Housing Appreciation and Marginal Land Supply in Monocentric Cities with Topography

We revisit the celebrated relationship between supply constraints and home price growth. Augmenting existing models, we distinguish the roles of average versus marginal constraints in a dynamic monocentric city. In both theory and the panel of U.S. metropolitan areas, housing appreciates more where land availability decreases more with distance from downtown. Similarly, prices rise faster in cities with steeper rent gradients. Empirically, the parameter we estimate that governs marginal availability is not as strongly correlated with demand factors as average availability.

1 Introduction

In the United States, housing appreciation has been notably persistent in coastal regions where development is difficult. This persistence has motivated economists to document an empirical relationship between growth rates of housing prices and constraints on housing supply. In any model with constant and finitely elastic demand, reducing supply raises the level of prices. However, it is not obvious that supply constraints raise the growth rate of housing prices holding constant the growth rate of housing demand. For supply constraints to cause price growth, they presumably must become increasingly restrictive as the city grows.

We present models of urban growth that distinguish the effects of static versus dynamic supply constraints on housing price growth. We then augment existing empirical models of land availability within metropolitan areas so that they are governed by two parameters: a static parameter that affects land availability everywhere, and a dynamic parameter that governs the rate of change of land availability as the metropolitan area expands outward. The second parameter is more tightly linked in our model to price growth than the first. Similarly, we show that all else equal, price growth should be greater where land value declines more sharply with distance from downtown. We then provide empirical estimates of the relevant parameters from geographic data and estimate their relationship with the panel of repeated-sale home price growth across U.S. metropolitan areas.

Supply constraints can be man-made or physical, and can affect the intensity of both new construction and redevelopment of existing properties. The densities of both new development on raw land and redevelopment of existing properties are commonly restricted by zoning. All else equal, stricter zoning will increase the price per square foot of structures, and might increase or decrease the value of urban land depending on the nature of the constraints and demand and supply elasticities. How unchanging zoning restrictions affect price growth as demand grows is not obvious, but when allowable densities are increased, the supply of residential land is effectively increased. This reduces housing prices, measured per unit of quantity or quality. Thus the rate of price appreciation will depend on changes in zoning.

New housing is also built on previously undeveloped land. In many metropolitan areas, a substantial share of that construction is concentrated in new and unfinished neighborhoods not far from suburban outer edges. There, land is relatively inexpensive and available for large subdivsions. Large subdivisions are preferred by large builders for multiple reasons: greater control, more flexibility, and economies of scale. As the urban area expands outward, it can encounter obstacles to continued growth, including land with steep slopes, wetlands, and water. With less land available in new neighborhoods, some large builders must focus on more remote subdivisions. This increases both sprawl and commuting costs to the core. Thereby, rates of both sprawl and appreciation can depend on the rate at which the fraction of buildable land decreases with additional distance from the core.

To distinguish the average level from the growth of supply constraints, we augment empirical models of physical supply constraints that were pioneered by Saiz (2010) and Kolko (2008). In our baseline model, following Saiz, the fraction of buildable land $F(r)$ at each radial distance r from the city's center is exogenous. We generalize prior work by allowing this fraction to equal $\lambda r^{-\zeta}$. In previous work, ζ has been held constant at zero, so that $F(r) = \lambda$. This constraint is both proportional and static. It is proportional because only a percentage of all land at each radial distance r is buildable. It is static because the buildable percentage λ is constant over time in a sprawling city. Alternatively, when ζ is non-zero, the supply constraint can be both proportional and dynamic: $d\ln F(r)/dr = -\zeta d\ln r/dr$. In this case, ζ is a constraint on the growth rate of housing supply when the outer edge of the city expands at a constant rate.

We show in a baseline model, where development only occurs at the urban fringe, that price growth falls with the marginal availability elasticity $-\zeta$, but not with the standard static availability measure λ. Our baseline model is closely related to Capozza and Helsley (1990). Perfectly competitive landowners with perfect foresight sell their rural land to perfectly competitive developers, who immediately build and sell houses to the public. In equilibrium landowners at each radial distance r maximize the present value of their land by selling when the outer edge of the city expands to their radial distance.

The essence of urban land models is the gradient of land value with distance to downtown. Depending on the functional form of that gradient, the extent to which land values decline may also affect price growth. With a steeper rent gradient, new homes on the urban fringe are a worse substitute for existing homes, and an equivalent growth in demand leads to greater price growth where prices are higher. We show that when land rent has a constant elasticity in distance, price growth increases in that elasticity.

Realistically, unbuildable land is hard to define and endogenous. With higher prices some housing is built on steeper slopes inside expanding cities. For example, in coastal California expensive homes are built on very steep slopes at very high unit costs for foundations. For these reasons we introduce a second model that incorporates construction on previously unbuildable, steeper slopes at progressively higher unit costs. Housing on steeper slopes can also be more valuable with better views or less valuable with more difficult access. In the resulting equilibrium houses are built at two boundaries: the previous outer edge of the city and an endogenous upper edge on steeper slopes inside the city.

All results from our baseline model hold with minor modifications when the physical difficulty of development is endogenized. The appreciation rate is increasing in the marginal cost of construction on slopes of a given steepness and decreasing in the premium paid for lots on slopes. Otherwise, the previous results are unchanged. Similar results would apply to development near other amenities, like lakes and seashores, when the density or quality of new construction is endogenous.

We test the theoretical results using topographical data and a panel of home prices for 302 U.S. metropolitan areas. A consistent estimator is derived for the two constants, ζ and λ and on our estimate of the land rental gradient. Housing appreciation for each metropolitan area is then regressed on our estimates of the two constants and multiple demand factors. As predicted by the model, the measures of the marginal unavailablity of land and the rental gradient are positively associated with housing appreciation between 1980 and 2010. Consistent with prior studies, average availability λ is also associated with housing appreciation, conditional on available demand controls. This result is not predicted by the initial model and could relate to correlation with unobserved demand factors. A relationship between price growth and λ is also consistent with the enriched model of endogenous development on slopes.

Previous papers focus on the average availablity of buildable land throughout a metropolitan area measured by the single parameter λ. Here, two parameters, λ and ζ, must be estimated simultaneously. In our two-parameter model, λ cannot be interpreted as average availability unless ζ is held constant. A two-parameter model of land availability must imperfectly approximate land availability at any point in a metropolitan area. We show in that the introduction of ζ significantly improves approximation of land availability at different radii relative to a λ-only approximation in the sense that a degrees-of-freedom-adjusted goodness of fit improves when we relax the standard assumption that $\zeta = 0$.

The next section contextualizes our paper relative to similar theoretical and empirical exercises. The third section sketches our baseline model and the extension to endogenous development on steep slopes. The initial model is then introduced formally in subsection 3.1 and its equilibrium is identified in 3.2. Subsection 3.3 covers the corresponding equilibrium with endogenous development on slopes. The estimator of the parameters of the power function for buildable land is explained in the Section 4.1 and all empirical results are presented in Section 4.2. Major results are summarized in the final section. All technical details appear in the Appendix.

2 Background

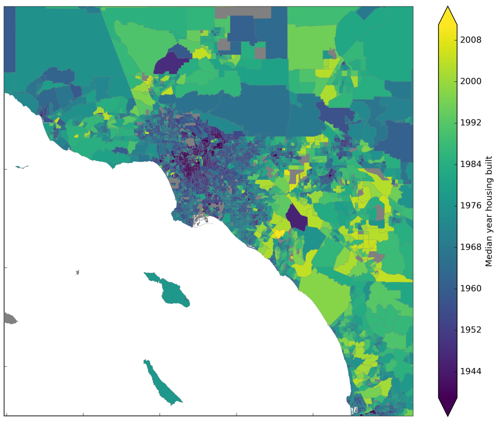
The models of this paper are most closely related to Capozza and Helsley (1990) CH was the first paper to apply results in real options to urban economics. It identified for the first time an equilibrium in which landowners exercise their options to develop land for housing at the expanding outer edge of a monocentric city. This innovation was important because much new construction is concentrated near the outer edges of major metropolitan areas: Bogin, Doerner, and Larson (2016) This pattern is illustrated in Figure 1 using data from the American Community Survey., 2011-2015.

CH and the initial model of this paper have the similarities identified in the introduction. The first major difference is also discussed in the introduction: topography with its implications for suburban sprawl and housing appreciation. The focus here on the relationship between housing appreciation and supply-side constraints motivates the remaining two material differences. In CH the dynamic model is additive; here, it is proportional. In CH elasticities of demand and supply are suppressed; here, elasticities are highlighted.

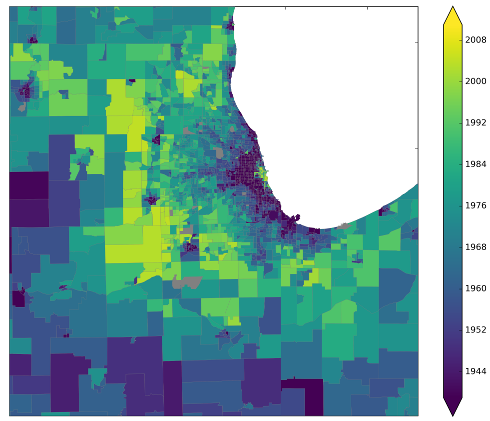
This paper is focused on the relationship between housing appreciation and the two proportional, supply-side constraints, static and dynamic, that are identified in the introduction. This suggests a variant of a standard, proportional model from the large literature on real options. Proportional models are characterized by isoelastic aggregate demand and supply and stationary growth rates of exogenous variables. In proportional models prices and other endogenous variables depend on levels of exogenous variables, whereas growth rates of prices and other endogenous variables depend on elasticities and growth rates of exogenous

Figure 1: Median age of housing stock by Census tract for six large metropolitan areas as of the 2011–2015 American Community Surveys.

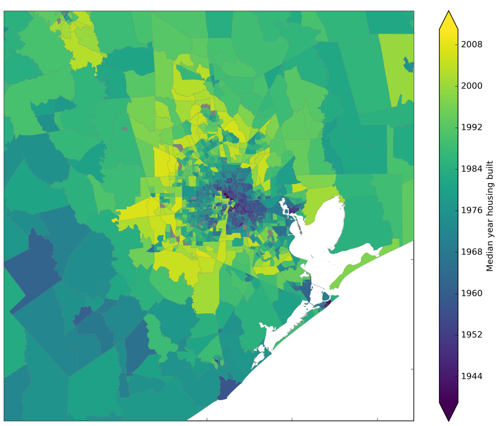
(a) Los Angeles



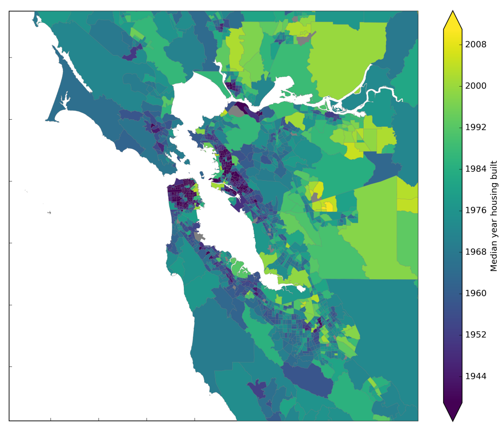
(b) Chicago



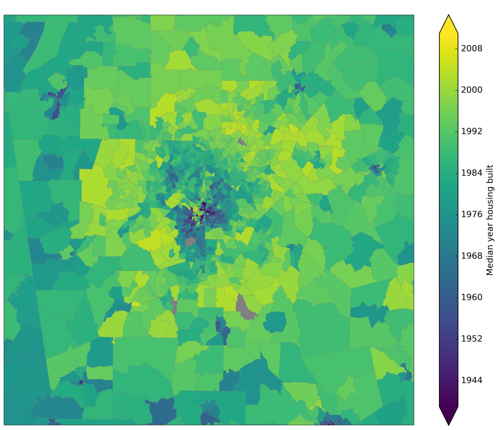
(c) Houston



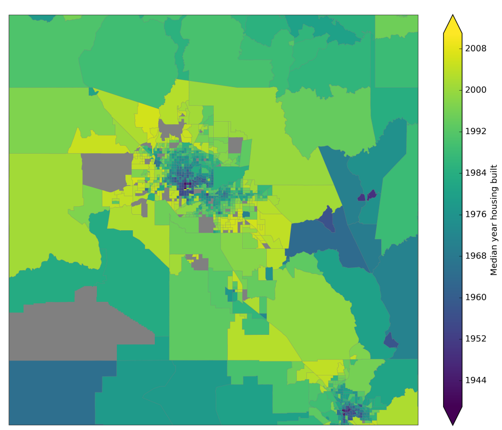
(d) San Francisco



(e) Atlanta



(f) Phoenix



variables. Thereby, growth rates of endogenous variables are stationary and constant if exogenous growth rates are constant. This plausible simplicity, independent of scale, facilitates the analysis of dynamic, supply-side constraints. It differs from additive models, like CH, where differences are easily identified but growth rates are complicated. Also, proportional models are often better approximations of more realistic, more complicated models than equally tractable, additive models.[1] The latter issues in this problem are identified below.

The third significant difference is related to the second. In CH rent for developed urban land is driven by the time-path of the representative household's utility. That utility disappears in their equilibrium rental function. Here, the aggregate demand for housing is driven by its exogenous component. Households are heterogeneous; their utility functions are suppressed; and their aggregate demand is specified exogenously. That aggregate demand is imperfectly inelastic–isoelastic with finite elasticities. These elasticities, which appear in the equilibrium pricing function for housing, contribute significantly to the empirical implications of the model. The distinction between prices and rent is moot in models with perfect foresight.

The supply-side effects of buildable land are central to both this paper and its second predecessor, Saiz (2010). Saiz has been widely cited for his sophisticated estimates of the average fractions of buildable land in multiple metropolitan areas throughout the United States. His model is static with a fixed fraction of buildable land λ. Housing appreciation is inferred from the elasticity e of the average housing price $\bar{p}$ with respect to the driver of aggregate demand x. With an additive pricing equation, that elasticity e is decreasing in the buildable fraction λ.

This static result can be restated as follows. Using the notation of this paper, combine the first two equations in Saiz:

$$\bar{P}(x) = \gamma + \frac{\kappa_0}{\sqrt{\lambda}} \left(x - \kappa_1\right).$$

This additively separable pricing function has four parameters: the unit construction cost $\gamma > 0$, two composite constants, $\kappa_0, \kappa_1 > 0$, and the buildable fraction, $0 < \lambda < 1$. Suppose that this static equation holds over some interval of time t. In this case, the growth rate of the average price $\bar{p}$ is proportional to the elasticity e:

$$\frac{d \ln \bar{p}}{dt} = e\frac{\dot{x}}{x}, \qquad e \equiv \frac{\bar{P}'(x)}{x\bar{P}(x)} = \frac{\kappa_0}{\gamma\sqrt{\lambda} + \kappa_0(x - \kappa_1)}.$$

[1] Indeed, it can be shown that our results apply when utility is Cobb-Douglas over other goods, land consumption, and "iceberg" commuting disutility. The main results do not rely on properties of irreversible supply and would apply to repeated rent of raw land.

In this case, the appreciation rate of housing is decreasing in the fraction of buildable land λ, holding constant the growth rate of exogenous demand, $\dot{x}/x > 0$.

This supply-side result holds because the average price of housing is additively separable in its common cost of construction γ and the average capitalized commuting costs to the core. The latter term includes the constant $\kappa_0/\sqrt{\lambda}$. This constant is decreasing in the buildable fraction λ and proportional to the common commuting cost per unit of radial distance for all households in the city. Thereby, larger fractions λ reduce suburban sprawl, shorten average commutes, and reduce average commuting costs. In turn, this decreases average housing prices and increases the above elasticity e. Finally, this reduces the appreciation rate of housing, conditional on the growth rate of exogenous demand.

In both CH and Saiz, the additive separation in housing prices follows from the linear pricing gradient. The gradient is linear in radial distance because the cost of commuting is proportional to commiting distance with the same unit costs for all households. In this proportional model the price gradient is assumed to be log-linear in radial distance. Both specifications are approximations of more complicated, more realistic models. Which approximation is more accurate? Average commuting speeds are faster farther from downtown.Also, strict convexity follows from separation and ordering of households by their costs of commuting. Finally, empirical pricing gradients are mostly decreasing and strictly convex.

Previous papers ignore the endogeneity of development on slopes. Evidence of that endogeneity appears in Table 1. There, the 95$^{\text{th}}$ percentile of slope with housing is regressed on housing prices in 1990. All values are in logarithms. If the maximum buildable slope is exogenous, that slope should be unrelated to prices. As shown in the table the coefficient of price is both positive and highly significant. Exogenous development on slopes is rejected at a high level of statistical significance.

Development on steeper slopes is progressively more costly. Gentle slopes with grades less than 10% have the lowest unit costs on site. Moderate slopes up to 20% require more grading and more expensive foundatione Utah Governor's Office of Planning and Budget (n.d.). Still steeper slopes require even more costly cut and fill and stabilization to reduce the risk of erosion and landslides Highland (2008). Local governments have rules related to risks of earthquakes and landslides Rosenberg and Papurello (2013), drainage and erosion Ohio Balanced Growth Program (2014), protection of wildlife City of Riverside (1998), and aesthetics The Marin County Community Development Agency (n.d.). In California houses are built at high cost on extreme slopes of 50% or more. Housing on steep slopes can also have higher costs off site of extending roads, sewers, and water to the property. These issues make it difficult to identify a maximum buildable slope.

Development on or near water has analogous costs. Residential development over water

Table 1: Regression of 95th percentile of slope with housing on 1990 housing prices. All values in logarithms.

	log(slope)	
Price	0.6802***	0.6643***
	(0.0962)	(0.1305)
Coastal		0.0371
		(0.2055)
Constant	−5.3070***	−5.1326***
	(1.0728)	(1.4447)
Observations	396	396
Adjusted R^2	0.1103	0.1081
Notes:	***Significant at the 1 percent level.	
	**Significant at the 5 percent level.	
	*Significant at the 10 percent level.	

or wetlands includes houseboats and housing on piers, wharfs, and landfill. All are more costly than development on dry land. Land near water can have poor drainage, poor soils, and subsurface water Building Advisor (2013). Development on that land may require compliance with coastal commissions concerned about environmental issues and public access. Housing built on that land has additional risks from floods and other hazards like liquification during earthquakes.

3 Theory

In this section the two models are presented and their equilibria are identified. In the first the maximum buildable slope is exogenous; in the second the maximum built slope is endogenized. The major assumptions are identified. For both the preliminary and enriched models the major results are also summarized.

A monocentric city is surrounded by both topography and an infinite supply of buildable land. In the initial model the fraction of buildable land at each radial distance is an exogenous power function of that radial distance. If the exponent or constant elasticity of that power function is zero, as in previous papers, then topography is independent of radial distance. If the exponent is negative, the fraction of buildable land at each radial distance is decreasing in that distance. The unit cost of developing buildable vacant land is constant inside the city and nondecreasing with radial distance beyond its outer boundary. The latter costs increase with distance if developers must pay to extend streets and utilities from the boundary to the property. Homes are priced in a spot market like consumer durables. Aggregate demand for housing is isoelastic That demand is driven by an exogenous component that increases over time at a constant rate. Housing prices decrease with radial distance at a constant rate that depends more on relative radial distance from the urban core than the relative supply of housing at those radial distances. Rural parcels are priced by perfectly competitive landowners as real options to build housing. Landowners exercise their options by selling their properties to perfectly competitive developers who then build and sell homes to the public without delay.

In the resulting equilibrium all development occurs at the outer edge of the city. Development of more remote, rural land is not optimal because the unit costs of development are nondecreasing in radial distance beyond the outer boundary and the market price of completed homes is decreasing in radial distance. At the boundary between suburban and rural land, the price of housing equals the unit price at which owners of local land optimally exercise their options to build. That constant price exceeds the constant cost of construction. The percentage premium that landowners at the boundary demand to exercise their options does not change as the boundary expands outward over time. That constant premium is increasing in the endogenous growth rate of housing prices and decreasing in landowners' constant discount rate. With a higher percentage premium the city has less housing and less sprawl.

The growth rate of housing prices is also constant in equilibrium. It equals the constant elasticity of the housing price gradient with respect to radial distance multiplied by the endogenous expansion rate of the outer boundary. This should not be surprising. Housing appreciates at each fixed radial distance inside the city because its negatively sloped, isoelastic price gradient shifts outward with the boundary. As a result, housing appreciates more rapidly in cities with steeper price gradients or more rapid sprawl. Both are greater with a more negative exponent of the fraction of buildable land with respect to radial distance. In this sense, the appreciation rate of housing is decreasing in the marginal supply of buildable land. More rapidly growing aggregate demand induces more rapid sprawl and thereby more rapid housing appreciation.

Alternatively, if the fraction of buildable land does not depend on radial distance, then the appreciation rate of housing does not depend on that fixed faction. Nor does the rate of suburban sprawl. Instead, the fixed fraction of buildable land at each radial distance affects only the level of housing prices and the area of the city. Larger fixed fractions are associated with lower prices and less sprawl. In this model where the fraction of buildable land is a power function of radial distance, both the appreciation rate of housing and the rate of suburban sprawl are independent of the coefficient of the power function.

The above result is generalized in the second, more realistic model with endogenous development on slopes. On the previously unbuildable land with steep slopes, unit construction costs are now a convex power function of relative slope. The coefficient of that power function is the previous unit cost of building at the outer boundary. In the resulting equilibrium all development occurs at the outer boundary and a second upper boundary. The upper boundary is a continuum of maximum slopes that are developed at different radial distances inside the city. At each distance inside the outer boundary, the maximum developed slope is a product of two components: the exogenous fraction of buildable land from the initial model and an endogenous residual. The residual decreases with greater radial distance from the center, more rapidly with larger premiums paid for better views or smaller construction costs on slopes, and disappears at the outer edge of the city. Thereby, steeper slopes are developed closer the urban core with the difference disappearing only at the city's outer edge. In this sense, development on slopes deviates systematically from physical measures of developable land.

Endogenous development on slopes has other effects. Most importantly, it decreases proportionally both the elasticity of the housing price gradient and the rate of growth of housing prices. Cities then have steeper price gradients and more rapid housing appreciation with higher construction costs on slopes or smaller premiums for views. Cities also have more sprawl and higher housing prices with either attribute.

More rapid housing appreciation with smaller premiums for views can help to explain a negative relationship between housing appreciation and the coefficient of the power function for buildable land. Relatively more buildable land can be associated with a smaller supply of potential lots with views relative to lots without views and thereby a larger premium for views. This induces more construction on slopes relative to the periphery, which flattens the price gradient and, in turn, reduces the appreciate rate of housing. Thereby, cities with larger coefficients of the power function can have slower housing appreciation.

3.1 Initial model

A circular city has a central business district with unit radius. All housing is located in the remaining residential band surrounding the CDB. That housing is distinguished solely by its radial distance r from the center of the city: $1 < r \leq \bar{r}$. The city is much larger than its CBD: $\bar{r} \gg 1$. Over time the outer boundary $\bar{r}$ expands with the development of new housing.

To simplify the model, all housing is always developed at a constant density, conveniently normalized at 1. Development is instantaneous once started. Once constructed housing never depreciates or otherwise obsolesces. Also, existing housing is never redeveloped at higher densities.

Beyond the outer boundary of the city, all land is rural. Each rural parcel located at any radial distance, $r \geq \bar{r}$, can be permitted for one house. To simplify the subsequent notation, rural land generates no net cash inflow. Thereby, each rural parcel is an option to develop a permitted and finished, fully serviced lot with one house. The exercise price of this option is the unit cost of building: $b = B(r)$ for $r > 1$. Both inside the city and at its outer edge, this unit cost is constant, independent of radial distance from the center: $B(r) = \beta > 0$ for $1 < r \leq \bar{r}$. Beyond the outer boundary the unit cost B is nondecreasing in the radial distance $r - \bar{r}$ between the boundary and property. With the latter assumption and a negative price gradient from the core outward, housing is built in the subsequent equilibrium only at the outer boundary of the city.

Houses can be constructed only on an exogenous fraction of all land at each radial distance. The remaining land has difficult topography: steep slopes, soft soils, or water. The fraction of buildable land $F(r)$ at each radial distance r changes at a constant rate: $F\prime/F = -\zeta > -2$. That rate can be zero, $\zeta = 0$, in which case the fraction of buildable land is an exogenous constant: $F(r) = \lambda$ with $0 < \lambda \leq 1$. These restrictions produce the power function: $F(r) = \lambda r^{-\zeta}$. This power function has two advantanges. It generalizes the constant fraction λ in previous papers. It also makes possible an explicit, stationary equilibrium. With it and subsequent assumptions, the growth rate of housing prices is constant in equilibrium.

If the elasticity $-\zeta$ is negative, the city is surrounded by smaller shares of buildable land at greater radial distances. Figure 2 shows the buildable share as a function of radial distance together with the buildable share predicted by the best-fit values of λ and ζ for selected metro areas. The estimated elasticities range from $-xx$ to $-xx$.

Under the above assumptions the existing housing stock is proportional to the buildable area inside the city. At radial distance r the city then has the marginal housing stock: $H'(r) = 2\lambda\pi r^{1-\zeta}$ for $1 < r \leq \bar{r}$. In this situation the city has the approximate total housing stock:

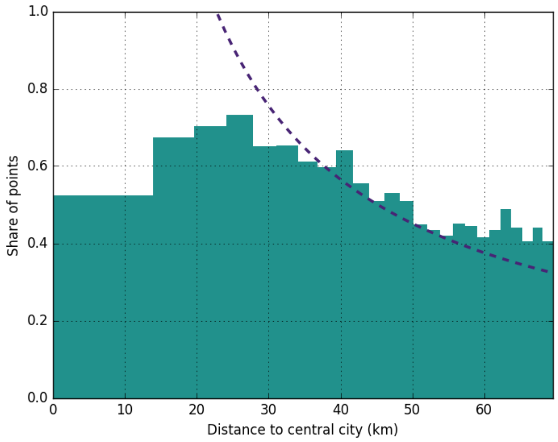
$$h = H(\bar{r}) = \int_1^{\bar{r}} H'(r)dr = \frac{2\lambda\pi}{2-\zeta}\left(\bar{r}^{2-\zeta} - 1\right) \approx \frac{2\lambda\pi}{2-\zeta}\bar{r}^{2-\zeta}, \tag{1}$$

with the outer boundary $\bar{r} \gg 1$. The housing stock (1) is an increasingly accurate approximation as $\bar{r} \to \infty$. Henceforth, the approximation is suppressed.

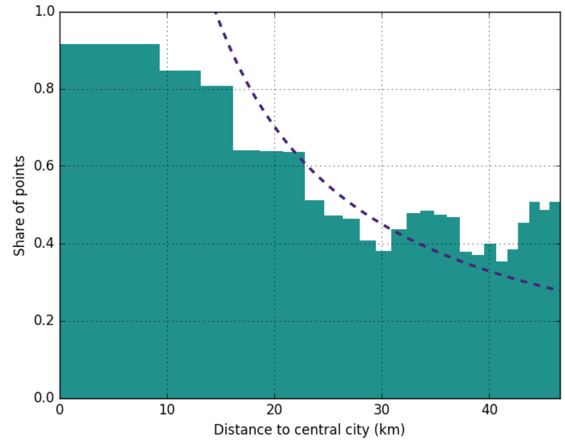
As explained and motivated in the previous section, the model is dynamic and propor-

Figure 2: Sample plots of buildable area as a function of radial distance for selected metro areas (bars) with buildable share as predicted by best-fit values of λ and ζ (dashed line).

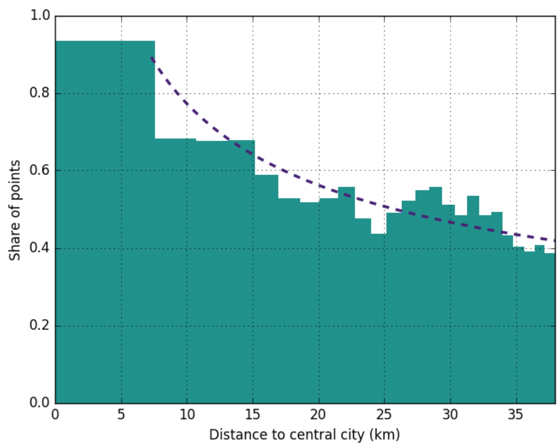
(a) Boston, MA



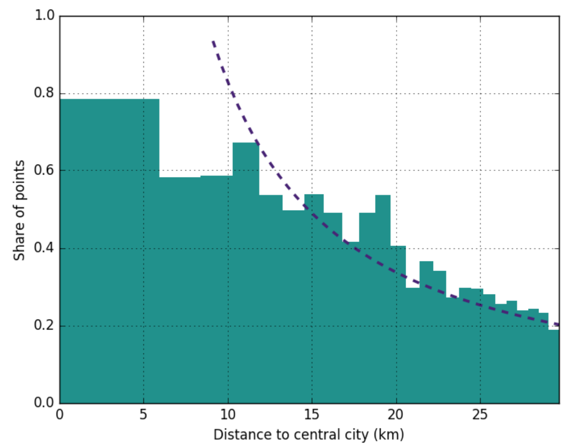
(b) Las Vegas, NV



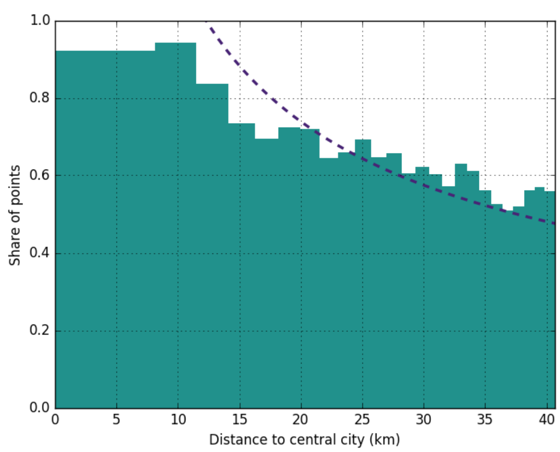
(c) Santa Fe, NM



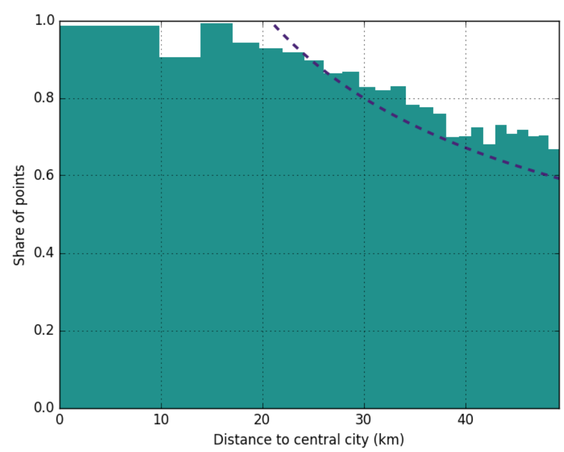
(d) Eugene, OR



(e) Port St. Lucie, FL



(f) Greenville, SC



tional with a stationary equilibrium. This requires that the unit pricing function for housing P be isoelastic everywhere. In other words, the inverse demand for housing and thereby the aggregate demand for housing must be a power function. This power function P depends on two variables. The first two are the radial distances to the property r and the outer boundary $\bar{r}$. The third variable is the exogenous component of housing demand: $x \geq 0$. That demand that grows over time at the constant rate: $\rho > 0$. This single statistic x summarizes the impact on aggregate demand of familiar variables like local employment, average wages, and nonhousing costs of living. Again, the constant rate of growth ρ preserves the proportionality of the model that makes possible the subsequent stationary equilibrium.

Without additional loss of generality, the isoelastic inverse aggregate demand for housing at any radial distance $P(r, \bar{r}, x)$ can be decomposed into two components. The first is the isoelastic inverse demand at any single radial distance, including the expanding outer edge of the city. Here, that radial distance is the expositionally convenient outer edge $\bar{r}$ with the associated price $P(\bar{r}, \bar{r}, x)$. This choice is motivated below. The second is the isoelastic pricing gradient over all remaining radial distances: $P(r.\bar{r}, x)/P(r, \bar{r}, x) = (\bar{r}/r)^{\phi}[H'(1)/H'(r)]^{\chi}$ for all $1 < r \leq \bar{r}$. The constant elasticities, $-\phi$ and $-\chi$, satisfy the inequalities: $-\infty < -\chi < -\phi < 0$. The indicated independence of the pricing gradient from the remaining variables, $\bar{r}$ and x, is a property of the power function P.

The elasticity of the price gradient with respect to radial distance $-\phi$ is easily motivated. With this negative constant the price of housing is everywhere decreasing and strictly convex in radial distance r. This convexity holds in monocentric cities with average commuting speeds that increase with radial distance. It is also consistent with heterogeneous households who are separated and ordered in radial bands by their costs of commuting between suburban homes and urban jobs. The constant elasticity can be viewed either as an analytically convenient approximation or a reduced form from a model with isoelastic household utilities.

The elasticity of the price gradient with respect to the relative supply of housing $-\chi$ is less familiar. If $\chi = 0$, this elasticity does not depend on the relative supply of housing at different radial distances. This is plausible only if households are either identical or completely mixed by their heterogeneous attributes. If, however, households are heterogenous and at least partly separated into radial bands by their heterogenous attributes, then in each radial band the price paid by residents must be greater than all bids by nonresidents. In this case, the housing price within the band can decrease in relative housing supply. As this partition becomes increasingly fine, it approaches in the limit a radial continuum of households distinguished by their heterogenous attributes where relative housing prices decrease everywhere in relative supply: $-\chi < 0$. With the restriction $-\phi < -\chi < 0$, relative radial distance affects relative housing prices more than relative housing supply. In other words, prices depend more on commuting costs than relative housing supply. This complication with the elasticity, $-\chi < 0$, is essential for the results in Proposition 2 below.

The isoelastic pricing function P is anchored above by its value at the outer boundary $\bar{r}$. Only this price $P(\bar{r}, \bar{r}, x)$ affects the aggregate demand for housing. As explained below, the pricing function P can be anchored at any single radial distance, as it is in both Capozza and Helsley (1990) and Saiz (2010). Prices at other radial distances are redundant. This simplification generates the isoelastic aggregate demand for housing: $P(\bar{r}, \bar{r}, x)^{-\eta}$with the constant price-elasticity: $-\infty < -\eta < 0$. The unitary elasticity with respect to exogenous aggregate demand x is merely a notational simplification because the variable x can be replaced by its power function.

The above aggregate demand for housing is motivated as follows. If all households are identical, they must be indifferent in equilibrium between housing at all radial distances both inside the city and at its outer edge. In this case, the housing price at any radial distance inside the city, including its expanding outer edge, can anchor the pricing function P. With heterogeneous households some entrants into the housing market may prefer to buy existing homes. If so, their sellers then buy other homes, existing or new. This creates a sequence of sellers that terminates eventually with sellers who buy new homes at the expanding outer edge. If the mix of entrants is stationary, then the pricing gradient must be stationary in equilibrium. In a proportional model the pricing function P must then be isoelastic.

In equilibrium the equality of aggregate demand and supply determines the price of housing at the outer boundary $\bar{r}$. With aggregate supply (1) and the above price gradient, the unit price has the form:

$$p = P(r, \bar{r}, x) = \left(\frac{\bar{r}}{r}\right)^{\phi+\chi-\zeta\chi} \left[\frac{x}{H(\bar{r})}\right]^{1/\eta}, \tag{2}$$

for $1 < r \leq \bar{r}$ and $x \geq 0$. This pricing function can be extended to all rural land beyond the outer boundary of the city: $\bar{r} < r < \infty$. As such it can be interpreted as the implicit price of rural housing that could be built, but is not in the subsequent equilibrium.

Before the landowner's problem can be specified, some preliminaries are necessary. All landowners exercise their options to develop housing by selling their properties to perfectly competitive, identical developers who immediately finish lots and build houses. Once started that development is instantaneous. For a landowner at radius r, the exercise price of this option to develop is the unit cost of building: $b = B(r)$. The price of the underlying asset is the unit price of a finished house and lot: $p = P(r, \bar{r}, x)$ in (2). These variables enter the developer's problem only through payout on the option, p–b, at its future exercise date. Also, the rate of change over time in the unit price (2) does not depend on exogenous demand x.

As a result, each developer always prices or values rural land $V(p,r)$ at each radial distance r conditional only on the current price of housing, p in (2). In other words, the endogenous value of land in the subsequent equilibrium always depends on aggregate demand and supply only through the price of housing in that equilibrium.

The market for residential land is perfectly competitive. Each owner of rural land at radius r takes as given both the current price of housing, p from (2), and the unit cost of building, $b = B(r)$, and solves the problem:

$$V(p,r) = \max\left\{p-b,\ e^{-\delta\Delta t}V(p+\Delta p,r)\right\}, \tag{3}$$

for $\bar{r} \leq r < \infty$ and $0 < x < \infty$. In (3) the current value of land is the maximum of two separate values: the value of immediate development, $p-b$, and the present value of deferred development. Development deferred from time t to time $t+\Delta t$ has the future value of undeveloped land when housing has the price $p+\Delta p$. This future value is discounted to the present at the constant rate δ over the interval of time Δt.

The solution to problem (3) is a stopping rule. Each landowner at radius r defers the option to develop until the price of land (2) first reaches a critical value: $p^* = P^*(r)$. This stopping price, which is identified in the subsequent equilibrium, can be interpreted as the landowner's reservation price for sales to developers. The optimal price p^* depends partly on the appreciation rate of housing. To solve this problem, each owner conjectures, correctly in the subsequent equilibrium, that the unit price p at each radial distance r always grows at the same constant rate g over each very small interval of time Δt: $\Delta p/p = g\Delta t + o(\Delta t)$. The residual $o(\Delta t)$ represents terms of smaller order than Δt. The endogenous, constant growth rate g^* is also determined in the subsequent equilibrium.

Equilibrium in the housing market is determined by two conditions. All landowners solve problem (3) by exercising their options to develop when the price of housing at their rural radial distance $P(q,r)$ reaches their reservation value $P^*(r)$. Second, the rate at which landowners exercise their options must supply sufficient land for new housing to meet the aggregate demand for new housing. Thereby, housing demand and supply must always grow at the same rate.

3.2 Initial equilibrium

The equilibrium of the initial model is identified in this section. First, the landlord's problem is rewritten as follows. Expand the right side of (3) in Δt; subtract V from both sides of

(3); divide the resulting right side by Δt; and let $\Delta t \to 0$. This produces the problem:

$$0 = \max\left\{p - b - V(p,r),\ gpV_p(p,r) - \delta V(p,r)\right\}. \tag{4}$$

In the absorbing state, $p = 0$, rural land has no present value from its alternative use:

$$V(0,r) = 0. \tag{5}$$

Finally, the optimal exercise price p^* must satisfy the smooth-pasting condition:

$$V_p(p^*,r) = 1. \tag{6}$$

Conditions (4) through (6) hold for all for $\bar{r} \le r < \infty$ and $0 < x < \infty$. The solution to (4) though (6) determines the landlord's optimal exercise price, $p^* = P^*(r)$, and resulting value of raw land V.

In the subsequent equilibrium all housing is developed at the outer boundary. Development beyond the outer boundary is precluded by the argument at the end of this section. Development at the outer boundary requires that the solution to (4) through (6) satisfies the following market clearing condition. At the outer radius $R^*(x)$, the optimal price of housing at which landowners exercise their option to develop, $P^*[R^*(x)]$ in (4) through (6), always equals the market clearing price for housing in (2): $P^*[R^*(x)] = P[R^*(x), x]$ for all $x \ge 0$. This equality determines the city's endogenous outer radius $R^*(x)$ for all $x \ge 0$ and thereby its housing stock (1). The growth rate of housing prices g^* follows in turn from the pricing function (2) and the rate at which the outer radius $R^*(x)$ expands with the growth of exogenous demand x. These properties, combined with the solution to (4) through (6), characterize of equilibrium.

Proposition 1: *The housing market characterized by (1), (2), and (4) through (6) has a unique equilibrium if $g^* < \delta$. In this case, all development occurs at the outer boundary,*

$$R^*(x) = \left(\frac{2-\zeta}{2\lambda\pi} x p^{*-\eta}\right)^{1/(2-\zeta)}, \tag{7}$$

with the associated housing supply,

$$H^*(x) = xp^{*-\eta}. \tag{8}$$

Housing has the unit price,

$$P(r,x) = p^* \left[\frac{r}{R^*(x)}\right]^{-\phi-\chi+\zeta\chi}, \tag{9}$$

for $0 < r < \infty$, *with the value at the outer boundary,*

$$p^* = \frac{\beta\delta}{\delta - g^*}. \tag{10}$$

At all fixed radial distances r housing prices grow at the constant rate:

$$g^* = \rho\frac{\phi+\chi-\zeta\chi}{2-\zeta}. \tag{11}$$

Rural land has the unit value,

$$V^*(r,x) = \frac{g^*}{\delta}P^*(r)\left[\frac{P(r,x)}{P^*(r)}\right]^{\delta/g^*}, \tag{12}$$

with the optimal exercise price,

$$P^*(r) = \frac{\delta}{\delta - g^*}B(r), \tag{13}$$

for $R^*(x) \le r < \infty$. *All results hold for* $0 \le x < \infty$.

This proposition is proved in the Appendix. Its main result is the endogenous growth rate of housing prices (11). That growth rate g^* is determined by the equality in (A.3) of the two prices, $P^*[R^*(x)]$ in (13) and $P^*[R^*(x), x]$ in (2), with the outer boundary, $R^*(x)$ in (7), and the associated housing stock, $H^*(x)$ in (8). This equality also appears in the text above the proposition. Differentiating this equality with respect to time t generates the growth rate, g^* in (11). Thereby, the constant growth rate of housing prices (11) clears the housing market continuously through time. This growth rate must be less than the discount rate δ if the landlord's problem is to have a finite solution.

The appreciation rate of housing (11) has the following properties. It is the product of two factors: the rate of sprawl, $\rho/(2-\zeta)$, from (7) and the elasticity of the housing price gradient, $\phi+\chi-\zeta\chi$ in (9). This result is not surprising. As the city sprawls, the negatively sloped price gradient to the outer edge expands outward and upward. The sprawl or expansion of the outer boundary (7) is distinct from the expansion of the housing stock (8), which is independent of the elasticity ζ of the fraction of buildable land with respect to radial distance. The same product (11) increases in the constant ζ if, as previously assumed, commuting costs affect relative housing prices more than relative housing supply at different radial distances, $\phi > \chi$. In this case, the growth rate of housing prices is greater with smaller marginal shares of buildable land farther from the urban core.

By contrast, the appreciation rate (11) is independent of the coefficient λ. This second constant is the fixed fraction of land available for development when land lost to topography is independent of radial distance: $\zeta = 0$. Larger values of the latter constant λ reduce the outer radius (7) and thereby the unit housing price (9), but alter neither the growth rate of the boundary (7), the price of housing (10) at the boundary, nor the appreciation rate of housing (11). Cities with smaller fixed fractions of buildable land have higher housing prices at fixed radial distances and larger sizes, but no other differences.

The price of housing at the outer boundary is not exogenous. Nor is it determined solely by the cost of construction. Instead, the endogenous unit price, p^* in (10), at the outer boundary (7) reflects the self-interested behavior of landowners who sell to builders at the optimal times to develop their properties. With higher growth rates of housing prices g^*, landowners defer development or, equivalently, raise their reservation prices and wait longer for higher bids from builders. The resulting higher unit price at the boundary (10) reduces the outer radius (7) and thereby the housing stock (1). By this process all factors that accelerate the appreciation rate of housing also reduce the size of the city. Other constants that affect the unit price (10) are familiar from the literature on real options.

The unit value of rural land in (12) is largely familiar from models of real options. Only its novel properties are discussed here. Housing has the unit price (9) everywhere inside the city. Again, that price can be extended everywhere outside the city, $R^*(x) \leq r < \infty$, as the price of housing that could be built, but is not in equilibrium. This extended pricing function (9) is everywhere decreasing in radial distance r. By contrast, the optimal price at which the option is optimally exercised, $P^*(r)$ in (13), is increasing in r. For both reasons, the option to develop is worth more in (12) not exercised than exercised at all rural radial distances beyond the outer boundary, $R^*(x) < r < \infty$. In other words, the option to develop is in the money only at the outer boundary of the city.

In equilibrium all development must occur at the outer boundary. To see this, suppose that landlords exercise their options to build only at the boundary of the city at all times before some time, $t > 0$, when the exogenous demand reaches the value x. In this case, the city has at time t the outer radius, $R^*(x)$ in (7). By the above argument landlords then optimally exercise at time t their options to build only at the boundary (7). At each radial distance beyond the boundary, the housing price at which they would exercise their options (13) exceeds the implicit price of housing (9) at that radial distance: $P^*(r) > P(r, x)$ for all $r > R^*(x)$ and all $x > 0$. The same argument also applies at all previous times, including the initial time 0 when development of the city starts. Therefore, development starts at the outer boundary and continues thereafter. This is the only equilibrium.

3.3 Endogenous development on slopes

In the model of Section 3.2 the fraction of land that can be developed is exogenous and independent of radial distance. In this section land is no longer characterized merely as buildable or not. Instead, it is ordered at each radial distance by its unit cost of development. The endogenous boundary for building on topography is then determined for each radial distance from the city center.

To simplify the exposition, construction is constrained only by topography. Also, topography is summarized by a single state variable s_0, conveniently called slope. Slope is continuous across all radial angles and distances from the city center. At each radius r, slope is uniformly distributed on the interval: $0 \leq s_0 \leq S_0(r)$.[2] The maximum slope S_0 can differ across both radial distances r and metropolitan areas. Each slope s_0 has the percentile or rank order: $s_1 = s_0/S_0(r)$.

In this model housing can be constructed at higher unit costs on previously unbuildable slopes. On sufficiently shallow slopes, the unit cost of construction $C(r,s)$ is unchanged: $C(r,s) = B(r)$ for all $0 \leq s_0 \leq \sigma$. On all steeper slopes, the unit costs are greater: $C(r,s) > B(r)$ for all $\sigma < s_0 \leq S_0(r)$. The constant, $\sigma > 0$, is the previous maximum buildable slope: $s_0/\sigma = s_1/\lambda r^{-\zeta}$. In other words, the slope divided by its buildable maximum equals the slope's percentile divided by the percentile of the previous buildable maximum. For example, the buildable maximum and its associated percentile have the respective values, .15 and λ, in Saiz (2010). This specification of the slope σ links the current analysis with endogenous slopes to the previous analysis with exogenous slopes and allows comparisons between the two. It is possible with the uniform distribution of slopes at each radial distance.

In this proportional model the unit costs of construction must also be isoelastic. Specifically, the higher unit costs of construction on steeper slopes must be homogeneous in slope: $B(r)\,(s_0/\sigma)^\gamma$ for $\sigma < s_0 < 1$ with the new constant, $\gamma > 1$. Because all steeper slopes have the relative values, $s_0/\sigma = s_1/\lambda r^\zeta$, this generates the isoelastic costs: $B(r)\left(s_1/\lambda r^\zeta\right)^\gamma$ for $\lambda r^\zeta < s_1 < 1$. Henceforth, slopes are identified by their percentile ranks: $s = s_1$. With this new notation, the costs of development can be summarized as follows:

$$C(r,s) = B(r)\max\left\{1, \left(s/\lambda r^{-\zeta}\right)^\gamma\right\}, \tag{14}$$

[2]Alternatively, slope can be Pareto or power law at each radial distance. In this case, the subsequent results have one additional parameter: the exponent of the power law.

for all radial distances, $1 < r < \infty$, and all feasible slopes, $0 \le s \le 1$. As in the previous section, the unit cost B has the values: $B(r) = \beta$ for $1 < r \le \bar{r}$ and $B(r) > \beta$ for $\bar{r} < r < \infty$.

The cost of construction (14) is an analytically convenient generalization of the initial model. The unit cost is the previous minimum, $C(r, s) = \beta$, on all previously buildable slopes, $0 \le s \le \lambda r^{-\zeta}$, both inside the city and at its outer boundary, $0 < r \le \bar{r}$. On all remaining, previously unbuildable, steeper slopes, $\lambda r^{-\zeta} < s \le 1$, the marginal costs are positive and increasing: $\gamma > 1$. This convexity, combined with an additional assumption below, guarantees that the city has an equilibrium with two endogenous boundaries: the previous outer boundary (7) on all smaller slopes, $0 \le s \le \lambda r^{-\zeta}$, and, inside the city, an additional upper boundary on all steeper slopes, $\lambda r^{-\zeta} < s \le 1$. The latter boundary was previously exogenous with the percentile or rank order slope $\lambda r^{-\zeta}$ equal to the fraction of buildable land at radius r. Both endogenous boundaries are identified in the subsequent solution. The isoelastic unit costs (14) make possible explicit solutions for both boundaries. With the uniform distribution the equilibria from the two models can be compared.

As before, all housing has an exogenous, unit density that does not change over time with either depreciation or redevelopment. In this case, the housing stock is again proportional to the developed area with one modification. At each radial distance, $0 < r \le \bar{r}$, the exogenous fraction of developed area $\lambda r^{-\zeta}$ is replaced by the endogenous fraction $\bar{S}(r)$, which is the maximum developed slope at radial distance r. This generates the marginal housing stock: $H'(r) = 2\pi r \bar{S}(r)$, and thereby the total housing stock:

$$h = H(r) = 2\pi \int_1^{\bar{r}} r\bar{S}(r)dr \approx 2\pi \int_0^{\bar{r}} r\bar{S}(r)dr, \tag{15}$$

with the outer boundary $\bar{r} \gg 1$. This housing stock (15) replaces the previous housing stock (1).

The premium paid for slopes, if any, is modeled simply as follows. On all previously buildable land, $0 \le s \le \lambda r^{-\zeta}$, the previous pricing function (2) with no premium for slopes again applies. On all remaining, previously unbuildable land with steeper slopes, the unit price of housing is also homogenous in relative slope s. For the latter land this produces the isoelastic prices: $p = P(r, s, x) = (\bar{r}/r)^{\phi}[H'(\bar{r})/H'(r)]^{\chi}\left(s/\lambda r^{-\zeta}\right)^{\psi} P(\bar{r}, s, x)$ for $\lambda r^{-\zeta} < s \le 1$ The new constant elasticity ψ is positive with a premium for views and negative with a discount for difficult access on slopes. Its upper bound, $\psi < \gamma$, is motivated below. Across all slopes this generates the unit price:

$$p = P(r, s, x) = \left(\frac{\bar{r}}{r}\right)^{\phi} \left[\frac{\bar{r}\bar{S}(\bar{r})}{r\bar{S}(r)}\right]^{\chi} \max\left\{1, \left(\frac{s}{\lambda r^{-\zeta}}\right)^{\psi}\right\} \left[\frac{x}{H(r)}\right]^{\eta}, \tag{16}$$

for $1 < r \leq \bar{r}$, $0 \leq s \leq 1$, and $x \geq 0$. For notational convenience, the dependence of the the new pricing function P on both boundaries, $\bar{r}$ and $\bar{S}$, is also suppressed. This new pricing function (16) replaces (2).

The housing equilibrium is derived much like before. Under the above assumptions, including (14), all housing is built at the two boundaries of the city: outer $\bar{r}$ and upper $\bar{S}$. In equilibrium these two boundaries have the respective values: $R^*(x)$ and $S^*(r,x)$. In the initial model the optimal exercise price at which development occurs depends on radial distance $P^*(r)$. Here, the same exercise price also depends on slope $P^*(r,s)$. Much like the initial model, the optimal price at which development occurs at the outer boundary must equal in equilibrium the market-clearing price (16) at the outer boundary: $P^*[R^*(x), s] = P[R^*(x), s, x]$ for all $0 \leq s \leq \lambda R^*(x)^{-\zeta}$. Similarly, the optimal price at which development occurs at the upper boundary must equal in equilibrium the market-clearing price (16) at the upper boundary: $P^*[r, S^*(r,x)] = P[r, S^*(r,x), x]$ for all $1 < r \leq R^*(x)$. The first equality determines the outer boundary $R^*(x)$, while the second determines the upper boundary $S^*(r,x)$.

The second proposition is presented much like the first. It uses the new notation: $\nu \equiv (\phi+\chi-\zeta\chi)/(\gamma+\chi-\psi) > 0$ and $\xi \equiv (\gamma-\psi)/(\gamma+\chi-\psi) > 0$. Again, all calculations appear in the Appendix.

Proposition 2: *The housing equilibrium characterized by (4) through (6), (15), and (16) has a unique solution if $g^* < \delta$. In this second case, all development occurs at the two boundaries:*

$$R^*(x) = \left(\frac{2-\zeta-\nu}{2\lambda\pi p^{*\eta}}x\right)^{1/(2-\zeta)}, \tag{17}$$

and

$$S^*(r,x) = \lambda r^{-\zeta}\left[\frac{R^*(x)}{r}\right]^{\nu}, \tag{18}$$

for $1 < r \leq R^*(x)$. Housing has the aggregate supply (1) and the unit price,

$$P(r,s,x) = p^*\left[\frac{R^*(x)}{r}\right]^{\xi(\phi+\chi-\zeta\chi)} \max\left\{1, \left(\frac{s}{\lambda r^{-\zeta}}\right)^{\psi}\right\}, \tag{19}$$

with the value, p^* in *(10), at the outer boundary and the growth rate,*

$$g^* = \xi\rho\frac{\phi+\chi-\zeta}{2-\zeta}. \tag{20}$$

rural land has the unit value,

$$V^{*}(r,s,x) = \frac{g^{*}}{\delta}P^{*}(r,s)\left[\frac{P(r,s,x)}{P^{*}(r,s)}\right]^{\delta/g^{*}}, \tag{21}$$

with the optimal exercise price,

$$P^{*}(r,s) = \frac{\delta}{\delta - g^{*}}C(r,s), \tag{22}$$

for $R^{*}(x) \leq r < \infty$ *and* $0 \leq s \leq 1$. *All results hold for* $0 < x < \infty$.

Endogenous development on slopes reduces proportionally both the elasticity of the price gradient in (19) and the growth rate of housing prices (20). Each value with endogenous development, $\xi(\phi + \chi - \zeta\chi)$ in (19) and g^{*} in (20), equals the respective value with exogenous development, (9) and (11), multiplied by the constant, $0 < \xi < 1$. This composite constant ξ is increasing in the difference, $\gamma - \psi$, between the elasticity of construction costs γ and the elasticity of the pricing premium ψ, both for housing on steep slopes, $\lambda r^{-\zeta} < s \leq 1$. This result holds because the relative supply of built land at each radial distance r inside the city decreases in the same difference, $\gamma - \psi$, under the previous assumptions. In other words, cities with with higher marginal construction costs on slopes or smaller marginal premiums paid for buildable lots on slopes have, other things equal, less endogenous development on slopes and more rapid appreciation of housing.

Endogenous development on slopes has other properties. The maximum developed slope (18) is the product of twocon components: the exogenous factor from the previous proposition $\lambda r^{-\zeta}$ and the endogenous residual, $S^{*}(r,x)/\lambda r^{-\zeta} > 1$. The residual decreases toward 1 as the cost elasticity γ increases without limit, $\gamma \to \infty$. It also decreases in radial distance r. Here, steeper slopes are developed at higher marginal costs closer to the center because unit housing prices are higher closer to the center. At the outer edge of the city (17), only smaller slopes are developed: $S^{*}[R^{*}(x), x] = \lambda R^{*}(x)^{-\zeta}$ with $\psi > \chi$. The last result holds because owners of land at the outer boundary prefer not to exercise their more costly options to develop steeper slopes when the unit price at the outer boundary p_1^{*} makes the same owners indifferent at the margin between exercising or not their option on shallow slopes. Not surprisingly, the steepest developed slope also increases with exogenous demand x. Finally, endogenous development on steeper slopes, $\lambda r^{-\zeta} < s \leq S^{*}(r,x)$, inside the city, $0 < r < R^{*}(x)$, reduces the outer boundary (17) below its previous value (7). Thereby, suburban sprawl is less with endogenous development on slopes, while the supply of housing (8) is unchanged.

In other aspects the growth rate of housing prices is unchanged by endogenous development on slopes. Much like (11), the growth rate (20) is greater with larger elasticities ζ and thereby less relatively flat land, $0 \leq s \leq \lambda r^{-\zeta}$, at greater radial distances r. It is also greater with more negative price gradients in (19), which, in turn, are more negative with larger elasticities ζ. If $\zeta = 0$, the parameter λ is the constant fraction of land with shallow slopes. As before, this parameter affects neither the price gradient in (19) nor the growth rate of housing prices, g^* in (20).

Although the parameter λ has no direct effect on the appreciation rate of housing, it can have an indirect effect. Suppose that cities with relatively more flat, buildable land have relatively fewer potential lots with views and thereby larger premiums for those views. In this case, cities with larger coefficients λ have larger parameters ψ, relatively more built housing on slopes everywhere inside the city, a more elastic negative price gradient for housing, and thereby less rapid appreciation of housing. Thereby, a negative relationship between λ and housing appreciation is consistent with endogenous development on slopes.

Steep slopes and water have similar effects on supply and demand. The costs of development are increasing and convex costs both on steeper slopes and closer to water. Also, views from and of slopes are an amenity much like views of water and easy access to shorelines. With either amenity the unit price of land is higher. With endogenous development either on steep slopes or near water, the parameter λ should then have a negative indirect effect on housing appreciation through the negative relationship between λ and the availability of land on or near slopes or shorelines.

In practice, the density of development can also be controlled at boundaries on steep slopes or near water. If the cost of development is increasing and convex in the quality or quantity of housing, then these measures of density are greater near either amenity, more so with higher prices of those amenities. The latter result follows from a minor modification of the second model This additional density increases the supply of housing inside the city, which, by the above argument, further reduces the rate of housing appreciation. Therefore, the parameter λ should have a more negative, indirect effect on housing appreciation.

4 Empirics

4.1 Estimating parameters

Whether the maximum built slope is exogenous or endogenous the appreciation rate of housing depends on the marginal supply of buildable land. Tests of the model predictions require a tractable, consistent estimator of the two parameters ζ and λ of the power function $\lambda r^{-\zeta}$ for the fraction of buildable land at all radial distances r in each metro area. Because the empirical distribution of buildable land can match the power distribution more closely in some cities than in others, the tests include a measure of the "goodness of fit" of the power law.

The estimator is calculated by minimizing the asymptotic mean integrated squared error (AMISE) of the power distribution relative to the empirically observed distribution of buildable land in each metropolitan area[3]. The latter distribution is denoted by D_m for metro area m. For each metro area m, the estimator $\left(\hat{\zeta}_m, \hat{\lambda}_m\right)$ minimizes the AMISE:

$$\left(\hat{\zeta}_m, \hat{\lambda}_m\right) = \arg\min \left\{ \frac{1}{\pi\left(\bar{r}^2 - 1\right)} \int_0^{2\pi} \int_1^{\bar{r}} \left[D_m(r) - \lambda r^{-\zeta}\right]^2 r \,\mathrm{d}r \,\mathrm{d}\theta \right\}. \tag{23}$$

The minimand in (23) is the squared distance between the predicted and actual buildable share, averaged over the annulus $1 \leq r \leq \bar{r}$. The first-order conditions for (23) generate the two equations, (A.5) and (A.6) in the Appendix. In the subsequent exposition the subscripts m are deleted.

In the data each distribution D_m is sampled. Suppress the subscript m and identify each observation by its number: $n = 1, ..., N$. Each observation n is a pixel at some radius, $1 \leq r_n \leq \bar{r}$, that is either buildable, $d_n = 1$, or not, $d_n = 0$. With this notation the finite-sample analogues of (A.5) and A.6) are

$$\frac{\frac{1}{N}\sum_{n=1}^{N} d_n \log(r_n) r_n^{-\hat{\zeta}}}{\frac{1}{N}\sum_{n=1}^{N} d_n r_n^{-\hat{\zeta}}} = \frac{1}{2\left(1-\hat{\zeta}\right)} \frac{\bar{r}^{2\left(1-\hat{\zeta}\right)}\left[2\left(1-\hat{\zeta}\right)\log(\bar{r}) - 1\right] + 1}{\bar{r}^{2\left(1-\hat{\zeta}\right)} - 1} \tag{24}$$

and.

$$\hat{\lambda} = \frac{\frac{1}{N}\sum_{n=1}^{N} d_n r_n^{-\hat{\zeta}}}{\frac{1}{N}\sum_{n=1}^{N} r_n^{-2\hat{\zeta}}}. \tag{25}$$

The estimate $\hat{\zeta}$ is the numerical solution to (24). The estimate $\hat{\lambda}$ then follows from (25).

This completes the estimation. The procedure produces an estimate of the mean integrated squared error:

$$\widehat{MISE} = \frac{1}{N}\sum_{n=1}^{N}\left[d_n - \hat{\lambda} r_n^{-\hat{\zeta}}\right]^2 \tag{26}$$

In the regressions of price growth on λ and ζ below, this estimated MISE is used as the weight in weighted least-squared regressions.

Observations are obtained as follows. As in Saiz (2010), a pixel is unbuildable if it

[3] The mean integrated error provides a global criterion of fit closeness Hart (2013). Scott (2001) investigates its application in parametric estimation.

is covered by water or its slope is greater than 15%; otherwise, it is buildable. Water is identified from the USGS National Hydography Dataset and rasterized using QGIS. Slopes are identified from elevations in the SRTM data of NASA and converted to slopes using the GDAL Digital Elevation Model utility. Additional details appear in the Appendix.

For each metropolitan area, an urban extent is identified as described in the Appendix. As described in the Appendix, a large sample of coordinate pairs is randomly generated on a disk centered on the principal city of the metropolitan area. The radius of this disk corresponds to the spatial scale of the urban extent. The land at each coordinate pair is either buildable or not by the above criteria. The estimates, $\hat{\zeta}$ and $\hat{\lambda}$, are then calculated from (24) and (25) with the associated mean integrated squared error (26). Small changes to these definitions do not appreciably alter the estimated values of ζ and λ.

4.2 Housing appreciation and land supply

In the initial model with exogenous development on slopes, housing appreciation depends on the marginal availability of land ζ but not the normalization factor λ. With either endogenous development on slopes or endogenous density at boundaries, both parameters affect housing appreciation. To test these theoretical prediction, housing appreciation rates are first regressed on the estimates of ζ and λ, a measure of growing labor demand, and additional covariates. All regressions are weighted by the inverse of the mean estimated squared error (26). In the panel regressions all standard errors are clustered at the level of the metropolitan area.

Two measures of housing appreciation are used below. To assess the medium-term relationship between appreciation and available land, appreciation is computed over each of three periods: 1980–1990, 1990–2000, and 2000–2010. This is the above panel specification. For a longer-run relationship, appreciation is measured over the single period: 1987 through 2014[4]. In both cases housing prices come from the FHFA index for the metropolitan area. The body of the text presents the decadal results and the long period results are in the Appendix. Throughout, coefficients are similar in magnitude and significance across the two specifications.

Growth in labor demand is estimated by the statistic Bartik[5]. This statistic, which was popularized in the economic literature by Bartik (1991) and Blanchard et al. (1992), is now

[4]The starting date 1987 maximizes the product of the number of metropolitan areas with available data multiplied by the length of the time period.

[5]For city m Bartik is the sum over all industries k of the share of workers in industry k within city m multiplied by the employment growth in industry k over all cities other than m. Let σ_{mk} the share of workers in city m employed in industry k and let y_{mk} be employment growth in city m in industry k. Then, Bartik is defined by $\sum_{-m} \sum_k sigmamky_{-mk}$.

widely used in urban economics. It is a proxy for local growth in labor demand under the assumption that employment in each industry is driven by factors outside the metropolitan area. All Bartik shocks are calculated over the appropriate time periods using three-digit industry codes from IPUMS.

In Table 1 the indicator variable Coastal identifies a metropolitan area in coastal California or the Northeast Corridor from Washington, DC to Boston. These areas have consistently experienced the most price growth among all metropolitan areas in the United States. Slope is the median slope of all pixels in the metropolitan area. Gradient is the elasticity of the housing price gradient with respect to radial distance from the city center. Immigrant is the percentage of immigrants in the population. Degree is the percentage of the population with post-secondary degrees. These demographic variables are calculated from the beginning of each decade in the decadal regressions.

Table 2: Summary statistics for regression variables.

Statistic	N	Mean	St. Dev.	Min	Max
Appreciation	871	0.486	0.305	0.001	2.654
Zeta	871	0.759	0.349	−1.206	1.987
Lambda	871	1.217	0.408	0.036	2.039
Bartik	871	24.767	24.868	−25.016	121.747
Coastal	871	0.096	0.295	0	1
Slope	869	3.955	4.290	0.688	30.076
Gradient	757	−0.025	0.197	−0.765	0.711
Immigrant	871	0.066	0.066	0.007	0.405
Degree	871	0.118	0.051	0.024	0.383

We estimate λ and ζ using three different sets of boundaries for the extent of the metropolitan area. In our baseline specification, we calculate the radial distance from the metro area centroid to each pixel in the extent of the built-up area. Then, we define a disc from the 25th percentile to the 99th percentile of radial distance to the metro area centroid as our inner and outer radii and estimate λ and ζ over that disc. However, because urban extents are endogenously determined, it is possible that using actual extents might bias the parameter estimates. Accordingly, we also estimate λ and ζ for a specification where the 1970 population of each metro area is used to predict these baseline boundaries. These are the boundaries that would have been realized had 1970 population grown at the national average rate and been dispersed as in a standard monocentric model. Finally, we consider a fully exogenous specification where the inner and outer radii are fixed at 1 km and 50 km for all metro areas irrespective of their actual spatial extent.

Table 3 confirms, not surprisingly, that our estimators of λ and ζ that use data concerning metropolitan area size (the baseline, first column) or historical population (second column) perform better in fitting data than using exogenous boundaries. The measure of goodness of fit (low AMISE) falls by almost 40% going from the third to the first column. We find that the boundaries over which λ and particularly ζ are estimated can substantially modify our estimates. The estimates of ζ are significantly, but highly imperfectly, correlated across specifications.

Table 3

	Baseline boundaries	Predicted boundaries	Constant boundaries
Correlation of λ with constant boundaries λ	0.642	0.779	
Correlation of λ with predicted boundaries λ	0.791		
Correlation of ζ with constant boundaries ζ	0.367	0.584	
Correlation of ζ with predicted boundaries ζ	0.467		
Mean AMISE	0.163	0.190	0.266
Median AMISE	0.166	0.201	0.275

Table 4 presents our main results from regressing 1980-2014 price growth on our estimates of static constraint λ, dynamic constraint ζ, and Gradient ϕ. Consistent with model predictions, dynamic land constraint ζ and the Gradient are positively associated with price growth in all specifications. The coefficient on the static land availability measure λ is always significantly negatively associated with price growth. This is consistent with past empirical work and not inconsistent with the enriched model with endogenous building on slopes, but inconsistent with the baseline model where development occurs only at the urban fringe.

Additional demand controls are introduced between specifications (1) and (2) of Table 4 and do not reduce the estimated relationship between the dynamic constraint ζ or the price gradient ϕ and long-run price growth. In the case of both λ and ζ, modifying the boundaries over which these land constraint measures are estimated has ambiguous effects on their relationship with price growth. The baseline model, estimated between the 25th and 99th percentiles of built-up extent yields coefficients that are roughly double those estimatd based on predicted boundaries with 1970 population, but half those estimated using the same distance from downtown across all metropolitan areas.

4.3 Average and marginal land availability

The preceding regressions establish a relationship between price appreciation and land availability, as measured by the parameters λ and ζ. The interpretation of these results is complicated by the correlation between the average share of land lost to oceans and mountains and a host of demand-side factors Davidoff (2014). Land lost at the margin might be less correlated with natural amenities and thereby other demand-side factors. This possibility is explored below and two features of the analysis stand out.

Table 4: Regressions of FHFA price growth 1980-2014 on land availability parameters λ and ζ and land price gradient ϕ and controls. Observations are inverse-AMISE-weighted metropolitan areas.

	(1)	(2)	(3)	(4)
Bartik (baseline boundaries)	0.0026***	0.0011	0.0007	0.0014*
	(0.0005)	(0.0007)	(0.0007)	(0.0007)
Lambda (baseline boundaries)	−0.1432**	−0.1545**		
	(0.0661)	(0.0653)		
Zeta	0.1635**	0.1829**		
	(0.0731)	(0.0724)		
Lambda (predicted boundaries)			−0.0806**	
			(0.0393)	
Zeta (predicted boundaries)			0.0922***	
			(0.0264)	
Lambda (constant boundaries)				−0.0941***
				(0.0282)
Zeta (constant boundaries)				0.3602***
				(0.0947)
Gradient	0.2168***	0.1414*	0.1230	0.2232***
	(0.0795)	(0.0821)	(0.0851)	(0.0744)
Coastal		1.3091**	2.1061***	0.8605
		(0.5650)	(0.4512)	(0.5853)
Immigrant		3.0569*	3.4002**	3.0745*
		(1.5528)	(1.5797)	(1.6447)
Degree	0.0119	0.0095	0.0166***	0.0140***
	(0.0082)	(0.0082)	(0.0063)	(0.0054)
Slope	0.4057***	0.3872***	0.5192***	0.3622***
	(0.0937)	(0.0926)	(0.0822)	(0.0923)
Constant	−0.3317***	−0.3036***	−0.3033***	−0.3467***
	(0.0809)	(0.0815)	(0.0766)	(0.0762)
Observations	271	271	271	271
Adjusted R^2	0.3451	0.3632	0.4350	0.3915

Notes: ***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level.

First, the estimated values of the two parameters, λ and ζ, and the monocentric gradient are regressed on 1980 values of local attributes correlated with demand growth. Table 5 contains the resulting coefficients. The monocentric gradient and the parameter related to average builable share λ have a significant relationship with the Coastal indicator and the 1980 share of the population with degrees. By contrast, the marginal-share parameter ζ has no statistically significant relationship with these variables.

Table 5: Regression of estimated λ, ζ, and monocentric gradient on attributes of metropolitan areas.

	λ		ζ		Gradient	
Immigrant	-0.7625^{***}	-0.4181^{*}	-0.2331	-0.2237	0.1753	0.2795^{**}
	(0.2136)	(0.2158)	(0.1903)	(0.1967)	(0.1131)	(0.1160)
Degree	-1.9946^{***}	-1.4032^{***}	0.1580	0.1742	0.1473	0.3211^{*}
	(0.3185)	(0.3250)	(0.2838)	(0.2963)	(0.1664)	(0.1721)
Bartik	-0.0032^{***}	-0.0018^{***}	0.0009	0.0009	0.0002	0.0007^{*}
	(0.0006)	(0.0006)	(0.0006)	(0.0006)	(0.0003)	(0.0003)
Coastal		-0.3187^{***}		-0.0087		-0.0920^{***}
		(0.0500)		(0.0456)		(0.0257)
Constant	1.5817^{***}	1.4850^{***}	0.7346^{***}	0.7320^{***}	-0.0615^{**}	-0.0902^{***}
	(0.0464)	(0.0479)	(0.0414)	(0.0436)	(0.0249)	(0.0259)
Observations	871	871	871	871	757	757
Adjusted R^2	0.0829	0.1229	0.0009	-0.0002	0.0025	0.0179

Notes: ***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level.

Second, the estimated marginal increases of unbuildable land ζ have a substantially different ordering than the average unbuildable shares in Saiz (2010). Table 6 displays for metropolitan areas with the highest and lowest unbuildable shares in Saiz (2010) their rank in the distribution of estimated marginal increase in unbuildable land ζ.[6] The average and marginal ranks are substantially different. For example, Miami-Miami Beach-Kendall, FL

[6]Because metropolitan areas are defined differently in the two studies, the areas are matched by their principal cities. If a metropolitan area with the same principal city is absent from either ranking, it is excluded from the following tables.

has a high average share of unbuildable land and a low marginal rate of increase in unbuildable land. Its buildable land is tightly constrained on average and weakly constrained at the margin. Table **??** displays the metropolitan areas with the highest and lowest estimated ζ with their rank in the distribution of unbuildable land in Saiz (2010). Both lists contain some cities with ample amenities and persistently high productivity growth (as documented in Davidoff (2014)) as well as other cities without these characteristics. While there is an obvious negative correlation between amenity value and λ, the relationship with amenities is not clear in the case of ζ.

Table 6: Rank of values of ζ for the ten metropolitan areas with the highest and lowest average share of unbuildable land in Saiz (2010).

Highest average unbuildable land		Lowest average unbuildable land	
Metropolitan area	Rank	Metropolitan area	Rank
Miami-Miami Beach-Kendall, FL	374	Wichita, KS	294
Los Angeles-Long Beach-Glendale, CA	2	Fort Wayne, IN	118
Fort Lauderdale-Pompano Beach-Deerfield Beach, FL	210	Indianapolis-Carmel-Anderson, IN	223
San Francisco-Redwood City-South San Francisco, CA	382	Dayton, OH	275
San Diego-Carlsbad, CA	56	McAllen-Edinburg-Mission, TX	248
Oakland-Hayward-Berkeley, CA	88	Omaha-Council Bluffs, NE-IA	192
Salt Lake City, UT	7	Tulsa, OK	73
Oxnard-Thousand Oaks-Ventura, CA	384	Oklahoma City, OK	186
New York-Jersey City-White Plains, NY-NJ	31	Kansas City, MO-KS	61
San Jose-Sunnyvale-Santa Clara, CA	49	Greensboro-High Point, NC	127

4.4 Slopes versus water

Steep slopes are not distinguished from water in the previous regressions of housing appreciation on buildable land. In fact, steep slopes and water are concentrated in different metropolitan areas: Figure 3. Many metro areas have extremely low shares of pixels with either steep slopes or water, but not both. To quantify this, all metro areas are ranked by their ratios of pixels with steep slopes relative to pixels with water. Cities in the bottom quartile have at most 0.6% of their pixels covered by steep slopes, while cities in the top quartile have no more than 6.7% of their pixels covered by water. The correlation between pixels covered by steep slopes and water is 0.45 for the low-slope quartile and 0.21 for the low-water quartile. For all metro areas the corresponding correlation is -0.11. All correlations are calculated over the radial distances indicated in the previous subsection. All are statistically significant at $p < 0.05$.

For each of the two quartiles, low-slope and low-water, the paired parameters, λ and ζ, are estimated as before. Next, housing appreciation for metro areas in each is regressed on the corresponding estimates and the previous covariates other than Slope. Median slope is excluded from these two small subsamples that are the extreme quartiles of metro areas ranked by relative slope. Specifics appear in the Appendix. The results are presented in Table 7 for the low-water quartile and 8 for the low-slope quartile. With one exception, the estimated coefficients of both variables, $\hat{\lambda}$ and $\hat{\zeta}$, are statistically significant with the correct signs.

When combined with Table 1, these results are consistent with endogenous development both on slopes and near water — possibly more so near water. These results are also consistent with a relationship bewteen the coefficient λ and drivers of demand in both quartiles.

5 Conclusion

In the initial version of our model the fraction of buildable land at each radial distance from the center of a monocentric city is a power function of that distance. In its extension steeper slopes can be developed at higher unit costs. In both versions vacant land is valued as a real option to develop housing. Each equilibrium includes a negative relationship between the appreciation rate of housing and the marginal supply of buildable land and a positive relationship between price growth and the elasticity of land value with respect to distance from downtown. That marginal supply is measured by the elasticity of the buildable share of land with respect to radial distance. The corresponding relationship between appreciation and the parameter affecting average supply is zero in the basic model and negative in its

Table 7: Regression results for long-period price growth on λ and ζ for the low-water subsample of metro areas.

	Price growth (decade)		
Bartik	0.0017	0.0011	0.0015
	(0.0014)	(0.0014)	(0.0021)
Lambda	-0.1314^{**}	-0.0978	0.1148
	(0.0627)	(0.0621)	(0.1783)
Zeta	0.3756^{**}	0.3042^{**}	0.1262
	(0.1417)	(0.1401)	(0.2128)
Coastal		0.6024^{**}	0.6270^{**}
		(0.2590)	(0.3090)
Slope			0.0256
			(0.0213)
Gradient			0.2435
			(0.2824)
Immigrant			-3.1557
			(2.0853)
Degree			6.8524
			(5.2392)
Constant	-0.1981	-0.0710	-0.3382
	(0.1891)	(0.1904)	(0.2227)
Observations	63	63	55
Adjusted R^2	0.1338	0.1940	0.2501

Notes: ***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level.

Table 8: Regression results for long-period price growth on λ and ζ for the low-slope subsample of metro areas.

	Price growth (decade)		
Bartik	0.0005	0.0004	0.0003
	(0.0008)	(0.0008)	(0.0010)
Lambda	-0.1537^{*}	-0.1520^{*}	-0.2571^{***}
	(0.0856)	(0.0865)	(0.0884)
Zeta	0.1537	0.1528	0.2376^{**}
	(0.1272)	(0.1281)	(0.1151)
Coastal		0.0621	0.1973
		(0.2720)	(0.2568)
Slope			−0.0838
			(0.0522)
Gradient			0.1402
			(0.1541)
Immigrant			0.6560
			(0.7475)
Degree			−3.3985
			(4.0012)
Constant	−0.0314	−0.0311	0.1712
	(0.1358)	(0.1366)	(0.1511)
Observations	81	81	74
Adjusted R^2	0.0116	−0.0007	0.1089

Notes: ***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level.

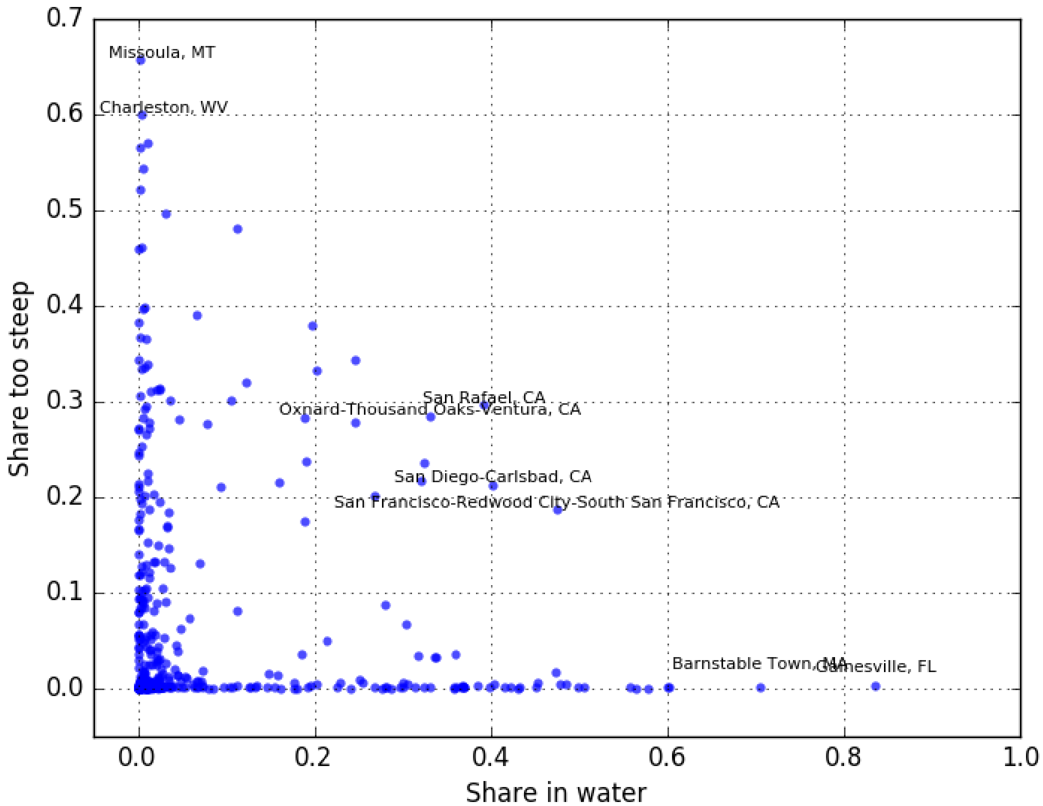


Figure 3: Share of cities covered by water and slope pixels. Metro areas with extreme values are noted.

extension. Estimators of these parameters are then derived to test the predicted relationships on a panel of US metropolitan areas.

The empirical results are consistent with the theoretical predictions. Most importantly, the relationship between housing appreciation and marginal supply is negative, significant, and robust across a range of empirical specifications over different time periods. The estimated parameter values for marginal supply appear to be largely uncorrelated with cross-sectional differences in demand. The relationship between appreciation and average supply is also negative. The latter result is inconsistent with the basic model, but consistent with its extensions to endogenous development both on slopes and near water. The marginal supply is also uncorrelated with several factors that influence demand growth: historical immigrant share of population, historical education levels, and location on the Pacific or Northeastern coasts.

We also demonstrate that the urban land rent gradient is associated both theoretically and empirically with greater price appreciation. The rental and land availability gradients play similar roles on the demand and supply side. There is more scope for price growth where the relative availability and desirability of land is greater close to the urban center.

Appendix

Proof of Proposition 1

On any interval in which the option to develop is not exercised, the first-order differential equation in (4) has the general solution:

$$V(p, r) = ap^{\delta/g}, \tag{A.1}$$

with the undetermined constant a for all $0 \leq p < \infty$. With $g > 0$, this solution satisfies the lower-boundary condition (5) for all constants, $a \geq 0$. The maximand in (4) requires that the valuation function V be continuous at the optimal exercise price p^*:

$$V(p^*, r) = p^* - b. \tag{A.2}$$

With $g < \delta$, the smooth-pasting condition (6) and (A.2) yield the unit value of rural land (12) and the optimal exercise price (13) with its value (10) at the outer boundary. The housing supply (9) follows from the unit price (2) and the equilibrium condition:

$$p^* = P^*[R^*(x)] = P[R^*(x), x] = \left[\frac{x}{H^*(x)}\right]^{1/\eta}. \tag{A.3}$$

Together, (1) and (9) generate the outer boundary (7), while (2) and (9) produce the unit price (9). Differentiate with respect to time t the price (9) with the outer boundary (7). This gives the growth rate (11).

Proof of Proposition 2

The general solution is again (A.1). The continuity condition is (A.2) with the cost, $b = B(r)$, replaced by the cost, $c = C(r, s)$ in (14). This and the smooth-pasting condition (6) yield the unit value of rural land (21) and the optimal exercise price (22) with its value (10) at the outer boundary. At the outer boundary, $r = R^*(x)$, the optimal exercise prices (22) and (13) are equal: $P^*[R^*(x), s] = P^*[R^*(x)]$, on all smaller slopes, $0 \leq s \leq \lambda r^{-\zeta}$. This follows from the unit costs (14). At the outer boundary on the same smaller slopes, the pricing functions (16) and (2) are also equal: $P[R^*(x), s, x] = P[R^*(x), x]$. As a result, the housing supply (8) again follows from (A.3). With this housing supply, the upper boundary, $S^*(r, x)$ in (18), follows from the optimal exercise price (22) with the unit cost (14) and the unit price (16) with (10):

$$
\begin{aligned}
p^* \left[\frac{S^*(r,x)}{\lambda r^{-\zeta}}\right]^\gamma &= P^*[r, S^*(r,x)] \\
&= P[r, S^*(r,x), x] \\
&= p^* \left[\frac{R^*(x)}{r}\right]^\phi \left[\frac{\lambda R^*(x)^{1-\zeta}}{r S^*(r,x)}\right]^\chi \left[\frac{S^*(r,x)}{\lambda r^{-\zeta}}\right]^\psi .
\end{aligned}
\tag{A.4}
$$

Together, (10) and (15) with (18) generate the outer boundary (17). Finally, (18) and (A.4) generate the unit price (19) with its attributes (10) and (20).

Derivation of estimators in (24) and (25)

Assume the empirical distribution of buildable land D_m is absolutely continuous. In this case, differentiate (23) with respect to both parameters, ζ and λ, and rearrange terms. This yields two equations:

$$
\frac{\frac{1}{\pi(\bar{r}^2-1^2)}\int_0^{2\pi}\int_1^{\bar{r}} D_m(r)\log(r) r^{-\hat{\zeta}} r \,\mathrm{d}r\,\mathrm{d}\theta}{\frac{1}{\pi(\bar{r}^2-1^2)}\int_0^{2\pi}\int_1^{\bar{r}} D_m(r) r^{-\hat{\zeta}} r \,\mathrm{d}r\,\mathrm{d}\theta} = \frac{1}{2\left(1-\hat{\zeta}\right)} \frac{\bar{r}^{2\left(1-\hat{\zeta}\right)}\left[2\left(1-\hat{\zeta}\right)\log(\bar{r})-1\right]+1}{\bar{r}^{2(1-\bar{r})}-1}
\tag{A.5}
$$

and

$$
\hat{\lambda} = \frac{\frac{1}{\pi(\bar{r}^2-1^2)}\int_0^{2\pi}\int_1^{\bar{r}} D_m(r) r^{-\hat{\zeta}} r \,\mathrm{d}r\,\mathrm{d}\theta}{\frac{1}{\pi(\bar{r}^2-1^2)}\int_0^{2\pi}\int_1^{\bar{r}} r^{-2\hat{\zeta}} r \,\mathrm{d}r\,\mathrm{d}\theta}.
\tag{A.6}
$$

These two equations have the discrete analogues (24) and (25), respectively

Estimation procedure

The built-up extent of metropolitan areas may not match closely the boundaries of their constituent counties: Rozenfeld and Rybski (2011) and others. For this reason, urban extents are identified from the Global Rural-Urban Mapping Project (GRUMP) data set **GRUMP11** as described in **Balk**. Specifically, the center of the metro area is specified as the centroid of its principal city, as defined by the US Census Gazetteer. From this initial location, pixels are iteratively assigned to the urban extents of the metropolitan area. Any pixel is assigned to the urban extent of the metropolitan area if three conditions are satisfied. (1) GRUMP classifies the pixel as urban. (2) The pixel is within the boundaries of the metropolitan area according to shapefiles provided by the US Census. (3) The pixel shares at least one face with a pixel already assigned to the urban extent of the metropolitan area. This procedure allows for the designation of an urban extent for every metropolitan area with the exception of The Villages, FL; this metropolitan area is almost entirely newer than the GRUMP data set and accordingly no urban extents can sensibly be defined.

For each metropolitan area, ten thousand coordinate pairs are randomly generated over a disk with a radius of the ninety-fifth percentile of the distance from pixels in the urban extent. Then, λ and ζ are estimated as described in the text by considering the pixels in an annulus between an inner radius (normalized to unity) of the twenty-fifth percentile of the distance from the generated coordinate pairs to the centre of the principal city and an outer radius of the ninety-ninth percentile of the distance from the generated coordinate pairs to the centre of the principal city. This annulus comprises the salient radii for potential construction of new housing on the periphery of the built-up area.

In Tables 11 and 12, the regressions are run on two small subsamples: the extreme quartiles of all metro areas ranked by pixels with steep slopes relative to pixels with water. In these regressions the independent variable, Slope, is excluded. Not surprisingly, the median slope has little measurable effect in the low-slope quartile. With and without Slope, the estimated coefficients of the principal variables, $\hat{\lambda}$ and $\hat{\zeta}$ are statistically significant and stable. In the low-water quartile, the median slope appears to be colinear with the two principal variables. The estimated coefficients of $\hat{\lambda}$ and $\hat{\zeta}$ are insignificant with Slope and unstable–the wrong sign for $\hat{\lambda}$ and much closer to zero for $\hat{\zeta}$. The variance inflation factor (VIF) is 3.27 for the low-water quartile, 1.99 for the whole sample, and 1.08 for the low-slope quartile.

Estimated parameter values

Table 9 shows the estimated values for the parameter ζ for each metropolitan statistical area and metropolitan division.

Long-period regression results

Table 10 and Table 11 contain regression coefficient results analogous to Table **??** and Table **??** for price appreciation over the entire 1980–2010 period.

Table 9: Estimated values of the parameter ζ.

Metro area	$\hat{\zeta}$	Metro area	$\hat{\zeta}$ Metro area	$\hat{\zeta}$		
Carson City, NV	1.987	Florence-Muscle Shoals, AL	0.967	St. Cloud, MN	0.741	Binghamton, NY
Los Angeles-Long Beach-Glendale, CA	1.728	York-Hanover, PA	0.963	Atlantic City-Hammonton, NJ	0.74	Rockingham County-Strafford Co
Medford, OR	1.698	Jackson, MI	0.961	Dutchess County-Putnam County, NY	0.738	Philadelphia, PA
Bremerton-Silverdale, WA	1.669	Evansville, IN-KY	0.958	Rochester, MN	0.737	Fayetteville-Springdale-Rogers, A
Mankato-North Mankato, MN	1.563	Bloomington, IL	0.949	Parkersburg-Vienna, WV	0.736	Duluth, MN-WI
Denver-Aurora-Lakewood, CO	1.498	Ann Arbor, MI	0.949	Wausau, WI	0.732	Grand Rapids-Wyoming, MI
Salt Lake City, UT	1.477	Barnstable Town, MA	0.947	Crestview-Fort Walton Beach-Destin, FL	0.732	Hanford-Corcoran, CA
Santa Maria-Santa Barbara, CA	1.46	Lexington-Fayette, KY	0.945	Charlotte-Concord-Gastonia, NC-SC	0.732	Birmingham-Hoover, AL
Jacksonville, FL	1.443	Morristown, TN	0.943	Fort Worth-Arlington, TX	0.727	Goldsboro, NC
Beckley, WV	1.412	Coeur d'Alene, ID	0.942	Pensacola-Ferry Pass-Brent, FL	0.727	Grand Forks, ND-MN
Fargo, ND-MN	1.408	Sioux Falls, SD	0.942	Hammond, LA	0.725	Iowa City, IA
Lake Havasu City-Kingman, AZ	1.384	Buffalo-Cheektowaga-Niagara Falls, NY	0.937	Fort Lauderdale-Pompano Beach-Deerfield Beach, FL	0.724	Boise City, ID
Roanoke, VA	1.38	Hinesville, GA	0.936	Albany, OR	0.724	Salem, OR
Missoula, MT	1.336	Pueblo, CO	0.935	Midland, MI	0.722	Bridgeport-Stamford-Norwalk, C
Columbia, MO	1.335	Bakersfield, CA	0.93	Amarillo, TX	0.722	Akron, OH
Eugene, OR	1.321	Columbia, SC	0.93	Saginaw, MI	0.72	Burlington, NC
Morgantown, WV	1.306	Lawton, OK	0.93	Dover, DE	0.72	Naples-Immokalee-Marco Island,
Grand Island, NE	1.303	Kingsport-Bristol-Bristol, TN-VA	0.929	Florence, SC	0.719	Fond du Lac, WI
Grants Pass, OR	1.297	Fort Wayne, IN	0.925	Scranton–Wilkes-Barre–Hazleton, PA	0.718	Sheboygan, WI
Ogden-Clearfield, UT	1.289	Lima, OH	0.923	Sierra Vista-Douglas, AZ	0.718	Brownsville-Harlingen, TX
Salinas, CA	1.286	Wilmington, DE-MD-NJ	0.922	Augusta-Richmond County, GA-SC	0.717	Battle Creek, MI
Knoxville, TN	1.286	Hattiesburg, MS	0.917	Pocatello, ID	0.714	Auburn-Opelika, AL
Cambridge-Newton-Framingham, MA	1.285	Boston, MA	0.914	College Station-Bryan, TX	0.714	Weirton-Steubenville, WV-OH
Richmond, VA	1.284	Columbus, OH	0.909	Tampa-St. Petersburg-Clearwater, FL	0.714	Rockford, IL
Logan, UT-ID	1.284	Champaign-Urbana, IL	0.895	Indianapolis-Carmel-Anderson, IN	0.711	North Port-Sarasota-Bradenton,
Seattle-Bellevue-Everett, WA	1.274	Warner Robins, GA	0.895	Utica-Rome, NY	0.71	Watertown-Fort Drum, NY
Portland-Vancouver-Hillsboro, OR-WA	1.269	Reading, PA	0.893	El Centro, CA	0.708	Youngstown-Warren-Boardman,
Cleveland, TN	1.258	Greensboro-High Point, NC	0.892	Odessa, TX	0.708	Corvallis, OR
Minneapolis-St. Paul-Bloomington, MN-WI	1.257	Cape Girardeau, MO-IL	0.889	Trenton, NJ	0.706	Virginia Beach-Norfolk-Newport
Santa Rosa, CA	1.254	Monroe, LA	0.88	Farmington, NM	0.703	Decatur, AL
New York-Jersey City-White Plains, NY-NJ	1.253	Dothan, AL	0.878	Dubuque, IA	0.693	Palm Bay-Melbourne-Titusville,
Lincoln, NE	1.253	Anniston-Oxford-Jacksonville, AL	0.874	Tallahassee, FL	0.686	Manhattan, KS
Olympia-Tumwater, WA	1.25	San Antonio-New Braunfels, TX	0.873	Austin-Round Rock, TX	0.685	Harrisburg-Carlisle, PA
Walla Walla, WA	1.24	Laredo, TX	0.871	Port St. Lucie, FL	0.684	Lakeland-Winter Haven, FL
Jefferson City, MO	1.23	Bloomington, IN	0.871	Sumter, SC	0.682	Michigan City-La Porte, IN
Bismarck, ND	1.227	Yuma, AZ	0.871	Terre Haute, IN	0.678	Madera, CA
Riverside-San Bernardino-Ontario, CA	1.225	Kokomo, IN	0.87	Hickory-Lenoir-Morganton, NC	0.677	Brunswick, GA
Salisbury, MD-DE	1.212	Baltimore-Columbia-Towson, MD	0.866	El Paso, TX	0.677	Manchester-Nashua, NH
Rocky Mount, NC	1.207	Chambersburg-Waynesboro, PA	0.862	Norwich-New London, CT	0.672	Camden, NJ
Chico, CA	1.193	St. Louis, MO-IL	0.859	Savannah, GA	0.67	Springfield, OH
Decatur, IL	1.186	California-Lexington Park, MD	0.858	Kennewick-Richland, WA	0.67	St. Joseph, MO-KS
Springfield, MO	1.185	Harrisonburg, VA	0.857	Raleigh, NC	0.668	Janesville-Beloit, WI
Anaheim-Santa Ana-Irvine, CA	1.17	Lewiston-Auburn, ME	0.857	Billings, MT	0.661	Elgin, IL
Asheville, NC	1.167	Hilton Head Island-Bluffton-Beaufort, SC	0.856	Lancaster, PA	0.66	Provo-Orem, UT
Cheyenne, WY	1.162	Davenport-Moline-Rock Island, IA-IL	0.851	Springfield, MA	0.65	Portland-South Portland, ME
Redding, CA	1.161	Fresno, CA	0.85	Lake Charles, LA	0.648	Merced, CA
Johnson City, TN	1.147	Spokane-Spokane Valley, WA	0.848	Erie, PA	0.644	Bay City, MI
Muskegon, MI	1.147	Albany, GA	0.848	Joplin, MO	0.631	Corpus Christi, TX
San Jose-Sunnyvale-Santa Clara, CA	1.144	Alexandria, LA	0.846	Yuba City, CA	0.629	Appleton, WI
Jackson, TN	1.13	Lewiston, ID-WA	0.843	McAllen-Edinburg-Mission, TX	0.628	Charlottesville, VA

South Bend-Mishawaka, IN-MI	1.127	Cedar Rapids, IA	0.841	Canton-Massillon, OH	0.627	Lebanon, PA
Reno, NV	1.123	Clarksville, TN-KY	0.833	Boulder, CO	0.626	Worcester, MA-CT
Staunton-Waynesboro, VA	1.119	Pittsburgh, PA	0.831	Sebastian-Vero Beach, FL	0.624	Santa Cruz-Watsonville, CA
Athens-Clarke County, GA	1.117	Orlando-Kissimmee-Sanford, FL	0.828	Winchester, VA-WV	0.623	Chicago-Naperville-Arlington He
Gadsden, AL	1.114	Homosassa Springs, FL	0.828	Owensboro, KY	0.616	Monroe, MI
San Diego-Carlsbad, CA	1.113	Flint, MI	0.825	Houma-Thibodaux, LA	0.614	Mobile, AL
Topeka, KS	1.11	Colorado Springs, CO	0.825	New Haven-Milford, CT	0.613	Shreveport-Bossier City, LA
Lubbock, TX	1.101	Ocala, FL	0.823	Dalton, GA	0.61	Silver Spring-Frederick-Rockville
Rome, GA	1.091	Washington-Arlington-Alexandria, DC-VA-MD-WV	0.819	Hagerstown-Martinsburg, MD-WV	0.606	Napa, CA
Des Moines-West Des Moines, IA	1.09	Wichita Falls, TX	0.819	Williamsport, PA	0.606	Nashville-Davidson–Murfreesboro
Kansas City, MO-KS	1.078	Sioux City, IA-NE-SD	0.817	Danville, IL	0.606	Spartanburg, SC
St. George, UT	1.077	Gary, IN	0.816	Bowling Green, KY	0.606	Gulfport-Biloxi-Pascagoula, MS
Valdosta, GA	1.069	Louisville/Jefferson County, KY-IN	0.815	Lafayette-West Lafayette, IN	0.605	Kingston, NY
Greeley, CO	1.068	Rochester, NY	0.813	Waco, TX	0.603	Altoona, PA
Casper, WY	1.065	Waterloo-Cedar Falls, IA	0.813	Ithaca, NY	0.603	Vallejo-Fairfield, CA
Chattanooga, TN-GA	1.064	Ames, IA	0.808	Yakima, WA	0.6	West Palm Beach-Boca Raton-De
Cleveland-Elyria, OH	1.059	Mansfield, OH	0.807	Greenville-Anderson-Mauldin, SC	0.598	New Bern, NC
Grand Junction, CO	1.057	Atlanta-Sandy Springs-Roswell, GA	0.806	Elkhart-Goshen, IN	0.598	Springfield, IL
Macon, GA	1.052	Montgomery County-Bucks County-Chester County, PA	0.806	Bellingham, WA	0.597	Kankakee, IL
Madison, WI	1.051	Panama City, FL	0.803	Muncie, IN	0.596	Syracuse, NY
Milwaukee-Waukesha-West Allis, WI	1.05	Memphis, TN-MS-AR	0.803	Prescott, AZ	0.593	Carbondale-Marion, IL
Jonesboro, AR	1.047	Lake County-Kenosha County, IL-WI	0.801	Tuscaloosa, AL	0.592	Wilmington, NC
Tulsa, OK	1.046	Lawrence, KS	0.796	Midland, TX	0.59	Las Cruces, NM
Las Vegas-Henderson-Paradise, NV	1.046	Punta Gorda, FL	0.796	Longview, WA	0.589	Killeen-Temple, TX
Tucson, AZ	1.045	Glens Falls, NY	0.796	Idaho Falls, ID	0.588	Hot Springs, AR
Phoenix-Mesa-Scottsdale, AZ	1.042	Detroit-Dearborn-Livonia, MI	0.794	Visalia-Porterville, CA	0.586	Ocean City, NJ
Rapid City, SD	1.037	Green Bay, WI	0.794	Dayton, OH	0.582	Miami-Miami Beach-Kendall, FL
Vineland-Bridgeton, NJ	1.034	Columbus, IN	0.789	Cape Coral-Fort Myers, FL	0.58	Blacksburg-Christiansburg-Radfo
San Angelo, TX	1.031	Montgomery, AL	0.788	Beaumont-Port Arthur, TX	0.575	Bloomsburg-Berwick, PA
Durham-Chapel Hill, NC	1.029	Winston-Salem, NC	0.788	Toledo, OH	0.573	New Orleans-Metairie, LA
Fayetteville, NC	1.025	State College, PA	0.788	Bend-Redmond, OR	0.571	Nassau County-Suffolk County, N
Bangor, ME	1.019	Gettysburg, PA	0.784	Eau Claire, WI	0.57	Gainesville, GA
La Crosse-Onalaska, WI-MN	1.019	Stockton-Lodi, CA	0.783	Deltona-Daytona Beach-Ormond Beach, FL	0.569	Huntington-Ashland, WV-KY-OH
Lansing-East Lansing, MI	1.016	Jacksonville, NC	0.781	Pine Bluff, AR	0.567	Wenatchee, WA
Great Falls, MT	1.014	Fort Collins, CO	0.78	Texarkana, TX-AR	0.562	San Francisco-Redwood City-Sou
Columbus, GA-AL	1.012	Tyler, TX	0.778	Little Rock-North Little Rock-Conway, AR	0.562	Johnstown, PA
Fort Smith, AR-OK	1.01	Oklahoma City, OK	0.771	Longview, TX	0.554	Oxnard-Thousand Oaks-Ventura
Oakland-Hayward-Berkeley, CA	1.002	Jackson, MS	0.768	Sebring, FL	0.552	Oshkosh-Neenah, WI
Huntsville, AL	1.001	Charleston-North Charleston, SC	0.767	Niles-Benton Harbor, MI	0.552	San Luis Obispo-Paso Robles-Ar
Newark, NJ-PA	0.992	Warren-Troy-Farmington Hills, MI	0.767	Elizabethtown-Fort Knox, KY	0.551	Providence-Warwick, RI-MA
Peoria, IL	0.987	Kalamazoo-Portage, MI	0.765	Pittsfield, MA	0.551	East Stroudsburg, PA
Tacoma-Lakewood, WA	0.984	Daphne-Fairhope-Foley, AL	0.765	Victoria, TX	0.551	Wheeling, WV-OH
Houston-The Woodlands-Sugar Land, TX	0.983	Omaha-Council Bluffs, NE-IA	0.762	Sherman-Denison, TX	0.549	Elmira, NY
Sacramento–Roseville–Arden-Arcade, CA	0.977	Racine, WI	0.755	Santa Fe, NM	0.541	San Rafael, CA
Hartford-West Hartford-East Hartford, CT	0.97	Albany-Schenectady-Troy, NY	0.755	Burlington-South Burlington, VT	0.54	Cumberland, MD-WV
Lynchburg, VA	0.97	Allentown-Bethlehem-Easton, PA-NJ	0.753	Wichita, KS	0.535	Charleston, WV
Mount Vernon-Anacortes, WA	0.97	Albuquerque, NM	0.751	Modesto, CA	0.532	Gainesville, FL
Cincinnati, OH-KY-IN	0.968	Greenville, NC	0.745	Baton Rouge, LA	0.531	Myrtle Beach-Conway-North My
Dallas-Plano-Irving, TX	0.967	Flagstaff, AZ	0.741	Lafayette, LA	0.53	

Table 10: Regression results for long-period price growth on average and marginal land availability parameters λ and ζ.

	Price growth (decade)			
Bartik	0.0030***	0.0026***	0.0029***	0.0026***
	(0.0005)	(0.0005)	(0.0005)	(0.0005)
Lambda	−0.2502***	−0.2275***	−0.1598**	−0.1532**
	(0.0388)	(0.0398)	(0.0657)	(0.0655)
Zeta	0.2721***	0.2528***	0.1909***	0.1852**
	(0.0638)	(0.0640)	(0.0721)	(0.0719)
Coastal		0.1812**		0.1499*
		(0.0811)		(0.0819)
Slope			0.0153**	0.0133*
			(0.0077)	(0.0078)
Constant	−0.3799***	−0.3330***	−0.3873***	−0.3471***
	(0.0776)	(0.0799)	(0.0783)	(0.0810)
Observations	303	303	302	302
Adjusted R^2	0.2458	0.2557	0.2601	0.2659

Notes: ***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level.

Table 11: Regression results for long-period price growth on average and marginal land availability parameters λ and $zeta$ as well as monocentric price gradient and additional demand controls.

	Price growth (decade)			
Bartik	0.0031***	0.0026***	0.0011	0.0011
	(0.0005)	(0.0005)	(0.0007)	(0.0007)
Lambda	−0.1488**	−0.1432**	−0.1598**	−0.1545**
	(0.0669)	(0.0661)	(0.0655)	(0.0653)
Zeta	0.1704**	0.1635**	0.1901***	0.1829**
	(0.0739)	(0.0731)	(0.0726)	(0.0724)
Coastal		0.2168***		0.1414*
		(0.0795)		(0.0821)
Slope	0.0159*	0.0119	0.0114	0.0095
	(0.0082)	(0.0082)	(0.0081)	(0.0082)
Gradient	0.3674***	0.4057***	0.3613***	0.3872***
	(0.0937)	(0.0937)	(0.0917)	(0.0926)
Immigrant			1.5164***	1.3091**
			(0.5541)	(0.5650)
Degree			3.5984**	3.0569*
			(1.5263)	(1.5528)
Constant	−0.3970***	−0.3317***	−0.3381***	−0.3036***
	(0.0783)	(0.0809)	(0.0793)	(0.0815)
Observations	271	271	271	271
Adjusted R^2	0.3292	0.3451	0.3584	0.3632

Notes: ***Significant at the 1 percent level.
**Significant at the 5 percent level.
*Significant at the 10 percent level.

References

Bartik, Timothy (1991). *Who Benefits From State and Local Economic Development Policies*. Kalamazoo, Mich.: W.E. Upjohn Institute for Employment Research.

Blanchard, Olivier Jean et al. (1992). "Regional evolutions". In: *Brookings papers on economic activity* 1992.1, pp. 1–75.

Bogin, Alexander N, William M Doerner, and William D Larson (2016). *Local house price dynamics: New indices and stylized facts*. Tech. rep.

Building Advisor (Oct. 2013). "Evaluating Slope, Soils, Drainage, and Radon". In: *Buying Land*. URL: `http://www.planning.utah.gov/CriticalLands/Critical%20Lands%20PDFs/toolkit.pdf`.

Capozza, Dennis and Robert Helsley (1990). "The Stochastic City". In: *Journal of Urban Economics* 28, pp. 187–203.

City of Riverside (1998). "Minimum Grading Standards and General Requirements". In: *City of Riverside Municipal Code*. Chap. 17.28. URL: `https://www.riversideca.gov/municode/pdf/17/17-28.pdf`.

Davidoff, Thomas (2014). *Supply Constraints Are Not Valid Instrumental Variables for Home Prices Because They Are Correlated with Many Demand Factors*. Working Paper. Sauder School of Business, University of British Columbia.

Hart, J. (2013). *Nonparametric Smoothing and Lack-of-Fit Tests*. Springer Series in Statistics. Springer New York, pp. 61–62. ISBN: 9781475727227. URL: `https://books.google.com/books?id=dU3jBwAAQBAJ`.

Highland, Lynn M. (2008). "Introduction to Landslide Stabilization and Mitigation". In: *The Landslide Handbook — A Guide to Understanding Landslides*. Chap. Appendix C.

Kolko, Jed (2008). *Did God make San Francisco expensive? Topography, regulation, and housing prices*. Working Paper. Public Policy Institute of California.

Ohio Balanced Growth Program (2014). "Steep Slope Protection". In: *Best Local Land Use Practices*. Chap. 11. URL: `http://balancedgrowth.ohio.gov/Portals/0/BG%20Documents/BLLUP%20Final%20Public%20Review%2010_15_12/Chapter%2011%20Steep%20Slope%20Protection.pdf`.

Rosenberg, Lewis I and Brian E Papurello (2013). *County Guidelines for Engineering Geology Reports*. Guidelines. Planning and Building Department, County of San Luis Obipso. URL: `http://www.marincounty.org/~/media/files/departments/cd/planning/currentplanning/publications/singlefamilydesignguidelines/sfrdg_pt1_bos_final.pdf`.

Rozenfeld, Hernán D and Diego Rybski (2011). "The Area and Population of Cities: New Insights from a Different Perspective on Cities". In: *The American Economic Review* 101, pp. 2205–2225.

Saiz, Albert (2010). "The Geographic Determinants of Housing Supply". In: *The Quarterly Journal of Economics* 125.3, pp. 1253–1296.

Scott, David W (2001). "Parametric statistical modeling by minimum integrated square error". In: *Technometrics* 43.3, pp. 274–285.

The Marin County Community Development Agency (n.d.). *Single-family Residential Design Guidelines*. Guidelines. Marin County Board of Supervisors, URL: http://www.marincounty.org/~/media/files/departments/cd/planning/currentplanning/publications/singlefamilydesignguidelines/sfrdg_pt1_bos_final.pdf.

Utah Governor's Office of Planning and Budget (n.d.). "Steep Slopes". In: *Critical Lands Planning Toolkit for the State of Utah*. P. 14. URL: http://www.planning.utah.gov/CriticalLands/Critical%20Lands%20PDFs/toolkit.pdf.