Ethical Voting in Multicandidate Elections*

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Abstract

We study the behavior of ethical voters in multicandidate elections. We consider two common electoral rules: plurality and majority runoff. Our model delivers crisper predictions than those of the pivotal voter model. An equilibrium always exists, and is unique for a broad range of parameter values. There are two types of equilibria: (i) the sincere voting equilibrium (voters vote for their most-preferred candidate), and (ii) Duverger's Law equilibria (two candidates attract all the votes). These never coexist. We identify the features of an election that favor sincere voting. Consistent with evidence, incentives to vote sincerely are stronger under majority runoff.

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1 Introduction

Voters ought to be a crucial piece of most political economy models. Yet, more often than not, they take the backstage: they are modeled as irrational and naive agents, who support the alternative they like without pondering the consequences of their actions. They are *sincere*. While clearly contradicted by the facts[1], this curious approach to voters' behavior is not clearly dominated by the fully rational workhorse model of strategic voting, i.e. the pivotal voter model. This model is also afflicted by serious weaknesses: the most lethal being the absurdly low turnout levels it predicts–highlighting the so-called *paradox of voting*.[2] Political economy as a field is impeded by this lack of consensus about how to approach and model voters' behavior.

Obviously, the challenge of finding a better approach to voters' behavior has not been left unanswered: various alternatives have been proposed, including the so-called *ethical voter* model (Harsanyi 1977, 1992, Feddersen and Sandroni 2006a, 2006b, and Coate and Conlin 2004).[3] It predicts high turnout level (no paradox of voting), and produces comparative statics that are empirically sound (Feddersen and Sandroni 2006a) in two-candidate elections. Moreover, direct tests of the model (both on observational and experimental data) are quite conclusive (Coate and Conlin 2004, Feddersen, Gailmard, and Sandroni 2009, Morton and Tyran 2012).

While promising, all these results concern elections with two alternatives. But, most real-life elections involve more than two candidates or parties (Jones 2001). Thus, to have

[1]There are various pieces of evidence that go against the sincere voting assumptions. First, the literature comparing each voter's actual vote to her preferences finds a fraction of voters casting *misaligned votes* (i.e. voting for another candidate than their most-preferred) ranging from 3% to 17% (see, e.g., Alvarez and Nagler 2000, Blais, Nadeau, Gidengil, and Nevitte 2001 and papers cited therein). Second, recent studies (Kawai and Watanabe 2013, Spenkuch 2016) find that a larger fraction of voters are strategic (estimates range from 40% to 85%), but they do not necessarily cast a misaligned vote. Third, the experimental literature finds evidence of strategic voting both in two-candidate elections (for the costly voting side, see e.g., Palfrey and Levine 2007; for the Condorcet jury theorem side, see e.g., Guarnaschelli et al. 2000, Battaglini et al. 2008, 2010, Goeree and Yariv 2011, Bhattachary et al. 2014, Bouton et al. 2017a) and in multicandidate elections (see e.g. Forsythe et al. 1993 and 1996, Fisher and Myatt 2001, Morton and Rietz 2008, Bouton et al. 2016, 2017b). Fourth, studies also find evidence of strategic voting through its implications on the number of "serious" candidates under different electoral systems (see, e.g., Fujiwara 2011). Last but not least, the empirical literature that studies voter turnout uncovers voting behavior that is not coherent with the sincere voting assumption (see, e.g., Coate and Conlin 2004, and Coate, Conlin and Moro 2008).

[2]Feddersen (2004) defines this paradox eloquently: "If each person only votes for the purpose of influencing the election outcome, then even a small cost to vote (...) should dissuade anyone from voting. Yet, it seems that many people will put up with long lines, daunting registration requirements and even the threat of physical violence or arrest in order to vote" (p. 99).

[3]For other group-based models, see, e.g., Shachar and Nalebuff (1999), and Levine and Mattozzi (2016).

a chance of establishing itself as the canonical model of voting, the ethical voter model must also perform well when applied to the study of multicandidate elections.

In this paper, we develop a model of voting in multicandidate elections with ethical voters. We study two of the most-widely used electoral rules around the world: the plurality rule[4] and the majority runoff rule.[5] Our results confirm the promises of the ethical voter model: the predictions are empirically sound and, due to equilibrium uniqueness for a broad set of parameter values, much crisper than those of the pivotal voter model.

In an ethical voter model (Feddersen and Sandroni 2006a, 2006b, and Coate and Conlin 2004), voters are assumed to be "rule-utilitarian": they understand that there is an endogenously determined (group-level) *rule* that they need to follow in order for the utility of the group to be maximized. They get a payoff if they act according to this rule, and that payoff is assumed larger than any cost of doing ones' part. This is the approach we embrace in the model presented here. To ease comparison with the previous literature on strategic voting in multicandidate elections, we focus on voters' decisions for whom to vote, not whether to vote.

The main novelty is that we consider a setup with more than two candidates.[6] In our baseline case, we consider the so-called *divided majority* setting.[7] There are three candidates (A, B, and C) and three types of voters (a, b, and c). A majority of voters prefer both A and B (the majority candidates) over C (the minority candidate). The majority is divided: a-voters prefer candidate A over candidate B, but b-voters prefer

[4] In plurality elections (a.k.a. first-past-the-post), voters can vote for one of the candidates in the running, and the candidate with the largest number of votes wins. This system is used in many countries, e.g. the U.K. and the U.S., to elect members of the lower house of their legislature.

[5] In a majority runoff election, there is up to two rounds of voting. In each round, voters can vote for one of the participating candidates. In the first round, all candidates participate, and a candidate wins outright if she obtains an absolute majority of the votes. If no candidate wins in the first round, then a second round is held between the top-two vote getters. The winner of that round wins the election. This is the most-widely used system to elect presidents around the world (Bromann and Golder 2013).

[6] In an independently developed paper, Li and Pique (2016) explore ethical motives for strategic voting in multicandidate elections. Their approach is fundamentally different: they work under the assumption that it is costly for voters to cast a misaligned vote but that ethical motives may nonetheless lead them to do so. One crucial difference between their results and ours is about the effect of the strength of an ethical voter's least preferred candidate on her propensity to cast a misaligned vote. Their model predicts that this effect is non-monotonic whereas ours predicts it is monotonic. As we discuss below, empirical evidence seems to support our finding. Another important difference is that they focus on plurality elections, while we consider both plurality and runoff elections.

[7] The divided majority setting is tractable but still captures some of the fundamental coordination problems that voters face in multicandidate elections. As a consequence, this setting is ubiquitous in the literature on strategic voting in multicandidate elections. See, e.g., Palfrey (1989), Myerson and Weber (1993), Cox (1997), Fey (1997), Piketty (2000), Myerson (2002), Dewan and Myatt (2007), Myatt (2007), Bouton and Castanheira (2012), and Bouton (2013).

candidate B over candidate A. The support for candidate C is sufficiently large so that she is a serious threat. In an extension, we show that our results are robust, at least qualitatively, to more sophisticated structures of preferences and to the presence of more than three candidates.

Our analysis focuses first on (pure strategy) equilibria under the plurality rule. There are two types of equilibria: (i) the sincere voting equilibrium (in which voters vote for their most-preferred candidate), and (ii) Duverger's Law equilibria (in which all majority voters vote for the same majority candidate, either A or B). We prove that an equilibrium always exists, and that it is unique for a broad range of parameter values. Moreover, equilibrium multiplicity only happens when the two Duverger's law equilibria exist. In other words, the sincere voting equilibrium never coexists with Duverger's law equilibria. Thus, our model uniquely predicts the number of candidates receiving a positive share of the votes.

Our equilibrium characterization allows us to identify the features of an election that favor sincere voting–that is, when Duverger's law should fail.[8] Quite intuitively, the incentives to vote sincerely are stronger when (i) the utility differential between the two majority candidates is large, (ii) the utility differential between the less preferred majority candidate and the minority candidate is small, (iii) the minority group is small, and (iv) the majority is evenly divided. The importance of cardinal utilities for equilibrium behavior is a distinguishing feature of our ethical voter model. These results find support in the empirical literature (see, e.g., Blais and Nadeau 1996, and Bouton, Castanheira, and Llorente-Saguer 2016).

We also characterize the set of equilibria under the majority runoff rule. In the first round, voters incentives are qualitatively similar as under plurality, but quantitatively different. Comparing the two systems, we find that the incentives to vote sincerely are stronger under majority runoff.[9] In particular, if the sincere voting equilibrium exists under plurality, it also exists (and is unique) under majority runoff. Conversely, if a Duverger's law equilibrium exists under majority runoff, then it also exists under plurality.

[8]Duverger's law says that "the simple-majority single-ballot system [the plurality electoral system] favors the two-party systems" (Duverger 1954).

[9]We also analyze French-style runoff elections, in which candidates obtaining a number of first-round votes larger than a threshold number of votes are allowed to participate in the second round. Perhaps surprisingly, in such runoff elections, the incentives to vote sincerely are stronger than under plurality but weaker than under majority runoff.

This result is consistent with the empirical findings that Duverger's law forces are sometimes (see Fujiwara 2011, on Brazilian data), but not always (see Bordignon et al. 2016, on Italian data) stronger under plurality than under runoff. Moreover, we identify the characteristics of the election for which the difference in the strengths of the Duverger's law forces should be noticeable: that is, when Duverger's law and hypothesis should both hold.[10] This presents a need to revisit the data having in mind the characteristics of the race for which the incentives to vote sincerely are different in plurality and runoff elections.[11]

There are several dimensions along which the ethical voter model fares better than the pivotal voter model. First, the ethical voter model produces *crisper* predictions: the equilibrium is unique for a broad set of parameter values and, even when there are multiple equilibria, there is no ambiguity about the number of "serious" candidates, i.e. those receiving a positive vote share. By contrast, the predictive power of the pivotal voter model is greatly weakened by issues of equilibrium multiplicity. In plurality elections, the pivotal voter model always produces multiple Duverger's law equilibria[12], as well as a non-Duverger's law equilibrium, in which more than two candidates are serious. Beyond the lack of predictive power, equilibrium multiplicity is a substantial hurdle to the inclusion of pivotal voters in political economy models that focus on the choice of candidates or parties.[13]

Second, the ethical voter model produces *sounder* predictions. First, while the coexistence of multiple Duverger's law equilibria is not undesirable per se (it can for instance capture the risk of coordination failure that exists in multicandidate elections (see, e.g., Myerson and Weber 1993, Fey 1997, and Bouton and Castanheira 2012)), it is clearly excessive in the case of the pivotal voter model. That model indeed predicts that essen-

[10]Duverger's hypothesis says that "simple majority with a second ballot [the runoff electoral system] favors multipartyism" (Duverger 1954).

[11]Such an analysis should take into account the findings of models with sincere voters and strategic candidates that also identify situations in which Duverger's law and hypothesis both hold (Osborne and Slivinski 1996 and Callander 2005).

[12]This is so because the pivotal voting logic gives full power to self-fulfilling prophecies: in order not to waste their votes, voters have an incentive to abandon their most-preferred candidate if they expect her not to have a serious shot at winning.

[13]In a companion paper (Bouton and Ogden 2017), we exploit the existence of a unique equilibrium for a broad set of parameter values in our model to analyze the behavior of candidates in multicandidate elections with strategic voters. In particular, we revisit classic models of candidates behavior (entry and positioning along the real line) in a model with ethical voters. We find that the presence of strategic voters affects dramatically the behavior of candidates.

tially any candidate can be a serious contender for victory in equilibrium, even one that is ranked behind another given candidate by **all** voters. By contrast, the ethical voter model produces multiple Duverger's law equilibria only in elections in which majority coordination is absolutely necessary (i.e. the minority candidate is a serious threat), and there is no majority candidate clearly stronger than the other (i.e. the majority is evenly divided). Second, in the pivotal voter model, the non-Duverger's law equilibrium is such that some supporters of the stronger majority candidate give their support to the weaker majority candidate. The fraction of them doing so is such that both majority candidates are (almost) equally likely to defeat the minority candidate. Moreover, supporters of the majority candidates split their votes even if it means that the probability that any of the majority candidates wins is essentially null.[14] By contrast, in the ethical voter model, the only possible non-Duverger's law equilibrium is such that voters vote for their most-preferred candidate. That equilibrium exists only when the minority candidate is not too much of a threat and both majority candidates are strong. Last but not least, even when the electorate is large, the intensity of voter preferences influences equilibrium behavior in the ethical voter model. By contrast, in the pivotal voter model, preference intensities do not shape equilibrium behavior in large elections. Yet, empirical evidence, and common sense, suggests that preference intensities play an important role in shaping voters' behavior in real-life elections (see, e.g., Blais and Nadeau 1996).

2 The Model

We model an election with 3 candidates, A, B, and C, and a continuum of voters.[15] The electoral rule is plurality (we consider majority runoff in section 4). There is an unique round of voting in which all voters are called to cast a vote for one of the candidates in the running. The action set is denoted by $\Psi = \{A, B, C\}$. The candidate with the largest number of votes wins.

Each agent has preferences over the set of candidates. The distribution of preferences

[14]Fey (1997) argues that this equilibrium is not reasonable because it is not expectationally stable.

[15]Our ethical voting model accommodates easily more than three candidates. When there are N>3 candidates, we can still identify situations where the sincere voting equilibrium exists (in this case, N candidates receive a positive vote share when voters vote sincerely). As in the baseline model, the sincere voting equilibrium never coexist with the Duverger's law equilibria (where two candidates receive a positive vote share).

in the population follows the so-called divided majority case. There are *majority agents* who prefer A and B over C and *minority agents* who prefer C over A and B (and, for the sake of simplicity, are indifferent between A and B).[16] The majority is divided into two groups: a-agents who prefer A over B, and b-agents who prefer B over A. There are thus three types of voters: $\Theta = \{a, b, c\}$.[17]

Formally, agents' preferences can be represented by the utility function $u : \Theta \times \Psi \to \mathbb{R}$. For majority agents, we assume:

$$\begin{aligned} u(A,a) &= W = u(B,b), \\ u(B,a) &= 0 = u(A,b), \text{ and} \\ u(C,a) &= -Y = u(C,b), \end{aligned}$$

with $W, Y > 0$. For minority agents, we assume:

$$\begin{aligned} u(C,c) &= Z > 0, \\ u(A,c) &= 0 = u(B,c). \end{aligned}$$

This directly implies that voting for C is a weakly dominant strategy for c-agents.

The size of the minority (i.e. the proportion of c-agents in the population) is fixed at $k \in \left[\frac{1}{3}, \frac{1}{2}\right]$.[18] Obviously, this means that the size of the majority (i.e. the proportion of a-agents and b-agents in the population) is $1-k$. The relative size of the two groups forming the majority is given by α: the share of a-agents in the population is thus $\alpha(1-k)$, and the share of b-agents is $(1-\alpha)(1-k)$. Without loss of generality, we assume that $\alpha \leq \frac{1}{2}$.

Agents of type $\theta \in \Theta$ turn out with an (exogenously given) probability q_θ, drawn from a Uniform distribution with support $[0, 1]$.[19] The distributions of these random variables are common knowledge but, at the time of the vote, their realizations are not.

[16] In section 5.1, we relax that assumption and prove that, in the real-line setting, voting for C is a dominant strategy for $c-$voters as soon as the group size is larger than $1/3$.

[17] We can easily extend the type space to accommodate any preference ordering over the set of candidates. As we show in Section 5.1 and Appendix A4, our results remain qualitatively similar.

[18] This assumption's purpose is to exclude uninteresting cases where candidate C is not a threat.

[19] One way to think about the exogenous turnout is as the product of a (simple) costly voting model in the vein of Feddersen and Sandroni 2006a: each voter faces a (fixed) cost $\bar{c}$ if they vote. A proportion q_θ are *ethical* agents who receive a benefit from doing one's duty to the group $D > \bar{c}$, while the remainder do not, and therefore abstain (since they have infinitesimal impact upon the election). Therefore, the exogenous turnout is the same as exogenous duty with fixed cost.

Following the ethical voter model (Feddersen and Sandroni 2006a, 2006b, and Coate and Conlin 2004), we assume that agents are "rule-utilitarian": they understand that there is a (group-level) *rule* (to be determined endogenously) that they need to follow in order for the utility of the group to be maximized, and they get a payoff if they act according to this rule (if they are "doing their part"). That payoff is assumed larger than any cost of doing ones part. As a consequence, and since the probability that one vote influences the outcome of the election is zero, voting decisions are as if made at the group level.[20]

Given the dominant strategy of c-agents, the game simplifies to a two-player game (one player choosing group-a action and one player choosing group-b action). Denoting by $\lambda_i \in [0,1]$ the probability that a member of group i votes for A[21], a Bayesian Nash Equilibrium is a tuple $\{\lambda_i^*\}_{i=a,b}$ such that

$$\lambda_a^* = \arg\max_{\lambda_a} p_A(\lambda_a, \lambda_b^*) W - (1 - p_A(\lambda_a, \lambda_b^*) - p_B(\lambda_a, \lambda_b^*))Y, \text{ and} \tag{1}$$

$$\lambda_b^* = \arg\max_{\lambda_b} p_B(\lambda_a^*, \lambda_b) W - (1 - p_A(\lambda_a^*, \lambda_b) - p_B(\lambda_a^*, \lambda_b))Y, \tag{2}$$

where $p_\psi(\lambda_a, \lambda_b)$ is the probability that candidate ψ wins when the strategy profile is (λ_a, λ_b). See Lemma 3 (in Appendix A1) for detail about these probabilities.

3 Equilibrium Analysis

In this section, we fully characterize the set of (pure strategy) equilibria, i.e. equilibria in which agents of a given group all votes for the same candidate with probability 1, under plurality.[22] Doing so, we prove that a (pure strategy) equilibrium always exists, and that,

[20] Alternatively, we could assume that a fraction γ of the electorate is rule-utilitarian and that the other voters are "sincere", in the sense that they always vote for their most preferred-candidate, no matter what they expected the other voters to do. Our results are robust to such an alternative specification.

[21] Our focus on equilibria in which all members of a given group adopt the same strategy is not totally innocuous. When members of a group are allowed to adopt different strategies, e.g. 40% of $a-$voters vote for A and 60% vote for B, then the existence of a pure strategy equilibrium is not guaranteed (but the existence of a mixed-strategy equilibrium is). However, we believe that the coordination required for asymmetric strategies within groups is quite demanding. Typically, in large groups, asymmetric strategies are implemented through mixed strategies (which is what our assumption allows for). Moreover, we can easily capture situations in which members of a group adopt different strategies, by splitting that particular group into two groups. These two groups are then free to adopt different strategies. As we show in section 5.2, our results are robust to an increase in the number of groups.

[22] In Appendix A2, we show that a mixed strategy equilibrium may exist. This equilibrium involves mixing by both group-a and group-b, in contrast to a pivotal voting model, which would imply that only

for a broad set of parameter values, there is an unique (pure strategy) equilibrium. In the remainder of the paper, we drop the qualifier "pure strategy" and refer to "pure strategy equilibria" as "equilibria".

There are two type of equilibria: (i) the sincere voting equilibrium, and (ii) Duverger's Law equilibria.[23]

Definition 1 *In the* **sincere voting equilibrium***, majority agents vote for their most preferred candidate (strategy profile* $(1, 0)$*).*

Definition 2 *In a* **Duverger's Law equilibrium***, all majority agents vote for the same candidate: either candidate* A *(strategy profile* $(1, 1)$*), or candidate* B *(strategy profile* $(0, 0)$*).*

3.1 Existence

Our proof of existence is constructive. First, we identify the necessary and sufficient condition under which the sincere voting equilibrium exists:

Proposition 1 *Under Plurality, the sincere voting equilibrium exists if and only if*

$$\frac{W}{Y} \geq \frac{p_A (1, 1) - p_A (1, 0) - p_B (1, 0)}{p_A (1, 0)}.$$

The condition in Proposition 1 guarantees that, when they expect the other group to vote sincerely, neither group prefers voting for their second choice candidate instead of voting for their first choice. This depends both on the utility function of groups a and b (W and Y) and the probabilities of winning of candidates A and B when agents are sincere or when they all vote for the same majority candidate ($p_A (1, 0)$, $p_B (1, 0)$, and $p_A (1, 1)$). Voters are more inclined to vote for their most-preferred candidate when they care a lot about electing their preferred majority candidate (large W) and when they do not dislike the minority candidate too intensely (small Y). Sincere voting is also more appealing when candidate A is a serious contender when voters are sincere (large $p_A (1, 0)$)

group-b (the larger majority group) would mix.

[23] It is easy to see that $(0, 1)$ cannot be an equilibrium: both group-a and group-b prefer to vote for their most-preferred candidate when the other majority group vote for that candidate.

and coordinating behind one majority candidate does not increase much the likelihood of defeating the minority candidate (small $p_A(1,1) - p_A(1,0) - p_B(1,0)$).

Using Lemma 3 (in Appendix A1), it is easy to rewrite the condition in Proposition 1 as a function of only primitives of the model: W, Y, α and k. We can then prove the following Lemma:

Lemma 1 $\frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_A(1,0)}$ *is increasing in* k *and decreasing in* α.

This Lemma shows that the sincere voting equilibrium is more likely when (i) the minority group is small (small k), then majority voters have weak incentives to coordinate behind one majority candidate to defeat the minority candidate; (ii) the majority is evenly divided (α large), then both types of majority voters realize that their champion has a significant chance to win, and they do not want to abandon that chance.

Moving to the second part of the proof of existence, we identify the necessary and sufficient condition under which a Duverger's Law equilibrium $(0,0)$ exists:

Proposition 2 *There exists a Duverger's Law equilibrium* $(0,0)$ *if and only if*

$$\frac{W}{Y} \leq \frac{p_A(1,1) - p_A(1,0) - p_B(1,0)}{p_A(1,0)}$$

The condition in Proposition 2 guarantees that group a prefers to vote for B (and thus that group b also wants to do so). This is the flip side of the condition in Proposition 1. This has two consequences. First, Lemma 1 applies and informs us that the Duverger's Law equilibrium $(0,0)$ is more likely when (i) the minority group is large (large k), then majority voters have strong incentives to coordinate behind one majority candidate since it is the only way to defeat the minority candidate; (ii) the majority is unevenly divided (α small), then a-voters realize that their champion is unlikely to win, and they thus prefer to increase the probability of defeating the minority candidate by coordinating behind candidate b. Second, the sincere voting equilibrium and the Duverger's Law equilibrium $(0,0)$ never coexist.

Existence of an equilibrium follows as a corollary of Propositions 1 and 2:

Corollary 1 *An equilibrium always exists.*

3.2 Uniqueness

From Propositions 1 and 2, we have that the sincere voting equilibrium and the Duverger's Law equilibrium $(0,0)$ never coexist under Plurality. To show uniqueness, it is thus sufficient to identify conditions under which the Duverger's Law equilibrium $(1,1)$ does not exist. This is the role of the following Proposition, that identifies the necessary and sufficient conditions for that equilibrium to exist:

Proposition 3 *There exists a Duverger's Law equilibrium* $(1,1)$ *if and only if*

$$\frac{W}{Y} \leq \frac{p_A(1,1) - p_A(1,0) - p_B(1,0)}{p_B(1,0)}$$

The condition that guarantees the existence the Duverger's Law equilibrium $(1,1)$ is more demanding than the condition for the Duverger's Law equilibrium $(0,0)$. This is so because, if all groups vote sincerely, candidate B is more likely to win than candidate A (i.e. $p_B(1,0) > p_A(1,0)$). Hence, b-voters have weaker incentives than a-voters to coordinate behind their second choice candidate.

As for the condition in Proposition 1, we can use Lemma 3 (in Appendix A1), to rewrite the condition in Proposition 3 as a function of the primitives of the model. We can then prove the following Lemma:

Lemma 2 $\frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_B(1,0)}$ *is increasing both in* k *and in* α.

This Lemma shows that the Duverger's Law equilibrium $(1,1)$ is more likely to exist when (i) the minority group is large (large k), then majority voters have strong incentives to coordinate behind one majority candidate since it is the only way to defeat the minority candidate; (ii) the majority is evenly divided (α large). At first sight, this second part might seem counterintuitive. Indeed, when the majority is evenly divided, it's possible that both majority candidates should have a chance to win, and voters should thus not want to abandon their champion. Yet, when the majority is evenly divided, it also might well be that both majority candidates are unlikely to win. In such a case, coordinating behind any of the majority candidate is appealing because it is the only way to defeat the minority candidate.

As a corollary of Propositions 1, 2, and 3, we have a necessary and sufficient condition for equilibrium uniqueness:

Corollary 2 *The equilibrium is unique if and only if* $\frac{W}{Y} > \frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_B(1,0)}$.

To better understand when the equilibrium is unique, it is useful to consider sufficient conditions on the primitives of the model. First, we can identify a sufficient condition on the utility function of majority agents:

Proposition 4 *If* $W \geq Y$*, then* $\forall k, \alpha$ *there exists a unique equilibrium. Moreover,* $\exists \overline{\alpha}$ *(increasing in* k*) such that, if* $\alpha > \overline{\alpha}$*, the unique equilibrium is the sincere voting equilibrium.*

Proposition 4 first shows that as soon as majority voters care relative more about the choice between the two majority candidates than about preventing a victory of the minority (W larger than Y), then the equilibrium is unique. This is so because coordinating behind one majority candidate leads to limited gains in terms of decreasing the likelihood that C wins. Therefore, $b-$voters are willing to abandon their champion only if they care relatively more about the defeating the minority candidate than about choosing their preferred majority candidate.

The second part of Proposition 4 shows that, if the majority is evenly divided, then the unique equilibrium is the sincere voting equilibrium. Otherwise, the unique equilibrium is the Duverger's Law equilibrium $(0,0)$. We can illustrate that result with a numerical example. Suppose that $\frac{W}{Y} = 2$. Figure 1 plots the equilibrium type for all possible combinations of α ($x-$axis) and k ($y-$axis). It also distinguishes between two types of cases when the sincere voting is the unique equilibrium: values of the parameters for which candidate B is the likely winner (i.e., the candidate who wins with the highest probability), and those for which candidate C is the likely winner. Figure 1 clearly illustrates that Duverger's Law fails when the majority is sufficiently evenly divided.

It is also useful to consider a sufficient conditions for the two Duverger's Law equilibria to coexist:

Proposition 5 *If* $W < Y$*, then for* α *sufficiently large,* $\exists \overline{k}$ *(decreasing in* α*) such that, if* $k > \overline{k}$*, both Duverger's Law equilibria exist.*

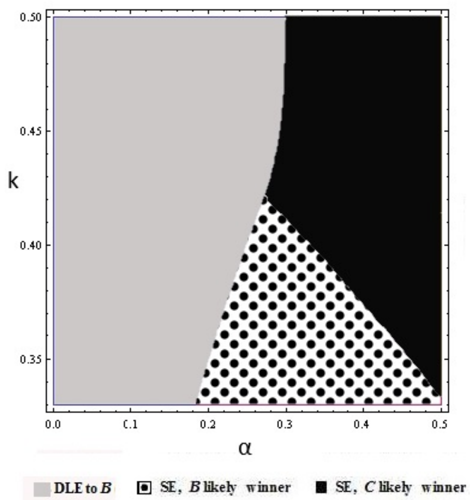

Figure 1: Equilibria with $\frac{W}{Y} = 2$. The light colored area represents parameter values for which the unique equilibrium is the Duverger's Law equilibrium $(0, 0)$, such that both types of voter vote for B. The remaining areas represent paramater values for which the unique equilibrium is sincere voting, such that each voter type votes for their first choice, with the black area representing those values for which the Condorcet loser is the likely winner.

Proposition 5 shows that both Duverger's Law equilibria exist when majority voters care relatively more about preventing the victory of the minority candidate than about electing their most-preferred majority candidate, and when the minority candidate is sufficiently strong. In such cases, even $b-$voters are willing to abandon their champion: this is the only way to defeat a candidate they dislike a lot. Again, we can illustrate the result with a numerical example. To do so we produce Figure 2, which is similar to Figure 1 but for $\frac{W}{Y} = \frac{3}{4}$. Figure 2 clearly shows that the Duverger's law equilibria coexist only when k is large enough. It also highlights that α must be large enough.

Together, Propositions 4 and 5 highlight a non-monotonicity with respect to α. If k is sufficiently large, then there will be equilibrium multiplicity (the two Duverger's Law equilibria) once the two groups become sufficiently evenly divided (α large). However, if k is sufficiently small, then a large α will lead to sincere voting.

Similarly, there is a corresponding discontinuity with respect to k. If α is relatively evenly divided, when k is small an increase in the size of the minority will still retain sincere voting as a unique equilibrium, while potentially making the Condorcet loser C the likely winner. However, there exists a point at which it will shift the equilibrium to a Duverger's Law equilibrium, meaning there exists a point at which an increase in k actually *decreases* the probability of C winning by incentivizing strategic behavior on the part of the majority.

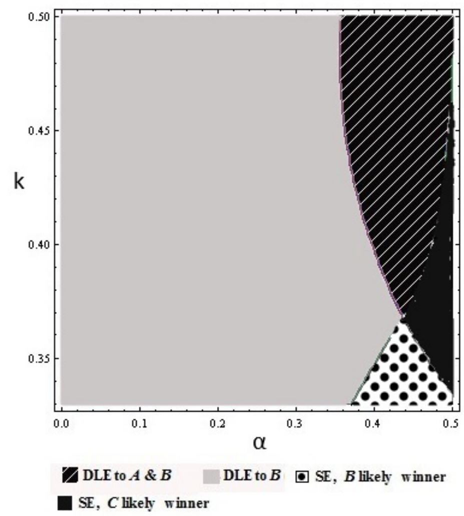

Figure 2: Equilibria with $\frac{W}{Y} = \frac{3}{4}$. The light colored area represents parameter values for which the unique equilibrium is the Duverger's Law equilibrium $(0, 0)$, such that both types of voter vote for B. The striped area represents parameter values for which both Duverger's Law equilibrium $(0, 0)$ and $(1, 1)$ exist. The remaining areas represent paramater values for which the unique equilibrium is sincere voting, such that each voter type votes for their first choice, with the black area representing those values for which the Condorcet loser is the likely winner.

4 Majority Runoff Elections

In this section, we study the behavior of voters in majority runoff elections. Our main focus is on the comparison with the behavior in plurality elections. We also study a variant of the majority runoff rule in which more than two candidates are allowed in the second round if they obtain more than a pre-defined fraction of the registered voters. We call this variant of the majority runoff rule the "French-style runoff".

Under the majority runoff rule, there is up to two rounds of voting. In the first round, $(\rho = 1)$, all voters are called to cast a ballot in favor of one of the candidates. The action set is denoted by $\Psi^1 = \{A, B, C\}$. If the candidate who ranks first obtains more than 50% of the votes, she wins outright and there is no second round. Otherwise, there is a second round $(\rho = 2)$ opposing the two candidates who received the most votes in the first round (the top-two candidates). In that round, all voters are again called to cast a vote for one of the participating candidates. The action set is denoted by $\Psi^2 = \{P, Q\}$, where P and Q refer to the candidates who ranked first and second in the first round, respectively. The candidate with the largest number of votes in that round wins the election.

Following Bouton (2013) and Bouton and Gratton (2015), we work under the assumption that the set of voters *going to the polls* in the first and the second rounds may differ. In particular, we assume that in round ρ, agents of type $\theta \in \Theta$ turn out with an (exogenously given) probability q_θ^ρ, drawn from a Uniform distribution with support $[0,1]$.

Formally, the strategy of a given type of voters has to specify the actions played in each subgame. Nevertheless, as we discuss in the next paragraph, in the case of majority runoff, it proves useless to change the notation in order to accommodate for these more complex strategies. This is so because majority voters have an obvious dominant strategy in every possible second round. It is thus not necessary to keep track of their strategy in that round. Instead, one can lighten notation by focusing on the behavior of voters in the first round. In particular, we denote by $\sigma_i \in [0,1]$ the probability that group i votes for A in the first round of a runoff election.

There are three possible second rounds in majority runoff elections: A vs. B, A vs. C, and B vs. C. When the second round pits one majority candidate against the minority candidate, both $a-$voters and $b-$voters strictly prefer to vote for the participating majority candidate. Otherwise, they increase the probability that their least-preferred candidate wins. For the same reason, when the second round features the two majority candidates, $a-$voters strictly prefer to vote for candidate A, and $b-$voters strictly prefer to vote for candidate B. It remains to be determined what $c-$voters do in that case. Given that they are indifferent between A and B, we assume that they abstain. This is for the sake of simplicity: we can relax that assumption without affecting the results qualitatively (see Section 5.1).

To deal with potential issues of sequential rationality, we focus on (pure strategy) Weak Perfect Bayesian Equilibria (WPBE).[24] Per the discussion above, a WPBE under majority runoff is a tuple $\{\sigma_i\}_{i=a,b}$ such that

$$\begin{aligned}
\sigma_a^* &= \arg\max_{\lambda_a} p_A^R(\sigma_a, \sigma_b^*) W - (1 - p_A^R(\sigma_a, \sigma_b^*) - p_B^R(\sigma_a, \sigma_b^*))Y, \text{ and} \\
\sigma_b^* &= \arg\max_{\lambda_b} p_B^R(\sigma_a^*, \sigma_b) W - (1 - p_A^R(\sigma_a^*, \sigma_b) - p_B^R(\sigma_a^*, \sigma_b))Y,
\end{aligned}$$

where $p_\psi^R(\sigma_a, \sigma_b)$ is the probability that candidate ψ wins when the strategy profile is (σ_a, σ_b), and taking into account the dominant strategies in the second round. See the proof of Lemma 5 (in Appendix A5) for details about these probabilities.

The set of pure strategy WPBE majority under runoff share many features with the

[24]Importantly, under plurality, the set of pure WPBE corresponds to the set of pure Bayesian equilibria.

set of equilibria under plurality (see Appendix A5 for details): (i) the same two types of equilibria exist, (ii) the sincere voting equilibrium never coexists with a Duverger's law equilibrium, and (iii) the equilibrium is unique for a broad set of parameter values (either the sincere voting equilibrium, or the Duverger's law $(0,0)$). Yet, the conditions under which those different situations occur are different under plurality and runoff. In particular, we can prove that sincere voting is more prevalent under runoff than under plurality:

Proposition 6 *Given a set of parameter values W, Y, k, and α,*

1. *The sincere voting equilibrium always exists in the first round of a runoff election if it exists under plurality;*

2. *There exist a set of parameter values W, Y, k, and α, the sincere voting equilibrium exists in the first round of a runoff election even if it does not exist under plurality.*

This result is rather intuitive. Voters have weaker incentives to coordinate behind one of the majority candidates in the first round of a majority runoff election because the second round offers another opportunity for majority voters to defeat the minority candidate. She is thus less of a threat to majority voters, who then feel freer to vote for their most-preferred candidate in the first round.

4.1 French-Style Runoff

The French runoff system for parliamentary and local elections differs from the "standard" majority runoff system described and analyzed in the previous section. The difference is that, on top of the 50%-threshold for first-round victory, it features another threshold at 12.5% of the registered voters. If a candidate finishes third (or lower), but receives more than that threshold number of voters, then she will also advance to the second round. In our setup, this opens the door for a second round with three candidates.

We consider any French-style runoff rule with a second-round qualification threshold $\tau \in (0, 0.5)$. Under such a system, there are four possible second rounds: A vs. B or C, B vs. C, and the opposition of all three candidates. The set of possible second round is thus $\{AB, AC, BC, ABC\}$. Given that a second round featuring three candidates is possible, it is important that strategies specify the actions played in each possible second (i.e. in each subgame). We thus have to adapt the notation. Yet, since majority voters have a dominant strategies in all second rounds involving two candidates, we only need to keep track of their strategy in the ABC second round. We denote by $\sigma_i^{ABC} \in [0,1]$ the probability that group i votes for A in the ABC second round. As before, we denote by $\sigma_i \in [0,1]$ the probability that group i votes for A in the first round of a runoff election.

As for majority runoff, we focus on (pure strategy) WPBE. A WPBE under French-style runoff is a tuple $\{\sigma_i, \sigma_i^{ABC}\}_{i=a,b}$.

We start with the analysis of the second-round voting behavior. When there are two candidates participating in the second round (for $\omega \in \{AB, AC, BC\}$), the behavior of voters is the same as the one in the second round of a majority runoff election. That is, if candidate C is pitted against one of the majority candidates, then it is a dominant strategy for all majority voters to vote for the participating majority candidate, and for $c-$voters vote for C. If candidates A and B are the only candidates, then it is a dominant strategy for majority voters vote for their most-preferred candidates and $c-$voters are assumed to abstain. When there are three candidates participating in the second round (for $\omega = ABC$), voters behave as in a plurality election (see Propositions 1, 2, and 3).

We are now in position to analyze the behavior of voters in the first round. We focus on the comparison with plurality and majority runoff. We can prove that sincere voting is more prevalent under French-style runoff than under plurality, but less prevalent under French-style runoff than under majority runoff.:

Proposition 7 *Given a set of parameter values W, Y, k, and α:*

- *if the sincere voting equilibrium exists under plurality, it also exists in the first round of a French-style runoff election, but the converse is not true;*
- *if the sincere voting equilibrium exists in the first round of a French-style runoff election, it is also an equilibrium in the first round of a majority runoff election, but the converse is not true*

The intuition of the first part of the Proposition is the same as for majority runoff: Voters have weaker incentives to coordinate behind one majority candidate in the first round of a french-style runoff election because the second round offers another opportunity for majority voters to defeat the minority candidate.

The second part of the Proposition is perhaps surprising. Even if it is easier for a candidate to qualify for the second round in a French-style runoff election than in a majority runoff election, voters are more inclined to abandon their preferred candidate in the former. This is so because, under French-style runoff, a majority candidate qualified for a second round against the minority candidate is not guaranteed of the support of all majority voters. Even worse, by sending one's preferred candidate to the second round, one may increase the risk that the minority candidate wins (if majority voters split their votes).

Proposition 7 is coherent with the empirical findings of Pons and Tricaud (2017). They find that the vote share of the top two candidates (as a percentage of the number of registered voters) is lower in second rounds with three candidates than with two candidates. This is due to two reasons: (i) some "loyal" voters abstain when their most preferred candidate, the third one, does not participate in the second round, and (ii) some "switchers" vote for their preferred among the top two candidates when their most preferred candidate, again, the third one, is not in the race. This is exactly what our model predicts if one considers equilibria such that voters vote sincerely both in the first round and in the ABC second round. In that case, the qualification of the third candidate to the second round leads to a substantial reduction in the vote share of the top two candidates. This is so because the supporters of the third candidate, say A, switch away from supporting the other majority candidate, say B, to instead support A, their preferred candidate.[25]

Finally, Proposition 7 leads to the following extension:

Proposition 8 *The prevalence of sincere voting is increasing in the level of the threshold* τ.

This result is somehow surprising in that it does not square with the common reason justifying the introduction of a second-round qualification threshold. The common argument is that the purpose of such a threshold is to balance two desirable but contradictory characteristics of the electoral systems. On the one hand, a higher threshold is deemed

[25] Given that turnout is assumed exogenous, our model does not deliver any interesting prediction about the behavior of "loyal" voters.

desirable because, by reducing the number of candidates in the second round, it increases the likelihood that the victor is supported by a large plurality of the population. On the other hand, a higher threshold is deemed undesirable because it reduces the diversity of choices offered to voters in the second round. Therefore, the common belief is that this legitimacy vs. pluralism trade-off tilts in favor of legitimacy for higher thresholds, and in favor of pluralism for lower ones. Our result suggests that this trade-off may actually tilt in the exact opposite direction: by increasing the second-round qualification threshold, one may actually increase the probability of a second round featuring three candidates (because voters have stronger incentives to vote sincerely in the first round). Increasing the threshold then tilts the balance in favor of pluralism, not legitimacy.

Obviously, this reversal of the legitimacy vs. pluralism trade-off is empirically relevant only if a substantial fraction of the electorate are strategic (in the way modeled here). Ultimately, it is thus an empirical question: how does an increase of the threshold influence the number of candidate participating in the second round? It should be possible to explore this question by exploiting the changes in the level of this threshold over the past 60 years in France. Since the beginning of the Fifth Republic in France, the second-round qualification threshold has raised significantly: from 5% in 1958, to 10% and then 12.5% since 1988 (with a one-election switch to proportional representation in 1986). Interestingly, a cursorily look at the history of french parliamentary and local elections does not suggest that second round opposing more than two parties have become less frequent with the increase of the threshold.

5 Extensions

5.1 Preferences: Real Line

In this section, we show that our results are robust to a more sophisticated structure of preferences. We focus on the widely studied case of a unidimensional policy space. In particular, we assume that the three candidates and the three groups of voters are positioned along the real line. This is a quite flexible case that allows us to relax the assumptions that (i) $c-$voters are indifferent between A and B, and (ii) majority voters all rank C last.

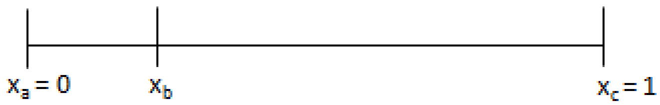

Figure 3: Example of group preferences on a real line.

For the sake of simplicity, the position of group-a is assumed identical to the position of candidate A (and similarly for the two other group-candidate pairs). The three positions are denoted by x_a, x_b, and x_c. These are the bliss points of the different types of voters.[26] Voters' utility is decreasing in the absolute value of the distance between their bliss point and the position of the winning candidate. To ensures comparability with the baseline model where candidate C is the Condorcet loser, we assume that $x_a = 0$, $x_b \in (0, 1/2)$, and $x_c = 1$. One can see an example of relative distances that satisfy this assumption in Figure 3.

This directly implies that $W_a = W_b = x_b = Y_c$ and $Y_a = W_c = 1 - x_b > 1 - 2x_b = Y_b$. Finally, we keep the assumption that, together, $a-$ and $b-$voters form a majority, but that $c-$voters are sufficiently numerous (i.e. more than 1/3 of the electorate) so that C is a serious threat.

First, we can prove that $c-$voters always prefer to vote for C :

Proposition 9 *Under plurality, in equilibrium, $c-$voters always vote for candidate C.*

The intuition is as follows. For $c-$voters, the gain of voting for candidate B instead of candidate C is to avoid the victory of candidate A. The cost of voting for candidate B is that their favorite candidate loses any chance of winning. Given that candidate B is not very appealing to $c-$voters (because $x_b \in (0, 1/2)$), and that candidate A is relatively unlikely to win when $c-$voters vote sincerely (because $k > 1/3$), $c-$voters have no incentive to vote for B.

Knowing that $c-$voters necessarily vote for C, we can fully characterize the set of equilibria:

[26] We have also considered an extension of the model in which there is preference heterogeneity within groups. In particular, we considered a distribution of preferences within each group, such that $x_i^a \sim f_a[0,1]$ and $x_i^b \sim f_b[0,1]$, with $\frac{1}{2} > E[x_i^b] > E[x_i^a]$. Assuming that candidate A's ideal point is $E[x_i^a]$ and similarly for B, we can show that heterogeneity within groups makes $a-$voters more inclined to vote sincerely. The effect on group $b-$voters depends on the specific shape of the preference distribution.

Proposition 10 *Under plurality:*

- *If* $\frac{x_b}{p_B(1,1)-p_A(1,0)-p_B(1,0)} > max\{\frac{1-x_b}{p_A(1,0)}, \frac{1}{p_B(1,0)}\}$*, then sincere voting is the unique equilibrium,*
- *If* $\frac{x_b}{p_B(1,1)-p_A(1,0)-p_B(1,0)} \in [\frac{1-x_b}{p_A(1,0)}, \frac{1}{p_B(1,0)}]$*, then Duverger's law (1,1) is the unique equilibrium,*
- *If* $\frac{x_b}{p_B(1,1)-p_A(1,0)-p_B(1,0)} \in [\frac{1}{p_B(1,0)}, \frac{1-x_b}{p_A(1,0)}]$*, then Duverger's law (0,0) is the unique equilibrium,*
- *Otherwise, both Duverger law equilibria exist.*

An interesting implication of Proposition 10 is that the supporters of the centrist candidate (i.e. candidate B) are more "committed" to their candidate than the other majority voters. In particular, if we were to swap the size of groups a and b, keeping everything else constant, there would be cases such that all majority voters vote for candidate B before the swap, but they all vote sincerely after the swap. By contrast, it is never the case that all majority voters vote for candidate A before the swap, but then all vote sincerely after the swap. This result leads to the testable prediction that centrist voters should have a higher propensity to vote for their most-preferred candidate than other voters. Figure 4 shows how this changes the equilibria in the case where $x_b = \frac{2}{5}$.

5.2 More Groups

In this section, we study how the size of the groups at which the ethical voting decisions are made influences equilibrium behavior. To do so, we consider an extension of the model, such that the a and b groups are split into n identical subgroups. Each group makes its own ethical voting decision. The following Proposition shows that, for n sufficiently large, our ethical voting model delivers the same set of equilibria as the pivotal voter model (without aggregate uncertainty, see Myerson and Weber 1993, Myerson 2002, Bouton, Castanheira and Llorente-Saguer 2017).

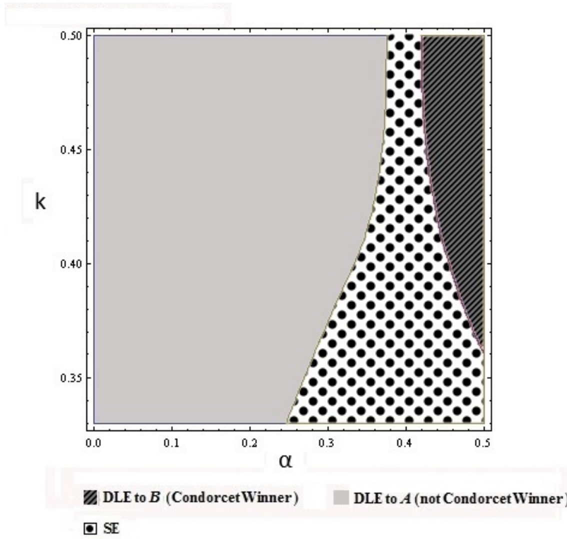

Figure 4: Equilibria with real line extension when $x_b = \frac{2}{5}$, and group a (the left-wing group) is larger than group b (the moderate group which supports the Condorcet winner); α represents the size of group b. The light colored area represents parameter values for which the unique equilibrium is the Duverger's Law equilibrium $(1, 1)$, such that both types of voter vote for A. The dark striped area represents parameter values for which the unique equilibrium is the Duverger's Law equilibrium $(0, 0)$, such that both types of voters vote for B. The remaining area represents paramater values for which the unique equilibrium is sincere voting, such that each voter type votes for their first choice.

Proposition 11 *Under plurality, for n sufficiently large:*

- *both Duverger's law equilibria exist for any values of W, Y, k, and α;*
- *the sincere voting equilibrium never exists.*

The intuition is exactly the same as under the pivotal voter model. Sincere voting is not an equilibrium because supporters of the weak majority candidate realize that the probability they make their favorite candidate win by voting for her is orders of magnitude smaller than the probability they make their second-preferred candidate win by voting for her. Thus, voting for their most-preferred candidate would be wasting a useful vote. Duverger's law equilibria exist for exactly the same reason.

As discussed in Grofman, Blais and Bowler (2009), some countries using plurality rule feature a stable two-party system (e.g., the US), while others do not (e.g. India). Proposition 11 suggests a novel explanation for such mixed empirical evidence about Duverger's law. Countries with stable two-party systems are those for which ethical groups are sufficiently fragmented. This prediction is not in contradiction with Duverger's original idea, which finds support in the data (Clark and Golder 2006), "[...] that social heterogeneity should increase the number of parties only once the electoral system is sufficiently permissive." (Clark and Golder 2006, p. 704). It actually refines it by suggesting that the nature of social heterogeneity is crucial: a fragmentation of groups with similar interests should have a negative effect on the number of "serious" candidates, whereas the fragmentation of society into more groups with different interests should have a positive effect.

6 Conclusions

The existence of a unique equilibrium for a broad range of parameter values is a very appealing feature of the ethical voting model with multiple candidates. As we show in a companion paper (Bouton and Ogden 2017), this uniqueness opens the door for a tractable model including both strategic voters and strategic candidates. Such a model is highly valuable because understanding political institutions and their influence on policies requires a good understanding of the strategic behavior of politicians and voters, and how they interact.[27] Yet, most of the political economy literature focuses either on strategic candidates, or strategic voters. For situations with more than two candidates, joint analyses are almost nonexistent.[28] And indeed, the tractability and predictive power of extant models is affected by the existence of multiple equilibria at the voting stage. In Bouton and Ogden (2017), we revisit classic models of candidates behavior (entry and positioning along the real line) in a model with ethical voters. We find that the presence of strategic voters affects dramatically the behavior of candidates.

[27] See, e.g., the discussion in Myerson (1993).

[28] We are only aware of a handful of exceptions: Austen-Smith and Banks 1988, Myerson and Weber 1993, Besley and Coate 1997, Castanheira 2003, Dellis and Oak 2006, and Dellis 2009.

Appendices

Appendix A1: Preliminaries

The following Lemma computes the probabilities of winning of the different candidates for all the possible pure strategy profiles.

Lemma 3 *If $k > (1-\alpha)(1-k)$ (i.e., group c is the largest), we have:*

$$\begin{aligned}
p_A(1,0) &= \frac{(1-k)\alpha^2}{3k(1-\alpha)}; \\
p_B(1,0) &= \frac{1-k}{2k}[1-\alpha-\frac{1}{3(1-a)}\alpha^2]; \\
p_A(1,1) &= p_B(0,0) = \frac{1}{2k}[1-k-\frac{(1-2k)^3}{3(1-\alpha)\alpha(1-k)^2}];
\end{aligned}$$

If $k < (1-\alpha)(1-k)$ (i.e., group b is the largest), we have:

$$\begin{aligned}
p_A(1,0) &= \frac{(1-k)\alpha^2}{3k(1-\alpha)}; \\
p_B(1,0) &= \frac{3(2-3k)k-\alpha^2(1-k)^2-6\alpha(1-k)k}{6k(1-\alpha)(1-k)}; \\
p_A(1,1) &= p_B(0,0) = \frac{1}{2(1-\alpha)}[\frac{2-3k}{(1-k)}-\alpha-\frac{\alpha^2(1-k)}{3k}].
\end{aligned}$$

Proof. For a given strategy profile, the only uncertainty comes from the turnout of the different groups. The Lemma focuses on three strategy profiles: $(1,0)$ –i.e. group-a votes for A and group-b votes for B, and $(1,1)$ or $(0,0)$ –i.e. group-a and group-b vote for the same majority candidate (either A $(1,1)$ or B $(0,0)$).

First, let us consider the strategy profile $(1,0)$. Candidate A wins if she obtains more votes than candidate B (i.e. q_a and q_b are such that $\alpha q_a > (1-\alpha)q_b$) and more votes than candidate C (i.e. q_a and q_c are such that $(1-k)\alpha q_a > kq_c$). Therefore, the probability that candidate A wins for the strategy profile $(1,0)$ is:

$$\begin{aligned}
p_A(1,0) &= \int_0^1 \int_0^{\frac{\alpha}{1-\alpha}q_a} \int_0^{\frac{1-k}{k}\alpha q_a} \partial q_c \partial q_b \partial q_a \\
&= \frac{(1-k)\alpha^2}{3k(1-\alpha)}.
\end{aligned}$$

The probability that candidate B wins for the strategy profile $(1,0)$ depends on which of groups b and c is largest. If group-c is the largest $(k > (1-\alpha)(1-k))$, then this probability is:

$$\begin{aligned} p_B(1,0) &= \int_0^1 \int_{\frac{\alpha}{1-\alpha}q_a}^1 \int_0^{\frac{1-k}{k}(1-\alpha)q_b} \partial q_c \partial q_b \partial q_a \\ &= \frac{1-k}{2k}[1-\alpha-\frac{1}{3(1-a)}\alpha^2]. \end{aligned}$$

If group-b is the largest $(k < (1-\alpha)(1-k))$, this probability is more easily computed as $1 - p_A(1,0) - p_C(1,0)$. Therefore, we need $p_C(1,0)$, which is:

$$\begin{aligned} p_C(1,0) &= \int_0^1 \int_{\frac{1-k}{k}\alpha q_a}^1 \int_0^{\frac{k}{(1-k)(1-\alpha)}q_c} \partial q_b \partial q_c \partial q_a \\ &= \frac{1}{2(1-\alpha)}[\frac{k}{(1-k)} - \frac{\alpha^2(1-k)}{3k}]. \end{aligned}$$

Thus, we have

$$\begin{aligned} p_B(1,0) &= 1 - \frac{(1-k)\alpha^2}{3k(1-\alpha)} - \frac{1}{2(1-\alpha)}[\frac{k}{(1-k)} - \frac{\alpha^2(1-k)}{3k}] \\ &= \frac{3(2-3k)k - \alpha^2(1-k)^2 - 6\alpha(1-k)k}{6k(1-\alpha)(1-k)}. \end{aligned}$$

Second, let us consider the strategy profile $(1,1)$ (the $(0,0)$ case is identical). We are interested in the probability that the candidate supported by all majority agents, A in the case under consideration), wins. This probability depends on which of groups b and c is largest. If group-c is the largest, we can show that:

$$p_A(1,1) = \frac{1}{2k}[1-k-\frac{(1-2k)^3}{3(1-\alpha)\alpha(1-k)^2}].$$

If group-b is the largest, then:

$$p_A(1,1) = \frac{1}{2(1-\alpha)}[\frac{2-3k}{(1-k)} - \alpha - \frac{\alpha^2(1-k)}{3k}].$$

■

The following Lemma proves that, when all voters vote sincerely, candidate B is more likely to win than candidate A.

Lemma 4 $p_B(1,0) > p_A(1,0)$.

Proof. Straightforward from Lemma 3. ■

Appendix A2: Proofs

Proposition 1 Under Plurality, the sincere voting equilibrium exists if and only if

$$\frac{W}{Y} \geq \frac{p_A(1,1) - p_A(1,0) - p_B(1,0)}{p_A(1,0)}.$$

Proof. The strategy $(1,0)$ is an equilibrium under Plurality if and only if neither group prefers voting for their second choice candidate. Formally, this requires

$$p_A(1,0)\,W \geq (p_B(0,0) - 1)Y + (1 - p_A(1,0) - p_B(1,0))Y, \tag{3}$$

for group a, and

$$p_B(1,0)\,W \geq (p_A(1,1) - 1)Y + (1 - p_A(1,0) - p_B(1,0))Y, \tag{4}$$

for group b. Given that $p_A(1,1) = p_B(0,0)$, the RHS of these two conditions are identical. From Lemma 4, we have $p_B(1,0) > p_A(1,0)$. Therefore, condition (4) is necessarily satisfied when condition (3) is. The result follows from rearranging condition (3).

■

Lemma 1 $\frac{p_A(1,1) - p_A(1,0) - p_B(1,0)}{p_A(1,0)}$ is increasing in k and decreasing in α.

Proof. Using Lemma 3 (in Appendix A1), we have that $\frac{p_A(1,1) - p_A(1,0) - p_B(1,0)}{p_A(1,0)}$ boils down to

$$\frac{3}{2\alpha} - \frac{(1-2k)^3}{2\alpha^3(1-k)^3} - 2,$$

if $k > (1-\alpha)(1-k)$, and to

$$\frac{3k}{2(1-k)\alpha} - 1,$$

if $k < (1-\alpha)(1-k)$.

Clearly, $\frac{3k}{2(1-k)\alpha} - 1$ is increasing in k and decreasing in α. It thus remains to prove that the same is true for $\frac{3}{2\alpha} - \frac{(1-2k)^3}{2\alpha^3(1-k)^3} - 2$. Taking the derivative with respect to k, we obtain

$$\frac{3}{2\alpha^3}\frac{(2k-1)^2}{(k-1)^4} > 0.$$

Taking the derivative with respect to α, we obtain the following condition:

$$\frac{3}{2\alpha^2}\left[\frac{(1-2k)^3}{\alpha^2(1-k)^3} - 1\right] < 0.$$

This simplifies to

$$(1-2k)^3 < \alpha^2 (1-k)^3.$$

Since $\alpha > \frac{1-2k}{1-k}$ in the relevant area, this is necessarily satisfied. ■

Proposition 2 There exists a Duverger's Law equilibrium $(0,0)$ if and only if

$$\frac{W}{Y} \leq \frac{p_A(1,1) - p_A(1,0) - p_B(1,0)}{p_A(1,0)}$$

Proof. Following a similar argument as the one in the proof of Proposition 1, we have that the strategy profile $(0,0)$ is an equilibrium if and only if

$$p_A(1,0)\,W \leq (p_B(0,0) - 1)Y + (1 - p_A(1,0) - p_B(1,0))Y,$$

It is obvious that group b does not want to deviate. The result follows from rearranging the condition and using the equality $p_B(0,0) = p_A(1,1)$.

■

Proposition 3 There exists a Duverger's Law equilibrium $(1,1)$ if and only if

$$\frac{W}{Y} \leq \frac{p_A(1,1) - p_A(1,0) - p_B(1,0)}{p_B(1,0)}$$

Proof. The proof is similar to the proof of Proposition 2. The only difference is that we need to determine the condition for b-voters to prefer voting for A (it is obvious that group a does not want to deviate). Doing so, we have that the strategy profile $(1,1)$ is an equilibrium if and only if

$$p_B(1,0)\,W \leq (p_A(1,1) - 1)Y + (1 - p_A(1,0) - p_B(1,0))Y,$$

The result follows from rearranging the condition. ■

Lemma 2 $\frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_B(1,0)}$ is increasing both in k and in α.

Proof. $\frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_B(1,0)}$ can be re-written as $\frac{p_A(1,1)-p_A(1,0)}{p_B(1,0)} - 1$.

The LHS can be re-written as $\frac{3(1-k)^3(1-\alpha)\alpha - 2(1-k)^3\alpha^3 - (1-2k)^3}{[3(1-k)(1-\alpha)^2 - (1-k)\alpha^2]\alpha(1-k)^2}$ if $k > (1-\alpha)(1-k)$.

Alternatively, it is $\frac{3(2-3k)k - 3\alpha(1-k)k - 3\alpha^2(1-k)^2}{6(1-k)k(1-\alpha) - (1-k)^2\alpha^2 - 3k^2}$ if $k < (1-\alpha)(1-k)$,

In both cases, the LHS is increasing in both k and α:

In the first case,

$\frac{\partial}{\partial k} = \frac{(2(-(1-2k)^3 + 3(1-a)a(1-k)^3 - 2a^3(1-k)^3))}{(a(3(1-a)^2(1-k) - a^2(1-k))(1-k)^3)} + \frac{(6(1-2k)^2 - 9(1-a)a(1-k)^2 + 6a^3(1-k)^2)}{(a(3(1-a)^2(1-k) - a^2(1-k))(1-k)^2)}$
$- \frac{((-3(1-a)^2 + a^2)(-(1-2k)^3 + 3(1-a)a(1-k)^3 - 2a^3(1-k)^3))}{(a(3(1-a)^2(1-k) - a^2(1-k))^2(1-k)^2)}$ and

$\frac{\partial}{\partial\alpha} = \frac{(3(1-a)(1-k)^3-3a(1-k)^3-6a^2(1-k)^3)}{(a(3(1-a)^2(1-k)-a^2(1-k))(1-k)^2)} - \frac{((-6(1-a)(1-k)-2a(1-k))(-(1-2k)^3+3(1-a)a(1-k)^3-2a^3(1-k)^3))}{(a(3(1-a)^2(1-k)-a^2(1-k))^2(1-k)^2)} - \frac{(-(1-2k)^3+3(1-a)a(1-k)^3-2a^3(1-k)^3)}{(a^2(3(1-a)^2(1-k)-a^2(1-k))(1-k)^2)}$.

In the second case,

$\frac{\partial}{\partial k} = \frac{-(((6(1-a)(1-k)+2a^2(1-k)-6k-6(1-a)k)(-3a^2(1-k)^2+3(2-3k)k-3a(1-k)k))}{(-a^2(1-k)^2+6(1-a)(1-k)k-3k^2)^2)} + \frac{(3(2-3k)-3a(1-k)+6a^2(1-k)-9k+3ak)}{(-a^2(1-k)^2+6(1-a)(1-k)k-3k^2)}$

and

$\frac{\partial}{\partial\alpha} = \frac{((-2a(1-k)^2-6(1-k)k)(-3a^2(1-k)^2+3(2-3k)k-3a(1-k)k))}{(-a^2(1-k)^2+6(1-a)(1-k)k-3k^2)^2} + \frac{(-6a(1-k)^2-3(1-k)k)}{(-a^2(1-k)^2+6(1-a)(1-k)k-3k^2)}$.

One can check that these derivatives are all positive in the relevant ranges. ■

Proposition 4 If $W \geq Y$, then $\forall k, \alpha$ there exists a unique equilibrium. Moreover, $\exists\overline{\alpha}$ (increasing in k) such that, if $\alpha > \overline{\alpha}$, the unique equilibrium is the sincere voting equilibrium.

Proof. The first part of the proposition comes from the fact that $\frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_B(1,0)}$ is maximized at $k = \frac{1}{2}$ and $\alpha = \frac{1}{2}$ (from Lemma 2), where it takes the value $\frac{\frac{1}{2}-\frac{1}{6}-\frac{1}{6}}{\frac{1}{6}} = 1$. Therefore, if $\frac{W}{Y} > 1$, it is a dominant strategy for group b to vote for B for all k and α. The equilibrium is then uniquely pinned down by the behavior of group a.

The second part of the proposition is straightforward from Lemma 1. Recall that the condition for sincere voting is $\frac{W}{Y} \geq \frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_A(1,0)}$, the RHS of which is increasing in k and decreasing in α. Therefore, the necessary threshold $\overline{\alpha}$ is in turn increasing in k. ■

Lemma 5 If $W < Y$, then for α sufficiently large, $\exists\overline{k}$ (decreasing in α) such that, if $k > \overline{k}$, both Duverger's Law equilibria exist.

Proof. This is direct from Lemma 2. ■

Proposition 6 Given a set of parameter values W, Y, k, and α:

- the sincere voting equilibrium always exists in the first round of a runoff election if it exists under plurality;
- the sincere voting equilibrium may exist in the first round of a runoff election even if it does not exist under plurality.

Proof. As shown in Proposition 14 (in Appendix A5), the necessary and sufficient condition for the existence of the sincere voting equilibrium under runoff is

$$\frac{W}{Y} \geq \frac{p_C^R(1,0) - p_C^R(1,1)}{p_A^R(1,0)}. \tag{5}$$

To prove the first part of the proposition, we need to show that

$$\frac{p_C(1,0) - p_C(1,1)}{p_A(1,0)} \geq \frac{p_C^R(1,0) - p_C^R(1,1)}{p_A^R(1,0)} \tag{6}$$

First, note that $p^R_C(1,1) = p^R_C(0,0) = p_C(1,1)$. Indeed, when all majority voters vote for the same majority candidate, only two candidates receive a positive fraction of the votes. As a consequence, one of those two candidates must receive more than 50% of the votes, and win outright in the first round. As under plurality, the candidate receiving the largest number of votes in the first round wins the elections.

Second, we have from Lemma 5 (in Appendix A5) that, if $p^R_A(1,0) > p_A(1,0)$, then $p^R_C(1,0) < p_C(1,0)$. Therefore, to show that condition (6) holds, it is sufficient to show that $p^R_A(1,0) > p_A(1,0)$ holds. Using mathematica, we can see that this is true for $k > \frac{1}{3}$. Therefore, if parameter values are such that the sincere voting equilibrium exists (and it thus unique) under plurality, then it is also the unique equilibrium under runoff.

We know from Proposition 14 (in Appendix A5) that the necessary and sufficient condition for the existence of the Duverger's law equilibrium $(0,0)$ is the complement of condition (5), we also have that if the Duverger's law equilibrium (0,0) exists under runoff, then it also exists under plurality.

Given that condition (6) is satisfied with strict inequality, there exist a range of parameter values for which the sincere voting equilibrium is the unique equilibrium under runoff but it does not exist under plurality. ■

Proposition 7 Given a set of parameter values W, Y, k, and α:

- if the sincere voting equilibrium exists under plurality, it also exists in the first round of a French-style runoff election, but the converse is not true;
- if the sincere voting equilibrium exists in the first round of a French-style runoff election, it is also an equilibrium in the first round of a majority runoff election, but the converse is not true.

Proof. The proof of the first part is a straightforward extension of the proof of Proposition 6.

The proof of the second part is in several steps. First, from the first part of the Proposition and from Proposition 6, we know that if a sincere voting equilibrium exists under plurality, it also exists under both majority runoff and French-style runoff; hence, if $\sigma^{ABC}_a \neq \sigma^{ABC}_b$, then sincere voting equilibrium exists under plurality, majority runoff, and French-style runoff.

Similarly, we know that Duverger's law equilibrium $(1,1)$ exists only if Duverger's law equilibrium $(0,0)$ exists, and b-voters have a stronger incentive to vote sincerely. Therefore, we only need to check the cases for which the Duverger's law equilibrium $(0,0)$ exists in the ABC second round, i.e. $\sigma^{ABC}_a = 0 = \sigma^{ABC}_b$ and b-voters vote sincerely in the first round.

In addition, for ease, define the following term:

Definition 3 *Let $\boldsymbol{v_i}$ be the (realized)* **vote total** *for group i if everyone votes sincerely, such that*

$v_a = \alpha(1-k)q_a$ *and* $v_b = (1-\alpha)(1-k)q_b$.

When $\sigma_a^{ABC} = 0 = \sigma_b^{ABC}$, the sincere voting equilibrium exists under majority runoff if and only if

$$\frac{W}{Y} > \frac{Pr(v_c \in (min\{v_a, v_b\}, v_a + v_b))(1 - p(0,0)}{Pr(v_a > v_b + v_c) + Pr(v_b + v_c > v_a > v_b < v_c)p(0,0) + Pr(v_b + v_c > v_a > v_c < v_b)\frac{\alpha}{2(1-\alpha)}}. \tag{7}$$

The sincere voting equilibrium exists under French-style runoff if and only if

$$\frac{W}{Y} > \frac{Pr(v_c \in (min\{v_a, v_b, \tau\}, v_a + v_b))(1 - p(0,0))}{Pr(v_a > v_b + v_c) + Pr(v_b + v_c > v_a > v_b < min\{v_c, \tau\})p(0,0) + Pr(v_b + v_c > v_a > v_c < min\{v_b, \tau\})\frac{\alpha}{2(1-\alpha)}}. \tag{8}$$

Note that $Pr(v_c \in (min\{v_a, v_b, \tau\}, v_a + v_b)) > Pr(v_c \in (min\{v_a, v_b\}, v_a + v_b))$, and hence the numerator (the change in probability of C winning between strategic and sincere in the first round) is greater under French-style runoff, since C now only needs to beat the threshold to get a 2nd chance in the 2nd round.

In addition, note that $Pr(v_b + v_c > v_a > v_b < min\{v_c, \tau\}) < Pr(v_b + v_c > v_a > v_b < v_c)$ and $Pr(v_b + v_c > v_a > v_c < min\{v_b, \tau\}) < Pr(v_b + v_c > v_a > v_c < v_b)$, meaning the denominator (the probability of A winning if sincere in the first round) is smaller under French-style runoff, since candidate A only receives votes in the second round if it is a two-candidate outcome, and that is now less likely due to the threshold.

Therefore, there exists a range of parameter values for which SVE is an equilibrium under majority runoff, but DLE are the only potential equilibria under French-style runoff.

■

Proposition 8 The prevalence of sincere voting is increasing in the level of the threshold τ.

Proof.

$Pr(v_c \in (min\{v_a, v_b, \tau\}, v_a + v_b))$ is decreasing in the threshold τ, meaning the numerator of equation 8 is decreasing.

$Pr(v_b + v_c > v_a > v_b < min\{v_c, \tau\})$ and $Pr(v_b + v_c > v_a > v_c < min\{v_b, \tau\})$ are increasing in τ, meaning the denominator in equation 8 is increasing.

Therefore, by equation 8, the prevalence of sincere voting is increasing in the level of the threshold τ.

■

Proposition 9 Under plurality, in equilibrium, $c-$voters always vote for candidate C.

Proof. $\frac{W_C}{Y_C} = \frac{1-x_b}{x_b} > 1$.

For a c - voter to vote for B,

$$\frac{W_C}{Y_C} < \frac{Prob[v_a > max\{v_b, v_c\}] - Prob[v_a > v_b + v_c]}{Prob[v_c > max\{v_a, v_b\}]} \tag{9}$$

By assumption of a credible threat, $k \geq (1-k)\alpha$.

Therefore, $Prob[v_a > max\{v_b, v_c\}] \leq Prob[v_c > max\{v_a, v_b\}]$.

The RHS of equation 9 is less than one, and hence can never hold.

■

Proposition 10 Under plurality:

- If $\frac{x_b}{p_B(1,1)-p_A(1,0)-p_B(1,0)} > max\{\frac{1-x_b}{p_A(1,0)}, \frac{1}{p_B(1,0)}\}$, then sincere voting is the unique equilibrium,

- If $\frac{x_b}{p_B(1,1)-p_A(1,0)-p_B(1,0)} \in [\frac{1-x_b}{p_A(1,0)}, \frac{1}{p_B(1,0)}]$, then Duverger's law (1,1) is the unique equilibrium,

- If $\frac{x_b}{p_B(1,1)-p_A(1,0)-p_B(1,0)} \in [\frac{1}{p_B(1,0)}, \frac{1-x_b}{p_A(1,0)}]$, then Duverger's law (0,0) is the unique equilibrium,

- Otherwise, both Duverger law equilibria exist.

Proof. Straightforward extension of the baseline case. ■

Proposition 11 Under plurality, for n sufficiently large:

- both Duverger's law equilibria exist for any values of W, Y, k, and α;

- the sincere voting equilibrium never exists.

Proof. The existence of the Duverger's law equilibrium in which all voters voter for the same majority candidate, say A, is trivial to prove. If all $n-1$ other $b-$groups are voting for candidate A (hence all $a-$groups must also be voting for A), candidate B wins if and only if the following condition is satisfied.

$$\frac{1}{n}(1-k)(1-\alpha)q_b > \frac{n-1}{n}(1-k)(1-\alpha)q_b + (1-k)\alpha q_a.$$

But, for n sufficiently large, this condition cannot be satisfied. Hence, there is no incentive to vote for B.

Sincere voting is an equilibrium only if the following condition is satisfied (it guarantees that $a-$voters prefer to vote for a when they expect other voters to vote sincerely):

$$\frac{W}{Y} > \frac{P(v_c > max\{(1-k)(1-\alpha)q_b, (1-k)\alpha q_a\}) - P(v_c > max\{(1-k)((1-\alpha)q_b + \frac{1}{n}\alpha q_a), \frac{n-1}{n}(1-k)\alpha q_a\})}{P(\frac{n-1}{n}(1-k)\alpha q_a \in [max\{(1-k)((1-\alpha)q_b - \frac{1}{n}\alpha q_a, v_c - \frac{1}{n}\alpha q_a\}, max\{(1-k)((1-\alpha)q_b + \frac{1}{n}\alpha q_a, v_c\}])}.$$

It is easy to see that the right hand side tends to infinity when n grows large due to the convexity of the probabilities. Therefore, for n sufficiently large, this condition cannot be satisfied. ■

Appendix A3: Mixed Strategy Equilibrium

Concerning the mixed strategy, recall that such a mix would need to involve both groups being indifferent between voting sincerely or strategically. This is because if one group is playing a pure strategy, there generically exists a pure strategy best response (see sections 3.1 and 3.2).

In addition, the mixed strategy will only occur when $\frac{W}{Y} \leq \frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_B(1,0)}$ (i.e., when both DLE exist); otherwise, b-voters have a dominant strategy to vote for sincerely regardless of a's behavior.

Formally, a mixed strategy for a-voters this involves $\sigma_b(Wp_A(1,0) - Y(1 - p_A(1,0) - p_B(1,0))) + (1-\sigma_b)(Wp_A(1,1) - Y(1-p_A(1,1))) = \sigma_b(-Y(1-p_A(1,1))) + (1-\sigma_B)(Wp_B(1,0) - Y(1 - p_A(1,0) - p_B(1,0)))$, where σ_i is the probability that group i votes sincerely.

Note that this simplifies to $\sigma_B(W(p_A(1,1) - p_A(1,0) - p_B(1,0)) + 2Y(p_A(1,1) - p_A(1,0) - p_B(1,0))) = W(p_A(1,1) - p_B(1,0)) + Y(p_A(1,1) - p_A(1,0) - p_B(1,0)$.

Therefore, a mixed strategy equilibrium involves $\sigma_b = \frac{W(p_A(1,1)-p_B(1,0))+Y(p_A(1,1)-p_A(1,0)-p_B(1,0))}{(p_A(1,1)-p_A(1,0)-p_B(1,0))(W+2Y)}$ and $1 - \sigma_b = \frac{(p_A(1,1)-p_A(1,0)-p_B(1,0))Y-p_A(1,0)W}{(p_A(1,1)-p_A(1,0)-p_B(1,0))(W+2Y)}$.

Similarly, $\sigma_a = \frac{W(p_A(1,1)-p_A(1,0))+Y(p_A(1,1)-p_A(1,0)-p_B(1,0))}{(p_A(1,1)-p_A(1,0)-p_B(1,0))(W+2Y)}$, $1-\sigma_a = \frac{(p_A(1,1)-p_A(1,0)-p_B(1,0))Y-p_B(1,0)W}{(p_A(1,1)-p_A(1,0)-p_B(1,0))(W+2Y)}$

Hence, we have the following proposition:

Proposition 12 *Under plurality, if* $\frac{W}{Y} \leq \frac{p_A(1,1)-p_A(1,0)-p_B(1,0)}{p_B(1,0)}$, *there exists a mixed-strategy equilibrium with a voters voting sincerely with probability* $\frac{W(p_A(1,1)-p_A(1,0))+Y(p_A(1,1)-p_A(1,0)-p_B(1,0))}{(p_A(1,1)-p_A(1,0)-p_B(1,0))(W+2Y)}$ *and b voters voting sincerely with probability* $\frac{W(p_A(1,1)-p_B(1,0))+Y(p_A(1,1)-p_A(1,0)-p_B(1,0))}{(p_A(1,1)-p_A(1,0)-p_B(1,0))(W+2Y)}$.

Therefore, a mixed strategy equilibrium must involve the larger group (in this case, b) voting strategically at a higher rate than the smaller group. This result is a less stark form of what is found in a standard pivotal voting framework (see, e.g., Myerson & Weber

1993). In the pivotal voting framework, the larger majority group is the only one mixing, while the smaller group votes sincerely. In the ethical voting framework, by comparison, the larger group must vote sincerely at a lesser rate than the smaller group, but still will vote sincerely more than half of the time. The requirement that the larger group vote strategically at a greater rate comes directly from the fact that the smaller group needs more support to be willing to consider voting sincerely, and hence exists in both models; however, this type of coordination breakdown in the ethical voting model can only occur when both groups are still more likely to vote sincerely. Note that this means it is possible for the Condorcet loser to be more likely to win than in the world with a unique equilibrium if this mixed strategy equilibrium occurs.

Appendix A4: Full Type Set

As there are three candidates, there exist six possible ordinal preference orderings, identified by ij, where i is the voter's first choice and j is the voter's second choice. Therefore, a voter identified as type ab features the ordering $a \succ b \succ c$. Let α_{ij} be the proportion of the population of type ij, such that $\sum a_{ij} = 1$.

We use the following definition:

Definition 4 *Let* $\mathbf{v_{ij}}$ *be the (realized)* **vote total** *for voter type* ij *if everyone votes sincerely, such that* $v_{ij} = \alpha_{ij} q_{ij}$.

A voter of type ij will vote sincerely, given all other voters are voting sincerely, if and only if

$$\frac{W}{Y} \geq \frac{Pr[v_{ij} + v_{ik} > max\{v_{ji} + v_{jk}, v_{ki} + v_{kj}\}] - Pr[v_{ik} > max\{v_{ji} + v_{jk} + v_{ij}, v_{ki} + v_{kj}\}]}{Pr[v_{ki} + v_{kj} > max\{v_{ji} + v_{jk} + v_{ij}, v_{ij}\}] - Pr[v_{ki} + v_{kj} > max\{v_{ij} + v_{ik}, v_{ji} + v_{jk}\}]} \tag{10}$$

In other words, equation 10 tells us that a group will vote sincerely if and only if the ratio between their welfare from their first and second choice winning is greater than the ratio in the difference between their marginal impact on defeating their third-choice from voting strategically and their marginal impact on helping their first choice win by voting sincerely. This is a generalization of the equations in Proposition 1 and Proposition 3. Hence, a group will still never vote strategically when there is a larger mass of other voters supporting their first choice than their second choice; otherwise, whether they vote sincerely will depend upon their marginal utilities over the different candidates.

Note that by the definition of vote totals, one can use a mathematical package such as mathematica to show that the right hand side of equation 10 is decreasing in α_{ij} and α_{ik}, increasing in α_{ji} and α_{jk}, and increasing in α_{ki} and α_{kj}. See the online mathematica code.

Proposition 13 *With a full type set, a sincere voting equilibrium will exist if and only if*

$$\frac{W}{Y} \geq max\{\frac{Pr[v_{ij}+v_{ik} > max\{v_{ji}+v_{jk}, v_{ki}+v_{kj}\}] - Pr[v_{ik} > max\{v_{ji}+v_{jk}+v_{ij}, v_{ki}+v_{kj}\}]}{Pr[v_{ki}+v_{kj} > max\{v_{ji}+v_{jk}+v_{ij}, v_{ik}\}] - Pr[v_{ki}+v_{kj} > max\{v_{ij}+v_{ik}, v_{ji}+v_{jk}\}]}\}$$

Also, note that as in the standard framework, a sincere voting equilibrium will be most likely when all six types are relatively equal in size.

Appendix A5: Majority Runoff

Note that in general terms, the decision of whether to vote sincerely is the same under both plurality and runoff. Therefore,

Proposition 14 Under runoff, the sincere voting equilibrium exists if and only if $\frac{W}{Y} \geq \frac{p_A^R(1,1) - p_A^R(1,0) - p_B^R(1,0)}{p_A^R(1,0)}$. Otherwise, the Duverger's law equilibrium (0,0) exists.

Proof. Direct from σ_A^* and σ_B^*, and the fact that $p_A^R(1,0) < p_B^R(1,0)$. ■

The condition in Proposition 14 can be re-written in terms of probabilities in the two rounds:

$$\frac{W}{Y} \geq \frac{(Prob[\frac{k}{1-k}q_c > \alpha q_a + (1-\alpha)q_b] + Prob[\frac{k}{1-k}q_c \in (min\{\alpha q_a, (1-\alpha)q_b\}, \alpha q_a + (1-\alpha)q_b] Prob[\frac{k}{1-k}q_c > \alpha q_a + (1-\alpha)q_b]) - Prob[\frac{k}{1-k}q_c > \alpha q_a + (1-\alpha)q_b]}{Prob[\alpha q_a > \frac{k}{1-k}q_c + (1-\alpha)q_b] + Prob[\alpha q_a \in (min\{\frac{k}{1-k}q_c, (1-\alpha)q_b\}, \frac{k}{1-k}q_c + (1-\alpha)q_b)] Prob[\alpha q_a > \frac{k}{1-k}q_c + (1-\alpha)q_b]},$$

and thus

$$\frac{W}{Y} \geq \frac{Prob[\frac{k}{1-k}q_c \in (min\{\alpha q_a, (1-\alpha)q_b\}, \alpha q_a + (1-\alpha)q_b]}{1 + Prob[\alpha q_a \in (min\{\frac{k}{1-k}q_c, (1-\alpha)q_b\}, \frac{k}{1-k}q_c + (1-\alpha)q_b)]} \frac{Prob[\frac{k}{1-k}q_c > \alpha q_a + (1-\alpha)q_b]}{Prob[\alpha q_a > \frac{k}{1-k}q_c + (1-\alpha)q_b]}$$

There are two comments about this simplification. First, these probabilities can be written in terms of the fundamentals α and k in the same way as under plurality. For expositional clarity, we will not do so here. Second, this formulation leads to the following lemma:

Lemma 5 If $p_A^R(1,0) > p_A(1,0)$, then $p_B^R(1,0) > p_B(1,0)$.

Proof.

First, note that $p_A^R(1,0) = (1+Prob[\alpha q_a \in (min\{\frac{k}{1-k}q_c, (1-\alpha)q_b\}, \frac{k}{1-k}q_c+(1-\alpha)q_b)])Prob[\alpha q_a > \frac{k}{1-k}q_c + (1-\alpha)q_b]$.

Meanwhile, $p_A(1,0) = Prob[\alpha q_a > \frac{k}{1-k}q_c+(1-\alpha)q_b]+Prob[\alpha q_a \in max\{\frac{k}{1-k}q_c, (1-\alpha)q_b\}, \frac{k}{1-k}q_c+(1-\alpha)q_b]$.

Therefore, the relevant inequality can be re-written as: $Prob[\alpha q_a \in (min\{\frac{k}{1-k}q_c, (1-\alpha)q_b\}, \frac{k}{1-k}q_c+(1-\alpha)q_b)]Prob[\alpha q_a > \frac{k}{1-k}q_c + (1-\alpha)q_b] > Prob[\alpha q_a \in max\{\frac{k}{1-k}q_c, (1-\alpha)q_b\}, \frac{k}{1-k}q_c + (1-\alpha)q_b]$.

Now consider the following for B: $Prob[(1-\alpha)q_b \in (min\{\frac{k}{1-k}q_c, \alpha q_a\}, \frac{k}{1-k}q_c + \alpha q_a)]Prob[(1-\alpha)q_b > \frac{k}{1-k}q_c + \alpha q_a] > Prob[(1-\alpha)q_b \in max\{\frac{k}{1-k}q_c, \alpha q_a\}, \frac{k}{1-k}q_c + \alpha q_a]$.

As $1-\alpha > \alpha$, the left-hand side of this equation is larger than B, while the right-hand side is smaller.

Therefore, if the condition holds for A, it also holds for B.

■

A corollary of this lemma is that, if $p_A^R(1,0) > p_A(1,0)$, then $p_C^R(1,0) < p_C(1,0)$.

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