Stable Matching in Large Economies*

Yeon-Koo Che, Jinwoo Kim, Fuhito Kojima†

September 6, 2017

Abstract

We study stability of two-sided many-to-one matching in which firms' preferences for workers may exhibit complementarities. Although such preferences are known to jeopardize stability in a finite market, we show that a stable matching exists in a large market with a continuum of workers, provided that each firm's choice is convex and changes continuously as the set of available workers changes. We also study the existence and the structure of stable matchings under preferences exhibiting substitutability and indifferences in a large market. Building on these results, we show that an approximately stable matching exists in large finite economies. We extend our framework to ensure a stable matching with desirable incentive and fairness properties in the presence of indifferences in firms' preferences.

JEL Classification Numbers: C70, D47, D61, D63.

Keywords: two-sided matching, stability, complementarity, strategy-proofness, large economy

*We are grateful to Nikhil Agarwal, Nick Arnosti, Eduardo Azevedo, Péter Biró, Aaron Bodoh-Creed, David Delacretaz, Pradeep Dubey, Piotr Dworczak, Tadashi Hashimoto, John William Hatfield, Johannes Hörner, Yuichiro Kamada, Michihiro Kandori, Scott Kominers, Ehud Lehrer, Jacob Leshno, Bobak Pakzad-Hurson, Jinjae Park, Parag Pathak, Marcin Peski, Larry Samuelson, Ilya Segal, Rajiv Sethi, Alex Teytelboym, Bob Wilson, and seminar participants at Hanyang, Korea, Kyoto, Paris, Seoul National, Stanford, and Tokyo universities as well as the ASSA Meeting 2015, NBER Market Design Workshop, INFORMS 2014, International Conference on Game Theory at Stony Brook 2014, Monash Market Design Workshop, SAET 2014, Midwest Economic Theory Conference 2014, and the Jacob Marschak lecture at the ESAM 2016, for helpful comments. Taehoon Kim, Janet Lu and Xingye Wu provided excellent research assistance.

†Che: Department of Economics, Columbia University (email: yeonkooche@gmail.com); Kim: Department of Economics, Seoul National University (email: jikim72@gmail.com); Kojima: Department of Economics, Stanford University (email: fuhitokojima1979@gmail.com). We acknowledge financial support from the National Research Foundation through its Global Research Network Grant (NRF-2013S1A2A2035408). Kojima gratefully acknowledges financial support from the Sloan Foundation.

1 Introduction

Since the celebrated work by Gale and Shapley (1962), matching theory has emerged as a central tool for analyzing the design of matching markets. A key concept of the theory is "stability"—the requirement that there be no incentives for participants to "block" (i.e., side-contract around) a prescribed matching. Eliminating blocks keeps markets robust and promotes their long-term sustainability.[1] Even when strategic blocking is not a concern, as in the case of public school matching where schools systems exercise direct control, stability is desirable from the fairness standpoint because it eliminates so-called justified envy: given stability, an agent has no envy toward another unless the latter's partner prefers the envied. In the school choice application, if schools' preferences are determined by test scores or other priorities that a student feels entitled to, eliminating justified envy appears to be important.

Unfortunately, a stable matching exists only under restrictive conditions. It is well known that in two-sided many-to-one matching, stability is not guaranteed unless the preferences of participants—for example, firms—are *substitutable*.[2] In other words, failure of substitutability, or complementarity, can lead to instability. This is a serious problem given the pervasiveness of complementary preferences. Firms often seek to hire workers with complementary skills. For instance, in professional sports leagues, teams demand athletes that complement one another in skills and roles, etc. Some public schools in New York City seek diversity in their student bodies with respect to their skill levels.[3] US colleges tend to assemble classes that are complementary and diverse in terms of their aptitudes, life backgrounds, and demographics. To better organize such markets, one must understand the extent to which stability can be achieved in the presence of such complementarities, or else the applicability of matching theory will remain severely limited.[4]

This paper takes a step forward in accommodating complementarities and other forms

[1]Table 1 in Roth (2002) shows that unstable matching algorithms tend to die out while stable algorithms survive the test of time.

[2]Substitutability here means that a firm's demand for a worker never grows when more workers are available. More precisely, if a firm does not wish to hire a worker from a set of workers, then it never prefers to hire that worker from a larger (in the sense of set inclusion) set of workers. The existence of a stable matching under substitutable preferences is established by Kelso and Crawford (1982), Roth (1985), and Hatfield and Milgrom (2005), while substitutability was shown to be a maximal domain for existence by Sönmez and Ünver (2010), Hatfield and Kojima (2008), and Hatfield and Kominers (2017).

[3] The so-called *Educational option* programs in NYC high schools seek to fill 16% of of their seats with high reading performers (as measured by the score on the 7th grade standardized reading test), 68% of the seats with middle reading performers and the 16% remaining seats with the low reading performers (see Abdulkadiroğlu, Pathak, and Roth (2005)).

[4]In particular, this limitation is important for many decentralized markets that might otherwise benefit from centralization, such as the markets for college and graduate admissions. Decentralized college admissions may entail inefficiencies and lack of fairness (see Che and Koh (2016)). But to centralize such college admissions, one must know how to deal with potential instability arising from complementary preferences by colleges.

of general preferences. In light of the general impossibility, this requires us to weaken the notion of stability in some way. Our approach is to consider a large market. Specifically, we consider a market that consists of a continuum of workers/students on one side and a finite number of firms/colleges with continuum of capacities on the other. We then ask whether stability can be achieved in an "asymptotic" sense—i.e., whether participants' incentives for blocking disappear as the economy grows large and approaches the continuum economy in the limit. Such a weakening preserves the original spirit of stability: as long as the incentive for blocking is sufficiently weak, the instability and fairness concerns will not be serious enough to jeopardize the mechanism.

Large market models are also of interest since many real world matching markets are large. School choice in a typical urban setting involves tens of thousands of students. Medical matching involves about 35,000 and 9,000 doctors in the US and Japan, respectively. Aside from complementary preferences, a large market model also allows us to address some outstanding issues in finite markets. One such issue is multiplicity of stable matchings. While the set of stable matchings can be large in finite economies, there is a sense in which the set shrinks as the market grows large. Indeed, Azevedo and Leshno (2016) establish that a stable matching is generically unique in a continuum economy when firms have so-called *responsive* preferences, a special case of substitutable preferences. To what extent such a result generalizes to more general preferences is an interesting issue that can be explored in a large market setting.

Our main model considers a continuum economy with a finite number of firms and a continuum of workers. Each worker may match with at most one firm and has strict preference orders over alternative firms. Firms may match with a group (or mass) of workers, and we assume general preferences over groups of workers. Importantly, their preferences may exhibit complementarities. Our model includes the setup of Azevedo and Leshno (2016) as a special case, which assumes that firms have responsive preferences. In addition, we allow firms to be indifferent over different groups of workers. Indifferences may arise from firms' limited observation about workers' characteristics or their unwillingness/inability by law to distinguish workers based on some characteristics. Indifferences are particularly common in school choice, for schools apply coarse priorities to ration their seats,[5] in which case school preferences encoding the priorities will exhibit indifference over students. Formally, we represent a firm's preferences by a *choice correspondence* defined over measures of worker types that may be potentially infinite. A matching is then defined as measures of worker types assigned to alternative firms and is said to be stable if it is not blocked by any firms or workers by themselves or via a coalition.

[5]In the public school choice program in Boston prior to 2005, for instance, a student's priority at a school was based only on broad criteria, such as the student's residence and whether any siblings were currently enrolled at that school. Consequently, at each school, many students were assigned the same priority (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005).

Our first result is to characterize a stable matching as a fixed point of a suitably defined correspondence over measures of workers available to firms. The characterizing correspondence is reminiscent of the tâtonnement process, in that it iterates each profile of worker types (in measure) available to firms to a new profile of available workers after processing firms' optimal choice on the former profile. While a fixed point characterization is standard in matching theory, our characterization is distinguished by the domain as well as the form of the characterizing correspondence. Our correspondence is defined over measure space, a rich functional space, unlike the standard approach. Further, the indifferences allowed for within and across worker types present subtle issues in its construction, which causes the construction to differ from those used in the existing matching literature, including Adachi (2000), Hatfield and Milgrom (2005) and Echenique and Oviedo (2006).

Using our characterization, we establish existence of a stable matching in general environments. First, we show that a stable matching exists if firms' preferences exhibit continuity, more precisely if each firm's choice correspondence is upper hemicontinuous and convex-valued. This result is quite general because these conditions are satisfied by a rich class of preferences—including those exhibiting complementarities.[6] The existence is established by means of the Kakutani-Fan-Glicksberg fixed point theorem—a generalization of Kakutani's fixed point theorem to functional spaces—which is new to the matching literature to the best of our knowledge.

Second, we obtain existence under the assumption of substitutable preferences for firms. The logic of this result is familiar. Namely, substitutability means that firms reject more workers as more workers become available to them. This feature gives rise to monotonicity of our characterization map. While such monotonicity is known to admit a fixed point, the generality of our model with choice correspondence makes it nontrivial to identify the exact forms of substitutable preferences required for existence.[7] We identify two different types of substitutable preferences with indifferences—a weak form leading to existence of *a* stable matching and a strong form leading to existence of side-optimal (i.e., firm-optimal and worker-optimal) stable matchings. We also identify a condition under which a side-optimal stable matching can be found via a generalized Gale-Shapley algorithm. Finally, we also find a condition, richness, that guarantees uniqueness of the stable matching, thus generalizing the uniqueness result of Azevedo and Leshno (2016) beyond the special case of responsive preferences. The richness delivers uniqueness under a full support assumption when firms have responsive preferences but face general forms of group-specific quotas (e.g., affirmative

[6]For instance, it allows for Leontief-type preferences with respect to alternative types of workers, in which firms desire to hire each type of workers in equal size.

[7]If a firm's preferences are responsive, an arbitrary resolution of indifferences—or tie-breaking—preserves responsiveness and thus implies existence. For more general preferences, however, a random or arbitrary tie-breaking of indifferences does not necessarily lead to a choice function that possesses necessary properties for existence.

actions).

We next draw implications of our results from a continuum economy for "nearby" large finite economies, assuming that each firm has a continuous utility function over the measure of workers it matches with. Specifically, we show that any large finite economy that is sufficiently close to our continuum economy (in terms of the distribution of worker types) admits a matching that is *approximately stable* in the sense that the incentives for blocking are arbitrarily small. The converse also holds: namely, if any approximately stable matchings defined over a sequence of large finite economies converge to a matching in the limit continuum economy, then the limit matching constitutes an (exact) stable matching in the continuum economy. In addition, the structure of approximately stable matchings—side-optimal stable matchings as well as uniqueness—in large finite economies are well approximated by that of the stable matchings in the continuum economy. Our results thus suggest the usefulness of the continuum economy as a tool for studying large finite economies.

Finally, we study fairness and incentive properties of matching. Stability eliminates justified envy and as such protects workers from being discriminated by a firm against the workers it perceives as *less desirable*. But stability alone is silent on how fair a matching is in treating workers that are perceived by a firm as *equivalent*. This issue is particularly relevant in school choice since schools evaluate students based on coarse priorities. Kesten and Ünver (2014) show that, given responsive preferences by schools (i.e., firms in our model), it is possible to implement a matching that eliminates discrimination among students enjoying the same priority. We show that this stronger notion of fairness can be achieved *even with general preferences*, either in a large economy or in a finite but "time-share" model in which schools/firms and students/workers can share time or match probabilistically in a stable manner in a finite economy (see Sotomayor (1999), Alkan and Gale (2003), and Kesten and Ünver (2014), among others).

The remainder of this paper is organized as follows. Section 2 presents an example to illustrate our main contributions. Section 3 describes a matching model in the continuum economy. Section 4 provides a fixed-point characterization of stable matchings in the continuum economy. Sections 5 and 6 provide the existence of a stable matching under continuous and substitutable preferences, respectively. In Section 7, we explore implications of our existence results for approximately stable matchings in large finite economies. In Section 8, we investigate fairness and strategy-proofness. In Section 9, we discuss the related literature. Section 10 concludes.

2 Illustrative Example

Before proceeding, we illustrate the main contribution of our paper with an example. We first illustrate how complementary preferences may lead to the non-existence of a stable matching when there is a finite number of agents. To this end, suppose that there are two firms, f_1 and f_2, and two workers, θ and θ'. The agents have the following preferences:

$$\theta : f_1 \succ f_2; \qquad\qquad f_1 : \{\theta, \theta'\} \succ \emptyset;$$
$$\theta' : f_2 \succ f_1; \qquad\qquad f_2 : \{\theta\} \succ \{\theta'\} \succ \emptyset.$$

In other words, worker θ prefers f_1 to f_2, and worker θ' prefers f_2 to f_1; firm f_1 prefers employing both workers to employing neither, which the firm in turn prefers to employing only one of the workers; and firm f_2 prefers worker θ to θ', which it in turn prefers to employing neither. Firm f_1 has a "complementary" preference, which creates instability. To illustrate this, recall that stability requires that there be no blocking coalition. Due to f_1's complementary preference, it must employ either both workers or neither in any stable matching. The former case is unstable because worker θ' prefers firm f_2 to firm f_1, and f_2 prefers θ' to being unmatched, so θ' and f_2 can form a blocking coalition. The latter case is also unstable because f_2 will only hire θ in that case, which leaves θ' unemployed; this outcome will be blocked by f_1 forming a coalition with θ and θ' that will benefit all members of the coalition.

Can stability be restored if the market becomes large? If the market remains finite, the answer is no. To illustrate this proposition, consider a scaled-up version of the above model: there are q workers of type θ and q workers of type θ', and they have the same preferences as previously described. Firm f_2 prefers type-θ workers to type-θ' workers and wishes to hire in that order but at most a total of q workers. Firm f_1 has a complementary preference for hiring identical numbers of type-θ and type-θ' workers (with no capacity limit). Formally, if x and x' are the numbers of available workers of types θ and θ', respectively, then firm f_1 would choose $\min\{x, x'\}$ workers of each type.

When q is odd (including the original economy, where $q = 1$), a stable matching does not exist.[8] To illustrate this, first note that if firm f_1 hires more than $q/2$ workers of each type, then firm f_2 has a vacant position to form a blocking coalition with a type-θ' worker, who prefers f_2 to f_1. If f_1 hires fewer than $q/2$ workers of each type, then some workers will remain unmatched (because f_2 hires at most q workers). If a type-θ worker is unmatched, then f_2 will form a blocking coalition with that worker. If a type-θ' worker is unmatched, then firm f_1 will form a blocking coalition by hiring that worker and a θ worker (possibly matched with f_2).

Consequently, "exact" stability is not guaranteed, even in a large finite market. Nevertheless, one may hope to achieve approximate stability. This is indeed the case with the above example; the "magnitude" of instability diminishes as the economy grows large. To illustrate this, let q be odd and consider a matching in which f_1 hires $\frac{q+1}{2}$ workers of each

[8]We sketch the argument here; Section S.1 of Supplementary Material provides the argument in fuller form. When q is even, a matching in which each firm hires $\frac{q}{2}$ of each type of workers is stable.

type, whereas f_2 hires $\frac{q-1}{2}$ workers of each type. This matching is unstable because f_2 has one vacant position it wants to fill, and there is a type-θ' worker who is matched to f_1 but prefers f_2. However, this is the only possible block of this matching, and it involves only one worker. As the economy grows large, if the additional worker becomes insignificant for firm f_2 relative to its size, which is what the continuity of a firm's preference captures, then the payoff consequence of forming such a block must also become insignificant, which suggests that the instability problem becomes insignificant as well.

This can be seen most clearly in the limits of the above economy. Suppose there is a unit mass of workers, half of whom are type θ and the other half of whom are type θ'. Their preferences are the same as described above. Suppose firm f_1 wishes to maximize $\min\{x, x'\}$, where x and x' are the measures of type-θ and type-θ' workers, respectively. Firm f_2 can hire at most $\frac{1}{2}$ of the workers, and it prefers to fill as much of this quota as possible with type-θ workers and fill the remaining quota with type-θ' workers. In this economy, there is a (unique) stable matching in which each firm hires exactly one-half of the workers of each type. To illustrate this, note that any blocking coalition involving firm f_1 requires taking away a positive—and identical—measure of type-θ' and type-θ workers from firm f_2, which is impossible because type-θ' workers will object to it. Additionally, any blocking coalition involving firm f_2 requires that a positive measure of type-θ workers be taken away from firm f_1 and replaced by the same measure of type-θ' workers in its workforce, which is impossible because type-θ workers will object to it. Our analysis below will demonstrate that the continuity of firms' preferences, which will be defined more clearly, is responsible for guaranteeing the existence of a stable matching in the continuum economy and approximate stability in the large finite economies in this example.

3 Model of a Continuum Economy

Agents and their measures. There is a finite set $F = \{f_1, \ldots, f_n\}$ of firms and a mass of workers. Let $\varnothing$ be the null firm, representing the possibility of workers not being matched with any firm, and define $\tilde{F} := F \cup \{\varnothing\}$. The workers are identified with types $\theta \in \Theta$, where Θ is a compact metric space with metric d^{Θ}. Let Σ denote a Borel σ-algebra of space Θ. Let $\overline{\mathcal{X}}$ be the set of all nonnegative measures such that for any $X \in \overline{\mathcal{X}}$, $X(\Theta) \leqslant 1$. Assume that the entire population of workers is distributed according to a nonnegative (Borel) measure $G \in \overline{\mathcal{X}}$ on (Θ, Σ). In other words, for any $E \in \Sigma$, $G(E)$ is the measure of workers belonging to E. For normalization, assume that $G(\Theta) = 1$. To illustrate, the limit economy of the example from the previous section is a continuum economy with $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\{\theta\}) = G(\{\theta'\}) = 1/2$.[9] In the sequel, we shall use this as our leading example for

[9]Henceforth, given any measure X, $X(\theta)$ will denote a measure of the singleton set $\{\theta\}$ to simplify notation.

purposes of illustrating the various concepts we develop.

Any subset of the population or **subpopulation** is represented by a nonnegative measure X on (Θ, Σ) such that $X(E) \leqslant G(E)$ for all $E \in \Sigma$.[10] Let $\mathcal{X} \subset \overline{\mathcal{X}}$ denote the set of all subpopulations. We further say that a nonnegative measure $\tilde{X} \in \mathcal{X}$ is a **subpopulation of** $X \in \mathcal{X}$, denoted as $\tilde{X} \sqsubset X$, if $\tilde{X}(E) \leqslant X(E)$ for all $E \in \Sigma$. We let $\mathcal{X}_X$ denote the set of all subpopulations of X. Note that $(\mathcal{X}, \sqsubset)$ is a partially ordered set.[11]

Given the partial order $\sqsubset$, for any $X, Y \in \mathcal{X}$, we define $X \vee Y$ (join) and $X \wedge Y$ (meet) to be the supremum and infimum of X and Y, respectively.[12] Also, for any $\mathcal{X}' \subset \mathcal{X}$, let $\bigvee \mathcal{X}'$ and $\bigwedge \mathcal{X}'$ denote the supremum and infimum of $\mathcal{X}'$, which exist according to the next lemma.

Lemma 1. *The partially ordered set $(\mathcal{X}, \sqsubset)$ is a complete lattice.*

Proof. See Section S.2.1 of Supplementary Material. ▮

The join and meet of X and Y in $\mathcal{X}$ can be illustrated with examples. Let $X = (x, x')$ and $Y = (y, y')$ be two measures in our leading example, where x and x' are the measures of types θ and θ', respectively, under X, and likewise y and y' under Y. Then, their join and meet are respectively measures $X \vee Y = (\max\{x, y\}, \max\{x', y'\})$ and $X \wedge Y = (\min\{x, y\}, \min\{x', y'\})$.

Next, consider a continuum economy with type space $\Theta = [0, 1]$ and suppose the measure G admits a bounded density g for all $\theta \in [0, 1]$. In this case, it easily follows that for $X, Y \sqsubset G$, their densities x and y are well defined.[13] Then, their join $Z = (X \vee Y)$ and meet $Z' = (X \wedge Y)$ admit densities z and z' defined by $z(\theta) = \max\{x(\theta), y(\theta)\}$ and $z'(\theta) = \min\{x(\theta), y(\theta)\}$ for all θ, respectively. As usual, for any two measures $X, Y \in \mathcal{X}$, $X + Y$ and $X - Y$ denote their sum and difference, respectively.

Consider the space of all (signed) measures (of bounded variation) on (Θ, Σ). We endow this space with a weak-$*$ topology and its subspace $\mathcal{X}$ with the relative topology. Given a sequence of measures (X_k) and a measure X on (Θ, Σ), we write $X_k \xrightarrow{w*} X$ to indicate that (X_k) converges to X as $k \to \infty$ under weak-$*$ topology and simply say that (X_k) **weakly converges** to X.[14]

[10] In case of finitely many types, we will use "measure" and "mass" interchangeably.

[11] Reflexivity, transitivity and antisymmetry of the order are easy to check.

[12] For instance, $X \vee Y$ is the smallest measure of which both X and Y are subpopulations. It can be shown that, for all $E \in \Sigma$,

$$(X \vee Y)(E) = \sup_{D \in \Sigma} X(E \cap D) + Y(E \cap D^c).$$

[13] $|X([0, \theta']) - X([0, \theta])| \leqslant |G([0, \theta']) - G([0, \theta])| \leqslant \bar{g}|\theta' - \theta|$, where $\bar{g} := \sup_s g(s)$. Thus, X is Lipschitz continuous, and its density is well defined.

[14] We use the term "weak convergence" because it is common in statistics and mathematics, although weak-$*$ convergence is a more appropriate term from the perspective of functional analysis. As is well known, $X_k \xrightarrow{w*} X$ if $\int_\Theta h dX_k \to \int_\Theta h dX$ for all bounded continuous functions h. See Theorem 12 in Appendix A for some implications of this convergence.

Agents' preferences. We now describe agents' preferences. Each worker is assumed to have a strict preference over $\tilde{F}$. Let a bijection $P : \{1, ..., n+1\} \to \tilde{F}$ denote a worker's preference, where $P(j)$ denotes the identity of the worker's j-th best alternative, and let $\mathcal{P}$ denote the (finite) set of all possible worker preferences.

We write $f \succ_P f'$ to indicate that f is strictly preferred to f', according to P. (We sometimes write $f \succ_\theta f'$ to express the preference of a particular type θ.) For each $P \in \mathcal{P}$, let $\Theta_P \subset \Theta$ denote the set of all worker types whose preference is given by P, and assume that Θ_P is measurable and $G(\partial\Theta_P) = 0$, where $\partial\Theta_P$ denotes the boundary of Θ_P.[15] Because all worker types have strict preferences, Θ can be partitioned into the sets in $\mathcal{P}_\Theta := \{\Theta_P : P \in \mathcal{P}\}$.

We next describe firms' preferences. We do so indirectly by defining a firm f's **choice correspondence** $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$, where $C_f(X) \subset \mathcal{X}_X$ is a nonempty set of subpopulations of X for any $X \in \mathcal{X}$.[16] We assume that C_f satisfies the **revealed preference property**: for any $X, X' \in \mathcal{X}$ with $X \sqsubset X'$, if $C_f(X') \cap \mathcal{X}_X \neq \emptyset$, then $C_f(X) = C_f(X') \cap \mathcal{X}_X$.[17] Let $R_f : \mathcal{X} \rightrightarrows \mathcal{X}$ be a **rejection correspondence** defined by $R_f(X) := \{Y \in \mathcal{X} | Y = X - X' \text{ for some } X' \in C_f(X)\}$. By convention, we let $C_\emptyset(X) = \{X\}, \forall X \in \mathcal{X}$, meaning that $R_\emptyset(X)(E) = 0$ for all $X \in \mathcal{X}$ and $E \in \Sigma$. We will call C_f (resp,. R_f) a firm f's **choice** (resp., **rejection**) **function** if $|C_f(X)| = 1$ for all $X \in \mathcal{X}$. In this case, we slightly abuse notation to write a unique outcome of function without the set notation.

In our leading example, the choice functions of firms f_1 and f_2 are given respectively by

$$C_{f_1}(x_1, x_1') = (\min\{x_1, x_1'\}, \min\{x_1, x_1'\}) \tag{1}$$

$$C_{f_2}(x_2, x_2') = \left(x_2, \min\left\{\tfrac{1}{2} - x_2, x_2'\right\}\right), \tag{2}$$

when $x_i \in [0, \frac{1}{2}]$ of type-θ workers and $x_i' \in [0, \frac{1}{2}]$ of type-θ' workers are available to firm f_i, $i = 1, 2$.

In sum, a continuum economy is summarized as a tuple $\Gamma = (G, F, \mathcal{P}_\Theta, C_F)$.

Matchings, and their efficiency and stability requirements. A **matching** is $M = (M_f)_{f \in \tilde{F}}$ such that $M_f \in \mathcal{X}$ for all $f \in \tilde{F}$ and $\sum_{f \in \tilde{F}} M_f = G$. Firms' choice correspondences

[15]This is a technical assumption that facilitates our analysis. The assumption is satisfied if, for each $P \in \mathcal{P}$, Θ_P is an open set such that $G(\cup_{P \in \mathcal{P}} \Theta_P) = G(\Theta)$: all agents, except for a measure-zero set, have strict preferences, a standard assumption in matching theory literature. The assumption that $G(\partial\Theta_P) = 0$ is also satisfied if Θ is discrete. To see it, note that $\partial E := \overline{E} \cap \overline{E^c}$, where $\overline{E}$ and $\overline{E^c}$ are the closures of E and E^c, respectively. Then, we have $\overline{E} = E$ and $\overline{E^c} = E^c$, so $\overline{E} \cap \overline{E^c} = E \cap E^c = \emptyset$. Hence, the assumption is satisfied.

[16] Taking firms' choices as a primitive offers flexibility with regard to the preferences over alternatives that are not chosen. This approach is also adopted by other studies in matching theory, which include Alkan and Gale (2003) and Aygün and Sönmez (2013), among others.

[17]This property must hold if the choice is made by a firm optimizing with a well-defined preference relation. The property is often invoked in the matching theory literature (see Hatfield and Milgrom (2005), Fleiner (2003), and Alkan and Gale (2003)). Recently, Aygün and Sönmez (2013) have clarified the role of this property in the context of matching with contracts.

can be used to define a binary relation describing firms' preferences over matchings. For any two matchings, M and M', we say that firm f prefers M'_f to M_f if $M'_f \in C_f(M'_f \vee M_f)$, and write $M'_f \succeq_f M_f$.[18] We also say that f strictly prefers M'_f to M_f if $M'_f \succeq_f M_f$ holds while $M_f \succeq_f M'_f$ does not, and write $M'_f \succ_f M_f$. The resulting preference relation amounts to taking a minimal stance on the firms' preferences, limiting attention to those revealed via their choices. Given this preference relation, we denote $M' \succeq_F M$ if $M'_f \succeq_f M_f$ for all $f \in F$. Also, $M' \succ_F M$ if $M' \succeq_F M$ and $M'_f \succ_f M_f$ for some $f \in F$.

To discuss workers' welfare, fix any matching M and any firm f. Let

$$D^{\succeq f}(M) := \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \succeq_P f} M_{f'}(\Theta_P \cap \cdot) \text{ and } D^{\preceq f}(M) := \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap \cdot) \quad (3)$$

denote the measure of workers assigned to firm f or better (according to their preferences) and the measure of workers assigned to firm f or worse (again, according to their preferences), respectively, where $M_{f'}(\Theta_P \cap \cdot)$ denotes a measure that takes the value $M_{f'}(\Theta_P \cap E)$ for each $E \in \Sigma$. Starting from M as a default matching, the latter measures the number of workers who would rather match with f. Meanwhile, the former measure is useful for characterizing the workers' overall welfare. For any two matchings M and M', we say that $M' \succeq_\Theta M$ if $D^{\succeq f}(M) \sqsubset D^{\succeq f}(M'), \forall f \in \tilde{F}$ and $M' \succ_\Theta M$ if $M' \succeq_\Theta M$ and $D^{\succeq f}(M) \neq D^{\succeq f}(M')$ for some $f \in \tilde{F}$.[19] In other words, for each firm f, if the measure of workers assigned to f or better is larger in one matching than in the other, then we can say that the workers' overall welfare is higher in the former matching.

Equipped with these notions, we can define Pareto efficiency and stability.

Definition 1. A matching M is **Pareto efficient** if there is no matching $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$, and **weakly Pareto efficient** if there is no matching M' such that $M' \succ_F M$ and $M' \succ_\Theta M$.[20]

Definition 2. A matching M is **stable** if

1. (Individual Rationality) For each $f \in F$, $M_f \in C_f(M_f)$; for each $P \in \mathcal{P}$, $M_f(\Theta_P) = 0, \forall f \prec_P \emptyset$; and

2. (No Blocking Coalition) No $f \in F$ and $M'_f \in \mathcal{X}$ exist such that $M'_f \sqsubset D^{\preceq f}(M)$ and $M'_f \succ_f M_f$.

[18] This is known as the Blair order in the literature. See Blair (1984).

[19] Note that this comparison is made in the aggregate matching sense, without keeping track of the identities of workers who get better off with M'.

[20] In the definition of Pareto efficiency, the condition that $M' \succeq_\Theta M$ and $M' \neq M$ implies that at least some workers are strictly better off under M' since workers have strict preferences, and hence M' Pareto dominates M (though all firms may be indifferent between M and M').

Condition 1 requires that no firm wish to unilaterally drop any of its matched workers and that each matched worker prefer being matched to being unmatched.[21] Condition 2 requires that there be no firm and no set of workers who are not matched together but prefer to be. When Condition 2 is violated by f and M'_f, we say that f and M'_f **block** M.

Remark 1 (Equivalence to group stability)**.** We say that a matching M is **group stable** if Condition 1 of Definition 2 holds and,

2'. *There are no* $F' \subseteq F$ *and* $M'_{F'} \in \mathcal{X}^{|F'|}$ *such that* $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M)$ for all $f \in F'$.

This definition strengthens our stability concept because it requires that matching be immune to blocks by coalitions that potentially involve multiple firms. Such stability concepts with coalitional blocks are analyzed by Sotomayor (1999), Echenique and Oviedo (2006), and Hatfield and Kominers (2017), among others.[22] It is easy to see in our context that a matching is stable if and only if it is group stable.[23]

As in the standard finite market, stability implies Pareto efficiency:

Proposition 1. *Any stable matching is weakly Pareto efficient, and Pareto efficient if each* C_f *is a choice function.*

Proof. See Section S.2.2 of Supplementary Material. ∎

4 A Characterization of Stable Matching

This section characterizes stable matchings, which will serve as a tool for establishing their existence in the subsequent sections. Stability exhausts the opportunities for blocking for

[21] We note that the first part of Condition 1 (namely $M_f \in C_f(M_f)$ for each $f \in F$) is implied by Condition 2. To see this, suppose $M_f \notin C_f(M_f)$. Let $M'_f \in C_f(M_f)$. Then, $M'_f \sqsubset M_f \sqsubset D^{\preceq f}(M)$, and also $M'_f \succ_f M_f$, violating Condition 2. We opted to write that condition to follow the convention in the literature and ease the exposition.

[22] By requiring $M'_f \sqsubset D^{\preceq f}(M)$ for all $f \in F'$ in Condition 2', our group stability concept implicitly assumes that workers who consider joining a blocking coalition with $f \in F'$ use the current matching $(M_{f'})_{f' \neq f}$ as a reference point. This means that workers are available to firm f as long as they prefer f to their current matching. However, given that a more preferred firm $f' \in F'$ may be making offers to workers in $D^{\preceq f}(M)$ as well, the set of workers available to f may be smaller. Such a consideration would result in a weaker notion of group stability. Any such concept, however, will be equivalent to our notion of stability because, as shown in footnote 23, even the most restrictive notion of group stability—the concept using $D^{\preceq f}(M)$ in Condition 2'—is equivalent to stability, while stability is weaker than any group stability concept described above.

[23] Clearly, any group stable matching is stable, because if Condition 2 is violated by a firm f and M'_f, then Condition 2' is violated by a singleton set $F' = \{f\}$ and $M'_{\{f\}}$. The converse also holds. To see why, note that if Condition 2' is violated by $F' \subseteq F$ and $M'_{F'}$, then Condition 2 is violated by any $f \in F'$ and M'_f because $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M)$, by assumption.

all firms, which requires each firm to choose optimally from the workers "available" to that firm. Hence, to identify a stable matching, one must identify *the set of workers available to each firm*. But this is inherently of "fixed-point" character, since the availability of a worker to a firm depends on the set of firms willing to match with her, but that set depends in turn on firms' optimization given the workers "available" to them.

The preceding logic suggests that a stable matching is associated with a fixed point of a mapping—or more intuitively, a stationary point of a process that repeatedly revises the set of available workers to the firms based on the preferences of the workers and the firms. Formally, we define a map $T : \mathcal{X}^{n+1} \rightrightarrows \mathcal{X}^{n+1}$ such that for each $X \in \mathcal{X}^{n+1}$,

$$T(X) := \Big\{ \tilde{X} \in \mathcal{X}^{n+1} \,\Big|\, \text{there exists } (Y_f)_{f \in \tilde{F}} \text{ with } Y_f \in R_f(X_f), \forall f \in \tilde{F}, \text{ such that } \tilde{X}_f(\cdot) = \sum_{P:P(1)=f} G(\Theta_P \cap \cdot) + \sum_{P:P(1)\neq f} Y_{f_-^P}(\Theta_P \cap \cdot), \forall f \in \tilde{F} \Big\}, \quad (4)$$

where $f_-^P \in \tilde{F}$, called the **immediate predecessor** of f at P, is a firm that is ranked immediately above firm f according to P.[24] This mapping takes a profile X of available workers as input and returns a nonempty set of profiles of available workers. For each $X \in \mathcal{X}^{n+1}, T(X)$ is nonempty because $R_f(X)$ is nonempty for each $f \in \tilde{F}$ by assumption.

To explain, fix a firm f. Consider first the worker types Θ_P who rank f as their first-best choice (i.e., $f = P(1)$). All such workers are available to f, which explains the first term of (4). Consider next the worker types Θ_P who rank f as their second-best choice (i.e., $f = P(2)$). Within this group, only the workers rejected by their top-choice firm $P(1) = f_-^P$ are available to f, which explains the second term of (4). Now, consider the worker types Θ_P who rank f as their third-best choice (i.e., $f = P(3)$). Within this group, only the workers rejected by *both* their first- and second-choice firms, that is, $P(1)$ and $P(2)$, would be available to f. To calculate the measure of these workers, however, one may focus on those available to and rejected by $P(2) = f_-^P$, since, by the previous observation, the workers available to $P(2)$ are those who are already rejected by $P(1)$. This explanation analogously applies to all the firms going down workers' rank order lists.

The map T can be interpreted as a tâtonnement process in which an auctioneer iteratively quotes firms' "budgets" (in terms of the measures of available workers). As in a classical Walrasian auction, the budget quotes are revised based on the preferences of the market participants, reducing the budget for firm f (i.e., making a smaller work force available) when more workers are demanded by the firms ranked above f and increasing the budget otherwise. Once the process converges, one reaches a fixed point, having found the workers who are "truly" available to firms—those who are compatible with the preferences of all market participants.

[24]Formally, $f_-^P \succ_P f$ and $f' \succeq_P f_-^P$ for any $f' \succ_P f$.

Remark 2. The mapping T can be seen as mimicking Gale and Shapley's deferred acceptance algorithm (DA), in particular the worker-proposing one. To see this, consider the case in which each C_f is a choice function. Then, we can write T as a profile $(T_f)_{f\in\tilde{F}}$, where, for each $X \in \mathcal{X}^{n+1}$,

$$T_f(X) = \sum_{P:P(1)=f} G(\Theta_P \cap \cdot) + \sum_{P:P(1)\neq f} R_{f_-^P}(X_{f_-^P})(\Theta_P \cap \cdot). \tag{5}$$

For each firm f, this mapping returns the workers who are rejected by an immediate predecessor of f. These are analogous to the workers who propose to firm f in the worker-proposing DA algorithm, since they are those rejected by the immediate predecessor. Indeed, this analogy becomes precise when the firms' preferences are substitutable (that is, each R_f is monotonic): each iteration of the mapping T (starting from zero subpopulations) coincides with the cumulative measures of workers proposing at a corresponding step of worker-proposing DA. This result is shown in Section S.3 of Supplementary Material. Our fixed-point mapping resembles those developed in the context of finite matching markets (e.g., see Adachi (2000), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006)), but the construction here differs since a continuum of workers draw their types from a very rich space and they are treated in aggregate terms without being distinguished by their identities.

We now present our main characterization theorem.

Theorem 1. *There exists a stable matching M with $X_f = D^{\preceq f}(M), \forall f \in \tilde{F}$ if and only if $(X_f)_{f\in\tilde{F}}$ is a fixed point of T (i.e., $X \in T(X)$).*

Proof. See Appendix A. ▌

This characterization identifies the measures of workers available to firms as a fixed point of T. A stable matching is then obtained as firms' optimal choices from these measures.[25] This process is illustrated in the next example.

Example 1. Consider our leading example with a continuum of workers in Section 2. The candidate measures of available workers are denoted by a tuple $X = (X_{f_1}, X_{f_2}) = (x_1, x_1', x_2, x_2') \in [0, \frac{1}{2}]^4$, where $X_{f_i} = (x_i, x_i')$ is the measures of type θ and type θ' workers available to f_i. Since f_1 is the top choice for θ and f_2 is the top choice for θ', according to our T mapping, all of these workers are available to the respective firms. Thus, without loss we can set $x_1 = G(\theta) = \frac{1}{2}$ and $x_2' = G(\theta') = \frac{1}{2}$ and consider $(\frac{1}{2}, x_1', x_2, \frac{1}{2})$ as our candidate

[25]Importantly, an arbitrary selection from $C_f(X_f)$ for each $f \in F$ at the fixed point X need not lead to a matching, let alone a stable one. Care is needed to construct a stable matching. Equation (13) in Appendix A provides a precise formula to obtain a stable matching M from a fixed point X of T. We thank a referee for raising a question that led us to clarify this issue.

measures. The firms' choice functions are then given by (1) and (2) while the fixed-point mapping in (5) is given by

$$T_{f_1}(X) = \left(\tfrac{1}{2}, R_{f_2}(x_2, \tfrac{1}{2})(\theta')\right) = \left(\tfrac{1}{2}, x_2\right) \tag{6}$$
$$T_{f_2}(X) = \left(R_{f_1}(\tfrac{1}{2}, x_1')(\theta), \tfrac{1}{2}\right) = \left(\tfrac{1}{2} - x_1', \tfrac{1}{2}\right). \tag{7}$$

Thus, $(x_1, x_1'; x_2, x_2')$ is a fixed point of T if and only if $(x_1, x_1'; x_2, x_2') = (\frac{1}{2}, x_2; \frac{1}{2} - x_1', \frac{1}{2})$, or $x_1 = x_2' = \frac{1}{2}$ and $x_1' = x_2 = \frac{1}{4}$. The optimal choice by each firm from the fixed point then gives a matching

$$M = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{4}\theta + \frac{1}{4}\theta' & \frac{1}{4}\theta + \frac{1}{4}\theta' \end{pmatrix},$$

where the notation here indicates that each of the firms f_1 and f_2 is matched to a mass $\frac{1}{4}$ of worker types θ and θ' (we will use an analogous notation throughout). This matching M is stable.

In light of Theorem 1, existence of a stable matching reduces to the existence of a fixed point of T. The next two sections identify two sufficient conditions for the latter.

5 The Existence of a Stable Matching in the Continuum Economy

Based on our characterization result, we now present the main existence result under the standard continuity assumption on the firms' choice correspondences. We say that firm f's choice correspondence C_f is **upper hemicontinuous** if, for any sequences $(X^k)_{k\in\mathbb{N}}$ and $(\tilde{X}^k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ such that $X^k \xrightarrow{w^*} X$, $\tilde{X}^k \xrightarrow{w^*} \tilde{X}$, and $\tilde{X}^k \in C_f(X^k), \forall k$, we have $\tilde{X} \in C_f(X)$.[26] As suggested by the name, the upper hemicontinuity means that a firm's choice changes continuously with the distribution of available workers. We say that C_f is **convex-valued** if $C_f(X)$ is a convex set for any $X \in \mathcal{X}$.[27]

Definition 3. Firm $f \in F$ has a **continuous preference** if C_f is upper hemicontinuous and convex-valued.

Many complementary preferences are compatible with continuous preferences. Recall Example 1, for instance, in which firm f_1 has a Leontief-type preference: it wishes to hire an equal number of workers of types θ and θ' (specifically, the firm wants to hire type-θ workers

[26] This definition is often referred to as the "closed graph property," which implies (the standard definition of) upper hemicontinuity and closed-valuedness if the range space is compact, as is true in our case.

[27] By the familiar observation based on Berge's maximum theorem (see Ok (2011) for instance), an upper hemicontinuous and convex-valued choice correspondence arises when a firm has a utility function $u : \mathcal{X} \to \mathbb{R}$ that is continuous (in weak-$*$ topology) and quasi-concave.

only if type-θ' workers are also available, and vice versa). As Example 1 shows, a stable matching exists despite the extreme complementarity. Also note that firm's preferences are clearly continuous. This is not a mere coincidence, as we now show that continuity of firms' preferences implies the existence of a stable matching:

Theorem 2. *If each firm $f \in F$ has a continuous preference, then a stable matching exists.*

Proof. See Appendix A. ▮

Given the fixed-point characterization of stable matchings in Theorem 1, our proof approach is to show that T has a fixed point. To this end, we first demonstrate that the upper hemicontinuity of firm preferences implies that the mapping T is also upper hemicontinuous. We also verify that $\mathcal{X}$ is a compact and convex set. Upper hemicontinuity of T and compactness and convexity of $\mathcal{X}$ allow us to apply the Kakutani-Fan-Glicksberg fixed point theorem to guarantee that T has a fixed point.[28] Then, the existence of a stable matching follows from Theorem 1, which shows the equivalence between the set of stable matchings and the set of fixed points of T.

Although the continuity assumption is quite general, including preferences not allowed for in the existing literature, it is not without a restriction, as we illustrate next.

Example 2 (Role of upper hemicontinuity)**.** Consider the following economy modified from Example 1: There are two firms f_1 and f_2, and two worker types θ and θ', each with measure $1/2$. Firm f_1 wishes to hire *exactly* measure $1/2$ of each type and prefers to be unmatched otherwise. Firm f_2's preference is responsive subject to the capacity of measure $1/2$: it prefers type-θ to type-θ' workers, and prefers the latter to leaving a position vacant. Given this, C_{f_1} violates upper hemicontinuity while C_{f_2} does not. As before, we assume

$$\theta : f_1 \succ f_2;$$
$$\theta' : f_2 \succ f_1.$$

No stable matching exists in this environment, as shown in Section S.4 of Supplementary Material.

The upper hemicontinuity assumption is important for the existence of a stable matching; this example shows that nonexistence can occur even if the choice function of only one firm violates upper hemicontinuity. This example also suggests that non-existence can reemerge when some "lumpiness" is reintroduced into the continuum economy (i.e., one firm can only hire a minimum mass of workers). However, this kind of lumpiness may not be very natural

[28] For the Kakutani-Fan-Glicksberg fixed point theorem, refer to Theorem 16.12 and Corollary 16.51 in Aliprantis and Border (2006).

in a continuum economy, which is unlike a finite economy where lumpiness is a natural consequence of the indivisibility of each worker.

By comparison, the convex-valuedness may rule out some realistic case:

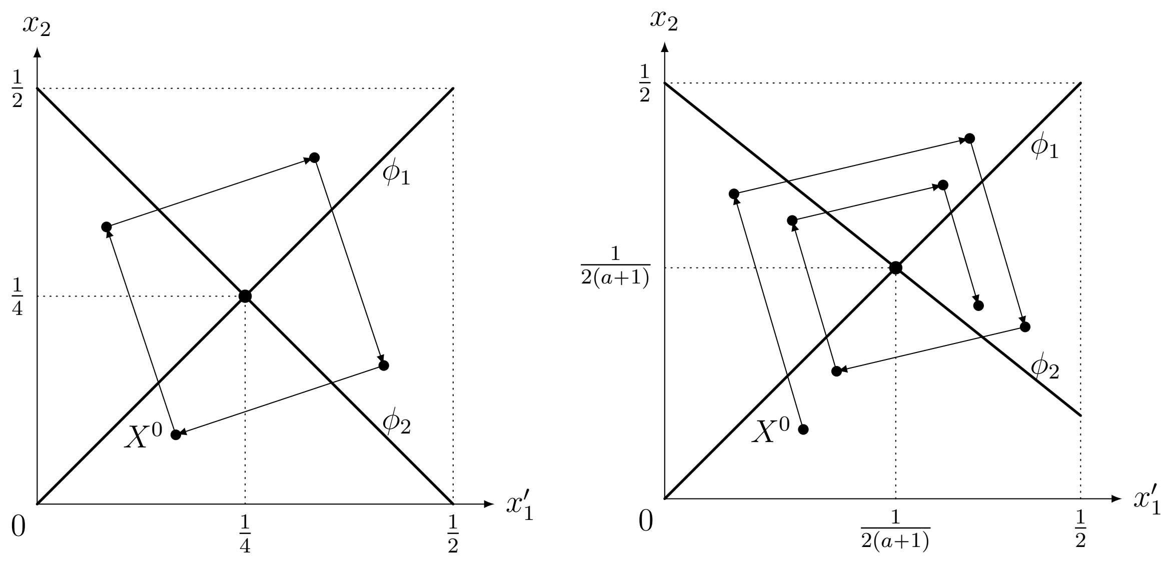
Example 3 (Role of convex-valuedness)**.** Let us modify again Example 1 as follows. The preferences of the firm f_1 as well as those of the two worker types remain the same, while the masses of type-θ and type-θ' workers are 0.6 and 0.4, respectively. Firm f_2 specializes in only one type of workers and prefers hiring as many workers as possible: If x and x' are the available masses of the two types, then the firm only hires mass x of type θ if $x > x'$ and only mass x' of type θ' if $x < x'$, but never wishes to mix the two types. If $x = x'$, the firm is indifferent between hiring either type of mass x (again without mixing types). It is straightforward to verify that the choice correspondence corresponding to this preference is upper hemicontinuous. However, it is not convex-valued since, for any $x = x' > 0$, the firm's choice set contains $(x, 0)$ and $(0, x)$ but not any (strict) convex combination of them. Consequently, a stable matching does not exist in this case (see Section S.4 of Supplementary Material).

Remark 3 (Algorithm to find a fixed point of T)**.** It will be useful to have an algorithm to find or at least approximate a stable matching, which is equivalent to approximating a fixed point of T. One such algorithm is the *tâtonnement process*, that is, to apply T iteratively starting from an initial point $X^0 \in \mathcal{X}^{n+1}$. Unfortunately, this algorithm does not always work. To see this, consider the mapping T in (6) and (7), and let $\phi_1(x_2) := R_{f_2}(x_2, \frac{1}{2})(\theta') = x_2$ and $\phi_2(x_1') := R_{f_1}(\frac{1}{2}, x_1')(\theta) = \frac{1}{2} - x_1'$. Then, T is effectively reduced to a mapping: $(x_1', x_2) \mapsto (\phi_1(x_2), \phi_2(x_1'))$, which is depicted as in Figure 1(a). While its fixed point exists (i.e, the intersection in Figure 1(a)), if one starts anywhere else, say a point X^0 in that figure, the algorithm gets trapped in a cycle.

The map T could work for other situations, however. For instance, modify Example 1 yet again so that, keeping the "Leontief" style of choice function, the firm f_1 would now like to hire mass $a < 1$ of type-θ workers per unit mass of type-θ' workers. Then, the mapping (ϕ_1, ϕ_2) changes to the one in Figure 1(b), where the tâtonnement process converges to a unique fixed point irrespective of the starting point, as can be seen in Figure 1(b).[29] In fact, the composite mapping $T^2 = T \circ T$ in this modified example is a contraction mapping, so the convergence result can be understood by invoking the following generalized version of contraction mapping theorem (see Ch. 3 of Ok (2017) for instance):

Proposition 2. *Suppose that T is singleton-valued, and let $\tilde{T} = T^m$ denote a function obtained from iterating T by m times. If $\tilde{T}$ is a contraction mapping, then, starting with any $X^0 \in \mathcal{X}^{n+1}$, $X^k := \tilde{T}(X^{k-1})$ converges to a unique fixed point of T as $k \to \infty$.*

[29] See Section S.4 of Supplementary Material for detailed analysis.



(a) Case of Cycle

(b) Case of Convergence ($a = \frac{4}{5}$)

Figure 1: Fixed Point of Mapping T

While the contraction mapping theorem provides a condition for our mapping T to serve as an algorithm for finding its fixed point, it need not be the only condition. We will later see another convergence result when firms have substitutable preferences (see Part (ii) of Theorem 4).

6 Substitutable Preferences

In this section, we study another class of preferences known as substitutable preferences in the framework of continuum economy. Although substitutable preferences have been studied extensively before, there are at least three reasons to study them in our context. First, substitutable preferences yield useful results beyond existence, such as side-optimal stable matchings and a constructive algorithm, and it is interesting to see if these results generalize to a large market. Further, as will be seen, substitutable preferences need not be continuous, so existence of a stable matching is not implied by Theorem 2. Second, most existing studies on substitutable preferences are confined to the domain of strict preferences.[30] However, indifferences are a prevalent feature of many markets (see for instance, Abdulkadiroğlu, Pathak, and Roth (2009)), and yet little is known on whether existence and other useful properties such as side-optimal stable matchings hold under substitutable preferences

[30]Sotomayor (1999) is a notable exception.

with indifferences.[31] Third, the large market setting raises another important question—uniqueness. Azevedo and Leshno (2016) offer sufficient conditions for a stable matching to be unique in the large economy. Their striking result is obtained in the restricted preference domain of "responsive" preferences, however, and it is interesting to ask if uniqueness extends to general substitutable preferences.

6.1 Existence and Side-Optimality

To define substitutable preferences in our general domain, we need a few definitions. Given a partial order $\sqsubset$, a correspondence $h : \mathcal{X} \rightrightarrows \mathcal{X}$ is said to be *weak-set monotonic* if it satisfies the following: (i) for any $X \sqsubset X'$ and $Z \in h(X)$, there is $Z' \in h(X')$ with $Z \sqsubset Z'$; (ii) for any $X \sqsubset X'$ and $Z' \in h(X')$, there is $Z \in h(X)$ with $Z \sqsubset Z'$.[32]

Definition 4. Firm f's preference is **weakly substitutable** if R_f is weak-set monotonic.

The current definition preserves the well-known property of a firm becoming more selective as more workers are available. The novelty here is that substitutability is defined for a rejection correspondence (instead of a rejection function, as in the literature). Indeed, it can be seen as a generalization of the standard notion: if $C_f(X)$ is singleton-valued for all $X \in \mathcal{X}$, this notion collapses to the requirement that R_f be monotonic in the underlying order $\sqsubset$: $R_f(X) \sqsubset R_f(X')$ whenever $X \sqsubset X'$.

We now establish the existence result in the domain of weakly substitutable preferences:

Theorem 3. *If each firm's preference is weakly substitutable and each C_f is closed-valued, then a stable matching exists.*[33]

Proof. See Appendix B. ▮

As before, this result rests on the existence of a fixed point of the correspondence T defined earlier. One can see that if firms have weakly substitutable preferences, then T is weak-set monotonic. While Zhou (1994) extends Tarski's well-known theorem to the case of correspondences, his monotonicity condition is stronger than ours, so we instead apply a recent result due to Li (2014) to prove the existence of a fixed point. The weakening of the required condition is not merely for generality. Weakly substitutable preferences allow

[31] Existence for the general substitutable preferences is not clear, unlike the case of responsive preferences. In the latter case, an arbitrary tie breaking (e.g., random tie breaking) preserves responsiveness, leading to existence. To our knowledge, there is no straightforward generalization of this method to the general class of substitutable preferences.

[32] The weak-set monotonicity is weaker than the strong-set monotonicity often used in the monotone comparative statics (e.g., Milgrom and Shannon (1994)).

[33] The closed-valuedness is a mild condition that may hold even if the choice correspondence fails to be upper hemicontinuous, as demonstrated by the example in footnote 34.

for indifferences that arise most naturally: for instance, consider a firm with a fixed quota which can be filled with any mixture of multiple types, as featured in the next example.

Example 4 (Weak Substitutability)**.** Suppose that there are three firms, f_1, f_2, and f_3, and two worker types, θ and θ', and that the capacity of each firm and the mass of each worker type are all equal to $\frac{1}{2}$. The workers' preferences are

$$\begin{aligned}\theta &: f_1 \succ f_2 \succ f_3\\ \theta' &: f_1 \succ f_3 \succ f_2.\end{aligned}$$

Firms f_2 and f_3 have responsive preferences: they both prefer θ to θ' (i.e., they wish to hire in that order up to the quota of $\frac{1}{2}$). Firm f_1 is indifferent between the two types of workers: its preference is described by a choice correspondence:

$$C_{f_1}(x, x') = \{(y, y') \in [0, x] \times [0, x'] \,|\, y + y' = \min\{x + x', \tfrac{1}{2}\}\}.$$

This choice correspondence satisfies the weak substitutability, as one can easily check. There exists a continuum of stable matchings[34]: for any $z \in [0, \frac{1}{2}]$, it is a stable matching for firm f_1 to hire mass z of type-θ workers and $\frac{1}{2} - z$ of type-θ' workers, for firm f_2 to hire mass $\frac{1}{2} - z$ of type-θ workers and for firm f_3 to hire mass z of type-θ' workers. Clearly, the higher z is, the worse off firm f_2 is and the better off firm f_3 is. Hence, the firm-optimal stable matching does not exist. Neither does the worker-optimal stable matching since firm f_1 hires type-θ and type-θ' workers in different proportions across different stable matchings.

We next introduce a stronger notion of substitutability that would restore side optimality. We say a set $\mathcal{X}' \subset \mathcal{X}$ of subpopulations is a *complete sublattice* if $\mathcal{X}'$ contains both $\bigvee \mathcal{Z}$ and $\bigwedge \mathcal{Z}$ for every set $\mathcal{Z} \subset \mathcal{X}'$.[35]

Definition 5. Firm f's preference is **substitutable** if (i) R_f is weak-set monotonic and (ii) for any $X \in \mathcal{X}$, $R_f(X)$ is a complete sublattice.[36]

[34]In this example, firms' preferences satisfy the conditions of Theorem 2, so Theorem 3 is not needed for showing existence of a stable matching. However, one can easily obtain an example where the latter theorem applies while the former does not. In Example 4, suppose firm f_1 is instead endowed with a choice correspondence defined as follows: for some $\bar{x} \in [0, 1/2]$,

$$C_{f_1}(x, x') = \begin{cases} \{(x, x')\} & \text{if } x' < \bar{x} \\ \{(0, y') \,|\, y' \in [\bar{x}, x']\} & \text{if } x' \geqslant \bar{x} \end{cases}.$$

This correspondence fails to be upper-hemicontinuous, rendering Theorem 2 inapplicable, but the conditions of Theorem 3 are satisfied, as can be checked easily.

[35]Authors use different terminologies for the same property: Topkis (1998) calls it *subcomplete sublattice* and Zhou (1994) calls it *closed sublattice*.

[36]This condition is weaker than Zhou (1994)'s which requires *strong-set monotonicity* in place of (i). Our substitutability guarantees side optimality but not a complete lattice, which Zhou's condition guarantees. See Example S1 in Section S.5 of Supplementary Material for the case in which our substitutability condition holds while the strong-set monotonicity fails, causing the lattice structure to fail.

When C_f is singleton-valued, the condition reduces to the standard notion of substitutability, so the distinction between the two different versions of substitutability disappears. Nevertheless, the requirements for substitutable preferences are stronger in the current weak preference domain. In particular, (ii) is a strong requirement that preferences such as those described by C_{f_1} in Example 4 fail.[37]

At the same time, substitutable preferences do accommodate some types of indifferences. Imagine, for instance, a school which has a selective program with limited quota and a general program with flexible quotas. For the selective program, the school admits students in the order of their scores up to its quota. Once the quota is reached, the school may admit students for the general program with flexible quotas and without consideration of their scores. To our knowledge, the next result is the first to establish the existence of side-optimal stable matchings in the weak preference domain[38]:

Theorem 4. *Suppose that each firm's preference is substitutable. Then, the following results hold: letting $\mathcal{M}^*$ denote the set of stable matchings,*

(i) (Side-Optimal Stable Matching) *There exist stable matchings, $\overline{M}, \underline{M} \in \mathcal{M}^*$, that are respectively firm-optimal/worker-pessimal and firm-pessimal/worker-optimal in the following senses: If $M \in \mathcal{M}^*$, then $M \geq_\Theta \overline{M} \geq_F M$ and $M \leq_\Theta \underline{M} \leq_F M$.*

(ii) (Generalized Gale-Shapley) *If, in addition, C_f is order continuous for each f,[39] then the limit of the algorithm that iteratively applies $\overline{T}$ starting with $X_f = G, \forall f \in \tilde{F}$, produces a firm-optimal stable matching, and the limit of the algorithm that iteratively applies $\underline{T}$ starting with $X_f = \mathbf{0}, \forall f \in \tilde{F}$, produces a worker-optimal stable matching, where $\overline{T}(X) := \bigvee T(X)$ and $\underline{T}(X) := \bigwedge T(X)$ for any $X \in \mathcal{X}^{n+1}$.*

Proof. See Appendix B. ∎

While the existence of firm-optimal and worker-optimal stable matchings is well-known for the strict preference domain, no such result is previously known for the case in which the firms' preferences involve indifferences. In fact, the received wisdom is that firms' indifferences are incompatible with the presence of side-optimal stable matchings even in a more restrictive domain such as responsive preferences. Theorems 3 and 4, taken together, clarify

[37] To see this, note $\mathcal{Z} = \{(\frac{1}{2}, 0), (0, \frac{1}{2})\} \subset R_{f_1}(\frac{1}{2}, \frac{1}{2})$, but $\bigvee \mathcal{Z} = (\frac{1}{2}, \frac{1}{2}) \notin R_{f_1}(\frac{1}{2}, \frac{1}{2})$, so R_{f_1} is not a sublattice (let alone a complete one).

[38] Theorem 4 does not require closed-valuedness of the choice correspondences, which Theorem 3 requires. It is often the case, however, that part (ii) of the substitutability (i.e., the complete sublattice property) implies the closed-valuedness. For instance, the relation holds if there are finitely many worker types so $\mathcal{X}$ is a subset of a finite dimensional Euclidean space.

[39] A correspondence C is *order-continuous* if $\overline{C}(X_k) \xrightarrow{w^*} \overline{C}(X)$ for any increasing sequence $X_k \xrightarrow{w^*} X$, and $\underline{C}(X_k) \xrightarrow{w^*} \underline{C}(X)$ for any decreasing sequence $\mathcal{X}_k \xrightarrow{w^*} X$, where $\overline{C}(X) = \bigvee C(X)$ and $\underline{C}(X) = \bigwedge C(X)$ for any $X \in \mathcal{X}$.

the types of indifferences that permit the existence of side-optimal stable matchings and those that do not. In particular, responsive preferences with indifferences (studied by Abdulkadiroğlu, Pathak, and Roth (2009) and Erdil and Ergin (2008) for instance) satisfy weak substitutability but fail substitutability and, consistent with Theorems 3 and 4, guarantee the existence of a stable matching but not a side-optimal one.

The second part of Theorem 4 shows that a generalized version of Gale-Shapley's deferred acceptance algorithm finds a side-optimal stable matching, but only with the additional (order) continuity assumption.[40] Without this continuity property, the algorithm may get "stuck" at an unstable matching. (Example S2 in Section S.5 of Supplementary Material illustrates this point.)

Next, we adapt another well-known condition to our context:

Definition 6. Firm f's preference exhibits the **law of aggregate demand (or LoAD)** if for any $X, X' \in \mathcal{X}$ with $X \sqsubset X'$, $\sup C_f(X)(\Theta) \leqslant \inf C_f(X')(\Theta)$.[41]

Given LoAD and substitutability, we show that the total measure of workers employed by each firm in any stable matching is uniquely pinned down:

Theorem 5 (Rural Hospital)**.** *If each firm's preference is substitutable and satisfies LoAD, then, for any $M \in \mathcal{M}^*$, we have $M_f(\Theta) = \overline{M}_f(\Theta), \forall f \in F$ and $M_\emptyset = \overline{M}_\emptyset$.*

Proof. See Appendix B. ▌

Remark 4 (Finite economy)**.** While the results are established for our continuum economy model, they apply to finite economy models with little modification. (Note for instance, the order-continuity required for Theorem 4-(ii) would be satisfied vacuously for the finite economy.) To the extent that these results were obtained in the extant literature for strict preferences, the current results would amount to their extensions to more general preferences in the finite-economy context.

6.2 Uniqueness of Stable Matching

Azevedo and Leshno (2016) established the uniqueness of a stable matching in a continuum economy when firms have responsive preferences. We now investigate the extent to which the uniqueness result extends to the general substitutable preferences environment. The uniqueness question is important not only for the continuum economy but also for the large

[40] This result is reminiscent of the well-known property of a supermodular game whereby, given the order continuity property, iterative deletion of strictly dominated strategies starting from the "largest" and "smallest" strategies produces largest and smallest Nash equilibria, respectively. See Milgrom and Roberts (1990) and Milgrom and Shannon (1994).

[41] This property is an adaptation of a property that appears in the literature such as Hatfield and Milgrom (2005), Alkan (2002), and Fleiner (2003).

finite one, as will be shown in the next section. Expanding the domain beyond responsive preferences helps to identify the underlying condition that drives uniqueness.

To begin, we assume each firm's choice is unique, i.e., each C_f is a choice function, and, for any matching M, firm f, and subset F' of firms, we let $M^f_{F'}$ be a subpopulation of workers defined by

$$M^f_{F'}(E) := \sum_{P \in \mathcal{P}} \sum_{f': f \succ_P f', f' \notin F'} M_{f'}(\Theta_P \cap E) \text{ for each } E \in \Sigma.$$

In words, this is the measure of workers who are matched outside firms F' and available to firm f under M (excluding those matched with f).[42] Consider the following property:

Definition 7 (Rich preferences)**.** The preferences are **rich** if for any individually rational matching $\hat{M} \neq \underline{M}$ such that $\hat{M} \succeq_F \underline{M}$, there exists $f^* \in F$ such that $\underline{M}_{f^*} \neq C_{f^*}((\underline{M}_{f^*} + \hat{M}^{f^*}_{\bar{F}}) \wedge G)$, where $\bar{F} := \{f \in F | \hat{M}_f \succ_f \underline{M}_f\}$.

The condition is explained as follows. Consider any individually rational matching $\hat{M}$ that is preferred to the worker-optimal stable matching $\underline{M}$ by all firms, strictly so by firms in $\bar{F} \subset F$. Then, the richness condition requires that, at matching $\underline{M}$, there must exist a firm f^* that would be happy to match with some workers who are not hired by the firms in $\bar{F}$ and are willing to match with f^* under $\hat{M}$. Since firms are more selective at $\hat{M}$ than at $\underline{M}$, it is intuitive that a firm would demand in the latter matching some workers that the more selective firms would not demand in the former matching. The presence of such worker types requires richness of the preference palette of firms as well as workers—hence the name. This point will be seen more clearly in the next section when one considers (a general class of) responsive preferences.

Theorem 6. *Suppose that each firm's preference is substitutable and satisfies LoAD. If the preferences are rich, then a unique stable matching exists.*

Proof. See Appendix B. ▮

Both richness and substitutability are necessary for the uniqueness result, as one can construct counterexamples without much difficulty. LoAD is also indispensable for the uniqueness, as demonstrated by Example S3 in Supplementary Material. (Recall the LoAD is trivially satisfied by the responsive preferences of Azevedo and Leshno (2016).)

While rich preference may not be easy to check, one can show that the condition is implied by a full-support condition in a general class of environments that nests Azevedo and Leshno (2016) as a special case, as demonstrated below.

[42] Note that this is a valid subpopulation, or a measure, since it is the sum of a finite number of measures.

Responsive Preferences with Submodular Quotas. Suppose firms have responsive preferences but may face quotas on the number of workers they can hire from different groups of workers. Such group-specific quotas, typically based on socio-economic status or other characteristics, may arise from affirmative action or diversity considerations. The resulting preferences (or choice functions) may violate responsiveness but they nonetheless satisfy substitutability.

Assume that there is a finite set $\mathcal{T}$ of "ethnic types" that describe characteristics of a worker such as ethnicity, gender, and socio-economic status, such that type θ is assigned an ethnic type $\tau(\theta)$ via some measurable function $\tau : \Theta \to \mathcal{T}$. For each $t \in \mathcal{T}$, a (measurable) set $\Theta^t := \{\theta \in \Theta | \tau(\theta) = t\}$ of agents has an ethnic type t. Each firm f faces a quota constraint given by function $\mathcal{Q}_f : 2^{\mathcal{T}} \to \mathbb{R}_+$ such that for each $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{Q}_f(\mathcal{T}')$ is a maximum quota (in terms of the measure of workers) the firm f can hire from the ethnic types in $\mathcal{T}'$. We assume that $\mathcal{Q}_f(\varnothing) = 0$, $\mathcal{Q}_f(\mathcal{T}) > 0$, and $\mathcal{Q}_f$ is submodular: for any $\mathcal{T}', \mathcal{T}'' \subset \mathcal{T}$,

$$\mathcal{Q}_f(\mathcal{T}') + \mathcal{Q}_f(\mathcal{T}'') \geqslant \mathcal{Q}_f(\mathcal{T}' \cup \mathcal{T}'') + \mathcal{Q}_f(\mathcal{T}' \cap \mathcal{T}'').$$

Submodularity allows for the most general form of group-specific quotas that encompasses all existing models: for instance, it holds if the firm faces arbitrary quotas on a *hierarchical family* of subsets of $\mathcal{T}$.[43] This case includes a familiar case studied by many authors (Abdulkadiroğlu and Sönmez (2003), for instance) in which the family forms a partition of $\mathcal{T}$. Subject to the quotas, each firm has *responsive preferences* given by a score function $s_f : \Theta \to [0, 1]$ such that f prefers type-θ to type-θ' worker if and only if $s_f(\theta) > s_f(\theta')$. For simplicity, we assume that no positive mass of types has an identical score.[44]

Clearly, this class of preferences subsumes pure responsive preferences considered by Azevedo and Leshno (2016) as a special case, but includes preferences that fail their condition. We can show that these preferences satisfy both substitutability and LoAD:

Lemma 2. *A firm f with responsive preferences facing submodular quotas exhibits a choice function that satisfies substitutability and LoAD.*[45]

Proof. See Section S.6.2 of Supplementary Material. ▮

Specifically, Section S.6 of Supplementary Material provides an algorithm that finds the choice function for a firm with this type of preferences, and shows that the choice function satisfies substitutability and LoAD. Given the prevalence of group-specific constraints, this lemma, which is highly nontrivial, may be of interest in its own right. For instance, because

[43] A family of sets is hierarchical if, for any sets $\mathcal{T}', \mathcal{T}''$, either $\mathcal{T}' \cap \mathcal{T}'' = \varnothing$, $\mathcal{T}' \subset \mathcal{T}''$, or $\mathcal{T}'' \subset \mathcal{T}'$. See Che, Kim, and Mierendorff (2013) for the proof of this result.

[44] This assumption is maintained by Azevedo and Leshno (2016), for instance.

[45] Section S.6.4 of Supplementary Material presents an example in which the substitutability fails due to the quota constraints which is not submodular.

the choice of each firm is a function, substitutability implies that the set of a stable matchings has a lattice structure, a conclusion that does not hold under general choice correspondence, even with substitutability.

Next, we generalize the full support condition of Azevedo and Leshno (2016) to the current setup:

Definition 8. The worker population has a **full support** if for each preference $P \in \mathcal{P}$, any ethnic type $t \in T$, and for any non-empty open cube set $S \subset [0,1]^n$, the worker types

$$\Theta_P^t(S) := \{\theta \in \Theta_P \cap \Theta^t \mid (s_f(\theta))_{f \in F} \in S\}$$

have a positive measure, i.e., $G(\Theta_P^t(S)) > 0$.

Note that this condition boils down to that of Azevedo and Leshno (2016) if $\mathcal{T}$ is a singleton set.

Proposition 3. *Suppose each firm $f \in F$ has responsive preferences and faces submodular quotas. Then, the full support condition implies the richness condition.*

Proof. See Section S.6.3 of Supplementary Material. ▮

Combining Lemma 2, Proposition 3, and Theorem 6, we conclude:

Corollary 1. *Suppose each firm $f \in F$ has responsive preferences and faces submodular quotas. If the full support condition holds, then a unique stable matching exists.*

7 Approximate Stability in Finite Economies

In Section 2, we have observed that a finite economy, however large it is, may not possess a stable matching while a large finite economy admits a matching that is stable in an approximate sense. Motivated by this and building on our findings in the continuum economy, we here formalize the notion of approximate stability and demonstrate that the set of approximately stable matchings in large finite economies inherits the desirable properties of stable matching in a continuum economy. Specifically, the set is nonempty, contains (approximately) firm-optimal and worker-optimal matchings, and consists of virtually unique matching, whenever the corresponding property is true for the continuum economy. This suggests that a continuum economy provides a good framework for analyzing large finite economies, which is useful since a continuum economy often permits a more tractable analysis, as demonstrated by Azevedo and Leshno (2016).

To analyze economies of finite sizes, we consider a sequence of economies $(\Gamma^q)_{q \in \mathbb{N}}$ indexed by the total number of workers $q \in \mathbb{N}$. In each economy Γ^q, there is a fixed set of n firms, $f_1, \ldots, f_n$, that does not vary with q. As before, each worker has a type in Θ. The worker distribution is normalized with the economy's size. Formally, let the (normalized) population G^q of workers in Γ^q be defined so that $G^q(E)$ represents the number of workers with types in E divided by q. A (discrete) measure X^q is **feasible** in economy Γ^q if $X^q \sqsubset G^q$, and it is a measure whose value for any E is a multiple of $1/q$. Let $\mathcal{X}^q$ denote the set of all feasible subpopulations in Γ^q. Note that G^q, and thus every $X^q \in \mathcal{X}^q$, belongs to $\overline{\mathcal{X}}$, although it need not be a subpopulation of G and thus may not belong to $\mathcal{X}$. Let us say that a sequence of economies $(\Gamma^q)_{q\in\mathbb{N}}$ **converges to a continuum economy** Γ if $G^q \xrightarrow{w^*} G$.

To formalize approximate stability, we first represent each firm f's preference by a cardinal utility function $u_f : \overline{\mathcal{X}} \to \mathbb{R}$ defined over normalized distributions of workers it matches with. And, this utility function represents a firm's preference for each finite economy Γ^q as well as for the continuum economy.[46] We assume that u_f is continuous in weak-$*$ topology.[47] Then, firm f chooses a feasible subpopulation that maximizes u_f in the respective economies: in the continuum economy Γ, the firm's choice correspondence is given by

$$C_f(X) = \arg\max_{X'\sqsubset X} u_f(X'), \forall X \in \mathcal{X}; \tag{8}$$

in each finite economy Γ^q, it is given by

$$C^q_f(X) := \arg\max_{X'\sqsubset X, X'\in\mathcal{X}^q} u_f(X'), \forall X \in \mathcal{X}^q. \tag{9}$$

All our results in this section rely on the existence of stable matching in the continuum economy, which holds if each u_f is such that C_f defined in (8) satisfies the conditions in Theorem 2 or in Theorem 3. For instance, the conditions in Theorem 2 are satisfied if each u_f is quasi-concave in addition to being continuous, since C_f is then convex-valued and upper hemicontinuous.[48]

A *matching* in finite economy Γ^q is $M^q = (M^q_f)_{f\in\tilde{F}}$ such that $M^q_f \in \mathcal{X}^q$ for all $f \in \tilde{F}$ and $\sum_{f\in\tilde{F}} M^q_f = G^q$. The measure of available workers for each firm f at matching $M^q \in (\mathcal{X}^q)^{n+1}$ is $D^{\preceq f}(M^q)$, where $D^{\preceq f}(\cdot)$ is defined as in (3).[49] Note that because each M^q_f is a multiple of $1/q$, $D^{\preceq f}(M^q)$ is feasible in Γ^q. We now define ϵ-stability in finite economy Γ^q.

Definition 9. For any $\epsilon > 0$, a matching $M^q \in (\mathcal{X}^q)^{n+1}$ in economy Γ^q is **ϵ-stable** if (i) for each $f \in F$, $M^q_f \in C^q_f(M^q_f)$; (ii) for each $P \in \mathcal{P}$, $M^q_f(\Theta_P) = 0, \forall f \prec_P \varnothing$; and (iii)

[46]The assumption that the same utility function applies to both finite and limit economies is made for convenience. The results in this section hold if, for instance, the utilities in finite economies converge uniformly to the utility in the continuum economy.

[47]To guarantee the existence of such a utility function, we may assume, as in Remark 16, that each firm is endowed with a complete, continuous preference relation. Then, because the set of alternatives $\overline{\mathcal{X}}$ is a compact metric space, such a preference can be represented by a continuous utility function according to the Debreu representation theorem (Debreu, 1954).

[48]The upper hemicontinuity is an implication of Berge's maximum theorem.

[49]To be precise, $D^{\preceq f}(M^q)$ is given as in (3) with G and M being replaced by G^q and M^q, respectively.

$u_f(\tilde{M}^q) < u_f(M^q_f) + \epsilon$ for any $f \in F$ and $\tilde{M}^q \in \mathcal{X}^q$ with $\tilde{M}^q \sqsubset D^{\preceq f}(M^q)$.[50]

Conditions (i) and (ii) of this definition are analogous to the corresponding conditions for exact stability, so ϵ-stability relaxes stability only with respect to condition (iii). Specifically, an ϵ-stable matching could be blocked, but if so, the gain from blocking must be small for any firm.[51] An ϵ-stable matching will be robust against blocks if a rematching process requires cost (at least of ϵ) for the firm initiating a block, which seems sensible when there are some frictions in the market.

Remark 5. For $\epsilon > 0$, we say that matching M is **ϵ-Pareto efficient** if there is no matching $M' \neq M$ and firm $f \in F$ such that $M' \succeq_F M$, $M' \succeq_\Theta M$, and $u_f(M'_f) \geqslant u_f(M_f) + \epsilon$. By an argument analogous to the Pareto efficiency of a stable matching presented in Section 3, it is easy to see that any ϵ-stable matching is ϵ-Pareto efficient.

Our main result follows:

Theorem 7. *Fix any sequence of economies $(\Gamma^q)_{q \in \mathbb{N}}$ that converges to a continuum economy Γ which admits a stable matching M. For any $\epsilon > 0$, there exists $Q \in \mathbb{N}$ such that for all $q > Q$, there is an ϵ-stable matching M^q in Γ^q.*[52]

Proof. See Appendix C. ∎

This result implies that a large finite market admits an approximately stable matching even with non-substitutable preferences. Interestingly, a converse of Theorem 7 also holds:

[50] Approximate stability might be defined slightly differently. Say a matching M^q is **ϵ-distance stable** if (i) and (ii) of Definition 9 hold and (iii') $d(\tilde{M}^q_f, M^q_f) < \epsilon$ for any coalition f and $\tilde{M}^q_f \in \mathcal{X}^q$ that blocks M^q in the sense that $\tilde{M}^q_f \sqsubset D^{\preceq f}(M^q)$ and $u_f(\tilde{M}^q_f) > u_f(M^q_f)$, where $d(\cdot,\cdot)$ is the Lévy-Prokhorov metric (which metrizes the weak-$*$ topology). In other words, if a matching M^q is ϵ-distance stable, then the distance of any alternative matching a firm proposes for blocking must be within ϵ from the original matching. One advantage of this concept is that it is ordinal, i.e., we need not endow the firms with cardinal utility functions to formalize the notion. Note that the notion also requires the ϵ bound for *any* blocking coalition, not just the "optimal" blocking coalition as defined in Definition 2-2, making the notion of ϵ-distance stability more robust. In Section S.7.2 of Supplementary Material, we prove the existence of ϵ-distance stable matching (under an additional mild assumption).

[51] Notice that the conditions (i) and (iii) are asymmetric in the sense that the matching should be precisely optimal against the blocking by an individual firm alone and only approximately optimal against the blocking by a coalition. We adopt this asymmetry because blocking with workers outside the firm is presumably harder for a firm to implement than retaining or firing its own workers.

[52] We note that M^q need not converge to M. In fact, there can be a stable matching in Γ that does not have any nearby approximate stable matching in large finite economy Γ^q (refer to Section S.7.3 of Supplementary Material for an example), meaning that the (approximately) stable matching correspondence is not "lower hemicontinuous." This is because the exact individual rationality, that is, condition (i) of Definition 9, can make a firm's choice in finite economy never close to a certain stable matching in the continuum economy. If this condition is relaxed analogously to the condition (iii), then any stable matching in the continuum economy can be approximated by ϵ-stable matchings in large finite economies.

Theorem 8. *Let $(M^q)_{q \in \mathbb{N}}$ be a sequence of matchings converging to M with the property that for every $\epsilon > 0$, there exists $Q \in \mathbb{N}$ such that for all $q > Q$, M^q is ϵ-stable in Γ^q. Then, M is stable in Γ.*[53]

Proof. See Appendix C. ▮

This result implies that the behavior of large finite economies is well approximated by the continuum economy in the sense that by studying the latter, we will not "miss" any approximately stable matching in the former.

Example 5. Recall the finite economy in Section 2, where there are q workers of each type.[54] Recall its limit economy admits a unique stable matching $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. If the index q is odd, then a stable matching does not exist. As we have already seen, the following matching is ϵ-stable in Γ^q for sufficiently large q and converges to the (unique) stable matching in Γ[55]:

$$\begin{pmatrix} f_1 & f_2 \\ \frac{q+1}{4q}\theta + \frac{q+1}{4q}\theta' & \frac{q-1}{4q}\theta + \frac{q-1}{4q}\theta' \end{pmatrix}.$$

Also, as Theorem 8 indicates, any ϵ-stable matching in Γ^q for sufficiently large q must be close to the stable matching in Γ. For instance, any matching where f_1's hiring of each type is bounded away from $\frac{1}{4}$ will be subject to a block that increases either firm's utility by more than a small ϵ.

An approximately stable matching established in Theorem 7 can be shown to possess other properties inherited from the structure of stable matchings in the continuum economy. To this end, we relax the notion of side optimality.

Definition 10. For $\epsilon > 0$, a matching M^q in Γ^q is an **ϵ-firm-optimal stable matching** if there is $\delta \in (0, \epsilon)$ such that

1. M^q is δ-stable in Γ^q, and
2. for any matching $\hat{M}^q$ which is δ-stable in Γ^q, $u_f(M^q_f) \geqslant u_f(\hat{M}^q_f) - \epsilon, \forall f \in F$.

Definition 11. For $\epsilon > 0$, a matching M^q in Γ^q is an **ϵ-worker-optimal stable matching** if there is $\delta \in (0, \epsilon)$ such that

[53] This result is reminiscent of the upper hemicontinuity of Nash equilibrium correspondence (see Fudenberg and Tirole (1991) for instance). But Theorem 8 establishes a more robust result in the sense that the convergence occurs even for "approximately" stable matchings in nearby economies.

[54] With a slight abuse of notation, this example assumes that there are a total of $2q$ workers (q workers of θ and θ' each) rather than q. Of course, this is done for purely expositional purposes.

[55] This matching is also ϵ-distance stable since the only profitable block involves f_2 taking a single worker of type θ' away from firm f_1.

1. M^q is δ-stable in Γ^q, and

2. for any matching $\hat{M}^q$ which is δ-stable in Γ^q,

$$D^{\succeq f}(M^q)(E^\epsilon) \geqslant D^{\succeq f}(\hat{M}^q)(E) - \epsilon, \forall f \in F, \forall E \in \Sigma,$$

where $E^\epsilon := \{\theta \in \Theta | \exists \theta' \in E \text{ such that } d^\Theta(\theta, \theta') < \epsilon\}$ is the ϵ-neighborhood of E.

The ϵ-firm-optimality requires that the matching itself be approximately stable and that there be no other approximately stable matching which makes any firm better off by more than ϵ. The ϵ-worker-optimality can be seen as a natural extension of worker optimality—i.e., $M^q \preceq_\Theta \hat{M}^q$—, for the concept collapses to the latter if $\epsilon = 0$. We now prove the existence of approximately side-optimal matchings in large finite economies.[56]

Theorem 9. *Suppose that a sequence of finite economies $(\Gamma^q)_{q \in \mathbb{N}}$ converges to a continuum economy Γ. Fix any $\epsilon > 0$.*

(i) If there is a firm-optimal stable matching in Γ, then there is $Q \in \mathbb{N}$ such that for all $q > Q$, an ϵ-firm-optimal stable matching in Γ^q exists.

(ii) If there is a worker-optimal stable matching $\underline{M}$ in Γ and $C_f(\underline{M}_f) = \{\underline{M}_f\}, \forall f \in F$ (i.e., for each firm f, $\underline{M}_f$ is its unique choice at $\underline{M}$), then there is $Q \in \mathbb{N}$ such that for all $q > Q$, an ϵ-worker-optimal stable matching in Γ^q exists.[57]

Proof. See Appendix C. ▌

Finally, we show that if there is a unique stable matching in the limit economy Γ, then the approximately stable matching is *virtually unique* in any sufficiently large finite economy.

Theorem 10. *Suppose that a sequence of finite economies $(\Gamma^q)_{q \in \mathbb{N}}$ converges to a continuum economy Γ which has a unique stable matching M. Then, the approximately stable matching of large finite economy is "virtually unique" in the following sense: for any $\epsilon > 0$, there are*

[56]This result will be particularly useful when preferences are substitutable in a continuum economy but not in finite economies that converge to that economy. Delacrétaz, Kominers, and Teytelboym (2016) offer one such example in their study of refugee resettlement. Translated into our setup, there are three types, θ, θ', and θ'', and a firm f with capacity κ (or κ units of seats) which has a responsive preference with $\theta \succ \theta' \succ \theta''$. Each of types θ and θ'' occupies one seat while type θ' occupies two seats. As Delacrétaz, Kominers, and Teytelboym (2016) show, the firm f's preference is not substitutable in finite economies, which is largely due to the integer problem that disappears in continuum economy. To see it, suppose that a continuum of workers $X = (x, x', x'')$ is available. Then, the firm f's choice function is given by $C_f(X)(\theta) = \min\{x, \kappa\}$, $C_f(X)(\theta') = \min\{x', \frac{\kappa - C_f(X)(\theta)}{2}\}$, and $C_f(X)(\theta'') = \min\{x'', \kappa - C_f(X)(\theta) - 2C_f(X)(\theta')\}$. It is straightforward to check that this choice function represents a substitutable preference.

[57]Section S.7.3 of Supplementary Material presents an example in which the result does not hold without the extra assumption, $C_f(\underline{M}_f) = \{\underline{M}_f\}, \forall f \in \tilde{F}$.

$Q \in \mathbb{N}$ *and* $\delta \in (0, \epsilon)$ *such that for every* $q > Q$ *and for every* δ*-stable matching* $\hat{M}^q$ *in* Γ^q*, we have* $d(M, \hat{M}^q) < \epsilon$.[58]

Proof. See Appendix C. ▮

This result, together with Theorem 6, leads to the following generalization of the convergence result (Theorem 2) in Azevedo and Leshno (2016).

Corollary 2. *Suppose that in the continuum economy* Γ*, the firm preferences are substitutable and satisfy LoAD while the preferences are rich. Then, the approximately stable matchings of any large finite economy* Γ^q *that converges to* Γ *are virtually unique.*

8 Strong Stability and Strategy-Proofness

Stability promotes fairness by eliminating justified envy for workers. However, stability alone may not guarantee fair treatment of workers if a firm is indifferent over worker types that are unobservable or regarded as indistinguishable by the firm. The following example illustrates the point.

Example 6. There are two firms f_1 and f_2, and a unit mass of workers with the following types:

$$\theta : f_1 \succ f_2 \succ \emptyset;$$
$$\theta' : f_2 \succ f_1 \succ \emptyset;$$
$$\theta'' : f_2 \succ \emptyset \succ f_1.$$

The type distribution is given by $G(\theta) = 1/2$ and $G(\theta') = 1/4 = G(\theta'')$. (Note that this example is the same as our leading example except that a mass of 1/4 of type-θ' workers now have a new preference P''.)

Both firms are indifferent between type-θ' and type-θ'' workers; they differ only in their own preferences for firms. Firm f_1 wishes to maximize $\min\{x, x' + x''\}$, where x, x' and x'' are the measures of workers with types θ, θ' and θ'', respectively. Firm f_2 has a responsive preference with a capacity of 1/2 and prefers type θ to type θ' or θ''.

Consider first a mechanism that maps G to matching

$$M = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{4}\theta + \frac{1}{4}\theta' & \frac{1}{4}\theta + \frac{1}{4}\theta'' \end{pmatrix}.$$

This matching is stable, which can be seen by the fact that the firms are matched with the same measures of productivity types as in the stable matching in Example 1. Observe,

[58] This implies that all stable matchings in any sufficiently large finite economy are also close to one another.

however, that this matching treats the type-θ' and type-θ'' workers differently—the former workers match with f_1 and the latter workers match with f_2 (which they both prefer)—despite the fact that the firms perceive them as equivalent. This lack of "fairness" leads to an incentive problem: type-θ' workers have an incentive to (mis)report their type as θ'' and thereby match with f_2 instead of f_1.

These problems can be addressed by another mechanism that maps G to a matching

$$\bar{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{6}\theta + \frac{1}{6}\theta' & \frac{1}{3}\theta + \frac{1}{12}\theta' + \frac{1}{12}\theta'' \end{pmatrix}.$$

Like M, this matching is stable, but in addition, firm f_2 treats type-θ' and type-θ'' workers identically in this matching. Further, neither type-θ' nor type-θ'' workers have incentives to misreport.

The fairness issue illustrated in this example is particularly relevant in school choice, for schools evaluate students based on coarse priorities. Fairness demands that students enjoying the same priorities be treated equally without any discrimination. This calls for what Kesten and Ünver (2014) labeled *strong stability*, a condition satisfied by the second matching in the above example. As illustrated, strong stability is closely related to strategy-proofness for workers in a large economy. We thus address both issues here.

8.1 Strong Stability and Strategy-Proofness in a Large Economy

We begin by adapting our model to address the issues at hand. First, we denote the type of each worker as a pair $\theta = (a, P)$, where a denotes the worker's productivity or skill and P describes her preferences over firms and the outside option, as above. We assume that worker preferences do not affect firm preferences and are private information, whereas productivity types may affect firm preferences and are observable to the firms (and to the mechanism designer). Let A and $\mathcal{P}$ be the sets of productivity and preference types, respectively, and $\Theta = A \times \mathcal{P}$. We assume that A is a finite set, which implies that Θ is a finite set, so the population G of worker types is a discrete measure.[59] We continue to assume that there is a continuum of workers.

The preferences of firms are also adapted for our environment. For each firm $f \in F$, worker types Θ are partitioned into $\mathbb{P}_f := \{\Theta_f^1, ..., \Theta_f^{K_f}\}$ such that f is indifferent across

[59]The finiteness of A is necessitated by our use of weak-$*$ topology as well as the construction of strong stability and strategy-proof mechanisms below. To illustrate the difficulty, suppose that A is a unit interval and G has a well defined density. Our construction below would require that the density associated with firms' choice mappings satisfy a certain population proportionality property. Convergence in our weak-$*$ topology does not preserve this restriction on density. Consequently, the operator T may violate upper hemicontinuity, which may result in the failure of the nonempty-valuedness of our solution. It may be possible to address this issue by strengthening the topology, but whether the resulting space satisfies the conditions that would guarantee the existence of a stable matching remains an open question.

all types within each indifference class $\Theta_f^k \subset \Theta$, for $k \in I_f := \{1, ..., K_f\}$. Since a firm differentiates workers based only on their productivity types, we require that if $(a, P) \in \Theta_f^k$ for some $P \in \mathbb{P}_f$, then $(a, P') \in \Theta_f^k$ for all $P' \in \mathbb{P}_f$. At the same time, a firm can be indifferent across multiple productivity types, in ways that are arbitrary and may differ across firms. We assume that each firm has a unique optimal choice in terms of the measure of workers in each indifference class, and let $\Lambda_f^k : \mathcal{X} \to \mathbb{R}_+$ denote firm f's unique choice of total measure of workers in each indifference class Θ_f^k, $k \in I_f$,[60] which induces a choice correspondence

$$C_f(X) = \{Y \sqsubset X \mid \sum_{\theta \in \Theta_f^k} Y(\theta) = \Lambda_f^k(X), \forall k \in I_f\} \tag{10}$$

for each $X \in \mathcal{X}$. Continuity and substitutability of preferences can be defined in terms of Λ_f^k. If $\Lambda_f^k(\cdot)$ is continuous for each $k \in I_f$ (in the Euclidean topology), then the induced correspondence C_f is upper hemicontinuous and convex valued. In that case, we simply say a firm f's preference is **continuous**. Another case of interest is when $\sum_{\theta \in \Theta_f^k} X(\theta) - \Lambda_f^k(X)$ is nondecreasing in $(X(\theta))_{\theta \in \Theta}$ for each $k \in I_f$. In this case, the induced correspondence C_f is weakly substitutable, and we simply call a firm f's preference to be **weakly substitutable**.

As before, a matching is described by a profile $M = (M_f)_{f \in \tilde{F}}$ of subpopulations of workers matched with alternative firms or the outside option. We assume that all workers of the same (reported) type are treated identically ex ante. Hence, given matching M, a worker of type (a, P) in the support of G is matched to $f \in \tilde{F}$ with probability $\frac{M_f(a,P)}{G(a,P)}$. Note that $\sum_{f \in \tilde{F}} \frac{M_f(a,P)}{G(a,P)} = 1$ holds by construction, giving rise to a valid probability distribution over $\tilde{F}$. A **mechanism** is a function φ that maps any $G \in \overline{\mathcal{X}}$ to a matching.

We now introduce a strong notion of stability proposed by Kesten and Ünver (2014):

Definition 12. A matching M is **strongly stable** if (i) it is stable and (ii) for any $f \in F$, $k \in I_f$, and $\theta, \theta' \in \Theta_f^k$, if $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$, then $\sum_{f' \in \tilde{F} : f' \prec_\theta f} M_{f'}(\theta) = 0$.

In other words, strong stability requires that, if a worker of type θ is assigned a firm f with strictly lower probability than another type θ' in the same indifference class for firm f, then the type-θ worker should never be assigned any firm f' that the worker ranks below f. In that sense, discrimination among workers in the same priority class should not occur.

Strategy-proofness can be defined via a stochastic dominance order, as proposed by Bogomolnaia and Moulin (2001).

Definition 13. A mechanism φ is **strategy-proof for workers** if, for each (reported) population $G \in \overline{\mathcal{X}}$, productivity type $a \in A$, preference types P and P' in $\mathcal{P}$ such that both

[60] Specifically, we assume that for each $X \sqsubset G$, $\Lambda_f^k(X) \in [0, \sum_{\theta \in \Theta_f^k} X(\theta)]$ and $\Lambda_f^k(X') = \Lambda_f^k(X)$ whenever $\sum_{\theta \in \Theta_f^{k'}} X'(\theta) = \sum_{\theta \in \Theta_f^{k'}} X(\theta)$ for all $k' \in I_f$. We also assume that $\Lambda_f^k(X') = \Lambda_f^k(X)$ whenever $\Lambda_f^{k'}(X) \leqslant \sum_{\theta \in \Theta_f^{k'}} X'(\theta) \leqslant \sum_{\theta \in \Theta_f^{k'}} X(\theta)$ for all $k' \in I_f$, which captures the revealed preference property.

(a, P) and (a, P') are in the support of G, and $f \in \tilde{F}$, we have

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} \geqslant \sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')}. \tag{11}$$

In words, strategy-proofenss means that a truthful reporting induces a random assignment for each worker that first-order stochastically dominates any random assignment that would result from untruthful reporting. Note that a worker can misreport only her preference type and not her productivity type (recall that a worker's productivity type determines firms' preferences regarding her).[61]

We are now ready to state our main result. Our approach is to demonstrate the existence of a stable matching that satisfies an additional property. Say a matching M is **population-proportional** if, for each $f \in F$ and $k \in I_f$, there is some $\alpha_f^k \in [0, 1]$ such that

$$M_f(\theta) = \min\{D^{\preceq f}(M)(\theta), \alpha_f^k G(\theta)\}, \forall \theta \in \Theta_f^k. \tag{12}$$

In other words, the measure of workers hired by firm f from the indifference class Θ_f^k is given by the same proportion α_f^k of $G(\theta)$ for all $\theta \in \Theta_f^k$, unless the measure of worker types θ available to f is less than the proportion α_f^k of $G(\theta)$, in which case the entire available measure of that type is assigned to that firm. In short, a population-proportional matching seeks to match a firm with workers of different types in proportion to their population sizes at G whenever possible, if they belong to the same indifference class of the firm. The stability and population proportionality of a mechanism translate into the desired fairness and incentive properties, as shown by the following result.

Lemma 3. *(i) If a matching is stable and population-proportional, then it is strongly stable.*

(ii) If a mechanism φ implements a strongly stable matching for every measure in $\overline{\mathcal{X}}$, then the mechanism is strategy-proof for workers.

Proof. See Section S.8 of Supplementary Material. ▌

We now present the main result of this section.

[61]Note also that unlike in finite population models, the worker cannot alter the population G by unilaterally misreporting her preferences because there is a continuum of workers. Further, we only impose restriction (11) for types (a, P) and (a, P') that are in the support of G. For the true worker type (a, P), this is the same assumption as in the standard strategy-proofness concept for finite markets. We do not impose any condition for misreporting a measure zero type because if φ is individually rational (which is the case for stable mechanisms), then the incentives for misreporting as a measure zero type can be eliminated by specifying the mechanism to assign a worker reporting such a type to the null firm with probability one.

Theorem 11. *If each firm's preference is continuous or if each firm's preference is weakly substitutable, then there exists a matching that is stable and population-proportional. Therefore, given the domain satisfying either property, there exists a mechanism that admits a strongly stable matching and is strategy-proof.*

Proof. See Section S.8 of Supplementary Material. ▌

Recall that the workers of the same reported type receive the same ex ante assignment. By Lemma 3, strong stability and strategy-proofness will be achieved if each firm's choice were to respect population proportionality. A key step of proof is therefore to select an optimal choice $\tilde{C}_f \in C_f$ that induces population proportionality for each f. The selection $\tilde{C}_f$ is then shown to satisfy the conditions of Theorems 2 and 3 given the continuity or substitutability conditions. Thus, a stable matching exists in the hypothetical continuum economy in which firms have preferences represented by the choice functions $\tilde{C}_f$. The final step is to show that the stable matching of the hypothetical economy is stable in the original economy and satisfies population proportionality.

This result establishes the existence of a matching mechanism that satisfies strong stability and strategy-proofness for workers in a large economy environment.[62] In contrast to the existing literature, our result holds under general firm preferences that may involve indifferences and/or complementarities.

8.2 Applications to Time Share/Probabilistic Matching Models

Our model introduced in Section 8.1 has a connection with time share and probabilistic matching models. In these models, a *finite* set of workers contracts with a finite set of firms for time shares or for probabilities with which they match. Probabilistic matching is often used in allocation problems without money, such as school choice, while time share models have been proposed as a solution to labor matching markets in which part-time jobs are available (see Biró, Fleiner, and Irving (2013) for instance).

Our model in Section 8.1 can be reinterpreted as a time share model. Let Θ be the *finite* set of workers whose shares firms may contract for, *as opposed to the finite types of a continuum of workers.* The measure $G(\theta)$ represents the total endowment of time or the probability that a worker θ has available for matching. A matching M describes the time or probability $M_f(\theta)$ that a worker θ and a firm f are matched.

The partition $\mathbb{P}_f$ then describes firm f's set of indifference classes, where each class describes the set of workers that the firm considers equivalent. The function $\Lambda_f = (\Lambda_f^k)_{k \in I_f}$ describes the time shares that firm f wishes to choose from available time shares in alternative indifference classes. On the worker side, for each worker $\theta \in \Theta_P$, the first-order

[62] Even with a continuum of workers, no stable mechanism is strategy-proof for firms. See an example in Section S.8.2 of Supplementary Material.

stochastic dominance induced by P describes the preference of the worker in evaluating lotteries. With this reinterpretation, Definition 12 provides an appropriate notion of a strongly stable matching.[63] The following result is immediate:

Corollary 3. *The (reinterpreted) time share model admits a strongly stable—and thus stable—matching if either each firm's preference is continuous or it is weakly substitutable.*[64]

This result generalizes the existence of a strongly stable matching in the school choice problem studied by Kesten and Ünver (2014), where schools may regard multiple students as having the same priority. They show their existence of a strongly stable probabilistic matching (which they call strong ex ante stability) under the assumption that schools have responsive preferences with ties. Our contribution is to extend the existence to general preferences that may violate responsiveness. Our result might be useful for school choice environments in which schools may need a balanced student body in terms of gender, ethnicity, income, or skill (recall footnote 3).

9 Relationship with the Literature

The present paper is connected with several strands of literature. Most importantly, it is related to the growing literature on matching and market design. Since the seminal contributions of Gale and Shapley (1962) and Roth (1984), stability has been recognized as the most compelling solution concept in matching markets.[65] As argued and demonstrated by Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kominers (2017) in various situations, the substitutability condition is necessary and sufficient to guarantee the existence of a stable matching with a finite number of agents. Our paper contributes to this line of research by showing that substitutability is not necessary for the existence of a stable matching when there is a continuum of agents on one side of the market, and that an approximately stable matching exists in a large finite market.

[63]The notion of strong stability in Definition 12 requires the proportion of time spent with a firm out of total endowment to be equalized among workers that the firm considers equivalent. This notion is sensible in the context of a time share model, particularly when $G(\theta)$ is the same across all workers, as with school choice (where each student has a unit demand). When $G(\theta)$ is different across θs, however, one could consider an alternative notion, such as one that equalizes the absolute amount of time (not divided by $G(\theta)$) that a worker spends with a firm. Our analysis can be easily modified to prove the existence of matching that satisfies this alternative notion of strong stability.

[64]Unlike the continuum model, population proportionality does not guarantee strategy-proofness. As is shown by Kesten and Ünver (2014), strategy-proofness is generally impossible to attain in the time share/probabilistic models with finite numbers of workers.

[65]See Roth (1991) and Kagel and Roth (2000) for empirical and experimental evidence on the importance of stability in labor markets and Abdulkadiroğlu and Sönmez (2003) for the interpretation of stability as a fairness concept in school choice.

Our study was inspired by recent research on matching with a continuum of agents by Abdulkadiroğlu, Che, and Yasuda (2015) and Azevedo and Leshno (2016).[66] As in the present study, these authors assume that there are a finite number of firms and a continuum of workers. In particular, Azevedo and Leshno (2016) show the existence and uniqueness of a stable matching in that setting. However, as opposed to the present study, these authors assume that firms have responsive preferences—which is a special case of substitutability. Our contribution is to show that the almost universally invoked restrictions on preferences (such as responsiveness or even substitutability) are not necessary for the existence of a stable matching in the continuum economy.

An independent and contemporaneous study by Azevedo and Hatfield (2015) (henceforth, AH) also analyzes matching with a continuum of agents.[67] Consistent with our study, these authors find that a stable matching exists even when not all agents have substitutable preferences. However, the two studies have several notable differences. First, AH consider a continuum of firms each employing a finite number of workers; thus, they consider a continuum of agents on both sides of the market. By contrast, the present paper considers a finite number of firms each employing a continuum of workers. These two models thus provide complementary approaches for studying large markets, and they are applicable to different environments.[68]

Second, AH assume that there is a finite number of both firm and worker types, which enables them to use Brouwer's fixed point theorem to demonstrate the existence of a stable matching. By contrast, we place no restriction on the number of workers' types and thus allow for both finite and infinite numbers of types, and this generality in type spaces requires a topological fixed point theorem from functional analysis. To the best of our knowledge, this type of mathematics has never been applied to two-sided matching, and we view the introduction of these tools into the matching literature as one of our methodological contributions. Our model also has the advantage of subsuming the previous work by Azevedo and Leshno (2016) as well as many of the other studies mentioned above that assume a continuum of worker types. Finally, the substantive issues studied in these papers are sig-

[66]Various recent studies on large matching markets are also related but formally different, such as Roth and Peranson (1999), Immorlica and Mahdian (2005), Kojima and Pathak (2009), Kojima and Manea (2010), Manea (2009), Che and Kojima (2010), Lee (2017), Liu and Pycia (2013), Che and Tercieux (2017, 2015), Ashlagi, Kanoria, and Leshno (2017), Miralles (2008), Miralles and Pycia (2017), Kojima, Pathak, and Roth (2013), and Hatfield, Kojima, and Narita (2016).

[67]Although not as closely related, our study is also analogous to Azevedo, Weyl, and White (2013), who demonstrate the existence of competitive equilibrium in an exchange economy with a continuum of agents and indivisible objects.

[68]For example, in the context of school choice, many school districts consist of a small number of schools that each admit hundreds of students, which fits well with our approach. However, in a large school district such as New York City, the number of schools admitting students is also large, and the AH model may offer a good approximation.

nificantly different. Indifferences in preferences, substitutable preferences, incentives, and fairness are studied only by the present paper, while many-to-many matching, core, and general equilibrium are studied only by AH. While they focus on the existence of various solution concepts under complementarities, we offer a comprehensive study of a variety of large matching markets. Overall, these points make the two papers substantially different.

Our methodological contribution is also related to another recent advance in matching theory based on the monotone method. In the context of one-to-one matching, Adachi (2000) defines a certain operator whose fixed points are equivalent to stable matchings. His work has been generalized in many directions by Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Ostrovsky (2008), and Hatfield and Kominers (2017), among others, and we also define an operator whose fixed points are equivalent to stable matchings. However, these previous studies also impose restrictions on preferences (e.g., responsiveness or substitutability) so that the operator is monotone, and utilize Tarski's fixed point theorem to ensure a stable matching. By contrast, a significant part of our paper does not impose responsiveness or substitutability restrictions on preferences; instead, we rely on the continuum of workers—along with continuity in firms' preferences—to guarantee the continuity of the operator (in an appropriately chosen topology). This approach allows us to use a generalization of the Kakutani fixed point theorem, a more familiar tool in traditional economic theory that is used in existence proofs of general equilibrium and Nash equilibrium in mixed strategies. Even for substitutable preferences, we are able to weaken the condition for existence and other properties of interest by accommodating indifferences.

The present paper is related to the literature on general equilibrium, especially models with clubs. To our knowledge, the closest contributions are two related papers by Ellickson, Grodal, Scotchmer, and Zame (1999, 2001).[69] Like the present paper, these papers consider large finite economies as well as continuum economies. They show the existence of a general equilibrium in large markets using Kakutani's fixed point theorem. Despite these similarities, there are also a number of notable differences. First, Ellickson, Grodal, Scotchmer, and Zame (1999, 2001) assume the existence of private goods and those private goods are divisible, while our model does not presume the existence of a private good. Second, they assume that it is possible to make transfer among club members. Third, in their model, the size of clubs (groups) as well as the number of agents' types are finite. In this respect, their model is closer to the large matching market models in which a continuum of firms each hire finite number of workers as in AH. By contrast, in our model a finite number of firms each hire a continuum of workers, which makes the analysis quite different. Due to these differences, our results and theirs are logically unrelated, and it seems impossible to make a direct comparison.

The current paper is also related to the literature on matching with couples. Like a firm

[69] Although not as closely related to our paper, other notable contributions include Ellickson (1979), Scotchmer and Wooders (1987), Gilles and Scotchmer (1997), and Scotchmer and Shannon (2015). See Sandler and Tschirhart (1997) for a survey.

in our model, a couple can be seen as a single agent with complementary preferences over contracts. Roth (1984) and unpublished work by Sotomayor show that a stable matching does not necessarily exist in the presence of a "couples" problem. Klaus and Klijn (2005) provide a condition for the existence of stable matchings in such a context. Pycia (2012) and Echenique and Yenmez (2007) study many-to-one matching with complementarities as well as with peer effects. These papers allow for complementarities like our paper, but they do not study large economies.

Closer to our study, several recent papers study couples problem in the context of large economies. Kojima, Pathak, and Roth (2013) provide conditions under which the probability that a stable matching exists even in the presence of couples converges to one as the market becomes infinitely large. Similar conditions have been further analyzed by Ashlagi, Braverman, and Hassidim (2014). Nguyen and Vohra (2014) study how one can minimally modify firms' quotas to guarantee a stable matching in a problem with couples.[70] Like our paper, these studies suggest finding a stable matching becomes easier in a large market even in the presence of complementarities. However, there is an important difference. It is crucial for their results that the only complementarity is caused by couples, meaning that the complementarity is between only two positions.[71] In other words, their results are relevant primarily for cases in which the magnitude of complementarities is small. By contrast, we allow for firms to have complementarity over arbitrarily large groups of workers.

10 Conclusion

Complementarity, although prevalent in matching markets, has been known as a source of difficulties for designing desirable mechanisms. The present paper took a step toward addressing the difficulties by considering large economies. We find that complementarity need not jeopardize stability in a large market. First, as long as preferences are continuous or substitutable, a stable matching exists in a limit continuum economy. Second, with such preferences, there exists an approximately stable matching in a large finite economy. We used this framework to show that there is a stable mechanism that is strategy-proof for workers and satisfies an additional fairness property, strong stability.

The scope of our analysis can be extended in a couple of directions. First, we can introduce "contracts," namely to allow each firm-worker pair to match under alternative contracts, as has been done by Hatfield and Milgrom (2005) in the context of substitutable preferences. Just as in our baseline model, we focus on the measures of workers matched with

[70] Nguyen, Peivandi, and Vohra (2016) also study preference complementarity. Their contribution is not as close to our study, however, as they study (stochastic) object-allocation problems rather than two-sided matching.

[71] These papers study more general complementarities as well, but their results become weaker under general complementarities.

alternative firms as basic unit of analysis. But unlike our main model, a *vector* of measures of workers matches with a firm under *alternative contract terms*. With this enrichment of the underlying space, our method can be extended to yield existence in this setup.[72] This result is provided in Section S.9 of the Supplementary Material.

Second, while we have considered the model in which a finite number of "large" firms match with a continuum of workers, we can extend our framework to study a model in which a continuum of "small" firms match with a continuum of workers, as has been studied by AH. Take their main model and for simplicity consider the pure matching case (i.e., without contracts) in which each worker demands at most one position. Assume as have been by AH that the set of firm types is finite. Then, one can interpret the entire mass of firms of each given type as a single "large" firm and "build" an *aggregate choice correspondence* for that fictitious large firm from optimal choices of infinitesimal firms (of the same type), say by maximizing their aggregate welfare. The aggregate choice correspondence constructed in this way is shown to satisfy the continuity condition required for the existence of a stable matching in Theorem 2. Therefore, our model can recover the existence result for a certain special case of AH. The specific result is described in Section S.10 of Supplementary Material.

To the best of our knowledge, this paper is the first to analyze matching in a continuum economy with the level of generality presented here. As such, our paper may pose as many questions as it answers. One issue worth pursuing is the computation of a stable matching. The existence of a stable matching, as established in this paper, is clearly necessary to find a desired mechanism, but practical implementation requires an algorithm. Although our fixed point mapping provides one such algorithm in the case it is contractionary or monotonic (i.e., preferences are substitutable), studying the computationally efficient and generally applicable algorithms to find stable matchings would be an interesting and challenging future research topic.

Appendix A Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. **("Only if" part):** Suppose that M is a stable matching, and let $X = (X_f)_{f\in\tilde{F}}$ with $X_f = D^{\preceq f}(M), \forall f \in \tilde{F}$. We prove that X is a fixed point of T. Let us first show that for each $f \in \tilde{F}$, $X_f \in \mathcal{X}$. It is clear that as each $M_{f'}(\Theta_P \cap \cdot)$ is nonnegative and countably additive, so is $X_f(\cdot)$. It is also clear that since $(M_f)_{f\in\tilde{F}}$ is a matching, $X_f \sqsubset G$. Thus, we have $X_f \in \mathcal{X}$.

We next claim that $M_f \in C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \emptyset$ since $M_\emptyset \sqsubset X_\emptyset = C_\emptyset(X_\emptyset)$. To prove the claim for $f \neq \emptyset$, suppose for a contradiction that

[72] Nevertheless, the generalization is more than mechanical. Since the measure of workers a firm matches with under a contract term depends on the measure of workers the same firm matches with under a different contract term, special care is needed to define the choice function as well as the measure of available workers to a firm under a particular contract term.

$M_f \notin C_f(X_f)$, which means that there is some $M'_f \in C_f(X_f)$ such that $M'_f \neq M_f$. Note that $M_f \sqsubset X_f$ and thus $(M'_f \vee M_f) \sqsubset X_f$. Then, by revealed preference, we have $M_f \notin C_f(M'_f \vee M_f)$ and $M'_f \in C_f(M'_f \vee M_f)$ or $M'_f \succ_f M_f$, which means that M is unstable since $M'_f \sqsubset X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X \in T(X)$. The fact that $M_f \in C_f(X_f), \forall f \in \tilde{F}$ means that $X_f - M_f \in R_f(X_f), \forall f \in \tilde{F}$. We set $Y_f = X_f - M_f$ for each $f \in \tilde{F}$ and obtain for any $E \in \Sigma$

$$\begin{aligned}
&\sum_{P:P(1)=f} G(\Theta_P \cap E) + \sum_{P:P(1)\neq f} Y_{f_-^P}(\Theta_P \cap E) \\
&= \sum_{P:P(1)=f} G(\Theta_P \cap E) + \sum_{P:P(1)\neq f} \Big(X_{f_-^P}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E) \Big) \\
&= \sum_{P:P(1)=f} G(\Theta_P \cap E) + \sum_{P:P(1)\neq f} \left(\sum_{f' \in \tilde{F}: f' \preceq_P f_-^P} M_{f'}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E) \right) \\
&= \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E) = X_f(E),
\end{aligned}$$

where the second and fourth equalities follow from the definition of $X_{f_-^P}$ and X_f, respectively, while the third from the fact that f_-^P is an immediate predecessor of f and $\sum_{f' \in \tilde{F}: f' \preceq_P P(1)} M_{f'}(\Theta_P \cap E) = G(\Theta_P \cap E)$. The above equation holds for every firm $f \in \tilde{F}$, so we conclude that $X \in T(X)$, i.e. X is a fixed point of T.

("If" part): Let us first introduce some notation. Let f_+^P denote an **immediate successor** of $f \in \tilde{F}$ at $P \in \mathcal{P}$: that is, $f_+^P \prec_P f$, and for any $f' \prec_P f$, $f' \preceq_P f_+^P$. Note that for any $f, \tilde{f} \in \tilde{F}$, $f = \tilde{f}_-^P$ if and only if $\tilde{f} = f_+^P$.

Suppose now that $X \in \mathcal{X}^{n+1}$ is a fixed point of T. For each firm $f \in \tilde{F}$ and $E \in \Sigma$, define

$$M_f(E) = X_f(E) - \sum_{P:P(n+1)\neq f} X_{f_+^P}(\Theta_P \cap E), \tag{13}$$

where $P(n+1)$ is the least preferred firm according to P.

We first verify that for each $f \in \tilde{F}$, $M_f \in \mathcal{X}$. First, it is clear that for each $f \in \tilde{F}$, M_f is countably additive as both $X_f(\cdot)$ and $X_{f_+^P}(\Theta_P \cap \cdot)$ are countably additive. It is also clear that for each $f \in \tilde{F}$, $M_f \sqsubset X_f$. To see that $M_f(E) \geqslant 0, \forall E \in \Sigma$, observe from (13) that

$$\begin{aligned}
M_f(E) &= \sum_{P:P \in \mathcal{P}} X_f(\Theta_P \cap E) - \sum_{P:P(n+1)\neq f} X_{f_+^P}(\Theta_P \cap E) \\
&\geqslant \sum_{P:P(n+1)\neq f} \Big(X_f(\Theta_P \cap E) - X_{f_+^P}(\Theta_P \cap E) \Big) \geqslant 0,
\end{aligned}$$

where the inequality holds since $X \in T(X)$ means that there is some $Y_f \in R_f(X_f)$ such that $X_{f_+^P}(\Theta_P \cap \cdot) = Y_f(\Theta_P \cap \cdot)$ for each $P \in \mathcal{P}$, and thus $X_{f_+^P}(\Theta_P \cap \cdot) \sqsubset X_f(\Theta_P \cap \cdot)$.

Let us next show that for all $f \in \tilde{F}$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_f(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \tag{14}$$

which means that $X_f = D^{\preceq f}(M)$. To do so, fix any $P \in \mathcal{P}$ and consider first a firm $f = P(n+1)$ (i.e., a firm ranked lowest at P). By (13), $M_f(\Theta_P \cap E) = X_f(\Theta_P \cap E)$ and thus (14) holds for such f. Consider next any $f \neq P(n+1)$, and assume for an inductive argument that (14) holds true for f_+^P, so $X_{f_+^P}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f_+^P} M_{f'}(\Theta_P \cap E)$. Then, by (13), we have

$$\begin{aligned} X_f(\Theta_P \cap E) = M_f(\Theta_P \cap E) + X_{f_+^P}(\Theta_P \cap E) &= M_f(\Theta_P \cap E) + \sum_{f' \in \tilde{F}: f' \preceq_P f_+^P} M_{f'}(\Theta_P \cap E) \\ &= \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \end{aligned}$$

as desired.

To show that $M = (M_f)_{f \in \tilde{F}}$ is a matching, let $f = P(1)$ and note that by definition of T, if $\tilde{X} \in T(X)$, then $\tilde{X}_f(\Theta_P \cap \cdot) = G(\Theta_P \cap \cdot)$. Since $X \in T(X)$, we have for any $E \in \Sigma$

$$G(\Theta_P \cap E) = X_f(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}} M_{f'}(\Theta_P \cap E),$$

where the second equality follows from (14). Since the above equation holds for every $P \in \mathcal{P}$, M is a matching.

We now prove that $(M_f)_{f \in \tilde{F}}$ is stable. As noted by footnote 21, the first part of Condition 1 is implied by Condition 2, which we check below. To prove the second part of Condition 1 of Definition 2, note first that $C_\emptyset(X_\emptyset) = \{X_\emptyset\}$ and thus $R_\emptyset = 0$. Fix any $P \in \mathcal{P}$ and assume $\emptyset \neq P(n+1)$, since there is nothing to prove if $\emptyset = P(n+1)$. Consider a firm f such that $f_-^P = \emptyset$. Then, X being a fixed point of T means $X_f(\Theta_P) = R_{f_-^P}(\Theta_P) = 0$, which implies by (14) that $0 = X_f(\Theta_P) = \sum_{f': f' \preceq_P f} M_{f'}(\Theta_P) = \sum_{f': f' \prec_P \emptyset} M_{f'}(\Theta_P)$, as desired.

It only remains to check Condition 2 of Definition 2. Suppose for a contradiction that it fails. Then, there exist f and M'_f such that

$$M'_f \sqsubset D^{\preceq f}(M),\ M'_f \in C_f(M'_f \vee M_f),\ \text{and } M_f \notin C_f(M'_f \vee M_f). \tag{15}$$

So $M'_f \sqsubset D^{\preceq f}(M) = X_f$. Since then $M_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$ and $M_f \in C_f(X_f)$, the revealed preference property implies $M_f \in C_f(M'_f \vee M_f)$, contradicting (15). We have thus proven that M is stable. ▮

Proof of Theorem 2. We establish the compactness of $\mathcal{X}$ and the upper hemicontinuity of T in Lemma 4 and Lemma 5 below. To do so, recall that $\mathcal{X}$ is endowed with weak-∗

topology. The notion of convergence in this topology, i.e. weak convergence, can be stated as follows: A sequence of measures $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ weakly converges to a measure $X \in \mathcal{X}$, written as $X_k \xrightarrow{w^*} X$, if and only if $\int_\Theta h dX_k \to \int_\Theta h dX$ for all $h \in C(\Theta)$, where $C(\Theta)$ is the space of all continuous functions defined on Θ. The next result provides some conditions that are equivalent to weak convergence.

Theorem 12. *Let X and $(X_k)_{k\in\mathbb{N}}$ be finite measures on Σ.*[73] *The following conditions are equivalent:*[74]

(a) $X_k \xrightarrow{w^*} X$;

(b) $\int_\Theta h dX_k \to \int_\Theta h dX$ *for all $h \in C_u(\Theta)$, where $C_u(\Theta)$ is the space of all uniformly continuous functions defined on Θ;*

(c) $\liminf_k X_k(A) \geqslant X(A)$ *for every open set $A \subset \Theta$, and $X_k(\Theta) \to X(\Theta)$;*

(d) $\limsup_k X_k(A) \leqslant X(A)$ *for every closed set $A \subset \Theta$, and $X_k(\Theta) \to X(\Theta)$;*

(e) $X_k(A) \to X(A)$ *for every set $A \in \Sigma$ such that $X(\partial A) = 0$ (∂A denotes the boundary of A).*

Lemma 4. *The space $\mathcal{X}$ is convex and compact. Also, for any $X \in \mathcal{X}$, $\mathcal{X}_X$ is compact.*

Proof. Convexity of $\mathcal{X}$ follows trivially. To prove the compactness of $\mathcal{X}$, let $C(\Theta)^*$ denote the dual (Banach) space of $C(\Theta)$ and note that $C(\Theta)^*$ is the space of all (signed) measures on (Θ, Σ), given Θ is a compact metric space.[75] Then, by Alaoglu's Theorem, the closed unit ball of $C(\Theta)^*$, denoted U^*, is weak-$*$ compact.[76] Clearly, $\mathcal{X}$ is a subspace of U^* since for any $X \in \mathcal{X}$, $\|X\| = X(\Theta) \leqslant G(\Theta) = 1$. The compactness of $\mathcal{X}$ will thus follow if $\mathcal{X}$ is shown to be a closed set. To prove this, we prove that for any sequence $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ and

[73] We note that this result can be established without having to assume that X is nonnegative, as long as all X_k's are nonnegative.

[74] This theorem is a modified version of "Portmanteau Theorem" that is modified to deal with any finite (i.e. not necessarily probability) measures. See Theorem 2.8.1 of Ash and Doléans-Dade (2009) for this result, for instance.

[75] More precisely, $C(\Theta)^*$ is isometrically isomorphic to the space of all signed measures on (Θ, Σ) according to the Riesz Representation Theorem (see Royden and Fitzpatrick (2010) for instance).

[76] The closed unit ball is defined as $U^* := \{X \in C^*(\Theta) : \|X\| \leqslant 1\}$, where $\|X\|$ is the dual norm, i.e.,

$$\|X\| = \sup\{\left|\int_\Theta h dX\right| : h \in C(\Theta) \text{ and } \max_{\theta\in\Theta} |h(\theta)| \leqslant 1\}.$$

If X is a nonnegative measure, then the supremum is achieved by taking $h \equiv 1$, and thus $\|X\| = X(\Theta)$. It is well known (see Royden and Fitzpatrick (2010) for instance) that if $C(\Theta)^*$ is infinite dimensional, then U^* is not compact under the norm topology (i.e., the topology induced by the dual norm). On the other hand, U^* is compact under the weak-$*$ topology, which follows from Alaoglu's Theorem (see Royden and Fitzpatrick (2010) for instance).

$X \in C(\Theta)^*$ such that $X_k \xrightarrow{w^*} X$, we must have $X \in \mathcal{X}$, which will be shown if we prove that $0 \leqslant X(E) \leqslant G(E)$ for any $E \in \Sigma$. Let us first make the following observation: every finite (Borel) measure X on the metric space Θ is normal,[77] which means that for any set $E \in \Sigma$,

$$X(E) = \inf\{X(O) : E \subset O \text{ and } O \in \Sigma \text{ is open}\} \tag{16}$$
$$= \sup\{X(F) : F \subset E \text{ and } F \in \Sigma \text{ is closed}\}. \tag{17}$$

To show first that $X(E) \leqslant G(E)$, consider any open set $O \in \Sigma$ such that $E \subset O$. Then, since $X_k \in \mathcal{X}$ for every k, we must have $X_k(O) \leqslant G(O)$ for every k, which, combined with (c) of Theorem 12 above, implies that $X(O) \leqslant \liminf_k X_k(O) \leqslant G(O)$. Given (16), this means that $X(E) \leqslant G(E)$.

To show next that $X(E) \geqslant 0$, consider any closed set $F \in \Sigma$ such that $F \subset E$. Since $X_k \in \mathcal{X}$ for every k, we must have $X_k(F) \geqslant 0$, which, combined with (d) of Theorem 12 above, implies that $X(F) \geqslant \limsup_k X_k(F) \geqslant 0$. Given (17), this means $X(E) \geqslant 0$.

The proof for the compactness of $\mathcal{X}_X$ is analogous and hence omitted. ▌

Lemma 5. *The map T is a correspondence from $\mathcal{X}^{n+1}$ to itself. Also, it is nonempty- and convex-valued, and upper hemicontinuous.*

Proof. To show that T maps from $\mathcal{X}^{n+1}$ to itself, observe that for any $X \in \mathcal{X}^{n+1}$ and $\tilde{X} \in T_f(X)$, there is $Y_f \in R_f(X_f)$ for each $f \in \tilde{F}$ such that for all $E \in \Sigma$,

$$\tilde{X}(E) = \sum_{P \in \mathcal{P}} Y_{f_-^P}(\Theta_P \cap E) \leqslant \sum_{P \in \mathcal{P}} X_{f_-^P}(\Theta_P \cap E) \leqslant \sum_{P \in \mathcal{P}} G(\Theta_P \cap E) = G(E),$$

which means that $\tilde{X} \in \mathcal{X}$, as desired.

As noted earlier, the correspondence T is nonempty-valued. To prove that T is convex-valued, it suffices to show that for each $f \in \tilde{F}$, R_f is convex-valued. Consider any $X \in \mathcal{X}$ and $Y', Y'' \in R_f(X)$. There are some $X', X'' \in C_f(X)$ satisfying $Y' = X - X'$ and $Y'' = X - X''$. Then, the convexity of $C_f(X)$ implies that for any $\lambda \in [0, 1]$, $\lambda X' + (1 - \lambda)X'' \in C_f(X)$ so $\lambda Y' + (1 - \lambda)Y'' = X - (\lambda X' + (1 - \lambda)X'') \in R_f(X)$.

To establish the upper hemicontinuity of T, we first establish the following claim:

Claim 1. *For any sequence $(X_k)_{k\in\mathbb{N}} \subset \mathcal{X}$ that weakly converges to $X \in \mathcal{X}$, a sequence $(X_k(\Theta_P \cap \cdot))_{k\in\mathbb{N}}$ also weakly converges to $X(\Theta_P \cap \cdot)$ for all $P \in \mathcal{P}$.*

Proof. Let X^P and X_k^P denote $X(\Theta_P \cap \cdot)$ and $X_k(\Theta_P \cap \cdot)$, respectively. Note first that for any $X \in \mathcal{X}$, we have $X^P \in \mathcal{X}$ for all $P \in \mathcal{P}$. Due to Theorem 12, it suffices to show that X^P and $(X_k^P)_{k\in\mathbb{N}}$ satisfy the condition (c) of Theorem 12. To do so, consider any open set $O \subset \Theta$. Then, letting Θ_P° denote the interior of Θ_P,

$$\liminf_k X_k^P(O) = \liminf_k X_k(\Theta_P^\circ \cap O) + X_k(\partial\Theta_P \cap O)$$

[77] See Theorem 12.5 of Aliprantis and Border (2006).

$$= \liminf_k X_k(\Theta_P^\circ \cap O) \geqslant X(\Theta_P^\circ \cap O) = X^P(O),$$

where the second equality follows from the fact that $X_k(\partial\Theta_P \cap O) \leqslant X_k(\partial\Theta_P) \leqslant G(\partial\Theta_P) = 0$, the lone inequality from $X_k \xrightarrow{w^*} X$, (c) of Theorem 12, and the fact that $\partial\Theta_P^\circ \cap O$ is an open set, and the last equality from repeating the first two equalities with X instead of X_k. Also, we have

$$X_k^P(\Theta) = X_k(\Theta_P) \to X(\Theta_P) = X^P(\Theta),$$

where the convergence is due to $X_k \xrightarrow{w^*} X$, (e) of Theorem 12, and the fact that $X(\partial\Theta_P) \leqslant G(\partial\Theta_P) = 0$. Thus, the two requirements in condition (c) of Theorem 12 are satisfied, so $X_k^P \xrightarrow{w^*} X^P$, as desired. ▌

It is also straightforward to observe that if C_f is upper hemicontinuous, then R_f is also upper hemicontinuous.

We now prove the upper hemicontinuity of T by considering any sequences $(X_k)_{k\in\mathbb{N}}$ and $(\tilde{X}_k)_{k\in\mathbb{N}}$ in $\mathcal{X}^{n+1}$ weakly converging to some X and $\tilde{X}$ in $\mathcal{X}^{n+1}$, respectively, such that $\tilde{X}_k \in T_f(X_k)$ for each k. To show that $\tilde{X} \in T(X)$, let $X_{k,f}$ and $\tilde{X}_{k,f}$ denote the components of X_k and $\tilde{X}_k$, respectively, that correspond to $f \in \tilde{F}$. Then, we can find $Y_{k,f} \in R_f(X_{k,f})$ for each k and f such that $\tilde{X}_{k,f}(\cdot) = \sum_{P\in\mathcal{P}} Y_{k,f_-^P}(\Theta_P \cap \cdot)$, which implies that for all $f \in \tilde{F}$ and $P \in \mathcal{P}$, $\tilde{X}_{k,f_+^P}(\Theta_P \cap \cdot) = Y_{k,f}(\Theta_P \cap \cdot)$ since f is the immediate predecessor of f_+^P at P. As $\tilde{X}_{k,f} \xrightarrow{w^*} \tilde{X}_f, \forall f \in \tilde{F}$, by assumption, we have $\tilde{X}_{k,f_+^P}(\Theta_P \cap \cdot) \xrightarrow{w^*} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$ by Claim 1, which means that $Y_{k,f}(\Theta_P \cap \cdot) \xrightarrow{w^*} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$ for all $f \in \tilde{F}$. This in turn implies that $Y_{k,f}(\cdot) = \sum_{P\in\mathcal{P}} Y_{k,f}(\Theta_P \cap \cdot) \xrightarrow{w^*} \sum_{P\in\mathcal{P}} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$. Now let $Y_f(\cdot) = \sum_{P\in\mathcal{P}} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$. We then have $\tilde{X}_f(\Theta_P \cap \cdot) = Y_{f_-^P}(\Theta_P \cap \cdot)$ and thus $\tilde{X}_f(\cdot) = \sum_{P\in\mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot)$. Also, since $X_{k,f} \xrightarrow{w^*} X_f$ and $Y_{k,f} \xrightarrow{w^*} Y_f$, we must have $Y_f \in R_f(X_f)$ by the upper hemicontinuity of R_f, which means $\tilde{X} \in T(X)$, as desired. ▌

Lemmas 4 and 5 show that T is nonempty- and convex-valued, and upper hemicontinuous while it is a mapping from convex, compact space $\mathcal{X}^{n+1}$ into itself, which implies that T is also closed-valued. Thus, we can invoke Kakutani-Fan-Glicksberg's fixed point theorem to conclude that the mapping T has a nonempty set of fixed points. ▌

Appendix B Proofs for Section 6

Proof of Theorem 3. Recall from Lemma 1 that the partially ordered set $(\mathcal{X}, \sqsubset)$, and thus partially ordered set $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$, is a complete lattice, where $X_{\tilde{F}} \sqsubset_{\tilde{F}} X'_{\tilde{F}}$ if $X_f \sqsubset X'_f$ for all $f \in \tilde{F}$. If each C_f is closed-valued, so are each R_f and T, as one can easily verify. Also, if each R_f is weak-set monotonic, so is T in the ordered set $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$. Note also that $\mathcal{X}^{n+1}$ is a compact set due to Lemma 4. Thus, if all firms have weakly substitutable preferences with closed-valued choice correspondences, then T has a fixed point according to Corollary 3.7 of Li (2014), which implies existence of a stable matching due to Theorem 1. ∎

Proof of Theorem 4. *Proof of Part (i):* Note first that by substitutability, each R_f is weak-set monotonic while $R_f(X)$ is a complete sublattice for any $X \in \mathcal{X}$, and that these properties are inherited by T. Given this, the proof of Theorem 1 in Zhou (1994) shows that the set of fixed points of T, denoted $\mathcal{X}^*$, contains the largest and smallest elements, $\overline{X} = \sup_{\sqsubset_{\tilde{F}}} \mathcal{X}^*$ and $\underline{X} = \inf_{\sqsubset_{\tilde{F}}} \mathcal{X}^*$.[78] Let $\overline{M}$ and $\underline{M}$ be the stable matchings associated with $\overline{X}$ and $\underline{X}$, respectively, given by Theorem 1. We only establish that $\overline{M}$ is firm-optimal and worker-pessimal, since the result for $\underline{M}$ can be established analogously. Recall from our characterization theorem that for any stable matching M, there is some $X \in \mathcal{X}^*$ such that $X_f = D^{\preceq f}(M)$ and $M_f \in C_f(X_f)$ for all $f \in \tilde{F}$. We thus have $M_f \sqsubset X_f \sqsubset \overline{X}_f$, which implies that $\overline{M}_f \in C_f(M_f \vee \overline{M}_f)$ by revealed preference since $\overline{M}_f \in C_f(\overline{X}_f)$ and $(M_f \vee \overline{M}_f) \sqsubset \overline{X}_f$. Thus, $\overline{M}_f \succeq_f M_f$ for each $f \in F$, as desired. To show that $\overline{M} \preceq_\Theta M, \forall M \in \mathcal{M}^*$, fix any $M \in \mathcal{M}^*$ and consider $X \in \mathcal{X}^*$ such that $X_f = D^{\preceq f}(M)$ for all $f \in \tilde{F}$. Then, for each $P \in \mathcal{P}$ and $E \in \Sigma$,

$$X_{f_+^P}(\Theta_P \cap E) = D^{\preceq f_+^P}(M)(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} M_{f'}(\Theta_P \cap E),$$

where f_+^P is an immediate successor of $f \in \tilde{F}$ at $P \in \mathcal{P}$, as defined earlier. Similarly, for $\overline{X}$, we have $\overline{X}_{f_+^P}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} \overline{M}_{f'}(\Theta_P \cap E)$. Given this and the fact that $X \sqsubset_{\tilde{F}} \overline{X}$,

$$\begin{aligned}\sum_{f' \in \tilde{F}: f' \succeq_P f} \overline{M}_{f'}(\Theta_P \cap E) =& G(\Theta_P \cap E) - \overline{X}_{f_+^P}(\Theta_P \cap E) \\ \leqslant& G(\Theta_P \cap E) - X_{f_+^P}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \succeq_P f} M_{f'}(\Theta_P \cap E) \qquad (18)\end{aligned}$$

for all $P \in \mathcal{P}$, $E \in \Sigma$, and $f \in \tilde{F}$, as desired.

Proof of Part (ii): Note that for any $X \in \mathcal{X}$, each $R_f(X)$, and thus $T(X)$, is a complete sublattice. Then, $\overline{T}$ must be monotonic since, for any $X \sqsubset X'$, we have $\overline{T}(X) \in T(X)$ and $\overline{T}(X') \in T(X')$, which implies by upper weak-set monotonicity that there exits $Y \in T(X')$ such that $\overline{T}(X) \sqsubset Y$, and then $\overline{T}(X) \sqsubset \overline{T}(X')$ by definition of $\overline{T}(X')$. Now let $X^0 = (X_f^0)_{f \in \tilde{F}}$ with $X_f^0 = G, \forall f \in \tilde{F}$. Define recursively $X^n = \overline{T}(X^{n-1})$ for each $n \geqslant 1$. The sequence $(X^n)_{n \in \mathbb{N}}$ is decreasing since $X^1 \sqsubset X^0$ and $X^2 = \overline{T}(X^1) \sqsubset \overline{T}(X^0) = X^1$ and so on, which implies that it has a limit point, denoted X^*. Because each C_f is oder-continuous, we have $\overline{R}_f(X_f^n) = X_f^n - \underline{C}_f(X_f^n) \xrightarrow{w^*} X_f^* - \underline{C}_f(X_f^*) = \overline{R}_f(X_f^*)$, which implies

[78] Zhou (1994)'s theorem requires the strong set monotonicity, but some inspection of its proof reveals that the weak-set monotonicity is sufficient for existence of largest and smallest fixed points.

that $X^{n+1} = \overline{T}(X^n) \xrightarrow{w^*} \overline{T}(X^*)$. Since $X^{n+1} \xrightarrow{w^*} X^*$, we must have $\overline{T}(X^*) = X^*$. To show that $X^* = \overline{X}$, consider any $X \in \mathcal{X}^*$. Then, $X \sqsubset X^0$ and thus $X \sqsubset \overline{T}(X) \sqsubset \overline{T}(X^0) = X^1$. Repeating this way, we obtain $X \sqsubset X^n, \forall n$, which implies that $X \sqsubset X^*$ and thus $X^* = \overline{X}$. By the proof of part (i), a stable matching associated with $\overline{X}$ is firm-optimal. The proof for worker optimal stable matching is analogous and thus omitted. ▌

Proof of Theorem 5. Let M be any stable matching. Then, by Theorem 1, there exists $X \in \mathcal{X}^*$ such that $M_f \in C_f(X_f)$ for each $f \in F$. Since $X \sqsubset_{\tilde{F}} \overline{X}$, LoAD implies

$$\overline{M}_f(\Theta) \geqslant \inf C_f(\overline{X}_f)(\Theta) \geqslant \sup C_f(X_f)(\Theta) \geqslant M_f(\Theta), \forall f \in F. \tag{19}$$

Next since $\overline{M}$ is worker pessimal, (18) holds for any $f \in \tilde{F}$. Let $w_P := \varnothing_-^P$ be the immediate predecessor of $\varnothing$ (i.e., the worst individually rational firm) for types in Θ_P. Then, setting $f = w_P$ in (18), we obtain

$$\begin{aligned}\sum_{f' \in F} \overline{M}_{f'}(\Theta_P \cap E) &= \sum_{f' \in \tilde{F}: f' \succeq_P w_P} \overline{M}_{f'}(\Theta_P \cap E) \\ &\leqslant \sum_{f' \in \tilde{F}: f' \succeq_P w_P} M_{f'}(\Theta_P \cap E) = \sum_{f' \in F} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma,\end{aligned}$$

or equivalently

$$\sum_{f' \in F} \overline{M}_{f'}(E) \leqslant \sum_{f' \in F} M_{f'}(E), \forall E \in \Sigma. \tag{20}$$

Since this inequality must hold with $E = \Theta$, combining it with (19) implies that $M_f(\Theta) = \overline{M}_f(\Theta)$ for all $f \in F$, as desired.

Further, we must have $\sum_{f \in F} \overline{M}_f = \sum_{f \in F} M_f$. Otherwise, by (20), we must have $\sum_{f' \in F} \overline{M}_{f'}(E) < \sum_{f' \in F} M_{f'}(E)$ for some $E \in \Sigma$. Also, by (20), $\sum_{f' \in F} \overline{M}_{f'}(E^c) \leqslant \sum_{f' \in F} M_{f'}(E^c)$. Combining these two inequalities, we obtain $\sum_{f' \in F} \overline{M}_{f'}(\Theta) < \sum_{f' \in F} M_{f'}(\Theta)$, which contradicts (19). Lastly, that $\sum_{f \in F} \overline{M}_f = \sum_{f \in F} M_f$ means $\overline{M}_\varnothing = M_\varnothing$. ▌

Proof of Theorem 6. Suppose otherwise. Then there exists a stable matching M that differs from the worker-optimal stable matching $\underline{M}$. Let X and $\underline{X}$ be respectively fixed points of T such that $M_f = C_f(X_f)$, $\underline{M}_f = C_f(\underline{X}_f)$ and $\underline{X}_f \sqsubset X_f$, for each $f \in F$.

First of all, since $\underline{X}_f \sqsubset X_f$ for each $f \in F$, we have $(\underline{M}_f \vee M_f) \sqsubset X_f$. Revealed preference then implies that, for each $f \in F$,

$$M_f = C_f(\underline{M}_f \vee M_f)$$

or $M \succeq_F \underline{M}$. Moreover, since $M \neq \underline{M}$, the set $\bar{F} := \{f \in F | M_f \succ_f \underline{M}_f\}$ is nonempty. But then by the rich preferences, there exists $f^* \in F$ such that

$$\underline{M}_{f*} \neq C_{f*}((\underline{M}_{f*} + M_{\bar{F}}^{f^*}) \wedge G).$$

For each $f \in F \backslash \bar{F}$, $M_f = \underline{M}_f$, by definition of $\bar{F}$, and Theorem 5 guarantees that $M_{\varnothing} = \underline{M}_{\varnothing}$. Consequently, we have for each $E \in \Sigma$, that

$$M_{\bar{F}}^{f^*}(E) = \sum_{P \in \mathcal{P}} \sum_{f': f^* >_P f', f' \notin \bar{F}} M_{f'}(\Theta_P \cap E) = \sum_{P \in \mathcal{P}} \sum_{f': f^* >_P f', f' \notin \bar{F}} \underline{M}_{f'}(\Theta_P \cap E) = \underline{M}_{\bar{F}}^{f^*}(E).$$

It then follows that $(\underline{M}_{f^*} + M_{\bar{F}}^{f^*}) \wedge G = (\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*}) \wedge G = \underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*}$ (since $\underline{M}$ is a matching), so

$$\underline{M}_{f^*} \neq C_{f^*}(\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*}). \tag{21}$$

Letting $\hat{M}_{f^*} := C_{f^*}(\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*})$, we have $\hat{M}_{f^*} \sqsubset (\underline{M}_{f^*} \vee \hat{M}_{f^*}) \sqsubset (\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*})$. Revealed preference then implies that

$$\hat{M}_{f^*} = C_{f^*}(\underline{M}_{f^*} \vee \hat{M}_{f^*}).$$

Then, by (21), we have $\hat{M}_{f^*} \succ_{f^*} \underline{M}_{f^*}$. Further, $\hat{M}_{f^*} \sqsubset (\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*}) \sqsubset D^{\leq f^*}(\underline{M})$. We therefore have a contradiction to the stability of $\underline{M}$. ▌

Appendix C Proofs for Section 7

Proof of Theorem 7. Let Γ be the limit continuum economy which the sequence $(\Gamma^q)_{q \in \mathbb{N}}$ converges to. For any population G, fix a sequence $(G^q)_{q \in \mathbb{N}}$ of finite-economy populations such that $G^q \xrightarrow{w^*} G$. Let $\Theta^q = \{\theta_1^q, \theta_2^q, \ldots, \theta_{\bar{q}}^q\} \subset \Theta$ be the support for G^q.[79] For each firm $f \in \tilde{F}$, define Θ_f to be the set of types that find firm f acceptable, i.e., $\Theta_f := \cup_{P \in \mathcal{P}: f >_P \varnothing} \Theta_P$ (let $\Theta_{\varnothing} = \Theta$ by convention). Let $\overline{\Theta}_f$ denote the closure of Θ_f with respect to the topology on Θ. We first prove a few preliminary results, whose proofs are provided in Section S.7.1 of Supplementary Material.

Lemma 6. *For any $r > 0$, there is a finite number of open balls, $B_1, \ldots, B_L$, in Θ that have radius smaller than r with a boundary of zero measure (i.e. $G(\partial B_\ell) = 0, \forall \ell$) and cover $\overline{\Theta}_f$ for each $f \in F$.*

Lemma 7. *Consider any $X, Y \in \mathcal{X}$ such that $X(\Theta \backslash \Theta_f) = 0$ for some $f \in \tilde{F}$ and $X \sqsubset Y$, and consider any sequence $(Y^q)_{q \in \mathbb{N}}$ such that $Y^q \in \mathcal{X}^q$ and $Y^q \xrightarrow{w^*} Y$.[80] Then, there exists a sequence $(X^q)_{q \in \mathbb{N}}$ such that $X^q \in \mathcal{X}^q$, $X^q \xrightarrow{w^*} X$, $X^q \sqsubset Y^q$, and $X^q(\Theta \backslash \Theta_f) = 0$ for all q.*

Lemma 8. *For any two sequences $(X^q)_{q \in \mathbb{N}}$ and $(Y^q)_{q \in \mathbb{N}}$ such that $X^q, Y^q \in \mathcal{X}^q$, $X^q \sqsubset Y^q, \forall q$, $X^q \xrightarrow{w^*} X$, and $Y^q \xrightarrow{w^*} Y$, we have $X \sqsubset Y$.*

Using these lemmas, we establish the following two lemmas:

[79]Note that we allow for the possibility that there are more than one worker of the same type even in finite economies, so $\bar{q}$ may be strictly smaller than q.

[80]Note that if $f = \varnothing$, then $\Theta \backslash \Theta_f = \emptyset$. Thus, the restriction that $X(\Theta \backslash \Theta_f) = 0$ becomes vacuous.

Lemma 9. *For any stable matching M in Γ and $\epsilon > 0$, there is $Q \in \mathbb{N}$ such that for any $q > Q$, one can construct a matching $M^q = (M_f^q)_{f \in \tilde{F}}$ that is feasible and individually rational in Γ^q, and satisfies*

$$u_f(M_f) < u_f(M_f^q) + \frac{\epsilon}{2}, \forall f \in F. \tag{22}$$

Proof. In any finite economy Γ^q, let us construct a matching $\tilde{M}^q = (\tilde{M}_f^q)_{f \in \tilde{F}}$ as follows: order the firms in F by $f_1, \ldots, f_n$, and

1. define $\tilde{M}_{f_1}^q$ as X^q in Lemma 7 with $X = M_{f_1}$, $Y = G$, and $Y^q = G^q$;[81]
2. define $\tilde{M}_{f_2}^q$ as X^q in Lemma 7 with $X = M_{f_2}$, $Y = G - M_{f_1}$, and $Y^q = G^q - \tilde{M}_{f_1}^q$ (this is possible since $G^q - \tilde{M}_{f_1}^q \xrightarrow{w^*} G - M_{f_1}$);
3. in general, for each $f_k \in \tilde{F}$, define inductively $\tilde{M}_{f_k}^q$ as X^q in Lemma 7 with $X = M_{f_k}$, $Y = G - \sum_{k' < k} M_{f_{k'}}$, and $Y^q = G^q - \sum_{k' < k} \tilde{M}_{f_{k'}}^q$;

and define $\tilde{M}_\varnothing^q = G^q - \sum_{f \in \tilde{F}} \tilde{M}_f^q$.

By Lemma 7, $\tilde{M}^q$ is feasible in Γ^q and individually rational for workers while $\tilde{M}^q \xrightarrow{w^*} M$. To ensure the individual rationality for firms, we construct another matching $M^q = (M_f^q)_{f \in \tilde{F}}$ as follows: for each $f \in F$, select any $M_f^q \in C_f^q(\tilde{M}_f^q)$, and then set $M_\varnothing^q = G^q - \sum_{f \in F} M_f^q$. By revealed preference, we have $M_f^q \in C_f^q(M_f^q)$ and thus M^q is individually rational for firms. Also, the individual rationality of M^q for workers follows immediately from the individual rationality of $\tilde{M}^q$ and the fact that $M_f^q \sqsubset \tilde{M}_f^q$ for all $f \in F$. By the continuity of u_f's and the fact $\tilde{M}_f^q \xrightarrow{w^*} M_f$, we can find sufficiently large $Q \in \mathbb{N}$ such that for all $q > Q$,

$$u_f(M_f) < u_f(\tilde{M}_f^q) + \frac{\epsilon}{2} \leqslant u_f(M_f^q) + \frac{\epsilon}{2}, \forall f \in F,$$

where the second inequality holds since $M_f^q \in C_f^q(\tilde{M}_f^q)$. ∎

Lemma 10. *The matching M^q constructed in Lemma 9 is ϵ-stable in Γ^q for all $q > Q$, where Q is identified in Lemma 9.*

Proof. Let $D^{\preceq f}(M^q)$ be the subpopulation of workers in economy Γ^q who weakly prefer f to their match in M^q.[82] Since $M^q \xrightarrow{w^*} M$, we have $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$.[83] Choose any $\tilde{M}_f^q \in C_f(D^{\preceq f}(M^q))$. In words, $\tilde{M}_f^q$ is the most profitable block of M^q for f in the continuum economy, that is, the optimal deviation in a situation where the current matching is M^q, but

[81] Note that $M_f(\Theta \backslash \Theta_f) = 0$ for all $f \in F$ since M is individually rational, so Lemma 7 can be applied.

[82] To be precise, $D^{\preceq f}(M^q)$ is given as in (3) with G and X being replaced by G^q and M^q, respectively.

[83] This convergence can be shown using an argument similar to that which we have used to establish the continuity of Ψ in the proof of Lemma 5.

the firm can deviate to any subpopulation, not just a discrete distribution. Then, we must have

$$u_f(\tilde{M}_f^q) < u_f(M_f) + \frac{\epsilon}{2}, \tag{23}$$

for any sufficiently large q. Otherwise, we could find some subsequence $(\hat{M}_f^q)_{q\in\mathbb{N}}$ of sequence $(\tilde{M}_f^q)_{q\in\mathbb{N}}$ for which

$$u_f(\hat{M}_f^q) \geqslant u_f(M_f) + \frac{\epsilon}{2}, \forall q. \tag{24}$$

We can assume that $(\hat{M}_f^q)_{q\in\mathbb{N}}$ is converging to some $\hat{M}_f$ (by choosing further subsequence if necessary). Then, the above-mentioned property that $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$ and upper hemicontinuity of C_f imply $\hat{M}_f \in C_f(D^{\preceq f}(M))$ and thus $u_f(\hat{M}_f) = u_f(M_f)$ since $M_f \in C_f(D^{\preceq f}(M))$ (due to stability of M), which contradicts (24).

Now let M'_f be the most profitable block of M^q for f in economy Γ^q. Then, M'_f is the optimal deviation facing the same population G^q and matching M^q as when computing $\tilde{M}_f^q$ but with an additional restriction that the deviation is feasible in Γ^q (multiples of $1/q$), so $u_f(M'_f) \leqslant u_f(\tilde{M}_f^q)$. This and inequality (23) imply

$$u_f(M'_f) < u_f(M_f) + \frac{\epsilon}{2}. \tag{25}$$

Combining inequalities (22) and (25), we get $u_f(M'_f) < u_f(M_f^q) + \epsilon$, completing the proof. ▌

Theorem 7 then follows from the existence of stable matching M in Γ and Lemmas 9 and 10. ▌

Proof of Theorem 8. The proof that M is a matching in Γ is straightforward and thus omitted. We first show that M is individually rational. First of all, since M^q is individually rational for workers, we have $M_f^q(\Theta_P) = 0$ for all $f \in F$ and $P \in \mathcal{P}$ such that $\emptyset \succ_P f$, which implies that $M_f(\Theta_P) = 0$ since $M_f^q \xrightarrow{w^*} M_f$ and $M_f(\partial\Theta_P) \leqslant G(\partial\Theta_P) = 0$. Thus, M is also individually rational for workers. To show that M is individually rational for firms, suppose for a contradiction that there are some $f \in F$ and $\hat{M}_f \in \mathcal{X}$ such that $\hat{M}_f \sqsubset M_f$ and $u_f(\hat{M}_f) - u_f(M_f) = 3\epsilon$ for some $\epsilon > 0$. We then prove the following claim:

Claim 2. *For all sufficiently large q, there exists a subpopulation $\hat{M}_f^q$ in Γ^q such that $\hat{M}_f^q \sqsubset D^{\preceq f}(M^q)$ and $u_f(\hat{M}_f^q) > u_f(\hat{M}_f) - \epsilon$.*

Proof. We use Lemma 7 with $Y = D^{\preceq f}(M)$, $Y^q = D^{\preceq f}(M^q)$, and $X = \hat{M}_f$. By the continuity of $D^{\preceq f}(\cdot)$ and the assumption that $M^q \xrightarrow{w^*} M$, we have $Y^q \xrightarrow{w^*} Y$. Also, we have $X = \hat{M}_f \sqsubset M_f \sqsubset D^{\preceq f}(M) = Y$. Lemma 7 then implies that there exists a sequence $(\hat{M}_f^q)_{q\in\mathbb{N}}$ such that $\hat{M}_f^q \in \mathcal{X}^q$, $\hat{M}_f^q \xrightarrow{w^*} X = \hat{M}_f$, and $\hat{M}_f^q \sqsubset Y^q = D^{\preceq f}(M^q)$. Then, by the continuity of u_f, we have $u_f(\hat{M}_f^q) > u_f(\hat{M}_f) - \epsilon$ for all sufficiently large q. ▌

Since $M^q \xrightarrow{w^*} M$ and u_f is continuous, we have that, for all sufficiently large q,

$$u_f(M_f^q) < u_f(M_f) + \epsilon = u_f(\hat{M}_f) - 2\epsilon < u_f(\hat{M}_f^q) - \epsilon, \tag{26}$$

where the second inequality follows from Claim 2. This contradicts ϵ-stability of M^q in Γ^q.

To prove that there is no blocking coalition, suppose for a contradiction that there exist a firm $f \in F$ and subpopulation $\hat{M}_f$ such that $\hat{M}_f \sqsubset D^{\leq f}(M)$ and $u_f(\hat{M}_f) - u_f(M_f) = 3\epsilon$ for some $\epsilon > 0$. By Claim 2, for all sufficiently large q, there exists a subpopulation $\hat{M}_f^q$ in Γ^q such that $\hat{M}_f^q \sqsubset D^{\leq f}(M^q)$ and $u_f(\hat{M}_f^q) > u_f(\hat{M}_f) - \epsilon$. Then, the same inequality as in (26) establishes the desired contradiction. ▌

Let us here state a variant of Theorem 8 for later use, whose proof is essentially the same as that of Theorem 8:

Theorem 8'. Let $(M^{q_k})_{k\in\mathbb{N}}$ be a sequence of matchings converging to M such that for every $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k > K$, M^{q_k} is ϵ-stable in Γ^{q_k}. Then, M is stable in Γ.

Proof of Theorem 9. First let us state a mathematical fact:

Lemma 11 (Heine-Cantor Theorem)**.** *Let $h : A \to B$ be a continuous function between two metric spaces A and B, and suppose A is compact. Then, h is uniformly continuous.*

Since the space of all subpopulations of G is metrizable by the Lévy-Prokhorov metric, and it is compact, the Heine-Cantor theorem applies to our setting.

We also need the following result:

Lemma 12. *For every $\epsilon > 0$, there exist $\delta \in (0, \epsilon)$ and $Q' \in \mathbb{N}$ such that for every $q > Q'$ and every matching M^q that is δ-stable in Γ^q, there exists a stable matching M in Γ such that $d(M^q, M) < \epsilon$, where $d(\cdot, \cdot)$ is the product Lévy-Prokhorov metric.*[84]

Proof. Suppose for contradiction that the conclusion of the statement does not hold. Then there exists $\epsilon > 0$ with the following property: for every $\delta \in (0, \epsilon)$ and $Q' \in \mathbb{N}$, there exist $q > Q'$ and M^q that is δ-stable in Γ^q such that $d(M^q, M) \geqslant \epsilon$ for every M that is stable in Γ. This implies there exists a decreasing sequence $(\delta^k)_k$ which converges to 0 and $(M^{q^k})_k$ such that M^{q^k} is δ^k-stable in Γ^{q^k}, $d(M^{q^k}, M) \geqslant \epsilon$ for every stable matching M in Γ, and $\lim_k q^k = \infty$. Without loss of generality, assume M^{q^k} converges to some matching $\hat{M}$

[84]The Lévy-Prokhorov metric on space $\mathcal{X}$ is defined as follows: for any $X, Y \in \mathcal{X}$,

$$d(X, Y) := \inf\{\varepsilon > 0 \mid X(E) \leqslant Y(E^\varepsilon) + \varepsilon \text{ and } Y(E) \leqslant X(E^\varepsilon) + \varepsilon \text{ for all } E \in \Sigma\},$$

where $E^\epsilon := \{\theta \in \Theta | \exists \theta' \in E \text{ such that } d^\Theta(\theta, \theta') < \epsilon\}$ with d^Θ being a metric for the space Θ. Here, we abuse notation since d is used to denote both the Lévy-Prokhorov metric and its product metric. Note that the choice of product metric is inconsequential since it is defined on a finite-dimensional space.

(because the sequence lies in a sequentially compact space). Then $d(\hat{M}, M) \geqslant \epsilon$ for every stable matching $M \in \Gamma$, so $\hat{M}$ is not stable in Γ. This is a contradiction to Theorem 8'. ▌

Proof of Part (i): Given an arbitrary $\epsilon > 0$, let $\eta > 0$ be a constant such that, for any two matchings M and M', $d(M, M') < \eta$ implies $|u_f(M_f) - u_f(M'_f)| < \epsilon/2$ for every $f \in F$. (Recall that u_f is continuous. Therefore it is uniformly continuous by the Heine-Cantor theorem.) Without loss, one can assume $\eta < \epsilon$.

For $\eta > 0$ defined in the last paragraph, choose $\delta \in (0, \eta)$ and Q' as described in the statement of Lemma 12. (Note that $\delta < \epsilon$ since $\delta < \eta < \epsilon$.) More precisely, δ and Q' have the property that for every $q > Q'$ and every matching $\hat{M}^q$ that is δ-stable in Γ^q, there exists a stable matching M in Γ such that $d(\hat{M}^q, M) < \eta$. Given this δ, by Lemma 9 and Lemma 10, there is $Q > Q'$ such that for all $q > Q$, there exists a matching M^q in Γ^q which is δ-stable in Γ^q and satisfies

$$u_f(M^q_f) > u_f(\overline{M}_f) - \frac{\delta}{2} > u_f(\overline{M}_f) - \frac{\epsilon}{2}. \tag{27}$$

Claim 3. *$u_f(\overline{M}_f) > u_f(\hat{M}^q_f) - \epsilon/2$ for any δ-stable matching $\hat{M}^q$ in Γ^q.*

Proof. By the argument in the last paragraph, there exists a stable matching M in Γ with $d(\hat{M}^q, M) < \eta$. So, by construction of η (and uniform continuity of u_f), we obtain $u_f(M_f) > u_f(\hat{M}^q_f) - \epsilon/2$. Meanwhile, by firm optimality of $\bar{M}$, we have $u_f(M_f) \leqslant u_f(\overline{M}_f)$. Combining these inequalities, we obtain the desired inequality. ▌

Then, the desired conclusion holds for any $q > Q$ since, by (27) and Claim 3, we have $u_f(M^q_f) > u_f(\overline{M}_f) - \epsilon/2 > u_f(\hat{M}^q_f) - \epsilon$.

Proof of Part (ii): Note first that each mapping $D^{\succeq f}(\cdot)$ is continuous, and hence uniformly continuous (see footnote 83). Thus, given an arbitrary $\epsilon > 0$, one can choose $\eta \in (0, \epsilon)$ such that for any $M, M' \in \mathcal{X}^{n+1}$, $d(M, M') < \eta$ implies $d(D^{\succeq f}(M), D^{\succeq f}(M')) < \frac{\epsilon}{2}$ for all $f \in \tilde{F}$. By Lemma 12, for the chosen η, one can find $\delta \in (0, \eta)$ and $Q' \in \mathbb{N}$ such that for every $q > Q'$ and every δ-stable matching $\hat{M}^q$ in Γ^q, there is a stable matching $\tilde{M}^q$ in Γ such that $d(\hat{M}^q, \tilde{M}^q) < \eta$. By definition of η, we must have $d(D^{\succeq f}(\tilde{M}^q), D^{\succeq f}(\hat{M}^q)) < \frac{\epsilon}{2}$.

Next, given that $C_f(\underline{M}_f) = \{\underline{M}_f\}$ for each $f \in F$, Lemma S4 of Supplementary Material implies that there is a sequence $(M^q)_{q \in \mathbb{N}}$ such that $M^q \xrightarrow{w^*} \underline{M}$, where M^q is a feasible and individually rational matching in Γ^q. Choose now $\epsilon_\delta > 0$ such that for any subpopulations $M, M' \in \mathcal{X}$, $d(M, M') < \epsilon_\delta$ implies $|u_f(M) - u_f(M')| < \delta$. By Lemma S5 of Supplementary Material, one can find $Q'' \in \mathbb{N}$ such that for all $q > Q''$, M^q is ϵ_δ-distance stable: that is, for any $M' \in \mathcal{X}^q$ such that $M' \sqsubset D^{\preceq f}(M^q)$ and $u_f(M') > u_f(M^q_f)$, we have $d(M', M^q_f) < \epsilon_\delta$. This implies by the definition of ϵ_δ that $u_f(M^q_f) + \delta > u_f(M')$. In other words, M^q is δ-stable for all $q > Q''$, as required by condition 1 of Definition 11. To satisfy condition 2, using the fact that M^q converges to $\underline{M}$, we can choose $Q > \max\{Q', Q''\}$ such that for all $q > Q$, we have $d(D^{\succeq f}(M^q), D^{\succeq f}(\underline{M})) < \frac{\epsilon}{2}$ for all $f \in \tilde{F}$, which implies

$$D^{\succeq f}(\underline{M})(E) \leqslant D^{\succeq f}(M^q)(E^{\frac{\epsilon}{2}}) + \frac{\epsilon}{2}, \forall E \in \Sigma, \forall f \in \tilde{F}, \tag{28}$$

by the fact that d is the Lévy-Prokhorov metric (refer to footnote 84 for the definition of d and E^{ε}). Then, for any $q > Q$ and for any $f \in \tilde{F}$ and $E \in \Sigma$,

$$\begin{aligned}\big(D^{\succeq f}(\hat{M}^q)(E) - \frac{\epsilon}{2}\big) - \frac{\epsilon}{2} \leqslant & D^{\succeq f}(\tilde{M}^q)(E^{\frac{\epsilon}{2}}) - \frac{\epsilon}{2} \\ \leqslant & D^{\succeq f}(\underline{M})(E^{\frac{\epsilon}{2}}) - \frac{\epsilon}{2} \leqslant D^{\succeq f}(M^q)((E^{\frac{\epsilon}{2}})^{\frac{\epsilon}{2}}) \leqslant D^{\succeq f}(M^q)(E^{\epsilon}),\end{aligned}$$

where the first inequality follows since $d(D^{\succeq f}(\tilde{M}^q), D^{\succeq f}(\hat{M}^q)) < \frac{\epsilon}{2}$, the second inequality from the worker-optimality of $\underline{M}$ and stability of $\tilde{M}^q$ in Γ, the third inequality from (28), and the last inequality from the fact that $(E^{\frac{\epsilon}{2}})^{\frac{\epsilon}{2}} \subset E^{\epsilon}$ (which can be easily verified). ▌

Proof of Theorem 10. Suppose not for contradiction. Then, there must be a sequence $(\delta_k, q_k)_{k \in \mathbb{N}}$ with $\delta_k \searrow 0$ and $q_k \nearrow \infty$ such that $\hat{M}^{q_k}$ is δ_k-stable and $d(M, \hat{M}^{q_k}) \geqslant \epsilon$ for all k. Then, one can find a subsequence $(q_{k_m})_{m \in \mathbb{N}}$ such that $\hat{M}^{q_{k_m}}$ converges to some $\hat{M}$ (since the sequence lies in a sequentially compact space), which must be stable in Γ due to Theorem 8'. Since $d(M, \hat{M}^{q_{k_m}}) \geqslant \epsilon$ for all m, we must have $d(M, \hat{M}) \geqslant \epsilon$, which contradicts the uniqueness of stable matching in Γ. ▌

References

Abdulkadiroğlu, A., Y.-K. Che, and Y. Yasuda (2015): "Expanding Choice in School Choice," *American Economic Journal: Microeconomics*, 7, 1–42.

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2005): "The New York City High School Match," *American Economic Review Papers and Proceedings*, 95, 364–367.

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2009): "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, 99, 1954–1978.

Abdulkadiroğlu, A., P. A. Pathak, A. E. Roth, and T. Sönmez (2005): "The Boston Public School Match," *American Economic Review Papers and Proceedings*, 95, 368–372.

Abdulkadiroğlu, A., and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Adachi, H. (2000): "On a Characterization of Stable Matchings," *Economics Letters*, 68, 43–49.

Aliprantis, C. D., and K. C. Border (2006): *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, Berlin.

Alkan, A. (2002): "A Class of Multipartner Matching Markets with a Strong Lattice Structure," *Economic Theory*, 19, 737–746.

Alkan, A., and D. Gale (2003): "Stable Schedule Matching Under Revealed Preferences," *Journal of Economic Theory*, 112, 289–306.

Ash, R. B., and C. A. Doléans-Dade (2009): *Real Analysis and Probability*. Academic Press, San Diego.

Ashlagi, I., M. Braverman, and A. Hassidim (2014): "Stability in Large Matching Markets with Complementarities," *Operations Research*, 62, 713–732.

Ashlagi, I., Y. Kanoria, and J. Leshno (2017): "Unbalanced Random Matching Markets: The Stark Effect of Competition," *Journal of Political Economy*, 125, 69–98.

Aygün, O., and T. Sönmez (2013): "Matching with Contracts: Comment," *American Economic Review*, 103, 2050–2051.

Azevedo, E. M., and J. W. Hatfield (2015): "Complementarity and Multidimensional Heterogeneity in Large Matching Markets," Discussion paper.

Azevedo, E. M., and J. D. Leshno (2016): "A Supply and Demand Framework for Two-Sided Matching Markets," *Journal of Political Economy*, 124, 1235–1268.

Azevedo, E. M., E. G. Weyl, and A. White (2013): "Walrasian Equilibrium in Large, Quasilinear Markets," *Theoretical Economics*, 8, 281–290.

Biró, P., T. Fleiner, and R. Irving (2013): "Matching Couples with Scarf's Algorithm," *Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and its Applications*, pp. 55–64.

Blair, C. (1984): "Every Finite Distributive Lattice Is a Set of Stable Matchings," *Journal of Combinatorial Theory*, 37, 353–356.

Bogomolnaia, A., and H. Moulin (2001): "A New Solution to the Random Assignment Problem," *Journal of Economic Theory*, 100, 295–328.

Che, Y.-K., J. Kim, and K. Mierendorff (2013): "Generalized Reduced-Form Auctions: A Network-Flow Approach," *Econometrica*, 81, 2487–2520.

Che, Y.-K., and Y. Koh (2016): "Decentralized College Admissions," *Journal of Political Economy*, 124, 1295–1338.

Che, Y.-K., and F. Kojima (2010): "Asymptotic Equivalence of Probabilistic Serial and Random Priority Mechanisms," *Econometrica*, 78, 1625–1672.

Che, Y.-K., and O. Tercieux (2015): "Efficiency and Stability in Large Matching Markets," forthcoming, *Theoretical Economics*.

——— (2017): "Payoff Equivalence of Efficient Mechanisms in Large Matching Markets," Discussion paper.

Debreu, G. (1954): "Representation of a Preference Ordering by a Numerical Function," *Decision Processes*, pp. 159–165.

Delacrétaz, D., S. D. Kominers, and A. Teytelboym (2016): "Refugee Resettlement," Discussion paper.

Echenique, F., and J. Oviedo (2004): "Core Many-to-One Matchings by Fixed Point Methods," *Journal of Economic Theory*, 115, 358–376.

——— (2006): "A theory of stability in many-to-many matching," *Theoretical Economics*, 1, 233–273.

Echenique, F., and M. B. Yenmez (2007): "A Solution to Matching with Preferences over Colleagues," *Games and Economic Behavior*, 59, 46–71.

Ellickson, B. (1979): "Competitive equilibrium with local public goods," *Journal of Economic Theory*, 21(1), 46–61.

Ellickson, B., B. Grodal, S. Scotchmer, and W. R. Zame (1999): "Clubs and the Market," *Econometrica*, 67, 1185–1217.

——— (2001): "Clubs and the market: large finite economies," *Journal of Economic Theory*, 101, 40–77.

Erdil, A., and H. Ergin (2008): "What's the matter with tie-breaking? Improving efficiency in school choice," *American Economic Review*, 98, 669–689.

Fleiner, T. (2003): "A Fixed-Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 28, 103–126.

Fudenberg, D., and J. Tirole (1991): *Game Theory*. MIT Press, Cambridge, Massachusetts.

GALE, D., AND L. S. SHAPLEY (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

GILLES, R. P., AND S. SCOTCHMER (1997): "Decentralization in replicated club economies with multiple private goods," *Journal of Economic Theory*, 72(2), 363–387.

HATFIELD, J. W., AND F. KOJIMA (2008): "Matching with Contracts: Comment," *American Economic Review*, 98, 1189–1194.

HATFIELD, J. W., F. KOJIMA, AND Y. NARITA (2016): "Improving schools through school choice: A market design approach," *Journal of Economic Theory*, 166, 186–211.

HATFIELD, J. W., AND S. D. KOMINERS (2017): "Contract Design and Stability in Many-to-Many Matching," *Games and Economic Behavior*, 101, 78–97.

HATFIELD, J. W., AND P. MILGROM (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

IMMORLICA, N., AND M. MAHDIAN (2005): "Marriage, Honesty, and Stability," *SODA 2005*, pp. 53–62.

KAGEL, J. H., AND A. E. ROTH (2000): "The Dynamics of Reorganization in Matching Markets: A Laboratory Experiment Motivated by a Natural Experiment," *The Quarterly Journal of Economics*, 115, 201–235.

KELSO, A., AND V. CRAWFORD (1982): "Job matching, coalition formation, and gross substitutes," *Econometrica*, 50, 1483–1504.

KESTEN, O., AND U. ÜNVER (2014): "A Theory of School Choice Lotteries," forthcoming, *Theoretical Economics*.

KLAUS, B., AND F. KLIJN (2005): "Stable Matchings and Preferences of Couples," *Journal of Economic Theory*, 121, 75–106.

KOJIMA, F., AND M. MANEA (2010): "Incentives in the Probabilistic Serial Mechanism," *Journal of Economic Theory*, 145, 106–123.

KOJIMA, F., AND P. A. PATHAK (2009): "Incentives and Stability in Large Two-Sided Matching Markets," *American Economic Review*, 99, 608–627.

KOJIMA, F., P. A. PATHAK, AND A. E. ROTH (2013): "Matching with Couples: Stability and Incentives in Large Markets," *Quarterly Journal of Economics*, 128, 1585–1632.

LEE, S. (2017): "Incentive compatibility of large centralized matching markets," *Review of Economic Studies*, 84, 444–463.

LI, J. (2014): "Fixed point theorems on partially ordered topological vector spaces and their applications to equilibrium problems with incomplete preferences," *Fixed Point Theory and Applications*, 2014, 192.

LIU, Q., AND M. PYCIA (2013): "Ordinal Efficiency, Fairness, and Incentives in Large Markets," Discussion paper.

MANEA, M. (2009): "Asymptotic Ordinal Inefficiency of Random Serial Dictatorship," *Theoretical Economics*, 4, 165–197.

MILGROM, P., AND J. ROBERTS (1990): "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58, 1255–1277.

MILGROM, P., AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180.

MIRALLES, A. (2008): "School choice: The case for the Boston mechanism," Discussion paper.

MIRALLES, A., AND M. PYCIA (2017): "Large vs. Continuum Assignment Economies," Discussion paper.

NGUYEN, T., A. PEIVANDI, AND R. VOHRA (2016): "Assignment problems with complementarities," *Journal of Economic Theory*, 165, 209–241.

NGUYEN, T., AND R. VOHRA (2014): "Near feasible stable matchings with couples," Discussion paper.

OK, E. (2011): *Real Analysis with Economic Applications*. Princeton University Press, Princeton.

——— (2017): *Applied Topology*. mimeo.

OSTROVSKY, M. (2008): "Stability in Supply Chain Networks," *American Economic Review*, pp. 897–923.

PYCIA, M. (2012): "Stability and Preference Alignment in Matching and Coalition Formation," *Econometrica*, 80, 323–362.

ROTH, A. E. (1984): "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 92, 991–1016.

——— (1985): "The college admission problem is not equivalent to the marriage problem," *Journal of Economic Theory*, 36, 277–288.

Roth, A. E. (1991): "A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom," *American Economic Review*, pp. 415–440.

Roth, A. E. (2002): "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics," *Econometrica*, 70, 1341–1378.

Roth, A. E., and E. Peranson (1999): "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review*, 89, 748–780.

Royden, H., and P. Fitzpatrick (2010): *Real Analysis*. Pearson, New Jersey.

Sandler, T., and J. Tschirhart (1997): "Club theory: Thirty years later," *Public choice*, 93(3-4), 335–355.

Scotchmer, S., and C. Shannon (2015): "Verifiability and group formation in markets," Discussion paper.

Scotchmer, S., and M. H. Wooders (1987): "Competitive equilibrium and the core in club economies with anonymous crowding," *Journal of Public Economics*, 34(2), 159–173.

Sönmez, T., and U. Ünver (2010): "Course Bidding at Business Schools," *International Economic Review*, 51, 99–123.

Sotomayor, M. (1999): "Three Remarks on the Many-to-Many Stable Matching Problem," *Mathematical Social Sciences*, 38, 55–70.

Topkis, D. M. (1998): *Supermodularity and Complementarity*. Princeton University, Princeton, New Jersey.

Zhou, L. (1994): "The Set of Nash Equilibria of a Supermodular Game is a Complete Lattice," *Games and Economic Behavior*, 7, 295–300.

Supplementary Material for "Stable Matching in Large Economies"

(Not for Publication)

Yeon-Koo Che, Jinwoo Kim, Fuhito Kojima

S.1 Analysis of the Example in Section 2

Let r be the number of workers with each of the two types who are matched to f. We consider the following cases:

1. Suppose $r > q/2$. For any such matching, at least one position is vacant at firm f' because f' has q positions, but strictly more than q workers are matched to f out of the total of $2q$ workers. Thus such a matching is blocked by f' and a type θ' worker who is currently matched to f.

2. Suppose $r < q/2$. Consider the following cases.

   (a) Suppose that there exists a type θ worker who is unmatched. Then such a matching is unstable because that worker and firm f' block it (note that f' prefers θ most).

   (b) Suppose that there exists no type θ worker who is unmatched. This implies that there exists a type θ' worker who is unmatched (because there are $2q$ workers in total, but firm f is matched to strictly fewer than q workers by assumption, and f' can be matched to at most q workers in any individually rational matching). Then, since f is the most preferred by all θ workers, a θ' worker prefers f to $\emptyset$, and there is some vacancy at f because $r < q/2$, the matching is blocked by a coalition of a type θ worker, a type θ' worker, and f.

S.2 Preliminaries for the Continuum Economy Model

S.2.1 Proof of Lemma 1

For any subset $\mathcal{Y} \subset \mathcal{X}$, define

$$\overline{Y}(E) := \sup\{\sum_i Y_i(E_i) \,|\, \{E_i\} \text{ is a finite partition of } E \text{ in } \Sigma \text{ and } \{Y_i\} \text{ is a finite collection of measures in } \mathcal{Y}, \forall i\}, \forall E.$$

and $\underline{Y}$ analogously (by replacing "sup" with "inf"). We prove the lemma by showing that $\overline{Y} = \sup \mathcal{Y} \in \mathcal{Y}$ and $\underline{Y} = \inf \mathcal{Y} \in \mathcal{X}$.

First of all, note that $\overline{Y}$ and $\underline{Y}$ are monotonic, i.e. for any $E \subset D$, we have $\overline{Y}(D) \geqslant \overline{Y}(E)$ and $\underline{Y}(D) \geqslant \underline{Y}(E)$, whose proof is straightforward and thus omitted.

We next show that $\overline{Y}$ and $\underline{Y}$ are measures. We only prove the countable additivity of $\overline{Y}$, since the other properties are straightforward to prove and also since a similar argument applies to $\underline{Y}$. To this end, consider any countable collection $\{E_i\}$ of disjoint sets in Σ and let $D = \cup E_i$. We need to show that $\overline{Y}(D) = \sum_i \overline{Y}(E_i)$. For doing so, consider any finite partition $\{D_i\}$ of D and any finite collection of measures $\{Y_i\}$. Letting $E_{ij} = E_i \cap D_j$, for any i, the collection $\{E_{ij}\}_j$ is a finite partition of E_i in Σ. Thus, we have

$$\sum_i Y_i(D_i) = \sum_i \sum_j Y_i(E_{ij}) \leqslant \sum_i \overline{Y}(E_i).$$

Since this inequality holds for any finite partition $\{D_i\}$ of D and collection $\{Y_i\}$, we must have $\overline{Y}(D) \leqslant \sum_i \overline{Y}(E_i)$. To show that the reverse inequality also holds, for each E_i, we consider any finite partition $\{E_{ij}\}_j$ of E_i in Σ and collection of measures $\{Y_{ij}\}_j$ in $\mathcal{Y}$. We prove that $\overline{Y}(D) \geqslant \sum_i \sum_j Y_{ij}(E_{ij})$, which would imply $\overline{Y}(D) \geqslant \sum_i \overline{Y}(E_i)$ as desired since the partition $\{E_{ij}\}_j$ and collection $\{Y_{ij}\}_j$ are arbitrarily chosen for each i. Suppose not for contradiction. Then, we must have $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij})$ for some k. Letting $E := \cup_{i=1}^k (\cup_j E_{ij})$, this implies $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij}) \leqslant \overline{Y}(E)$, where the second inequality holds by the definition of $\overline{Y}$. This contradicts with the monotonicity of $\overline{Y}$ since $E \subset D$.

We now show that $\overline{Y}$ and $\underline{Y}$ are the supremum and infimum of $\mathcal{Y}$, respectively. It is straightforward to check that for any $Y \in \mathcal{Y}$, $Y \sqsubset \overline{Y}$ and $\underline{Y} \sqsubset Y$. Consider any $X, X' \in \mathcal{X}$ such that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$ and $X' \sqsubset Y$. We show that $\overline{Y} \sqsubset X$ and $X' \sqsubset \underline{Y}$. First, if $\overline{Y} \not\sqsubset X$ to the contrary, then there must be some $E \in \Sigma$ such that $\overline{Y}(E) > X(E)$. This means there are a finite partition $\{E_i\}$ of E and a collection of measures $\{Y_i\}$ in $\mathcal{Y}$ such that $\overline{Y}(E) \geqslant \sum Y_i(E_i) > X(E) = \sum X(E_i)$. Thus, for at least one i, we have $Y_i(E_i) > X(E_i)$, which contradicts the assumption that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$. An analogous argument can be used to show $X' \sqsubset \underline{Y}$.

S.2.2 Proof of Proposition 1

Suppose that matching M is not weakly Pareto efficient. Then, by definition of weak Pareto efficiency, there exists M' and $f \in F$ such that $M' \succ_\Theta M$ and $M'_f \succ_f M_f$.

Next, since $M' \succ_\Theta M$, for each $\tilde{f}$, we have $D^{\succeq \tilde{f}}(M') \sqsupset D^{\succeq \tilde{f}}(M)$, or

$$\sum_{f' : f' \succeq_P \tilde{f}} M'_{f'}(\Theta_P \cap E) \geqslant \sum_{f' : f' \succeq_P \tilde{f}} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

This implies that

$$\sum_{f' : f' \succeq_P f^P_-} M'_{f'}(\Theta_P \cap E) \geqslant \sum_{f' : f' \succeq_P f^P_-} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma,$$

where f^P_- refers to the firm that is ranked immediately above f according to P (whenever it is well defined),[1] or equivalently

$$\sum_{f' : f' \succ_P f} M'_{f'}(\Theta_P \cap E) \geqslant \sum_{f' : f' \succ_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

This in turn implies that, for each P,

$$\sum_{f' : f' \preceq_P f} M'_{f'}(\Theta_P \cap E) \leqslant \sum_{f' : f' \preceq_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma,$$

or equivalently,

$$D^{\preceq f}(M') \sqsubset D^{\preceq f}(M).$$

By definition, $M'_f \sqsubset D^{\preceq f}(M')$, so we have $M'_f \sqsubset D^{\preceq f}(M)$.

Collecting the observations so far, we conclude that f and M'_f block M, implying that M is not stable. We have thus established that stability implies weak Pareto efficiency.

Suppose now that each C_f is a choice function and that a stable matching M is not Pareto efficient. Then, there is another matching $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$. Choose any firm $f \in F$ with $M_f \neq M'_f$ and note that since C_f is a choice function, we have $C_f(M_f \vee M'_f) = M'_f \neq M_f$, which means $M'_f \succ_f M_f$. Given this, a contradiction can be drawn following the same argument as above.

S.3 Equivalence with Worker-Proposing DA

In this section, we establish the equivalence between a repeated application of our fixed point mapping and the worker-proposing DA process when firms have substitutable preferences. To do so, we assume that each firm's choice is always unique, i.e., C_f is a choice

[1] This is defined later as an immediate predecessor. Formally, $f^P_- \succ_P f$ and if $f' \succ_P f$, then $f' \succeq_P f^P_-$.

function. Then, the substitutability of firm f's reference becomes

$$R_f(X) \sqsubset R_f(X') \text{ whenever } X \sqsubset X'. \tag{SUB}$$

Let $\hat{X}_f^t$ denote the cumulative measure of workers proposing to the firm f from round 1 through t of the worker-proposing DA process. Let $\hat{A}_f^t$ denote the measure of workers (tentatively) accepted by f in round t. Let $(\hat{X}_f^0, \hat{A}_f^0) = (\mathbf{0}, \mathbf{0})$. In the first round, all workers propose to their most preferred firms, which means that for any $P \in \mathcal{P}$ and $E \subset \Theta_P$,

$$\hat{X}_f^1(E) = \begin{cases} G(E) & \text{if } f \succ_P f', \forall f' \neq f \\ 0 & \text{otherwise.} \end{cases} \tag{S1}$$

Given this,

$$\hat{A}_f^1 = C_f(\hat{X}_f^1). \tag{S2}$$

For $t \geqslant 2$, the pair $(\hat{X}_f^t, \hat{A}_f^t)$ is recursively defined as follows: For any $P \in \mathcal{P}$ and $E \subset \Theta_P$,

$$\hat{X}_f^t(E) = \begin{cases} G(E) & \text{if } f \succ_P f', \forall f' \neq f \\ R_{f_-^P}\left(\hat{X}_{f_-^P}^{t-1} - \hat{X}_{f_-^P}^{t-2} + \hat{A}_{f_-^P}^{t-2}\right)(E) + \hat{X}_f^{t-1}(E) & \text{otherwise} \end{cases} \tag{S3}$$

$$\hat{A}_f^t = C_f\left(\hat{X}_f^t - \hat{X}_f^{t-1} + \hat{A}_f^{t-1}\right). \tag{S4}$$

The first expression in (S3) is straightforward, given that all workers who most prefer f propose to f in the first round. To understand the second expression, the cumulative measure of workers proposing to f (which is not most preferred according to P) from round 1 through t is obtained by adding to $\hat{X}_f^{t-1}$—that is, measure of workers proposing to f from round 1 through $t-1$—the measure of workers who newly propose to f in round t. The latter workers then coincide with those rejected by f's immediate predecessor (i.e., f_-^P) in round $t-1$, whose measure is equal to $R_{f_-^P}\left(\hat{X}_{f_-^P}^{t-1} - \hat{X}_{f_-^P}^{t-2} + \hat{A}_{f_-^P}^{t-2}\right)$. To see this, note that in round $t-1$, the firm f_-^P considers and accepts/rejects among those tentatively accepted by f_-^P in round $t-2$ (their measure being equal to $\hat{A}_{f_-^P}^{t-2}$) and those newly proposing to f_-^P in round $t-1$ (their measure being equal to $\hat{X}_{f_-^P}^{t-1} - \hat{X}_{f_-^P}^{t-2}$). The expression in (S4) can be understood similarly.

Let $\tilde{X}^0$ denote a profile of zero measures (that is, the profile has one zero measure for each firm in $\tilde{F}$). Define iteratively $\tilde{X}^t = T(\tilde{X}^{t-1})$ for each $t \geqslant 1$, where T is our fixed-point mapping.

Proposition S1. *If (SUB) holds for all $f \in F$, then $\hat{X}_f^t = \tilde{X}_f^t$ in each round $t \geqslant 1$.*

Before starting the proof, we establish the following lemma:

Lemma S1. *Given RP, (SUB) is equivalent to the path independence:*

$$C_f(X') = C_f(C_f(X) + X' - X), \forall X \sqsubset X'. \tag{PI}$$

Proof. That (PI) implies (SUB) follows immediately from noting that

$$C_f(X') = C_f(C_f(X) + X' - X) \sqsubset C_f(X) + X' - X,$$

and thus $X - C_f(X) \sqsubset X' - C_f(X')$ or $R_f(X) \sqsubset R_f(X')$.

To prove the converse, for any subpopulations X and X' with $X \sqsubset X'$, let $Z = C_f(X) + X' - X$. Then, by SUB, we have $C_f(X') \sqsubset Z$. Since $Z \sqsubset X'$, RP implies $C_f(Z) = C_f(X')$, which is equivalent to (PI), as desired. ▌

Proof of Proposition S1. We need to show that for each $s \geqslant 1$ and for each $P \in \mathcal{P}$ and $E \subset \Theta_P$,

$$\hat{X}^s_f(E) = \tilde{X}^s_f(E) = T_f(\tilde{X}^{s-1})(E) = \begin{cases} G(E) & \text{if } f \succ_P f', \forall f' \neq f \\ R_{f^P_-}(\tilde{X}^{s-1}_{f^P_-})(E) & \text{otherwise.} \end{cases} \tag{S5}$$

Let us first establish that for all $s \geqslant 1$, $\hat{A}^s_f = C_f(\hat{X}^s_f)$. This holds for $s = 1$ due to (S2). Assuming inductively that this holds for all $s \leqslant t-1$, we have

$$\hat{A}^t_f = C_f\left(\hat{X}^t_f - \hat{X}^{t-1}_f + \hat{A}^{t-1}_f\right) = C_f\left(\hat{X}^t_f - \hat{X}^{t-1}_f + C_f(\hat{X}^{t-1}_f)\right) = C_f(\hat{X}^t_f),$$

where the last equality holds due to (PI) and the fact that $\hat{X}^{t-1}_f \sqsubset \hat{X}^t_f$.

To show (S5), consider $s = 1$ and note that if f is not most preferred according to P, then

$$\tilde{X}^1_f(E) = T_f(\tilde{X}^0)(E) = R_{f^P_-}(\tilde{X}^0_{f^P_-})(E) = R_{f^P_-}(\mathbf{0})(E) = 0$$

while, if f is most preferred, then $\tilde{X}^1_f(E) = G(E)$. This means that $\tilde{X}^1_f$ coincides with $\hat{X}^1_f$ given in (S1), so (S5) holds for $s = 1$. Assume inductively that (S5) holds for all $s \leqslant t-1$. To show that it holds for $s = t$, for any $P \in \mathcal{P}$ and $E \subset \Theta_P$, letting $g = f^P_-$ (to simplify notation), we have

$$\begin{aligned} \hat{X}^t_f(E) &= R_g\left(\hat{X}^{t-1}_g - \hat{X}^{t-2}_g + \hat{A}^{t-2}_g\right)(E) + \hat{X}^{t-1}_f(E) \\ &= R_g\left(\hat{X}^{t-1}_g - \hat{X}^{t-2}_g + C_g(\hat{X}^{t-2}_g)\right)(E) + \hat{X}^{t-1}_f(E) \\ &= \hat{X}^{t-1}_g(E) - \hat{X}^{t-2}_g(E) + C_g(\hat{X}^{t-2}_g)(E) - C_g\left(\hat{X}^{t-1}_g - \hat{X}^{t-2}_g + C_g(\hat{X}^{t-2}_g)\right)(E) + \hat{X}^{t-1}_f(E) \end{aligned}$$

$$= \hat{X}_g^{t-1}(E) - \hat{X}_g^{t-2}(E) + C_g(\hat{X}_g^{t-2})(E) - C_g(\hat{X}_g^{t-1})(E) + \hat{X}_f^{t-1}(E)$$
$$= R_g(\hat{X}_g^{t-1})(E) - R_g(\hat{X}_g^{t-2})(E) + \hat{X}_f^{t-1}(E)$$
$$= R_g(\hat{X}_g^{t-1})(E) - R_g(\hat{X}_g^{t-2})(E) + R_g(\tilde{X}_g^{t-2})(E) = R_g(\tilde{X}_g^{t-1})(E)$$

as desired, where the fourth equality holds due to Lemma S1 while the last two equalities hold due to the inductive assumption that for all $s \leqslant t-1$, $\hat{X}_f^s(E) = R_{f_-^P}(\tilde{X}_{f_-^P}^{s-1})(E)$ and $\hat{X}_g^s = \tilde{X}_g^s$. ∎

S.4 Analysis of the Examples in Section 4

S.4.1 Example for Remark 3

Let us modify Example 1 by assuming that f_1 has a "Leontief" preference and would like to hire mass $a < 1$ of type-θ workers per unit mass of type-θ' workers, while keeping preferences of all other players unchanged. Thus, f_1's choice function becomes

$$C_{f_1}(X_{f_1}) = \left(a \min\{\tfrac{x_1}{a}, x_1'\}, \min\{\tfrac{x_1}{a}, x_1'\}\right), \tag{S6}$$

where $X_{f_1} = (x_1, x_1')$ is the measures of type θ and type θ' workers available to f_1. As in Example 1, without loss, we can set $x_1 = G(\theta) = \frac{1}{2}$ and $x_2' = G(\theta') = \frac{1}{2}$, and consider $X = (\frac{1}{2}, x_1', x_2, \frac{1}{2})$ as our candidate measures. Using this with (5), (S6), and (2), the fixed-point mapping is given as follows: for any $X = (\frac{1}{2}, x_1', x_2, \frac{1}{2})$,

$$T_{f_1}(X) = \left(\tfrac{1}{2}, R_{f_2}(x_2, \tfrac{1}{2})(\theta')\right) = \left(\tfrac{1}{2}, x_2\right) \tag{S7}$$
$$T_{f_2}(X) = \left(R_{f_1}(\tfrac{1}{2}, x_1')(\theta), \tfrac{1}{2}\right) = \left(\tfrac{1}{2} - a x_1', \tfrac{1}{2}\right). \tag{S8}$$

Letting $\phi_1(x_2) = x_2$ and $\phi_2(x_1') = \frac{1}{2} - a x_1'$ and assuming $q \leqslant \frac{1}{4}$, the mapping $(x_1', x_2) \mapsto (\phi_1(x_2), \phi_2(x_1'))$ can be depicted as in Figure S1. The unique fixed point of T is given as $x_1' = x_2 = \frac{1}{2(a+1)}$, which yields the corresponding stable matching

$$M = \begin{pmatrix} f_1 & f_2 \\ \frac{a}{2(a+1)}\theta + \frac{1}{2(a+1)}\theta' & \frac{1}{2(a+1)}\theta + \frac{a}{2(a+1)}\theta' \end{pmatrix}.$$

To show that the tâtonnement process with any initial point converges to the fixed point, it suffices to show that $T^2 = T \circ T$ is a contraction mapping, and to invoke Proposition 2. To do so, consider any $X = (\frac{1}{2}, x_1', x_2, \frac{1}{2})$ and $Y = (\frac{1}{2}, y_1', y_2, \frac{1}{2})$. Then, $T^2(X) = (\frac{1}{2}, \frac{1}{2} - a x_1', \frac{1}{2} - a x_2, \frac{1}{2})$ and $T^2(Y) = (\frac{1}{2}, \frac{1}{2} - a y_1', \frac{1}{2} - a y_2, \frac{1}{2})$. Thus,

$$\|T^2(X) - T^2(Y)\| = \|(0, -a(x_1' - y_1'), a(x_2 - y_2), 0)\| = a\|X - Y\|,$$

which implies that T^2 is a contraction mapping, since $a < 1$.

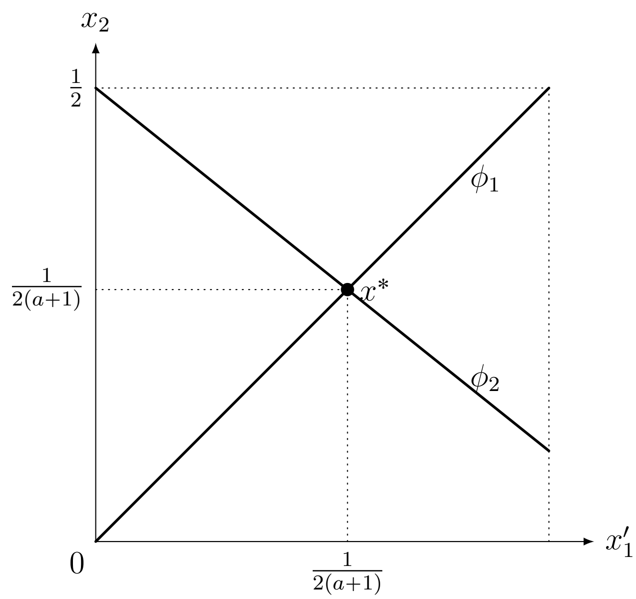


Figure S1: Fixed Point of Mapping T

S.4.2 Analysis of Example 2

Consider the following two cases:

1. Suppose f_1 hires measure 1/2 of each type of workers (i.e., all workers). In such a matching, none of the capacity of f_2 is filled. Thus, such a matching is blocked by f_2 and type-θ' workers (note that every type-θ' worker is currently matched with f_1, so they are willing to participate in the block).

2. Suppose f_1 hires no worker. Then, the only candidate for a stable matching is one in which f_2 hires measure 1/2 of the type-θ workers (otherwise f_2 and unmatched workers of type θ would block the matching). Then, because f_1 is the top choice of all type-θ workers and type-θ' workers prefer f_1 to $\emptyset$, the matching is blocked by a coalition of 1/2 of the type-θ workers, 1/2 of the type-θ' workers, and f_1.

S.4.3 Analysis of Example 3

Consider first a matching in which f_2 hires a positive mass of type-θ' workers. Then, it must hire type-θ' workers only and hire all of them. (Recall that type θ' prefers f_2 while f_2 has the capacity of 0.5.) Then, f_1 hires no one, implying that mass 0.6 of type-θ workers are all unmatched. Then, f_2 could form a blocking coalition with mass 0.6 of type-θ workers.

Consider second a matching in which f_2 hires zero mass of type-θ' workers. Then, f_1 must hire the entire type-θ' workers and the same mass of type-θ workers (since all type-θ' workers are available and the type-θ prefers f_1 to f_2.) This would only leave the mass 0.2 of type-θ workers for f_2 to hire. Then, f_2 could form a blocking coalition with the mass 0.4 of type-θ' workers (since the type θ' prefers f_2 to f_1).

S.5 Omitted Examples from Section 6:

Example S1. [Substitutable Preference] Consider Example 1 again and assume that the preference of firm f_2 as well as those of workers remains the same, but f_1's preference is changed as follows: it has a capacity equal to 1 (which is large enough to hire the entire workers); for the first quarter of its capacity, it hires workers according to the responsive preference: $\theta > \theta'$; for the remaining capacity, it is indifferent to hiring any number of additional workers. The resulting choice correspondence is

$$C_{f_2}(x, x') = \begin{cases} \{(x, x')\} & \text{if } x + x' \leqslant \frac{1}{4} \\ \{x\} \times [\frac{1}{4} - x, x'] & \text{if } x + x' > \frac{1}{4} \text{ and } x \leqslant \frac{1}{4} \\ [\frac{1}{4}, x] \times [0, x'] & \text{if } x + x' > \frac{1}{4} \text{ and } x > \frac{1}{4}. \end{cases}$$

One can verify that this preference is substitutable, and the set of stable matchings is

$$\mathcal{M}^* = \{(x_i, x_i')_{i=1,2} \,|\, x_1 \in [\tfrac{1}{4}, \tfrac{1}{2}], x_1' \in [0, \tfrac{1}{2} - x_1], \text{ and } (x_2, x_2') = (\tfrac{1}{2} - x_1, x_1)\}.$$

Observe M^* contains side-optimal matchings: the firm-optimal/worker-pessimal matching is $(x_1, x_1') = (\frac{1}{4}, \frac{1}{4})$ and $(x_2, x_2') = (\frac{1}{4}, \frac{1}{4})$, and the worker-optimal/firm-pessimal matching is $(x_1, x_1') = (\frac{1}{2}, 0)$ and $(x_2, x_2') = (0, \frac{1}{2})$. It can be seen easily, however, that M^* is not a lattice while C_{f_2} fails the strong-set monotonicity.

Example S2. [The role of order continuity in Theorem 4-(ii):] Consider our leading example with two types of workers, each of mass $\frac{1}{2}$, with the same preferences as before. As before, the measures of available workers can be described succinctly by (x_1', x_2), where x_1' is the measure of type θ' workers available to firm 1 and x_2 is the measure of type θ workers available to firm 2. (As before, the measure of type θ workers available to firm 1 and that of type θ' workers available to firm 2 are always $\frac{1}{2}$.) Suppose firms' preferences are given by two choice functions:

$$C_{f_1}(\tfrac{1}{2}, x_1') = \begin{cases} (\frac{1}{4}, x_1') & \text{if } x_1' \leqslant \frac{1}{3}; \\ (\frac{1}{4} - \frac{1}{4}x_1', x_1') & \text{if } x_1' > \frac{1}{3}; \end{cases} \quad \text{and } C_{f_2}(x_2, \tfrac{1}{2}) = \begin{cases} (x_2, \frac{1}{4}) & \text{if } x_2 \leqslant \frac{1}{3}; \\ (x_2, \frac{1}{4} - \frac{1}{4}x_2) & \text{if } x_2 > \frac{1}{3}, \end{cases}$$

where we set $x_1 = x_2' = 1/2$ as in other examples. As can be seen, the choice function fails to be order-continuous. Letting $\phi_1(x_2) = R_{f_2}(x_2, \frac{1}{2})$ and $\phi_2(x_1') = R_{f_1}(\frac{1}{2}, x_1')$, Figure S2 depicts ϕ_1 and ϕ_2 in (x_1', x_2) plane, whose intersection gives a fixed point of T. As can be seen, there exists a unique fixed point $(\frac{1}{4}; \frac{1}{4})$. Yet, if we iterate T from the largest point of the space $(\frac{1}{2}, \frac{1}{2})$, the algorithm gets "stuck" at $(\frac{1}{3}, \frac{1}{3}) = \lim_{k\to\infty} T^k(\frac{1}{2}, \frac{1}{2})$, which does not correspond to a stable matching.

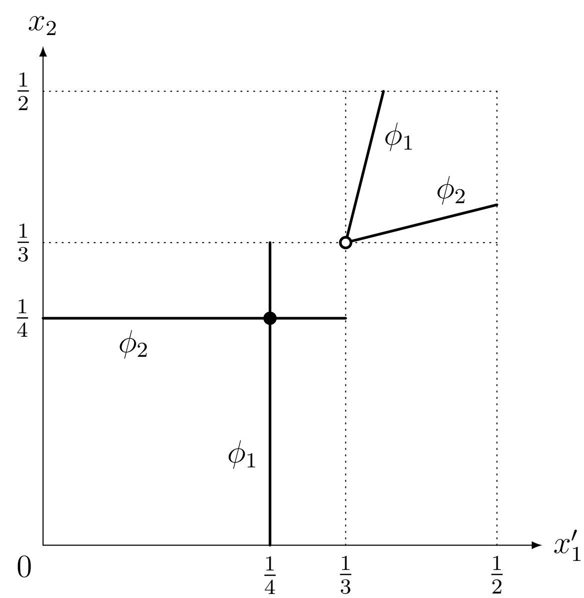


Figure S2: Order continuity fails at $(x_1', x_2) = (1/3, 1/3)$.

Example S3 (The role of LoAD for Theorem 6)**.** Consider a continuum economy with worker types θ_1 and θ_2 (each with measure 1/2) and firms f_1 and f_2. Preferences are as follows:

1. Firm f_1 wants to hire as many workers of type θ_2 as possible if no worker of type θ_1 is available, but if any positive measure of type-θ_1 workers is available, then f_1 wants to hire only type-θ_1 workers and no type-θ_2 workers at all, and f_1 wants to hire only up to measure 1/3 of type-θ_1 workers.

2. The preference of firm f_2 is symmetric, changing the roles of worker types θ_1 and θ_2. More specifically, Firm f_2 wants to hire as many workers of type θ_1 as possible if no worker of type θ_2 is available, but if any positive measure of type-θ_2 workers is available, then f_2 wants to hire only type-θ_2 workers and no type-θ_1 workers at all, and f_2 wants to hire only up to measure 1/3 of type-θ_2 workers.

3. Worker preferences are as follows:

$$\theta_1 : f_2 \succ f_1 \succ \emptyset,$$
$$\theta_2 : f_1 \succ f_2 \succ \emptyset.$$

Clearly, the firm preferences are substitutable. Note also that the worker optimal stable matching is

$$\underline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta_2 & \frac{1}{2}\theta_1 \end{pmatrix},$$

where the notation is such that measure 1/2 of type-θ_1 workers are matched to f_2 and measure 1/2 of type-θ_2 workers are matched to f_1.[2] Given this, it is straightforward to check the rich preferences hold.[3] Finally, firm preferences violate LoAD because, for instance, the choice of f_1 from measure 1/2 of θ_2 is to hire all of them, but even adding a measure $\epsilon < 1/2$ of type-θ_1 workers would cause f_1 to reject all θ_2 workers. As it turns out, there is a firm-optimal stable matching that is different from $\underline{M}$ and given as follows:

$$\overline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{3}\theta_1 & \frac{1}{3}\theta_2 \end{pmatrix}.$$

S.6 Analysis for Section 6

S.6.1 Preliminary Analysis

Throughout this section, we study the choice function of any individual firm with responsive preference while omitting the firm index from all notations for simplicity.

We begin by characterizing the choice function induced by the preferences. To this end, note first that, given a measure X of available workers, the quota constraint imposes the following constraint on any choice $X' \sqsubset X$:

$$X'(E) \leqslant \inf_{E' \subset E,\, E' \in \Sigma} X(E \backslash E') + \mathcal{Q}(\{t \in \mathcal{T} | E' \cap \Theta^t \neq \varnothing\}), \forall E \in \Sigma. \tag{S9}$$

[2] That this is a worker-optimal stable matching follows from the fact that the worker-proposing DA ends after the first round where each worker applies to and is accepted by her preferred firm.

[3] Under any matching $\hat{M} \neq \underline{M}$ that satisfies $\hat{M}_f = C_f(\hat{M}_f \vee \underline{M}_f)$ for all f, some firm, say f_1, must be matched with a positive measure of θ_1 workers. Given that $\hat{M}$ is individually rational, this implies that f_1 is not matched with any θ_2 workers. Also, since f_2 is matched with no more than measure 1/3 workers of θ_2 under any individual rational matching, at least measure 1/6 of θ_2 workers are unemployed under $\hat{M}$, which means that these workers belong to $\hat{M}^{f_2}_{\bar{F}}$ since they prefer f_2 to $\emptyset$ and $\emptyset \notin \bar{F}$. If they are available to f_2 in addition to $\underline{M}_{f_2}$, then f_2 would choose not to be matched with any θ_1 workers, to whom it is matched under $\underline{M}_{f_2}$. Thus, the rich preference condition is satisfied.

Then, the firm's optimization problem becomes

$$[P] \qquad \max_{X'} \int_0^1 s_f(\theta) dX'(\theta) \text{ subject to (S9).}$$

We identify a (unique) solution to $[P]$ via *Greedy Algorithm* defined below, which consists of multiple steps at each of which the firm hires workers with highest scores (among remaining workers) until the quota constraint becomes binding for some subset of ethnic types.

Greedy Algorithm (GA)**.** Set $T_0 = \varnothing$. For each Step $k \geqslant 1$, define T_k as a maximal element (in the set inclusion sense) of

$$\arg \max_{T' \subset \mathcal{T} \setminus (\cup_{j=0}^{k-1} T_j)} \inf \Big\{ s \in [0,1] \Big| X(\{\theta \in \Theta | \tau(\theta) \in T' \text{ and } s_f(\theta) \in [s,1]\}) < \mathcal{Q}_f((\cup_{j=0}^{k-1} T_j) \cup T') - \mathcal{Q}_f(\cup_{j=0}^{k-1} T_j) \Big\}, \quad \text{(S10)}$$

and s_k as the resulting maximum.[4] If $\cup_{j=1}^{k} T_j = \mathcal{T}$, stop; otherwise iterate to Step $k+1$.

Each step iteratively identifies the cutoff score for a group of workers whose residual quota is most binding. Let m denote the last step of this procedure, at which $\cup_{j=1}^{m} T_j = \mathcal{T}$.

Below, we first show that GA yields a unique profile $(s_k, T_k)_{k=1}^m$ (Lemma S3), and use this profile to identify a unique solution to $[P]$ (Proposition S2).

To begin, from any subpopulation X, one can obtain a corresponding score distribution for each ethnic type $t \in \mathcal{T}$, denoted F_t, as follows: for any (Borel) set $S \subset [0,1]$,

$$F_t(S) = X(\{\theta \in \Theta^t | s(\theta) \in S\}).$$

By abuse of notation, we denote for each $s \in [0,1]$

$$F_t(s) = F_t([0,s]) \text{ and } \bar{F}_t(s) = F_t([s,1]).$$

For any profile of sets $(S_t)_{t \in \mathcal{T}} \subset [0,1]^{|\mathcal{T}|}$ and $T' \subset \mathcal{T}$, let $S_{T'} := (S_t)_{t \in T'}$ and

$$F_{T'}(S_{T'}) := \sum_{t \in T'} F_t(S_t),$$

[4] We assume the infimum of an empty set is 1. Note that s_k is strictly decreasing in k since otherwise there would exist a k such that $s_k \geqslant s_{k-1}$ and

$$\bar{F}_{T_k}(s_{k-1}) + \bar{F}_{T_{k-1}}(s_{k-1}) \geqslant \bar{F}_{T_k}(s_k) + \bar{F}_{T_{k-1}}(s_{k-1}) = \mathcal{Q}((\cup_{j=1}^{k-2} T_j) \cup T_k \cup T_{k-1}) - \mathcal{Q}(\cup_{j=1}^{k-2} T_j),$$

contradicting the fact that T_{k-1} is the maximal element of the maximizer in Step $k-1$.

and, for each $s \in [0,1]$, let

$$F_{T'}(s) := \sum_{t \in T'} F_t(s) \text{ and } \bar{F}_{T'}(s) := \sum_{t \in T'} \bar{F}_t(s).$$

Let $F_{\varnothing}(\cdot) = \bar{F}_{\varnothing}(\cdot) = 0$.

Given a measure X of available workers, any choice $X' \sqsubset X$ of the firm must satisfy the following constraint:

$$X'(E) \leqslant \inf_{E' \subset E,\, E' \in \Sigma} X(E \backslash E') + \mathcal{Q}(\{t \in T | E' \cap \Theta^t \neq \varnothing\}), \forall E \in \Sigma. \tag{S11}$$

Lemma S2. *Let $F = (F_t)_{t \in \mathcal{T}}$ and $F' = (F'_t)_{t \in \mathcal{T}}$ be the score distributions corresponding to X and $X' \sqsubset X$, respectively. Then, the constraint (S11) holds if and only if*

$$F'_{\mathcal{T}}(S_{\mathcal{T}}) \leqslant \Psi_F(S_{\mathcal{T}}) := \min_{T' \subset \mathcal{T}} F_{\mathcal{T} \backslash T'}(S_{\mathcal{T} \backslash T'}) + \mathcal{Q}(T'), \forall S_{\mathcal{T}} = (S_t)_{t \in \mathcal{T}} \subset [0,1]^{|\mathcal{T}|}. \tag{S12}$$

Proof. To prove that (S11) implies (S12), for any $S_{\mathcal{T}} = (S_t)_{t \in \mathcal{T}}$, let $E_t = s^{-1}(S_t) \cap \Theta^t$ for each $t \in \mathcal{T}$. Fix $T' \subset \mathcal{T}$ and set $E = \cup_{t \in \mathcal{T}} E_t$ and $E' = \cup_{t \in T'} E_t$ in (S11). Then,

$$X'(E) = \sum_{t \in \mathcal{T}} X'(E_t) = \sum_{t \in \mathcal{T}} F'_t(S_t) = F'_T(S_T) \tag{S13}$$

$$X(E \backslash E') = \sum_{t \in \mathcal{T} \backslash T'} X(E_t) = \sum_{t \in \mathcal{T} \backslash T'} F_t(S_t) = F_{\mathcal{T} \backslash T'}(S_{\mathcal{T} \backslash T'}). \tag{S14}$$

That (S11) implies (S12) thus follows from observing that $\{t \in \mathcal{T} | E' \cap \Theta^t \neq \varnothing\} \subset T'$.

To prove the converse, if (S11) fails, then there must be E and $E' \subset E$ such that

$$X'(E) > X(E \backslash E') + \mathcal{Q}(\{t \in \mathcal{T} | E' \cap \Theta^t \neq \varnothing\}). \tag{S15}$$

Let $E_t = \Theta^t \cap E$ and $S_t = s(E_t)$ for each $t \in \mathcal{T}$, and $T' = \{t \in \mathcal{T} | E' \cap \Theta^t \neq \varnothing\}$. Then, (S13) easily holds. Also, (S14) holds since $E \backslash E' = E \cap (\cup_{t \in \mathcal{T} \backslash T'} \Theta^t) = \cup_{t \in \mathcal{T} \backslash T'} (E \cap \Theta^t) = \cup_{t \in \mathcal{T} \backslash T'} E_t$. Thus, (S15) means that the inequality (S12) fails. ▌

Given Lemma S2 and the fact that the firm's preference depends only on the score of workers, the firm's optimization problem $[P]$ can be rewritten as

$$[P'] \qquad \max_{(F'_t)_{t \in \mathcal{T}}} \int_0^1 s dF'_{\mathcal{T}}(s) \text{ subject to (S12)}.$$

Once the solution to $[P']$ is obtained, it will be straightforward to find a corresponding solution to the original problem $[P]$, as will be seen later.

Given the definition of $\bar{F}_t$, the set T_k in Greedy Algorithm is a maximal element of

$$\arg \max_{T' \subset \mathcal{T} \backslash (\cup_{j=0}^{k-1} T_j)} \inf \left\{ s \in [0,1] \middle| \bar{F}_{T'}(s) < \mathcal{Q}((\cup_{j=0}^{k-1} T_j) \cup T') - \mathcal{Q}(\cup_{j=0}^{k-1} T_j) \right\}, \tag{S16}$$

while s_k is the resulting maximum. (Recall $T_0 = \varnothing$.)

Lemma S3. *GA yields a unique profile* $(s_k, T_k)_{k=1}^m$.

Proof. Suppose that there are two profiles given by the Greedy Algorithm: $(s_k, T_k)_{k=1}^m$ and $(s'_k, T'_k)_{k=1}^{m'}$. Let $s_0 = s'_0 = 1$ and $T_0 = T'_0 = \varnothing$. Assume wlog that $m \leqslant m'$. For an inductive argument, fix any $k \leqslant m$ and assume that $(s_j, T_j) = (s'_j, T'_j), \forall j < k$. We aim to show that $(s_k, T_k) = (s'_k, T'_k)$. Given the inductive assumption and GA, it is clear that $s_k = s'_k$. Suppose for contradiction that $T_k, T'_k \subset \mathcal{T} \backslash (\cup_{j=0}^{k-1} T_j)$ and $T_k \neq T'_k$. By GA, letting $s_k = s'_k = s$ and $\tilde{T} = \cup_{j=1}^{k-1} T_k$, we have

$$\sum_{t \in T_k} \bar{F}_t(s) \leqslant \mathcal{Q}(\tilde{T} \cup T_k) - \mathcal{Q}(\tilde{T}) \text{ and } \sum_{t \in T'_k} \bar{F}_t(s) \leqslant \mathcal{Q}(\tilde{T} \cup T'_k) - \mathcal{Q}(\tilde{T}) \tag{S17}$$

with equality if $k < m$. Also, we must have

$$\sum_{t \in T_k \cap T'_k} \bar{F}_t(s) \leqslant \mathcal{Q}(\tilde{T} \cup (T_k \cap T'_k)) - \mathcal{Q}(\tilde{T}). \tag{S18}$$

By definition of T_k and the fact that $T_k \subsetneq T_k \cup T'_k$, we have

$$\begin{aligned}
\sum_{t \in T_k \cup T'_k} \bar{F}_t(s) &< \mathcal{Q}(\tilde{T} \cup (T_k \cup T'_k)) - \mathcal{Q}(\tilde{T}) \\
&\leqslant \mathcal{Q}(\tilde{T} \cup T_k) - \mathcal{Q}(\tilde{T}) + \mathcal{Q}(\tilde{T} \cup T'_k) - \mathcal{Q}(\tilde{T}) + Q(\tilde{T}) - \mathcal{Q}(\tilde{T} \cup (T_k \cap T'_k)) \\
&\leqslant \sum_{t \in T_k} \bar{F}_t(s) + \sum_{t \in T'_k} \bar{F}_t(s) + \mathcal{Q}(\tilde{T}) - \mathcal{Q}(\tilde{T} \cup (T_k \cap T'_k)),
\end{aligned}$$

where the weak inequality follows from submodularity of $\mathcal{Q}$ while the equality from (S17). Rearranging this equation, we obtain

$$\mathcal{Q}(\tilde{T} \cup (T_k \cap T'_k)) - \mathcal{Q}(\tilde{T}) < \sum_{t \in T_k \cap T'_k} \bar{F}_t(s),$$

which contradicts (S18).

Lastly, the inequality $m \leqslant m'$ must hold as equality, since we have $\cup_{k=1}^m T'_k = \cup_{k=1}^m T_k = \mathcal{T}$ by the above induction argument and the definition of m. ∎

Using the profile $(s_k, T_k)_{k=1}^m$ obtained from GA, let us define $F^* = (F_t^*)_{t \in \mathcal{T}}$ as follows: for each $t \in T_k$ and $S \subset [0,1]$,

$$F_t^*(S) = F_t(S \cap [s_k, 1]), \tag{S19}$$

that is, the firm hires a worker of ethnic type $t \in T_k$ if and only if her score is above s_k. This score distribution can be generated by the following subpopulation: for any $E \in \Sigma$ and $t \in T_k$,

$$X^*(E \cap \Theta^t) = X(\{\theta \in E \,|\, \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\})$$

and

$$X^*(E) = \sum_{k=1}^{m} \sum_{t \in T_k} X^*(E \cap \Theta^t). \tag{S20}$$

Proposition S2. *The subpopulation X^* in (S20) is a unique solution to $[P]$.*

Proof. We first prove that $F^* = (F_t^*)_{t\in\mathcal{T}}$ is a solution to $[P']$, which means that X^* is a solution to $[P]$. Afterward, we prove the uniqueness.

We first show that F^* satisfies the feasibility constraint (S12), that is, for any $S_{\mathcal{T}} = (S_t)_{t\in\mathcal{T}}$,

$$F_{\mathcal{T}}^*(S_{\mathcal{T}}) \leqslant F_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}) + \mathcal{Q}(T'), \forall T' \subset \mathcal{T}. \tag{S21}$$

Fix any $T' \subset \mathcal{T}$ and let $T_k' := T' \cap T_k$. for each $k = 1, \ldots, m$. Let $s_t = s_k$ for each $t \in T_k$. Note first that

$$F_{\mathcal{T}\setminus T'}^*(S_{\mathcal{T}\setminus T'}) = \sum_{t\in\mathcal{T}\setminus T'} F_t(S_t \cap [s_t, 1]) \leqslant \sum_{t\in\mathcal{T}\setminus T'} F_t(S_t) = F_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}). \tag{S22}$$

Next,

$$\begin{aligned}
\sum_{t\in T_k'} F_t^*(S_t) &= \sum_{t\in T_k'} F_t(S_t \cap [s_k, 1]) \\
&\leqslant \sum_{t\in T_k'} F_t([s_k, 1]) \leqslant \mathcal{Q}((\cup_{i=1}^{k-1} T_i) \cup T_k') - \mathcal{Q}(\cup_{i=1}^{k-1} T_i) \\
&\qquad \leqslant \mathcal{Q}((\cup_{j=1}^{k-2} T_j) \cup T_{k-1}' \cup T_k') - \mathcal{Q}((\cup_{j=1}^{k-2} T_j) \cup T_{k-1}') \\
&\qquad \ldots \\
&\qquad \leqslant \mathcal{Q}(\cup_{j=1}^{k} T_j') - \mathcal{Q}(\cup_{j=1}^{k-1} T_j'),
\end{aligned} \tag{S23}$$

where the second inequality holds since $T_k' \subset T_k$ while the third to last inequalities hold due to the submodularity. By (S22) and (S23), we get

$$\begin{aligned}
F_{\mathcal{T}}^*(S_{\mathcal{T}}) &= \sum_{t\in\mathcal{T}\setminus T'} F_t^*(S_t) + \sum_{k=1}^{m} \sum_{t\in T_k'} F_t^*(S_t) \\
&\leqslant \sum_{t\in\mathcal{T}\setminus T'} F_t(S_t) + \sum_{k=1}^{m} \left(\mathcal{Q}(\cup_{j=1}^{k} T_j') - \mathcal{Q}(\cup_{j=1}^{k-1} T_j')\right)
\end{aligned}$$

$$= \sum_{t \in \mathcal{T} \setminus T'} F_t(S_t) + \mathcal{Q}(\cup_{j=1}^m T'_k) = F_{\mathcal{T} \setminus T'}(S_{\mathcal{T} \setminus T'}) + \mathcal{Q}(T'),$$

which proves (S21).

To prove the optimality of F^*, note first that (S19) implies

$$\bar{F}^*_{\mathcal{T}}(s) = \begin{cases} \bar{F}_{\mathcal{T}}(s) & \text{if } s \geqslant s_1 \\ \bar{F}_{\mathcal{T} \setminus (\cup_{j=1}^{k-1} T_j)}(s) + \mathcal{Q}(\cup_{j=1}^{k-1} T_j) & \text{if } s \in [s_k, s_{k-1}), k = 2, ..., m \\ \mathcal{Q}(\mathcal{T}) & \text{if } s < s_m, \end{cases} \tag{S24}$$

which in turn implies

$$\bar{F}^*_{\mathcal{T}}(s) = \Psi_F([s,1]^{|\mathcal{T}|}), \forall s \in [0,1], \tag{S25}$$

that is, the constraint (S12) is binding with $S_t = [s,1]$ for all $t \in \mathcal{T}$ and $s \in [0,1]$. This can be easily seen by setting T' in (S12) as follows: $T' = \varnothing$ if $s \geqslant s_1$; $T' = \cup_{j=1}^{k-1} T_j$ if $s \in [s_k, s_{k-1})$ for some $k \in \{2, \ldots, m\}$; and $T' = \mathcal{T}$ if $s < s_m$. Now, (S25) implies that for any $F' = (F'_t)_{t \in \mathcal{T}}$ satisfying (S12), we have $\bar{F}'_{\mathcal{T}}(s) \leqslant \bar{F}^*_{\mathcal{T}}(s), \forall s \in [0,1]$. Using this, we obtain

$$\begin{aligned} \int_0^1 s dF^*_{\mathcal{T}}(s) &= -s\bar{F}^*_{\mathcal{T}}(s)\big|_{s=0}^1 + \int_0^1 \bar{F}^*_{\mathcal{T}}(s) ds \\ &= \int_0^1 \bar{F}^*_{\mathcal{T}}(s) ds \geqslant \int_0^1 \bar{F}'_{\mathcal{T}}(s) ds = \int_0^1 s dF'_{\mathcal{T}}(s), \end{aligned} \tag{S26}$$

which means that F^* is a solution to $[P]$.

To prove the uniqueness, let X' be any solution to $[P]$ and $F' = (F'_t)_{t \in \mathcal{T}}$ be the corresponding score distribution, which must thus be a solution to $[P']$. Then, we must have $\bar{F}'_{\mathcal{T}}(s) = \bar{F}^*_{\mathcal{T}}(s) = \Psi_F([s,1]^{|\mathcal{T}|})$ for all $s \in [0,1]$, since otherwise the inequality in (S26) would hold strictly. Next, we prove the following claim:

Claim S1. *For all k and $t \in T_k$, $F'_t([s_k, 1]) = F_t([s_k, 1])$ and $F'_t([0, s_k]) = 0$.*

Proof. Assume that this statement is true up to $k-1$. To show that it also holds for k, observe first that

$$\begin{aligned} \sum_{t \in \mathcal{T} \setminus (\cup_{j=1}^{k-1} T_j)} F'_t([s_k, 1]) &= F'_{\mathcal{T}}([s_k, 1]) - \sum_{j=1}^{k-1} F'_{T_j}([s_k, 1]) \\ &= F^*_{\mathcal{T}}([s_k, 1]) - \sum_{j=1}^{k-1} F_{T_j}([s_j, 1]) = \sum_{t \in \mathcal{T} \setminus (\cup_{j=1}^{k-1} T_j)} F_t([s_k, 1]), \end{aligned} \tag{S27}$$

where the second equality holds since $\bar{F}'_{\mathcal{T}} = \bar{F}^*$ and since the induction hypothesis together with the fact that $s_j < s_k, \forall j < k$ implies $F'_{T_j}([s_k, 1]) = F'_{T_j}([s_j, 1]) = F_{T_j}([s_j, 1]), \forall j < k$, while the second equality holds since (S16) and (S24) imply

$$\sum_{j=1}^{k-1} \bar{F}_{T_j}(s_j) = \sum_{j=1}^{k-1} \left(\mathcal{Q}(\cup_{i=0}^{j} T_i) - \mathcal{Q}(\cup_{i=0}^{j-1} T_i)\right) = \mathcal{Q}(\cup_{i=1}^{k-1} T_i) = \bar{F}^*_{\mathcal{T}}(s_k) - \bar{F}_{\mathcal{T}\setminus(\cup_{i=1}^{k-1} T_j)}(s_k).$$

Since $F'_t([s_k, 1]) \leqslant F_t([s_k, 1]), \forall t$, the equality (S27) implies $F'_t([s_k, 1]) = F_t([s_k.1])$ for all $t \in T_k$. Also, if $F'_t([0, s_k]) > 0$ for some $t \in T_k$, then we have

$$\begin{aligned}\sum_{t \in \cup_{j=1}^{k} T_j} F'_t([0,1]) &= \sum_{t \in \cup_{j=1}^{k} T_j} F'_t([0, s_k]) + \sum_{t \in \cup_{j=1}^{k} T_j} F'_t([s_k, 1]) \\ &> \sum_{t \in \cup_{j=1}^{k} T_j} F'_t([s_k, 1]) = \sum_{t \in \cup_{j=1}^{k} T_j} F_t([s_k, 1]) = \mathcal{Q}(\cup_{j=1}^{k} T_j),\end{aligned}$$

which contradicts (S12). ▮

For uniqueness, it suffices to prove that for any $E \in \Sigma$ and $t \in \mathcal{T}$, $X'(E \cap \Theta^t) = X^*(E \cap \Theta^t)$. Suppose not for contradiction, and suppose $t \in T_k$. Then, since $F'_t([0, s_k]) = 0$ by Claim S1, we must have

$$X'(E \cap \Theta^t) < X(\{\theta \in E | \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\}) = X^*(E \cap \Theta^t). \tag{S28}$$

Also,

$$X'(E^c \cap \Theta^t) \leqslant X(\{\theta \in E^c | \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\}). \tag{S29}$$

Adding up (S28) and (S29) side by side, we obtain

$$F'_t([s_k, 1]) = X'(\Theta^t) < X(\{\theta \in \Theta | \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\}) = F_t([s_k, 1]),$$

which contradicts Claim S1. ▮

S.6.2 Proof of Lemma 2

Consider any subpopulations X and Y with $Y \sqsubset X$ and corresponding score distributions $F = (F_t)_{t \in \mathcal{T}}$ and $G = (G_t)_{t \in \mathcal{T}}$. Note that for any $t \in \mathcal{T}$, Borel set $S \subset [0, 1]$, and $s \in [0, 1]$, we have $F_t(S) \geqslant G_t(S)$ and $\bar{F}_t(s) \geqslant \bar{G}_t(s)$. Let $(s_t)_{t \in \mathcal{T}}$ and $(s'_t)_{t \in \mathcal{T}}$ be the cutoff profiles from GA under F and G, respectively.

We first prove substitutability, for which it suffices to show that $s_t \geqslant s'_t$ for all ethnic types $t \in \mathcal{T}$. To show this suppose the contrary, i.e., there exists an ethnic type $t \in \mathcal{T}$ such that $s_t < s'_t$. Then the set $T^* := \{t \in \mathcal{T} : s_t < s'_t\}$ is nonempty. Fix an ethnic type t^* in this set T^* that has the highest cutoff among those in T^*, that is,

1. $t^* \in T^*$, and

2. $s'_{t^*} \geqslant s'_{t'}$ for every $t' \in T^*$.

Now, let k be the step of GA such that $t^* \in T_k$ under F, and k' be the step of GA such that $t^* \in T_{k'}$ under G, respectively. That is, k and k' are the steps at which some constraint related to type t^* becomes binding under F and G, respectively (or the last step of the algorithm if no constraint related to t^* becomes binding in any step of the algorithm).

Now, note that because t^* satisfies the property in (2) as described above, for every ethnic type t whose constraint is already binding by the beginning of step k' under G, a constraint for that type t is also binding by the beginning of step k under F. More formally, we have $\bar{T}' \subseteq \bar{T}$ for $\bar{T} := \cup_{j=1}^{k-1} T_j$ and $\bar{T}' := \cup_{j=1}^{k'-1} T'_j$, where T_j and T'_j are the maximal sets that solve the problem given as (S16) in step j of GA under F and G, respectively.[5]

Let T' be the set which is the maximal solution to (S16) at step k' under G. Then, $s^* := s'_{t^*}$ is strictly positive by our maintained assumption $s^* > s_{t^*}$ and the fact $s_{t^*} \geqslant 0$. Thus, it follows that

$$\bar{G}_{T'}(s^*) = \mathcal{Q}(\bar{T}' \cup T') - \mathcal{Q}(\bar{T}'). \tag{S30}$$

We also note that

$$\bar{G}_{T' \cap \bar{T}}(s^*) \leqslant \mathcal{Q}(\bar{T}' \cup (T' \cap \bar{T})) - \mathcal{Q}(\bar{T}'), \tag{S31}$$

because T' is a solution of the maximization problem described in (S16), and s^* is the associated time at which a constraint becomes binding in this step. Subtracting (S31) from (S30), we obtain

$$\bar{G}_{T'}(s^*) - \bar{G}_{T' \cap \bar{T}}(s^*) \geqslant \mathcal{Q}(\bar{T}' \cup T') - \mathcal{Q}(\bar{T}' \cup (T' \cap \bar{T})). \tag{S32}$$

Note that the left hand side of (S32) satisfies

$$\begin{aligned} \bar{G}_{T'}(s^*) - \bar{G}_{T' \cap \bar{T}}(s^*) &= \bar{G}_{T' \backslash \bar{T}}(s^*) \\ &\leqslant \bar{F}_{T' \backslash \bar{T}}(s^*), \end{aligned} \tag{S33}$$

where the equality follows from modularity of $\bar{G}$ (with respect to sets of ethnic types) and identity $T' \backslash (T' \cap \bar{T}) = T' \backslash \bar{T}$, while the inequality follows from the assumption that $G \sqsubseteq F$. Note also that the right hand side of (S32) satisfies

$$\mathcal{Q}(\bar{T}' \cup T') - \mathcal{Q}(\bar{T}' \cup (T' \cap \bar{T})) = \mathcal{Q}([\bar{T}' \cup (T' \cap \bar{T})] \cup (T' \backslash \bar{T})) - \mathcal{Q}([\bar{T}' \cup (T' \cap \bar{T})])$$

[5] In case $k = 1$ or $k' = 1$, we take T or T' to be an empty set.

$$\geqslant \mathcal{Q}(\bar{T} \cup (T' \backslash \bar{T})) - \mathcal{Q}(\bar{T}), \tag{S34}$$

where the equality is an identity and the inequality follows from the fact that $[\bar{T}' \cup (T' \cap \bar{T})] \subseteq \bar{T}$ (which in tern follows from the fact that $\bar{T}'$ is a subset of $\bar{T}$) and submodularity of $\mathcal{Q}$.

Substituting (S33) and (S34) into (S32), we obtain

$$\bar{F}_{T' \backslash \bar{T}}(s^*) \geqslant \mathcal{Q}(\bar{T} \cup (T' \backslash \bar{T})) - \mathcal{Q}(\bar{T}),$$

which implies $s_t \geqslant s^* = s'_t$, a contradiction.

To next prove LoAD, consider any subpopulations X and Y with $Y \sqsubset X$ and corresponding score distributions F and G. Let F^* and G^* denote the solution of $[P']$ under F and G, resp. The result is then immediate from observing that the total mass hired by the firm is

$$\sum_{t \in \mathcal{T}} F_t^*([0,1]) = \bar{F}_{\mathcal{T}}^*(0) = \Psi_F([0,1]^{|T|}) \geqslant \Psi_G([0,1]^{|T|}) = \bar{F}_{\mathcal{T}}^*(0) = \sum_{t \in \mathcal{T}} G_t^*([0,1]),$$

where the inequality follows from the definition of Ψ_F, Ψ_G in (S12) and the fact that $\sum_{t \in \mathcal{T} \backslash T'} F_t([0,1]) \geqslant \sum_{t \in \mathcal{T} \backslash T'} G_t([0,1]), \forall T' \subset \mathcal{T}$.

S.6.3 Proof of Proposition 3

To simplify notation, let $M = \underline{M}$, i.e., the worker-optimal matching. Fix any individually rational matching $\hat{M}$ such that $\hat{M} \succeq_F M$ and assume that $\bar{F} := \{f' \in F | \hat{M}_{f'} \succ_{f'} M_{f'}\}$ is nonempty. For any f, t, let $M_f^t := M_f(\Theta^t \cap \cdot)$ and $\hat{M}_f^t := \hat{M}_f(\Theta^t \cap \cdot)$. Since G is absolutely continuous, for any f, t, both M_f^t and $\hat{M}_f^t$, being its subpopulations, admit densities, denoted respectively by m_f^t and $\hat{m}_f^t$.

By Proposition S2 in Supplementary Material, Greedy Algorithm yields a unique optimal choice for each firm. Given this and the fact that $M_f = C_f(M_f)$ and $\hat{M}_f = C_f(M_f \vee \hat{M}_f)$, we may let s_f^t and $\hat{s}_f^t$ denote the cutoffs for each type $t \in T$ for M_f and $\hat{M}_f$ in the sense that $s_f^t = \inf\{s_f(\theta) | \theta \in \Theta^t \text{ and } m_f^t(\theta) > 0\}$ and $\hat{s}_f^t = \inf\{s_f(\theta) | \theta \in \Theta^t \text{ and } \hat{m}_f^t(\theta) > 0\}$.[6]

Because C_f satisfies LoAD by Lemma 2, $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ and $M_f = C_f(M_f)$ imply $M_f(\Theta) \leqslant \hat{M}_f(\Theta)$ for each $f \in F$. Then, Proposition 3 follows from proving a sequence of claims.

[6]These cutoffs are obtained from running Greedy Algorithm with M_f and $M_f \vee \hat{M}_f$ as measures of available workers, respectively. More precisely, we have $s_f^t = s_k$ if $t \in T_k$ in Greedy Algorithm run with M_f as measure of available workers, for instance.

Claim S2. *$M_\emptyset = \hat{M}_\emptyset$. Thus, $\sum_{f\in F} M_f = \sum_{f\in F} \hat{M}_f$ and $M_f(\Theta) = \hat{M}_f(\Theta), \forall f \in F$.*

Proof. Suppose to the contrary that $M_\emptyset \neq \hat{M}_\emptyset$. Then, with their densities denoted by $m_\emptyset$ and $\hat{m}_\emptyset$, $E_\emptyset = \{\theta \in \Theta \,|\, m_\emptyset(\theta) > \hat{m}_\emptyset(\theta)\}$ must be a non-empty set of positive (Lebesgue) measure, due to the fact that $M_\emptyset(\Theta) = G(\Theta) - \sum_{f\in F} M_f(\Theta) \geqslant G(\Theta) - \sum_{f\in F} \hat{M}_f(\Theta) = \hat{M}_\emptyset(\Theta)$. Also, letting $\hat{E}_f = \{\theta \in \Theta \,|\, \hat{m}_f(\theta) > m_f(\theta)\}$, there must be at least one firm f for which $E_\emptyset \cap \hat{E}_f$ is a non-empty set of positive measure, since otherwise we would have $\sum_{f'\in\tilde{F}} m_{f'}(\theta) \geqslant \sum_{f'\in\tilde{F}} \hat{m}_{f'}(\theta)$ for all $\theta \in E_\emptyset$, a contradiction. Now fixing such a firm f and letting $\tilde{E} = E_\emptyset \cap \hat{E}_f$, define

$$\tilde{m}_f(\theta) = \begin{cases} \min\{m_f(\theta) + m_\emptyset(\theta), \hat{m}_f(\theta)\} & \text{if } \theta \in \tilde{E} \\ m_f(\theta) & \text{otherwise.} \end{cases}$$

and let $\tilde{M}_f$ denote the corresponding measure. Note that $\tilde{m}_f(\theta) > m_f(\theta)$ for all $\theta \in \tilde{E}$, and also that $(M_f \vee \tilde{M}_f) = \tilde{M}_f \neq M_f$ and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$. Letting $M'_f = C_f(\tilde{M}_f)$, we show below that f and M'_f are a blocking coalition for M, contradicting the stability of M.

First of all, it follows from revealed preference that $C_f(M_f \vee M'_f) = M'_f$. To show that $M'_f \neq M_f$, note first that $\hat{m}_f(\theta) > m_f(\theta), \forall \theta \in \tilde{E}$ means $(\hat{M}_f \vee M_f)(\tilde{E}) = \hat{M}_f(\tilde{E})$, so

$$R_f(M_f \vee \hat{M}_f)(\tilde{E}) = (M_f \vee \hat{M}_f)(\tilde{E}) - C_f(M_f \vee \hat{M}_f)(\tilde{E}) = \hat{M}_f(\tilde{E}) - \hat{M}_f(\tilde{E}) = 0.$$

Then, since f has a substitutable preference and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$, we have $R_f(\tilde{M}_f)(\tilde{E}) = 0$, which means $M'_f(\tilde{E}) = C_f(\tilde{M}_f)(\tilde{E}) = \tilde{M}_f(\tilde{E}) \neq M_f(\tilde{E})$. It only remains to show that $M'_f \sqsubset D^{\preceq f}(M)$. For this, note that since $\hat{M}$ is individually rational and $\hat{m}_f(\theta) > 0, \forall \theta \in \tilde{E}$, we have $f \succ_\theta \emptyset, \forall \theta \in \tilde{E}$. Given the definition of $\tilde{M}_f$ (i.e., only those added to f are unmatched under M), this implies that $\tilde{M}_f \sqsubset D^{\preceq f}(M)$ and thus $M'_f \sqsubset \tilde{M}_f \sqsubset D^{\preceq f}(M)$. ▌

We then prove the next claim.

Claim S3. *For each $f \in \bar{F}$, there must be some t such that $s^t_f < \hat{s}^t_f$.*

Proof. Suppose to the contrary that $\hat{s}^t_f \leqslant s^t_f$ for all $t \in T$. Since $\sum_{t\in T} M^t_f(\Theta) = M_f(\Theta) = \hat{M}_f(\Theta) = \sum_{t\in T} \hat{M}^t_f(\Theta)$ and $M_f \neq \hat{M}_f$, there must exist $t \in T$ such that the set $\{\theta \in \Theta^t | s_f(\theta) > s^t_f \geqslant \hat{s}^t_f$ and $m^t_f(\theta) > \hat{m}^t_f(\theta)\}$ has a positive measure. A contradiction then arises since, due to the fact that C_f selects all workers of type t whose scores are above the cutoff $\hat{s}^t_f$ and that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$, the measure of workers of type $\theta \in \Theta^t$ selected when $\hat{M}_f \vee M_f$ is available is equal to $\hat{m}^t_f(\theta) = \max\{\hat{m}^t_f(\theta), m^t_f(\theta)\}$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geqslant \hat{s}^t_f$, which cannot be smaller than $m^t_f(\theta)$. ▌

Claim S4. *For any $f \in \bar{F}$ and $t \in T$, if $\hat{s}^t_f = 0$, then $\hat{M}_f(\Theta^t \cap \cdot) = M_f(\Theta^t \cap \cdot)$.*

Proof. Let us first observe that for any $f \in \bar{F}$ and t, if $\hat{M}_f(\Theta^t) < M_f(\Theta^t)$, then we have $\hat{s}^t_f > s^t_f$ since, as we argued in the proof of Claim S3, the fact that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ implies that $\hat{m}^t_f(\theta) = \max\{\hat{m}^t_f(\theta), m^t_f(\theta)\} \geqslant m^t_f(\theta)$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geqslant \hat{s}^t_f$, so if $\hat{s}^t_f \leqslant s^t_f$, then we would have a contradiction..

Fix now any $f \in \bar{F}$ and $t \in T$ for which $\hat{s}^t_f = 0$. Since it means $\hat{s}^t_f \leqslant s^t_f$, we must have $\hat{M}_f(\Theta^t) \geqslant M_f(\Theta^t)$ according to the above argument. We next show that $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$. Suppose to the contrary that $\hat{M}_f(\Theta^t) > M_f(\Theta^t)$. Then, the fact that $\hat{M}_f(\Theta) = M_f(\Theta)$ by Claim S2 implies that there must exist t' such that $\hat{M}_f(\Theta^{t'}) < M_f(\Theta^{t'})$ and no constraint for t' is binding at $\hat{M}_f$, i.e., $\hat{s}^{t'}_f = 0$. To show this, note first that for any $k \in \{1, \ldots, m-1\}$,

$$\sum_{t'' \in T_k} \hat{M}_f(\Theta^{t''}) = \mathcal{Q}(\cup_{j=0}^{k} T_j) - \mathcal{Q}(\cup_{j=0}^{k-1} T_j),$$

where m and T_k are as defined in the Greedy Algorithm when f chooses $\hat{M}_f$ (given $M_f \vee \hat{M}_f$). Adding up these equalities from $k = 1$ to $m-1$, we obtain

$$\sum_{t'' \in T^*} \hat{M}_f(\Theta^{t''}) = \mathcal{Q}(T^*), \tag{S35}$$

where $T^* := \cup_{k=0}^{m-1} T_k$ represents the set of all ethnic types at least one of whose constraints is binding at $\hat{M}_f$. Also note that, because $\mathcal{Q}$ gives upper-bound constraints for any matching by assumption, we have

$$\sum_{t'' \in T^*} M_f(\Theta^{t''}) \leqslant \mathcal{Q}(T^*), \tag{S36}$$

so combining (S35) and (S36), we obtain

$$\sum_{t'' \in T^*} \hat{M}_f(\Theta^{t''}) \geqslant \sum_{t'' \in T^*} M_f(\Theta^{t''}). \tag{S37}$$

(S37) and the assumption that $\hat{M}_f(\Theta^t) > M_f(\Theta^t)$, together with the fact that $M_f(\Theta) = \hat{M}_f(\Theta)$ by Claim S2, imply that

$$\sum_{t'' \in T^{**}} \hat{M}_f(\Theta^{t''}) < \sum_{t'' \in T^{**}} M_f(\Theta^{t''}), \tag{S38}$$

where $T^{**} := T \backslash (T^* \cup \{t\})$ represents the set of ethnic types other than t whose constraints are not binding at $\hat{M}_f$. (S38) implies that there is at least one ethnic type $t' \in T^{**}$ such that

$$\hat{M}_f(\Theta^{t'}) < M_f(\Theta^{t'}), \tag{S39}$$

as desired.

Since $t' \in T^{**}$, i.e., t' is unconstrained at $\hat{M}$, all workers of ethnic type t' who are available to f at $\hat{M}$ are hired by f. Furthermore, the firm is faced with a weakly larger measure of workers of ethnic type t' when choosing $\hat{M}$ than at M (recall $\hat{M}_f \succeq_f M_f$). So (S39) cannot hold, a contradiction. Hence, $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$.

Given $\hat{s}^t_f = 0$ (i.e. the lowest possible score), we must have $\max\{\hat{m}^t_f(\theta), m^t_f(\theta)\} = \hat{m}^t_f(\theta)$ for all $\theta \in \Theta^t$. In order that $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$, we must then have $\hat{m}^t_f(\theta) = m^t_f(\theta)$ for (almost) all $\theta \in \Theta^t$. ▮

Claim S5. *For any $t \in T$, if there is some $f \in \bar{F}$ such that $\hat{s}^t_f > s^t_f$, then we must have $\hat{s}^t_{f'} > 0, \forall f' \in \bar{F}$.*

Proof. Fix a firm $f \in \bar{F}$ with $\hat{s}^t_f > s^t_f$. Suppose to the contrary that the set $\bar{F}_0 = \{f' \in \bar{F} | \hat{s}^t_{f'} = 0\}$ is nonempty, and note that $f \notin \bar{F}_0$. Then, let us define $\bar{F}_+ = \bar{F} \backslash \bar{F}_0$ and consider the set

$$\{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (s^t_f, \hat{s}^t_f), \text{ and } s_{f'}(\theta) < \hat{s}^t_{f'} \, \forall f' \in \bar{F}_+ \backslash \{f\}\}.$$

Since M is stable, all worker types in this set must be matched with f under M, which implies that they cannot be matched with any firm in $\tilde{F} \backslash \bar{F}$ under $\hat{M}$ since $\hat{M}_{f'} = M_{f'}$ for each $f' \in F \backslash \bar{F}$ by assumption and also since $\hat{M}_\varnothing = M_\varnothing$ by Claim S2. Moreover, these workers cannot be matched with any firm $f' \in \bar{F}_+$ under $\hat{M}$ since their scores are below $\hat{s}^t_{f'}$. It thus follows that they must be matched with firms in $\bar{F}_0$ under $\hat{M}$ while being matched with $f \notin \bar{F}_0$ under M, which contradicts Claim S4. ▮

Claim S6. *Rich preferences hold.*

Proof. Fix any $f \in \bar{F}$ and $t \in T$ such that $s^t_f < \hat{s}^t_f$ (given by Claim S3), and let

$$\tilde{\Theta}^t_f := \{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (s^t_f, \hat{s}^t_f), \text{ and } s_{f'}(\theta) < \hat{s}^t_{f'} \, \forall f' \in \bar{F} \backslash \{f\}\}$$

be the set of ethnic type-t workers who prefer f to all other firms and have scores that will make them employable at f under M but not under $\hat{M}$ and not employable at any other firm in $\bar{F}$ under $\hat{M}$. Let $M' := \sum_{t \in T} G(\tilde{\Theta}^t_f \cap \cdot)$ denote the measure of these workers. The full support assumption and the fact (given by Claim S5) that $\hat{s}^t_{f'} > 0, \forall f' \in \bar{F}$ implies that $M'(\Theta) > 0$.

We show that these workers are not employed by any firm in $\bar{F}$ under either $\hat{M}$ or M. It is easy to see that these workers are not employed by any firm in $\bar{F}$ under $\hat{M}$ since their scores are below the cutoffs of these firms at $\hat{M}$. Since $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$, and since

$M_f = \hat{M}_f$ for each $f \in F \backslash \bar{F}$, we must have $\sum_{f \in \bar{F}} M_f = \sum_{f \in \bar{F}} \hat{M}_f$. It thus follows that these workers are not employed by firms in $\bar{F}$ under matching M either.

It follows that M' measures the workers who are employed outside $\bar{F}$ under M but available to firm f. Hence, $M' \sqsubset \hat{M}^f_{\bar{F}}$. Since $\hat{s}^t_f > s^t_f$, firm f will wish to replace some of its workers with these workers under M. Hence, $M_f \neq C_f((M_f + \hat{M}^f_{\bar{F}}) \wedge G)$, so the rich preferences property follows. ▌

The above claims complete the proof of the proposition.

S.6.4 (Counter)Example for Lemma 2: Role of Submodularity

Suppose that $\mathcal{T} = \{t_1, t_2, t_3\}$ and that $\mathcal{Q}(\{t_1, t_3\}) = \mathcal{Q}(\{t_2, t_3\}) = \mathcal{Q}(\{t_i\}) = 1/2, \forall i$ and $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\{t_1, t_2\}) = 1$. It is straightforward to check that this constraint violates the submodularity. Suppose that the subpopulations of available workers are given such that F_{t_i} is uniform on $[0, 1]$ for $i = 1, 3$ while $F_{t_2} = 0$. Clearly, the optimal cutoffs are $s_{t_1} = s_{t_3} = 3/4$ and $s_{t_2} = 0$. Consider next larger subpopulations whose score distributions are uniform on $[0, 1]$ for all three types. We argue that the optimal cutoffs are $s_{t_1} = s_{t_2} = 1/2$ and $s_{t_3} = 1$, which means that the preference of the firm is not substitutable since the cutoff s_{t_1} decreases from $3/4$ to $1/2$ as more workers of type t_2 become available. To prove this, let us set up the firm's optimization problem as

$$\max_{(s_{t_i})} \sum_{i=1}^{3} \int_{s_{t_i}}^{1} s ds$$

subject to

$$(1 - s_{t_1}) + (1 - s_{t_3}) \leqslant 1/2 \tag{S40}$$

$$(1 - s_{t_2}) + (1 - s_{t_3}) \leqslant 1/2. \tag{S41}$$

Note that we ignore all other constraints that can later be verified to be nonbinding. The corresponding Langrangian is

$$\sum_{i=1}^{3} \int_{s_{t_i}}^{1} s ds + \lambda_1 [s_{t_1} + s_{t_3} - 3/2] + \lambda_2 [s_{t_2} + s_{t_3} - 3/2],$$

which yields the first-order conditions given as

$$-s_{t_i} + \lambda_i \geqslant (=) 0 \text{ (if } s_{t_i} < 1) \quad \text{for } i = 1, 2$$
$$-s_{t_3} + \lambda_1 + \lambda_2 \geqslant (=) 0 \text{ (if } s_{t_3} < 1) \ .$$

Clearly, $\lambda_1, \lambda_2 > 0$ so (S40) and (S41) must be binding at the optimum. If $s_{t_3} < 1$, then (S40) and (S41) being binding implies $s_{t_1} = s_{t_2} = \frac{3}{2} - s_{t_3} > \frac{1}{2}$ and thus $s_{t_3} = \lambda_1 + \lambda_2 \geqslant s_{t_1} + s_{t_2} > 1$, a contradiction. So we must have $s_{t_3} = 1$ and thus $s_{t_1} = s_{t_2} = \frac{3}{2} - s_{t_3} = \frac{1}{2}$.

S.7 Results for Section 7

S.7.1 Omitted Proofs for Section 7

Proof of Lemma 6. Let $B(\theta, r) = \{\theta' \in \Theta \,|\, d^{\Theta}(\theta', \theta) < r\}$ and $S(\theta, r) = \{\theta' \in \Theta \,|\, d^{\Theta}(\theta', \theta) = r\}$ (recall d^{Θ} is a metric for the space Θ). For all $\theta \in \overline{\Theta}_f$ and $r > 0$, there must be some $r_\theta \in (0, r)$ such that $G(S(\theta, r_\theta)) = 0$.[7] This means that $\partial B(\theta, r_\theta) = S(\theta, r_\theta)$ has a zero measure. Consider now a collection $\{B(\theta, r_\theta) \,|\, \theta \in \Theta\}$ of open balls that covers $\overline{\Theta}_f$. Since $\overline{\Theta}_f$ is a closed subset of the compact set Θ, it is compact and thus has a finite cover. ▌

Proof of Lemma 7. Consider a decreasing sequence $(\epsilon_k)_{k \in \mathbb{N}}$ of real numbers converging to 0. Fix any k. Then, by Lemma 6, we can find a finite cover $\{B_\ell^k\}_{\ell=1,\ldots,L_k}$ of $\overline{\Theta}_f$ for each k such that for each ℓ, B_ℓ^k has a radius smaller than ϵ_k and $G(\partial B_\ell^k) = 0$. Define $A_1^k = B_1^k \cap \Theta_f$ and $A_\ell^k = \left(B_\ell^k \backslash (\cup_{\ell'=1}^{\ell-1} B_{\ell'}^k)\right) \cap \Theta_f$ for each $\ell \geqslant 2$. Then, $\{A_\ell^k\}_{\ell=1,\ldots,L_k}$ constitutes a partition of Θ_f. It is straightforward to see that $G(\partial A_\ell^k) = 0, \forall \ell$, since $G(\partial B_\ell^k) = 0, \forall \ell$, and that $G(\partial \Theta_f) = 0$.[8] This implies that $Y(\partial A_\ell^k) = 0, \forall \ell$. Given this and the assumption that $Y^q \xrightarrow{w^*} Y$, condition (e) of Theorem 12 implies that there exists sufficiently large q, denoted q_k, such that for all $q \geqslant q_k$

$$\frac{1}{q} < \frac{\epsilon_k}{L_k} \text{ and } |Y(A_\ell^k) - Y^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}, \forall \ell = 1, \ldots, L_k. \tag{S42}$$

Let us choose $(q_k)_{k \in \mathbb{N}}$ to be a sequence that strictly increases with k.

We construct X^q as follows: (i) $X^q(\theta) \leqslant Y^q(\theta), \forall \theta \in \Theta^q$; (ii) for each $q \in \{q_k, \ldots, q_{k+1} - 1\}$,

$$X^q(A_\ell^k) = \max \left\{ \frac{m}{q} \,\middle|\, m \in \mathbb{N} \cup \{0\} \text{ and } \frac{m}{q} \leqslant \min\{X(A_\ell^k), Y^q(A_\ell^k)\} \right\} \text{ for each } \ell = 1, \ldots, L_k.$$

[7] To see this, note first that $B(\theta, r) = \cup_{\tilde{r} \in [0,r)} S(\theta, \tilde{r})$ and $G(B(\theta, r)) < \infty$. Then, $G(S(\theta, \tilde{r})) > 0$ for at most countably many $\tilde{r}$'s, since otherwise the set $R_n \equiv \{\tilde{r} \in [0, r) \,|\, G(S(\theta, \tilde{r})) \geqslant 1/n\}$ has to be infinite for at least one n, which yields $G(B(\theta, r)) \geqslant G(\cup_{\tilde{r} \in R_n} S(\theta, \tilde{r})) \geqslant \frac{\infty}{n}$, a contradiction.

[8] The latter fact holds since $\Theta_f = \cup_{P \in \mathcal{P}: f > \emptyset} \Theta_P$ and thus $\partial \Theta_f \subset \cup_{P \in \mathcal{P}: f >_P \emptyset} \partial \Theta_P$, which implies

$$G(\partial \Theta_f) \leqslant G(\cup_{P \in \mathcal{P}: f >_P \emptyset} \partial \Theta_P) \leqslant \sum_{P \in \mathcal{P}: f >_P \emptyset} G(\partial \Theta_P) = 0.$$

It is straightforward to check the existence of X^q that satisfies both (i) and (ii). Note that (i) ensures that $X^q \sqsubset Y^q$ and $X^q(\Theta \backslash \Theta_f) \leqslant Y^q(\Theta \backslash \Theta_f) = 0$.

We show that for all $q \in \{q_k, \ldots, q_{k+1} - 1\}$, we have

$$|X(A_\ell^k) - X^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}. \tag{S43}$$

To see this, consider first the case where $X(A_\ell^k) < Y^q(A_\ell^k)$. Then, by definition of X^q and (S42), we have $0 \leqslant X(A_\ell^k) - X^q(A_\ell^k) < \frac{1}{q} < \frac{\epsilon_k}{L_k}$. In the case where $X(A_\ell^k) \geqslant Y^q(A_\ell^k)$, we have $X^q(A_\ell^k) = Y^q(A_\ell^k) \leqslant X(A_\ell^k) \leqslant Y(A_\ell^k)$, which implies by (S42)

$$|X(A_\ell^k) - X^q(A_\ell^k)| \leqslant |Y(A_\ell^k) - Y^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}.$$

Let us now prove that $X^q \xrightarrow{w^*} X$. We do so by invoking (b) of Theorem 12, according to which $X^q \xrightarrow{w^*} X$ if and only if $|\int h dX^q - \int h dX| \to 0$ as $q \to \infty$, for any uniformly continuous function $h \in C_u(\Theta)$.

Hence, to begin, fix any $h \in C_u(\Theta)$, and fix any $\epsilon > 0$. Next we define for each k and $q \in \{q_k, \ldots, q_{k+1} - 1\}$

$$\bar{h}_\ell^{q,k} \equiv \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta) h(\theta)}{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)} = \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta) h(\theta)}{X^q(A_\ell^k)}$$

if $X^q(A_\ell^k) > 0$, and if $X^q(A_\ell^k) = 0$, then define $\bar{h}_\ell^{q,k} \equiv h(\theta)$ for some arbitrarily chosen $\theta \in A_\ell^k$.

Note that $C_u(\Theta)$ is endowed with the sup norm $\|\cdot\|_\infty$ and $\|h\|_\infty$ is finite for any $h \in C_u(\Theta)$. Thus, there exists sufficiently large $K \in \mathbb{N}$ that for all $k > K$ and $q \in \{q_k, \ldots, q_{k+1} - 1\}$,

$$\|h\|_\infty \epsilon_k < \frac{\epsilon}{2} \text{ and } \sum_{\ell=1}^{L_k} \Big(\sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| \Big) X(A_\ell^k) < \frac{\epsilon}{2}, \tag{S44}$$

where the latter inequality is possible since the expression in the parenthesis can be made arbitrarily small by choosing sufficiently large k due to the uniform continuity of h and the fact that $A_\ell^k \subset B_\ell^k$ while B_ℓ^k has a radius smaller than ϵ_k with $\epsilon_k \to 0$ as $k \to \infty$.

Then, for any $q > Q := q_K$, there exists $k > K$ satisfying $q \in \{q_k, \ldots, q_{k+1} - 1\}$ such that

$$\begin{aligned} &\left| \int h dX^q - \int h dX \right| \\ = &\left| \int_{\theta \in \Theta_f} h dX^q - \int_{\theta \in \Theta_f} h dX \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X^q(A_\ell^k) - \int_{\theta \in \Theta_f} h dX \right| \\
&\leqslant \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} (X^q(A_\ell^k) - X(A_\ell^k)) \right| + \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X(A_\ell^k) - \int_{\theta \in \Theta_f} h dX \right| \\
&\leqslant \sum_{\ell=1}^{L_k} \|h\|_\infty |X^q(A_\ell^k) - X(A_\ell^k)| + \left| \sum_{\ell=1}^{L_k} \int_{\theta \in \Theta_f} \bar{h}_\ell^{q,k} \mathbb{1}_{A_\ell^k} dX - \sum_{\ell=1}^{L_k} \int_{\theta \in \Theta_f} h \mathbb{1}_{A_\ell^k} dX \right| \\
&\leqslant \|h\|_\infty \epsilon_k + \sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| X(A_\ell^k) \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}
$$

where the first equality holds since $X(\Theta \backslash \Theta_f) = X^q(\Theta \backslash \Theta_f) = 0$ while the third and fourth inequalities follow from (S43) and (S44), respectively. ▌

Proof of Lemma 8. Letting $Z^q = Y^q - X^q$ and $Z = Y - X$, we have $Z^q \xrightarrow{w*} Z$ because of the fact that for any $h \in C_u(\Theta)$,

$$\int_\Theta hdZ^q = \int_\Theta hdY^q - \int_\Theta hdX^q \to \int_\Theta hdY - \int_\Theta hdX = \int_\Theta hdZ$$

and (b) of Theorem 12. Since $Z^q = Y^q - X^q \in \mathcal{X}$, $Z^q \xrightarrow{w*} Z$, and $\mathcal{X}$ is compact, we have $Z \in \mathcal{X}$, which implies that $Z(E) = Y(E) - X(E) \geqslant 0$ for all $E \in \Sigma$, as desired. ▌

S.7.2 Proof for the Existence of ϵ-Distance Stable Matching

Let us reiterate the definition of ϵ-distance stability[9]: A matching $M^q \in (\mathcal{X}^q)^{n+1}$ in economy Γ^q is **ϵ-distance stable** if (i) for each $f \in F$, $M_f^q \in C_f^q(M_f^q)$; (ii) for each $P \in \mathcal{P}$, $M_f^q(\Theta_P) = 0, \forall f \prec_P \varnothing$; and (iii') $d(\tilde{M}_f^q, M_f^q) < \epsilon$ for any coalition f and $\tilde{M}_f^q \in \mathcal{X}^q$ that blocks M^q in the sense that $\tilde{M}_f^q \sqsubset D^{\leq f}(M^q)$ and $u_f(\tilde{M}_f^q) > u_f(M_f^q)$.

Proposition S3. *Suppose that there exists a stable matching in Γ such that $C_f(M_f) = \{M_f\}, \forall f \in F$. Then, for any $\epsilon > 0$, there is $Q \in \mathbb{N}$ such that for all $q > Q$, there exists an ϵ-distance stable matching.*

This result follows directly from combining the following two lemmas.[10]

[9] The definition of ϵ-distance stability is introduced in footnote 50 of the main paper.
[10] These lemmas are also used to prove Theorem 9

Lemma S4. *Consider any stable matching M in Γ such that $C_f(M_f) = \{M_f\}, \forall f \in F$. Then, there exists a sequence $(M^q)_{q\in\mathbb{N}}$ such that $M^q \xrightarrow{w^*} M$ while $M^q = (M_f^q)_{f\in\tilde{F}}$ is a feasible and individually rational matching in Γ^q.*

Proof. Given M, let us construct the matchings $\tilde{M}^q$ and M^q as in the proof of Lemma 9. It suffices to show that M_f^q converges to M_f since M^q is feasible and individually rational in Γ^q. To do so, we use the following fact: If every subsequence of sequence $(M_f^q)_{q\in\mathbb{N}}$ has a further subsequence that converges to M_f, then M_f^q converges to M_f. Consider any subsequence $(M_f^{k_m})_{m\in\mathbb{N}}$, which must then have a further subsequence, denoted $(M_f^{\ell_m})_{m\in\mathbb{N}}$, converging to some $\hat{M}_f$ since the sequence $(M_f^{k_m})_{m\in\mathbb{N}}$ lies in the compact space $\mathcal{X}$. Suppose for a contradiction that $\hat{M}_f \neq M_f$. Note first that $M_f^{\ell_m} \sqsubset \tilde{M}_f^{\ell_m}, \forall m \in \mathbb{N}$ (since $M_f^q \in C_f^q(\tilde{M}_f^q), \forall q \in \mathbb{N}$) and $\tilde{M}_f^{\ell_m} \xrightarrow{w^*} M_f$, which implies by Lemma 8 that $\hat{M}_f \sqsubset M_f$. Thus, we must have $u_f(\hat{M}_f) = u_f(M_f) - \epsilon$ for some $\epsilon > 0$ since $\hat{M}_f \neq C_f(M_f) = M_f$. By Lemma 9, we can find $Q \in \mathbb{N}$ such that for all $q > Q$,

$$u_f(M_f) < u_f(M_f^q) + \frac{\epsilon}{2}. \tag{S45}$$

Also, since $M_f^{\ell_m} \xrightarrow{w^*} \hat{M}_f$, we can find a sufficiently large $\ell_m > Q$ such that $u_f(M_f^{\ell_m}) < u_f(\hat{M}_f) + \frac{\epsilon}{2} = u_f(M_f) - \frac{\epsilon}{2}$, which contradicts (S45). ▮

Lemma S5. *Consider the sequence $(M^q)_{q\in\mathbb{N}}$ in Lemma S4. For any $\epsilon > 0$, there is $Q \in \mathbb{N}$ such that for all $q > Q$, M^q is an ϵ-distance stable matching.*

Proof. Let $\mathcal{B}_f^q$ denote the set of all blocking coalitions involving f under M^q: that is, $\mathcal{B}_f^q = \{\tilde{M} \in \mathcal{X}^q \mid \tilde{M} \sqsubset D^{\preceq f}(M_f^q) \text{ and } u_f(\tilde{M}) > u_f(M^q)\}$. Since $\mathcal{B}_f^q$ is finite for each q, the set $\mathcal{B}_f := \cup_{q\in\mathbb{N}}\mathcal{B}_f^q$ is countable. One can index the blocking coalitions in $\mathcal{B}_f$ to form a sequence $(\tilde{M}^k)_{k\in\mathbb{N}}$ such that for any $\tilde{M}^k \in \mathcal{B}_f^q$ and $\tilde{M}^{k'} \in \mathcal{B}_f^{q'}$ with $q < q'$, we have $k' > k$. Define $q(k)$ to be such that $\tilde{M}^k \in \mathcal{B}_f^{q(k)}$. We show that $\tilde{M}^k \xrightarrow{w^*} M_f$. If not, there must be a subsequence $(\tilde{M}^{k_m})_{m\in\mathbb{N}}$ that converges to some $M' \in \mathcal{X}$ with $M' \neq M_f$. To draw a contradiction, note first that since $D^{\preceq f}(\cdot)$ is continuous and $M^q \xrightarrow{w^*} M$, we have $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$. Combining this with the fact that $\tilde{M}^{k_m} \xrightarrow{w^*} M'$ and $\tilde{M}^{k_m} \sqsubset D^{\preceq f}(M^{q(k_m)})$, and invoking Lemma 8, we obtain $M' \sqsubset D^{\preceq f}(M)$, which implies that $u_f(M_f) - \epsilon' > u_f(M') + \epsilon'$ for some $\epsilon' > 0$, since C_f chooses a uniquely utility-maximizing subpopulation. Since $\tilde{M}^{k_m} \xrightarrow{w^*} M'$ and $M^q \xrightarrow{w^*} M_f$, we can find sufficiently large m such that $u_f(M_f^{q(k_m)}) > u_f(M_f) - \epsilon' > u_f(M') + \epsilon' > u_f(\tilde{M}^{q(k_m)})$, which contradicts with the fact that $\tilde{M}^{q(k_m)} \in \mathcal{B}_f^{q(k_m)}$. This establishes that $\tilde{M}^k \xrightarrow{w^*} M_f$. Using this and the fact that $M_f^q \xrightarrow{w^*} M_f$, one can choose sufficiently large K such that for all $k > K$, we have $d(\tilde{M}^k, M_f) < \frac{\epsilon}{2}$ and $d(M_f, M_f^{q(k)}) < \frac{\epsilon}{2}$, which implies that $d(\tilde{M}^k, M_f^{q(k)}) < d(\tilde{M}^k, M_f) + d(M_f, M_f^{q(k)}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This means that for all $q > q(K)$ and $\tilde{M} \in \mathcal{B}_f^q$, we have $d(\tilde{M}, M_f^q) < \epsilon$, showing that M^q is an ϵ-distance stable matching. ▌

S.7.3 (Counter)Example for Theorem 9

In this section, we provide an example that shows the assumption, $C_f(\underline{M}_f) = \{\underline{M}_f\}, \forall f \in \tilde{F}$, is necessary for Part 2 of Theorem 9.

Assume that $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and that $G^q(\theta_1) = G^q(\theta_2) = \frac{n_q}{q}$ and $G^q(\theta_3) = \frac{q-2n_q}{q}$, where n_q is a positive integer satisfying $\frac{n_q}{q} < \frac{1}{3}$ and $\lim_{q\to\infty} \frac{n_q}{q} = \frac{1}{3}$, which implies $G(\theta_i) = \frac{1}{3}, \forall i$. Assume also that in any finite economy Γ^q and limit economy Γ, there is a single firm f which is acceptable to all three types of workers and whose utility function is given as

$$u_f(x_1, x_2, x_3) = \max\{x_1, x_2\} - x_1 x_2 (\tfrac{1}{3} - x_1)(\tfrac{1}{3} - x_2) + x_3, \tag{S46}$$

where x_i is the measure of type θ_i. Given this, we have $\underline{M}_f(\theta_i) = \frac{1}{3}, \forall i$ while $C_f(\underline{M}_f) = \{(x_1, x_2, x_3) \mid \max\{x_1, x_2\} = x_3 = \frac{1}{3} \text{ and } x_1, x_2 \geqslant 0\}$ so the assumption $C_f(\underline{M}_f) = \{\underline{M}_f\}$ fails. In the finite economy Γ^q, the δ-stability requires that either

$$G^q(\theta_1) + G^q(\theta_3) - \delta \leqslant M_f^q(\theta_1) + M_f^q(\theta_3) \leqslant G^q(\theta_1) + G^q(\theta_3) \text{ and } M_f^q(\theta_2) = 0 \tag{S47}$$

or

$$G^q(\theta_2) + G^q(\theta_3) - \delta \leqslant M_f^q(\theta_2) + M_f^q(\theta_3) \leqslant G^q(\theta_2) + G^q(\theta_3) \text{ and } M_f^q(\theta_1) = 0, \tag{S48}$$

while $M_f^q(\theta_i) \geqslant 0, \forall i$. To see this, note that if both $M_f^q(\theta_2)$ and $M_f^q(\theta_1)$ were positive, then the firm could drop the entire mass of either type-θ_1 or type-θ_2 workers to (strictly) increase the second term in (S46) without affecting any other terms. If, for instance, $M_f^q(\theta_1) = 0$, then the firm's utility becomes $M_f^q(\theta_1) + M_f^q(\theta_3)$, so the δ-stability requires (S47). Observe now that for any δ-stable matching M^q satisfying (S47), there is another δ-stable matching $\tilde{M}^q$ satisfying (S48) such that $M_f^q(\theta_1) = \tilde{M}_f^q(\theta_2)$ and $M_f^q(\theta_3) = \tilde{M}_f^q(\theta_3)$. However, for small ϵ, neither matching is ϵ-worker optimal stable in Γ^q since the interests of types θ_1 and θ_2 are sharply opposed across the two matchings.

S.8 Analysis for Section 8.1

S.8.1 Proofs

Proof of Theorem 3. To prove (i), suppose a matching M is stable and population-proportional. We shall show that M satisfies the property (ii) of Definition 12. The population-proportionality of M, equivalently equality (12), implies that, if $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$ for any $\theta, \theta' \in \Theta_f^k$, then we must have $M_f(\theta) = D^{\leq f}(M)(\theta)$, or else $\frac{M_f(\theta)}{G(\theta)} = \alpha_f^k$, but in that case, we have a contradiction since $\alpha_f^k \geqslant \frac{M_f(\theta')}{G(\theta')}$. Then, by definition of $D^{\leq f}$,

$$M_f(\theta) = D^{\leq f}(M)(\theta) = \sum_{f' \in \tilde{F}: f' \leq f} M_{f'}(\theta) = M_f(\theta) + \sum_{f' \in \tilde{F}: f' < f} M_{f'}(\theta),$$

so $\sum_{f' \in \tilde{F}: f' < f} M_{f'}(\theta) = 0$. We have thus proven that M is strongly stable.

To prove (ii), fix any mechanism φ that implements a strongly stable matching for any measure. Suppose for contradiction that inequality (11) fails for some measure $G \in \overline{\mathcal{X}}$, for some a, P, P', with (a, P) and (a, P') in the support of G, and for some f. Then, let f be the most preferred firm (or the outside option) at P among those for which inequality (11) fails. Then,

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} < \sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')}, \tag{S49}$$

while

$$\sum_{f': f' \succeq_P f_-^P} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} \geqslant \sum_{f': f' \succeq_P f_-^P} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')},$$

so it follows that

$$\frac{\varphi_f(G)(a, P)}{G(a, P)} < \frac{\varphi_f(G)(a, P')}{G(a, P')}. \tag{S50}$$

By the strong stability of $\varphi(G)$ and the fact that (a, P) and (a, P') are in the same indifference class for firm f by assumption, inequality (S50) holds only if $\sum_{f': f \succ_P f'} \varphi_{f'}(G)(a, P) = 0$. Thus, because $\sum_{f' \in \tilde{F}} \frac{\varphi_{f'}(G)(a,P)}{G(a,P)} = 1$ as $\varphi(G)$ is a matching, we obtain

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} = 1.$$

This equality contradicts inequality (S49) because the right hand side of inequality (S49) cannot be strictly larger than 1 as $\varphi(G)$ is a matching, which completes the proof. ▌

Proof of Theorem 11 requires several lemmas.

Lemma S6. *The correspondence defined in* (10) *is convex-valued and upper hemicontinuous, and satisfies the revealed preference property.*

Proof. To first show that C_f is convex-valued, for any given X, consider any $X', X'' \in C_f(X)$. Note first that $X', X'' \sqsubseteq X$ implies $\lambda X' + (1-\lambda)X'' \sqsubseteq X$. Also, for any $\lambda \in [0,1]$ and $k \in I_f$,

$$\sum_{\theta \in \Theta_f^k} (\lambda X' + (1-\lambda)X'')(\theta) = \lambda \sum_{\theta \in \Theta_f^k} X'(\theta) + (1-\lambda) \sum_{\theta \in \Theta_f^k} X''(\theta) = \Lambda_f^k(X),$$

where the second equality holds since the assumption that $X', X'' \in C_f(X)$ implies $\Lambda_f^k(X) = \sum_{\theta \in \Theta_f^k} X'(\theta) = \sum_{\theta \in \Theta_f^k} X''(\theta)$. Thus, $\lambda X' + (1-\lambda)X'' \in C_f(X)$.

To next show the upper hemicontinuity, consider two sequences $(X^\ell)_{\ell \in \mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell \in \mathbb{N}}$ converging to some X and $\tilde{X}$, respectively, such that for each ℓ, $\tilde{X}^\ell \in C_f(X^\ell)$, i.e., $\tilde{X}^\ell \sqsubseteq X^\ell$ and $\Lambda_f^k(X^\ell) = \sum_{\theta \in \Theta_f^k} \tilde{X}^\ell(\theta), \forall k \in I_f$. Since Λ_f is continuous, we have $\Lambda_f^k(X) = \lim_{\ell \to \infty} \Lambda_f^k(X^\ell) = \lim_{\ell \to \infty} \sum_{\theta \in \Theta_f^k} \tilde{X}^\ell(\theta) = \sum_{\theta \in \Theta_f^k} \tilde{X}(\theta)$, which, together with the fact that $\tilde{X} \sqsubseteq X$, means that $\tilde{X} \in C_f(X)$, establishing the upper hemicontinuity of C_f.[11] To show the revealed preference property, let $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, and suppose $C_f(X) \cap \mathcal{X}_{X'} \neq \varnothing$. Consider any $Y \in C_f(X)$ such that $Y(\theta) \leqslant X'(\theta)$ for all θ. Then, $\Lambda_f^k(X) = \sum_{\theta \in \Theta_f^k} Y(\theta) \leqslant \sum_{\theta \in \Theta_f^k} X'(\theta)$ for all $k \in I_f$. By the revealed preference property of Λ_f, it follows that $\Lambda_f(X') = \Lambda_f(X)$. Therefore, Y satisfies $\sum_{\theta \in \Theta_f^k} Y(\theta) = \Lambda_f^k(X) = \Lambda_f^k(X')$ for all $k \in I_f$, which implies that $Y \in C_f(X')$ and thus $C_f(X) \cap \mathcal{X}_{X'} \subseteq C_f(X')$. To show $C_f(X) \cap \mathcal{X}_{X'} \supseteq C_f(X')$, consider any $Y \in C_f(X')$ and $\tilde{X} \in \mathcal{X}_{X'}$ such that $\tilde{X} \in C_f(X)$. By the previous argument, we have $\tilde{X} \in C_f(X')$, which implies that for each $f \in F$ and $k \in I_f$, $\sum_{\theta \in \Theta_f^k} Y(\theta) = \sum_{\theta \in \Theta_f^k} \tilde{X}(\theta)$. Since $\tilde{X} \in C_f(X)$, this means that $Y \in C_f(X)$ and thus $Y \in C_f(X) \cap \mathcal{X}_{X'}$. Therefore, we conclude that $C_f(X') = C_f(X) \cap \mathcal{X}_{X'}$ as desired. ∎

From now, we establish a couple of lemmas (Lemmas S7 and S8) and use them to prove Theorem 11. To do so, define a correspondence B_f from $\mathcal{X}$ to itself as follows:

$$B_f(X) := \{X' \sqsubset X \mid \text{ for each } k \in I_f, \text{ there is some } \alpha^k \in [0,1] \text{ such that } X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\} \text{ for all } \theta \in \Theta_f^k\}. \tag{S51}$$

We then modify the choice correspondence C_f in (10) to

$$\tilde{C}_f(X) = C_f(X) \cap B_f(X), \tag{S52}$$

for every $f \in F$ while we let $\tilde{C}_\varnothing = C_\varnothing$.

[11] The argument for $\tilde{X} \sqsubseteq X$ is that for each $\theta \in \Theta$, $\tilde{X}^\ell(\theta) \leqslant X^\ell(\theta)$, so taking the limit with respect to ℓ yields $\tilde{X}(\theta) \leqslant X(\theta)$.

Lemma S7. *For any $X \sqsubset G$, $\tilde{C}_f(X)$ is nonempty and a singleton set (i.e., $\tilde{C}_f$ is a function). Also, $\tilde{C}_f$ satisfies the revealed preference property.*

Proof. We first establish that for X, $\tilde{C}_f(X)$ is a singleton set. To do so, for any $X \in \mathcal{X}$, $f \in F$, $k \in I_f$, and $\alpha^k \in [0,1]$, define $\zeta_f^k(\alpha^k) := \sum_{\theta \in \Theta_f^k} \min\{X(\theta), \alpha^k G(\theta)\}$. From now on, we assume $C_f(X) \neq \{X\}$ since, if $C_f(X) = \{X\}$, then we have $\tilde{C}_f(X) = \{X\}$, a singleton set as desired. We show that there exists a unique $\hat{\alpha}^k$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Lambda_f^k(X)$, which means that $\tilde{C}_f(X)$ is a singleton set. First, we must have $\hat{\alpha}^k < \max_{\theta \in \Theta_f^k} X(\theta)$ since otherwise $\zeta_f^k(\hat{\alpha}^k) = \sum_{\theta \in \Theta_f^k} X(\theta) > \Lambda_f^k(X)$ (which follows from the assumption that $C_f(z) \neq \{X\}$ and thus, for any $X' \in C_f(X)$, $X' \sqsubset X$ and $X' \neq X$). Next, observe that $\zeta_f^k(\cdot)$ is strictly increasing in the range $[0, \max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)})$. Then, the continuity of ζ_f^k, along with the fact that $\zeta_f^k(0) = 0$ and $\zeta_f^k(\max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)}) > \Lambda_f^k(X)$, implies that there is a unique $\hat{\alpha}^k \in [0, \max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)})$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Lambda_f^k(X)$.

To show the revealed preference property, consider any $X, X', X'' \in \mathcal{X}$ such that $\tilde{C}_f(X) = \{X'\}$ and $X' \sqsubset X'' \sqsubset X$. Since we already know that $C_f(\cdot)$ satisfies the revealed preference property, we have $X' \in C_f(X'')$. It suffices to show that $X' \in B_f(X'')$, since it means $\tilde{C}_f(X'') = \{X'\}$, from which the revealed preference property follows. To do so, note that $X' \in B_f(X)$ means that $X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}$ for each k and $\theta \in \Theta_f^k$. Then, since $X(\theta) \geqslant X''(\theta) \geqslant X'(\theta)$ and $\alpha^k G(\theta) \geqslant X'(\theta)$, we have

$$X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\} \geqslant \min\{X''(\theta), \alpha^k G(\theta)\} \geqslant X'(\theta),$$

so $X'(\theta) = \min\{X''(\theta), \alpha^k G(\theta)\}$ as desired. ▮

Lemma S8. *Any stable matching in the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ is stable and population-proportional in the economy $(G, F, \mathcal{P}_\Theta, C_F)$.*[12]

Proof. Consider a stable matching $M = (M_f)_{f \in \tilde{F}}$ in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ and let $X_f = D^{\preceq f}(M)$ for each $f \in \tilde{F}$. We first show that M is stable in $(G, F, \mathcal{P}_\Theta, C_F)$. It is straightforward, thus omitted, to check the individual rationality. To check the condition of no blocking coalition, suppose to the contrary that there is a blocking pair f and M'_f, which means that $M'_f \sqsubset X_f$, $M'_f \in C_f(M'_f \vee M_f)$, and $M_f \notin C_f(M'_f \vee M_f)$. Given this, by Lemma S7, there exists $\tilde{M}_f$ such that $\tilde{C}_f(M'_f \vee M_f) = \{\tilde{M}_f\}$. First, by the revealed preference property of $\tilde{C}_f$ and the fact that $\tilde{M}_f \sqsubset (\tilde{M}_f \vee M_f) \sqsubset (M'_f \vee M_f)$, we have $\tilde{M}_f \in \tilde{C}_f(\tilde{M}_f \vee M_f)$ and $M_f \notin \tilde{C}_f(\tilde{M}_f \vee M_f)$. Second, since $M_f \sqsubset X_f$ and $M'_f \sqsubset X_f$, we have $\tilde{M}_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$. In sum, f and $\tilde{M}_f$ form a blocking pair in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, which is a contradiction.

[12]The economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ is a hypothetical economy that is identical to the original economy, except that the firms' choice correspondences C_F are replaced by $\tilde{C}_F$, which is defined in (S52).

To show the population-proportionality of M, observe that since M is stable in the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, we have $M_f = \tilde{C}_f(D^{\preceq f}(M)) = C_f(D^{\preceq f}(M)) \cap B_f(D^{\preceq f}(M))$ for each $f \in F$. Thus, $M_f \in B_f(D^{\preceq f}(M))$, that is, there is some α^k for each $k \in I_f$ such that (12) holds. ▮

Proof of Theorem 11. First, consider the case in which each firm's preference satisfies continuity. Given Lemma S8, it suffices to establish the existence of stable matching in the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$. For doing so, we prove the continuity of $\tilde{C}_f$ and invoke Theorem 2. The continuity of $\tilde{C}_f = C_f \cap B_f$ follows if both C_f and B_f are shown to be upper hemicontinuous, since the intersection of a family of closed-valued upper hemicontinuous correspondences, one of which is also compact-valued, is upper hemicontinuous (see 16.25 Theorem of Aliprantis and Border (2006) for instance), implying that $\tilde{C}_f$, which is a singleton-valued correspondence by Lemma S7, is continuous.

Since C_f is upper hemicontinuous by Lemma S6, it remains to show that B_f is upper hemicontinuous. Consider sequences $(X^\ell)_{\ell\in\mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell\in\mathbb{N}}$ with $\tilde{X}^\ell \in B_f(X^\ell), \forall \ell$, converging weakly to X and $\tilde{X}$, respectively. So, for each $k \in I_f$, there is a sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ such that $\tilde{X}^\ell(\theta) = \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}, \forall \theta \in \Theta^k_f$. For each k, let α^k be a limit to which a subsequence of the sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ converges. We claim that $\tilde{X}(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}, \forall \theta \in \Theta^k_f$. If $\tilde{X}(\theta) > \min\{X(\theta), \alpha^k G(\theta)\}$, then one can find sufficiently large ℓ to make $\tilde{X}^\ell(\theta)$, $X^\ell(\theta)$, and α^k_ℓ close to $\tilde{X}(\theta)$, $X(\theta)$, and α^k, respectively, so that $\tilde{X}^\ell(\theta) > \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}$, which is a contradiction. The same argument applies to the case where $\tilde{X}(\theta) < \min\{X(\theta), \alpha^k G(\theta)\}$.

Second, consider the case in which each firm's preference satisfies substitutability. Let $\tilde{C}_f$ be the augmented choice of f and $\tilde{R}_f$ the corresponding augmented rejection function. For each $f \in F$ and $k \in I_f$, let $\rho^k_f : \mathcal{X} \to \mathbb{R}_+$ denote firm f's rejection of total measure of workers in the indifference class Θ^k_f. Formally, define $\rho^k_f(X) := \sum_{\theta\in\Theta^k_f} X(\theta) - \Lambda^k_f(X)$ for each X.

Without loss of generality, fix $k \in I_f$ and consider X, X' with $X \sqsubseteq X'$ and $X \neq X'$ such that $X(\theta) = X'(\theta)$ for every $\theta \notin I^k_f$. First, consider $k' \neq k$. Then, by substitutability of Λ, we have $\rho^{k'}_f(X) \leqslant \rho^{k'}_f(X')$. Because $\sum_{\theta\in\Theta^{k'}_f} X(\theta) = \sum_{\theta\in\Theta^{k'}_f} X'(\theta)$ by assumption, it follows that

$$\Lambda^{k'}_f(X) = \sum_{\theta\in\Theta^{k'}_f} X(\theta) - \rho^{k'}_f(X) \geqslant \sum_{\theta\in\Theta^{k'}_f} X'(\theta) - \rho^{k'}_f(X') = \Lambda^{k'}_f(X').$$

Hence, $\alpha^{k'}_f \in [0,1]$ such that

$$\Lambda^{k'}_f(X) = \sum_{\theta\in\Theta^{k'}_f} \min\{X(\theta), \alpha^k_f G(\theta)\},$$

and $\bar{\alpha}_f^{k'} \in [0,1]$ such that

$$\Lambda_f^{k'}(X') = \sum_{\theta \in \Theta_f^{k'}} \min\{X'(\theta), \bar{\alpha}_f^k G(\theta)\},$$

have a relationship $\alpha_f^{k'} \geqslant \bar{\alpha}_f^{k'}$ (to see this, recall $X(\theta) = X'(\theta)$ for any $\theta \in \Theta_f^{k'}$ by assumption and note that the right hand sides of these equations are nondecreasing in $\alpha_f^{k'}$ and $\bar{\alpha}_f^{k'}$, respectively). This implies $\tilde{R}_f(X)(\theta) \leqslant \tilde{R}_f(X')(\theta)$ for all $\theta \in \Theta_f^{k'}$, as desired.

Second, consider k and investigate the following cases.

1. Suppose $\Lambda_f^k(X) = \sum_{\theta \in \Theta_f^k} X(\theta)$. Then, clearly $\rho_f^k(X) = \sum_{\theta \in \Theta_f^k} X(\theta) - \Lambda_f^k(X) = 0$, and thus $\tilde{R}_f(X)(\theta) = 0 \leqslant \tilde{R}_f(X')(\theta)$ for all $\theta \in \Theta_f^k$, as desired.

2. Suppose $\Lambda_f^k(X) < \sum_{\theta \in \Theta_f^k} X(\theta)$. Then,

   Claim S7. $\Lambda_f^k(X) = \Lambda_f^k(X')$.

   Proof. Suppose for contradiction that $\Lambda_f^k(X) \neq \Lambda_f^k(X')$. First, we cannot have $\Lambda_f^k(X') \in [0, \sum_{\theta \in \Theta_f^k} X(\theta)]$, since it would imply $\Lambda_f(X) \neq \Lambda_f(X') \leqslant (\sum_{\theta \in \Theta_f^k} X(\theta))_{k \in I_f}$, violating the revealed preference. So we must have $\Lambda_f^k(X') \in (\sum_{\theta \in \Theta_f^k} X(\theta), \sum_{\theta \in \Theta_f^k} X'(\theta)]$. We can then define $X^t := tX' + (1-t)X$ and find $t^* \in (0,1]$ such that $\sum_{\theta \in \Theta_f^k} X^{t^*}(\theta) = \Lambda_f^k(X')$. Since $X^{t^*} \leqslant X'$ and $\Lambda_f(X') \leqslant (\sum_{\theta \in \Theta_f^k} X^{t^*}(\theta))_{k \in I_f}$, the revealed preference implies $\Lambda_f^k(X^{t^*}) = \Lambda_f^k(X')$, which in turn implies $\rho_f^k(X^{t^*}) = \sum_{\theta \in \Theta_f^k} X^{t^*}(\theta) - \Lambda_f^k(X^{t^*}) = 0 < \sum_{\theta \in \Theta_f^k} X(\theta) - \Lambda_f^k(X) = \rho_f^k(X)$, contradicting the substitutability. ∎

   Given Claim S7, it follows that $\alpha_f^k \in [0,1]$ such that

   $$\Lambda_f^k(X) = \sum_{\theta \in \Theta_f^k} \min\{X(\theta), \alpha_f^k G(\theta)\},$$

   and $\bar{\alpha}_f^k \in [0,1]$ such that

   $$\Lambda_f^k(X') = \sum_{\theta \in \Theta_f^k} \min\{X'(\theta), \bar{\alpha}_f^k G(\theta)\},$$

   have a relationship $\alpha_f^k \geqslant \bar{\alpha}_f^k$ (recall $X(\theta) \leqslant X'(\theta)$ for any $\theta \in \Theta_f^k$ by assumption, and the right hand side of these equations are nondecreasing in the first arguments of the minimum operators). This implies $\tilde{R}_f(X)(\theta) \leqslant \tilde{R}_f(X')(\theta)$ for all $\theta \in \Theta_f^k$, as desired.

∎

S.8.2 Non-Strategy-Proofness for Firms

Even with a continuum of workers, no stable mechanism is strategy-proof for firms. Consider the following example.[13] Let $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\theta) = G(\theta') = 1/2$. Worker preferences are given as follows:

$$\begin{aligned} \theta :& f_2 \succ f_1 \succ \emptyset, \\ \theta' :& f_1 \succ f_2 \succ \emptyset. \end{aligned}$$

Firm preferences are responsive; f_1 prefers θ to θ' to vacant positions and wants to be matched with workers up to measure 1, while f_2 prefers θ' to θ to vacant positions and wants to be matched with workers up to measure 1/2.

Let φ be any stable mechanism. Given the above input, the following matching is the unique stable matching:

$$M \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta' & \frac{1}{2}\theta \end{pmatrix}.$$

Matching M is clearly stable because it is individually rational and every worker is matched to her most preferred firm. To see the uniqueness, note first that in any stable matching, every worker has to be matched to a firm (if there is a positive measure of unmatched workers, then there is also a vacant position in firm f_1, and they block the matching). All workers of type θ' are matched with f_1; otherwise, f_1 and θ' workers who are not matched with f_1 block the matching (note that f_1 has vacant positions to fill with θ' workers). Given this scenario, all workers of type θ are matched with f_2; otherwise, f_2 and θ workers who are not matched with f_2 block the matching (note that f_2 has vacant positions to fill with type θ workers).

Now, assume that f_1 misreports its preferences, declaring that θ is the only acceptable worker type, and it wants to be matched to them up to measure 1/2. Additionally, assume that preferences of other agents remain unchanged. Then, it is easy to verify that the unique stable matching is

$$M' \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta & \frac{1}{2}\theta' \end{pmatrix}.$$

Therefore, firm f_1 prefers its outcome at M' to the one at M, proving that no stable mechanism is strategy-proof for firms.

[13] This example is a continuum-population variant of an example in Section 3 of Hatfield, Kojima, and Narita (2014). See also Azevedo (2014), who shows that stable mechanisms are manipulable via capacities, even in markets with a continuum of workers.

S.9 Matching with Contracts

Our paper has assumed that the terms of employment contracts are exogenously given. In many applications, however, they are decided endogenously. To study such a situation, we generalize our basic model by introducing a continuum-population version of the "matching with contracts" model due to Hatfield and Milgrom (2005).

Let Ω denote a finite set of all available contracts with its typical element denoted as ω. Assume that Ω is partitioned into subsets, $\{\Omega_f\}_{f\in\tilde{F}}$, where Ω_f is the set of contacts for $f \in \tilde{F}$ and $\Omega_\emptyset = \{\omega_\emptyset\}$ (where $\omega_\emptyset$ denotes the option of not contracting with any firm). Each contract ω specifies contract terms a firm f may offer to a worker.[14] Let $f(\omega) \in \tilde{F}$ denote the firm associated with contract ω (or the outside option if $\omega = \omega_\emptyset$). Thus, $f(\omega) = f$ if and only if $\omega \in \Omega_f$. We use $P \in \mathcal{P}$ to denote workers' preference defined over Ω. Let $\omega_-^P \in \Omega$ denote a contract that is an immediate predecessor of ω according to preference P, that is, ω_-^P is the contract with the property $\omega_-^P \succ_P \omega$ and $\omega' \succeq_P \omega_-^P$ for all $\omega' \succ_P \omega$. As before, Θ_P denotes the subset of types in Θ whose preference is given by P.

In the current framework, the relevant unit of analysis is the measure of workers assigned to a particular contract. We let $X_\omega \in \mathcal{X}$ denote the subpopulation assigned to contract $\omega \in \Omega$ and $X_f = (X_\omega)_{\omega\in\Omega_f}$ denote a profile of subpopulations contracting with firm f. For any profiles $X, X' \in \mathcal{X}^{|\Omega_f|}$, we denote $X \sqsubset_f X'$ if $X_\omega \sqsubset X'_\omega$ for all $\omega \in \Omega_f$. Given a profile $X_f = (X_\omega)_{\omega\in\Omega_f}$, we use

$$X_f^{\preceq\omega}(\cdot) := \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega_f:\omega'\preceq_P\omega} X_{\omega'}(\Theta_P \cap \cdot), \tag{S53}$$

to denote the measure of workers hired by f under contract ω or worse; these are the workers who are willing to work for f under ω given their current contracts. We then let $X_f^{\preceq} = (X_f^{\preceq\omega})_{\omega\in\Omega_f}$.

For any $\omega \in \Omega_f$, let $X_\omega \in \mathcal{X}$ denote the subpopulation of workers who are available to firm f under the contract ω. Given any profile $X_f = (X_\omega)_{\omega\in\Omega_f} \in \mathcal{X}^{|\Omega_f|}$, each firm f's choice is described by a map $X_f \mapsto C_f(X_f) = (C_\omega(X_f))_{\omega\in\Omega_f} \in \mathcal{Y}_f(X_f)$, where

$$\mathcal{Y}_f(X_f) := \{Y_f \in \mathcal{X}^{|\Omega_f|} \,|\, Y_f^{\preceq\omega} \sqsubset X_\omega, \forall\omega \in \Omega_f\}.$$

For any profile of subpoulations in $\mathcal{Y}_f(X_f)$, the measure of workers who are hired by f under any contract $\omega \in \Omega_f$ or worse cannot exceed the measure of workers, X_ω, who are available under ω. The requirement that the output of C_f should belong to $\mathcal{Y}_f(X_f)$ is

[14] Note that the contract itself does not contain information about the associated worker type, and that each firm's preference is determined by what worker types it is matched with under what contracts.

based on the premise that each firm f is aware of workers' preferences and also believes (correctly) that only those workers who are available under $\omega \in \Omega_f$ can be hired under the contracts that are weakly inferior to ω, and thus put an upper bound on the measure of workers that can be hired under the latter contracts. As before, we let $C_{\omega_\emptyset}(X_{\omega_\emptyset}) = X_{\omega_\emptyset}$. We then assume the revealed preference property that for any $X, X' \in \mathcal{X}^{|\Omega_f|}$ with $X' \sqsubset_f X$ and for $M_f = C_f(X)$, if $M_f \in \mathcal{Y}_f(X')$, then $M_f = C_f(X')$.

An **allocation** is $M = (M_\omega)_{\omega \in \Omega}$ such that $M_\omega \in \mathcal{X}$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} M_\omega = G$. Let $M_f = (M_\omega)_{\omega \in \Omega_f} \in \mathcal{X}^{|\Omega_f|}$ denote a profile of subpopulations who are matched with f. Given $M_f = (M_\omega)_{\omega \in \Omega_f}$, define $M_f^{\precsim\omega}$ by (S53) and let $M_f^{\precsim} = (M_f^{\precsim\omega})_{\omega \in \Omega_f}$. Note that $M_f^{\precsim\omega}$ corresponds to a subpopulation of workers already hired by firm f who are willing to work for f under ω given their current contracts. In other words, $M_f^{\precsim}$ does *not* include the workers available to firm f who are currently matched with firms other than f. A subpopulation of *all* workers—not only those hired by firm f—who are available to $f \in \tilde{F}$ under contract $\omega \in \Omega_f$ is denoted as before by

$$D^{\preceq\omega}(M)(\cdot) = \sum_{P \in \mathcal{P}} \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap \cdot).$$

Let $D^{\preceq f}(M) = (D^{\preceq\omega}(M))_{\omega \in \Omega_f}$.

Definition S1. An allocation $M = (M_\omega)_{\omega \in \Omega}$ is **stable** if

1. (Individual Rationality) $M_\omega(\Theta_P) = 0$ for all $P \in \mathcal{P}$ and $\omega \in \Omega$ satisfying $\omega \prec_P \omega_\emptyset$; and for each $f \in F$, $M_f = C_f(M_f^{\precsim})$, and

2. (No Blocking Coalition) There exist no $f \in F$ and $\tilde{M}_f \in \mathcal{X}^{|\Omega_f|}, \tilde{M}_f \neq M_f$ such that

$$\tilde{M}_f = C_f(\tilde{M}_f^{\precsim} \vee M_f^{\precsim}) \text{ and } \tilde{M}_f^{\precsim} \sqsubset_f D^{\preceq f}(M).$$

Note that this definition reduces to the notion of stability in Definition 2 if each firm is associated with exactly one contract.

Let us now define a map $T = (T_\omega)_{\omega \in \Omega} : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$ by specifying, for each $\omega \in \Omega$ and $E \in \Sigma$,

$$T_\omega(X)(E) := \sum_{P : P(1) = \omega} G(\Theta_P \cap E) + \sum_{P : P(1) \neq \omega} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E). \qquad \text{(S54)}$$

Theorem S1. *$M = (M_\omega)_{\omega \in \Omega}$ is a stable allocation if and only if $M_f = C_f(X_f), \forall f \in \tilde{F}$, where $X = (X_\omega)_{\omega \in \Omega}$ is a fixed point of mapping T.*

Proof. **("Only if" part)** Suppose M is a stable allocation in $\mathcal{X}^{|\Omega|}$. We prove that $X = (D^{\preceq\omega}(M))_{\omega\in\Omega}$ is a fixed point of T. Let us first show that for each $\omega \in \Omega$, $X_\omega \in \mathcal{X}$. It is clear that as each M_ω is countably additive, so is $M_\omega(\Theta_P \cap \cdot)$, which implies that $X_\omega(\cdot) = D^{\preceq\omega}(M)(\cdot) = \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap \cdot)$ is also countably additive. It is also clear that since $(M_\omega)_{\omega\in\Omega}$ is an allocation, $X_\omega \sqsubset G$. Thus, we have $X_\omega \in \mathcal{X}$.

We next claim that $M_f = C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \emptyset$ since $M_\emptyset = X_\emptyset = C_\emptyset(X_\emptyset)$. To prove the claim for $f \neq \emptyset$, suppose for a contradiction that $M_f \neq C_f(X_f)$, and let us denote $\widetilde{M}_f = C_f(X_f)$. Since $C_f(X_f) \in \mathcal{Y}_f(X_f)$ by definition, we have $\widetilde{M}_f^{\precsim} \sqsubset_f X_f$ and thus $(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim}) \sqsubset_f X_f$. Given this and $\widetilde{M}_f \in \mathcal{Y}_f(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim})$, we have $\widetilde{M}_f = C_f(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim})$ by revealed preference, which means that M is not stable since $\widetilde{M}_f^{\precsim} \sqsubset_f X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X = T(X)$. The fact that $M_\omega = C_\omega(X_{f(\omega)}), \forall\omega \in \Omega$ means that $X_\omega - M_\omega = R_\omega(X_{f(\omega)}), \forall\omega \in \Omega$. Then, for each $\omega \in \Omega$ and $E \in \Sigma$, we obtain

$$\begin{aligned}
&\sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E) \\
&= \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \Big(X_{\omega_-^P}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E) \Big) \\
&= \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \left(\sum_{\omega'\in\Omega:\omega'\preceq_P\omega_-^P} M_{\omega'}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E) \right) \\
&= \sum_{P:P(1)=\omega} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap E) = X_\omega(E),
\end{aligned}$$

where the second and fourth equalities follow from the definition of $X_{\omega_-^P}$ and X_ω, respectively, while the third from the fact that ω_-^P is an immediate predecessor of ω and $\sum_{\omega'\in\Omega:\omega'\preceq_P P(1)} M_{\omega'}(\Theta_P \cap E) = G(\Theta_P \cap E)$. The above equation holds for every contract $\omega \in \Omega$, so we conclude that $X = T(X)$, i.e. X is a fixed point of T.

("If" part) Let us first introduce some notations. Let ω_+^P denote an **immediate successor** of $\omega \in \Omega$ at $P \in \mathcal{P}$: that is, $\omega_+^P \prec_P \omega$, and for any $\omega' \prec_P \omega$, $\omega' \preceq_P \omega_+^P$. Note that for any $\omega, \tilde{\omega} \in \Omega$, $\omega = \tilde{\omega}_-^P$ if and only if $\tilde{\omega} = \omega_+^P$.

Suppose now that $X = (X_\omega)_{\omega\in\Omega} \in \mathcal{X}^{|\Omega|}$ is a fixed point of T. For each contract $\omega \in \Omega$ and $E \in \Sigma$, define

$$M_\omega(E) = X_\omega(E) - \sum_{P:P(|\Omega|)\neq\omega} X_{\omega_+^P}(\Theta_P \cap E), \tag{S55}$$

where $P(|\Omega|) \neq \omega$ means that ω is not ranked lowest at P.

We first verify that for each $\omega \in \Omega$, $M_\omega \in \mathcal{X}$. First, it is clear that for each $\omega \in \Omega$, as both $X_\omega(\cdot)$ and $X_{\omega_+^P}(\Theta_P \cap \cdot)$ are countably additive, so is M_ω. It is also clear that for each $\omega \in \Omega$, $M_\omega \sqsubset X_\omega$.

Let us next show that for all $\omega \in \Omega$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \tag{S56}$$

which means that $X_\omega = D^{\leq \omega}(M)$. To do so, consider first a contract ω that is ranked lowest at P. By (S55) and the fact that $X_{\omega_+^P}(\Theta_P \cap \cdot) \equiv 0$, we have $M_\omega(\Theta_P \cap E) = X_\omega(\Theta_P \cap E)$. Hence, (S56) holds for such ω. Consider now any $\omega \in \Omega$ which is not ranked last, and assume for an inductive argument that (S56) holds true for ω_+^P, so $X_{\omega_+^P}(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E)$. Then, by (S55), we have

$$\begin{aligned} X_\omega(\Theta_P \cap E) = M_\omega(\Theta_P \cap E) + X_{\omega_+^P}(\Theta_P \cap E) &= M_\omega(\Theta_P \cap E) + \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E) \\ &= \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \end{aligned}$$

as desired.

To show that $M = (M_\omega)_{\omega \in \Omega}$ is an allocation, let $\omega = P(1)$. Then, the definition of T and the fact that X is a fixed point of T imply that for any $E \in \Sigma$,

$$G(\Theta_P \cap E) = X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E) = \sum_{\omega' \in \Omega} M_{\omega'}(\Theta_P \cap E),$$

where the second equality follows from (S56). Since the above equation holds for every $P \in \mathcal{P}$, M is an allocation.

We now prove that $(M_\omega)_{\omega \in \Omega}$ is stable. To prove the first part of Condition 1 of Definition S1, note first that $C_{\omega_\varnothing}(X_{\omega_\varnothing}) = \{X_{\omega_\varnothing}\}$ and thus $R_{\omega_\varnothing} = 0$. Fix any $P \in \mathcal{P}$ and assume $\varnothing \neq P(|\Omega|)$, since there is nothing to prove if $\varnothing$ is ranked lowest at P. Consider a contract ω such that $\omega_-^P = \omega_\varnothing$. Then, X being a fixed point of T means $X_\omega(\Theta_P) = R_{\omega_-^P}(\Theta_P) = R_{\omega_\varnothing}(\Theta_P) = 0$, which implies by (S56) that $0 = X_\omega(\Theta_P) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P) = \sum_{\omega' \in \Omega : \omega' \prec_P \omega_\varnothing} M_{\omega'}(\Theta_P)$, as desired.

To prove the second part of Condition 1 of Definition S1, we first show that $M_\omega = C_\omega(X_{f(\omega)})$, which is equivalent to showing $X_\omega - M_\omega = R_\omega(X_{f(\omega)})$. Since $X = T(X)$, we have $X_\omega(\Theta_P \cap \cdot) = R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap \cdot)$ for all $\omega \neq P(1)$, or $X_{\omega_+^P}(\Theta_P \cap \cdot) = R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot)$ for all $\omega \neq P(|\Omega|)$. Then, (S55) implies that for any $\omega \in \Omega$,

$$X_\omega(\cdot) - M_\omega(\cdot) = \sum_{P : P(|\Omega|) \neq \omega} X_{\omega_+^P}(\Theta_P \cap \cdot) = \sum_{P : P(|\Omega|) \neq \omega} R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) = R_\omega(X_{f(\omega)})(\cdot),$$

as desired. The last equality here follows from the fact that $R_\omega(\Theta_P \cap \cdot) = 0$ if $\omega = P(|\Omega|)$. To see this, note that if $\omega = P(|\Omega|) = \omega_\emptyset$, then $R_\omega(X_{f(\omega)}) = R_{\omega_\emptyset}(X_\emptyset) = 0$ by definition of $R_{\omega_\emptyset}$, and that if $\omega = P(|\Omega|) \prec_P \omega_\emptyset$, then the individual rationality of M for workers implies that $X_\omega(\Theta_P \cap \cdot) = M_\omega(\Theta_P \cap \cdot) = 0$, which in turn implies $R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) = 0$ since $R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) \sqsubset X_\omega(\Theta_P \cap \cdot)$. Given that $M_\omega = C_\omega(X_{f(\omega)})$ for all $\omega \in \Omega$ or $M_f = C_f(X_f)$ for all $f \in F$, $M_f = C_f(M_f^{\precsim})$ follows from the revealed preference and the fact that $M_f^{\precsim} \sqsubset_f X_f$.

It only remains to check Condition 2 of Definition S1. Suppose for a contradiction that it fails. Then, there exist f and $\widetilde{M}_f$ such that

$$M_f \neq \widetilde{M}_f = C_f(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim}) \text{ and } \widetilde{M}_f^{\precsim} \sqsubset_f D^{\preceq f}(M). \tag{S57}$$

Then, we have $M_f \in \mathcal{Y}_f(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim})$, $(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim}) \sqsubset_f D^{\preceq f}(M) = X_f$, and $M_f = C_f(X_f)$, which, by revealed preference, implies $M_f = C_f(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim})$, contradicting (S57). We have thus proven that M is stable. ▌

Given this characterization result, the existence of stable allocation follows from assuming that for each $f \in F$, $C_f : \mathcal{X}^{|\Omega_f|} \to \mathcal{X}^{|\Omega_f|}$ is continuous, since it guarantees the continuity of $T : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$:

Theorem S2. *If each firm's preference is continuous, then a stable allocation exists.*

S.10 Continuum of Firms: AH Model

Following AH, suppose that there is a continuum of firms. Each firm is infinitesimal and takes one of finitely many types, $1, \dots, n$. Let $N = \{1, \dots, n\}$ and $\overline{N} = N \cup \{\emptyset\}$. For each $i \in \overline{N}$, let m_i denote the mass of type-i firms in the economy with $m_\emptyset = \infty$. Assume for simplicity that there are finitely many types of workers so $\Theta = \{\theta_1, \dots, \theta_K\}$. We assume that each type-i firm has a strict preference over the sets in 2^Θ, denoted $\succeq_i$, which gives rise to a choice function $c_i : 2^\Theta \to 2^\Theta$.[15] For a null firm $i = \emptyset$, we let $E \succ_\emptyset E'$ for any $E' \subsetneq E$ and thus $c_\emptyset(E) = E, \forall E \in 2^\Theta$. We assume that $\succeq_i$ satisfies the standard axioms: completeness and transitivity. Each worker can be matched with only one firm (which may be a null firm) and is indifferent over firms of the same type while having strict preferences over different types of firms. We denote this economy as $\mathcal{E}$. This model is exactly the same

[15] An implicit assumption here is that each firm hires at most one worker per each worker type. However, our model can be extended in a straightforward manner to allow each firm to hire multiple workers of the same type.

as AH, except that there is no contracting issue (a firm and worker can contract under only one term) and we are considering a many-to-one matching environment.

A matching for type-i firms is a measure z_i defined on 2^Θ such that for each $E \in 2^\Theta$, $z_i(E)$ is the measure (or mass) of type-i firms matched with E. A profile $(z_i)_{i \in \overline{N}}$ is a matching if

$$\sum_{i \in \overline{N}} \sum_{E \in 2^\Theta : \theta \in E} z_i(E) = G(\theta), \forall \theta \in \Theta \tag{S58}$$

$$\sum_{E \in 2^\Theta} z_i(E) = m_i, \forall i \in N. \tag{S59}$$

Definition S2. A matching $z = (z_i)_{i \in \overline{N}}$ is stable for the economy $\mathcal{E}$ if the following properties hold:

1 (Individual rationality). $z_i(E) = 0$ for any $i \in N$ and $E \in 2^\Theta$ such that there is some $\theta \in E$ with $\emptyset \succ_\theta i$; For any $i \in N$ and $E \in 2^\Theta$, $z_i(E) > 0$ implies $c_i(E) = E$;

2 (No blocking coalition). there are no $i \in N$ and $E, E' \in 2^\Theta$ with $E \cap E' = \emptyset$ such that (i) $E' \subset c_i(E \cup E')$; (ii) $z_i(E) > 0$; and (iii) for each $\theta \in E'$, there are $j \in \overline{N}$ and $E'' \in 2^\Theta$ such that $i \succ_\theta j$, $\theta \in E''$, and $z_j(E'') > 0$.

Individual rationality condition is straightforward. No blocking coalition condition requires no positive mass of firms which can get better off by hiring workers away from their less preferred firms. This notion of stability coincides with that of AH, once their model of many-to-many matching with contracts is adapted to our setup.

To show the existence of stable matching, we map the current setting into our model of continuum economy by introducing a large firm representing all type-i firms for each type $i \in \overline{N}$ and defining the *aggregate choice correspondence* for this firm, denoted $C_i : \mathcal{X} \rightrightarrows \mathcal{X}$. To do so, suppose that $X_i \in \mathcal{X}$ is a subpopulation of workers available to the large type-i firm, which is a subpopulation defined on Θ. We then allocate these workers *efficiently* across type-i firms as follows: Endow each small type-i firm with an arbitrary utility function $v_i : 2^\Theta \to \mathbb{R}_+$ that represents $\succeq_i$ and satisfies $v_i(\emptyset) = 0$.[16] And assign a set of workers $E \subset \Theta$ to the mass $z_i(E)$ of type-i firms for each $E \in 2^\Theta$ to solve

$$\max_{z_i \in \mathbb{R}_+^{|2^\Theta|}} \sum_{E \in 2^\Theta} v_i(E) z_i(E) \tag{A}$$

[16]Existence of such v_i is guaranteed because the firms' preferences satisfy the standard axioms.

subject to

$$\sum_{E' \in 2^\Theta : \theta \in E'} z_i(E') \leqslant X_i(\theta), \forall \theta \in \Theta \tag{S60}$$

$$\sum_{E \in 2^\Theta} z_i(E) = m_i, \tag{S61}$$

where the constraint (S61) is dropped if $i = \varnothing$.[17] That is, the aggregate (utilitarian) welfare of type-i firms is maximized under the constraint that for each type θ, the measure of type-i firms hiring (some) type-θ workers cannot exceed the measure of available type-θ workers. Letting $S_i(X_i)$ denote the set of optimal solutions for (A), it is straightforward to see that $S_i(X_i)$ is nonempty.

The aggregate choice correspondence for the large firm i is then defined as

$$C_i(X_i) = \Big\{ X_i' \in \mathcal{X} \mid \exists z_i \in S_i(X_i) \text{ such that } X_i'(\theta) = \sum_{E' \in 2^\Theta : \theta \in E'} z_i(E'), \forall \theta \in \Theta \Big\}.$$[18]

It is worth noting that our method to build the aggregate choice differs from that of AH in which firms of the same type choose workers following *serial dictatorship*. We let Γ denote a hypothetical economy that consists of large firms $1, \ldots, n, \varnothing$, whose choice correspondences are given as $(C_i)_{i \in \overline{N}}$, and workers whose population is given as G. Since (A) is linear, and thus continuous, in z_i, by Berge's maximum theorem, each correspondence S_i is upper hemicontinuous and convex-valued, so is C_i. Hence, by Theorem 2, there exists a stable matching in economy Γ, which implies the existence of a stable matching in the original economy $\mathcal{E}$, as is shown next:

Proposition S4. *Let $M = (M_i)_{i \in \overline{N}}$ be a stable matching for the hypothetical economy Γ. Then, there is a profile of solutions $z = (z_i)_{i \in \overline{N}}$ for (A) with $X_i = M_i$, $\forall i \in \overline{N}$ that constitutes a stable matching for economy $\mathcal{E}$.*

Proof. First, there must be a solution of (A) with $X_i = M_i$ that satisfies (S60) as equality, since otherwise M_i would not be individually rational in economy Γ. Now let $z = (z_i)_{i \in \overline{N}}$ be a profile of such solutions. First of all, we check that z is a matching in economy $\mathcal{E}$. That (S60) is binding with $X_i(\theta) = M_i(\theta)$ implies (S58) is satisfied since M is a matching so $\sum_{i \in \overline{N}} M_i(\theta) = G(\theta)$. Also, (S59) follows directly from (S61).

[17] Recall that $m_\varnothing = \infty$. Note that the constraint (S60) must always be binding for $i = \varnothing$ at any optimum since $v_\varnothing(E) > v_\varnothing(\emptyset) = 0$ for any $E \neq \emptyset$, as implied by the earlier assumption.

[18] Since each $S_i(X_i)$ consists of optimal solutions, S_i satisfies the revealed preference. Given this, C_i also satisfies the revealed preference property.

Note next that since M is stable in economy Γ, we must have $M_i \in C_i(\tilde{X}_i)$ for $\tilde{X}_i = D^{\preceq i}(M)$, which implies that $(z_i)_{i\in\overline{N}}$ solves (A) with $X_i = \tilde{X}_i$.

To show the stability of z in economy $\mathcal{E}$, we first prove that it is individually rational. To see the individual rationality for workers, observe that for any $\theta \in \Theta$ and $\emptyset \succ_\theta i$, we have $M_i(\theta) = 0$, which follows from the stability of M in economy Γ. It therefore follows from (S60) with $X_i(\theta) = M_i(\theta)$ that $z_i(E) = 0$ for any E containing θ. To see individual rationality of $(z_i)_{i\in\overline{N}}$ for firms, supppose not. Then, there must be some firm $i \in N$ and $E \in 2^\Theta$ such that $z_i(E) > 0$ and $c_i(E) \subsetneq E$, which means that $v_i(E) < v_i(c_i(E))$. Given this, consider another matching for type-i firms which assigns the set of workers $c_i(E)$ to the type-i firms of mass $z_i(E)$ which are hiring E under z_i, while assigning the same set of workers to all other type-i firms in N. This alternative matching then achieves a higher value for (A), which contradicts with the optimality of z_i.

We next prove z satisfies the second requirement of stability in economy $\mathcal{E}$. Suppose for contradiction that $(z_i)_{i\in\overline{N}}$ admits a blocking coalition with the firm type i and E, E' as in Condition 2 of Definition S2. Let for each $\theta \in E'$

$$\bar{z}(\theta) = \max\left\{z_j(E'') \,|\, j \in \overline{N}, \theta \in E'', z_j(E'') > 0, \text{ and } i \succ_\theta j\right\}.$$

Then, at least $\bar{z}(\theta)$ of workers of type $\theta \in E'$ is not matched with type-i firms under $(z_i)_{i\in\overline{N}}$ but available to them under $\tilde{X}_i = D^{\preceq i}(M)$. Consider now an alternative matching z_i' for type-i firms given as follows: (1) mass $\min\{\min_{\theta\in E''} \bar{z}(\theta), z_i(E)\}$ of type-i firms which were matched with E under z_i are now each matched with the set $c_i(E \cup E')$ of workers; (2) all other type-i firms are matched with the same set of workers as under z_i. Note first that the workers matched with type-i firms under z_i' are a subpopulation of $\tilde{X}_i$, satisfying (S60) with $X_i = \tilde{X}_i$. Also, z_i' easily satisfies (S61). However, since $c_i(E \cup E') \neq E$, we have $v_i(c_i(E \cup E')) > v_i(E)$, which means that the type-i firms in (1) above enjoy a higher utility under z_i' than z_i while the type-i firms in (2) enjoy the same utility. This contradicts with the fact that $(z_i)_{i\in\overline{N}}$ solves (A) with $X_i = \tilde{X}_i$. ▌

Corollary S1. *There exists a stable matching for economy $\mathcal{E}$.*

Recall that the approach taken here to build the aggregate choice correspondence differs from that of AH based on the serial dictatorship. One advantage of the current approach is its extendability beyond finite types of workers. It is not difficult to extend (A) to allow for continuum of worker types. Since (A) is linear, its solution set (or correspondences) will satisfy the properties such as upper hemicontinuity and convex-valuedness (as long as v_i is a continuous function).[19]

[19]In the functional space, a linearity need not imply continuity. But in our case, as long as v_i is assumed

References

Aliprantis, C. D., and K. C. Border (2006): *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, Berlin.

Azevedo, E. M. (2014): "Imperfect Competition in Two-Sided Matching Markets," *Games and Economic Behavior*, 83, 207–223.

Hatfield, J. W., F. Kojima, and Y. Narita (2014): "Many-to-Many Matching with Max–Min Preferences," *Discrete Applied Mathematics*, 179, 235–240.

Hatfield, J. W., and P. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

to be continuous, the objective function of (A) is continuous in z_i in the weak-$*$ topology.