Cyclical Housing Prices in Flatland

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During the 2000s the most volatile housing markets in the United States were concentrated in Arizona, Florida, Nevada, and noncoastal California. These "Sand States" have sprawling cities surrounded by ample supplies of flat, buildable land. This puzzling combination of highly volatile housing prices and unlimited residential land is consistent with the predictions from this cyclic model of vacant land as an option to build. In the model a monocentric city has a negative rental gradient with development costs that do not depend on the radial distance of its expanding outer edge. All agents are equally informed about the uncertain, mean-reverting, future growth rates of housing demand. In equilibrium all development of rural land occurs during booms at the outer edge. Procyclical changes in land prices produce procyclical changes in housing price-rent ratios, which lead procyclical growth rates of housing rents. Land prices are more volatile than housing prices, which are more volatile than housing rents. During speculative booms housing prices can increase rapidly and exceed construction costs even at the rapidly expanding outer edge. These properties persist with a nearly flat housing price gradient.

Key words: volatile housing prices, price-rent ratios.

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"In Flatland, which occupies the middle of the country, it's easy to build houses. When the demand for houses rises, Flatland metropolitan areas, which don't really have traditional downtowns, just sprawl some more. As a result, housing prices are basically determined by the cost of construction. In Flatland a housing bubble can't even get started." Krugman (2005).

1. Introduction

Paul Krugman's provocative column highlights a question commonly posed by housing economists. How can sprawling cities with relatively few commuters to the core and ample supplies of flat, buildable land on the periphery, have highly volatile housing prices across booms and busts? In these markets the price gradient between the core and periphery is relatively flat and land prices on the periphery are constrained by competition among landowners. In this case, the procyclicality of housing prices is determined largely by the relatively small procyclical volatility of construction costs on the periphery. By this argument sprawling cities on flat land cannot have highly volatile prices over housing cycles.[1]

The above issues were further highlighted by the subsequent housing cycle of 2000-2011. During those years the most volatile housing prices were concentrated in Arizona, Florida, Nevada, and noncoastal California: Davidoff (2013). Housing markets in these four "Sand States" are characterized by sprawling cities surrounded by ample supplies of flat developable land. Early in the decade these metropolitan areas had rapid growth of both employment in residential construction and population, substantial speculation by investors in single-family homes, and ample use of affordable financing, such as hybrid, adjustable-rate mortgages. Later, as housing prices collapsed, foreclosures rose rapidly, eventually exceeding two-thirds of all residential resales in Las Vegas and Phoenix: Olesiuk and Kalser (2009). In a regression across cities, this collapse in prices was increasing in both cumulative construction and price appreciation during the previous boom: Nathanson and Zwick (2015). The procyclical volatility of land prices was also greater than the procyclical volatility of construction costs: Nathanson and Zwick (2015).

As shown in this paper, housing and land prices can be highly volatile across booms and busts in sprawling cities surrounded by endless supplies of flat, buildable land. The basic argument is simple. It starts with two observations. First, flat land located outside the city is distinguished by its radial distance to the outer edge of existing development. Rural land without streets and utilities is often more costly to develop than vacant land at the edge of the city. The additional cost is the developer's share of the total costs of extending streets and utilities to the property from the suburban edge. This cost is increasing in the distance

[1] See, for example, the responses to Shiller (2003) in Himmelberg et al (2005).

between the property and the outer edge. Second, cities generally have negative housing price gradients from centers of employment.[2] In a circular city the price of housing decreases with increasing radial distance from the urban core reflecting the costs of commuting between suburban homes and more centrally located jobs. That decreasing price function can be extended outside the city to identify the implicit price of housing that could be built at the current time, but is not in equilibrium. For the results of the model, it is sufficient that construction costs are nondecreasing in rural radial distance, housing prices are nonincreasing in radial distance, and at least one is strictly monotonic.

Owners of raw, rural land at each radial distance outside the city can choose when to sell their parcels to developers who then entitle the land, finish lots, and build houses. Thereby, rural land comes with an option to build housing. Completed houses in subdivisions are subsequently sold to the public at prices equal to the implicit price of housing at that radial distance. If, as indicated above, the price of housing and its cost of development are weakly monotonic in radial distance from the city's outer edge and at least one is strictly so, then options to build on rural land are exercised optimally only at the outer edge. Current development of more remote rural land is precluded by landowners' optimal reservation prices. In the resulting equilibrium, the current supply of rural lots for new houses is thereby restricted to the buildable share of land at the city's outer edge.

Like any real option, the value of raw, rural land increases in the expected appreciation rate of its underlying asset. In this case, the underlying asset can be viewed simply as a completed house on a finished lot. Landowners optimally exercise their options to develop at a percentage premium over the cost of development that increases in the expected growth rate of housing demand. Suppose that the growth rate of housing demand suddenly increases from one constant value to another. The underlying cause could be more employment opportunities or new, more affordable financing. In this case, landowners rationally expect that the future growth rate of housing prices has also increased. They then defer their options to develop at the outer edge until the housing price at the edge increases to the new, higher price at which they optimally exercise their options. Thereby, owners of land at the outer edge immediately raise their reservation prices when housing demand starts growing at a more rapid rate. Owners of other land also raise their reservation prices because they expect the city will sprawl more rapidly to their more rural properties.

If the growth rate of housing demand can change, informed, rational landowners anticipate this possibility and value accordingly their options to develop. To simplify the problem to its essentials, suppose that the housing market cycles at random times between two states: hot markets with growing aggregate demand for housing and cold markets with constant demand. These transitions are immediately observed by all agents. Also, no information about the timing of the next transition arrives before that transition. Housing rents, by contrast, are determined in a spot market by the current aggregate demand and supply of

[2] Elasticities of the housing price gradient are estimated for Chicago in McMillen (2003).

housing services delivered at different distances from downtown. As a result, rents at each location are continuous in time, changing only as aggregate demand and supply change over time.

Under these conditions landowners' optimal reservation prices are higher in hot markets than cold markets. Land prices and thereby housing prices jump up during transitions from cold to hot markets and down during the reverse transitions back to cold markets. Because rents do not change during instantaneous transitions between markets, procyclical price-rent ratios respond like procyclical prices. Between transitions price-rent ratios remain constant because no agent receives new information about future rents until the next transaction. This separates the procyclical volatility of housing prices into two components: procyclical changes in price-rent ratios followed by procyclical changes in rents. The former, which occur only during transitions between markets, reflect the changing beliefs of both landowners and homeowners about future growth rates of aggregate demand. As such, they anticipate the subsequent differences across hot and cold markets in the realized growth rates of rents. In this case, procyclical changes in price-rent ratios must lead procyclical changes in rents. Also, the intertemporal volatility of housing prices must exceed the intertemporal volatility of housing rents. In the numerical solutions of this paper, the differences are substantial.

This initial model has additional implications. Most importantly, the procyclical volatility of housing prices is less than procyclical volatility of land prices. Also, hot markets must have more speculation in housing with more marginal occupants than cold markets. Both properties reflect the dual value of housing as both a consumer durable and a speculative real asset. At all times investors in housing and land receive the same, perfectly competitive, expected rate of return. For housing, but not land, that total rate of return includes a percentage dividend of perishable housing services valued at the rent-price ratio. With higher rent-price ratios, housing is valued more like a consumer durable and less like a speculative asset. When housing is valued less like a speculative asset, relatively less of its total return comes from changes in its price, including changes during transitions between hot and cold markets. Thereby, housing has in equilibrium less procyclical volatility than land. Because rent-price ratios are countercyclical, the difference can be substantial in cold markets. Counter-cyclical rent-price ratios also require procyclical speculation. Additional properties of equilibrium are identified in Section 3.

More results follow when cold markets have decreasing aggregate demand. With contracting cold markets, also called busts, expanding hot markets have two phases: initial recoveries without construction followed by booms with construction. In this case, land is less valuable during expansions. It appreciates more rapidly during booms and exhibits more procyclical volatility during transitions between contractions and expansions. Price-rent ratios are again procyclical: higher during expansions than contractions. Contractions or busts are more abrupt than booms because booms are followed by busts, whereas busts are followed by recoveries before booms. Also, bigger average booms are associated with bigger average busts because both are dependent on the same parameters. Again, additional results are identified in Section 3.

Other implications are specific to sprawling cities in flatland. With flatter rental gradients from the core to the periphery, sprawling cities have less cyclical housing prices and rents. Nevertheless, rents and especially housing prices can be highly procyclical even in cities with relatively flat gradients. Both rents and prices can rise rapidly during booms with rapid rates of construction, but only inside the city. At the city's expanding outer edge, the unit price of housing during booms always equals the unit cost of construction plus the constant price of land. The unit price of land can be positive even in cities with relatively flat rental gradients, more so in more rapidly sprawling cities.

Construction costs in flatland and elsewhere are also procyclical. During booms when aggregate construction increases, legal entitlements and local factors of production become either more difficult or more costly: Saks (2008) and Nathanson and Zwick (2015). In the final version of the model, unit construction costs and aggregate construction are assumed to grow at proportional rates. Not surprisingly, this retards the rate of suburban sprawl and raises the appreciation rate of housing. In turn, this has multiple effects, including higher and more volatile price-rent ratios, more speculation during booms, and higher unit prices for both land and housing relative to construction costs at the city's expanding outer edge. As a result, housing prices exceed construction costs at the expanding outer edge even in cities with nearly flat rental gradients.

With the latter costs numerical solutions from the model approximate the volatilities observed in the Sand States during the boom and subsequent bust of 2000-2011. In the base case with calibrated parameter values, the average annual growth rate of housing prices is -15.1% during contractions or busts and 11.3% during expansions or, equivalently, recoveries followed by booms, with the annual difference of 26.4%. Also, the expected cumulative construction during booms is 27.5% of the housing stock at the beginning of booms. For metropolitan areas in the Sand States during the 2000s, the corresponding median values were 25.5% and 20%: Davidoff (2013). The elasticity of the rental gradient with respect to commuting distance contributes very little to these results. By contrast, the results depend very much on the relationship between the growth rates of construction costs and aggregate construction.

This paper also makes two methodological contributions to the broader literature on housing cycles. Low-frequency housing cycles are largely ignored in the theoretical literature on real options despite their obvious potential for sharp empirical implications. One problem is technical: generating relatively simple solutions to linked pairs of valuation equations for both housing and land. These differential equations for each state of the market, hot and cold, are linked by stochastic transitions between the two states. This problem is further complicated by another important issue. Price-rent ratios cannot be constant with discrete states distinguished only by the finite growth rates of state variables. In the equilibria of this paper, price-rent ratios must be higher in hot markets than cold markets because the two states, contracting and expanding aggregate demand, are distinguished only by their constant growth rates of demand. Here, the first problem is solved by exploiting a plausible property of the model. Houses are constructed if and only if aggregate demand is greater than their aggregate supply. The second is solved by valuing housing simultaneously as both a consumer durable and a speculative real asset.

The paper is organized as follows. After a brief discussion of the literature in the second section, the model is motivated in the third section. The formal analysis starts with a relatively simple, special case: constant housing demand during cold markets combined with growing demand during hot markets. This initial model is introduced in the fourth section and its equilibrium is identified in the fifth. The main model appears in the sixth section. It has contracting cold markets followed by expanding hot markets with two phases: initial recoveries from the previous bust and subsequent booms. Construction occurs only during booms. Construction costs that grow proportionally with aggregate construction are introduced in the seventh section and incorporated into the numerical calculations of the eighth section. Easy extensions and empirical implications are identified in the subsequent two sections. The major results are summarized in the final section. All derivations appear in the Appendix.

2. Literature

Housing volatility, broadly interpreted to include bubble and cycles, has attracted considerable academic attention. Recent models include Spiegel (2001), Nathanson and Zwick (2015), and Burnside et al (2016). Before the housing boom and bust of the 2000s, that volatility was linked largely to price-inelastic housing supply: Glaeser et al (2008). Inelastic supply can reflect difficult topography, including steep slopes and water, regulatory restrictions on development, or land set aside for public uses: Saiz (2010) and Davidoff (2013).

Housing markets are dynamic. Prices change over time in response to demand and supply that change over time. As a result, predictions about rates of housing appreciation follow naturally from the comparative dynamics of proportional dynamic models. Also, durable housing is developed at locations that change over time as the city expands outward. Much of that construction occurs at or near the expanding outer edge of metropolitan areas: Washington Post (2014) and Boglin, Doerner, and Lawson (2016). Finally, housing prices at or near the outer edge of cities depend on both the cost of construction and the price of land. The former is much less procyclical than the latter: Wheaton and Simonton (2007), Nichols et al (2013), and Nathanson and Zwick (2015).

Housing markets can be volatile. That volatility affects the value of the option to develop vacant land into housing and thereby the procyclical volatility of housing, both prices and supply. Option-pricing models of housing development are largely limited to partial equilibrium with housing prices determined exogenously by geometric Brownian motion: Bulan et al (2009) and the citations therein. The major exception in urban economics is Capozza and Helsley (1990). There, real options are embedded in a circular city with all development at its expanding outer edge. Equilibrium introduces additional complications. Land as an option to build must be priced together with housing at all times. This includes an endogenous price for housing that reflects the differences between periods without construction when housing has excess capacity and periods with construction and no excess capacity. It precludes housing prices modeled as exogenous Brownian motion. Also, Brownian motion is a poor fit for low-frequency housing cycles.

In this model vacant land is developed only at the city's suburban edge, while all rural land is priced as an option to build. Thereby, investors who wish to speculate on future housing prices can purchase rural land beyond the outer edge without competition from developers. This is a simple representation of sprawling metropolitan areas in Sand States. It contrasts with Nathanson and Zwick (2015) where investors compete with developers for a limited supply of rural land beyond the outer edge. As such the later model matches more closely metropolitan areas with redevelopment of infill properties or restrictions on rural development. In the western United States partial examples of the latter include metropolitan areas with urban growth boundaries, like Portland, or Las Vegas with its highly concentrated ownership of developable raw land. Both cities are discussed in Section 9. With short-sale constraints and advantages to owner-occupied housing, optimistic investors can then push up prices of raw land and thereby prices of new houses: Nathanson and Zwick (2015).

[More references]

3. Preview

The model is motivated in this section. The motivation includes a discussion of the critical assumptions, a description of the derivations, and an explanation of the main results.

This cyclic model is stripped to its barest bones. Uncertainty is limited to mean-reverting, randomly timed transitions between two states: cold and hot markets. The two variants of the model are distinguished only by the exogenous growth rate during cold markets of aggregate demand for housing services. In its introductory version, the exogenous component or driver of aggregate demand is constant. In the more realistic, main model, exogenous demand contracts at a constant rate during cold markets. Construction occurs only when aggregate demand expands and only then when the housing market has no excess capacity remaining from the previous contraction. Endogenous housing prices can depend on both the state of the market, hot or cold, the exogenous demand for housing, and the radial distance of the house from the center of the circular city. These variables operate through aggregate demand and supply as described below.

All agents are always fully informed about the current state of the housing market. Perfectly competitive landlords exercise optimally their options to sell their unlimited supplies of rural land to perfectly competitive developers of new homes. Each parcel of vacant land is priced like an option to develop housing. Houses are real assets with rents from tenants or implicit rents for homeowners. Real assets are priced at the expected present value of their future rents or implicit rents of homeowners. Households are distinguished only by their houses, which are distinguished only by their radial locations. Rents are current spot prices at different radial distances for perishable housing services produced by houses functioning as consumer durables. Spot prices for housing services at each radial distance depend the current aggregate demand and supply for housing services at that distance, but not future growth rates of either. At each radial distance throughout the city the resulting aggregate demand for housing services must always equal its aggregate supply.

The model has no behavioral biases, informational asymmetries, capital constraints, or urban growth boundaries. Instead, it relies largely on standard assumptions in the large literature on real options. That includes rational, self-interested behavior by fully informed investors. Novel results about state-dependent land prices follow from the removal of high-frequency Brownian motion and its replacement by low-frequency Poisson shifts between discrete states. Additional results about state-dependent housing prices follow from their decomposition into two components: state-dependent price-rent ratios determined by investors' expectations about future rents versus short-term rents determined in a spot market for perishable housing services.

This problem is unavoidably complex. With stochastic transitions between the two states, hot and cold, housing and land must be valued simultaneously in both states distinguished in both variants of the model by their different growth rates of aggregate demand. Nevertheless, each variant has an explicit, stationary equilibrium with clear empirical implications. This follows from three sets of simplifying assumptions. The first is familiar from the literature on real options. Aggregate demand is isoelastic and the growth rate of its exogenous component is constant. This proportionality in the model makes possible its relatively simple solution. The only equally tractable alternative is an additive model with less realistic assumptions. For example, empirical housing price gradients or, more generally, hedonic pricing functions are commonly specified as log-log or, equivalently, convex power functions.

The second simplification is the sole source of uncertainty: Poisson transitions between two fully observable states. With Poisson transitions, the time to the next transition has a negative exponential distribution that does not depend on the time since the last transition. In this case, investors learn nothing about the timing of the next transition until it occurs. Instead, information arrives only during instantaneous transitions between states. That information about discrete states is immediately observed by all investors and reflected fully in discrete changes of both housing and land prices. By contrast, short-term rents remain constant during instantaneous transitions because spot prices depend only on current demand

and supply that can change only over time. This has the empirical implications identified in the introduction. It also precludes inertia in the pricing of housing relative to land with the implications identified in Section 9.

The third simplification is nonstochastic transitions from recoveries to booms. Booms begin only when recoveries end. Recoveries end only when expanding aggregate demand absorbs the excess supply of housing from the previous boom. This deterministic transition during expanding hot markets differs from the stochastic transitions between contracting cold markets and expanding hot markets–booms to busts to recoveries. It simplifies the main model by restricting it to two states, contracting and expanding aggregate demand, separated by Poisson transitions. Also, the combination of busts and recoveries not only has no construction but also begins and ends with the same exogenous component of demand. Because this matches the initial model with its two states, constant and expanding aggregate demand, the solution to the main model can exploit the relatively simple solution to the initial model.

This simple solution can be sketched as follows. With two states connected by Poisson transitions, the value of either housing or land is determined by a pair of linked differential equations, one each for cold and hot markets. Because these equations are first-order and linear with constant coefficients, the pair can be solved explicitly, but the solution is complex. That complex solution can be simplified significantly by exploiting the special properties of the problem. With constant demand during cold markets, the differential equation for cold markets simplifies to a proportional relationship between hot and cold markets. With unchanging aggregate demand, that demand during cold markets begins and ends with the same value and no construction occurs in the interim. Using this price for cold markets, the remaining differential equation for hot markets is easily solved.

Contracting cold markets followed by deterministic recoveries are much the same. Aggregate demand begins and ends with the same value because recoveries end when aggregate demand returns to its last value during the previous boom. Also, no construction occurs in the interim while the housing market has excess capacity. From the perspective of a previous or subsequent housing boom, that combination of contraction and recovery is like a constant or stagnant cold market with only one exception. The duration of the combination also has a negative exponential distribution, but with a larger mean. Therefore, housing and land have the same values during booms associated with either constant cold markets or busts followed by recoveries, both with the same expected duration. With both housing and land, this solution for booms generates one differential equation for recoveries conditional on the initial price during booms. In turn, it generates another differential equation for busts conditional on the initial price during recoveries. The latter equations also have relatively simple, unique solutions.

With these simplifications the unique equilibria of the model's two variants are identified in two propositions. In the first proposition, the city stagnates during cold markets and sprawls during hot markets. Sprawl is measured by the radial distance at the city's outer edge: constant during cold markets and increasing at a constant rate during hot markets. At all times only during hot markets, land is sold for immediate development only at the expanding outer edge of the city. Both housing and land are priced in both markets at all feasible radial distances for all feasible values of the exogenous component of aggregate demand. Also, landlords' optimal exercise policy of their effective option to develop is identified.

This first equilibrium has the properties identified in the introduction and others. The additional properties further distinguish housing from land. During hot markets housing appreciates less rapidly than land. More rapidly growing demand is associated with not only more rapid housing appreciation but also higher, constant, price-rent ratios in both markets, relatively more so in hot markets. Hot markets with longer expected durations are associated with higher price-rent ratios in both markets, relatively more so in hot markets. Cold markets with longer average durations are associated with lower price-rent ratios in both markets, more so in cold markets. With higher price-rent ratios, investors regard housing less like a consumer durable and more like land, a speculative real asset without perishable housing services. For this reason hot housing markets with their higher price-rent ratios have more speculation and more marginal occupancy than cold markets. Both markets have more speculation and more marginal occupancy with more rapid appreciation during hot markets, longer hot markets, or shorter cold markets.

These results can be explained as follows. In equilibrium the essentially identical investors of the model must be indifferent at all times between buying housing or land at any location inside the city. This requires that all land must always have in both states the same expected appreciation rate equal to the common, constant discount rate of all investors in the model. It also requires that all housing always has for homeowners a total expected rate of return equal to the same discount rate. This total return is the sum of two components: the expected appreciation rate of housing plus a percentage dividend in the form of perishable housing services. That dividend is measured by the rent-price ratio.

Consider next the simple case of constant or stagnant cold markets. During these cold markets homeowners receive perishable housing services, while neither homeowners nor landowners realize any appreciation until the next transition from cold to hot markets. To make investors indifferent between houses and land, landowners must then realize a larger gain during that transition than homeowners. Because the price-multiple during the transition is the reciprocal of the price-multiple during the reverse transition back to cold markets, the price of land must be more volatile than the price of housing during transitions between markets. A similar argument applies to transitions between contractions and expansions, as does an analogous argument about housing appreciation during booms. Therefore, land prices must be more volatile than housing prices.

The main model has additional properties. During contracting cold markets, aggregate demand drops below its historic maximum. The resulting excess supply of housing must be then be absorbed during the subsequent recovery. Recoveries end and booms begin when the excess housing disappears and construction starts again. This second equilibrium is characterized in the second proposition. It has two significant differences from the previous proposition. Housing and land are priced differently during the three phases: contractions or busts, recoveries, and booms. During busts and recoveries, rent-price ratios are also different from both each other and the previous, stagnant cold markets.

Contracting cold markets have additional implications. The combination of contractions or busts followed by recoveries increases the average time between booms. This deepens the crash of housing and land prices during transitions from boom to bust. Both prices decrease by bigger percentages with more rapid contractions or less rapid expansions. Because housing appreciates more rapidly during recoveries without construction than booms with construction, recoveries also have a higher price-rent ratio than booms, which have a higher price-rent ratio than busts. Thereby, buyers value housing most as a speculative investment during recoveries, less in booms, and least in busts.

During periods without construction–busts and subsequent recoveries–housing prices change at rates determined by the price-elasticity of aggregate demand and the growth rates, negative and positive, of its exogenous component. Booms are very different. During booms the rate of housing appreciation equals the city's rate of sprawl multiplied by the elasticity of its housing-price gradient. As this elasticity approaches to zero, the appreciation rate of housing converges to zero. However, that rate of convergence can be slow. It is extremely slow if the total demand by all households for all housing in the city, not just buyers and sellers, is roughly proportional to the city's housing stock. This occurs when existing housing supply or factors correlated with housing supply induce housing demand. It could be associated with cities characterized by less turnover of homes, more established neighborhoods with more mature households, or even more diversified employers in larger cities with more housing.

The latter results are made much stronger by a minor modification of the main model. In that modification the unit costs of construction and aggregate construction are assumed to grow at proportional rates. This has the effects described in the introduction. Because the city sprawls less rapidly and its housing appreciates more rapidly, the numerical solutions in Section 7 match much more closely the data from the Sand States also described in the introduction.

4. Initial Model

A circular city has a central business district with unit radius. All housing is distinguished solely by its radial distance x from the urban center: $1 < x \leq b$. The outer edge b of the city expands over time with the development of new housing. Housing is developed at a constant density, conveniently normalized at one. Development is instantaneous once started. Once finished housing never depreciates or otherwise obsolesces. Also, existing housing is never redeveloped at higher densities. Endogenous density at the edge, buildable topography, and redevelopment inside the city are precluded in this model solely to simplify the analysis. Time both to build and then to sell houses is also ignored for the same reason.

Beyond the outer edge of the city, all land is rural. Rural land can an alternative use with a constant value conveniently normalized at zero. At each radial distance, houses can be constructed only on an exogenous fraction of all land: $0 < \lambda \leq 1$. The remaining land is either nonresidential or unbuildable. This circular city with constant density on residential land inside the outer boundary b has the total housing stock:

$$h = \lambda\pi\left(b^2 - 1\right) \approx \lambda\pi b^2. \tag{1}$$

The error in (1), calculated as a fraction of the city's total area πb^2, disappears rapidly as the city expands outward: $b \to \infty$. Henceforth, that error is ignored under the assumption that the city is large relative to its urban core: $b \gg 1$.

The housing market has two completely observable states: cold and hot. The two states are distinguished only by the growth rate of the exogenous component or driver q of the aggregate demand for housing services. In each state i this exogenous quantity q changes at a constant rate: $\dot{q}/q = \rho_i$ for $i = 0, 1$. In the introductory model, exogenous demand q is constant during cold markets: $\rho_0 = 0$. In the main model, exogenous demand decreases at a constant rate, $\rho_0 < 0$, during cold markets. In both variants of the model, demand grows at a constant rate during hot markets: $\rho_1 > 0$. The initial model has two benefits. It simplifies both the analysis and exposition of the main model.

Over time the market switches randomly between the two states, hot and cold. During the short interval of time Δt, the market switches from state i to the alternative state, $j \neq i$, with the probability: $\alpha_i \Delta t + o(\Delta t)$ for $i, j = 0, 1$. The residual $o(\Delta t)$ represents all terms of smaller order than Δt. With these Poisson shifts between states, the remaining time in state i has at all times an independent negative exponential distribution with the mean $1/\alpha_i$. Consistent with empirical evidence on business cycles, cold markets are shorter on average than hot markets: $0 < \alpha_1 < \alpha_0 < 1$. All agents can always observe the current state. The model has no other uncertainty.

Houses are both consumer durables and real assets. As consumer durables houses produce perishable housing services at a constant rate per unit of time for their occupants. Occupants can be either tenants or homeowners. Because housing is distinguished only by its radial distance, each otherwise identical unit of housing produces one unit of housing services per unit of time. Thereby, the aggregate production or supply of housing services equals the current housing stock, h in (1). Each unit of housing services has a market price equal to the rental rate of one unit of the consumer durable, housing, all measured per unit of time. For owner-occupied homes this rent can be interpreted as the implicit rent of marginal homeowners.

Housing services are priced in a spot market continuously through time. The current spot price or rent at each radial distance, $1 < x \leq b$, depends only on the current aggregate demand and supply of housing services at that radial distance. In turn, that demand and supply depend on the three state variables: the current size of the city measured in (1) by its outer boundary b, the current exogenous quantity q, and the property's radial distance q. This determines the spot rent: $R(b, q, x)$ for $1 < x \leq b$. Because the current spot rent does not depend on future values of the variables, b and q, it does not depend on the state of the market i.

For reasons indicated in the previous section, the model is dynamic, proportional, and stationary. In this case, the spot rent R must be isoelastic everywhere. In other words, the inverse demand for housing services and thereby the aggregate demand for housing services must be isoelastic at all radial distances. Without loss of additional generality, the isoelastic inverse aggregate demand for housing services any radial distance $R(b, q, x)$ can then be decomposed into two components. The first is the isoelastic demand at the expositionally convenient inner residential radius $R(b, q, 1)$. The second is the isoelastic rental gradient over all remaining radial distances: $R(b, q, x)/R(b, q, 1) = x^{-\zeta}$ for all $1 < x \leq b$. With the constant elasticity, $-\infty < -\zeta < 0$, housing rents are decreasing and strictly convex in radial distance x. The indicated independence of the rental gradient from the variables, b and q, is an immediate property of the isoelastic rents R.

Homes and households are distinguished in this model only by their radial distance. In this case, all households must be indifferent in equilibrium between purchasing the same rental services at different radial distances. Their indifference has two effects. It determines the elasticity $-\zeta$ of the radial gradient. It also allows households' aggregate demand for housing services to depend on the rental rate $R(b, q, x)$ at any fixed radial distance, $1 \leq x < b$. Here, that notationally convenient but otherwise arbitrary radial distance is the inner boundary, $x = 1$, with the rental rate $R(b, q, 1)$. Stated alternatively, the aggregate demand for housing services depends on the variables, b and q, only through the rental rate $R(b, q, 1)$ at the inner radial distance: $R(b, q, x) = R(b, q, 1)x^{-\zeta}$ for all $1 \leq x < b$. This generates the isoelastic aggregate demand for housing services: $qR(b, q, 1)^{-\eta}h^{\theta}$ with the housing stock h from (1).. It has the constant rent-elasticity, $-\infty < -\eta < 0$, and the constant size-elasticity, $0 \leq \theta < 1$. The quantity-elasticity is 1 without additional loss of generality because the exogenous quantity q can be replaced by its power function without altering the subsequent results. The size-elasticity θ is motivated below.

Households satisfy their demand for perishable housing services by buying or renting housing. The resulting derived demand for housing as a consumer durable can depend on the aggregate supply or stock of all homes for multiple reasons. In this parsimonious model, the housing supply or stock h summarizes all effects on aggregate demand of population, employment opportunities, net urban amenities, and other omitted factors related the size of the city. It also reflects inertia in the housing market. Households who choose not to move implicitly demand the housing services that their homes supply. In this proportional model the impact of housing supply on housing demand is restricted to the power function h^{θ}. Thereby, aggregate demand for housing services increases proportionally with the size of the city at the constant rate θ. This fraction, $0 \leq \theta < 1$, is closer to 1 if, for example, movers are smaller fractions of the housing stock h. The elasticity θ has an important role in both the numerical calculations and empirical implications.

Rental services are priced in a spot market by the intersection of aggregate demand and supply. With the aggregate supply (1), the isoelastic aggregate demand, and the isoelastic rental gradient, housing has the spot rents:

$$R(b,q,x) = x^{-\zeta}\left[\frac{q}{\left(\lambda\pi b^2\right)^{1-\theta}}\right]^{1/\eta}, \tag{2}$$

for $0 < q < \infty$, and $1 < x \leq b$. As indicated, current rents depend only on current aggregate demand and supply–not future demands or supplies. For this reason spot rents are independent of the state i. When the market switches between its states, the growth rate of rents ρ_i/η changes but the current level of rents R remains unchanged. This rental function can be extended to all rural land beyond the outer boundary of the city: $b < x < \infty$. As such it can be interpreted as the implicit rental rate of rural housing that could be built, but is not in the subsequent equilibrium.

Homes have prices in the market that depend on the aggregate demand and supply of homes and thereby the values of all state variables. In market i each unit of housing has the price $P^i(b,q,x)$. This price is calculated in the subsequent equilibria of both models. Each price has an associated rent-price ratio:

$$r_i = \frac{R(b,q,x)}{P^i(b,q,x)}, \tag{3}$$

for $0 < q < \infty$, and $1 < x \leq b$. In the equilibrium of each model, the rent-price ratio r_i depends only on the state of the market i. More precisely, the restricted rent-price ratio (3) is subsequently shown to be sufficient for a unique equilibrium with weakly contracting cold markets, $\rho_0 \leq 0$.

Housing is also a real asset with net cash inflows in the form of rents or implicit rents. Rents are received by landlords with tenants. Implicit rents that are reflected in prices of owner-occupied housing are received by homeowners. In this minimalist model, all expenses of ownership, mainly maintenance, repairs, and property taxes, are ignored. At all times the price of each home must then equal the expected present value of its future rents:

$$\begin{aligned} P^i(b,q,r) \;&=\; b^{-\delta\Delta t}\big\{R(b,q,r)\,\Delta t + P^i(b,q{+}\Delta q,r) + \\ &\quad \alpha_i\Delta t\big[P^j(b,q{+}\Delta q,r) - P^i(b,q{+}\Delta q,r)\big]\big\} + o(\Delta t), \end{aligned} \tag{4}$$

for $i \neq j \in \{0,1\}$, $\bar{r} \leq r < \infty$ and $0 < q < \infty$. The present value at time t is calculated by discounting the expected future value at time $t + \Delta t$ at the constant rate δ per unit of time.

The first component of this future value is the rent $R(b, q, r)\,\Delta t$ over the short interval of time Δt. The second is the future price conditional on the future quantity $q + \Delta q$ at the future time $t + \Delta t$. The remaining terms in the brackets are the expected change in price of switching from state i to the other state j within the same interval of time Δt. This expectation reflects the sole source of uncertainty in the model: the approximate probability $\alpha_i \Delta t$ of switching during time Δt.

The price of housing appears twice in the model. In (3) it capitalizes implicit rents that clear the spot market for housing services. In (4) it must satisfy the pricing equation conditional on the rents (2). Because the market can have only one price for each combination of state variables, these two prices must be equal. This equality determines the endogenous rent-price ratios r_i in cold and hot markets, $i = 0, 1$. The properties of these ratios have multiple empirical implications.

Perfectly competitive developers buy raw land from landowners, immediately finish lots, build houses, and then sell new homes to owner-occupiers. Thereby, each identical developer incurs with each property at each radial distance, $x > 1$, the constant construction costs in cold and hot markets: $0 < \gamma_0 \leq \gamma_1$. These procyclical constants cover all costs of development measured per property. Constant unit costs at all radial distances simplify the subsequent exposition. Constant costs are sufficient for almost all subsequent results because housing prices are assumed to be decreasing in radial distance x. Higher unit costs beyond the outer boundary are discussed in Section 8.

Perfectly competitive landowners exercise their options to develop by selling their land to developers. With instantaneous development and sale, each identical developer always pays per property the perfectly competitive price: $P^i(b, q, x) - \gamma_i$. Landowners anticipate this price and always time their sales to maximize the market values of their properties. Under this optimal exercise policy, each parcel of rural land on which developers can construct one house has the current market value $V^i(b, q, x)$. This valuation function is derived in the subsequent equilibrium. Like the pricing function, it depends on both aggregate demand and supply, which depend, in turn, on the same four variables: i, q, x, and b.

As specified above, each landowner solves the problem:

$$\begin{aligned} V^i(b, q, x) \;=\; \max\bigl\{&P^i(b, q, x) - \gamma_i,\; b^{-\delta \Delta t}[V^i(b, q{+}\Delta q, x) + \\ &\alpha_i \Delta t\bigl[V^j(b, q{+}\Delta q, x) - V^i(b, q{+}\Delta q, x)\bigr]]\bigr\} + o(\Delta t), \end{aligned} \tag{5}$$

for $i \neq j \in \{0, 1\}$, $b \leq x < \infty$ and $0 < q < \infty$. In (5) the owner chooses the more valuable of two alternatives: exercising the option immediately by selling the land it to a developer or retaining the option for a short interval of time Δt. The latter alternative has the expected present value on the right side of (5). The present value matches the corresponding present value of homes in (4) with one important exception: no rents on land. This expositional simplification focuses attention on the critical distinction here between land as a real asset versus housing as both a real asset and a consumer durable with perishable housing services.

The solution to the above problem determines each landlord's optimal exercise policy. That policy is a stopping rule: the critical exogenous demand $D(x)$ at which the option to develop is exercised by owners of land at radius x. With lesser quantities, $q < D(x)$, properties at radius x are not sold to developers. As indicated, it depends on both the current exogenous demand q and the property's radial distance x. It does not depend on the outer boundary b because unit construction costs γ_1 are constant everywhere. Higher unit costs beyond the outer edge are an easy extension identified in Section 8. The option is always exercised only at the outer boundary if the critical demand D has two properties: $D(b) = q < D(x)$ for all radial distances, $x > b$, and all feasible values, b and q. The equality and inequality respectively insure that development occurs on buildable land at the outer edge and not more remote rural land. These two properties are part of the subsequent equilibrium.

Equilibrium in the housing market has the following components. The rental rate, $R(b, q, x)$ in (2), clears the spot market for perishable housing services. The price of housing $P^i(b, q, x)$ equals its expected present value in (4), conditional on the rental rate (2). The value of land $V^i(b, q, x)$ equals its expected present value in (5), conditional on the optimal development point, $D(x)$ from (5). Finally, development occurs only at the expanding outer edge of the city:

5. Initial Equilibrium

The above equilibrium is characterized in this section. Again, this initial solution has constant aggregate demand during cold markets: $\rho_0 = 0$. As such, it is an introduction to the more complicated, more realistic solution with contracting cold markets in the subsequent section. To simplify the subsequent notation, the dependence of the housing price P^i and land value V^i on the outer boundary b are suppressed henceforth.

First, the previous problem is rewritten as follows. Expand the expected present value on the right side of (4) in Δt; ignore all terms of order $o(\Delta t)$; subtract P^i from both sides of (4); divide by Δt; and let $\Delta t \to 0$. This generates the two differential equations that price housing as a real asset:

$$0 = \rho_i q P^i_q(q, x) - (\alpha_i + \delta - r_i) P^i(q, x) + \alpha_i P^j(q, x), \tag{6}$$

for $i \neq j = 0, 1$. The expected return on the right side reflects the growth rate of housing demand ρ_i in the current state i, the possible transition at rate α_i from state i, the rent-price ratio r_i, and the discounting of those future events at the rate δ. Thereby, investments in housing always have the expected rate of return δ. This total return has two components: an effective dividend at the rate r_i and expected appreciation at the rate $\delta - r_i$.

The valuation equations for land are similar. The above calculation for housing applied to land produces the same differential equations, but without the percentage rents r_i for housing services. Subtracting V^i from both sides of the equation also generates the landlord's gain from trade on the left side of the maximand. Thereby, each landlord solves the problem:

$$0 = \max\left\{P^i(q,x) - \gamma_i - V^i(q,x),\ \rho_i q V^i_q(q,x) - (\alpha_i + \delta)V^i(q,x) + \alpha_i V^j(q,x)\right\}, \tag{7}$$

or $i \neq j = 0, 1$. As indicated, each owner either exercises the option to sell his property to a developer with the resulting gain on the left side of the maximand or defers that exercise and then expects the return on the right side. As a result, optimized investments in raw land must always have in both states i an expected rate of return equal to the discount rate δ.

The valuation equations in (7) have three boundary conditions. Exogenous aggregate demand q grows continuously during a hot market and never during a cold market. In this case, each landlord's optimal exercise policy for developing its land is a stopping rule. At each rural radius $x \geq b$, the landlord sells to developers only during hot markets and only then when the exogenous quantity q first reaches the development point $D(x)$. At this quantity the value of land must equal the price of housing minus the cost of construction in hot markets, $i = 1$, and exceed or equal the corresponding difference in cold markets:

$$V^0[D(x),x] \geq P^0[D(x),x] - \gamma_0, \qquad V^1[D(x),x] = P^1[D(x),x] - \gamma_1, \tag{8}$$

at all feasible radial distances $x > 1$. In hot markets the optimal quantity $D(x)$ must also satisfy the smooth-pasting condition:

$$V^1_q[D(x),x] = P^1_q[D(x),x], \tag{9}$$

for all $x > 1$.

The pairs of differential equations in (6) and (7) are solved as follows. Focus first on housing. Because cold markets have constant aggregate demand, $\rho_0 = 0$, the differential equation for housing in cold markets, (6) with $i = 0$, disappears. It is replaced by a simple proportionality between housing prices in hot and cold markets: (A.1) in the Appendix. With (A.1) the differential equation (6) for hot markets, $i = 1$, does not depend on the corresponding price in cold markets P^0. This single equation has the general solution (A.3). That solution must match the pricing function for housing as capitalized rents below (2). This equality determines the rent-price ratios, r_0 and r_1. Land has the same general solution as housing, but without rents: $r_0 = r_1 = 0$. It, combined with the continuity and smooth-pasting conditions in (8) and (9), generates the value of land in (15) with (16).

In the subsequent equilibrium development occurs only during hot markets and then only at the outer edge of the city. During hot markets development never stops. Because exogenous demand never contracts and housing never depreciates, the city then has an outer radius or boundary $B(q)$ for all feasible exogenous demands q. The outer boundary is constant during cold markets and increasing continuously with q during hot markets. At all times during hot markets, landlords exercise their options to develop at the expanding outer edge of the city. In this stationary equilibrium landlords' optimal quantity $D(x)$ at the expanding outer edge must always equal the current exogenous demand q: $D[B(q)] = q$ for all feasible q. In this case, development is restricted to the outer edge if the optimal quantity D is greater beyond the boundary: $D(x) > q$ for all $x > B(q)$. Both these conditions are satisfied in the subsequent equilibrium.

All properties of the first proposition follow from the above argument. Details appear in the Appendix.

Proposition 1: *With constant cold markets, $\rho_0 = 0 < \rho_1$, the unique housing equilibrium satisfying (1), (2), and (6) through (9) has the following properties. All development occurs at the outer boundary:*

$$B(q) = \left[\frac{q}{(\lambda\pi)^{1-\theta}\,(r_1 p_1)^{\eta}}\right]^{s/\rho_1}, \tag{10}$$

with the associated housing supply from (1). Housing has the unit prices:

$$P^i(q,x) = p_i\left[\frac{B(q)}{x}\right]^{\zeta}, \tag{11}$$

for $i = 0, 1$, with the values at the outer boundary,

$$p_0 = p_1\frac{\alpha_0}{\alpha_0+\delta-r_0}, \qquad p_1 = \frac{\gamma_1 g_L}{g_L-g_H}. \tag{12}$$

These results hold for all radial distances, $0 < x \le B(q)$. Housing also has the rent-price ratios:

$$r_0 = \delta - \frac{\alpha_0 g_H}{\alpha_0+\alpha_1+\delta-g_H}, \qquad r_1 = \frac{\alpha_0 r_0}{\alpha_0+\delta-r_0}. \tag{13}$$

During hot markets, $i = 1$, the prices housing and land grow at the respective rates, g_H and g_L, while the city sprawls at the rate s:

$$g_H = \zeta s, \qquad g_L = \delta\left(1+\frac{\alpha_1}{\alpha_0+\delta}\right), \qquad s = \frac{\rho_1}{\zeta\eta+2(1-\theta)}. \tag{14}$$

Rural land has the unit values:

$$V^i(q,x) = v_i\left[\frac{q}{D(x)}\right]^{g_L/\rho_1}, \tag{15}$$

for $i = 0, 1$, with the values at the outer boundary,

$$v_0 = v_1\frac{\alpha_0}{\alpha_0+\delta}, \qquad v_1 = \frac{\gamma_1 g_H}{g_L-g_H}. \tag{16}$$

Land located at radius x is developed when exogenous aggregate demand q reaches the value:

$$D(x) = (\lambda\pi)^{1-\theta} (r_1 p_1)^{\eta} x^{\rho_1/s}, \tag{17}$$

for $B(q) \leq x < \infty$.

The discussion starts with cold markets. During cold markets, $i = 0$, exogenous demand q is constant; no land is bought by builders; and no new homes are sold to owner-occupiers. In this situation all agents–homeowners, landowners, and developers–effectively wait for the next hot market. That wait has an independent negative exponential distribution with the mean $1/\alpha_0$. Thereby, the wait is independent of both calendar time and all previous history. With the proportionality in the model–constant elasticities and constant growth rates–the equilibrium must then be stationary. At all times during cold markets, each unit of land is then priced in (15) and (16) at the constant expected present value of its future value at the start of the next hot market. These future values are discounted by landowners at the constant rate δ. As a result, landowners always expect in both hot and cold markets a return on land at the constant rate δ.

Rent-price ratios are countercyclical: $0 < r_1 < r_0 < \delta$ in (13). As shown in the Appendix, both ratios, r_0 and r_1, decrease with two parameters: the rate of expansion of aggregate demand during hot markets ρ_1 and the expected duration of hot markets $1/\alpha_1$. Both reductions are more rapid in hot markets. By contrast, the same rent-price ratios increase with the expected duration of cold markets $1/\alpha_0$, less rapidly so for hot markets. Increments in either parameter, ρ_1 or $1/\alpha_1$, increase the procyclical volatility of housing prices, but do not alter the corresponding volatility of land values. In contrast, increments in the remaining parameter $1/\alpha_0$ increase the volatility of land values more than the volatility of housing prices. These volatilities are measured by the respective multiples, P^1/P^0, V^1/V^0, and their reciprocals.

During transitions between cold and hot markets, both housing and land change abruptly in value. When the market switches from cold to hot, houses jump in value less than land: $1 < P^1/P^0 < V^1/V^0$. When the market switches back to cold, houses drop in value less than land: $V^0/V^1 < P^0/P^1 < 1$. Thereby, housing has less procyclical volatility than land. This result follows from (11), (12), (15), and (16). There, it holds because only housing has implicit rents during cold markets: $r_0 > 0$. In other words, housing is valued partly as a consumer durable, whereas vacant land is purely a speculative asset. This result can also be seen from another perspective. If housing had the same or more volatility than land, then landlords could earn strictly higher expected returns in cold markets by selling their land to other investors, buying empty houses, and then renting those houses to tenants.

With minimal restrictions on the parameters, housing prices must be less volatile than land values and more volatile than construction costs during transitions between markets:

$$1 < \gamma_1/\gamma_0 < P^1/P^0 < V^1/V^0. \tag{18}$$

This result requires only that construction costs have less procyclical volatility than the land values in (16). The latter inequality is satisfied with plausible parameters–e.g., $1/\alpha_0 = 3$ years, $\gamma_0 = .85\gamma_1$, and $\delta \geq .06$ per year. Details appear in the Appendix.

Other properties of volatility follow immediately. Big increases in prices during transitions from cold to hot markets are associated with big decreases in prices during transitions back to cold markets. This is necessary because the ratios of housing and land prices during transitions to cold markets are reciprocals of the corresponding ratios during transitions to hot markets. The symmetry is specific to constant demand during cold markets, $\rho_0 = 0$. It does not hold with contracting cold markets, $\rho_0 < 0$.

During hot markets the city sprawls on buildable land. Its outer boundary (10) and thereby its area is decreasing in both the constant fraction of buildable land λ at each radial distance x and the unit price of housing at the outer boundary, p_1 in (12). By contrast, its rate of sprawl, s in (14), depends on neither constant. Instead, cities sprawl more rapidly with more rapid growth of demand ρ_1 or less negative rental gradients, ζ from (11). As indicated, this sprawl also depends on the remaining two elasticities in the model: one each with respect housing rent and city size. This is not surprising. In this dynamic, proportional model, as in others, levels depend on other levels, while rates of change depend elasticities and other rates of change.

Housing appreciation has similar properties during hot markets. Most notably, the price of housing (2) is decreasing in the fraction of buildable land λ through its affect on the outer edge (10), while its growth rate, g_H in (14), is independent of λ. Also, cities with flatter price gradients in (11) and thereby smaller constants ζ have, other things equal, more rapid sprawl in (10) and less rapid housing appreciation in (14). Both housing and land then have lower prices at the outer boundary $B(q)$. As the elasticity of the pricing gradient ζ converges to zero, the appreciation rate of housing, g_H in (14), converges to zero. At the outer boundary, the price of housing, p_1 in (12), then converges to the unit cost of construction γ_1, while the unit value of land, $V^1[q, B(q)]$ in (15), converges to zero. However, this convergence is extremely slow when the aggregate demand for housing is nearly proportional to its supply: $\theta \approx 1$. As a result, cities with relatively flat price gradients can have housing that appreciates rapidly during hot markets with prices at their outer edges that exceed costs of construction. Examples of this slow convergence appear in the numerical calculations.

Finally, owners of rural land beyond the outer edge do not exercise their options to build until the edge expands to their radial distance. At the outer boundary $B(q)$, the exercise quantity (16) at which landowners sell their properties to developers during hot markets always equals the current exogenous aggregate demand: $D[B(q)] = q$ for all $q > 0$. This equality holds because the outer boundary (10) clears the housing market at each radial distance: $B[D(x)] = x$ for all feasible x. Thereby, landowners' optimal exercise quantity D has a simple characterization: $D = B^{-1}$ for all $q > 0$. Beyond the outer boundary $B(q)$, no housing is developed in (16) because the exercise quantity is too large: $D(x) > q$ for all

$x > B(q)$. This inequality holds because the outer boundary (10) always expands with the exogenous demand q: $B' > 0$ for all feasible q. The optimality of no exercise beyond the outer boundary is evidenced by the higher value of rural land during booms (15) than its current value for housing: $V^1(q,x) > v_1$ for $q > D(x)$.

6. Contracting Cold Markets

The previous model with stagnant cold markets is extended in this section to cold markets with contracting demand: $\rho_0 < 0 < \rho_1$. As before, construction occurs only during hot markets. Here, however, construction occurs only when housing also has no excess supply. Housing has no excess supply when and only when the exogenous aggregate demand for housing q equals its historical high or running maximum:

$$\bar{q}_t = \max\left\{q_\tau : 0 \leq \tau \leq t\right\}.$$

With excess supply, $\bar{q}_t - q_t > 0$, no new housing is built even during hot markets. Construction restarts only when exogenous demand q_t returns to its historical high $\bar{q}_t$. The subscript t is omitted below.

This modification of the previous model motivates the following recharacterization of housing markets. In this section stagnant cold markets, $i = 0$, are replaced by contracting cold markets, $i = 0.1$, called contractions or busts. Also, expanding hot markets, $i = 1$, are split into two phases: an initial recovery, $i = 1.0$, when housing still has some excess supply from the previous bust, $\bar{q} - q > 0$, and the subsequent expansion or boom, $i = 1.1$, when housing has no excess supply, $\bar{q} = q$. Busts become recoveries with the approximate probability, $\alpha_{0.1} = \alpha_0$, measured per unit of time, while booms become busts with the corresponding constant probability, $\alpha_{1.1} = \alpha_1$. By contrast, recoveries never switch stochastically to busts: $\alpha_{1.0} = 0$. Instead, recoveries become booms when exogenous demand q first returns to its historical high $\bar{q}$. This deviation from the previous model greatly simplifies the subsequent analysis.

With this modification the main model is solved much like the initial model. Previously, markets switched from cold to hot with no intermediate period of recovery. No recovery was required because demand was constant during cold markets: $\rho_0 = 0$. During cold markets no land was sold to developers and no new homes were built. In this case, land at the beginning of cold markets was priced at its expected present value at the end of the cold markets. The same properties hold for contracting cold markets, $\rho_0 < 0$, combined with their subsequent recoveries. When the recovery ends and the new boom begins, exogenous aggregate demand q again equals its last value during the previous boom $\bar{q}$. Also, no land is sold to developers during the bust and subsequent recovery. Therefore, land at the beginning of busts is priced at the expected present value of its future value at the end of recoveries. The two problems differ in only one detail: their expected durations.

The above correspondence greatly simplifies the valuation of both housing and land as real assets. Focus first on housing. By the above argument It has the same price at the beginning of busts when $q = \bar{q}$ as it does during cold markets with the same expected duration. Previously, each cold market had the expected duration $1/\alpha_0$ at all times during a cold market. At the beginning of a bust, the bust and its subsequent recovery have the longer expected duration $1/\acute{\alpha}_0$ with the new parameter:

$$\acute{\alpha}_0 \equiv \frac{\alpha_0}{1-\rho_0/\rho_1} < \alpha_0. \tag{19}$$

This new duration is derived in the Appendix and discussed below the second proposition. Therefore, the housing price at the beginning of busts is its previous price at the start of cold markets in (12) with one substitution: the transition rate from cold to hot markets α_0 is replaced by $\acute{\alpha}_0$ in (19). The same argument also applies to land.

With these valuations at the beginning of busts, the valuation equations for housing and land in (6) and (6) are solved as follows. Again focus on housing. Represent by $P^{0.1}(q, \bar{q}, x)$, $P^{1.0}(q, \bar{q}, x)$, and $P^{1.1}(\bar{q}, x)$ the respective prices of housing during contractions, recoveries, and booms, for $1 < q \leq \bar{q}$ and $1 < x \leq B(\bar{q})$. The price of housing at the beginning of busts $P^{0.1}(\bar{q}, \bar{q}, x)$ is the previous price at the start of cold markets with the substitution (19). Insert this price into the differential equation into the pricing equation (6) for housing during expansions, $i = 1$, to calculate the price of housing during booms $P^{1.1}$. This is the previous price during hot markets with the same substitution (19). Given $P^{1.1}$, calculate next the price of housing during recoveries $P^{1.0}$ from (6) for hot markets, $i = 1$, with the substitution: $\alpha_{1.0} = 0$ for $\alpha_1 > 0$. The solution is the expected present value of housing during recoveries with known duration. Finally, calculate conditional on $P^{1.0}$ the price of housing during busts P^0 from (6) for cold markets, $i = 0$, with another substitution: $\rho_0 < 0$ for $\rho_0 = 0$. Again, the same calculations apply to land. Details appear in the Appendix.

The unique stationary equilibrium with contracting cold markets is identified below. All references in Proposition 2 to results in the previous proposition include the substitution: α_0 replaced by $\acute{\alpha}_0$. In other words, the parameter α_0 now has its new value $\acute{\alpha}_0$. In turn, this new value affects the values of the rent-price ratios, r_0 and r_1 in (13), and the growth rate of land prices during hot markets, g_L in (14). As before, all results are derived in the Appendix.

Proposition 2: *With contracting cold markets, $\rho_0 < 0 < \rho_1$, and no stochastic transitions from recoveries, $\alpha_{1.0} = 0$, the unique equilibrium satisfying (1), (2), and (6) through (9) has the following properties. All development occurs at the city's outer boundary, $B(\bar{q})$ from (10). Housing has the unit prices (11)* with $0 < q \leq \bar{q}$ *during contractions and recoveries,* $i = 0.1, 1.0$, and $q = \bar{q}$ *during booms,* $i = 1.1$. *During all states,* $i = 0.1, 1.0, 1.1$, *housing has at the outer boundary $B(\bar{q})$ the respective unit prices:*

$$P^{0.1}[q, B(\bar{q})] = \frac{\acute{\alpha}_0 p_{1.1}}{\acute{\alpha}_0 + \delta - r_0} \left(\frac{q}{\bar{q}}\right)^{1/\eta}, \tag{20}$$

$$P^{1.0}[q, B(\bar{q})] = p_{1.1}\left(\frac{q}{\bar{q}}\right)^{1/\eta}, \tag{21}$$

for $1 < q \leq \bar{q}$, and

$$p_{1.1} = P^{1.1}[\bar{q}, B(\bar{q})] = \frac{\gamma_1 g_L}{g_L - g_H}. \tag{22}$$

Housing also has the rent-price ratios:

$$r_{0.1} = \alpha_0 + \delta - \rho_0/\eta, \qquad r_{1.0} = \delta - \rho_1/\eta, \qquad r_{1.1} = r_1. \tag{23}$$

These results hold for all radial distances, $0 < x \leq B(\bar{q})$. Housing prices change at the constant rates: ρ_i/η during contractions and recoveries, $i = 0.1, 1.0$, and g_H in (14) during booms. During contractions, recoveries, and booms, $i = 0, 1.0, 1.1$, rural land has the respective unit values:

$$V^{0.1}(q, \bar{q}, x) = \frac{\acute{\alpha}_0}{\acute{\alpha}_0 + \delta} V^{1.1}(\bar{q}, x)\left(\frac{q}{\bar{q}}\right)^{(\acute{\alpha}_0 + \delta)/\rho_0}, \tag{24}$$

$$V^{1.0}(q, \bar{q}, x) = V^{1.1}(\bar{q}, x)\left(\frac{q}{\bar{q}}\right)^{\delta/\rho_1}, \tag{25}$$

and

$$V^{1.1}(q, x) = \frac{\gamma_1 g_H}{g_L - g_H}\left[\frac{q}{D(x)}\right]^{g_L/\rho_1}, \tag{26}$$

with the exercise quantity (17).

The second equilibrium is similar to the first with some significant differences . Again, housing is less volatile than land during transitions between markets. Again, this is due to housing's dual role as a consumer durable and a real asset with rents. Here, however, land is more volatile during transitions between markets than in the previous model with stagnant cold markets. Land also has less value during expansions with more rapid appreciation during booms. These results follow from (14) and (16) with the longer expected duration of busts plus recoveries $1/\acute{\alpha}_0$ than the previous cold markets with constant demand $1/\acute{\alpha}_0$. For housing the longer duration $1/\acute{\alpha}_0$ also raises the rent-price ratio during booms $r_{1.1}$ relative to its previous value r_1 in the initial model

With contracting cold markets and subsequent recoveries, decreases in value during transitions from boom to bust are also greater in magnitude than the increases during transitions from bust to recovery. For land the respective relative values are $V^{0.1}(\bar{q}, x)/V^{1.1}(\bar{q}, x)$ and $V^{1.0}(q, x)/V^{0.1}(q, x)$. The first ratio is less than the reciprocal of the second ratio because land is less valuable in (25) during recoveries than booms. Nearly identical results for housing follow from (10) and (18) through (20). This contrasts with the symmetric ratios in the previous model without recoveries. Therefore, transitions to busts affect prices more than transitions to recoveries because demand contracts during cold markets.

7. Increasing Costs

In both models construction costs depend only on the state of the market. In fact, construction costs can also depend on aggregate local construction. As local construction expands, more local, progessively less skilled workers must be hired and trained. Other local inputs, like concrete, must be purchased in greater volume. Legal entitlements for lots also become increasingly costly during booms when developers must wait for longer for regulatory approvals. These realistic costs are shown in this section to alter somewhat the previous results. In the next section this change is shown to improve the match between the numerical calcuations and the data on Sand States cited in the introduction.

The main model in previous section has the following properties. New homes n are built only during booms, at which time the growth rate of aggregate demand is constant. When the housing stock h grows, it always grows at a constant rate: $n = \dot{h} \propto h$. With (1) new and existing homes must then grow at the rate: $\dot{n}/n = \dot{h}/h = 2\dot{b}/b = 2s > 0$. The new constant rate of suburban sprawl s is determined below. It can differ from the previous rate, s in (10), derived with the constant unit cost of construction γ_1.

Suppose that unit construction costs and aggregate construction grow at proportional rates: $\dot{c}/c = \nu\dot{n}/n$. The constant of proportionality, $\nu > 0$, is exogenous. During booms unit construction costs must then grow at a constant rate: $\dot{c}/c = 2\nu s$. This generates the isoelastic unit costs during booms.

$$C_1(\bar{q}) = \gamma_1 \bar{q}^{2\nu s/\rho_1}, \tag{27}$$

with the endogenous constant, $s > 0$. With constant unit costs, $\nu = 0$, the isoelastic cost function (27) simplifies to the constant unit cost γ_1 in the main model.

This simple convex cost function has direct and indirect effects on the previous equilibrium. The direct effects are derived in the Appendix and presented below in the third and final proposition. The indirect effects are identified below the proposition.

Proposition 3: *With the isoelastic unit costs,* $\nu > 0$ in *(27), Proposition 2 is modified as follows. The unit price, $p_{1.1}$ in (22), is replaced by the unit price $p_{1.1}\bar{q}^{2\nu s/\rho_1}$. Also, the rates of housing appreciation g_H and suburban sprawl s in (14) have during booms the new constant values:*

$$g_H = (\zeta + 2\nu)s, \qquad s = \frac{\rho_1}{\zeta\eta + 2(1 - \theta + \eta\nu)}. \tag{28}$$

Otherwise, Proposition 2 is unchanged.

The increasing, convex unit costs of construction (27) have two direct effects on housing markets. During booms housing appreciates more rapidly and the city sprawls less rapidly, both in (28) relative to (14). In (28) larger constants of proportionality ν increase the growth rate of housing prices g_H and reduce the rate of sprawl s. The indirect effects are more subtle. More rapid housing appreciation g_H makes land more valuable at the outer edge of the city in (26), which raises the price of housing at the outer edge relative to the cost of construction in (22). More rapid housing appreciation also reduces both rent-price ratios, r_0 and r_1 in (13). This raises the volatility of housing prices and price-rent ratios in (20) and (21) during transitions between states and makes housing more valuable as a speculative asset during booms. Moreover, none of these results disappear as the elasticity of the rental gradient $-\zeta$ approaches zero.

8. Numerical Results

In this section additional implications of the main model with contracting cold markets are presented and used to calculate numerical results. Those results are then matched to housing prices in the Sand States during the decade 2000-09.

Focus first on the transition from peak to trough. The transition has two components. The market switches from boom to bust. This produces the price multiple: $P^{0.1}[\bar{q}, B(\bar{q})]/P^{1.1}[\bar{q}, B(\bar{q})]$ from (20) and (22). Next, housing prices decrease at the constant rate ρ_0/η until the market switches back from bust to recovery. That bust has a negative exponential distribution with the mean $1/\acute{\alpha}_0$. Thereby, housing markets have from peak to trough the expected multiple:

$$\frac{\acute{\alpha}_0}{\acute{\alpha}_0 + \delta - r_0} \frac{\alpha_0}{\alpha_0 - \rho_0/\eta}. \tag{29}$$

The multiple (29) is the product of two price ratios. The first, calculated from (20) with $q = \bar{q}$ and (22), is realized instantaneously during the transition from boom to bust. The second, calculated in (A.16), is realized over time during the subsequent bust. Because both ratios are less than one, the expected percentage change from peak to trough can be very negative. In the subsequent numerical solutions, the first ratio is much more smaller than the second.

The transition from trough to peak is similar. During transitions between bust and recovery, housing prices jump by the multiple: $P^{1.0}[q, B(\bar{q})]/P^{0.1}[\bar{q}, B(\bar{q})]$ from (20) and (21). During the subsequent recoveries and booms, housing prices always increase at the respective rates: ρ_1/η and g_H. The latter growth rate comes from (28). Also, future recoveries and booms have independent negative exponential distributions with the respective means: $-\rho_0/\alpha_0\rho_1$ and $1/\alpha_1$. As a result, recoveries and booms have the analogous expected multiples:

$$\frac{\acute{\alpha}_0 + \delta - r_0}{\acute{\alpha}_0} \frac{\alpha_0}{\alpha_0 + \rho_0/\eta} \quad \text{and} \quad \frac{\alpha_1}{\alpha_1 - g_H}. \tag{30}$$

The first ratio on the left side of (30) is the multiple of prices during the transition from bust to recovery. It is calculated from (20) and (21). The second ratio on the left side is the multiple of prices at any fixed radial distance that is expected during the recovery. The ratio on the right side of (30) is the multiple expected during the subsequent boom. The latter two ratios are (A.15) and (A16), respectively. All ratios are greater than one: finite if $g_H < \alpha_1$ and infinite otherwise. Thereby, the expected percentage change from trough to peak is also positive and potentially large.

The remaining multiple measures the development of housing during booms. The housing stock (1) grows with the outer edge (10) only during booms and then at the constant rate $2s$ from the previous section. Thereby, the housing stock grows from trough to peak by the expected multiple:

$$\frac{\alpha_1}{\alpha_1 - 2s}. \tag{31}$$

This multiple also exceeds one, infinitely so if $2s \geq \alpha_1$.

Using the above results from this and the previous section, housing prices are calculated for booms and busts and displayed in Table 2. The parameters for these calculations are constrained as follows. Consistent with empirical evidence on business cycles, contractions are shorter and sharper on average than expansions: $\alpha_0 > \alpha_1 > 0$ and $\rho_0 < 0 < \rho_1 < |\rho_0|$. During the 2000s the bust lasted three years, while the previous recovery and boom lasted about 7.5 years: Davidoff (2013). This matches the base case in Table 2. In metropolitan Chicago the housing price gradient had estimated elasticities $-\zeta$ between 0 and $-.08$: McMillen (2003). Finally, the long-run price-elasticity for owner-occupied housing $-\eta$ has been reported as -1.2: Anderson et al (1997).

The base case with the above parameter values appears in the first row of Table 2. There, the expected annual growth rate of housing prices for the base case is 11.3% during expansion and -15.1% during contraction. The difference between these two returns of 26.4% matches the median value of 25.5% for metropolitan areas in the Sand States during the 2000s. The cumulative growth of the housing stock expected during booms of 27.5% also approximates the corresponding median value of 20% for the Sand States. Again, the medians are reported in Davidoff (2013). These results require unit construction costs that increase in (27) with aggregate construction: $\nu = 1$ in the base case. With smaller values ν, the suburbs sprawl more rapidly and housing appreciates less rapidly.

The base case also illustrates the greater volatility of housing prices than housing rents. The difference, column two minus column one, is the total appreciation from trough to peak in housing prices that is attributable to changes in the price-rent ratio during transitions between hot and cold markets. That total growth in the price-rent ratio from trough to peak is 31%. The corresponding growth in housing prices between transitions is calculated from columns four through six as 183%. Because rents do not change during transitions between markets and price-rent ratios remain constant between transitions, rents also grow from trough to peak by 183%. This is smaller than the total growth of prices from trough to peak of 272%.

The remaining rows of Table 2 are comparative statics. As indicated, the rate of sprawl s is infinite with a constant unit cost of construction, $\nu = 0$. Also, the elasticity of the rental gradient $-\zeta$ has very little impact on suburban sprawl and almost none on housing appreciation. Housing appreciation is affected much more by absorption of excess housing during recoveries and increasing costs of construction during booms. With larger constants ν, the volatility of price-rent ratios during transitions between expansions and contractions is also important. For these reasons cities with flat housing price gradients that sprawl on flat land can have highly cyclic rates of housing appreciation.

9. Easy Extensions

Four easy extensions are identified in this section: more costly rural development, more complicated topography, heterogeneous housing and land, imperfectly competitive landowners, and limited land. Each extension has additional implications.

Rural Development: In the model the unit costs of development are constant beyond the outer edge of the city: $0 < \gamma_0 \leq \gamma_1$. More realistically, the unit costs increase with rural radial distance $r-b$. Suppose also that the unit cost is homothetic in the outer boundary b: $\gamma_i C(x/b)$ for $i = 0, 1$ and $x \geq b$. In this case, the unit values of land, v_i in (16), are replaced by the corresponding values: $v_i C[x/B(q)]$. This adds a second source of value for rural land in (15), greater for more remote rural land, that precludes its development beyond the outer edge of the city. Thereby, development is restricted to the outer boundary $B(q)$, even in cities with flat rental gradients.

Heterogeneity: In the model households and their housing are distinguished only by radial distance from the urban core. All houses have rents or implicit rents depend only on radial distance. In fact, households and housing are heterogenous. Different homeowners in the same category of homes can have different implicit rents. In this case, the implicit rent of all households in the previous analysis is replaced by the implicit rent of marginal households in the category. Different categories of homes can then be distinguished by their different implicit rents of marginal homeowners. Because buyers search for housing matches much more intensively than renters, owner-occupied homes should have higher implicit rents than otherwise equivalent rental homes. Among owner-occupied homes, distressed properties should have the lowest implicit rents, while custom homes should have the highest implicit rents.

Differences in rent-price ratios or, equivalently, price-rent ratios across categories of housing have multiple implications. From the model housing with higher price-rent ratios has more procyclical volatility during transitions between markets. Thereby, prices should have more procyclical volatility for rental homes than owner-occupied homes. Among owner-occupied homes procyclical volatility should be greatest for distressed homes and least for new homes with customized features, less so for more customized homes.

Land is also homogeneous in the model and heterogeneous in practice. In the model land has no net cash inflows. In practice, net cash inflows can be positive or negative. Parcels with agricultural leases and zoning generally have small, positive cash flows, while parcels without leases have negative cash inflows mostly from property taxes. Cash inflows can be significantly positive for land with leased underground mineral rights and significantly negative for bankrupted building lots with bonded improvements. By the above argument, land with more negative cash inflows has more procyclical volatility in prices.

Landowners: The model is developed for sprawling cities surrounded by flat land with perfectly competitive owners. It is easily extended to the same cities with imperfectly competitive or monopolistic landowners. A monopolistic owner sells its land at the outer edge of the city during booms to perfectly competitive developers at a price that maximizes the expected present value of both its sold land and its remaining land. An imperfectly competitive landowner does the same subject to its correct conjectures about the behavior of its competitors. In a symmetric equilibrium all landowners do the same subject to their correct conjectures that all other landowners behave identically. Otherwise, the previous problem is unchanged.

Not surprisingly, a monopolistic owner sells its land for development at a slower rate and a higher price than perfectly competitive owners. In a symmetric Nash equilibrium, sales are slower and the price is higher with fewer identical landowners. With slower sales and higher prices at the outer edge, housing has higher prices everywhere inside the edge. Also, sprawl is reduced and the remaining rural land is more valuable everywhere beyond the edge. These results are altered if problematic properties can be developed at higher unit costs inside the city or the density of development is endogenous. Depending on details, rates of housing appreciation and sprawl may be reduced during periods with construction.

The extension to a monopolistic owner of peripheral land has an important application in the Sand States: metropolitan Las Vegas. The United States Department of the Interior, Bureau of Land Management (BLM) controls almost all peripheral land around metropolitan Las Vegas not previously sold the public.[3] Under the Southern Nevada Public Land Management Act, which became public law in October 1988, the BLM established a disposal boundary surrounding 67,920 acres of vacant peripheral land adjacent to Las Vegas. Between October 1988 and September 2015, the BLM exchanged or sold 30.8% of that vacant land and reserved or otherwise conveyed another 21.4%. As a result, land inside the disposal boundary was sold to the public at a very slow rate: 0.78% per year. A substantial share of that land is still undeveloped. The public can also nominate for sale land controlled by the BLM outside the disposal boundary. By these metrics Las Vegas effectively has no urban growth boundary. Instead, it is distinguished by its monopolistic ownership of peripheral land suitable for large masterplans.

[3] In 2005 the BLM controlled 90% of all developable land in Clark County, Nevada: Snyder (2000).

Limited Land: The supply of rural land available for residential development can be limited by both topography and legal restrictions on the uses of land. In the model the fraction of buildable at each radial distance from the urban core is an exogenous constant: $0 < \lambda \leq 1$. That constant fraction of radial distance can be generalized to any feasible isoelastic function of radial distance, in which case the stationarity of the previous equilibria are preserved. If the fraction of buildable land decreases more rapidly with radial distance, then suburbs sprawl and housing appreciates more rapidly during booms: Williams et al (2017). In turn, this raises the price-rent ratio during booms.

A different legal restriction on the use of rural land is an urban growth boundary. Formal urban growth boundaries exist in Oregon and elsewhere. Portland's strong boundary, enacted in 1979, has been widely studied: Myung-Jin (2006) and references therein. Farmland in Oregon just outside the edge is priced on average much lower–more than 90% less–than comparable land just inside the edge. Nevertheless, Portland's edge has neither slowed local sprawl nor altered the appreciation rate of its housing. Instead, suburbanization has been pushed north into Clark County, Washington. Investors who wish to speculate on raw residential land around Portland can do so in both Clark County and adjacent counties of Washington.

As illustrated by Portland, even formal urban growth boundaries do not necessarily preclude the behavior predicted by this model and its extension immediately above. With suburban sprawl outside of urban Portland restricted to Washington, the fraction of buildable rural land is smaller. If that fraction is independent of radial distance, then it is represented in the model by a smaller value of the constant, $0 < \lambda \leq 1$. This smaller value expands the outer edge of the urban area in (11). Otherwise, the fraction λ appears nowhere in the previous results. In other words, metropolitan Portland sprawls farther north into Washington and much less everywhere else.

Other rural land in Oregon surrounding Portland's urban growth boundary is much less valuable than adjacent land in Washington but more valuable than more rural land in Oregon. These properties and others are predicted by an easy extension of the model. Rural land located just outside urban Portland is much like more remote rural land without an urban growth boundary. In both cases, the land is not developed until it is approached by urban sprawl and entitled for development. In other words, an urban growth boundary effectively lengthens rural radial distance outside the boundary by a metric that depends on details of the entitlement process. This additional distance reduces the value of the option to develop at each rural radial distance, but little else.

10. Empirical Implications

In this section the empirical implications of the model are summarized in three categories: cyclical prices in flatland, price-rent ratios, and limited buildable land. Additional empirical implications appear in Table 3. The model's major limitations and their empirical implications are identified in the fourth subsection. Existing empirical evidence appears in the last subsection.

Cyclical Prices: Housing prices can be highly procyclical even in sprawling cities surrounded by vast supplies of flat buildable land. Most of that volatility is driven by the even more volatile procyclical pricing of undeveloped land around such cities. This procyclical pricing of land persists even with perfectly competitive landowners who sell their raw land to perfectly competitive developers that subsequently sell finished lots to perfectly competitive home builders. Developers buy raw land located near the suburban edge of the city during periods with growing aggregate demand for housing. Other investors who wish to speculate on raw land, and thereby housing, can purchase more remote rural parcels farther from the urban core.

Housing prices have less procyclical volatility than land prices and more procyclical volatility than either construction costs or rents. The latter difference can be substantial. Rents rise during expansions when the aggregate demand for housing services increases and fall during contractions when demand decreases. During expansions rents rise more rapidly before construction begins. Once construction resumes suburbs sprawl more rapidly in cities on flat land with flatter rental gradients from the core to the periphery. In the same cities rents and thereby housing prices also rise less rapidly. At the expanding outer edge of the city, the constant unit price of housing equals the constant unit cost of construction plus the constant unit price of land. Even in cities with nearly flat rental gradients, the constant unit price of land is positive. Thereby, housing at the expanding outer edge is always priced at a positive premium over its construction costs during periods with rapid construction.

Prices inside the city also depend on the procyclical price of buildable land at the outer edge. In addition, those prices decrease during busts when the aggregate demand for housing decreases and increase during the subsequent recoveries when aggregate demand increases but construction has not yet resumed. Big busts are followed by big recoveries. Both are bigger with more price-elastic demand for housing. Booms begin when construction resumes. During booms housing appreciation does not depend on the price-elasticity of demand. It also depends very little on the elasticity of the rental gradient with respect to commuting distance in sprawling cities with relatively flat rental gradients. Instead, it depends mostly on the elasticity of unit construction costs with respect to aggregate construction. Even in cities with nearly flat rental gradients, housing prices exceed construction costs at the expanding outer edge the suburbs.

Price-Rent Ratios: Price-rent ratios are procyclical: lower during contractions and higher during expansions. Procyclical changes in price-rent ratios lead procyclical growth rates of rents, increasing during transitions from contractions to expansions and decreasing during the reverse transitions from expansions to contractions. During expansions price-rent ratios are higher during initial recoveries and lower during later booms. Recoveries are characterized by excess housing from the previous contraction, little or no construction, and relatively rapid appreciation of housing. The subsequent booms are characterized by no excess housing, rapid construction, and much less rapid appreciation of housing.

Housing prices have the least procyclical volatility in new neighborhoods with construction at the expanding outer edge of the city. There, house prices are driven largely by the procyclical prices of buildable land at the outer edge. This procyclicality determines the procyclicality of price-rent ratios at the outer edge and thereby elsewhere inside the city. That procyclical volatility is largely independent of the elasticity of the rent gradient from the core to the periphery.

Rent-price ratios are countercyclical. With lower rent-price ratios, housing has relatively less value as a consumer durable and relatively more value as a speculative real asset. As a result, speculation in housing must be procyclical. More properties are purchased by investors and rented to tenants in expanding hot markets than contracting cold markets. Hot markets have more speculation when rents grow more rapidly or hot markets last longer on average. Hotter markets are also characterized by lower rent-price ratios and more marginal tenants. In the hottest housing markets, homes held for speculation may not be rented to tenants.

Limited Buildable Land: Constraints on the supply of buildable land come in several forms. If the fraction of buildable land is an exogenous constant, as in the model, then the extent of suburban sprawl and the level of prices are smaller in cities with more buildable land. Neither the rates of suburban sprawl and housing appreciation during booms nor the volatility of housing prices during transitions between contracting and expanding markets can be attributed directly to cross-sectional differences in the constant fractions of buildable land. Instead, any such effects must be attributed to a more subtle source, such as demand-side factors.

Urban growth boundaries with limited coverage, like Portland, induce additional suburban sprawl in adjacent, unrestricted areas, like adjacent counties in Washington, and reduce land prices in restricted areas, like adjacent counties in Oregon. The latter land is priced like more remote rural land surrounding cities without urban growth boundaries. Otherwise, the pricing of land and housing is similar. So too is the behavior by developers and landowners.

Limitations: In the model the housing market cycles between contractions and expansions. The Poisson transitions between these two states are observed immediately by all owners of housing and land. This precludes prior information about the timing of the next transition and asymmetric information both before and after the transition. Together with other assumptions of the model, it eliminates inertia in housing prices. These assumptions are inconsistent with common behavior. For example, buyers' demand for homes depends on financing supported by appraisals based largely on historical comparables.

Inertia in housing prices has multiple implications, some more subtle than others. If professional developers and landowners anticipate expansions before homeowners, then land prices rise rapidly relative to housing prices during transitions from contraction to expansion. This puts considerable stress on home builders–a complaint commonly expressed by their principals and land buyers–and retards their construction of new homes. If the same professionals anticipate contractions before homeowners, land prices fall rapidly relative to housing prices during the reverse transitions from expansion to contraction. As a result, homes are built and sold during the early stages of contractions. If housing prices have sufficient inertia, investors in housing can mimic this behavior by builders. They buy early during expansions and sell early during contractions. The latter can include homeowners who sell and rent early during contractions, thereby inducing inertia in rents.

Evidence: These results are consistent with existing empirical evidence. The evidence of cyclic housing prices in the Sand States is cited in the introduction. Housing prices are less volatile than land prices: Nichols et al (2013) and Nathanson and Zwick (2015). Housing rents have very little volatility: Genesove (2003) and Shimizu et al (2010). Large, luxury homes arguably have lower price-elasticites of demand, higher rent-price ratios, and thereby less procyclical price-volatility than smaller homes, consistent with recent evidence: Liu et al (2016). Finally, more volatile housing prices are associated with more housing speculation in hot housing markets by both homeowners and investors with more marginal tenants. In a recent survey owners who expect to sell within three years attach relatively less value to their house as a consumer durable than owners who expect longer tenancy: Zillow (2016). Marginal home buyers financed by subprime loans and speculative purchases of single-family homes were observed concurrently with highly volatile housing prices in the Sand States during 2001-2013: Mian and Sufi (2009) and Olesiuk and Kalser (2009).

11. Conclusion

During the 2000s metropolitan areas in the United States with the most volatile procyclical housing prices were concentrated in the Sand States: Arizona, Florida, Nevada, and non-coastal California. This is puzzling. In the Sand States many cities have relatively flat housing price gradients with high fractions of flat buildable land beyond their outer boundaries. Such cities should have highly elastic housing supplies and thereby minimal housing cycles.

Surprisingly, this puzzle has an easy resolution. In metropolitan areas the housing rental gradient is decreasing from the urban core to its outer edge. Also, the unit cost of developing rural land is increasing in its distance from the outer edge. In the main model the market cycles randomly between busts or contracting cold markets when demand for housing decreases and expanding hot markets when demand increases. Hot markets are separated into two phases: recoveries until the excess supply of housing from the previous bust is absorbed and booms during which additional demand is supplied by new construction.

Under these circumstances rural land is developed only at the city's expanding outer edge. That development occurs when perfectly competitive landowners exercise their options to sell their land to perfectly competitive developers who immediately finish lots, construct houses, and sell properties to the public.

In the resulting equilibrium land prices are procyclical at the expanding outer edge of the city, increasing during randomly timed transitions from contraction to expansion and decreasing during reverse transitions. Concurrent changes in housing prices and thereby price-rent ratios are also procyclical because all transitions are observed simultaneously by all agents. Between transitions no new information about future rents arrives in the market. As a result, price-rent ratios remain constant while rents change, increasing during expansions and decreasing during contractions. Thereby, changes in price-rent ratios lead changes in rents. Also, prices are more volatile than rents. Finally, prices are less volatile for housing than land because only housing is both a consumer durable and a speculative real asset.

In calibrated numerical solutions, the procyclical volatility of housing prices and construction can match housing statistics for the Sand States during the decade 2000-2011 if unit construction costs and aggregate construction increase at proportional rates. In this case, housing prices exceed construction costs during booms at the expanding outer edge of cities. These results are nearly independent of the rental gradient between the city's core and its periphery.

Table 1: Notation

Functions and variables:

b	Radial distance from city center to outer boundary.
h	Housing stock or aggregate supply.
i	State of housing market.
	Introductory model: cold market, $i = 0$, or hot market, $i = 1$.
	Main model: contraction, $i = 0$, recovery, $i = 1.0$, or boom, $i = 1.1$.
P^i	Pricing function for housing in market i.
p_i	Price of housing at outer edge of city in market i.
q	Exogenous component of aggregate demand for housing services.
$\bar{q}$	Historic high of exogenous aggregate demand.
R	Rental function for housing services.
r_i	Rent-price ratio in market i.
V^i	Valuation function for rural land in market i.
x	Radial distance from city center to property.

Parameters and endogenous constants:

α_i	Transition probability per unit of time from state i to state j.
$\acute{\alpha}_0$	Transformed transition probability (19).
γ_i	Construction cost of one house and finished lot.
δ	Discount rate per unit of time.
$-\zeta$	Elasticity of housing rent with respect to radial distance r.
$-\eta$	Elasticity of aggregate demand for housing services wrt rent.
g_H	Growth rate of housing prices in (14) and (28).
g_L	Growth rate of land value in (14).
θ	Elasticity of aggregate demand with respect to housing stock.
λ	Fraction of buildable land at each radial distance.
ν	Growth rate of unit costs relative to aggregate construction.
ρ_i	Growth rate of exogenous component q of aggregate demand:
	Initial model, $\rho_0 = 0 < \rho_1$; main model, $\rho_{0.1} = \rho_0 < 0 < \rho_1 = \rho_{1.0} = \rho_{1.1}$.
s	Rate of suburban sprawl in (14) and (28).

Table 2: Housing appreciation and supply

	Total housing appreciation					Annualized housing appreciation during		Growth house supply during
	Transitions		Between transitions appreciation during					
	boom to bust	bust to recov	bust	recov	boom	bust	r&b	boom
Base Case	-.127	.146	-.275	.610	.276	-.151	.113	.275
$1/\alpha_0$ Expected length of bust								
2.5	-.121	.138	-.288	.455	.276	-.160	.110	.275
3.5	-.131	.151	-.304	.776	.276	-.144	.116	.275
$1/\alpha_1$ Expected length of boom								
2.5	-.113	.127	-.275	.610	.218	-.146	.113	.216
3.5	-.138	.160	-.275	.610	.333	-.155	.113	.331
$-\zeta$ Elasticity of rent gradient with respect to radial distance								
-.080	-.128	.147	-.275	.610	.278	-.151	.113	.265
-.001	-.127	.146	-.275	.610	.276	-.151	.113	.276
ν Proportional factor in elasticity of unit construction cost								
0.0	-.0043	.0043	-.275	.610	.0074	-.107	.064	∞
2.0	-.137	.159	-.275	.610	.304	-.155	.117	.132
ρ_0 Growth rate of exogenous housing demand during bust								
-.20	-.132	.153	-.336	1.02	.276	-.182	.120	.275
-.10	-.120	,137	-.202	.338	.276	-.116	.109	.275
ρ_1 Growth rate of demand during recovery and boom								
.05	-.070	.075	-.275	.610	.121	-.130	.055	.121
.10	-.180	.220	-.275	.610	.481	-.172	.176	.478

Notes: The percentage growth of housing prices, negative and positive, during transitions from boom to bust and bust to recovery, as calculated in Proposition 2, appear in the first two columns of Table 2. This growth comes entirely from changes in the price-rent ratio. The corresponding growth in columns three through five comes entirely from changes in rents. At any fixed radial distance inside the city, the total percentage appreciation of housing expected during busts comes from columns one and three. During recoveries the corresponding expected return comes from columns two and four. During booms the same total is column five. The corresponding total annual returns anywhere inside the city are column six for busts and column seven for recoveries and booms. The first row is the base case with the parameter values: $1/\alpha_0 = 3$, $1/\alpha_1 = 3$, $\gamma = 1.0$, $\delta = .08$, $\zeta = .01$, $\eta = 1.2$, $\theta = .8$, $\nu = 1.0$, $\rho_0 = -.15$, and $\rho_1 = .10$. In the remaining rows, only the value of the indicated parameter is altered from the base case.

Table 3: Comparative statics for booms with contracting cold markets, $\rho_0 < 0$

Increase in		Appreciation rate of homes g_H	Appreciation rate of land g_L	Housing price at edge $p_{1.1}$	Rent-price ratio $r_{1.1}$
Expected length of					
contraction	$1/\alpha_0$	0	$+$	$-$	$+$
expansion	$1/\alpha_1$	0	$-$	$+$	$-$
Elasticities of					
price wrt radial distance	$-\zeta$	$-$	0	$-$	$+$
aggregate demand wrt price	$-\eta$	$+$	0	$+$	0
Proportional cost	ν	$+$	0	$+$	$-$
Growth rate of exogenous aggregate demand during					
contraction	ρ_0	0	$+$	$-$	$+$
expansion	ρ_1	$+$	$-$	$+$	$-$

Notes: Comparative statics are calculated from Proposition 2 with contracting cold markets or busts: $\rho_0 < 0$. The first two column headings are the growth rates per unit of time of prices for homes and land during booms, g_H and g_L from (14). The third column heading is the price of housing during booms at the expanding outer edge of the city, $p_{1.1}$ from (22). The fourth and final column heading is the rent-price ratio during booms, $r_{1.1}$ from (23). The first two rows are the expected times to the end of the current contraction and expansion, $1/\alpha_0$ and $1/\alpha_1$, calculated from the corresponding probabilities per unit of time of switching from contraction to expansion α_0 and expansion to contraction α_1. The middle two rows are the elasticities of housing price with respect to radial distance $-\zeta$ and aggregate demand with respect to housing price $-\eta$, both above the pricing function for housing (2). The final two rows are the growth rates per unit of time for the exogenous component of aggregate demand during expansions and contractions, ρ_0 and ρ_1.

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Appendix

Proof of Proposition 1: Focus first on housing. With $\rho_0 = 0$, the valuation equation (6) for state, $i = 0$, satisfies

$$P^0(q,x) = \frac{\alpha_0}{\alpha_0 + \delta - r_0} P^1(q,x). \tag{A.1}$$

This generates the left side of (12). Insert (A.1) into the corresponding differential equation for state, $i = 1$, with $\rho_1 > 0$. This yields the differential equation:

$$0 = \rho_1 q P_q^1 - g_H P^1,$$

with the composite constant,

$$g_H \equiv \alpha_1 + \delta - r_1 - \frac{\alpha_0 \alpha_1}{\alpha_0 + \delta - r_0}. \tag{A.2}$$

This differential equation has the general solution:

$$P^1(q,x) = F(x) q^{g_H/\rho_1}, \tag{A.3}$$

with the undetermined factor of proportionality $F(x)$. This factor must make (2) and (A.3) equivalent: $F(x) \propto x^{-\zeta}$.

Because rents (2) do not change during transitions between hot and cold markets, the rent-price ratios, r_0 and r_1, must satisfy the first equality below:

$$\frac{r_1}{r_0} = \frac{P^0(q,x)}{P^1(q,x)} = \frac{\alpha_0}{\alpha_0 + \delta - r_0}. \tag{A.4}$$

The second equality follows from (A.1). Also, the growth rates of housing prices during hot markets, g_H in (14) and (A.2), must be equal:

$$\alpha_1 + \delta - r_1 - \frac{\alpha_0 \alpha_1}{\alpha_0 + \delta - r_0} = g_H = \frac{\zeta \rho_1}{\zeta \eta + 2(1 - \theta)}. \tag{A.5}$$

Together, (A.4) and (A.5) determine the ratios, r_0 and r_1 in (13).

The valuation of land is similar. The equation for $i = 0$ in (6) satisfies (A.1) with $r_0 = 0$ and P^i replaced by V^i for $i = 0, 1$. This generates the left side of (16). Inserting the same substitution for (A.1) into the corresponding differential equation for $i = 1$ in (6) yields the general solution (A.3) with P^1 replaced by V^1 and g_H replaced by g_L in (14). The latter constant is (A.2) without rents: $r_0 = r_1 = 0$.

The above general solution for the value of land must satisfy the continuity condition in (8) and the smooth-pasting condition (9). At the outer boundary b, these two conditions,

$$P^1(q_1, b) = g_H + V^1(q_1, b) \qquad \text{and} \qquad g_H P^1(q_1, b) = g_L V^1(q_1, b),$$

have the solution:

$$P^1(q_1, b) = \frac{\gamma_1 g_L}{g_L - g_H} \qquad \text{and} \qquad V^1(q_1, b) = \frac{\gamma_1 g_H}{g_L - g_H}. \tag{A.6}$$

The housing price, $P^1(b, q, b)$ from (2) and (3), must equal in equilibrium, $b = B(q)$, the unit price p_1 in (11). This equality generates the outer boundary (10). The price (11) then follows from the rental gradient with the constant elasticity $-\zeta$. The unit price, p_1 in (12), comes from in (A.4). The growth rates, g_H and g_L in (14), follow from (10) and (11) for housing and the above general solution for the value of land. The value of land, (15) with (16), comes from the general solution for the value of land and the boundary conditions (A.14). The development point (17) follows from (10) and the argument above (10): $D = B^{-1}$.

Properties below Proposition 1: The rent-price ratios, r_0 and r_1 in (13) and (A.4), have the properties:

$$\frac{p_0}{p_1} = \frac{r_1}{r_0} = \frac{\alpha_0 + \alpha_1 + \delta - g_H}{\alpha_0 + \alpha_1 + \delta}.$$

From (13) and this result , it follows that

$$\frac{\partial r_i}{\partial \alpha_0} < 0, \qquad \frac{\partial r_i}{\partial \alpha_1} > 0, \qquad \frac{\partial r_i}{\partial \rho_1} < 0,$$

for $i = 0, 1$, and

$$\frac{\partial}{\partial \alpha_0}\left(\frac{r_1}{r_0}\right) > 0, \quad \frac{\partial}{\partial \alpha_1}\left(\frac{r_1}{r_0}\right) > 0, \quad \frac{\partial}{\partial \rho_1}\left(\frac{r_1}{r_0}\right) < 0.$$

The two inequalities, $v_0/v_1 < \gamma_0/\gamma_1$, p_0/p_1, insure that

$$0 < \frac{p_0}{p_1} - \frac{v_0}{v_1} \le \frac{\gamma_0 + v_0}{\gamma_1 + v_1} - \frac{v_0}{v_1} = \frac{\gamma_1}{\gamma_1 + v_1}\left(\frac{\gamma_0}{\gamma_1} - \frac{v_0}{v_1}\right),$$

and thereby $p_0/p_1 < \gamma_0/\gamma_1$. This completes the derivation of the volatilities in (18).

Proof of Proposition 2: Focus first on booms. By the argument above Proposition 2, booms replace hot markets, while the combination of busts and recoveries replace cold markets. Booms must then satisfy the valuation equations, (6) and (6) for hot markets, $i = 1$, with the expected duration of cold markets $1/\alpha_0$ replaced by the expected durations of busts combined with subsequent recoveries $1/\acute{\alpha}_0$. This new mean is calculated below.

Suppose that exogenous demand has the value, $q \leq q$, at some time during a bust. If the the remainder of the bust has the duration y_0, exogenous demand then decreases from its current value q to its trough, $\underline{q} = q \exp(\rho_0 y_0)$. During the initial phase of the subsequent recovery with the duration y_1, exogenous demand returns to its previous value, $q = \underline{q} \exp(\rho_1 y_1)$. These two durations must satisfy the constraint: $0 = \rho_0 y_0 + \rho_1 y_1$. Thereby, the remaining bust and subsequent partial recovery must have the random duration or length: $y_0 + y_1 = (1-\rho_0/\rho_1) y_0$. The duration y_0 has a negative exponential distribution with the mean $1/\alpha_0$. Hence, the remaining bust and partial recovery must have a negative exponential distribution with the mean: $1/\acute{\alpha}_0 = (1-\rho_0/\rho_1)/\alpha_0$. This mean also applies at the beginning of the bust to the entire bust and complete recovery.

With the above substitutions, housing and land must have at the beginning of a bust the prices:

$$P^{0.1}(\bar{q}, \bar{q}, x) = \frac{\acute{\alpha}_0}{\acute{\alpha}_0 + \delta - r_0} P^{1.1}(\bar{q}, x), \qquad V^0(\bar{q}, \bar{q}, x) = \frac{\acute{\alpha}_0}{\acute{\alpha}_0 + \delta} V^{1.1}(\bar{q}, x). \tag{A.7}$$

This is (A.1) with $q = \bar{q}$ and α_0 replaced by $\acute{\alpha}_0$ in (19). Also, replace α_0 by $\acute{\alpha}_0$ in both (A.2) and the right side of (14). By the argument below (A.1), the differential equation for housing prices during booms and its analogue for land then have the general solutions:

$$P^{1.1}(\bar{q}, x) = F_H(x) \bar{q}^{g_H/\rho_1}, \qquad V^{1.1}(\bar{q}, x) = F_L(x) \bar{q}^{g_L/\rho_1}, \tag{A.8}$$

with the undetermined factors of proportionality F_H and F_L. Because the two prices of housing, (2) and (A.8), must be equal at all times during booms, their exponents must be equal, as indicated. This determines the rent-price ratios in (A.4) and (A.5). The value of land during booms (26) follows from (A.8).

Consider next housing and land during recoveries. By the argument above the proposition, the valuation equations during recoveries are (6) and the right side (7), both with $i = 1$ and $\alpha_1 = 0$. These differential equations have the unique solutions:

$$P^{1.0}(q, \bar{q}, x) = P^{1.0}(\bar{q}, \bar{q}, x) \left(\frac{q}{\bar{q}}\right)^{(\delta - r_{1.0})/\rho_1}, \qquad V^{1.0}(q, \bar{q}, x) = V^{1.0}(\bar{q}, \bar{q}, x) \left(\frac{q}{\bar{q}}\right)^{\delta/\rho_1}. \tag{A.9}$$

The appreciation rates of housing in (2) and (A.9) must be equal at all times during recoveries: $\rho_1/\eta = \delta - r_{1.0}$. This determines the rent-price ratio during recoveries: $r_{1.0} = \delta - \rho_1/\eta$. The value of land (25) is the right side of (A.9) since $V^{1.0}(\bar{q}, \bar{q}, x) = V^{1.1}(\bar{q}, \bar{q}, x)$.

Finally, focus on housing during busts. For busts the valuation equation is (6) with $i = 0$ and the substitutions: $\rho_0 = 0$ replaced by $\rho_0 < 0$ and P^1 replaced by $P^{1.0}$. Suppose that $\rho_1(\alpha_0 + \delta - r_{0.1}) \neq \rho_0(\delta - r_{1.0})$. This differential equation has the unique solution:

$$P^{0.1}(q, \bar{q}, x) = P^{1.0}(\bar{q}, \bar{q}, x) \left[\left(\frac{\acute{\alpha}_0}{\acute{\alpha}_0 + \delta - r_{0.1}} - \chi \right) \left(\frac{q}{\bar{q}}\right)^{(\alpha_0 + \delta - r_0)/\rho_0} + \chi \left(\frac{q}{\bar{q}}\right)^{(\delta - r_{1.0})/\rho_1} \right], \tag{A.10}$$

with the constant,

$$\chi \equiv \frac{\alpha_0\rho_1}{\rho_1(\alpha_0+\delta-r_{0.1}) - \rho_0(\delta-r_{1.0})}.$$

The two prices, (2) and (A.10), must be equal at all times during busts. With $r_{1.0} = \delta - \rho_1/\eta$ from above, this requires the relationship: $q^{(\alpha_0+\delta-r_{0.1})/\rho_0} \propto q^{(\delta-r_1)/\rho_{1.0}}$ and thereby $\rho_1(\alpha_0+\delta-r_{0.1}) = \rho_0(\delta-r_{1.0})$. This contradiction precludes the solution (A.10) and thereby the inequality above (A.10).

With the equality, $\rho_1(\alpha_0+\delta-r_{0.1}) = \rho_0(\delta-r_{1.0})$, the above valuation equation in (6) has the unique solution:

$$P^{0.1}(q,\bar{q},x) = \frac{\acute{\alpha}_0}{\acute{\alpha}_0+\delta-r_{0.1}} P^{1.0}(q,\bar{q},x)\left(\frac{q}{\bar{q}}\right)^{(\alpha_0+\delta-r_{0.1})/\rho_0}. \tag{A.11}$$

With $q = \bar{q}$ and the notational simplification, $P^{1.0}(\bar{q},\bar{q},x) = P^{1.0}(\bar{q},x)$, this matches (22) and (A.8). Land has the corresponding value with $r_{0.1} = 0$:

$$V^{0.1}(q,\bar{q},x) = \frac{\acute{\alpha}_0}{\acute{\alpha}_0+\delta} V^{1.0}(q,\bar{q},x)\left(\frac{q}{\bar{q}}\right)^{(\alpha_0+\delta)/\rho_0}. \tag{A.12}$$

This value is (24). The two prices of housing, (2) and (A.11), must be equal at all times during busts. This requires the rent-price ratio: $r_{0.1} = \alpha_0+\delta - \rho_0/\eta$ in (23).

The remaining results follow from analogous arguments in the proof of Proposition 1.

Properties below Proposition 2: Both housing and land are more volatile during transitions between markets than in the initial model with contracting cold markets than constant cold markets. For land this follows from two sets of ratios:

$$0 < \frac{V^{0.1}(q,x)}{V^{1.1}(q,x)} = \frac{\acute{\alpha}_0}{\acute{\alpha}_0+\delta} < \frac{\alpha_0}{\alpha_0+\delta} = \frac{V^0(q,x)}{V^1(q,x)} < 1,$$

for transitions from booms to contractions versus hot to cold markets, and

$$\frac{V^{1.0}(q,x)}{V^{0.1}(q,x)} > \frac{\alpha_0+\delta}{\alpha_0} = \frac{V^1(q,x)}{V^0(q,x)} > 1,$$

for contractions to recoveries versus cold to hot. These ratios are calculated from (24) through (26). Similar ratios for housing follow from (10) and (20) through (22).

Proof of Proposition 3: Given the unit cost (27), replace the unit price, $p_{1.1}$ in (22) by $p_{1.1}q^{2\nu s/\rho_1}$ with the undetermined rate of sprawl s. This new price must equal the price, $P^{1.1}(b,\bar{q},b)$ from (2) and (3) with $b = B(\bar{q})$. This equality generates the outer boundary:

$$B(\bar{q}) = \left[\frac{\bar{q}^{1-2\eta\nu s/\rho_1}}{(\lambda\pi)^{1-\theta}(r_{1.1}p_{1.1})^{\eta}}\right]^{1/[\zeta\eta+2(1-\theta)]}.$$

During booms the outer boundary $B(q)$ then grows at the rate:

$$s \;=\; \frac{\dot{b}}{b} \;=\; \frac{\rho_1 - 2\eta\nu s}{\zeta\eta + 2(1-\theta)}.$$

This equality determines the constant rate of sprawl, s in (28).

With the above outer boundary, $B(\bar{q}) \propto \bar{q}^{s/\rho_1}$, the new price from (2) and (3) has the property:

$$P^{1.1}(b, \bar{q}, b) \propto x^{-\zeta} \bar{q}^{(\zeta+2\eta)s/\rho_1}.$$

This generates the new growth rate, g_H in (28). The remainder of the derivation matches the corresponding parts of the proofs to Propositions 1 and 2.

Derivation of (29) through (31): With Poisson switching between hot and cold markets, interarrival times are negative exponential. In this case, the additional expected price multiples inside the city are

$$\alpha_0 \int_0^\infty \exp\left[(\rho_0/\eta - \alpha_0)\, y_0\right] dy_0 = \frac{\alpha_0}{\alpha_0 - \rho_0/\eta}, \tag{A.13}$$

during busts,

$$-\alpha_0 \frac{\rho_1}{\rho_0} \int_0^\infty \exp\left[\left(\rho_1/\eta + \alpha_0 \frac{\rho_1}{\rho_0}\right) y_0\right] dy_0 = \frac{\alpha_0}{\alpha_0 + \rho_0/\eta}, \tag{A.14}$$

during recoveries, and

$$\alpha_1 \int_0^\infty \exp[(\acute{g}_H - \alpha_1)\, y_1]\, dy_1 = \frac{\alpha_1}{\alpha_1 - \acute{g}_H}, \tag{A.15}$$

during booms. The total expected multiple of housing during booms (31) is the last result with $\acute{g}_H$ replaced by $2\acute{\beta}\rho_1$.