a_A(s-t) + b_A(s-t)v_t} ds$$

and in short-hand notation

$$J^{ARM}\left(\bar{Y}_{ht}, v_t, t\right) = -C\int_t^T D ds.$$

The relevant derivatives are found

$$\frac{\partial J}{\partial t} = C\int_t^T \left(-\bar{\delta}_h + a_A'(s-t) + b_A'(s-t)v_t\right) D ds + C$$

$$J_Y = \frac{1}{\bar{\tau}_h} C \int_t^T D ds, \qquad J_{YY} = -\frac{1}{\bar{\tau}_h^2} C \int_t^T D ds$$

$$J_v = \frac{R_1 F_h}{\bar{\tau}_h} C \int_t^T D ds - C\int_t^T b_A(s-t) D ds$$

$$J_{vv} = -\left(\frac{R_1 F_h}{\bar{\tau}_h}\right)^2 C\int_t^T D ds + 2\frac{R_1 F_h}{\bar{\tau}_h} C \int_t^T b_A(s-t) D ds - C\int_t^T b_A(s-t)^2 ds$$

$$J_{Yv} = \frac{1}{\bar{\tau}_h} C\left(-\frac{R_1 F_h}{\bar{\tau}_h}\int_t^T D ds + \int_t^T b_A(s-t) D ds\right).$$

Plugging these into equation $(A.5)$ and by the same arguments as for the FRM-case we get two ODE's for the functions b_A and a_A

$$b'_A(s-t) = \frac{1}{2}\frac{\left(\bar{\sigma}_h^2+\bar{\beta}_h^2\right)}{\bar{\tau}_h^2} - \frac{\bar{\kappa}_h}{\tau_h} - \frac{R_1F_h}{\bar{\tau}_h}\kappa_v + \frac{1}{2}\left(\frac{R_1F_h}{\bar{\tau}_h}\right)^2\sigma_v^2 + \frac{R_1F_h}{\tau_h^2}\sigma_v\sigma_y \tag{A.10}$$

$$-\left(\sigma_v\bar{\sigma}_h\frac{1}{\bar{\tau}_h} - \kappa_v + \frac{R_1F_h}{\bar{\tau}_h}\sigma_v^2\right)b_A(s-t) + \frac{1}{2}\sigma_v^2 b_A(s-t)^2 \tag{A.11}$$

$$a'_A(s-t) = -\frac{\bar{\mu}_h}{\bar{\tau}_h} - \frac{R_1F_h}{\bar{\tau}_h}\mu_v + \mu_v b_A(s-t) \tag{A.12}$$

with initial conditions $a_A(0) = b_A(0) = 0$. These are solved similarly as for FRMs using the results in Appendix A.2

$$b_A(s-t) = \frac{2\frac{1}{2}\sigma_v^2\left(e^{\omega_A(s-t)}-1\right)}{\left(\omega_A+\sigma_v\bar{\sigma}_h\frac{1}{\bar{\tau}_h}-\kappa_v+\frac{R_1F_h}{\bar{\tau}_h}\sigma_v^2\right)\left(e^{\omega_A(s-t)}-1\right)+2\omega_A},$$

where

$$\omega_A = \Bigg[\left(\sigma_v\bar{\sigma}_h\frac{1}{\bar{\tau}_h} - \kappa_v + \frac{R_1F_h}{\bar{\tau}_h}\sigma_v^2\right)$$
$$-4\frac{1}{2}\sigma_v^2\left(\frac{1}{2}\frac{\left(\bar{\sigma}_h^2+\bar{\beta}_h^2\right)}{\bar{\tau}_h^2} - \frac{\bar{\kappa}_h}{\bar{\tau}_h} - \frac{R_1F_h}{\bar{\tau}_h}\kappa_v + \frac{1}{2}\left(\frac{R_1F_h}{\bar{\tau}_h}\right)^2\sigma_v^2 + \frac{R_1F_h}{\bar{\tau}_h^2}\sigma_v\sigma_y\right)\Bigg]^{\frac{1}{2}},$$

and

$$a_A(s-t) = \left(-\frac{\bar{\mu}_h}{\bar{\tau}_h} - \frac{R_1F_h}{\bar{\tau}_h}\mu_v\right)(s-t)$$
$$+\mu_v\frac{1}{\frac{1}{2}\frac{\left(\bar{\sigma}_h^2+\bar{\beta}_h^2\right)}{\bar{\tau}_h^2} - \frac{\bar{\kappa}_h}{\bar{\tau}_h} - \frac{R_1F_h}{\bar{\tau}_h}\kappa_v + \frac{1}{2}\left(\frac{R_1F_h}{\bar{\tau}_h}\right)^2\sigma_v^2 + \frac{R_1F_h}{\bar{\tau}_h^2}\sigma_v\sigma_y}$$
$$\cdot\Bigg[\frac{1}{2}\left(\omega_A+\sigma_v\bar{\sigma}_h\frac{1}{\bar{\tau}_h}-\kappa_v+\frac{R_1F_h}{\bar{\tau}_h}\sigma_v^2\right)(s-t)$$
$$+\log\left(\frac{2\omega_A}{\left(\omega_A+\sigma_v\bar{\sigma}_h\frac{1}{\bar{\tau}_h}-\kappa_v+\frac{R_1F_h}{\bar{\tau}_h}\sigma_v^2\right)\left(e^{\omega_A(s-t)}-1\right)+2\omega_A}\right)\Bigg].$$

A.4 Proof of Lemma 3.1

When we clear the consumption market, we pin down the risk-free rate and market price of risk. The dynamics of optimal consumption strategies of the investors still hold as functions of r_t, λ_t, v_t and, therefore, the following must hold

$$\begin{aligned}
\mathbf{F}^a r_t =& \sum_{i=1}^{I} c_{it} \\
\Rightarrow \mathbf{F}^a dr_t =& \sum_{i=1}^{I} dc_{it} \\
=& \tau_\Sigma \left(r_t + \frac{1}{2}\lambda_t^2 - \left(R_0(0) - R_1(0)v_t + \frac{1}{2}L(0)v_t \right) \right) dt + \tau_\Sigma \left(\lambda_t - L(0)\sqrt{v_t} \right) dW_t.
\end{aligned}$$

First, conjecture that the risk-free rate is of the same form as in the basic model, namely,

$$r_t = R_0(\mathbf{F}^a) - R_1(\mathbf{F}^a)v_t$$

such that

$$\mathbf{F}^a dr_t = -R_1(\mathbf{F}^a)\mathbf{F}^a dv_t = -R_1(\mathbf{F}^a)\mathbf{F}^a \left(\mu_v + \kappa_v v_t\right) dt - R_1(\mathbf{F}^a)\mathbf{F}^a \sigma_v \sqrt{v_t} dW_t.$$

By matching the dW_t-terms, the market price of risk must equal

$$\lambda_t \equiv L(\mathbf{F}^a)\sqrt{(v_t)} = \left(L(0) - \frac{\sigma_v}{\tau_\Sigma}\mathbf{F}^a R_1(\mathbf{F}^a) \right) \sqrt{(v_t)},$$

and from dt-term, we have two equations such that the equation holds for all v_t

$$\begin{aligned}
R_0(\mathbf{F}^a) =& R_0(\mathbf{F}^a) - \mathbf{F}^a R_1(\mathbf{F}^a)\frac{\mu_v}{\tau_\Sigma} \\
-\mathbf{F}^a R_1(\mathbf{F}^a)\frac{\kappa_v}{\tau_\Sigma} =& -R_1(\mathbf{F}^a) + \frac{1}{2}L(\mathbf{F}^a)^2 + R_1(0) - \frac{1}{2}L(0)^2 \\
\Leftrightarrow R_1(\mathbf{F}^a) - R_1(0) =& R_1(\mathbf{F}^a)\frac{\mathbf{F}^a}{\tau_\Sigma}(\kappa_v - L(0)\sigma_v) + \frac{1}{2}\left(\frac{\sigma_v}{\tau_\Sigma}\right)^2 (\mathbf{F}^a)^2 R_1(\mathbf{F}^a)^2
\end{aligned}$$